GLOBAL PROPERTIES OF CONTINUOUS PIECEWISE-LINEAR VECTOR FIELDS PART II: SIMPLEST SYMMETRIC CASE IN $\Re^2$

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GLOBAL PROPERTIES OF CONTINUOUS PIECEWISE-LINEAR VECTOR FIELDS PART II: SIMPLEST SYMMETRIC CASE IN $\Re^2$. †

Robert Lum AND Leon O. Chua. ††

Abstract

Among nonlinear vector fields, the simplest of which can be studied are those which are continuous and piecewise linear. Among these nonlinear vector fields a large and important subset are those vector fields which are odd symmetric. Associated with these types of vector fields are partitions of the state-space into a finite number of regions. In each region the vector field is linear. On the boundary between regions it is required that the vector field be continuous from both regions in which it is linear. This presentation is devoted to the analysis in two dimensions of the simplest possible types of continuous piecewise linear vector fields with odd symmetry, namely those vector fields possessing a pair of symmetric boundary conditions.

As a practical concern, the analysis will attempt to ask and answer questions raised about the existence of steady-state solutions. Since the local theory of fixed points in a linear vector field is sufficient to determine stability of fixed points in a piecewise linear vector field, most of the steady state behaviour to be studied will be towards limit cycles. The results will present sufficient conditions for the existence, or nonexistence as the case may be, for limit cycles. Particular attention will be paid to the domain of attraction whenever possible.

With these results qualitative statements may be made for piecewise linear models of physical systems which have odd symmetry.

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§0. Introduction.

The determination of limit cycles is of great practical and theorectical importance. The work on Hilbert's 16th problem (a survey paper being that of Lloyd[4]) has shown that even for two dimensions and polynomial vector fields as simple as degree two, it is not even known the maximum number of limit cycles possible. This situation is symptomatic of the present intractability of the determination of limit cycles globally, i.e. in the entire plane $\Re^2$. However, it may be possible in certain cases to give results on the global determination of limit cycles. One such area has arisen from the solution of problems in electrical engineering.

With the advent of computer aided design and the subsequent increase of computer simulations of physical circuits, device modeling has emerged as an increasingly important area of research. In the modeling of electrical and electronic circuits an exemplary case of such work is the paper Chua and Deng[1] "Canonical piecewise linear modeling." In that paper a number of electronic circuits were shown to have concise representations as piecewise linear functions. The connection of one or more of such circuits in feedback naturally creates a dynamical system. If any of the constituent elements has a representation as a nonlinear function, the dynamical system is defined by a nonlinear vector field acting on the state space. For example, dynamic circuits with piecewise linear resistors give rise to such dynamical systems.

Conversely, in two and three dimensions, nonlinear vector fields which are piecewise linear may be emulated by equivalent physical circuits. Such emulation requires the use of piecewise linear resistors, capacitors and inductors.

Once a piecewise linear representation of a circuit has been created, the computer becomes a powerful tool with which to study the original circuit. Computer work with such models has suggested the possibility of proving qualitative results about certain classes of piecewise linear vector fields arising from such modeling.

This paper has been devoted to finding attractors in piecewise linear vector fields which possess odd symmetry about the origin. Section 1 will introduce the basic definitions and concepts to be used, then sections 2 through 9 will present the analysis of continuous piecewise linear vector fields with odd symmetry about the origin.

To conclude this introductory section, some examples of the variety of behaviour possible in a symmetric piecewise linear vector field will be presented, then a summary of the main results will end this section.

Example 1. (Figure 1.) Consider the vector field

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{cases}\begin{bmatrix}3&1\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}, & x<-1.\\ \begin{bmatrix}2&1\\-1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}, & x\leq 1;\\ \begin{bmatrix}3&1\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}1\\2\end{bmatrix}, & 1<x.\end{cases}$$

The only fixed point is at the origin. The fixed point is an unstable focus. By theorem 0.2 there are no limit cycles.

Example 2. (Figure 2.) Consider the vector field

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{cases}\begin{bmatrix}-3&1\\-11&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}-5\\-10\end{bmatrix}, & x<-1.\\ \begin{bmatrix}2&1\\-1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}, & x\leq 1;\\ \begin{bmatrix}-3&1\\-11&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}-5\\-10\end{bmatrix}, & 1<x.\end{cases}$$

The only fixed point is at the origin. The fixed point is an unstable focus. By theorem 0.2 there exists a globally attracting limit cycle.

Example 3. (Figure 3.) Consider the vector field

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{cases}\begin{bmatrix}-2&1\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}-4\\2\end{bmatrix}, & x<-1.\\ \begin{bmatrix}2&1\\-1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}, & x\leq 1;\\ \begin{bmatrix}-2&1\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}-4\\2\end{bmatrix}, & 1<x.\end{cases}$$

There is a fixed point at the origin which is an unstable focus. The other fixed points are saddle points. By theorem 0.3 this is an example of a vector field without any limit cycles.

Example 4. (Figure 4.) Consider the vector field

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{cases}\begin{bmatrix}-2&1\\-4&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}-3\\-6\end{bmatrix}, & x<-1.\\ \begin{bmatrix}1&1\\2&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}, & x\leq 1;\\ \begin{bmatrix}-2&1\\-4&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}-3\\-6\end{bmatrix}, & 1<x.\end{cases}$$

There are three lines of fixed points. Together, these lines form a partition of $\Re^2$ into two distinct regions. By theorem 0.4 there do not exist any limit cycles for this symmetric vector field.

EXAMPLE 5. (Figure 5.) Consider the vector field

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}, & x < -1. \\ \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}, & 1 < x. \end{cases}$$

The only fixed point is at the origin which is a saddle point. By theorem 0.6 there do not exist any limit cycles for this particular symmetric vector field.

EXAMPLE 6. (Figure 6.) Consider the vector field

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}, & x < -1. \\ \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \end{bmatrix}, & 1 < x. \end{cases}$$

The fixed point at the origin is a saddle point. The other fixed points are unstable nodes. By theorem 0.7 this is an example of a vector field without any limit cycles.

EXAMPLE 7. (Figure 7.) Consider the vector field

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & x < -1. \\ \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & 1 < x. \end{cases}$$

The only fixed point is at the origin which is a saddle point. By theorem 0.8, there do not exist limit cycles for this symmetric vector field.

EXAMPLE 8. (Figure 8.) Consider the vector field

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}, & x < -1. \\ \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, & x \leq 1; \\ \begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}, & 1 < x. \end{cases}$$

The only fixed point is at the origin which is an unstable node. By theorem 0.9 the symmetric vector field does not have any limit cycles.

Summary of main results.

The results in sections 2-9 may best be summarised in the following set of theorems that collect the main points of those sections.

Conjecture 0.1. *The symmetric vector field ξ (see definition 1.2) does not admit semi-stable annuli. If the symmetric vector field ξ admits a limit annulus (see definition 1.9), then the annulus is either an attracting limit cycle or a repelling limit cycle.*

Note that for the asymmetric piecewise linear vector field in [3] with one boundary condition which is obtained from ξ with the same defining constants, conjecture 0.1 implies that the associated vector field does not have any semi-stable annuli and that all limit annuli are either attracting limit cycles or repelling limit cycles. Under conjecture 0.1, the following results may be proved to hold true:

Theorem 0.2. *Let $0 < b, 0 < a + d, 0 < ad - bc, 0 < ad - bc + dk - bl$. Let (x, y) be the primary induced fixed point of the symmetric vector field with defining constants a, b, c, d, k, l. Define*

$$\begin{aligned} X_1(x, y) &= y - \frac{1}{b}(-d(1 - x) - a) \\ &= \frac{1}{b}(a + k + d)(1 - x). \end{aligned}$$

Consider the symmetric vector field ξ with defining constants a, b, c, d, k, l. If $X_1(x, y) < 0$ then ξ has an attracting limit cycle. If $0 \leq X_1(x, y)$ then ξ does not have any limit cycles.

PROOF. See theorems 3.5, 3.7, 4.2, 4.3, 3.2, 4.1. ∎

Theorem 0.3. *Let $0 < b, 0 < a+d, 0 < ad-bc, ad-bc+dk-bl < 0$. Let (x, y) be the primary induced fixed point of the symmetric vector field with defining constants a, b, c, d, kl. If $(a + d)^2/4 < ad - bc$ define*

$$X_2(x, y) = y - \overline{\chi}_2(\overline{\chi_1}^{-1}(x))$$

where $\overline{\chi}(y) = (\overline{\chi}_1(y), \overline{\chi}_2(y))$ is given in lemma 3.10. If $0 < ad - bc \leq (a + d)^2/4$ define

$$X_2(x, y) = y - \tilde{\chi}_2(\tilde{\chi}_1^{-1}(x))$$

where $\tilde{\chi}(y) = (\tilde{\chi}_1(y), \tilde{\chi}_2(y))$ is given in lemma 4.5. If $X_2(x, y) < 0$ then ξ does not have any limit cycles. If $0 < X_2(x, y)$ then ξ has an attracting limit cycle..

PROOF. See theorems 3.9, 3.15, 4.4, 4.8, 3.10, and 4.9. ∎

Theorem 0.4. *Let $0 < b, 0 < a + d, 0 = ad - bc, dk - bl \leq (a + k + d)^2/4$. The symmetric vector field ξ with defining constants a, b, c, d, k, l does not have any limit cycles.*

PROOF. See propositions 6.1 and 6.2. ∎

Theorem 0.5. *Let $0 < b, 0 < a + d, 0 = ad - bc, (a + k + d)^2/4 < dk - bl$. If $a + k + d < 0$ then the symmetric vector field ξ with defining constants a, b, c, d, k, l has a unique attracting limit cycle. If $0 \leq a + k + d$ then the symmetric vector field ξ with defining constants a, b, c, d, k, l does not have any limit cycles.*

PROOF. See propositions 6.3 and 6.4. ∎

Theorem 0.6. *Let $0 < b, 0 < a + d, ad - bc < 0, ad - bc + dk - bl \leq 0$. The symmetric vector field ξ with defining constants a, b, c, d, k, l does not have any limit cycles.*

PROOF. See proposition 7.1 and 7.2. ∎

Theorem 0.7. *Let $0 < b, 0 < a + d, ad - bc < 0, 0 < ad - bc + dk - bl$. If the symmetric vector field ξ with defining constants a, b, c, d, k, l does not have any homoclinic orbits then either (i) ξ does not have any limit cycles or, (ii) ξ has an attracting limit cycle and a pair of repelling limit cycles.*

PROOF. See proposition 7.3, theorems 7.5,7.8 and 7.9. ∎

Theorem 0.8. *Let $0 < b, 0 = a+d$. The symmetric vector field ξ with defining constants a, b, c, d, k, l does not have any limit cycles.*

PROOF. See proposition 8.1 and theorem 8.2. ∎

Theorem 0.9. *Let $0 = b$. The symmetric vector field ξ with defining constants a, b, c, d, k, l does not have any limit cycles.*

PROOF. See proposition 9.1. ∎

§1. Definitions.

In this section the basic definitions of the nonlinear vector fields to be studied are presented. As all the work to be presented lies in the plane, it will be taken that all vectors lie in $\Re^2$.

Definition 1.1. $\mathbf{L}$ is a linear † vector field $\Leftrightarrow$ there exists constants a, b, c, d, e, f such that

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} e \\ f \end{bmatrix}.$$

Definition 1.2. ξ is a symmetric continuous piecewise linear vector field $\Leftrightarrow$ there exists constants a, b, c, d, k, l with either $k \neq 0$ or $l \neq 0$, and

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}, & x < -1; \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, & -1 \leq x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x. \end{cases}$$

The usage of symmetric vector field will mean a symmetric continuous piecewise linear vector field.

A symmetric vector field is linear in each of the regions $\{(x,y) : x \leq -1\}, \{(x,y) : -1 < x < 1\}, \{(x,y) : 1 \leq x\}$. As either k or l is nonzero, the symmetric vector field is nonlinear. Furthermore, symmetric vector fields have the property that $\xi(-\mathbf{x}) = -\xi(\mathbf{x})$ so that as functions, they possess odd symmetry about the origin.

Definition 1.3. For the symmetric vector field ξ the function $\phi(t,(x_0,y_0))$ will denote the solution to $\phi(t,(x_0,y_0))' = \xi(\phi(t,(x_0,y_0))), \phi(0,(x_0,y_0)) = (x_0,y_0)$.

Definition 1.4. The point (x_0, y_0) is called a periodic point if there is a $0 < t_0 < \infty$ for which $\phi(t_0,(x_0,y_0)) = (x_0,y_0)$. The set $\{\phi(t,(x_0,y_0)) : 0 < t \leq t_0\}$ is called a cycle.

Definition 1.5. Let (x_0, y_0) be a point on a cycle. Consider a local transversal Σ through (x_0, y_0) and Poincare map $\mathbf{P} : \Sigma \to \Sigma$. If the point (x_0, y_0) is attracting (respectively repelling) for the map $\mathbf{P}$ then the cycle is said to be attracting (respectively repelling). If it is attracting from one side in positive time and repelling from the other side in positive time then the point is said to be semi-stable.

Definition 1.6. A limit cycle is a cycle that is either attracting, repelling or semi-stable. Hence, a cycle is a limit cycle if and only if it is isolated.

† An equivalent name is affine.

Definition 1.7. Define an ordering on the set of cycles by $C_1 \prec C_2$ if the cycle C_1 lies in the interior of the cycle C_2 and there are no fixed points in the region bounded by C_1 and C_2. Let $\{\cdots \prec C_{-1} \prec C_0 \prec C_1 \prec \cdots\}$ be a maximal chain of cycles bounded below and above by the cycles $C_{-\infty}, C_{\infty}$. The pair $(C_{-\infty}, C_{\infty})$ is an annulus with boundary cycles $C_{-\infty}, C_{\infty}$. The annulus will be identified with the closed region between its boundary cycles.

Definition 1.8. Let N be a set and ϕ be the solution to a symmetric vector field, then $\phi(t, N)$ is the set given by $\phi(t, N) = \{\phi(t, (x, y)) : (x, y) \in N\}$.

Definition 1.9. An attracting annulus A has a neighbourhood $N(A \subset N)$, such that for non-negative times $0 \le t_0 < t_1, A \subset \phi(t_1, N) \subset \phi(t_0, N), A = \cap_{t=0}^{\infty}\phi(t, N)$. A repelling annulus is an annulus which is attracting in reverse time. If the annulus is attracting from one side in positive time and repelling from the other side in positive time then it is said to be semi-stable. A limit annulus is an annulus that is either attracting or repelling.

§2. Simplifying assumptions for symmetric vector fields.

The following two propositions will allow some mild restrictions to be put on the defining constants for a symmetric vector field. This will allow the analysis of representative examples of symmetric vector fields.

Proposition 2.1. *Let ξ_1, ξ_2 be symmetric vector fields with defining constants a, b, c, d, k, l and $-a$,$-b$,$-c$,$-d$,$-k$,$-l$ respectively. Then the respective solutions $\phi_1(t, (x_0, y_0))$ and $\phi_2(t, (x_0, y_0))$ are related by*

$$\phi_1(t, (x_0, y_0)) = \phi_2(-t, (x_0, y_0)).$$

Proof. Let $\phi_1(t, (x_0, y_0)) = (x_1(t), y_1(t)), \phi_2(t, (x_0, y_0)) = (x_2(t), y_2(t))$.

(i) :For $x < -1$ and using $\phi_2(t, (x_0, y_0))' = \xi_2(\phi_2(t, (x_0, y_0)))$,

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} -a-k & -b \\ -c-l & -d \end{bmatrix} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} -k \\ -l \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_2(-t) \\ y_2(-t) \end{bmatrix}' = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x_2(-t) \\ y_2(-t) \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}.$$

By uniqueness of solutions, $\phi_1(t, (x_0, y_0)) = \phi_2(-t, (x_0, y_0))$.

(ii) :For $-1 \le x \le 1$ and using $\phi_2(t, (x_0, y_0))' = \xi_2(\phi_2(t, (x_0, y_0)))$,

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_2(-t) \\ y_2(-t) \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_2(-t) \\ y_2(-t) \end{bmatrix}.$$

By uniqueness of solutions, $\phi_1(t,(x_0,y_0)) = \phi_2(-t,(x_0,y_0))$.

(*iii*) :For $1 < x$ and using $\phi_2(t,(x_0,y_0))' = \xi_2(\phi_2(t,(x_0,y_0)))$,

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} -a-k & -b \\ -c-l & -d \end{bmatrix} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} -k \\ -l \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_2(-t) \\ y_2(-t) \end{bmatrix}' = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x_2(-t) \\ y_2(-t) \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

By uniqueness of solutions, $\phi_1(t,(x_0,y_0)) = \phi_2(-t,(x_0,y_0))$. ∎

Proposition 2.2. *Let ξ_1, ξ_2 be a symmetric vector fields with defining constants a, b, c, d, k, l and $a, -b, -c, d, k, -l$ respectively. With the respective solutions being $\phi_1(t,(x_0,y_0)) = (x_1(t), y_1(t))$ and $\phi_2(t,(x_0,y_0)) = (x_2(t), y_2(t))$, then $(x_1(t), y_1(t)) = (x_2(t), -y_2(t))$.*

Proof. (*i*) :For $x < -1$ and using $\phi_2(t,(x_0,y_0))' = \xi_2(\phi_2(t,(x_0,y_0)))$,

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} k \\ -l \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_2(t) \\ -y_2(t) \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_2(t) \\ -y_2(t) \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}.$$

By uniqueness of solutions $(x_1(t), y_1(t)) = (x_2(t), -y_2(t))$.

(*ii*) :For $x \leq 1$ and using $\phi_2(t,(x_0,y_0))' = \xi_2(\phi_2(t,(x_0,y_0)))$,

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_2(t) \\ -y_2(t) \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_2(t) \\ -y_2(t) \end{bmatrix}.$$

By uniqueness of solutions $(x_1(t), y_1(t)) = (x_2(t), -y_2(t))$.

(*iii*) :For $1 < x$ and using $\phi_2(t,(x_0,y_0))' = \xi_2(\phi_2(t,(x_0,y_0)))$,

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} k \\ -l \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x_2(t) \\ -y_2(t) \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_2(t) \\ -y_2(t) \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

By uniqueness of solutions $(x_1(t), y_1(t)) = (x_2(t), -y_2(t))$. ∎

By proposition 2.1 it may be assumed that $0 \leq a + d$. Using proposition 2.2 it may further be taken that $0 \leq b$. Note that proposition 2.2 leaves the value of $a + d$ unaffected.

The dynamics of ξ to the right of $x \equiv 1$ is determined by the linear vector field

$$\mathbf{L}_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

Similarly, the dynamics of ξ to the left of $x \equiv -1$ is determined by the linear vector field

$$\mathbf{L}_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}.$$

The fixed points of $\mathbf{L}_1, \mathbf{L}_2$ have special names.

Definition 2.3. Let ξ be a symmetric vector field with $ad - bc + dk - bl \neq 0$. The point (x_p, y_p) is called the primary induced fixed point of ξ if

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x_p \\ y_p \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

The point (x_s, y_s) is called the secondary induced fixed point of ξ if

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x_s \\ y_s \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}.$$

It is clear that if (x_p, y_p) is the primary induced fixed point of ξ then $(-x_p, -y_p)$ is the secondary induced fixed point of ξ and conversely. The use of primary and secondary fixed points is not to imply a preference for primary fixed points over secondary fixed points.

The relationship of the primary induced fixed point of the symmetric 2-boundary vector field ξ, and the induced fixed point of the 1-boundary piecewise linear vector field (as introduced in [3]) with the same defining constants is immediate. The relationship of the secondary induced fixed point of the symmetric vector field and the induced fixed point of the corresponding piecewise linear vector field is not as immediate.

Lemma 2.4. *There is a homeomorphism $m(x,y) = (x,y)$ between the primary induced fixed point of the symmetric vector field ξ and the induced fixed point of the piecewise linear vector field with the same defining constants a, b, c, d, k, l.*

PROOF. The primary induced fixed point and the induced fixed point, when they exist, are both given by the same formula

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc + dk - bl} \begin{bmatrix} dk - bl \\ -ck + al \end{bmatrix}. \qquad \blacksquare$$

This allows the following corollary to be stated.

Corollary 2.5. *For fixed $a, d, c, b, ad - bc \neq 0$ there exists a homeomorphism $\overline{h}(k,l) = (x,y)$ from the set of parameter values k, l satisfying $ad - bc + dk - bl \neq 0$ to the set of primary induced fixed points (x,y) with $x \neq 1$.*

PROOF. Let $\overline{h}(k,l) = m^{-1}(h(k,l))$ where h is the function defined in theorem 2.6 of [3] and m is the function defined in lemma 2.4. $\blacksquare$

The dynamics of the symmetric vector field to the left of $x \equiv -1$ and to the right of $x \equiv 1$ are determined by the eigenvalues of the defining matrices in the linear vector fields $\mathbf{L}_1, \mathbf{L}_2$.

Definition 2.6. The eigenvalues at the primary induced fixed point (respectively the secondary induced fixed point) are the eigenvalues of the matrix

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}.$$

Lemma 2.7. *Let ξ be a symmetric vector field with $ad-bc \neq 0, ad-bc+dk-bl \neq 0$. The product of the eigenvalues at the primary induced fixed point (respectively the secondary induced fixed point) is $(ad-bc)/(1-x_p)$.*

PROOF. Lemma 2.4 and corollary 2.8 of [3]. ∎

The lines $x \equiv -1$ and $x \equiv 1$ are the dividing boundaries of the linear regions and also have significance in symmetric vector fields.

Proposition 2.8. *Let ξ be a symmetric vector field with $0 < b$. The line $x \equiv 1$ is transversal to ξ at all points except $(1, y^*) = (1, -a/b)$. For points $(1, y)$ with $y^* < y$ the vectors point to the right, for points $(1, y)$ with $y < y^*$ the vectors point to the left. Similarly, the line $x \equiv -1$ is also transversal to the symmetric vector field everywhere except at the point $(-1, -y^*) = (-1, a/b)$. For points $(-1, y)$ with $-y^* < y$ the vectors point to the right and for points $(-1, y)$ with $y < -y^*$ the vectors point to the left.*

PROOF. That the line $x \equiv 1$ is transversal to ξ except $(1, y^*) = (1, -a/b)$ follows from the solution of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} a+by \\ c+dy \end{bmatrix}.$$

As $0 < b$, for $y < y^*$ the x-ordinate of the vector is $a + by < 0$ pointing to the left, and for $y^* < y$ the x-ordinate of the vector is given by $0 < a + by$ pointing to the right. By symmetry of $\xi(\mathbf{x}) = -\xi(-\mathbf{x})$ the line $x \equiv -1$ is also transversal to the symmetric vector field ξ but at the point $(-1, -y^*) = (-1, a/b)$. For points $(-1, y)$ with $-y^* < y$ the vector points right, being the rotation of π radians of the vector at $(1, -y)$ with $-y < y^*$. For points $(-1, y)$ with $y < -y^*$ the vector points left, being the rotation of π radians of the vector at $(1, -y)$ with $y^* < -y$. ∎

Definition 2.9. For $v < w$, the following notation will be used,

$$\overline{I}(v, w) = \{y : v < y < w\},$$

$$\overline{I}(v, w] = \{y : v < y \leq w\},$$

$$\overline{L}[v,w) = \{y : v \le y < w\},$$

$$\overline{L}[v,w] = \{y : v \le y \le w\}.$$

Definition 2.10. The following functions

$$\pi_{11} : L(y^*, \infty) \to L(-\infty, y^*],$$

$$\pi_{12} : L(-\infty, y^*] \to \overline{L}(-\infty, -y^*),$$

$$\pi_{22} : \overline{L}(-\infty, -y^*) \to \overline{L}[-y^*, \infty),$$

$$\pi_{21} : \overline{L}[-y^*, \infty) \to L(y^*, \infty),$$

are given by

$$(1, \pi_{11}(y_0)) = \phi(t_0, (1, y_0)), t_0 = \min\{t : 0 < t, \phi(t, (1, y_0)) \cap \{(x,y) : x = 1\} \neq \emptyset\},$$

$$(-1, \pi_{12}(y_0)) = \phi(t_0, (1, y_0)), t_0 = \min\{t : 0 < t, \phi(t, (1, y_0)) \cap \{(x,y) : x = -1\} \neq \emptyset\},$$

$$(-1, \pi_{22}(y_0)) = \phi(t_0, (-1, y_0)), t_0 = \min\{t : 0 < t, \phi(t, (-1, y_0)) \cap \{(x,y) : x = -1\} \neq \emptyset\},$$

$$(1, \pi_{21}(y_0)) = \phi(t_0, (-1, y_0)), t_0 = \min\{t : 0 < t, \phi(t, (-1, y_0)) \cap \{(x,y) : x = 1\} \neq \emptyset\},$$

whenever they are defined.

The subscript 1 refers to the *first* line $x \equiv 1$.The subscript 2 refers to the *second* line $x \equiv -1$. Thus, the function π_{12} maps from the first line $x \equiv 1$, to the second line $x \equiv -1$. Likewise for the functions $\pi_{22}, \pi_{21}, \pi_{11}$. The function π^2, where π is the return map of definition 2.11 of [3] when applied to $L(-\infty, y^*]$ has the decomposition $\pi^2 = \pi_{11} \circ \pi_{21} \circ \pi_{22} \circ \pi_{12}$.

§3. $0 < b$, $0 < a + d$, $(a + d)^2/4 < ad - bc$, $ad - bc + dk - bl \neq 0$.

Many of the results in this section have close similarities with results in section 3 of [3]. However, the existence of subtle differences prevents a direct application of the results in that section.

Lemma 3.1. *Let ξ be a symmetric vector field with $0 \leq a + k + d$ then there there are no cycles.*

PROOF. The only fixed point is at the origin. Any cycle must therefore pass through the region $\{(x, y) : -1 < x < 1\}$. The cycle cannot lie wholly in this region as $0 < a + d$ and linear vector fields with non-zero trace do not admit cycles. Thus, the cycle intersects either the line $x \equiv 1$ or $x \equiv -1$ or both of these lines.

If it only intersects the line $x \equiv 1$ then lemma 3.1 of [3] would be contradicted for the piecewise linear vector field with the same defining constants. If the cycle only intersected the line $x \equiv -1$ then by symmetry there also exists a cycle intersecting only the line $x \equiv 1$. Again, this would be a contradiction of lemma 3.1 of [3]. Any cycle must intersect both the lines $x \equiv 1$ and $x \equiv -1$.

By Stoke's theorem,

$$\oint_C \frac{dx}{dt}dy - \frac{dy}{dt}dx = \int_{\mathrm{int}(C)} \frac{d}{dx}\left(\frac{dx}{dt}\right) + \frac{d}{dy}\left(\frac{dy}{dt}\right) dx\ dy.$$

Breaking up the area integral into the three regions, $A = \mathrm{int}(C) \cap \{(x, y) : x < -1\}$, $B = \mathrm{int}(C) \cap \{(x, y) : -1 \leq x \leq 1\}$, and $D = \mathrm{int}(C) \cap \{(x, y) : 1 < x\}$, then

$$0 = \int_A \frac{d}{dx}((a + k)x + by - k) + \frac{d}{dy}((c + l)x + dy - l)dx\ dy +$$

$$\int_B \frac{d}{dx}(ax + by) + \frac{d}{dy}(cx + dy)dx\ dy + \int_D \frac{d}{dx}((a + k)x + by - k) + \frac{d}{dy}((c + l)x + dy - l)dx\ dy.$$

Thus,

$$0 = \int_A (a + k + d)dx\ dy + \int_B (a + d)dx\ dy + \int_D (a + k + d)dx\ dy.$$

The integral on the right in nonzero. By contradicton, limit cycles do not exist. ∎

Theorem 3.2. *If $(x, y), x < 1$, is the primary induced fixed point of the symmetric vector field ξ with*

$$\frac{1}{b}(-d(1 - x) - a) \leq y$$

then there are no cycles.

PROOF. Note by corollary 2.5 that $k = (ax + by)/(1 - x)$. Using lemma 3.1 it is sufficient that $0 \leq a + k + d$ for there to be no cycles. This inequality simplifies to $(-d(1 - x) - a)/b \leq y$.

The following lemmas will delimit under which conditions the eigenvalues at the primary induced fixed point are real. This will allow the proof of the subsequent theorem.

Lemma 3.3. *Let $(x, y), x < 1$, be the primary induced fixed point of the symmetric vector field ξ. Then it has complex eigenvalues $\Leftrightarrow$*

$$\frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a) < y < \frac{1}{b}(2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a).$$

PROOF. Lemma 2.4 and lemma 3.3 of [3].

Lemma 3.4. *Under $\pi_{22} \circ \pi_{12}$ then point $(-1, -y^*)$ has no pre-image.*

PROOF. The point $(-1, -y^*)$ has no preimage under the map π_{22}, thus it cannot have a pre-image under the composition map $\pi_{22} \circ \pi_{12}$. ∎

Theorem 3.5. *If $(x, y), x < 1$, is the primary induced fixed point for the symmetric vector field ξ with*

$$y \leq \frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a)$$

then there is a globally attracting annulus for $\Re^2 - \{(0,0)\}$. Furthermore, assuming conjecture 0.1 holds, the attracting annulus is an attracting limit cycle.

PROOF. As the primary induced fixed point corresponds to an induced fixed point with real eigenvalues, there are either one or two induced points $(1, v_1), (1, v_2)$ such that the line through those points, in the direction of the respective vectors, pass through (x, y). If there are two such points, then let $v = \max\{v_1, v_2\}$. By lemma 2.4 and proposition 3.6 of [3] it may be assumed that $v < y^*$.

At the origin $0 < a+d, (a+d)^2 < 4(ad-bc)$, so that the solutions have non-zero rotational speed. As the real part of the complex eigenvalues are non-negative, the values of $||\phi(t, (x_), y_0)) - (0,0)||$, being the radial distance from the origin, is bounded below by $\sqrt{1+y_0^2}$. If $\phi(t, (x_0, y_0))$ does not intersect the line $x \equiv -1$ then for some $0 < t_0$ $\phi(t_0, (x_0, y_0)) = (z_0, 0)$ with $-1 < z_0 \leq 0$. Then, $||(z_0, 0) - (0,0)|| = |z_0| < 1$. But, as $\sqrt{1+y_0^2} \leq |z|$ this means that $\sqrt{1+y_0^2} < 1$ which is impossible. Thus π_{12} is well-defined on $L(-\infty, y^*]$. By symmetry π_{21} is well-defined on $\overline{L}[-y^*, \infty)$.

The primary induced fixed point satisfies $y \leq (-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a)/b < (-d(1-x) - a)/b$, or $ax + by < (-a-d)(1-x)$ from which it follows that $a + k + d < 0$. Thus the primary induced fixed point is attracting, implying that π_{11} is well-defined. By symmetry π_{22} is also well-defined.

Consider the function $\pi_{22} \circ \pi_{12} : L(-\infty, y^*] \rightarrow \overline{L}[-y^*, \infty)$ and the image of the set $L[v, y^*]$ under this map. Under one application of $\pi_{22} \circ \pi_{12}$,

$$\pi_{22} \circ \pi_{12}(L[v, y^*]) = L[\pi_{22} \circ \pi_{12}(y^*), \pi_{22} \circ \pi_{12}(v)].$$

The invariant manifold of the secondary induced fixed point constraints $\pi_{22} \circ \pi_{12}(v) < -v$. As $\pi_{22} \circ \pi_{12}(y^*) \in \overline{L}[-y^*, \infty)$ and as $-y^*$ has no pre-image then $-y^* < \pi_{22} \circ \pi_{12}(y^*)$. Thus $\pi_{22} \circ$

$\pi_{12}(L[v,y^*]) \subset \overline{L}[-y^*,-v]$. By symmetry it is also true that $\pi_{11} \circ \pi_{21}(\overline{L}[-y^*,-v]) \subset L[v,y^*]$. Thus $\pi^2 = \pi_{11} \circ \pi_{21} \circ \pi_{22} \circ \pi_{12}$ satisfies $\pi^2(L[v,y^*]) \subset L[v,y^*]$.

By theorem 2.12 of [3] there exists a locally attracting annulus A. It is clear that A is attracting for all points in $L[v,y^*]$. Points in $L(-\infty,v)$ iterate under $\pi_{22} \circ \pi_{12}$ to $\overline{L}[-y^*,-v]$ and eventually to A. Points on $L(y^*,\infty)$ iterate to $L[v,y^*]$ and thus also to A. Thus, all points along $x \equiv 1$ iterate to A. Points on the line $x \equiv -1$ will meet the line $x \equiv 1$ under either the function $\pi_{21} \circ \pi_{22}$ or the function π_{21}. Thus all points along $x \equiv -1$ will also iterate towards A. Let $(x_0,y_0) \neq (0,0)$, then there is some $0 < t_0$ for which $\phi(t_0,(x_0,y_0))$ intersects either $x \equiv 1$ or $x \equiv -1$. Then, also (x_0,y_0) will converge to the point z. The annulus A is globally attracting for all points in $\Re^2 - \{(0,0)\}$. Assuming conjecture 0.1 holds, it is an immediate consequence that the attracting annulus is an attracting limit cycle. ∎

The following lemma and subsequent theorem will expand the region on which an attracting annulus is known to exist.

Lemma 3.6. *Let $(x_i,y_i), x_i < 1$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$\frac{1}{b}(-2\sqrt{1-x_i}\sqrt{ad-bc} - d(1-x_i) - a) < y_i < \frac{1}{b}(2\sqrt{1-x_i}\sqrt{ad-bc} - d(1-x_i) - a)$$

then there exists $v_0 < y^*, K_0$ such that for $v \leq v_0$ the map $\pi_{22} \circ \pi_{12} : L(-\infty,y^*] \to \overline{L}[-y^*,\infty)$ satisfies

$$\pi_{22} \circ \pi_{12}(v) < -e^{\frac{\lambda\pi}{\omega}} + K_0$$

where $\lambda \pm i\omega$ are the complex eigenvalues of the matrix

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}$$

for $k = (ax_i + by_i)/(1-x_i), l = (cx_i + dy_i)/(1-x_i)$.

Proof. Let $v < y^*$. The vector at the point $(1,v)$ has slope given by $(c+dv)/(a+bv)$. The line through $(1,v)$ with this slope intersects the line $x \equiv -1$ at a point below where the solution $\phi(t,(1,v))$ intersects the line $x \equiv -1$. The line through $(1,v)$ with slope $(c+dv)/(a+bv)$ has the equation

$$y = \frac{c+dv}{a+bv}(x-1) + v$$

and intersects the line $x \equiv -1$ at the point $(-1, v - 2(c+dv)/(a+bv)) = (-1,v')$. Consider the point $(-1,v')$. The slope at this point is $(-c+dv')/(-a+bv')$. The line through $(-1,v')$ intersects the line $x \equiv -x_i$ at a point below where $\phi(t_0,(1,v))$ intersects the line $x \equiv -x_i$ under the linear vector field

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}.$$

The line through $(-1, v')$ with slope $(-c+dv')/(-a+bv')$ is given by

$$y = \frac{-c+dv'}{-a+bv'}(x+1)+v'$$

and intersects the line $x \equiv -x_i$ at the point $(-x_i, v' + (-x_i+1)(-c+dv')/(-a+bv')) = (-x_i, v'')$. Under π/ω units of time the point $(-x_i, v'')$ iterates to the point $(-x_i, -e^{\frac{\lambda\pi}{\omega}}(v''+y_i) - y_i) = (-x_i, v''')$. Consider the point $(-x_i, v''')$. Under the linear vector field

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}$$

the point $(-x_i, v''')$ induces a vector with slope d/b. The line through $(-x_i, v''')$ with slope d/b is given by

$$y = \frac{d}{b}(x+x_i)+v'''.$$

This line intersects the line $x \equiv -1$ at the point $(-1, v''' + (d/b)(x_i - 1))$. This gives an upper bound for $\pi_{22}\circ\pi_{12}(v)$,

$$\pi_{22}\circ\pi_{12}(v) < -e^{\frac{\lambda\pi}{\omega}}\left[\frac{-c+d(v-2\frac{c+dv}{a+bv})}{-a+b(v-2\frac{c+dv}{a+bv})}(1-x_i) - 2\frac{c+dv}{a+bv} + y_i\right] - y_i + \frac{d}{b}(x_i-1)$$

$$= -e^{\frac{\lambda\pi}{\omega}}v + e^{\frac{\lambda\pi}{\omega}}\left[2\left(\frac{d}{b} - \frac{ad-bc}{b(a+bv)}\right) - \left(\frac{d}{b} + \frac{ad-bc}{b(-a+b(v-2\frac{c+dv}{a+bv}))}\right)(1-x_i) - y_i\right] - y_i + \frac{d}{b}(x_i-1).$$

Let $v_0 = \min\{-1/b - a/b, -1/b + a/b + 2d/b\}$. Then for $v \le v_0$ one has that $a + bv \le -1$ from which $-(ad-bc)/(b(a+bv))$ is bounded above by $-(ad-bc)/b$. Furthermore, since $a+bv \le -1$ then

$$\frac{c+dv}{a+bv} = \frac{d}{b} - \frac{ad-bc}{b(a+bv)} > \frac{d}{b}$$

Thus,

$$v \le -\frac{1}{b} + \frac{a}{b} + 2\frac{c+dv}{a+bv}$$

from which it follows that

$$-a + b\left(v - 2\frac{c+vd}{a+bv}\right) < -1.$$

Thus,

$$\pi_{22}\circ\pi_{12}(v) < -e^{\frac{\lambda\pi}{\omega}}v + e^{\frac{\lambda\pi}{\omega}}\left[2\left(\frac{d}{b} - \frac{ad-bc}{b}\right) - \left(\frac{d}{b} + \frac{ad-bc}{b}\right)(1-x_i) - y_i\right] - y_i + \frac{d}{b}(x_i-1).$$

Hence $\pi_{22}\circ\pi_{12}(v) < -e^{\frac{\lambda\pi}{\omega}}v + K_0$ where

$$K_0 = e^{\frac{\lambda\pi}{\omega}}\left[2\left(\frac{d}{b} - \frac{ad-bc}{b}\right) - \left(\frac{d}{b} + \frac{ad-bc}{b}\right)(1-x_i) - y_i\right] - y_i + \frac{d}{b}(x_i-1). \qquad \blacksquare$$

Theorem 3.7. *Let $(x,y), x<1$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$\frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a) < y < \frac{1}{b}(-d(1-x)-a)$$

then there is an attracting annulus in $\Re^2 - \{(0,0)\}$*. Furthermore, assuming conjecture 0.1 holds, the attracting annulus is an attracting limit cycle.*

PROOF. There exist $v_0 < y^*, K_0$ such that for $v \le v_0$,

$$\pi_{22} \circ \pi_{12}(v) < -e^{\frac{\lambda\pi}{\omega}} v + K_0.$$

Similarly, for $-y^* < -v_0 \le v$, by symmetry one has that

$$\pi_{11} \circ \pi_{21}(v) > -e^{\frac{\lambda\pi}{\omega}} v - K_0.$$

Let $v_1 = \min\{v_0, (\pi_{22} \circ \pi_{12})^{-1}(-v_0)\}$. Then for $v \le v_1$,

$$\begin{aligned}\pi^2(v) &= \pi_{11} \circ \pi_{21} \circ \pi_{22} \circ \pi_{12}(v)\\ &> -e^{\frac{\lambda\pi}{\omega}}(\pi_{22} \circ \pi_{12}(v)) - K_0\\ &> -e^{\frac{\lambda\pi}{\omega}}(-e^{\frac{\lambda\pi}{\omega}} v + K_0) - K_0\\ &= e^{\frac{2\lambda\pi}{\omega}} v - K_0(e^{\frac{\lambda\pi}{\omega}} + 1).\end{aligned}$$

If $\pi^2(v) > v$ it is sufficient that

$$e^{\frac{2\lambda\pi}{\omega}} v - K_0(e^{\frac{\lambda\pi}{\omega}} + 1) \ge v.$$

Remembering that if $y < (-d(1-x) - a)/b$ then $\lambda = (a + k + d)/2 < 0$, it is enough that

$$\frac{-K_0(e^{\frac{\lambda\pi}{\omega}} + 1)}{1 - e^{\frac{2\lambda\pi}{\omega}}} \ge v.$$

Thus, let $v_2 = \min\{v_1, -K_0(e^{\frac{\lambda\pi}{\omega}} + 1)/(1 - e^{\frac{2\lambda\pi}{\omega}})\}$. Then for $v \le v_2, \pi^2(v) > v$.

Consider the line segment $L[v_2, y^*]$. Under π^2 one has that

$$\pi^2(L[v_2, y^*]) = L[\pi^2(v), \pi^2(y^*)] \subset L[v_2, y^*].$$

By theorem 2.12 of [3] there is a locally attracting annulus A for the points in $L[v_2, y^*]$. Let C_1, C_2 be the boundary cycles of the annulus. The boundary cycles intersect the line at points $(1, r), (1, s), v_2 < s \le r < y^*$. The point $(1, r)$ is a limit for the point $(1, y^*)$ under iteration of π^2. The point $(1, s)$ is a limit for the point $(1, v_2)$ under iteration of π^2. Let $v \le v_2$. As before it can be shown that there is an attracting annulus A' for points in the interval $L[v, y^*]$. Let C_1', C_2' be the boundary cycles of the annulus A'. These boundary cycles intersect the line $x \equiv 1$ at the points $(1, r'), (1, s'), v < s' \le r' < y^*$. The point $(1, r')$ is a limit for $(1, y^*)$ under π^2. The point $(1, s')$ is a limit for the point $(1, v_2)$ under π^2. The point $(1, y^*)$ can have only one limit, thus $(1, r) = (1, r')$. Note that $v_2 < s'$ so that the point $(1, s')$ is also a limit for the point $(1, v_2)$. The point $(1, v_2)$ has only one limit, thus $(1, s) = (1, s')$. The annulus A' is formed from the same boundary cycles as the annulus A. It follows that the annulus A is attracting for all points along $L(-\infty, y^*]$. As points in $L(y^*, \infty)$

iterate to $L(-\infty, y^*]$, it follows that the annulus is attracting for all points along $x \equiv 1$. Let (x, y) be any point in $\Re^2 - \{(0,0)\}$. Under finite time the solution through $(x, y), \phi(t, (x, y))$ will intersect the line $x \equiv 1$ and iterate towards the annulus. The annulus is globally attracting for all points in $\Re^2 - \{(0,0)\}$. Assuming conjecture 0.1 holds, it is an immediate consequence that the attracting annulus is an attracting limit cycle. ∎

For the remainder of this section it will be taken that the primary induced fixed point lies to the right of the line $x \equiv 1$. The lemma and theorem that follows will outline a region in which there are no limit cycles. The subsequent theorem will give a region in which there are locally attracting annuli.

Lemma 3.8. *There is a C^∞ diffeomorphism $\overline{g}(x, y) = (v, w)$ from the set $\{(x, y) : 1 < x\}$ of primary induced fixed points to the set $\{(v, w) : v < y^* < w\}$.*

Proof. The function $\overline{g}$ is the composition of the diffeomorphism m (lemma 2.4) between primary induced fixed points and induced fixed points, and the diffeomorphism g (theorem 3.23 of [3]) between induced fixed points and the set $\{(v, w) : v < y^* < w\}$. ∎

Theorem 3.9. *Let $(x, y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. Let $y^{***} = \pi_{21}(-y^*)$. Then, if*

$$y \leq \left(\frac{c + dy^{***}}{a + by^{***}}\right)(x - 1) + y^{***}$$

there are no limit cycles.

Proof. Let $\overline{g}(x, y) = (v, w)$. As in the proof in the case $0 < b, 0 < a + d, (a + d)^2/4 < ad - bc$ with the induced fixed point satisfying

$$y \leq \left(\frac{c + dy^{**}}{a + by^{**}}\right)(x - 1) + y^{**}$$

it can be proved that $w \leq y^{***}$.

Any limit cycle must intersect the line $x \equiv 1$. Points on $L(w, \infty)$ cannot form cycles, being so prevented by the invariant manifold passing through $(1, w)$. The point $(1, w)$ cannot form any cycles, being attracted to the primary induced fixed point (x, y).

As $w \leq y^{***}$ then $L[y^{***}, \infty) \subseteq L[w, \infty)$. Under symmetry one has $\overline{L}(-\infty, -y^{***}] \subseteq \overline{L}(-\infty, -w]$. By symmetry the points in $\overline{L}(-\infty, -w]$ cannot form cycles.

As $y^{***} = \pi_{21}(-y^*)$, then by symmetry it is also true that $\pi_{12}(y^*) = -y^{***}$. Then one has that $\pi_{12}(L(-\infty, y^*]) \subseteq \overline{L}(-\infty, \pi_{12}(y^*)) = \overline{L}(-\infty, -y^{***}) \subseteq \overline{L}(-\infty, -w]$. Thus neither can points in $L(-\infty, y^*]$ form cycles. Finally, points in $L(y^*, w)$ map to $L(-\infty, y^*]$, so neither can they form cycles. In consequence, points along $x \equiv 1$ cannot form cycles, it follows that limit cycles do not exist. ∎

It would be useful to know when it is true that (x, y) satisfies $\pi(v) = w$ where $g(x, y) = (v, w)$. In the following lemmas a graph $\overline{\chi}$ will be determined that will separate the points for which $\pi(v) < w$ and $w < \pi(v)$. Then points for which $\pi(v) < w$ will be exactly those points that lie above the graph $\overline{\chi}$. It will be shown that the graph χ is continuously differentiable and extends infinitely to the right.

Lemma 3.10. *Define* $\overline{\chi} : (-\infty, y^*) \to \{(x, y) : 1 < x\}$ *by the formula* $\overline{\chi}(y) = \overline{g}^{-1}(\pi_{21}(-y), y)$. *Then* $\overline{\chi}$ *is a continuous curve with* $\lim_{y \to y^*} \overline{\chi}(y) = (1, y^{***})$.

Proof. Continuity of $\overline{\chi}$ on the interval $(-\infty, y^*)$ is immediate.

Let $0 < \epsilon$ and consider the ball $B((1, y^{***}), \epsilon) = \{(x, y) : \sqrt{(x-1)^2 + (y - y^{***})^2} < \epsilon\}$. Consider the set given by $V \times W = \overline{g}(B((1, y^{***}), \epsilon) \cap \{(x, y) : 1 < x\})$. Because $\overline{g}$ is a homeomorphism the set $V \times W$ is open. As open subsets of $x \equiv 1$ then $V = (s, y^*)$, $W = (y^{***}, t)$. Consider the set $-\pi_{21}^{-1}(W)$, this set is open and has the form (v, y^*). Let $X = -\pi_{21}^{-1}(W) \cap V = (u, y^*)$, this set is open. It will be shown that $\overline{\chi}(y) \subseteq B((1, y^{***}))$.

Say $u < y < y^*$, then $y \in -\pi_{21}^{-1}(W)$, and $\pi_{21}(-y) \in W$. Thus $\overline{\chi}(y) = \overline{g}^{-1}(\pi_{21}(-y), y) \in B((1, y^{***}), \epsilon)$. ∎

Lemma 3.11. *Let* $\overline{\chi}(y) = (\overline{\chi}_1(y), \overline{\chi}_2(y))$ *where* $\overline{\chi}(y) = \overline{g}^{-1}(\pi_{21}(-y), y)$. *Then* $\overline{\chi}_1(y)$ *is a decreasing function of* y.

Proof. The formula for $\overline{\chi}_1(y)$ is given by

$$\overline{\chi}_1(y) = 1 - \frac{(a + by)(a + b\pi_{21}(-y))}{ad - bc}.$$

As ξ is C^0 continuous, π_{21} is differentiable and thus,

$$\overline{\chi}_1'(y) = -\frac{b(a + b\pi_{21}(-y)) - b\pi_{21}'(-y)(a + by)}{ad - bc}.$$

Now $y^* < \pi_{21}(-y)$ so that $0 < a + b\pi_{21}(-y)$. Also, $0 < \pi_{21}'(-y)$ and since $y < y^*$ then $a + by < 0$. Thus $\overline{\chi}_1'(y) < 0$ and is monotonically decreasing.

As $\lim_{y \to y^*} \overline{\chi}_1(y) = 1$, it follows that $1 < \overline{\chi}_1(y)$ for all $y < y^*$. By monotonicity, an inverse of $\overline{\chi}_1(y)$ exists. It is then possible to write $\overline{\chi}_2(y) = \overline{\chi}_2(\overline{\chi}_1^{-1}(\overline{\chi}_1(y)))$. Thus, $\overline{\chi}_2(y) = \overline{F}(\overline{\chi}_1(y))$ for $\overline{F}(y) = \overline{\chi}_2(\overline{\chi}_1^{-1}(y))$. ∎

Lemma 3.12. *Let* $y < y^*$. *Then* $\overline{\chi}(y) - \overline{\chi}(y^*) = \mathrm{D}\overline{\chi}(\eta)K$ *where* $\eta \in (y, y^*)$.

Proof. Writing $\overline{\chi}(y) = (\overline{\chi}_1(y), \overline{F}(\chi_1(y)))$ where $\overline{F}$ is as given in lemma 3.11, then by the mean value theorem,

$$\overline{\chi}_2(y) - \overline{\chi}_2(y^*) = \overline{F}'(\eta^*)(\overline{\chi}_1(y) - \overline{\chi}_1(y^*)),$$

where $\eta^* \in (\overline{\chi}_1(y^*), \overline{\chi}_1(y))$. As $\overline{\chi}_1$ is monotonic then $\eta^* = \overline{\chi}_1(\eta)$ for some $\eta \in (y, y^*)$. Substituting in the above formula,

$$\overline{\chi}(y) - \overline{\chi}(y^*) = \begin{bmatrix} 1 \\ \overline{F}'(\overline{\chi}_1(\eta)) \end{bmatrix} (\overline{\chi}_1(y) - \overline{\chi}_1(y^*)).$$

And since $\overline{\chi}_1'(\eta) \neq 0$,

$$\overline{\chi}(y) - \overline{\chi}(y^*) = \left[\frac{\overline{\chi}_1'(\eta)}{\overline{F}'(\overline{\chi}_1(\eta))\overline{\chi}_1'(\eta)}\right] \frac{\overline{\chi}_1(y) - \overline{\chi}_1(y^*)}{\overline{\chi}_1'(\eta)}.$$

Thus, $\overline{\chi}(y) - \overline{\chi}(y^*) = \mathrm{D}\overline{\chi}(\eta)K$ where $K = (\overline{\chi}_1(y) - \overline{\chi}_1(y^*))/\overline{\chi}_1'(\eta)$ and $\eta \in (y, y^*)$. ∎

Lemma 3.13. *The function $\overline{\chi}$ is C^1 on the interval $(-\infty, y^*)$. If $\pi_{21}'(-y^*)$ exists then*

$$\lim_{y \to y^*} \frac{\overline{\chi}_2(y) - y^{***}}{\overline{\chi}_1(y) - 1} = \frac{(c + dy^{***}) - \pi_{21}'(-y^*)(c + dy^*)}{a + by^{***}}.$$

PROOF. As $\overline{\chi}$ is the composition of two C^1 functions defined on the interval $(-\infty, y^*)$ it is also C^1 on $(-\infty, y^*)$.

Because $\overline{\chi}(y) = \overline{g}^{-1}(\pi_{21}(-y), y)$ then $\mathrm{D}\overline{\chi}(y) = \mathrm{D}(\overline{g}^{-1}(\pi_{21}(-y), y))\mathrm{D}(\pi_{21}(-y), y)$. Thus,

$$\mathrm{D}\overline{\chi}(y) = \frac{1}{ad - bc}\begin{bmatrix} -(a + by)b & -(a + b\pi_{21}(-y))b \\ -(c + dy)b & -(c + d\pi_{21}(-y))b \end{bmatrix} \begin{bmatrix} -\pi_{21}'(-y) \\ 1 \end{bmatrix}.$$

Thus $\mathrm{D}\overline{\chi}(y)$ is a vector with slope,

$$\frac{\pi_{21}'(-y)(c + dy)b - (c + d\pi_{21}(-y))b}{\pi_{21}'(-y)(a + by)b - (a + b\pi_{21}(-y))b},$$

which can be simplified to the form

$$\frac{\pi_{21}'(-y)(c + dy) - (c + d\pi_{21}(-y))}{\pi_{21}'(-y)(a + by) - (a + b\pi_{21}(-y))}.$$

Using lemma 3.12 that $\overline{\chi}(y) - \overline{\chi}(y^*) = \mathrm{D}\overline{\chi}(\eta)K$ for some $\eta \in (y, y^*)$, then

$$\begin{bmatrix} \overline{\chi}_1(y) - 1 \\ \overline{\chi}_2(y) - y^{***} \end{bmatrix} = \mathrm{D}\overline{\chi}(\eta)K.$$

Hence,

$$\lim_{y \to y^*} \frac{\overline{\chi}_2(y) - y^{***}}{\overline{\chi}_1(y) - 1} = \lim_{y \to y^*} \frac{\pi_{21}'(-\eta)(c + d\eta) - (c + d\pi_{21}(-\eta))}{\pi_{21}'(-\eta)(a + b\eta) - (a + b\pi_{21}(-\eta))}.$$

So that,

$$\lim_{y \to y^*} \frac{\overline{\chi}_2(y) - y^{***}}{\overline{\chi}_1(y) - 1} = \frac{\pi_{21}'(-y^*)(c + dy^*) - (c + d\pi_{21}(-y^*))}{\pi_{21}'(-y^*)(a + by^*) - (a + b\pi_{21}(-y^*))}.$$

As $y^* = -a/b$ the denominator reduces to $a + by^{***} > 0$ and finally,

$$\lim_{y \to y^*} \frac{\overline{\chi}_2(y) - y^{***}}{\overline{\chi}_1(y) - 1} = \frac{(c + dy^{***}) - \pi_{21}'(-y^*)(c + dy^*)}{a + by^{***}}.$$ ∎

The final results in this section will be regions in which limit cycles do and do not exist.

Theorem 3.14. *Let $(x, y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$\overline{\chi}_2(\overline{\chi}_1^{-1}(x)) < y.$$

then there is a locally attracting annulus. Furthermore, assuming conjecture 0.1 holds, the attracting annulus is an attracting limit cycle.

PROOF. Consider the line $L[v, y^*]$. Under the map $\pi_{22} \circ \pi_{12}$ the set goes to $\pi_{22} \circ \pi_{12}(L[v, y^*]) = L[\pi_{22} \circ \pi_{12}(y^*), \pi_{22} \circ \pi_{12}(v)]$. Now $\pi_{22} \circ \pi_{12}(y^*) \in \overline{L}[-y^*, \infty)$. As $-y^*$ has no pre-image then $-y^* < \pi_{22} \circ \pi_{12}(y^*)$. Observe that $\overline{\chi}_2(\overline{\chi}_1^{-1}(x)) < y$ implies $\pi_{21}(-v) < w$. Because $\pi_{21}(-v) < w$ then under symmetry it happens that $-w < \pi_{12}(v)$. As $-w < \pi_{12}(v)$ then under π_{22}, the point $\pi_{22} \circ \pi_{12}(v)$ is constrained to lie below the other invariant manifold of the secondary induced fixed point, thus $\pi_{22} \circ \pi_{12}(v) < -v$.

Then $\pi_{22} \circ \pi_{12}(L[v, y^*]) \subset \overline{L}[-y^*, -v]$. By symmetry it can be shown that $\pi_{11} \circ \pi_{21}(\overline{L}[-y^*, -v]) \subset L[v, y^*]$. Then,

$$\pi^2(L[v, y^*]) = \pi_{11} \circ \pi_{21} \circ \pi_{22} \circ \pi_{12}(L[v, y^*]) \subset L[v, y^*].$$

By theorem 2.12 of [3] there is a locally attracting annulus for all points in $L[v, y^*]$.

The point $(1, w)$ is on the invariant manifold that passes through (x, y), the annulus is not globally attracting. Points along $L[w, \infty)$ cannot induce cycles. Along $L(y^*, w)$ the points map under π_{11} into $L[v, y^*]$, and thus iterate to z. By symmetry, points along $\overline{L}(-\infty, -w]$ cannot induce cycles, as neither can the points along $\pi_{12}^{-1}(\overline{L}(-\infty, -w]) = L(-\infty, \pi_{12}^{-1}(-w)]$. Points in $L(\pi_{12}^{-1}(-w), v)$ map under π_{12} into $\overline{L}(-w, -y^*)$ and thence under π_{22} to $\overline{L}[-y^*, -v]$, so these points iterate to A. The annulus is unique. Assuming conjecture 0.1 holds, it follows immediately that the attracting annulus is an attracting limit cycle. ∎

Theorem 3.15. *Let $(x, y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$\left(\frac{c + dy^{***}}{a + by^{***}}\right)(x - 1) + y^{***} < y < \overline{\chi}_2(\overline{\chi}_1^{-1}(x))$$

then there are no limit annuli. Furthermore, assuming conjecture 0.1 holds, then there are no cycles.

PROOF. Assume a limit annulus exists. Let C_1, C_2 be the boundary cycles of the annulus. Say the boundary cycles intersect the line $x \equiv 1$ at the points $(1, r), (1, s)$ where $y^{***} < r \leq s < w$. As the annulus encircles the origin, which is repelling, the annulus is attracting.

Let $\overline{g}(x, y) = (v, w)$. As $\overline{\chi}_2(\overline{\chi}_1^{-1}(x))$ then $w < \pi_{21}(-v)$. Consider the line segment $L[s, w]$ under application of $\pi_{22}^{-1} \circ \pi_{21}^{-1}$, then

$$\begin{aligned}
\pi_{22}^{-1} \circ \pi_{21}^{-1}(L[s, w]) &= \pi_{22}^{-1}(L[\pi_{21}^{-1}(s), \pi_{21}^{-1}(w)]) \\
&\subset \pi_{22}^{-1}(L[\pi_{21}^{-1}(s), -v)) \\
&= L(\pi_{22}^{-1}(-v), \pi_{22}^{-1} \circ \pi_{21}^{-1}(s)] \\
&\subset L[-w, -s].
\end{aligned}$$

By symmetry, $\pi_{11}^{-1} \circ \pi_{12}^{-1}(L[-w,-s]) \subset L[s,w]$. Thus,

$$\begin{aligned}\pi^{-2}(L[s,w]) &= \pi_{11}^{-1} \circ \pi_{12}^{-1} \circ \pi_{22}^{-1} \circ \pi_{21}^{-1}(L[s,w]) \\ &\subset \pi_{11}^{-1} \circ \pi_{12}^{-1}(L[-w,-s]) \\ &\subset L[s,w].\end{aligned}$$

Note that $(1,s)$ is the only fixed point for π^{-2} in the line segment $L[s,w]$. If another fixed point existed then maximality of the annulus would be violated.

Thus, the point $(1,s)$ is attracting for π^{-2}. The cycle through the point $(1,s)$ is then repelling under forward time. The annulus is repelling. A limit annulus cannot be both attracting and repelling. Limit annuli do not exist.

Assume an annulus exists whose boundary cycles both intersect the line $x \equiv 1$. By the same argument as before, the annulus has to be semi-stable. By conjecture 0.1, semi-stable annuli do not exist. Thus, if annuli exist then its boundary cycles cannot both intersect the line $x \equiv 1$.

If an annulus existed then one of its boundary cycles does not intersect the line $x \equiv 1$. Because of the nature of the vector field in the region $\{(x,y) : -1 < x < 1\}$, if the boundary cycle intersected the line $x \equiv -1$ then the cycle would also intersect the line $x \equiv 1$. Thus the boundary cycle does not intersect the line $x \equiv -1$. This boundary cycle must then lie in one of the regions $\{(x,y) : x \leq -1\}$, $\{(x,y) : -1 < x < 1\}$ or $\{(x,y) : 1 \leq x\}$. However, the trace of the symmetric vector field in each of these regions is nonzero, preventing cycles from forming. Thus, the annulus does not exist and it follows that cycles do not exist. ∎

§4. $0 < b$, $0 < a + d$, $0 < ad - bc \leq (a+d)^2/4$, $ad - bc + dk - bl \neq 0$.

The results in this section are virtually identical to those obtained in section 3. The salient point to note is that the maps $\pi_{12} : L(-\infty, y^*] \to \overline{L}(-\infty, y^*)$, $\pi_{21} : \overline{L}[y^*, \infty) \to L(y^*, \infty)$ are well defined. Because of the similarity of proof, only references to the corresponding proofs in section 3 will be given.

Theorem 4.1. *If $(x,y), x < 1$, is the primary induced fixed point of the symmetric vector field ξ with*

$$\frac{1}{b}(-d(1-x) - a) \leq y$$

then there are no cycles.

Proof. See theorem 3.2. ∎

Theorem 4.2. *If $(x,y), x < 1$, is the primary induced fixed point of the symmetric vector field ξ with*

$$y \leq \frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a)$$

then there is a globally attracting annulus for $\Re^2 - \{(0,0)\}$. Furthermore, assuming conjecture 0.1 holds, the attracting annulus is an attracting limit cycle.

PROOF. See theorem 3.5. ∎

Theorem 4.3. *Let $(x, y), x < 1$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$\frac{1}{b}(-2\sqrt{1-x}\sqrt{ad-bc} - d(1-x) - a) < y < \frac{1}{b}(-d(1-x) - a)$$

then there is an attracting annulus in $\Re^2 - \{(0,0)\}$. Furthermore, assuming conjecture 0.1 holds, the attracting annulus is an attracting limit cycle.

PROOF. See theorem 3.7. ∎

Theorem 4.4. *Let $(x, y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. Let $y^{***} = \pi_{21}(-y^*)$. Then, if*

$$y \leq \left(\frac{c + dy^{***}}{a + by^{***}}\right)(x - 1) + y^{***}$$

there are no limit cycles.

PROOF. See theorem 3.9. ∎

The following lemmas will consider properties of the curve $\tilde{\chi}$ that separate point for which $\pi_{21}(-v) < w$ and $\pi_{21}(-v) > w$. Points in the former set will lie above $\tilde{\chi}$ while points in the latter set will lie below $\tilde{\chi}$.

Lemma 4.5. *Define $\tilde{\chi} : (-\infty, y^*) \to \{(x, y) : 1 < x\}$ by the formula $\tilde{\chi}(y) = \overline{g}^{-1}(\pi_{21}(-y), y)$. Then $\tilde{\chi}$ is a continuous curve with $\lim_{y \to y^*} \tilde{\chi}(y) = (1, y^{***})$.*

PROOF. See lemma 3.10. ∎

Lemma 4.6. *Let $\tilde{\chi}(y) = (\tilde{\chi}_1(y), \tilde{\chi}_2(y))$ where $\tilde{\chi}(y) = \overline{g}^{-1}(\pi_{21}(-y), y)$. Then $\tilde{\chi}_1(y)$ is a decreasing function of y.*

PROOF. See lemma 3.11. ∎

Lemma 4.7. *The function $\tilde{\chi}$ is C^1 on the interval $(-\infty, y^*)$. If $\pi_{21}'(-y^*)$ exists then*

$$\lim_{y \to y^*} \frac{\tilde{\chi}_2(y) - y^{***}}{\tilde{\chi}_1(y) - 1} = \frac{(c + dy^{***}) - \pi_{21}'(-y^*)(c + dy^*)}{a + by^{***}}.$$

PROOF. See lemma 3.13. ∎

Theorem 4.8. *Let $(x, y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$\tilde{\chi}_2(\tilde{\chi}_1^{-1}(x)) < y.$$

then there is a locally attracting annulus. Furthermore, assuming conjecture 0.1 holds, the attracting annulus is an attracting limit cycle.

PROOF. See theorem 3.14.

Theorem 4.9. *Let $(x, y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$\left(\frac{c+dy^{***}}{a+by^{***}}\right)(x-1)+y^{***} < y < \tilde{\chi}_2(\tilde{\chi}_1^{-1}(x))$$

then there are no limit annuli. Furthermore, assuming conjecture 0.1 holds, then there are no cycles.

PROOF. See theorem 3.15. ∎

§5. $0 < b$, $0 < a + d$, $0 < ad - bc$, $ad - bc + dk - bl = 0$.

The only result so far in this direction has been the following corollary.

Corollary 5.1. *If $0 \leq a + k + d$ then there are no cycles.*

PROOF. See lemma 3.1. ∎

§6. $0 < b$, $0 < a + d$, $ad - bc = 0$.

The analysis is most easily facilitated by considering the values of $dk - bl$. The following results will expand on this study.

Proposition 6.1. *If $dk - bl \leq 0$ then there are no cycles.*

PROOF. The symmetric vector field bounded by the lines $x \equiv \pm 1$ does not admit cycles. If cycles existed then they must cross either the lines $x \equiv \pm 1$ or lie outside of the region bounded by the lines $x \equiv \pm 1$. Firstly, consider the case that a cycle either intersects the lines $x \equiv 1$ or lies to the right of the line $x \equiv 1$. To the right of the line $x \equiv 1$ the symmetric vector field has the form of the linear vector field

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

The stability at the primary induced fixed point $(1, y^*)$ is given by the determinant of the defining matrix in the linear vector field $\mathbf{L}$. The value of this determinant is $dk - bl$.

If $dk - bl < 0$ then the primary induced fixed point is a saddle point to which can be identified two linear invariant manifolds, at least one of which bisects the region $\{(x, y) : 1 \leq x\}$. This bisection prevents cycles from forming. If $dk - bl = 0$ then a line of fixed points given by $0 = (a+k)x + by - k$ bisects the region $\{(x, y) : 1 \leq x\}$ preventing cycles from forming.

By symmetry, cycles cannot intersect the line $x \equiv -1$ nor lie to the left of the line $x \equiv -1$. Thus cycles do not exist. ∎

Proposition 6.2. *If $0 < dk - bl \leq (a + k + d)^2/4$ then there are no cycles.*

PROOF. The symmetric vector field bounded by the lines $x \equiv \pm 1$ does not admit cycles. If cycles existed then they must cross either the lines $x \equiv \pm 1$ or lie outside of the region bounded by the lines $x \equiv \pm 1$. Firstly, consider the case that a cycle either intersects the lines $x \equiv 1$ or lies to the right of the line $x \equiv 1$. As in proposition 6.1, the stability at the fixed point $(1, y^*)$ is determined by the value of the determinant of the corresponding linear vector field

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

If $0 < dk - bl < (a+k+d)^2/4$ the primary induced fixed point is either an attracting or repelling fixed point to which are two linear invariant manifolds, one of which will bisect the region $\{(x, y) : 1 \leq x\}$. In this case, cycles do not exist. When $dk - bl = (a+k+d)^2/4$ a linear invariant manifold lies in the region $\{(x, y) : 1 \leq x\}$ preventing cycles from forming.

Since cycles cannot intersect the line $x \equiv -1$ nor lie to the left of the line $x \equiv -1$ they do not exist. ∎

Proposition 6.3. *If $(a + k + d)^2 < dk - bl$ and $0 \leq a + k + d$ then there are no limit cycles. If $0 < a + k + d$ then there are no cycles.*

PROOF. The first part of the proposition is an immediate application of lemma 3.1. Cycles do not exist in the region bounded by the lines $x \equiv \pm 1$. If cycles existed then they must cross either the line $x \equiv 1$ or the line $x \equiv -1$ or lie outside the region bounded by these two lines. Firstly, consider the case of a cycle intersecting the line $x \equiv 1$ or being in the region to the right of the line $x \equiv 1$. By using Stoke's theorem, the case of a cycle intersecting the line $x \equiv 1$ can be excluded. As $0 < a+k+d$ the primary induced fixed point is repelling, cycles do not exist in the region right of the line $x \equiv 1$. Similarly, cycles can neither intersect the line $x \equiv -1$ nor lie to the left of the line $x \equiv -1$. Thus, cycles do not exist. ∎

Proposition 6.4. *If $(a+k+d)^2 < dk - bl$ and $a+k+d < 0$ then there is a unique attracting cycle.*

PROOF. The function $\pi_{11} : L(y^*, \infty) \to L(-\infty, y^*]$ is given by $\pi_{11}(v) = -e^{\frac{\lambda\pi}{\omega}}(v - y^*) + y^*$ where $\lambda \pm i\omega$ are the eigenvalues of the matrix

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix},$$

i.e, $\lambda = (a+k+d)/2 < 0, \omega = \sqrt{4(dk-bl) - (a+k+d)^2/4}/2$. The function $\pi_{22} : \overline{L}(-\infty, -y^*) \to \overline{L}[-y^*, \infty)$ is given by $\pi_{22}(v) = -e^{\frac{\lambda\pi}{\omega}}(v + y^*) - y^*$. For the vector field given by

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

the solution through the point (x_0, y_0) is given by

$$x(t) = \frac{ax_0 + by_0}{a+d}e^{(a+d)t} + \frac{dx_0 - by_0}{a+d},$$
$$y(t) = \frac{d}{b}\frac{ax_0 + by_0}{a+d}e^{(a+d)t} - \frac{a}{b}\frac{dx_0 - by_0}{a+d}.$$

For points in $L(-\infty, y^*]$ a non-parametric solution through the point $(1, v)$ is given by

$$y = \frac{d}{b}x - \frac{d - bv}{b}$$

where it is remembered that as $t \to \infty$ then $x \to \infty$. Thus, the point $(1, v)$ under the solution meets the line $x \equiv -1$ at the point $(-1, v - 2d/b)$. Hence $\pi_{12} : L(-\infty, y^*] \to \overline{L}(-\infty, y^*)$ is given by $\pi_{21}(v) = v + 2d/b$. By symmetry, the map $\pi_{21} : \overline{L}[-y^*, \infty) \to L(y^*, \infty)$ is given by $\pi_{21}(v) = v + 2d/b$.

The function $\pi^2 = \pi_{11} \circ \pi_{21} \circ \pi_{22} \circ \pi_{12} : L(-\infty, y^*] \to L(-\infty, y^*]$ is thus given by

$$\pi^2(v) = -e^{\frac{\lambda\pi}{\omega}} \left[-e^{\frac{\lambda\pi}{\omega}} (v - \frac{2d}{b} + y^*) - 2y^* + \frac{2d}{b} \right] + y^*.$$

Solving for $\pi^2(v) = v$ one has that

$$v = \frac{1}{1 - e^{\frac{2\lambda\pi}{\omega}}} \left[y^*(1 + 2e^{\frac{\lambda\pi}{\omega}} + e^{\frac{2\lambda\pi}{\omega}}) - \frac{2d}{b} e^{\frac{\lambda\pi}{\omega}} (e^{\frac{\lambda\pi}{\omega}} + 1) \right].$$

Notice that the following inequalities hold,

$$\begin{aligned}
\Rightarrow \qquad & -\frac{a+d}{b} < 0 \\
\Rightarrow \qquad & 2y^* - \frac{2d}{b} < 0 \\
\Rightarrow \qquad & y^* 2e^{\frac{\lambda\pi}{\omega}} (e^{\frac{\lambda\pi}{\omega}} + 1) - \frac{2d}{b} e^{\frac{\lambda\pi}{\omega}} (e^{\frac{\lambda\pi}{\omega}} + 1) < 0 \\
\Rightarrow \qquad & y^*(1 + 2e^{\frac{\lambda\pi}{\omega}} + e^{\frac{2\lambda\pi}{\omega}}) - \frac{2d}{b} e^{\frac{\lambda\pi}{\omega}} (e^{\frac{\lambda\pi}{\omega}} + 1) < y^*(1 - e^{\frac{2\lambda\pi}{\omega}}) \\
\Rightarrow \qquad & v < y^*.
\end{aligned}$$

Thus, indeed π^2 has a unique fixed point in $L(-\infty, y^*]$. As $|(\pi^2)'(v)| = e^{\frac{2\lambda\pi}{\omega}} < 1$ then the fixed point is attracting. The cycle through the fixed point gives the unique attracting cycle. ∎

§7. $0 < b,\ 0 < a + d,\ ad - bc < 0$.

When these conditions hold, the possibility of attracting annuli are very remote. This section will depend on results obtained in section 7 of [3].

Proposition 7.1. *If $ad - bc + dk - bl = 0$ then there are no cycles.*

PROOF. Consider the linear vector field given by

$$\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}.$$

As the determinant is zero, the number of fixed points is either zero or infinite. There are two cases to consider.

(i) :If $l \neq (d/b)k$ then there are no induced fixed points. Any cycles must contain the origin. The origin is a saddle with index -1, thus cycles cannot exist.

(ii) :If $l = (d/b)k$ then $ad - bc = 0$, this case cannot occur. ∎

For the remainder of the section it will be taken that $ad - bc + dk - bl \neq 0$. It will be proved that attracting cycles are not very likely.

Proposition 7.2. *Let $(x, y), x < 1$, be the primary induced fixed point of the symmetric vector field ξ. Then ξ has no cycles.*

PROOF. Any cycle must enclose fixed points whose indexes sum to 1. As the only fixed point is at the origin, with index -1, cycles do not exist. ∎

Proposition 7.3. *Let $(x, y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$y \leq \frac{1}{b}(-d(1-x) - a)$$

then there are no cycles.

PROOF. By lemma 3.1 it is sufficient that $0 \leq a + k + d$ for there to be no cycles. This means $0 \leq a + (ax + by)/(1 - x) + d$ which reduces to $y \leq (-d(1-x) - a)/b$. ∎

Lemma 7.4. *Let $(x, y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. Then (x, y) has complex eigenvalues $\Leftrightarrow$*

$$\frac{1}{b}(-2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a) < y < \frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a).$$

PROOF. The eigenvalues at the primary induced fixed point are determined by the roots of the characteristic equation of the matrix

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}.$$

The characteristic equation is $\lambda^2 + (a+k+d)\lambda + ad - bc + dk - bl = 0$. Substituting the values $k = (ax+by)/(1-x), l = (cx+dy)/(1-x)$ into the characteristic equation the above reduces to

$$\lambda^2 - \left(\frac{a+d-dx+by}{1-x}\right)\lambda + \frac{ad-bc}{1-x} = 0.$$

The eigenvalues are complex if and only if the discriminant is negative,

$$\left(\frac{a+d-dx+by}{1-x}\right)^2 - 4\left(\frac{ad-bc}{1-x}\right) < 0.$$

Thus,

$$\frac{1}{b}(-2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a) < y < \frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a). \qquad \blacksquare$$

Theorem 7.5. *Let $(x,y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$\frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a) \le y$$

then there are no cycles.

Proof. The primary induced fixed point has eigenvalues whose product is $(ad-bc)/(1-x) > 0$. Thus, the primary induced fixed point has index 1. Similarly the secondary induced fixed point also has index 1. Any cycle must contain within its interior points whose indices sum to 1. Thus, either the primary or secondary induced fixed point are in the interior of the cycle. Since the primary induced fixed point has at least one real eigenvalue, there can be associated an eigenvector through which lies a linear invariant manifold. Similarly, through the secondary induced fixed point lies a linear invariant manifold. As linear invariant manifolds cannot be bounded, it follows that cycles do not exist. $\blacksquare$

Lemma 7.6. *Let $(x,y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ such that*

$$\frac{1}{b}(-2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a) < y < \frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a).$$

To the primary induced fixed point corresponds unique values of k, l. Let $\lambda \pm i\omega$ be the complex eigenvalues of the matrix given by

$$\begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}.$$

Then there exist $-y^* < v_2, K_2$ such that for $v_2 \le v$ the map $\pi_{11} \circ \pi_{21} : \overline{L}[-y^*, \infty) \to L[y^*, \infty)$ satisfies

$$-e^{\frac{\lambda\pi}{\omega}} v + K_2 < \pi_{11} \circ \pi_{21}(v).$$

Proof. The eigenvalues of the origin are the eigenvalues of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The eigenvalues are given by

$$\lambda_1 = \frac{a+d+\sqrt{(a+d)^2-4(ad-bc)}}{2}$$
$$\lambda_2 = \frac{a+d-\sqrt{(a+d)^2-4(ad-bc)}}{2}$$

where $\lambda_2 < 0 < \lambda_1$. The vector corresponding to the larger of the two eigenvalues is

$$\begin{bmatrix} 1 \\ \overline{v} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-a+d+\sqrt{(a+d)^2-4(ad-bc)}}{2b} \end{bmatrix}.$$

Let $-y^* \leq v$. The solution through $(-1, v)$ lies inside of the line through $(-1, v)$ parallel to the linear manifold corresponding to the positive eigenvalue λ_1. The line through $(-1, v)$ in the direction of the eigenvector corresponding to λ_1 is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ v \end{bmatrix} + t \begin{bmatrix} 1 \\ \overline{v} \end{bmatrix}.$$

This line intersects $x \equiv 1$ at the point $(1, v+2\overline{v})$. Thus, $\pi_{21}(v) < v + 2\overline{v}$.

By a proof analogous to lemma 3.11 of [3], there exists $y^* < v_1, K_1$ such that for $v_1 \leq v$,

$$-e^{\frac{\lambda\pi}{\omega}} v + K_1 < \pi_{11}(v).$$

Let $v_2 = \max\{-y^*, v_1 - 2\overline{v}\}$. Then for $v_2 \leq v$ one has that $\pi_{21}(v) < v + 2\overline{v}$ where $v_1 \leq v + 2\overline{v}$. Thus,

$$\begin{aligned} \pi_{11} \circ \pi_{21}(v) &> -e^{\frac{\lambda\pi}{\omega}} \pi_{21} + K_1 \\ &> -e^{\frac{\lambda\pi}{\omega}} (v + 2\overline{v}) + K_1 \\ &= -e^{\frac{\lambda\pi}{\omega}} v + K_1 - 2\overline{v} e^{\frac{\lambda\pi}{\omega}}. \end{aligned}$$

Hence, $\pi_{11} \circ \pi_{21}(v) > -e^{\frac{\lambda\pi}{\omega}} v + K_2$ where $K_2 = K_1 - 2\overline{v} e^{\frac{\lambda\pi}{\omega}}$. ∎

Lemma 7.7. *Let $(x, y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$\frac{1}{b}(-d(1-x) - a) < y < \frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a)$$

then there exist $v_3 \leq y^*$ such that for $v \leq v_3$ the return map satisfies $v < \pi^2(v)$.

Proof. By lemma 7.6, there exist $-y^* \leq v_2, K_2$ such that for $v_2 \leq v$,

$$\pi_{11} \circ \pi_{21}(v) > -e^{\frac{\lambda\pi}{\omega}} v + K_2.$$

By symmetry of the symmetric vector field, for $v \leq -v_2$,

$$\pi_{22} \circ \pi_{12}(v) < -e^{\frac{\lambda\pi}{\omega}} v - K_2.$$

Let $v' = \min\{-v_2, (\pi_{22} \circ \pi_{12})^{-1}(v_2)\}$. Then for $v \leq v_3$, we have

$$\begin{aligned} \pi^2(v) &= \pi_{11} \circ \pi_{21} \circ \pi_{22} \circ \pi_{12}(v) \\ &> -e^{\frac{\lambda\pi}{\omega}} (\pi_{22} \circ \pi_{12}(v)) + K_2 \\ &> -e^{\frac{\lambda\pi}{\omega}} (-e^{\frac{\lambda\pi}{\omega}} v - K_2) + K_2 \\ &= e^{\frac{2\lambda\pi}{\omega}} v + K_2(e^{\frac{\lambda\pi}{\omega}} + 1). \end{aligned}$$

If $\pi^2(v) > v$ it is sufficient that

$$e^{\frac{2\lambda\pi}{\omega}} v + K_2(e^{\frac{\lambda\pi}{\omega}} + 1) \geq v.$$

As $(-d(1-x) - a)/b < y$ then $a + k + d < 0$ and $\lambda < 0$. Then it is required that

$$\frac{K_2(e^{\frac{\lambda\pi}{\omega}} + 1)}{1 - e^{\frac{2\lambda\pi}{\omega}}} \geq v.$$

Let $v_3 = \min\{v', K_2(e^{\frac{\lambda\pi}{\omega}} + 1)/(1 - e^{\frac{2\lambda\pi}{\omega}})\}$. Then for $v \leq v_3$ it happens that $v < \pi^2(v)$. ∎

Theorem 7.8. *Let $(x, y), 1 < x$, be the primary induced fixed point of the symmetric vector field ξ. If*

$$\frac{1}{b}(-d(1-x) - a) < y < \frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a)$$

and $\pi(\overline{w}) < \overline{v}$ then there is a pair of repelling annuli, each of which intersects only one of the lines $x \equiv 1$ or $x \equiv -1$, and an attracting annulus which intersects both the lines $x \equiv 1$ and $x \equiv -1$. Assuming conjecture 0.1 holds, the repelling annuli are repelling limit cycles and the attracting annulus is an attracting limit cycle.

Proof. By theorem 7.7 of [3], there is a pair of repelling annuli, each annulus intersecting only one of the lines $x \equiv 1$ or $x \equiv -1$. Assuming conjecture 0.1 holds, the repelling annuli are repelling limit cycles.

Consider the line segment $L[v_3, \pi_{11}(w)]$ where v_3 is given in lemma 7.7. Note that $\pi^2 \circ \pi_{11}(w) < \pi_{11}(w)$. Thus,

$$\begin{aligned} \pi^2(L[v_3, \pi_{11}(w)]) &= L[\pi^2(v_3), \pi^2 \circ \pi_{11}(w)] \\ &\subset L[v_3, \pi_{11}(w)]. \end{aligned}$$

By theorem 2.12 of [3] there is a locally attracting annulus for points in $L[v_3, \pi_{11}(w)]$. The annulus intersects both lines $x \equiv 1$ and $x \equiv -1$. Assuming conjecture 0.1 holds, the attracting annulus is an attracting limit cycle. ∎

Theorem 7.9. *Let $(x, y), 1 < x$, be the induced fixed point of the symmetric vector field ξ. If*

$$\frac{1}{b}(-d(1-x) - a) < y < \frac{1}{b}(2\sqrt{x-1}\sqrt{-(ad-bc)} - d(1-x) - a)$$

and $\overline{v} < \pi(\overline{w})$ then there are no limit annuli. Furthermore, assuming conjecture 0.1 holds, then there are no cycles.

PROOF. By theorem 7.8 of [3], there are no limit annuli which intersect only the lines $x \equiv 1$ or $x \equiv -1$.

Say a limit annulus exists. It must then intersect both lines $x \equiv 1$ and $x \equiv -1$. Let the boundary cycles of the annulus intersect the line $x \equiv 1$ at the points $(1, r), (1, s)$ where $v_3 < s \leq r < v$. The value v_3 is given in lemma 7.7 and $(1, v)$ is the point of intersection of the stable manifold through the origin and the line $x \equiv 1$. Note that as $\pi^2(v) < v$ the annulus is repelling. However, $\pi^2(L[v_3, s]) = L[\pi^2(v_3), \pi^2(s)] \subset L[v_3, s]$. The point $(1, s)$ is the only fixed point of the π^2 in the line segment $L[v_3, s]$. If another fixed point existed then maximality of the annulus would be contradicted. The cycle through $(1, s)$ is attracting. The annulus is attracting. A limit annulus cannot be both repelling and attracting. Limit annuli do not exist.

Assume an annulus exists whose boundary cycles both intersect the lines $x \equiv \pm 1$. By the same argument as before, the annulus has to be semi-stable. By conjecture 0.1, semi-stable annuli do not exist. Thus, if annuli exist then its boundary cycles cannot both intersect the lines $x \equiv \pm 1$.

If an annulus existed then one of its boundary cycles does not intersect both the lines $x \equiv \pm 1$. If this boundary cycle intersected only one of the lines $x \equiv \pm 1$ then a contradiction to theorem 7.8 of [3] would arise from the piecewise linear vector field with the same defining constants as ξ. The boundary cycle does not intersect either of the lines $x \equiv \pm 1$. This boundary cycle must then lie in one of the regions $\{(x, y) : x \leq -1\}$, $\{(x, y) : -1 < x < 1\}$ or $\{(x, y) : 1 \leq x\}$. However, the trace of the symmetric vector field in each of these regions is nonzero, preventing cycles from forming. Thus, the annulus does not exist, it then follows that cycles do not exist. ∎

§8. $0 < b$, $0 = a + d$.

For these choices of the defining constants there are no limit cycles.

Proposition 8.1. *If $k \neq 0$ then there are no limit cycles.*

PROOF. If a limit cycle existed that only intersected the line $x \equiv 1$ then proposition 8.1 of [3] would be violated for the piecewise linear vector field with the same defining constants to the right of $x \equiv -1$. If the limit cycle only intersected the line $x \equiv -1$, then under symmetry there would be a limit cycle that only intersected the line $x \equiv 1$. This would contradict proposition 8.1 of [3] for the piecewise linear vector field with the same defining constants, limit cycles that only intersect $x \equiv -1$ do not exist.

Thus, any limit cycle that exist must intersect the lines $x \equiv -1$ and $x \equiv 1$. Let C be this limit cycle, by Stokes theorem,

$$\oint_C \frac{dx}{dt}dy - \frac{dy}{dt}dx = \int_{\text{int}(C)} \frac{d}{dx}\left(\frac{dx}{dt}\right) + \frac{d}{dy}\left(\frac{dy}{dt}\right) dx\ dy.$$

Breaking up the area integral into the three regions, $A = \text{int}(C) \cap \{(x, y) : x < -1\}$, $B = \text{int}(C) \cap \{(x, y) : -1 \leq x \leq 1\}$, and $D = \text{int}(C) \cap \{(x, y) : 1 < x\}$, then

$$0 = \int_A \frac{d}{dx}((a+k)x + by - k) + \frac{d}{dy}((c+l)x + dy - l)dx\ dy +$$

$$\int_B \frac{d}{dx}(ax + by) + \frac{d}{dy}(cx + dy)dx\ dy + \int_D \frac{d}{dx}((a+k)x + by - k) + \frac{d}{dy}((c+l)x + dy - l)dx\ dy$$

$$= \int_A (a + k + d)dx\ dy + \int_B (a + d)dx\ dy + \int_D (a + k + d)dx\ dy$$

$$= \int_A k dx\ dy + \int_D k dx\ dy.$$

As $k \neq 0$ the integral on the right is nonzero. By contradiction, limit cycles do not exist. ∎

Theorem 8.2. *If $k = 0$ then there are no limit cycles.*

PROOF. Under the change of variables given by

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{a}{b} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

the symmetric vector field given by

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}, & x < -1; \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & -1 \leq x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x, \end{cases}$$

becomes

$$\xi\begin{bmatrix}X\\Y\end{bmatrix}=\begin{cases}\begin{bmatrix}0 & b\\-(ad-bc)/b+l & 0\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}+\begin{bmatrix}0\\l\end{bmatrix}, & X<-1;\\ \begin{bmatrix}0 & b\\-(ad-bc)/b & 0\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}, & -1\le X\le 1;\\ \begin{bmatrix}0 & b\\-(ad-bc)/b+l & 0\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}-\begin{bmatrix}0\\l\end{bmatrix}, & 1<X.\end{cases}$$

Thus,

$$\frac{dY}{dX}=\begin{cases}\dfrac{\left(-\frac{ad-bc}{b}+l\right)X+l}{bY},\\ \dfrac{\left(-\frac{ad-bc}{b}\right)X}{bY},\\ \dfrac{\left(-\frac{ad-bc}{b}+l\right)X-l}{bY}.\end{cases}$$

Thus,

$$bY^2+\left(\frac{ad-bc}{b}-l\right)X^2-2lX+c=0, \qquad X<-1,$$
$$bY^2+\left(\frac{ad-bc}{b}\right)X^2+c+l=0, \qquad -1\le X\le 1,$$
$$bY^2+\left(\frac{ad-bc}{b}-l\right)X^2+2lX+c=0, \qquad 1<X.$$

Assume that there is a limit cycle. The cycle must be on a level curve in the X,Y plane for some particular value c. By X–axis symmetry of the defining relations between X and Y, the cycle intersects the X–axis at some point $(X_0,0)$. Because of Y–axis symmetry the cycle also intersects the X–axis at the point $(-X_0,0)$. Say the cycle is attracting, the repelling case is handled in an analogous manner. Then the point $(X_1,0)$ close to $(X_0,0)$ will, under the symmetric vector field, approach the limit cycle. Consider the level curve on which the point $(X_1,0)$ lies. This level curve has Y–axis symmetry and X–axis symmetry. The path of the point $(X_1,0)$ is continuous and must intersect the X–axis at some point $(X_2,0)$ close to $(-X_0,0)$ if it is to approach the cycle. But, by Y–axis symmetry of the level curve through $(X_1,0)$ it follows that $(X_2,0)=(-X_1,0)$. The path through $(X_1,0)$ joins up with the point $(-X_1,0)$. Then by X–axis symmetry the level curve will also join up the points $(-X_1,0),(X_1,0)$. Thus, a cycle exists through the point $(X_1,0)$ so that the point cannot approach the claimed limit cycle. Thus, limit cycles do not exist. ∎

§9. $0 = b$.

There is a short proof as to why cycles cannot exist under the condition that $b = 0$.

Proposition 9.1. *There are no cycles.*

PROOF. The symmetric vector field ξ has the following form,

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a+k & 0 \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}, & x < -1; \\ \begin{bmatrix} a & 0 \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & -1 \leq x \leq 1; \\ \begin{bmatrix} a+k & 0 \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x. \end{cases}$$

Assume a cycle exists. Within the cycle lies a fixed point. But through the fixed point lies a linear invariant manifold parallel to the y-coordinate axis. Since cycles cannot encircle linear invariant manifolds they do not exist. ∎

References.

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Appendix A.

In this appendix, the relationship between continuous piecewise linear vector fields of [2,3] and the symmetric vector fields will be discussed. First, the respective definitions will be presented.

Definition A.1. ξ is a continuous piecewise linear vector field in canonical form $\Leftrightarrow$ there exists an integer $1 \leq n$, matrix $\mathbf{B}$, vectors $\alpha, \alpha_i, \beta_i$, $1 \leq i \leq n$ and constants γ_i, $1 \leq i \leq n$ for which $\xi(\mathbf{x}) = \alpha + \mathbf{B}\mathbf{x} + \sum_{i=1}^{n} \alpha_i | < \beta_i, \mathbf{x} > -\gamma_i |$.

The following definition is repeated from the main body of the text:

Definition 1.2. ξ is a symmetric vector field $\Leftrightarrow$ there exists constants a, b, c, d, k, l with either $k \neq 0$ or $l \neq 0$, and

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}, & x < -1; \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & -1 \leq x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x. \end{cases}$$

The following lemma shows how the defining constants in the two types of representation are related.

Lemma A.2. *(i) :Let* $\xi(\mathbf{x}) = \mathbf{B}\mathbf{x} + |[1\ 0]\mathbf{x} - 1|\alpha - |[1\ 0]\mathbf{x} + 1|\alpha$ *with*

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Then

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}, & x < -1; \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & -1 \leq x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x \end{cases}$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b_{11} - 2\alpha_1 & b_{12} \\ b_{21} - 2\alpha_2 & b_{22} \end{bmatrix}, \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 2\alpha_1 \\ 2\alpha_2 \end{bmatrix}.$$

(ii) :Let

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}, & x < -1; \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & -1 \leq x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x \end{cases}$$

be a symmetric vector field with k, l not both zero. Then $\xi(\mathbf{x}) = \mathbf{B}\mathbf{x} + |[1\ 0]\mathbf{x} - 1|\alpha - |[1\ 0]\mathbf{x} + 1|\alpha$ *with*

$$\alpha = \begin{bmatrix} \frac{1}{2}k \\ \frac{1}{2}l \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}.$$

Proof. The continuous piecewise linear vector field ξ in canonical form has the following decomposition,

$$\begin{aligned}
\xi(\mathbf{x}) &= \mathbf{B}\mathbf{x} + (1-[1\ 0]\mathbf{x})\alpha + ([1\ \ 0]\mathbf{x}+1)\alpha && \mathbf{x} \in \{(x,y) : x < -1\} \\
&= \mathbf{B}\mathbf{x} + 2\alpha \\
\xi(\mathbf{x}) &= \mathbf{B}\mathbf{x} + (1-[1\ 0]\mathbf{x})\alpha - ([1\ 0]\mathbf{x}+1)\alpha && \mathbf{x} \in \{(x,y) : -1 \le x \le 1\} \\
&= \mathbf{B}\mathbf{x} - 2\alpha[1\ 0]\mathbf{x} \\
\xi(\mathbf{x}) &= \mathbf{B}\mathbf{x} - (1-[1\ 0]\mathbf{x})\alpha - ([1\ 0]\mathbf{x}+1)\alpha && \mathbf{x} \in \{(x,y) : 1 < x\} \\
&= \mathbf{B}\mathbf{x} - 2\alpha.
\end{aligned}$$

Thus matching with the corresponding decomposition for a symmetric vector field,

$$\begin{aligned}
\xi\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix} && \mathbf{x} \in \{(x,y) : x < -1\} \\
\xi\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} && \mathbf{x} \in \{(x,y) : -1 \le x \le 1\} \\
\xi\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix} && \mathbf{x} \in \{(x,y) : 1 < x\}
\end{aligned}$$

gives,

$$\begin{aligned}
\mathbf{B}\mathbf{x} + 2\alpha &= \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix} && \mathbf{x} \in \{(x,y) : x < -1\} \\
(\mathbf{B} - 2\alpha[1\ 0])\mathbf{x} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} && \mathbf{x} \in \{(x,y) : -1 \le x \le 1\} \\
\mathbf{B}\mathbf{x} - 2\alpha &= \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix} && \mathbf{x} \in \{(x,y) : 1 < x\}.
\end{aligned}$$

(i) :Given the values of

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then by using the equivalences above,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b_{11} - 2\alpha_1 & b_{12} \\ b_{21} - 2\alpha_2 & b_{22} \end{bmatrix}, \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 2\alpha_1 \\ 2\alpha_2 \end{bmatrix}.$$

(ii) :Given the values of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}, \begin{bmatrix} k \\ l \end{bmatrix},$$

then by using the same set of equivalences,

$$\alpha = \begin{bmatrix} \frac{1}{2}k \\ \frac{1}{2}l \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix}. \qquad \blacksquare$$

The concept of the induced fixed point occurs frequently and in many of the results proved in the text. The next two lemmas relate the primary induced fixed point to the defining constants in the canonical representation of a symmetric vector field.

Lemma A.3. *Let the symmetric vector field ξ be given as*

$$\xi(\mathbf{x}) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} |[1\ 0]\mathbf{x} - 1| - \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} |[1\ 0]\mathbf{x} + 1|.$$

If $b_{11}b_{22} - b_{21}b_{12} \neq 0$ then the primary induced fixed point of ξ is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{2}{b_{11}b_{22} - b_{21}b_{12}} \begin{bmatrix} \alpha_1 b_{22} - \alpha_2 b_{12} \\ -\alpha_1 b_{21} + \alpha_2 b_{11} \end{bmatrix}.$$

PROOF. Using lemma A.2 the symmetric vector field in canonical form can be rewritten as

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}, & x < -1; \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & -1 \leq x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x \end{cases}$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b_{11} - 2\alpha_1 & b_{12} \\ b_{21} - 2\alpha_2 & b_{22} \end{bmatrix}, \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 2\alpha_1 \\ 2\alpha_2 \end{bmatrix}.$$

The primary induced fixed point is the solution (if it is uniquely defined) to

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2\alpha_1 \\ 2\alpha_2 \end{bmatrix}.$$

If $b_{11}b_{22} - b_{21}b_{12} \neq 0$ then a unique solution exists and is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{2}{b_{11}b_{22} - b_{21}b_{12}} \begin{bmatrix} \alpha_1 b_{22} - \alpha_2 b_{12} \\ -\alpha_1 b_{21} + \alpha_2 b_{11} \end{bmatrix}. \blacksquare$$

Lemma A.4. *Let a, b, c, d be constants. If the symmetric vector field has $(x_i, y_i), x_i \neq 1$ as the primary induced fixed point then*

$$\xi(\mathbf{x}) = \begin{bmatrix} a + \frac{ax_i+by_i}{1-x_i} & b \\ c + \frac{cx_i+dy_i}{1-x_i} & d \end{bmatrix} + \frac{1}{2(1-x_i)} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix} |[1\ 0]\mathbf{x} - 1| - \frac{1}{2(1-x_i)} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix} |[1\ 0]\mathbf{x} + 1|.$$

PROOF. The symmetric vector field

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k \\ l \end{bmatrix}, & x < -1; \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & -1 \leq x \leq 1; \\ \begin{bmatrix} a+k & b \\ c+l & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}, & 1 < x \end{cases}$$

for

$$\begin{bmatrix} k \\ l \end{bmatrix} = \frac{1}{1-x_i} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix}$$

has the point (x_i, y_i) as the primary induced fixed point. Using lemma A.2 to convert the representation into canonical form gives

$$\alpha = \frac{1}{2(1-x_i)} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix}, \mathbf{B} = \begin{bmatrix} a + \frac{ax_i+by_i}{1-x_i} & b \\ c + \frac{cx_i+dy_i}{1-x_i} & d \end{bmatrix}$$

and thus

$$\xi(\mathbf{x}) = \begin{bmatrix} a + \frac{ax_i+by_i}{1-x_i} & b \\ c + \frac{cx_i+dy_i}{1-x_i} & d \end{bmatrix} + \frac{1}{2(1-x_i)} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix} |[1\ 0]\mathbf{x} - 1| - \frac{1}{2(1-x_i)} \begin{bmatrix} ax_i + by_i \\ cx_i + dy_i \end{bmatrix} |[1\ 0]\mathbf{x} + 1| \blacksquare$$

Figure captions.

Figure 1. The symmetric vector field is conjugate to a linear vector field with an unstable focus at the origin.

Figure 2. The symmetric vector field has an attracting limit cycle, as indicated in bold type. The effect of symmetry of the vector field is clearly evident in the symmetry of the limit cycle.

Figure 3. A pair of saddle-node connections is another possibility for a symmetric vector field as indicated by this phase portrait. The invariant manifolds of the saddle points are indicated in bold type.

Figure 4. The three lines of fixed points, indicated in bold type, divides $\Re^2$ into two disjoint regions. In each region there are no attractors.

Figure 5. The symmetric vector field is conjugate to a linear vector field with a saddle point at the origin. The invariant manifolds through the origin are indicated in bold type.

Figure 6. At the origin is a saddle point. The invariant manifolds through the origin are indicated in bold type. Furthermore, there also exists a pair of unstable nodes. The saddle point at the origin and this pair of unstable nodes together allow the possibility of saddle-node connections. The saddle-node connections occur on the stable manifolds through the origin.

Figure 7. The symmetric vector field is conjugate to a linear vector field with a saddle point at the origin. The invariant manifolds through the origin are indicated in bold type.

Figure 8. The symmetric vector field is conjugate to a linear vector field with an unstable node at the origin. Note that the y-axis is invariant under the vector field, this prevents the formation of cycles.

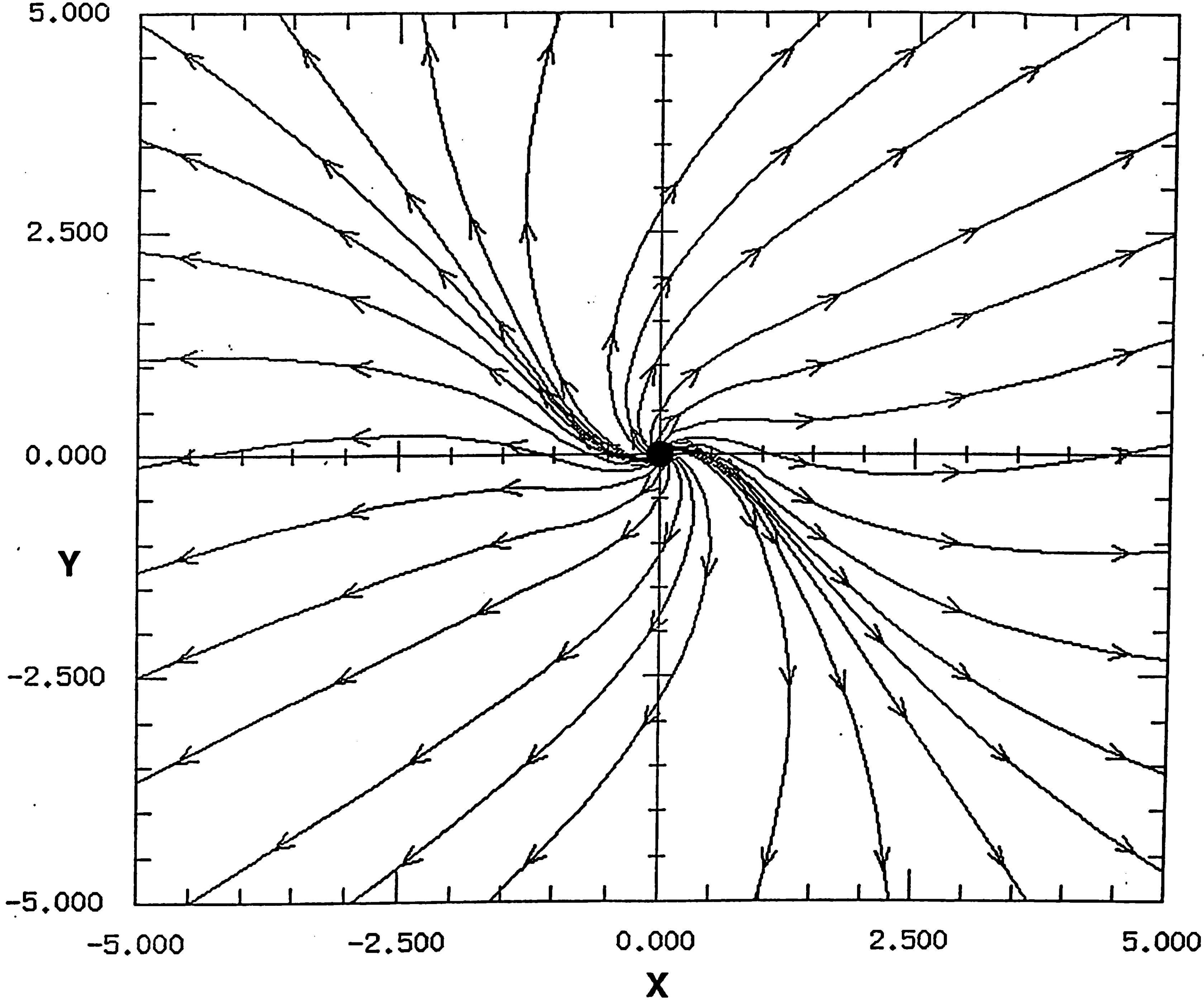

FIGURE: 1

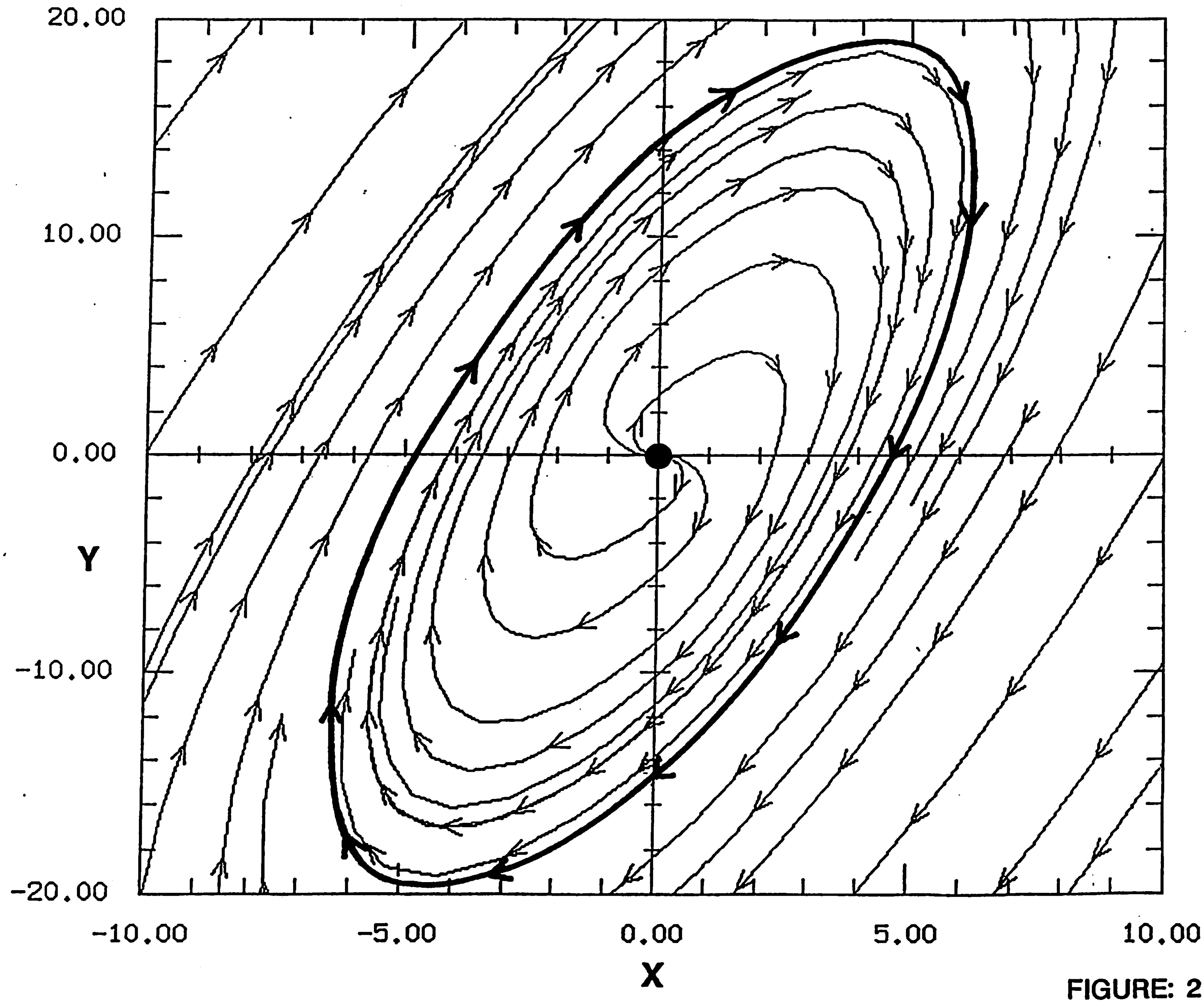

FIGURE: 2

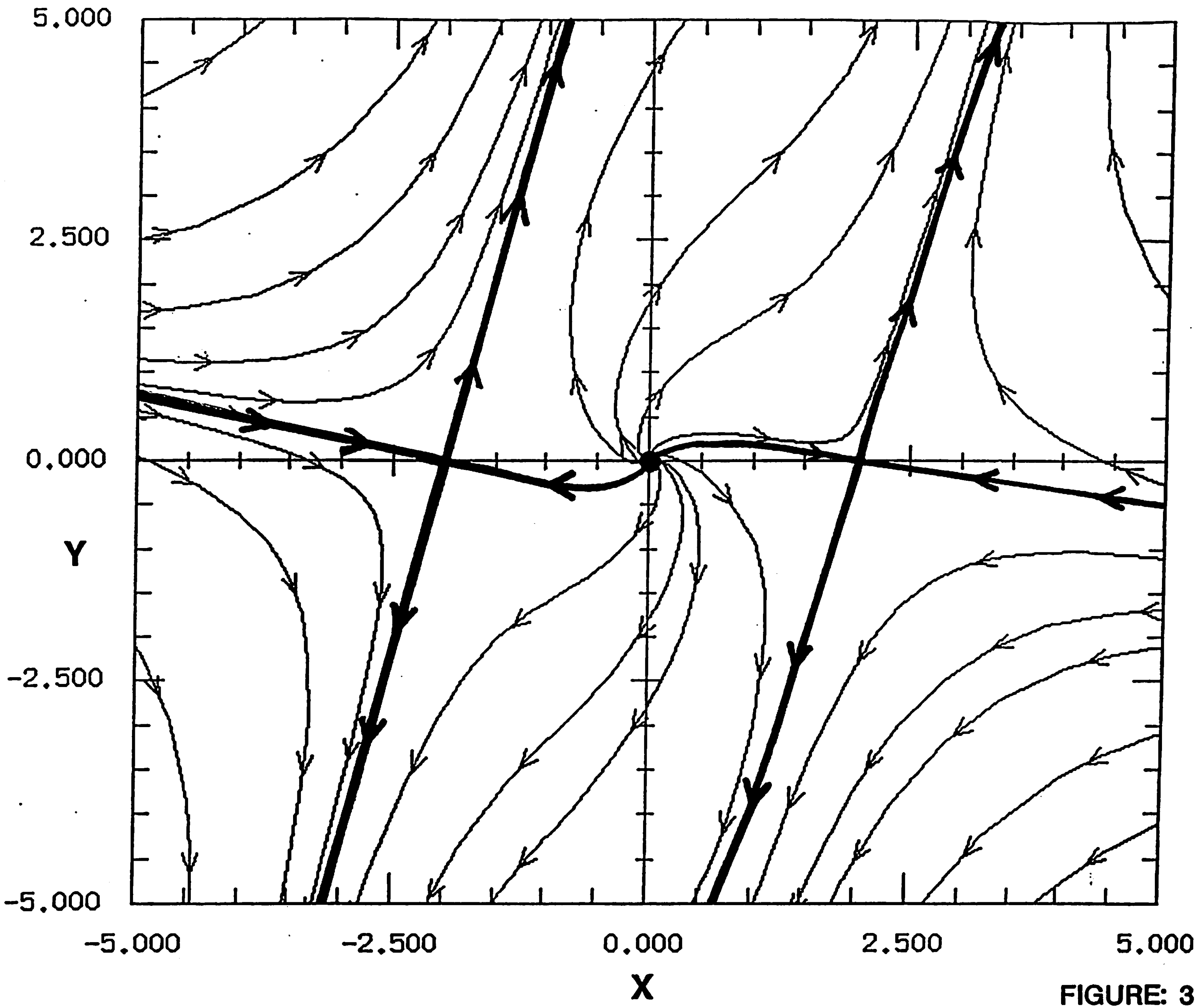

FIGURE: 3

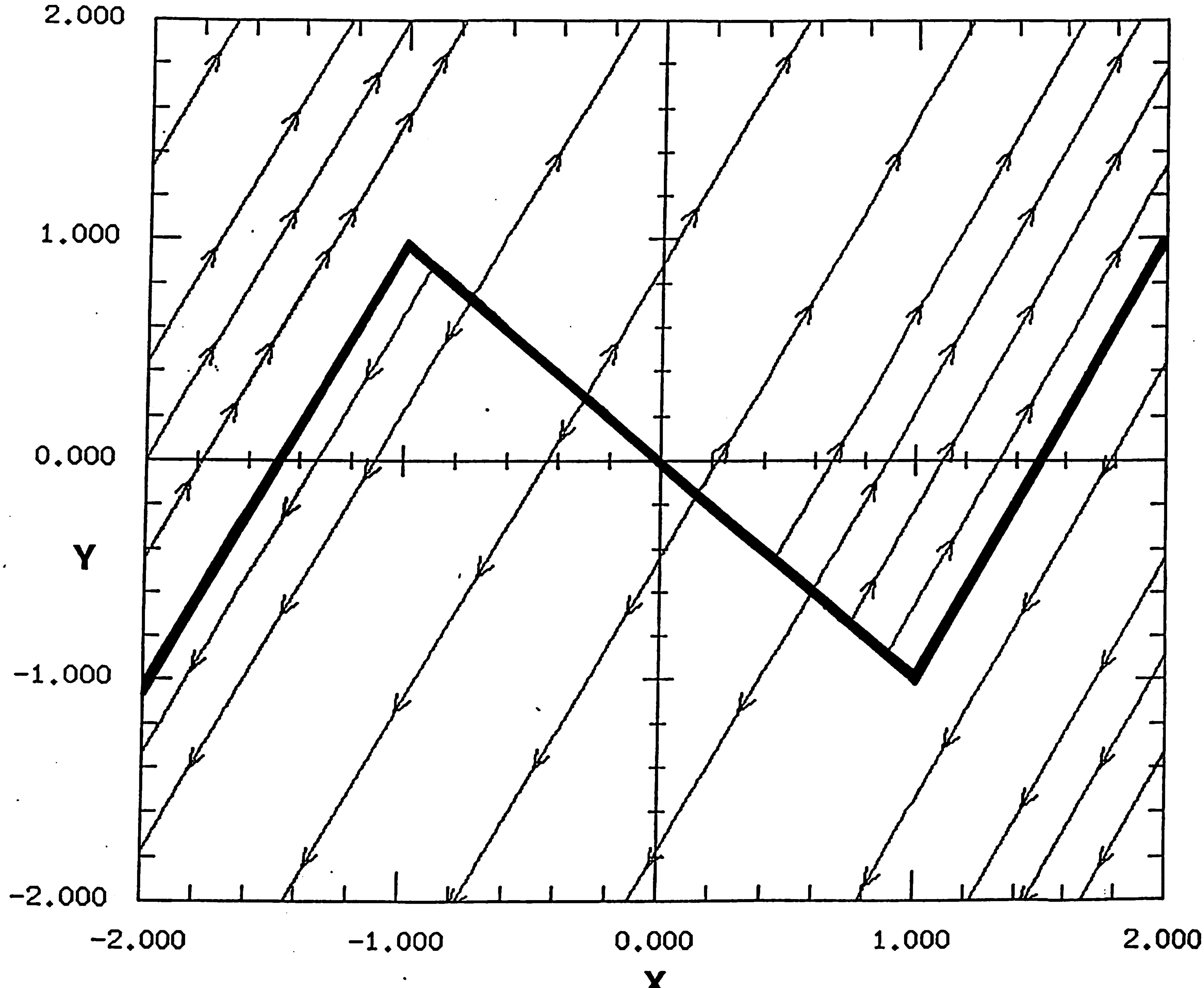

FIGURE: 4

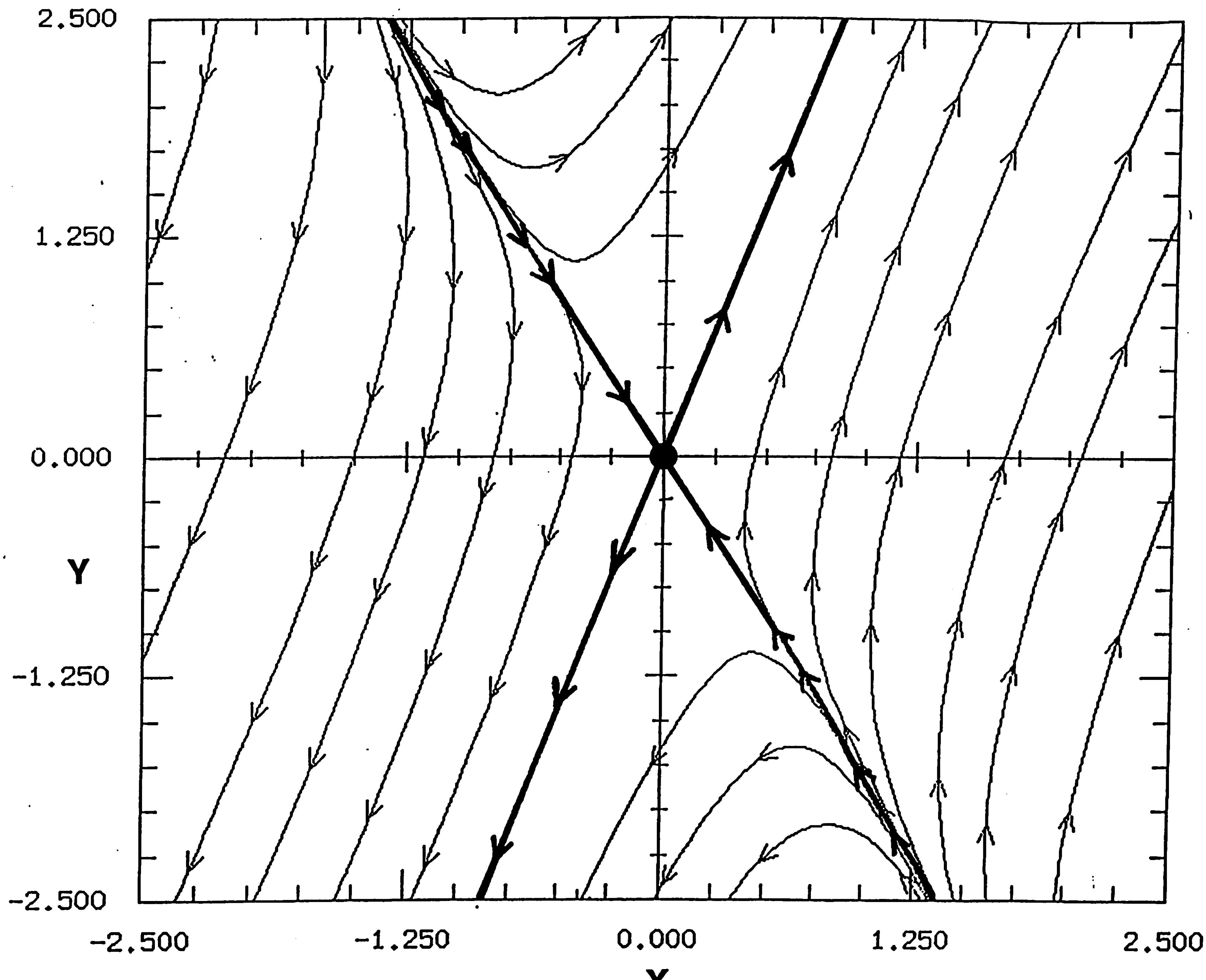

FIGURE: 5

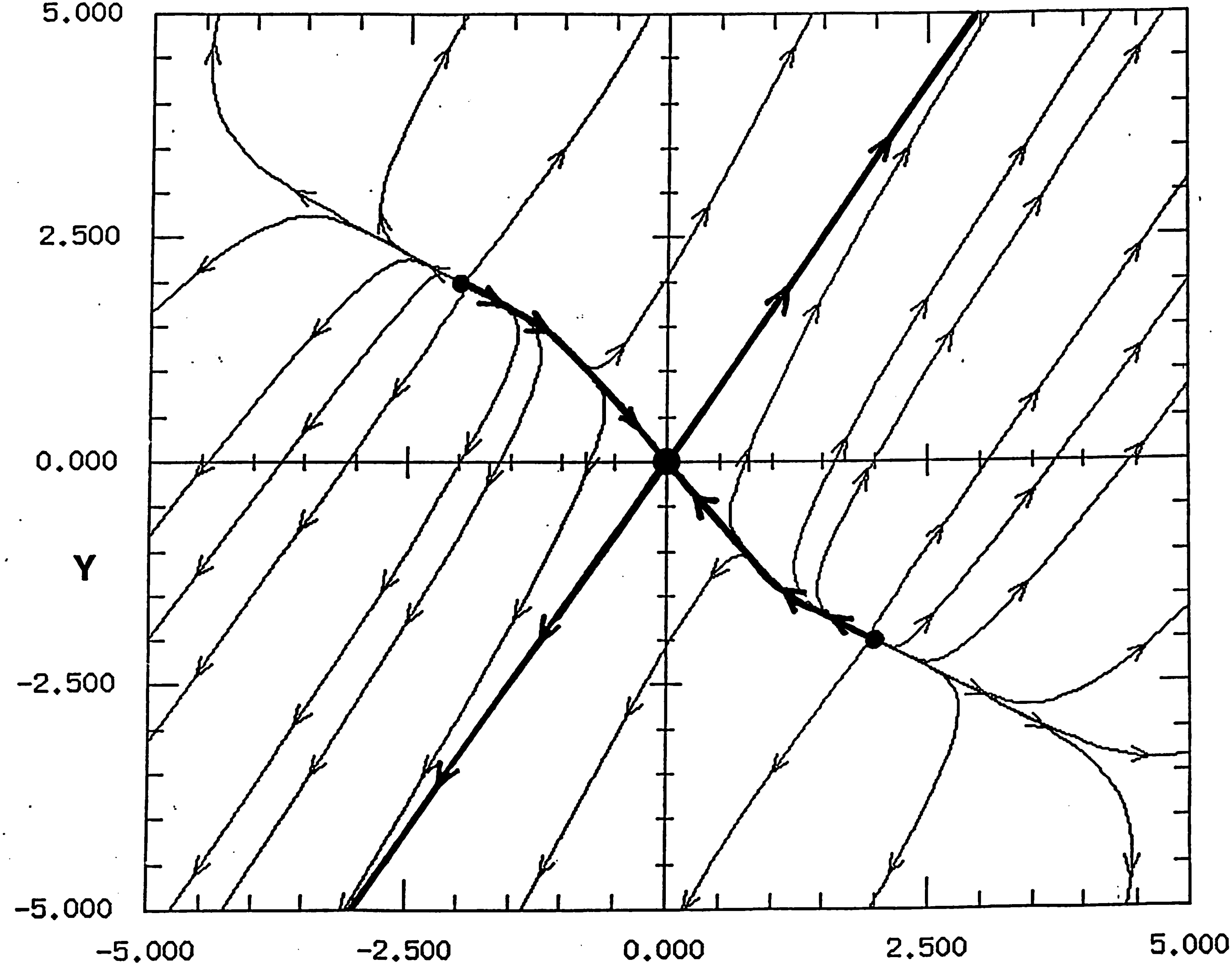

FIGURE: 6

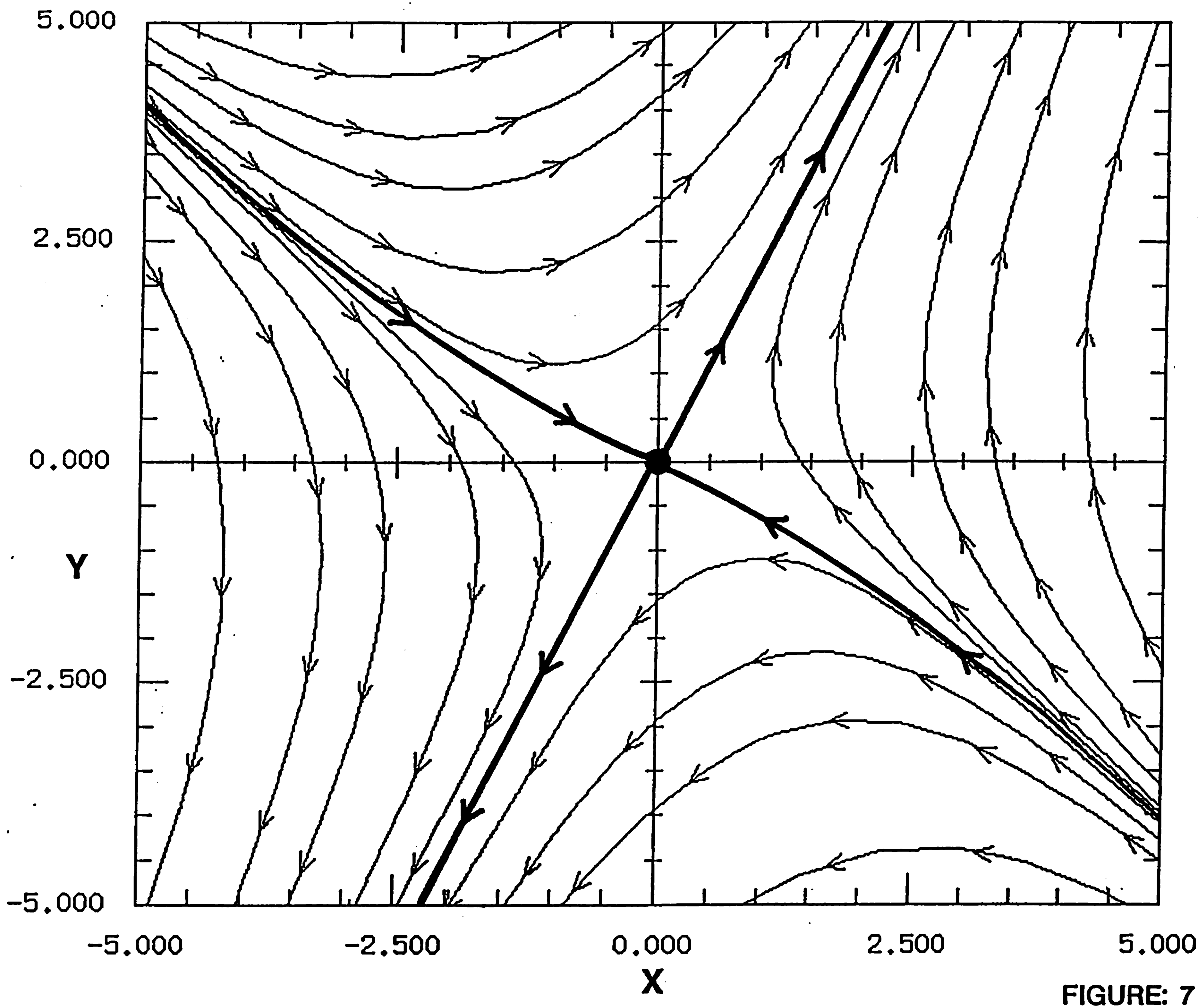

FIGURE: 7

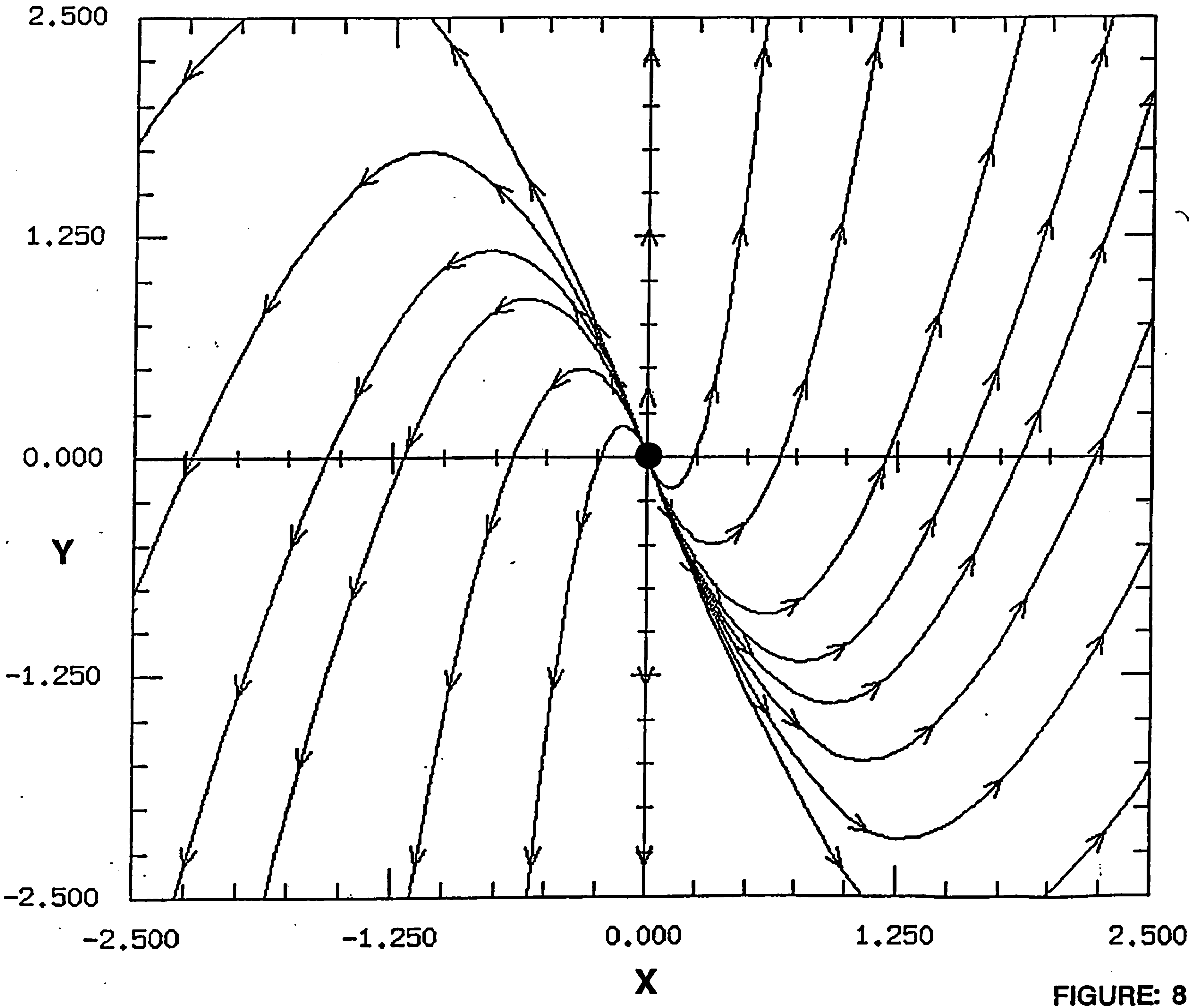

FIGURE: 8