INVARIANCE PROPERTIES OF CONTINUOUS PIECEWISE-LINEAR VECTOR FIELDS

by

Robert Lum and Leon O. Chua

Memorandum No. UCB/ERL M90/38

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INVARIANCE PROPERTIES OF CONTINUOUS PIECEWISE-LINEAR VECTOR FIELDS. †

Robert Lum AND Leon O. Chua. ††

Abstract

In the application of continuous piecewise-linear vector fields to the modelling of systems, it is often desirable that the model preserve certain properties of the system that it is supposed to emulate. These properties may represent fluid incompressibility, conservation of energy and symmetries. In this papar necessary and sufficient conditions are stated for the identification and imposition of several such properties.

The first part of the paper addresses continuous piecewise-linear vector fields that are divergence free, gradient systems, and Hamiltonian systems. The second half of the paper determines possible relationships between a lattice piecewise-linear vector field and a transformation matrix. This will facilitate the identification of symmetries of a lattice vector field and the classification of lattice vector fields possessing certain symmetry properties.

With these results, the modelling process via piecewise-linear vector fields will have the capacity to preserve intrinsic structure of the modelled system.

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§0. Introduction.

The modelling of physical systems may require certain salient properties of the original system to be preserved. For example, in modelling a reciprocal RC or RL circuit, the vector field must be a gradient system. In the modelling of fluid flow, the model should be divergence free to reflect the incompressibility of the original fluid. In buckling plate problems, with the deformations of a square plate under increaing pressure, the symmetry of the situation has to be taken into account. In this paper, the identification of several important types of continuous piecewise-linear vector fields is undertaken.

The first half of the paper is devoted to giving necessary and sufficient conditions for the identification of divergence free, gradient, and Hamiltonian piecewise-linear vector fields. With these results a researcher will be able to validate whether certain properties of the original vector field are maintained under a piecewise-linear vector field modelling. On the other hand, the researcher may a priori require that a piecewise linear vector field have certain properties, and determine whether or not it is possible to model the original vector field under the condition of possesing this property.

The second half of this paper determines some results between lattice piecewise-linear vector fields and symmetry. Given such a vector field and a matrix acting on the state space there are several questions that can be asked about the relation between the lattice piecewise-linear vector field and the matrix. Are the vectors of the lattice piecewise-linear vector field invariant under the transformation induced by the matrix? Is the lattice piecewise-linear vector field invariant under transformation of the state space by the matrix? Do the lattice piecewise-linear vector field and matrix commute as functions? Answers to these questions form the second half of the paper.

§1. Definitions.

In this section the definition of a continuous piecewise-linear vector field and lattice piecewise-linear vector field are presented.

Definition 1.1. A continuous piecewise linear vector field ξ in n independent variables is given by

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|$$

where $0 < \alpha_{j1}^2 + \ldots + \alpha_{jn}^2$, $0 < \beta_{j1}^2 + \ldots + \beta_{jn}^2$ for $j = 1 \ldots m$. Henceforth, continuous piecewise linear vector fields will be called vector fields.

Definition 1.2. The signature of a point $\mathbf{x}$ is the m-tuple given by

$$\mathrm{sig}(\mathbf{x}) = (\mathrm{sgn}(\beta_1^t\mathbf{x} - \gamma_1), \ldots, \mathrm{sgn}(\beta_m^t\mathbf{x} - \gamma_m))$$

where $\mathrm{sgn}(x)$ is $-1, 0, 1$ depending on whether $x < 0, x = 0, 0 < x$ respectively.

Definition 1.3. Given a vector field ξ in n independent variables there is an associated partition of $\Re^n$ where

$$\mathrm{Part}(\xi) = \{\overline{\mathrm{A}_{i_1,\ldots,i_m}} : \mathrm{A}_{i_1,\ldots,i_m} = \{\mathbf{x} : \mathrm{sig}(\mathbf{x}) = (i_1, \ldots, i_m)\}, (i_1, \ldots, i_m) \in \{-1, 1\}^m\}$$

and $\overline{\mathrm{A}}_{i_1,\ldots,i_m}$ is the closure of the set $\mathrm{A}_{i_1,\ldots,i_m}$. The set $\mathrm{A}_{i_1,\ldots,i_m}$ is open, being the intersection of half-planes of the form $0 < \beta_i^t\mathbf{x} - \gamma_i$ or $\beta_i^t\mathbf{x} - \gamma_i < 0$.

Lemma 1.4. *For every $\mathrm{A}_{i_1,\ldots,i_m} \in \mathrm{Part}(\xi)$ of a piecewise-linear vector field ξ there is a unique linear vector field $\xi_{i_1,\ldots,i_m}$ such that $\xi_{i_1,\ldots,i_m}|_{\mathrm{A}_{i_1,\ldots,i_m}} = \xi|_{\mathrm{A}_{i_1,\ldots,i_m}}$.*

PROOF. For $\mathbf{x} \in \mathrm{A}_{i_1,\ldots,i_m}$ the vector $\xi(\mathbf{x})$ is given by

$$\begin{aligned}\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right| \\ &= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} i_j\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left(\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right) \\ &= \begin{bmatrix} \alpha_1 - \sum_{j=1}^m i_j\alpha_{j1}\gamma_j \\ \vdots \\ \alpha_n - \sum_{j=1}^m i_j\alpha_{jn}\gamma_j \end{bmatrix} + \begin{bmatrix} b_{11} + \sum_{j=1}^m i_j\alpha_{j1}\beta_{j1} & \dots & b_{1n} + \sum_{j=1}^m i_j\alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ b_{n1} + \sum_{j=1}^m i_j\alpha_{jn}\beta_{j1} & \dots & b_{nn} + \sum_{j=1}^m i_j\alpha_{jn}\beta_{jn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.\end{aligned}$$

It is immediately clear that there exists a unique linear vector field $\xi_{i_1,\ldots,i_m}$ with the property that $\xi_{i_1,\ldots,i_m}|_{\mathrm{A}_{i_1,\ldots,i_m}} = \xi|_{\mathrm{A}_{i_1,\ldots,i_m}}$. The linear vector field $\xi_{i_1,\ldots,i_m}$ will be written as $\xi_{i_1,\ldots,i_m}(\mathbf{x}) = \mathbf{d}_{i_1,\ldots,i_m} + \mathbf{M}_{i_1,\ldots,i_m}\mathbf{x}$. ∎

Definition 1.5. A lattice continuous piecewise linear vector field ξ in n independent variables is given by

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{i=1}^{n} \sum_{j=1}^{n_i} \begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} |x_i - \gamma_{ij}|$$

where $0 < \alpha_{ij1}^2 + \ldots + \alpha_{ijn}^2$, $\gamma_{i1} < \ldots < \gamma_{in_i}$ for $i = 1 \ldots n$. Henceforth, lattice continuous piecewise linear vector fields will be called lattice vector fields.

Definition 1.6. Given a lattice vector field ξ in n independent variables there is an associated partition of $\Re^n$ where

$$\mathrm{Part}(\xi) = \{\mathrm{A}_{k_1,\ldots,k_n} = \mathrm{L}_{1k_1} \times \ldots \times \mathrm{L}_{nk_n} : 0 \le k_1 \le n_1, \ldots, \ 0 \le k_n \le n_n\}$$

with

$$\begin{aligned} \mathrm{L}_{i0} &= (-\infty, \gamma_{i1}] \\ \mathrm{L}_{ij} &= [\gamma_{ij}, \gamma_{ij+1}] & \qquad 1 \le j < n_i \\ \mathrm{L}_{in_i} &= [\gamma_{in_n}, \infty). \end{aligned}$$

Lemma 1.7. *For every $\mathrm{A}_{k_1,\ldots,k_n} \in \mathrm{Part}(\xi)$ of a lattice vector field ξ, there is a unique linear vector field $\xi_{k_1,\ldots,k_n}$ such that $\xi_{k_1,\ldots,k_n}|_{\mathrm{A}_{k_1,\ldots,k_n}} = \xi|_{\mathrm{A}_{k_1,\ldots,k_n}}$.*

Proof. For $\mathbf{x} \in \mathrm{A}_{k_1,\ldots,k_n}$ the vector $\xi(\mathbf{x})$ is given by

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{i=1}^{n} \sum_{j=1}^{n} \begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} |x_i - \gamma_{ij}|$$

$$= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{i=1}^{n} \left(\sum_{j=1}^{k_i} \begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} (x_i - \gamma_{ij}) + \sum_{j=k_i+1}^{n_i} \begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} (\gamma_{ij} - x_i) \right)$$

$$= \begin{bmatrix} \alpha_1 - \sum_{i=1}^{n} \left(\sum_{j=1}^{k_i} \alpha_{ij1}\gamma_{ij} - \sum_{j=k_i+1}^{n_i} \alpha_{ij1}\gamma_{ij} \right) \\ \vdots \\ \alpha_n - \sum_{i=1}^{n} \left(\sum_{j=1}^{k_i} \alpha_{ijn}\gamma_{ij} - \sum_{j=k_i+1}^{n_i} \alpha_{ijn}\gamma_{ij} \right) \end{bmatrix} +$$

$$\begin{bmatrix} b_{11} + \sum_{j=1}^{k_1} \alpha_{1j1} - \sum_{j=k_1+1}^{n_1} \alpha_{1j1} & \dots & b_{1n} + \sum_{j=1}^{k_n} \alpha_{nj1} - \sum_{j=k_n+1}^{n_1} \alpha_{nj1} \\ \vdots & & \vdots \\ b_{n1} + \sum_{j=1}^{k_1} \alpha_{1jn} - \sum_{j=k_1+1}^{n_1} \alpha_{1jn} & \dots & b_{nn} + \sum_{j=1}^{k_n} \alpha_{njn} - \sum_{j=k_n+1}^{n_1} \alpha_{njn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

It is immediately clear that there exists a unique linear vector field $\xi_{k_1,\ldots,k_n}$ with the property that $\xi_{k_1,\ldots,k_n}|_{\mathrm{A}_{k_1,\ldots,k_n}} = \xi|_{\mathrm{A}_{k_1,\ldots,k_n}}$. The linear vector field $\xi_{k_1,\ldots,k_n}$ will be written as $\xi_{k_1,\ldots,k_n}(\mathbf{x}) = \mathbf{d}_{k_1,\ldots,k_n} + \mathbf{M}_{k_1,\ldots,k_n}\mathbf{x}$. ∎

§2. The divergence free vector field.

In this section necessary and sufficient conditions are given for a vector field to be divergence free.

Theorem 2.1. *Let ξ be a vector field of the form*

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|.$$

The vector field ξ is divergence free if and only if

$$\sum_{i=1}^{n} b_{ii} = 0$$

and $\sum_{k=1}^{n}\alpha_{jk}\beta_{jk} = 0$ for $1 \le j \le m$.

Proof. First note that if a linear vector field $\zeta(\mathbf{x}) = \mathbf{d} + \mathbf{M}\mathbf{x}$ is divergence free then the trace of the matrix $\mathbf{M}$ is zero. Assume that the vector field ξ is a divergence free. For given signatures $(i_1, \dots, i_{j-1}, i_j, i_{j+1}, \dots, i_m), (i_1, \dots, i_{j-1}, -i_j, i_{j+1}, \dots, i_m)$ consider the linear vector fields $\xi_{i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m}$ and $\xi_{i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m}$ Thus, as divergence free linear vector fields, the traces of the matrices $\mathbf{M}_{i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m}$ and $\mathbf{M}_{i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m}$ corresponding to these two linear vector fields are zero. In particular, the matrix $\mathbf{M}_{i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m} - \mathbf{M}_{i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m}$ has zero trace from which it follows that

$$2\begin{bmatrix} \alpha_{j1}\beta_{j1} & \dots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \dots & \alpha_{jn}\beta_{jn} \end{bmatrix}$$

has zero trace. Thus $\sum_{k=1}^{n}\alpha_{jk}\beta_{jk} = 0$ for $1 \le j \le m$. Now consider the linear vector field $\xi_{1,\dots,1}$. The matrix $\mathbf{M}_{1,\dots,1}$ corresponding to this linear vector field also has zero trace, thus

$$\begin{bmatrix} b_{11} + \sum_{j=1}^{m}\alpha_{j1}\beta_{j1} & \dots & b_{1n} + \sum_{j=1}^{m}\alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ b_{n1} + \sum_{j=1}^{m}\alpha_{jn}\beta_{j1} & \dots & b_{nn} + \sum_{j=1}^{m}\alpha_{jn}\beta_{jn} \end{bmatrix}$$

has zero trace from which it follows that $\sum_{i=1}^{n} b_{ii} = 0$.

Conversely, assume that $\sum_{i=1}^{n} b_{ii} = 0$ and $\sum_{k=1}^{n}\alpha_{jk}\beta_{jk} = 0$ for $1 \le j \le m$. Then for the linear vector fields $\xi_{i_1,\dots,i_m}$ the matrices $\mathbf{M}_{i_1,\dots,i_m}$,

$$\begin{bmatrix} b_{11} + \sum_{j=1}^{m} i_j\alpha_{j1}\beta_{j1} & \dots & b_{1n} + \sum_{j=1}^{m} i_j\alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ b_{n1} + \sum_{j=1}^{m} i_j\alpha_{jn}\beta_{j1} & \dots & b_{nn} + \sum_{j=1}^{m} i_j\alpha_{jn}\beta_{jn} \end{bmatrix}$$

have trace

$$\sum_{k=1}^{n}\left(b_{kk} + \sum_{j=1}^{m} i_j\alpha_{jk}\beta_{jk}\right) = \sum_{i=1}^{n} b_{ii} + \sum_{j=1}^{m} i_j\sum_{k=1}^{n}\alpha_{jk}\beta_{jk} = 0.$$

Thus, the linear vector field $\xi_{i_1,\ldots,i_m}$ is divergence free, from which it follows that the original vector field ξ is divergence free. ∎

Example 2.2. (Figure 1.) A divergence free vector field in $\Re^2$ is that given by

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}3\\4\end{bmatrix}\left|\begin{bmatrix}4\\-3\end{bmatrix}^t\begin{bmatrix}x\\y\end{bmatrix}-2\right|.$$

It is clear that the defining constants of the vector field satisfy the conditions of theorem 2.1, thus implying that the vector field is divergence free.

Example 2.3. (Figure 2.) A divergence free vector field in $\Re^3$ is that given by

$$\xi\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}1\\2\\3\end{bmatrix}\left|\begin{bmatrix}1\\4\\-3\end{bmatrix}^t\begin{bmatrix}x\\y\\z\end{bmatrix}-1\right|+\begin{bmatrix}1\\4\\-3\end{bmatrix}\left|\begin{bmatrix}1\\2\\3\end{bmatrix}^t\begin{bmatrix}x\\y\\z\end{bmatrix}+1\right|.$$

This is another application of theorem 2.1 to construct a divergence free vector field, this time in $\Re^3$.

Corollary 2.4. *Let ξ be a lattice vector field of the form*

$$\xi\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}\alpha_1\\\vdots\\\alpha_n\end{bmatrix}+\begin{bmatrix}b_{11}&\ldots&b_{1n}\\\vdots&&\vdots\\b_{n1}&\ldots&b_{nn}\end{bmatrix}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}+\sum_{i=1}^{n}\sum_{j=1}^{n_i}\begin{bmatrix}\alpha_{ij1}\\\vdots\\\alpha_{ijn}\end{bmatrix}|x_i-\gamma_{ij}|.$$

The lattice vector field ξ is divergence free if and only if

$$\sum_{i=1}^{n}b_{ii}=0$$

and $\alpha_{iji}=0$ for $1\le i\le n,\ 1\le j\le n_i$.

Proof. Immediate conseqence of theorem 2.1. ∎

Example 2.5. (Figure 3.) A divergence free lattice vector field in $\Re^2$ is that given by

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}2&4\\5&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}|x-2|+\begin{bmatrix}1\\0\end{bmatrix}|y-1|.$$

It is clear that the defining constants of the lattice vector field satisfy the conditions of corollary 2.4, thus implying that the lattice vector field is divergence free.

Example 2.6. (Figure 4.) A divergence free lattice vector field in $\Re^3$ is that given by

$$\xi\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\1\\2\end{bmatrix}+\begin{bmatrix}2&0&1\\2&3&0\\1&1&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}0\\1\\3\end{bmatrix}|x-1|+\begin{bmatrix}0\\3\\1\end{bmatrix}|x+1|+\begin{bmatrix}1\\0\\-2\end{bmatrix}|y|.$$

This is another application of corollary 2.4 to construct a divergence free lattice vector field, this time in $\Re^3$.

§3. The gradient vector field.

In this section necessary and sufficient conditions are given for a vector field to be a gradient system.

Lemma 3.1. *Let ξ be the linear vector field given by*

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} m^{11} & \dots & m^{1n} \\ \vdots & & \vdots \\ m^{n1} & \dots & m^{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d^1 \\ \vdots \\ d^n \end{bmatrix}.$$

Then ξ is a gradient system if and only if the matrix

$$\begin{bmatrix} m^{11} & \dots & m^{1n} \\ \vdots & & \vdots \\ m^{n1} & \dots & m^{nn} \end{bmatrix}$$

is symmetric.

Proof. Assume that ξ is a gradient system with gradient function $G(x_1, \dots, x_n)$. Then

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{\partial G}{\partial x_1} \\ \vdots \\ \frac{\partial G}{\partial x_n} \end{bmatrix}$$

from which it follows that

$$\mathbf{D}\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} m^{11} & \dots & m^{1n} \\ \vdots & & \vdots \\ m^{n1} & \dots & m^{nn} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 G}{\partial x_1^2} & \dots & \frac{\partial^2 G}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 G}{\partial x_1 \partial x_n} & \dots & \frac{\partial^2 G}{\partial^2 x_n} \end{bmatrix}.$$

Thus

$$\begin{bmatrix} m^{11} & \dots & m^{1n} \\ \vdots & & \vdots \\ m^{n1} & \dots & m^{nn} \end{bmatrix}$$

is a symmetric matrix.

Conversly, suppose that the matrix

$$\begin{bmatrix} m^{11} & \dots & m^{1n} \\ \vdots & & \vdots \\ m^{n1} & \dots & m^{nn} \end{bmatrix}$$

is symmetric, then a gradient function for ξ is given by

$$G(x_1, \dots, x_n) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m^{ij} x_i x_j + \sum_{i=1}^{n} d^i x_i + c$$

for some constant c. ∎

Theorem 3.2. *Let ξ be a vector field of the form*

$$\xi\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}+\begin{bmatrix}b_{11} & \dots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \dots & b_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}+\sum_{j=1}^{m}\begin{bmatrix}\alpha_{j1}\\ \vdots\\ \alpha_{jn}\end{bmatrix}\left|\begin{bmatrix}\beta_{j1}\\ \vdots\\ \beta_{jn}\end{bmatrix}^t\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}-\gamma_j\right|.$$

The vector field ξ is a gradient system if and only if

$$\begin{bmatrix}b_{11} & \dots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \dots & b_{nn}\end{bmatrix}$$

is symmetric and

$$\begin{bmatrix}\alpha_{j1}\\ \vdots\\ \alpha_{jn}\end{bmatrix}=k_j\begin{bmatrix}\beta_{j1}\\ \vdots\\ \beta_{jn}\end{bmatrix}$$

for $1\le j\le m$.

PROOF. Assume that the vector field ξ is a gradient system with gradient function G. For given signatures $(i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m),(i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m)$ consider the linear vector fields $\xi_{i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m}$ and $\xi_{i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m}$. By lemma 3.1, the matrices $\mathrm{M}_{i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m}$ and $\mathrm{M}_{i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m}$ corresponding to these two linear vector fields are symmetric. In particular, the matrix $\mathrm{M}_{i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m}-\mathrm{M}_{i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m}$ is symmetric from which it follows that

$$2\begin{bmatrix}\alpha_{j1}\beta_{j1} & \dots & \alpha_{j1}\beta_{jn}\\ \vdots & & \vdots\\ \alpha_{jn}\beta_{j1} & \dots & \alpha_{jn}\beta_{jn}\end{bmatrix}$$

is symmetric. Thus $\alpha_{jk}\beta_{jl}=\alpha_{jl}\beta_{jk}$. As $0<\beta_{j1}^2+\dots+\beta_{jn}^2$ then $\beta_{jl}\neq 0$ for some $1\le l\le n$. Assume that $\beta_{j1}\neq 0$, then $\alpha_{jk}=(\alpha_{j1}/\beta_{j1})\beta_{jk}$. Let $k_j=(\alpha_{j1}/\beta_{j1})$, then

$$\begin{bmatrix}\alpha_{j1}\\ \vdots\\ \alpha_{jn}\end{bmatrix}=k_j\begin{bmatrix}\beta_{j1}\\ \vdots\\ \beta_{jn}\end{bmatrix}.$$

Now consider the linear vector field $\xi_{1,\dots,1}$. The matrix $\mathrm{M}_{1,\dots,1}$ corresponding to this linear vector field is symmetric, thus

$$\begin{bmatrix}b_{11}+\sum_{j=1}^m k_j\beta_{j1}\beta_{j1} & \dots & b_{1n}+\sum_{j=1}^m k_j\beta_{j1}\beta_{jn}\\ \vdots & & \vdots\\ b_{n1}+\sum_{j=1}^m k_j\beta_{jn}\beta_{j1} & \dots & b_{nn}+\sum_{j=1}^m k_j\beta_{jn}\beta_{jn}\end{bmatrix}$$

is symmetric from which it follows that

$$\begin{bmatrix}b_{11} & \dots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \dots & b_{nn}\end{bmatrix}$$

is a symmetric matrix.

Conversely, assume that

$$\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

is symmetric and

$$\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} = k_j \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}$$

for $1 \le j \le m$. Then for the linear vector fields $\xi_{i_1,\ldots,i_m}$ the matrices $\mathbf{M}_{i_1,\ldots,i_m}$,

$$\begin{bmatrix} b_{11} + \sum_{j=1}^{m} i_j \alpha_{j1}\beta_{j1} & \cdots & b_{1n} + \sum_{j=1}^{m} i_j \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ b_{n1} + \sum_{j=1}^{m} i_j \alpha_{jn}\beta_{j1} & \cdots & b_{nn} + \sum_{j=1}^{m} i_j \alpha_{jn}\beta_{jn} \end{bmatrix} =$$

$$\begin{bmatrix} b_{11} + \sum_{j=1}^{m} i_j k_j \beta_{j1}\beta_{j1} & \cdots & b_{1n} + \sum_{j=1}^{m} i_j k_j \beta_{j1}\beta_{jn} \\ \vdots & & \vdots \\ b_{n1} + \sum_{j=1}^{m} i_j k_j \beta_{jn}\beta_{j1} & \cdots & b_{nn} + \sum_{j=1}^{m} i_j k_j \beta_{jn}\beta_{jn} \end{bmatrix}$$

are symmetric. By lemma 3.1, there exists a gradient function $G_{i_1,\ldots,i_m}(x_1,\ldots,x_n)$ defined on $\mathbf{A}_{i_1,\ldots,i_m}$. Finally define

$$G(x_1,\ldots,x_n) = G_{i_1,\ldots,i_m}(x_1,\ldots,x_n) \qquad (x_1,\ldots,x_n) \in \mathbf{A}_{i_1,\ldots,i_m}.$$

It is immediate that G is a gradient function for ξ. ∎

EXAMPLE 3.3. (Figure 5.) A gradient vector field in $\Re^2$ is that given by

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\left|\begin{bmatrix} 3 \\ 1 \end{bmatrix}^t \begin{bmatrix} x \\ y \end{bmatrix} + 1\right| + \begin{bmatrix} 1 \\ 3 \end{bmatrix}\left|\begin{bmatrix} -1 \\ -3 \end{bmatrix}^t \begin{bmatrix} x \\ y \end{bmatrix}\right|.$$

By theorem 3.2 this is an example of a gradient vector field. The symmetry that the theorem demands is clearly evident.

EXAMPLE 3.4. (Figure 6.) A gradient vector field in $\Re^3$ is that given by

$$\xi\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\left|\begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}^t \begin{bmatrix} x \\ y \\ z \end{bmatrix} + 1\right| + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\left|\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}^t \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 1\right|.$$

This example was also constructed by the use of theorem 3.2. Again, the symmetry required for a gradient vector field is clearly evident.

Definition 3.5. The vector $\mathbf{e}_i$ is the ith coordinate vector where all the entries are zero except for a one in the ith row.

Corollary 3.6. *Let ξ be a lattice vector field of the form*

$$\xi\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}+\begin{bmatrix}b_{11} & \dots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \dots & b_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}+\sum_{i=1}^{n}\sum_{j=1}^{n_i}\begin{bmatrix}\alpha_{ij1}\\ \vdots\\ \alpha_{ijn}\end{bmatrix}|x_i-\gamma_{ij}|.$$

The lattice vector field ξ is a gradient system if and only if

$$\begin{bmatrix}b_{11} & \dots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \dots & b_{nn}\end{bmatrix}$$

is symmetric and

$$\begin{bmatrix}\alpha_{ij1}\\ \vdots\\ \alpha_{ijn}\end{bmatrix}=\alpha_{iji}\mathbf{e}_i$$

for $1\le i\le n, 1\le j\le n_i$.

Proof. Immediate consequence of theorem 3.2. ∎

Example 3.7. (Figure 7.) A gradient lattice vector field in $\Re^2$ is that given by

$$\xi\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ 4\end{bmatrix}+\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}5\\ 0\end{bmatrix}|x-1|+\begin{bmatrix}4\\ 0\end{bmatrix}|x|+\begin{bmatrix}0\\ 2\end{bmatrix}|y+1|.$$

By corollary 3.6 this is an example of a gradient lattice vector field. The symmetry that the corollary demands is clearly evident.

Example 3.8. (Figure 8.) A gradient lattice vector field in $\Re^3$ is that given by

$$\xi\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 1\\ -2\end{bmatrix}+\begin{bmatrix}1 & 4 & -1\\ 4 & 2 & 0\\ -1 & 0 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}|x+3|+\begin{bmatrix}0\\ 6\\ 0\end{bmatrix}|y-3|+\begin{bmatrix}0\\ 0\\ 2\end{bmatrix}|z-2|+\begin{bmatrix}0\\ 0\\ 3\end{bmatrix}|z+2|.$$

This example was also constructed by the use of corollary 3.6. Again, the symmetry required for a gradient lattice vector field is clearly evident.

Closely related to gradient systems are pseudo-gradient systems. Pseudo-gradient systems include gradient systems as a proper subset and thus may be considered as an extension of gradient systems. The remainder of this section is devoted to finding necessary and sufficient conditions for the determination of psuedo-gradient systems.

Definition 3.9. A pseudo-gradient vector field ξ, is a vector field for which there exists a matrix $\mathbf{X}$ and gradient vector field ζ such that either $(\mathbf{X}\circ\xi)(\mathbf{x})=\zeta(\mathbf{x})$ or $\xi(\mathbf{x})=(\mathbf{X}\circ\zeta)(\mathbf{x})$.

Definition 3.10. Given a matrix $\mathbf{A}$, define the set

$$\mathrm{Pg}(\mathbf{A})=\{\mathbf{X}:\mathbf{XA}=\mathbf{A}^t\mathbf{X}^t\}.$$

The matrix $\mathbf{X}$ is such that $\mathbf{XA}$ is a symmetric matrix.

Lemma 3.11. *Considering a matrix $\mathbf{X}$ written in the form of a $n \times n$-tuple*

$$\begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{nn} \end{bmatrix}$$

there exists a finite set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \Re^{n \times n}$ such that

$$\mathrm{Pg}(\mathbf{A}) = \{t_1\mathbf{v}_1 + \ldots + t_p\mathbf{v}_p : t_1, \ldots, t_p \in \Re\}.$$

PROOF. To solve the equation $\mathbf{XA} = \mathbf{A}^t\mathbf{X}^t$ is the same as solving

$$\begin{gathered}
\sum_{k=1}^{n} x_{1k}a_{k1} = \sum_{k=1}^{n} a_{k1}x_{1k} \\
\vdots \\
\sum_{k=1}^{n} x_{1k}a_{kn} = \sum_{k=1}^{n} a_{k1}x_{nk} \\
\vdots \\
\sum_{k=1}^{n} x_{nk}a_{k1} = \sum_{k=1}^{n} a_{kn}x_{1k} \\
\vdots \\
\sum_{k=1}^{n} x_{nk}a_{kn} = \sum_{k=1}^{n} a_{kn}x_{nk}
\end{gathered}$$

which can be rewritten in the form

$$\begin{bmatrix} 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{1n} & \ldots & a_{nn} & \ldots & -a_{11} & \ldots & -a_{n1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ -a_{1n} & \ldots & -a_{nn} & \ldots & a_{11} & \ldots & a_{n1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus, $\mathbf{X} \in \mathrm{Pg}(\mathbf{A})$ if and only if it solves the above equation. This means that

$$\begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{nn} \end{bmatrix}$$

is in the kernel of the matrix in the right-handside of the above equation. By linear algebra, the kernel is a linear subspace of $\Re^{n\times n}$ which can written as the span of the linearly independent vectors $\mathbf{v}_1,\ldots,\mathbf{v}_p$. Thus,

$$\mathrm{Pg}(\mathrm{A}) = \{t_1\mathbf{v}_1+\ldots+t_p\mathbf{v}_p : t_1,\ldots,t_p \in \Re\}. \qquad \blacksquare$$

Lemma 3.12. *Given two linear subspaces spanned by the vectors $\mathbf{v}_1,\ldots,\mathbf{v}_p$ and $\mathbf{w}_1,\ldots,\mathbf{w}_q$ respectively, the intersection of the two subspaces is given by the span of some vectors $\mathbf{u}_1,\ldots,\mathbf{u}_r$ with $r\le p,q$.*

PROOF. Let $\mathbf{x}\in\{t_1\mathbf{v}_1+\ldots+t_p\mathbf{v}_p : t_1,\ldots,t_p\in\Re\}\cap\{s_1\mathbf{w}_1+\ldots+s_q\mathbf{w}_q : s_1,\ldots,s_q\in\Re\}$. Then

$$[\mathbf{v}_1,\ldots,\mathbf{v}_p,\mathbf{w}_1,\ldots,\mathbf{w}_q]\begin{bmatrix} t_1\\ \vdots\\ t_p\\ -s_1\\ \vdots\\ -s_q\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix}.$$

By linear algebra, the solution for $t_1,\ldots,t_p,-s_1,\ldots,-s_q$ is in the kernel of the matrix in the left of the above equality. Let the kernel be spanned by the vectors $\mathbf{y}^1,\ldots,\mathbf{y}^r$. Thus,

$$\begin{bmatrix} t_1\\ \vdots\\ t_p\end{bmatrix} = \begin{bmatrix} y_1^1 & \cdots & y_1^r\\ \vdots & & \vdots\\ y_p^1 & \cdots & y_p^r\end{bmatrix}\begin{bmatrix} t_1'\\ \vdots\\ t_r'\end{bmatrix}$$

from which it follows that a spanning set of vectors for the intersection of the two subspaces is given by

$$[\mathbf{u}_1,\ldots,\mathbf{u}_r] = [\mathbf{v}_1,\ldots,\mathbf{v}_p]\begin{bmatrix} y_1^1 & \cdots & y_1^r\\ \vdots & & \vdots\\ y_p^1 & \cdots & y_p^r\end{bmatrix}.$$

Without loss of generality, the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_r$ may be assumed to be independent and form a basis. The dimension of the intersection cannot exceed the dimension of the subspaces that it intersects, thus $r\le p,q$. $\blacksquare$

Theorem 3.13. *Let ξ be a vector field of the form*

$$\xi\begin{bmatrix} x_1\\ \vdots\\ x_n\end{bmatrix} = \begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \cdots & b_{nn}\end{bmatrix}\begin{bmatrix} x_1\\ \vdots\\ x_n\end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix}\alpha_{j1}\\ \vdots\\ \alpha_{jn}\end{bmatrix}\left|\begin{bmatrix}\beta_{j1}\\ \vdots\\ \beta_{jn}\end{bmatrix}^t\begin{bmatrix} x_1\\ \vdots\\ x_n\end{bmatrix} - \gamma_j\right|.$$

There exists a matrix $\mathbf{X}$ such that $(\mathbf{X}\circ\xi)(\mathbf{x})$ is a gradient vector field if and only if

$$\mathbf{X}\in\mathrm{Pg}\left(\begin{bmatrix} b_{11} & \cdots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \cdots & b_{nn}\end{bmatrix}\right)\cap\left(\bigcap_{j=1}^{m}\mathrm{Pg}\left(\begin{bmatrix}\alpha_{j1}\beta_{j1} & \cdots & \alpha_{j1}\beta_{jn}\\ \vdots & & \vdots\\ \alpha_{jn}\beta_{j1} & \cdots & \alpha_{jn}\beta_{jn}\end{bmatrix}\right)\right).$$

PROOF. Assume that there exists a matrix $\mathbf{X}$ such that $(\mathbf{X}\circ\xi)(\mathbf{x})$ is a gradient vector field. As in the proof of theorem 3.2, it is necessary and sufficient that

$$\mathbf{X}\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

and

$$\mathbf{X}\begin{bmatrix} \alpha_{j1}\beta_{j1} & \cdots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \cdots & \alpha_{jn}\beta_{jn} \end{bmatrix}$$

to be symmetric matrices for $(\mathbf{X}\circ\xi)(\mathbf{x})$ to be a gradient vector field. Thus

$$\mathbf{X}\in \mathrm{Pg}\left(\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}\right)\bigcap\left(\bigcap_{j=1}^{m}\mathrm{Pg}\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \cdots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \cdots & \alpha_{jn}\beta_{jn} \end{bmatrix}\right)\right).$$ ∎

Definition 3.14. For vectors $\mathbf{v},\mathbf{w}$ define the set

$$\mathrm{Ph}(\mathbf{v},\mathbf{w}) = \cup_{k\in\Re}\{\mathbf{X} : \mathbf{v} = k\mathbf{X}\mathbf{w}\}.$$

Theorem 3.15. *Let ξ be a vector field of the form*

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|.$$

There exists matrix $\mathbf{X}$ and gradient vector field $\zeta(\mathbf{x})$ such that $\xi(\mathbf{x}) = (\mathbf{X}\circ\zeta)(\mathbf{x})$ if and only if

$$\zeta\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{bmatrix} + \begin{bmatrix} b'_{11} & \cdots & b'_{1n} \\ \vdots & & \vdots \\ b'_{1n} & \cdots & b'_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}k_j\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|,$$

$$\mathbf{X}\in\bigcap_{j=1}^{m}\mathrm{Ph}\left(\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix},\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}\right)$$

and

$$\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} = \mathbf{X}\begin{bmatrix} b'_{11} & \cdots & b'_{1n} \\ \vdots & & \vdots \\ b'_{1n} & \cdots & b'_{nn} \end{bmatrix},\ \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{X}\begin{bmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{bmatrix}.$$

PROOF. If $\zeta(\mathbf{x})$ is a gradient vector field such that $\xi(\mathbf{x}) = (\mathbf{X}\circ\zeta)(\mathbf{x})$ then ζ in not differentiable along the same points that ξ is not differentiable. As ζ is also a gradient vector field then it has the form

$$\zeta\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{bmatrix} + \begin{bmatrix} b'_{11} & \cdots & b'_{1n} \\ \vdots & & \vdots \\ b'_{1n} & \cdots & b'_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}k_j\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|.$$

From the equalities

$$\xi_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = \xi_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} + 2\begin{bmatrix}\alpha_{j1}\\\vdots\\\alpha_{jn}\end{bmatrix}\left(\begin{bmatrix}\beta_{j1}\\\vdots\\\beta_{jn}\end{bmatrix}^t\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} - \gamma_j\right)$$

$$(\mathbf{X}\circ\zeta)_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = (\mathbf{X}\circ\zeta)_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} +$$

$$2k_j\mathbf{X}\begin{bmatrix}\beta_{j1}\\\vdots\\\beta_{jn}\end{bmatrix}\left(\begin{bmatrix}\beta_{j1}\\\vdots\\\beta_{jn}\end{bmatrix}^t\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} - \gamma_j\right)$$

$$\xi_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m}(\mathbf{x}) = (\mathbf{X}\circ\zeta)_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m}(\mathbf{x})$$

$$\xi_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m}(\mathbf{x}) = (\mathbf{X}\circ\zeta)_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m}(\mathbf{x})$$

it follows that

$$\begin{bmatrix}\alpha_{j1}\\\vdots\\\alpha_{jn}\end{bmatrix} = k_j\mathbf{X}\begin{bmatrix}\beta_{j1}\\\vdots\\\beta_{jn}\end{bmatrix}.$$

Thus

$$\mathbf{X} \in \bigcap_{j=1}^{m} \mathrm{Ph}\left(\begin{bmatrix}\alpha_{j1}\\\vdots\\\alpha_{jn}\end{bmatrix}, \begin{bmatrix}\beta_{j1}\\\vdots\\\beta_{jn}\end{bmatrix}\right)$$

Now consider the two linear vector fields

$$\xi_{1,\dots,1}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = \begin{bmatrix}\alpha_1\\\vdots\\\alpha_n\end{bmatrix} + \begin{bmatrix}b_{11} & \dots & b_{1n}\\\vdots & & \vdots\\ b_{n1} & \dots & b_{nn}\end{bmatrix}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix}\alpha_{j1}\\\vdots\\\alpha_{jn}\end{bmatrix}\left(\begin{bmatrix}\beta_{j1}\\\vdots\\\beta_{jn}\end{bmatrix}^t\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} - \gamma_j\right)$$

and

$$(\mathbf{X}\circ\zeta)_{1,\dots,1}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = \mathbf{X}\begin{bmatrix}\alpha_1\\\vdots\\\alpha_n\end{bmatrix} + \mathbf{X}\begin{bmatrix}b'_{11} & \dots & b'_{1n}\\\vdots & & \vdots\\ b'_{1n} & \dots & b'_{nn}\end{bmatrix}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} + \sum_{j=1}^{m}k_j\mathbf{X}\begin{bmatrix}\beta_{j1}\\\vdots\\\beta_{jn}\end{bmatrix}\left(\begin{bmatrix}\beta_{j1}\\\vdots\\\beta_{jn}\end{bmatrix}^t\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} - \gamma_j\right)$$

which agree on the set $A_{1,\dots,1}$. Equating the derivatives

$$\frac{\partial}{\partial x_i}\xi_{1,\dots,1}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = \begin{bmatrix}b_{1i}\\\vdots\\b_{ni}\end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix}\alpha_{j1}\\\vdots\\\alpha_{jn}\end{bmatrix}\beta_{ji}$$

$$\frac{\partial}{\partial x_i}(\mathbf{X}\circ\zeta)_{1,\dots,1}\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = \mathbf{X}\begin{bmatrix}b'_{1i}\\\vdots\\b'_{ni}\end{bmatrix} + \sum_{j=1}^{m}k_j\mathbf{X}\begin{bmatrix}\beta_{j1}\\\vdots\\\beta_{jn}\end{bmatrix}\beta_{ji}$$

gives

$$\begin{bmatrix}b_{1i}\\\vdots\\b_{ni}\end{bmatrix} = \mathbf{X}\begin{bmatrix}b'_{1i}\\\vdots\\b'_{ni}\end{bmatrix}$$

for $1 \le i \le n$. Thus,

$$\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} = \mathbf{X} \begin{bmatrix} b'_{11} & \cdots & b'_{1n} \\ \vdots & & \vdots \\ b'_{1n} & \cdots & b'_{nn} \end{bmatrix}.$$

Equating the constant terms in the linear vector fields requires that

$$\mathbf{d}(\xi)_{1,\ldots,1} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} - \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \gamma_j$$

$$\mathbf{d}(\mathbf{X} \circ \zeta)_{1,\ldots,1} = \mathbf{X} \begin{bmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{bmatrix} - \sum_{j=1}^{m} k_j \mathbf{X} \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix} \gamma_j$$

are identical from which it follows that

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{X} \begin{bmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{bmatrix}.$$

Conversely, assuming that the stated equalities hold, then it is an easy matter to check that $\xi(\mathbf{x}) = (\mathbf{X} \circ \zeta)(\mathbf{x})$ where $\zeta(\mathbf{x})$ is a gradient vector field. ∎

In many cases it is desirable to determine if a vector field ξ can be decomposed as the product of a matrix $\mathbf{X}$ premultiplying a gradient vector field ζ. The most desirable situation would be that the matrix $\mathbf{X}$ is invertible and symmetric.

In the case that there is an invertible and symmetric matrix $\mathbf{X}^{-1}$ such that $(\mathbf{X}^{-1} \circ \xi)(\mathbf{x}) = \zeta(\mathbf{x})$ is a gradient vector field then it is immediate that $\xi(\mathbf{x}) = (\mathbf{X} \circ \zeta)(\mathbf{x})$. Thus, by theorem 3.13 it is sufficient to determine if there exist such matrices $\mathbf{X}^{-1}$ from the linear subspace of all matrices that can cause $(\mathbf{X}^{-1} \circ \xi)(\mathbf{x})$ to be a gradient system. At the moment, determining the intersection of a linear subspace and the nonlinear manifold of invertible symmetric matrices is still an open problem.

If the matrix $\mathbf{X}$ cannot be both invertible and symmetric, then an invertible matrix would be the next most desirable situation. This reduces to finding an invertible matrix such that $(\mathbf{X}^{-1} \circ \xi)(\mathbf{x})$ is a gradient system. A sufficient condition would be to determine if the linear subspace of matrices such that $(\mathbf{X} \circ \xi)(\mathbf{x})$ is a gradient system intersects the nonlinear manifold of invertible matrices. Again, this is still an open problem.

Lastly, if no such invertible matrix $\mathbf{X}$ exists then theorem 3.15 needs to be invoked to determine if the decomposition $\xi(\mathbf{x}) = (\mathbf{X} \circ \zeta)(\mathbf{x})$ can be effected. By the necessary and sufficient conditions of the theorem, if such a matrix exists then the given conditions must be satisfied. Testing the conditions is tantamount to determining the intersection of three manifolds, one of which is a linear subspace and thus a linear manifold.

§4. The Hamiltonian vector field.

In this section necessary and sufficient conditions are given for a vector field to be a Hamiltonian system.

Lemma 4.1. *Let ξ be the linear vector field given by*

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} m^{1\ 1} & \dots & m^{1\ n} & m^{1\ n+1} & \dots & m^{1\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m^{n\ 1} & \dots & m^{n\ n} & m^{n\ n+1} & \dots & m^{n\ 2n} \\ m^{n+1\ 1} & \dots & m^{n+1\ n} & m^{n+1\ n+1} & \dots & m^{n+1\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m^{2n\ 1} & \dots & m^{2n\ n} & m^{2n\ n+1} & \dots & m^{2n\ 2n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} d^1 \\ \vdots \\ d^n \\ d^{n+1} \\ \vdots \\ d^{2n} \end{bmatrix}.$$

The vector field ξ is a Hamiltonian system if and only if the matrix

$$\begin{bmatrix} -m^{n+1\ 1} & \dots & -m^{n+1\ n} & -m^{n+1\ n+1} & \dots & -m^{n+1\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -m^{2n\ 1} & \dots & -m^{2n\ n} & -m^{2n\ n+1} & \dots & -m^{2n\ 2n} \\ m^{1\ 1} & \dots & m^{1\ n} & m^{1\ n+1} & \dots & m^{1\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m^{n\ 1} & \dots & m^{n\ n} & m^{n\ n+1} & \dots & m^{n\ 2n} \end{bmatrix}$$

is symmetric.

Proof. If there exists a Hamiltonian function $H(x_1, \dots, x_n, y_1, \dots, y_n)$ then

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial y_1} \\ \vdots \\ \frac{\partial H}{\partial y_n} \\ -\frac{\partial H}{\partial x_1} \\ \vdots \\ -\frac{\partial H}{\partial x_n} \end{bmatrix}$$

from which it follows that

$$D\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} m^{1\ 1} & \dots & m^{1\ n} & m^{1\ n+1} & \dots & m^{1\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m^{n\ 1} & \dots & m^{n\ n} & m^{n\ n+1} & \dots & m^{n\ 2n} \\ m^{n+1\ 1} & \dots & m^{n+1\ n} & m^{n+1\ n+1} & \dots & m^{n+1\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m^{2n\ 1} & \dots & m^{2n\ n} & m^{2n\ n+1} & \dots & m^{2n\ 2n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial^2 H}{\partial x_1 \partial y_1} & \dots & \frac{\partial^2 H}{\partial x_n \partial y_1} & \frac{\partial^2 H}{\partial y_1^2} & \dots & \frac{\partial^2 H}{\partial y_n \partial y_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2 H}{\partial x_1 \partial y_n} & \dots & \frac{\partial^2 H}{\partial x_n \partial y_n} & \frac{\partial^2 H}{\partial y_1 \partial y_n} & \dots & \frac{\partial^2 H}{\partial y_n^2} \\ -\frac{\partial^2 H}{\partial x_1^2} & \dots & -\frac{\partial^2 H}{\partial x_n \partial x_1} & -\frac{\partial^2 H}{\partial y_1 \partial x_1} & \dots & -\frac{\partial^2 H}{\partial y_n \partial x_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ -\frac{\partial^2 H}{\partial x_1 \partial x_n} & \dots & -\frac{\partial^2 H}{\partial x_n^2} & -\frac{\partial^2 H}{\partial y_1 \partial x_n} & \dots & -\frac{\partial^2 H}{\partial y_n \partial x_n} \end{bmatrix}.$$

Thus

$$\begin{bmatrix} -m^{n+1\ 1} & \dots & -m^{n+1\ n} & -m^{n+1\ n+1} & \dots & -m^{n+1\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -m^{2n\ 1} & \dots & -m^{2n\ n} & -m^{2n\ n+1} & \dots & -m^{2n\ 2n} \\ m^{1\ 1} & \dots & m^{1\ n} & m^{1\ n+1} & \dots & m^{1\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m^{n\ 1} & \dots & m^{n\ n} & m^{n\ n+1} & \dots & m^{n\ 2n} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial^2 H}{\partial x_1^2} & \dots & \frac{\partial^2 H}{\partial x_n \partial x_1} & \frac{\partial^2 H}{\partial y_1 \partial x_1} & \dots & \frac{\partial^2 H}{\partial y_n \partial x_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2 H}{\partial x_1 \partial x_n} & \dots & \frac{\partial^2 H}{\partial x_n^2} & \frac{\partial^2 H}{\partial y_1 \partial x_n} & \dots & \frac{\partial^2 H}{\partial y_n \partial x_n} \\ \frac{\partial^2 H}{\partial x_1 \partial y_1} & \dots & \frac{\partial^2 H}{\partial x_n \partial y_1} & \frac{\partial^2 H}{\partial y_1^2} & \dots & \frac{\partial^2 H}{\partial y_n \partial y_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2 H}{\partial x_1 \partial y_n} & \dots & \frac{\partial^2 H}{\partial x_n \partial y_n} & \frac{\partial^2 H}{\partial y_1 \partial y_n} & \dots & \frac{\partial^2 H}{\partial y_n^2} \end{bmatrix}$$

is a symmetric matrix.

Conversely, suppose that the matrix

$$\begin{bmatrix} -m^{n+1\ 1} & \dots & -m^{n+1\ n} & -m^{n+1\ n+1} & \dots & -m^{n+1\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -m^{2n\ 1} & \dots & -m^{2n\ n} & -m^{2n\ n+1} & \dots & -m^{2n\ 2n} \\ m^{1\ 1} & \dots & m^{1\ n} & m^{1\ n+1} & \dots & m^{1\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m^{n\ 1} & \dots & m^{n\ n} & m^{n\ n+1} & \dots & m^{n\ 2n} \end{bmatrix}$$

is symmetric, then a Hamiltonian function for ξ is given by

$$H(x_1, \dots, x_n, y_1, \dots, y_n) =$$

$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m^{n+i\ j} x_i x_j - \sum_{i=1}^{n}\sum_{j=1}^{n} m^{n+i\ n+j} x_i y_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m^{i\ n+j} y_i y_j - \sum_{i=1}^{n} d^{n+i} x_i + \sum_{i=1}^{n} d^i y_i + c$$

for constant c. ∎

Theorem 4.2. *Let ξ be a vector field of the form*

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_{2n} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{2n} \end{bmatrix} + \begin{bmatrix} b_{1\ 1} & \dots & b_{1\ 2n} \\ \vdots & & \vdots \\ b_{2n\ 1} & \dots & b_{2n\ 2n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{2n} \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j\ 1} \\ \vdots \\ \alpha_{j\ 2n} \end{bmatrix} \left| \begin{bmatrix} \beta_{j\ 1} \\ \vdots \\ \beta_{j\ 2n} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_{2n} \end{bmatrix} - \gamma_j \right|.$$

The vector field ξ is a Hamiltonian system if and only if

$$\begin{bmatrix} -b_{n+1\ 1} & \dots & -b_{n+1\ 2n} \\ \vdots & & \vdots \\ -b_{2n\ 1} & \dots & -b_{2n\ 2n} \\ b_{1\ 1} & \dots & b_{1\ 2n} \\ \vdots & & \vdots \\ b_{n\ 1} & \dots & b_{n\ 2n} \end{bmatrix}$$

is symmetric and

$$\begin{bmatrix} \alpha_{j\ 1} \\ \vdots \\ \alpha_{j\ 2n} \end{bmatrix} = k_j \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix} \begin{bmatrix} \beta_{j\ 1} \\ \vdots \\ \beta_{j\ 2n} \end{bmatrix}$$

for $1 \le j \le m$.

PROOF. For the matrix

$$\mathbf{M}_{i_1,\ldots,i_m} = \begin{bmatrix} m^{1\ 1}_{i_1,\ldots,i_m} & \cdots & m^{1\ 2n}_{i_1,\ldots,i_m} \\ \vdots & & \vdots \\ m^{n\ 1}_{i_1,\ldots,i_m} & \cdots & m^{n\ 2n}_{i_1,\ldots,i_m} \\ m^{n+1\ 1}_{i_1,\ldots,i_m} & \cdots & m^{n+1\ 2n}_{i_1,\ldots,i_m} \\ \vdots & & \vdots \\ m^{2n\ 1}_{i_1,\ldots,i_m} & \cdots & m^{2n\ 2n}_{i_1,\ldots,i_m} \end{bmatrix}$$

define the matrix

$$\mathbf{N}_{i_1,\ldots,i_m} = \begin{bmatrix} -m^{n+1\ 1}_{i_1,\ldots,i_m} & \cdots & -m^{n+1\ 2n}_{i_1,\ldots,i_m} \\ \vdots & & \vdots \\ -m^{2n\ 1}_{i_1,\ldots,i_m} & \cdots & -m^{2n\ 2n}_{i_1,\ldots,i_m} \\ m^{1\ 1}_{i_1,\ldots,i_m} & \cdots & m^{1\ 2n}_{i_1,\ldots,i_m} \\ \vdots & & \vdots \\ m^{n\ 1}_{i_1,\ldots,i_m} & \cdots & m^{n\ 2n}_{i_1,\ldots,i_m} \end{bmatrix}.$$

Assume that the lattice vector field ξ is a Hamiltonian system with Hamiltonian function H. For given signatures $(i_1,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_m),(i_1,\ldots,i_{j-1},-i_j,i_{j+1},\ldots,i_m)$ Consider the linear vector fields $\xi_{i_1,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_m}$ and $\xi_{i_1,\ldots,i_{j-1},-i_j,i_{j+1},\ldots,i_m}$. By lemma 4.1, matrices $\mathbf{N}_{i_1,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_m}$ and $\mathbf{N}_{i_1,\ldots,i_{j-1},-i_j,i_{j+1},\ldots,i_m}$ corresponding to these two linear vector fields are symmetric. The matrix $\mathbf{N}_{i_1,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_m} - \mathbf{N}_{i_1,\ldots,i_{j-1},-i_j,i_{j+1},\ldots,i_m}$ is symmetric from which it follows that

$$2\begin{bmatrix} -\alpha_{j\ n+1}\beta_{j\ 1} & \cdots & -\alpha_{j\ n+1}\beta_{j\ 2n} \\ \vdots & & \vdots \\ -\alpha_{j\ 2n}\beta_{j\ 1} & \cdots & -\alpha_{j\ 2n}\beta_{j\ 2n} \\ \alpha_{j\ 1}\beta_{j\ 1} & \cdots & \alpha_{j\ 1}\beta_{j\ 2n} \\ \vdots & & \vdots \\ \alpha_{j\ n}\beta_{j\ 1} & \cdots & \alpha_{j\ n}\beta_{j\ 2n} \end{bmatrix}$$

is symmetric. As $0 < \beta_{j\ 1}^2 + \ldots + \beta_{j\ 2n}^2$ then $\beta_{j\ l} \neq 0$ for some $1 \le l \le 2n$. Assume that $\beta_{j\ 1} \neq 0$, then from the symmetry of the above matrix, $\alpha_{j\ k}\beta_{j\ 1} = -\alpha_{j\ n+1}\beta_{j\ n+k}$ and $-\alpha_{j\ k}\beta_{j\ 1} = -\alpha_{j\ n+1}\beta_{j\ k-n}$ for $1 \le k \le n, n+1 \le k \le 2n$ respectively. It follows that $\alpha_{j\ k} = (-\alpha_{j\ n+1}/\beta_{j\ 1})\beta_{j\ n+k}$ and $\alpha_{j\ k} = -(-\alpha_{j\ n+1}/\beta_{j\ 1})\beta_{j\ k-n}$ for $1 \le k \le n, n+1 \le k \le 2n$ respectively. Thus, with $k_j = -\alpha_{j\ n+1}/\beta_{j\ 1}$,

$$\begin{bmatrix} \alpha_{j\ 1} \\ \vdots \\ \alpha_{j\ 2n} \end{bmatrix} = k_j \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix} \begin{bmatrix} \beta_{j\ 1} \\ \vdots \\ \beta_{j\ 2n} \end{bmatrix}$$

for $1 \le j \le m$. Now consider the linear vector field $\xi_{n_1,\ldots,n_{2n}}$. The matrix $\mathbf{N}_{1,\ldots,1}$ corresponding to this linear vector field is symmetric, thus

$$\begin{bmatrix} -b_{n+1\ 1} + \sum_{j=1}^{m} k_j \beta_{j\ 1} \beta_{j\ 1} & \cdots & -b_{n+1\ 2n} + \sum_{j=1}^{m} k_j \beta_{j\ 1} \beta_{j\ 2n} \\ \vdots & & \vdots \\ -b_{2n\ 1} + \sum_{j=1}^{m} k_j \beta_{j\ n} \beta_{j\ 1} & \cdots & -b_{2n\ 2n} + \sum_{j=1}^{m} k_j \beta_{j\ n} \beta_{j\ 2n} \\ b_{1\ 1} + \sum_{j=1}^{m} k_j \beta_{j\ n+1} \beta_{j\ 1} & \cdots & b_{1\ 2n} + \sum_{j=1}^{m} k_j \beta_{j\ n+1} \beta_{j\ 2n} \\ \vdots & & \vdots \\ b_{n\ 1} + \sum_{j=1}^{m} k_j \beta_{j\ 2n} \beta_{j\ 1} & \cdots & b_{n\ 2n} + \sum_{j=1}^{m} k_j \beta_{j\ 2n} \beta_{j\ 2n} \end{bmatrix}$$

is symmetric from which it follows that

$$\begin{bmatrix} -b_{n+1\ 1} & \cdots & -b_{n+1\ 2n} \\ \vdots & & \vdots \\ -b_{2n\ 1} & \cdots & -b_{2n\ 2n} \\ b_{1\ 1} & \cdots & b_{1\ 2n} \\ \vdots & & \vdots \\ b_{n\ 1} & \cdots & b_{n\ 2n} \end{bmatrix}$$

is a symmetric matrix.

Conversely, assume that

$$\begin{bmatrix} -b_{n+1\ 1} & \cdots & -b_{n+1\ 2n} \\ \vdots & & \vdots \\ -b_{2n\ 1} & \cdots & -b_{2n\ 2n} \\ b_{1\ 1} & \cdots & b_{1\ 2n} \\ \vdots & & \vdots \\ b_{n\ 1} & \cdots & b_{n\ 2n} \end{bmatrix}$$

is symmetric and

$$\begin{bmatrix} \alpha_{j\ 1} \\ \vdots \\ \alpha_{j\ 2n} \end{bmatrix} = k_j \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix} \begin{bmatrix} \beta_{j\ 1} \\ \vdots \\ \beta_{j\ 2n} \end{bmatrix}$$

for $1 \leq j \leq m$. Then for the linear vector fields $\xi_{i_1,\ldots,i_m}$ the matrices $\mathbf{N}_{i_1,\ldots,i_m}$,

$$\begin{bmatrix} -b_{n+1\ 1} - \sum_{j=1}^{m} i_j \alpha_{j\ n+1} \beta_{j\ 1} & \cdots & -b_{n+1\ 2n} - \sum_{j=1}^{m} i_j \alpha_{j\ n+1} \beta_{j\ 2n} \\ \vdots & & \vdots \\ -b_{2n\ 1} - \sum_{j=1}^{m} i_j \alpha_{j\ 2n} \beta_{j\ 1} & \cdots & -b_{2n\ 2n} - \sum_{j=1}^{m} i_j \alpha_{j\ 2n} \beta_{j\ 2n} \\ b_{1\ 1} + \sum_{j=1}^{m} i_j \alpha_{j\ 1} \beta_{j\ 1} & \cdots & b_{1\ 2n} + \sum_{j=1}^{m} i_j \alpha_{j\ 1} \beta_{j\ 2n} \\ \vdots & & \vdots \\ b_{n\ 1} + \sum_{j=1}^{m} i_j \alpha_{j\ n} \beta_{j\ 1} & \cdots & b_{n\ 2n} + \sum_{j=1}^{m} i_j \alpha_{j\ n} \beta_{j\ 2n} \end{bmatrix} =$$

$$\begin{bmatrix} -b_{n+1\ 1} + \sum_{j=1}^{m} i_j k_j \beta_{j\ 1} \beta_{j\ 1} & \cdots & -b_{n+1\ 2n} + \sum_{j=1}^{m} i_j k_j \beta_{j\ 1} \beta_{j\ 2n} \\ \vdots & & \vdots \\ -b_{2n\ 1} + \sum_{j=1}^{m} i_j k_j \beta_{j\ n} \beta_{j\ 1} & \cdots & -b_{2n\ 2n} + \sum_{j=1}^{m} i_j k_j \beta_{j\ n} \beta_{j\ 2n} \\ b_{1\ 1} + \sum_{j=1}^{m} i_j k_j \beta_{j\ n+1} \beta_{j\ 1} & \cdots & b_{1\ 2n} + \sum_{j=1}^{m} i_j k_j \beta_{j\ n+1} \beta_{j\ 2n} \\ \vdots & & \vdots \\ b_{n\ 1} + \sum_{j=1}^{m} i_j k_j \beta_{j\ 2n} \beta_{j\ 1} & \cdots & b_{n\ 2n} + \sum_{j=1}^{m} i_j k_j \beta_{j\ 2n} \beta_{j\ 2n} \end{bmatrix}$$

are symmetric. By lemma 4.1, there exists a Hamiltonian function $H_{i_1,\dots,i_m}(x_1,\dots,x_{2n})$ defined on $\mathbf{A}_{i_1,\dots,i_m}$. Finally define

$$H(x_1,\dots,x_{2n}) = H_{i_1,\dots,i_m}(x_1,\dots,x_{2n}) \qquad (x_1,\dots,x_{2n}) \in \mathbf{A}_{i_1,\dots,i_m}.$$

It is immediate that H is a Hamiltonian function for ξ. ∎

EXAMPLE 4.3. (Figure 9.) A Hamiltonian vector field in $\Re^2$ is that given by

$$\xi\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}3 & 4\\-1 & -3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}1\\-3\end{bmatrix}\left|\begin{bmatrix}3\\1\end{bmatrix}^t\begin{bmatrix}x\\y\end{bmatrix} - 1\right| + \begin{bmatrix}2\\-1\end{bmatrix}\left|\begin{bmatrix}1\\2\end{bmatrix}^t\begin{bmatrix}x\\y\end{bmatrix} - 1\right|.$$

EXAMPLE 4.4. A Hamiltonian vector field in $\Re^4$ is that given by

$$\xi\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}-1\\-3\\1\\2\end{bmatrix} + \begin{bmatrix}-1 & -4 & 4 & -1\\3 & -2 & -1 & 5\\1 & 4 & -1 & 3\\4 & 3 & 4 & 2\end{bmatrix}\begin{bmatrix}w\\x\\y\\z\end{bmatrix} + \begin{bmatrix}-4\\-5\\1\\3\end{bmatrix}\left|\begin{bmatrix}1\\3\\4\\5\end{bmatrix}^t\begin{bmatrix}w\\x\\y\\z\end{bmatrix}\right| + \begin{bmatrix}3\\2\\-5\\-4\end{bmatrix}\left|\begin{bmatrix}5\\4\\3\\2\end{bmatrix}^t\begin{bmatrix}w\\x\\y\\z\end{bmatrix} - 2\right|.$$

This example was constructed with the aid of theorem 4.2.

Corollary 4.5. *Let ξ be a lattice vector field of the form*

$$\xi\begin{bmatrix}x_1\\\vdots\\x_{2n}\end{bmatrix} = \begin{bmatrix}\alpha_1\\\vdots\\\alpha_{2n}\end{bmatrix} + \begin{bmatrix}b_{1\,1} & \dots & b_{1\,2n}\\\vdots & & \vdots\\b_{2n\,1} & \dots & b_{2n\,2n}\end{bmatrix}\begin{bmatrix}x_1\\\vdots\\x_{2n}\end{bmatrix} + \sum_{i=1}^{2n}\sum_{j=1}^{n_i}\begin{bmatrix}\alpha_{ij1}\\\vdots\\\alpha_{ijn}\end{bmatrix}|x_i - \gamma_{ij}|.$$

The lattice vector field ξ is a Hamiltonian system if and only if

$$\begin{bmatrix}-b_{n+1\,1} & \dots & -b_{n+1\,2n}\\\vdots & & \vdots\\-b_{2n\,1} & \dots & -b_{2n\,2n}\\b_{1\,1} & \dots & b_{1\,2n}\\\vdots & & \vdots\\b_{n\,1} & \dots & b_{n\,2n}\end{bmatrix}$$

is symmetric and

$$\begin{bmatrix}\alpha_{ij1}\\\vdots\\\alpha_{ijn}\end{bmatrix} = \alpha_{ijn+i}\mathbf{e}_{n+i}$$

for $1 \le i \le n, 1 \le j \le n_i$,

$$\begin{bmatrix}\alpha_{ij1}\\\vdots\\\alpha_{ijn}\end{bmatrix} = \alpha_{iji-n}\mathbf{e}_{i-n}$$

for $n+1 \le i \le 2n, 1 \le j \le n_i$.

PROOF. This is an immediate consequence of theorem 4.2. ∎

Example 4.6. (Figure 10.) A Hamiltonian lattice vector field in $\Re^2$ is that given by

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}|x-2| + \begin{bmatrix} 0 \\ 3 \end{bmatrix}|x+4| + \begin{bmatrix} 5 \\ 0 \end{bmatrix}|y+1| + \begin{bmatrix} 3 \\ 0 \end{bmatrix}|y-2|.$$

Example 4.7. A Hamiltonian lattice vector field in $\Re^4$ is that given by

$$\xi\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 & -4 & 4 & -1 \\ 3 & -2 & -1 & 5 \\ 1 & 4 & -1 & 3 \\ 4 & 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}|w+1| + \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}|w-2| + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}|x+3| +$$
$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}|y-2| + \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}|y+4| + \begin{bmatrix} 0 \\ 6 \\ 0 \\ 0 \end{bmatrix}|z-3| + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}|z-5|.$$

This example was constructed with the aid of corollary 4.5. In cases when graphical techniques are not applicable, theorectical tools may provide insight into underlying structure.

§5. Vector fields invariant under post-composition with matrix multiplication.

A vector field ξ is invariant under post-composition with the matrix M if $(\mathrm{M}\circ\xi)(\mathrm{x}) = \xi(\mathrm{x})$. Lemma 5.1 proves the basic relationship between the vector field and the matrix. Given a vector field, theorem 5.2 determines the form of the matrices which the vector field is invariant under post-composition. It turns out that the matrices have a very restricted form. Conversely, given a matrix, theorem 5.3 determines the vector fields that are invariant under post-composition with the matrix. Examples at the end of the section will illustrate applications of the theorems.

Lemma 5.1. *Let* M *be a* $n\times n$ *matrix and* ξ *be a piecewise-linear vector field in* $\Re^2$. *The equality* $(\mathrm{M}\circ\xi)(\mathrm{x}) = \xi(\mathrm{x})$ *holds if and only if*

$$\mathrm{M}\begin{bmatrix} \alpha_1 & b_{11} & \dots & b_{1n} & \alpha_{11} & \dots & \alpha_{m1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_n & b_{n1} & \dots & b_{nn} & \alpha_{1n} & \dots & \alpha_{mn} \end{bmatrix} = \begin{bmatrix} \alpha_1 & b_{11} & \dots & b_{1n} & \alpha_{11} & \dots & \alpha_{m1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_n & b_{n1} & \dots & b_{nn} & \alpha_{1n} & \dots & \alpha_{mn} \end{bmatrix}.$$

Proof. Assume that the stated equality holds, then

$$\begin{aligned}(\mathrm{M}\circ\xi)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &= \mathrm{M}\left(\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|\right) \\ &= \mathrm{M}\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \mathrm{M}\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\mathrm{M}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right| \\ &= \xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.\end{aligned}$$

Conversely, assume that $(\mathrm{M} \circ \xi)(\mathbf{x}) = \xi(\mathbf{x})$ holds where

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right|$$

and

$$(\mathrm{M} \circ \xi) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathrm{M} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \mathrm{M} \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \mathrm{M} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right|.$$

Notice that $\mathrm{Part}(\mathrm{M} \circ \xi) = \mathrm{Part}(\xi)$ and considering the following equalities

$$\xi_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \xi_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + 2 \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left(\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right)$$

$$(\mathrm{M} \circ \xi)_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (\mathrm{M} \circ \xi)_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + 2\mathrm{M} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left(\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right)$$

$$\xi_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m}(\mathbf{x}) = (\mathrm{M} \circ \xi)_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m}(\mathbf{x})$$

$$\xi_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m}(\mathbf{x}) = (\mathrm{M} \circ \xi)_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m}(\mathbf{x})$$

it follows that

$$\mathrm{M} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} = \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}.$$

Now consider the two linear vector fields

$$\xi_{1,\dots,1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left(\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right)$$

and

$$(\mathrm{M} \circ \xi)_{1,\dots,1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathrm{M} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \mathrm{M} \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \mathrm{M} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left(\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right)$$

which agree on the set $\mathrm{A}_{1,\dots,1}$. Equating the derivatives

$$\frac{\partial}{\partial x_i} \xi_{1,\dots,1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_{1i} \\ \vdots \\ b_{ni} \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \beta_{ji}$$

$$\frac{\partial}{\partial x_i} (\mathrm{M} \circ \xi)_{1,\dots,1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathrm{M} \begin{bmatrix} b_{1i} \\ \vdots \\ b_{ni} \end{bmatrix} + \sum_{j=1}^{m} \mathrm{M} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \beta_{ji}$$

gives

$$\mathbf{M}\begin{bmatrix} b_{1i} \\ \vdots \\ b_{ni} \end{bmatrix} = \begin{bmatrix} b_{1i} \\ \vdots \\ b_{ni} \end{bmatrix}$$

for $1 \le i \le n$.

Equating the constant terms in the linear vector fields requires that

$$d(\xi)_{1,\ldots,1} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} - \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \gamma_j$$

$$d(\mathbf{M} \circ \xi)_{1,\ldots,1} = \mathbf{M}\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} - \sum_{j=1}^{m} \mathbf{M}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \gamma_j$$

are identical from which it follows that

$$\mathbf{M}\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}. \qquad \blacksquare$$

Theorem 5.2. *Let the vector field ξ be given. Let r be the rank of the matrix*

$$\begin{bmatrix} \alpha_1 & b_{11} & \ldots & b_{1n} & \alpha_{11} & \ldots & \alpha_{m1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_n & b_{n1} & \ldots & b_{nn} & \alpha_{1n} & \ldots & \alpha_{mn} \end{bmatrix}.$$

If $\mathbf{M}$ is a matrix for which $(\mathbf{M} \circ \xi)(\mathbf{x}) = \xi(\mathbf{x})$ then either (i) $r = n$ in which case $\mathbf{M} = \mathbf{I}_{n \times n}$ or (ii) $r < n$ in which case there exist an invertible matrix S such that

$$\mathbf{MS} = \mathbf{S}\begin{bmatrix} \mathbf{I}_{r \times r} & \mathbf{0}_{r \times n-r} \\ \mathbf{0}_{n-r \times r} & \mathbf{B}_{n-r \times n-r} \end{bmatrix}.$$

Proof. By lemma 5.1 the given matrix equality must hold if the vector field is to be invariant under post-composition with the matrix $\mathbf{M}$. The value of r determines the number of independent eigenvectors of eigenvalue 1 for the matrix $\mathbf{M}$. If $r = n$ then $\mathbf{M}$ has a full set of eigenvectors corresponding to the eigenvalue 1. By linear algebra there exists an invertible matrix S such that $\mathbf{MS} = \mathbf{I}_{n \times n}\mathbf{S}$ from which it follows that $\mathbf{M} = \mathbf{I}_{n \times n}$. If $r < n$ then $\mathbf{M}$ has a subspace of dimension at least r for the eigenvalue 1. Again, by linear algebra there exists an invertible matrix $\mathbf{S}$ for which

$$\mathbf{MS} = \mathbf{S}\begin{bmatrix} \mathbf{I}_{r \times r} & \mathbf{0}_{r \times n-r} \\ \mathbf{0}_{n-r \times r} & \mathbf{B}_{n-r \times n-r} \end{bmatrix}$$

for some matrix $\mathbf{B}_{n-r \times n-r}$. $\blacksquare$

Theorem 5.3. *Let the matrix $\mathbf{M}$ be given. If ξ is a vector field for which $(\mathbf{M} \circ \xi)(\mathbf{x}) = \xi(\mathbf{x})$ then the defining constants of ξ satisfy*

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \ldots, \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix}, \ldots, \begin{bmatrix} b_{1n} \\ \vdots \\ b_{nn} \end{bmatrix},$$

$$\begin{bmatrix}\alpha_{111}\\ \vdots\\ \alpha_{11n}\end{bmatrix},\ldots,\begin{bmatrix}\alpha_{1n_11}\\ \vdots\\ \alpha_{1n_1n}\end{bmatrix},\ldots,\begin{bmatrix}\alpha_{n11}\\ \vdots\\ \alpha_{n1n}\end{bmatrix},\ldots,\begin{bmatrix}\alpha_{nn_n1}\\ \vdots\\ \alpha_{nn_nn}\end{bmatrix}\in\left\{\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}:\mathrm{M}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}\right\}.$$

PROOF. This is an immediate consequence of lemma 5.1 and

$$\mathrm{M}\begin{bmatrix}\alpha_1 & b_{11} & \ldots & b_{1n} & \alpha_{11} & \ldots & \alpha_{m1}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \alpha_n & b_{n1} & \ldots & b_{nn} & \alpha_{1n} & \ldots & \alpha_{mn}\end{bmatrix}=\begin{bmatrix}\alpha_1 & b_{11} & \ldots & b_{1n} & \alpha_{11} & \ldots & \alpha_{m1}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \alpha_n & b_{n1} & \ldots & b_{nn} & \alpha_{1n} & \ldots & \alpha_{mn}\end{bmatrix}. \blacksquare$$

EXAMPLE 5.4. For the vector field given by

$$\xi\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}+\begin{bmatrix}2 & -1\\ 4 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}1\\ 2\end{bmatrix}|x+1|$$

the rank of the matrix

$$\begin{bmatrix}1 & 2 & -1 & 1\\ 2 & 4 & -2 & 2\end{bmatrix}$$

is one. Any matrix which satisfies $(\mathrm{M}\circ\xi)(\mathbf{x})=\xi(\mathbf{x})$ is conjugate to the matrix

$$\begin{bmatrix}1 & 0\\ 0 & \lambda\end{bmatrix},$$

in the sense that there exists an invertibe matrix S such that

$$\mathrm{MS}=\mathrm{S}\begin{bmatrix}1 & 0\\ 0 & \lambda\end{bmatrix}.$$

EXAMPLE 5.5. Let M be the matrix

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

which represents reflection about the line $y=x$. The eigenspace corresponding to the eigenvalue 1 is spanned by the vector

$$\begin{bmatrix}1\\ 1\end{bmatrix}.$$

A vector field for which $(\mathrm{M}\circ\xi)(\mathbf{x})=\xi(\mathbf{x})$ is given by

$$\xi\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\alpha\\ \alpha\end{bmatrix}+\begin{bmatrix}b_{11} & b_{12}\\ b_{11} & b_{12}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\sum_{j=1}^{m}\begin{bmatrix}\alpha_i\\ \alpha_i\end{bmatrix}\left|\begin{bmatrix}\beta_{i1}\\ \beta_{i2}\end{bmatrix}^t\begin{bmatrix}x\\ y\end{bmatrix}-\gamma_i\right|.$$

§6. Lattice vector fields invariant under pre-composition with matrix multiplication.

A lattice vector field ξ is invariant under pre-composition with the matrix M if $(\xi \circ \mathrm{M})(\mathbf{x}) = \xi(\mathbf{x})$. Theorem 6.1 proves the basic relationship between the lattice vector field and the matrix. Examples at the end of the section will illustrate applications of theorem 6.1.

Theorem 6.1. *Let* M *be a matrix and* ξ *be a lattice vector field. The equality* $(\xi \circ \mathrm{M})(\mathbf{x}) = \xi(\mathbf{x})$ *holds if and only if*

$$\mathrm{M}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (-1)^{l_1} x_{\pi_1} \\ \vdots \\ (-1)^{l_n} x_{\pi_n} \end{bmatrix}$$

for $l_i \in \{0,1\}$, *and* $(\pi_1, \ldots, \pi_n)$ *a permutation of* $(1, \ldots, n)$ *such that*

$$n_i = n_{\pi_i} \qquad 1 \le i \le n$$

$$\gamma_{ij} = (-1)^{l_i}\gamma_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)} \qquad 1 \le i \le n, 1 \le j \le n_i$$

$$\begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} = \begin{bmatrix} \alpha_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)\ 1} \\ \vdots \\ \alpha_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)\ n} \end{bmatrix} \qquad 1 \le i \le n, 1 \le j \le n_i$$

and

$$\begin{bmatrix} b_{1i} \\ \vdots \\ b_{ni} \end{bmatrix} = \begin{bmatrix} (-1)^{l_i} b_{1\pi_i} \\ \vdots \\ (-1)^{l_i} b_{n\pi_i} \end{bmatrix}.$$

PROOF. Assume that the stated equality holds, then

$$(\xi \circ \mathrm{M})\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \left(\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} \begin{bmatrix} (-1)^{l_1} x_{\pi_1} \\ \vdots \\ (-1)^{l_n} x_{\pi_n} \end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_i} \begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} |(-1)^{l_i} x_{\pi_i} - \gamma_{ij}| \right)$$

$$= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} (-1)^{l_1} b_{1\pi_1} & \ldots & (-1)^{l_n} b_{1\pi_n} \\ \vdots & & \vdots \\ (-1)^{l_1} b_{n\pi_1} & \ldots & (-1)^{l_n} b_{n\pi_n} \end{bmatrix} \begin{bmatrix} (-1)^{l_1} x_{\pi_1} \\ \vdots \\ (-1)^{l_n} x_{\pi_n} \end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_{\pi_i}} \begin{bmatrix} \alpha_{\pi_i j 1} \\ \vdots \\ \alpha_{\pi_i j n} \end{bmatrix} |x_{\pi_i} - \gamma_{\pi_i j}|$$

$$= \xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Conversely, assume that $(\xi \circ \mathrm{M})(\mathbf{x}) = \xi(\mathbf{x})$ holds where

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_i} \begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} |x_i - \gamma_{ij}|$$

and

$$(\xi\circ \mathrm{M})\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}+\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\mathrm{M}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}+\sum_{i=1}^{n}\sum_{j=1}^{n_i}\begin{bmatrix}\alpha_{ij1}\\ \vdots\\ \alpha_{ijn}\end{bmatrix}|m_{i1}x_1+\ldots+m_{in}x_n-\gamma_{ij}|.$$

As $(\xi\circ \mathrm{M})(\mathrm{x})=\xi(\mathrm{x})$, the points at which the two functions $\xi\circ\mathrm{M},\xi$ are not differentiable are the same. Thus, the points along which $\xi\circ\mathrm{M}$ is not differentiable are lines parallel to the coordinate axes. Hence,

$$[m_{i1}\ldots m_{in}]=(-1)^{l_i}\mathrm{e}^t_{\pi_i}$$

where $l_i\in\{0,1\}$ and $\pi_i\in\{1,\ldots,n\}$. The points at which ξ is not differentiable are given by

$$\begin{gathered}x_1\equiv\gamma_{11}<\ldots<\gamma_{1n_1}\\ \vdots\\ x_n\equiv\gamma_{n1}<\ldots<\gamma_{nn_n}\end{gathered}$$

while the points at which $\xi\circ\mathrm{M}$ are not differentiable are given by

$$\begin{gathered}x_{\pi_1}\equiv\begin{cases}\gamma_{11}<\ldots<\gamma_{1n_1} & l_1=0\\ -\gamma_{1n_1}<\ldots<-\gamma_{11} & l_1=1\end{cases}\\ \vdots\\ x_{\pi_n}\equiv\begin{cases}\gamma_{n1}<\ldots<\gamma_{nn_n} & l_n=0\\ -\gamma_{nn_n}<\ldots<-\gamma_{n1} & l_n=1\end{cases}.\end{gathered}$$

If the set of points at which $\xi\circ\mathrm{M},\xi$ are nondifferentiable are the same then $(\pi_1,\ldots,\pi_n)$ is a permutation of $(1,\ldots,n)$. Thus

$$\mathrm{M}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}(-1)^{l_1}x_{\pi_1}\\ \vdots\\ (-1)^{l_n}x_{\pi_n}\end{bmatrix}.$$

Furthermore, it is easily observed that

$$\begin{aligned}n_i&=n_{\pi_i} && 1\le i\le n\\ \gamma_{ij}&=\begin{cases}\gamma_{\pi_i j} & l_i=0,1\le i\le n,1\le j\le n_{\pi_i}\\ -\gamma_{\pi_i\ n_{\pi_i}+1-j} & l_i=1,1\le i\le n,1\le j\le n_{\pi_i}\end{cases}\\ &=(-1)^{l_i}\gamma_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)} && 1\le i\le n,1\le j\le n_{\pi_i}.\end{aligned}$$

Thus

$$\xi\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}+\begin{bmatrix}(-1)^{l_1}b_{1\pi_1}&\dots&(-1)^{l_n}b_{1\pi_n}\\ \vdots&&\vdots\\ (-1)^{l_1}b_{n\pi_1}&\dots&(-1)^{l_n}b_{n\pi_n}\end{bmatrix}\begin{bmatrix}(-1)^{l_1}x_{\pi_1}\\ \vdots\\ (-1)^{l_n}x_{\pi_n}\end{bmatrix}+\sum_{i=1}^{n}\sum_{j=1}^{n_{\pi_i}}\begin{bmatrix}\alpha_{\pi_i j1}\\ \vdots\\ \alpha_{\pi_i jn}\end{bmatrix}|x_{\pi_i}-\gamma_{\pi_i j}|$$

while

$$(\xi\circ \mathrm{M})\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}+\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\begin{bmatrix}(-1)^{l_1}x_{\pi_1}\\ \vdots\\ (-1)^{l_n}x_{\pi_n}\end{bmatrix}+\sum_{i=1}^{n}\sum_{j=1}^{n_{\pi_i}}\begin{bmatrix}\alpha_{ij1}\\ \vdots\\ \alpha_{ijn}\end{bmatrix}|(-1)^{l_i}x_{\pi_i}-\gamma_{ij}|.$$

Notice that $\mathrm{Part}(\xi \circ \mathrm{M}) = \mathrm{Part}(\xi)$ and considering the following equalities

$$\xi_{n_1,\ldots,n_{\pi_i}-1,j,n_{\pi_i+1},\ldots,n_n}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \xi_{n_1,\ldots,n_{\pi_i}-1,j-1,n_{\pi_i+1},\ldots,n_n}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + 2\begin{bmatrix} \alpha_{\pi_i j 1} \\ \vdots \\ \alpha_{\pi_i j n} \end{bmatrix}(x_{\pi_i} - \gamma_{\pi_i j})$$

$$(\xi \circ \mathrm{M})_{n_1,\ldots,n_{\pi_i}-1,j,n_{\pi_i+1},\ldots,n_n}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

$$(\xi \circ \mathrm{M})_{n_1,\ldots,n_{\pi_i}-1,j-1,n_{\pi_i+1},\ldots,n_n}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{cases} 2\begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix}(x_{\pi_i} - \gamma_{ij}) & l_i = 0 \\ 2\begin{bmatrix} \alpha_{i\ n_{\pi_i}+1-j\ 1} \\ \vdots \\ \alpha_{i\ n_{\pi_i}+1-j\ n} \end{bmatrix}(x_{\pi_i} - \gamma_{ij}) & l_i = 1 \end{cases}$$

$$\xi_{n_1,\ldots,n_{\pi_i}-1,j,n_{\pi_i+1},\ldots,n_n}(\mathbf{x}) = (\xi \circ \mathrm{M})_{n_1,\ldots,n_{\pi_i}-1,j,n_{\pi_i+1},\ldots,n_n}(\mathbf{x})$$
$$\xi_{n_1,\ldots,n_{\pi_i}-1,j-1,n_{\pi_i+1},\ldots,n_n}(\mathbf{x}) = (\xi \circ \mathrm{M})_{n_1,\ldots,n_{\pi_i}-1,j-1,n_{\pi_i+1},\ldots,n_n}(\mathbf{x})$$

it follows that

$$\begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} = \begin{cases} \begin{bmatrix} \alpha_{\pi_i j 1} \\ \vdots \\ \alpha_{\pi_i j n} \end{bmatrix} & l_i = 0, 1 \le i \le n, 1 \le j \le n_{\pi_i} \\ \begin{bmatrix} \alpha_{\pi_i\ n_{\pi_i}+1-j\ 1} \\ \vdots \\ \alpha_{\pi_i\ n_{\pi_1}+1-j\ n} \end{bmatrix} & l_i = 1, 1 \le i \le n, 1 \le j \le n_{\pi_i} \end{cases}$$
$$= \begin{bmatrix} \alpha_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)\ 1} \\ \vdots \\ \alpha_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)\ n} \end{bmatrix}.$$

Now consider the two linear vector fields

$$\xi_{n_1,\ldots,n_n}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$
$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} (-1)^{l_1} b_{1\pi_1} & \ldots & (-1)^{l_n} b_{1\pi_n} \\ \vdots & & \vdots \\ (-1)^{l_1} b_{n\pi_1} & \ldots & (-1)^{l_n} b_{n\pi_n} \end{bmatrix}\begin{bmatrix} (-1)^{l_1} x_{\pi_1} \\ \vdots \\ (-1)^{l_n} x_{\pi_n} \end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_{\pi_i}}\begin{bmatrix} \alpha_{\pi_i j 1} \\ \vdots \\ \alpha_{\pi_i j n} \end{bmatrix}(x_{\pi_i} - \gamma_{\pi_i j})$$

and

$$(\xi \circ \mathrm{M})_{n_1,\ldots,n_n}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}\begin{bmatrix} (-1)^{l_1} x_{\pi_1} \\ \vdots \\ (-1)^{l_n} x_{\pi_n} \end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_{\pi_i}}\begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix}(x_{\pi_i} - (-1)^{l_i}\gamma_{ij})$$

which agree on the set $A_{n_1,\ldots,n_n}$. Equating the derivatives

$$\frac{\partial}{\partial x_{\pi_i}}\xi_{n_1,\ldots,n_n}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}(-1)^{l_i}b_{1\pi_i}\\ \vdots\\ (-1)^{l_i}b_{n\pi_i}\end{bmatrix}+\sum_{j=1}^{n_{\pi_i}}\begin{bmatrix}\alpha_{\pi_i j1}\\ \vdots\\ \alpha_{\pi_i jn}\end{bmatrix}$$

$$\frac{\partial}{\partial x_{\pi_i}}(\xi\circ \mathrm{M})_{n_1,\ldots,n_n}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}b_{1i}\\ \vdots\\ b_{ni}\end{bmatrix}+\sum_{j=1}^{n_{\pi_i}}\begin{bmatrix}\alpha_{ij1}\\ \vdots\\ \alpha_{ijn}\end{bmatrix}$$

gives

$$\begin{bmatrix}b_{1i}\\ \vdots\\ b_{ni}\end{bmatrix}=\begin{bmatrix}(-1)^{l_i}b_{1\pi_i}\\ \vdots\\ (-1)^{l_i}b_{n\pi_i}\end{bmatrix}.$$

for $1\le i\le n$. ∎

EXAMPLE 6.2. Let M be the matrix

$$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}.$$

Lattice vector fields for which $(\xi\circ \mathrm{M})(\mathbf{x})=\xi(\mathbf{x})$ (i.e. $\xi(-\mathbf{x})=\xi(\mathbf{x})$) are given by

$$\xi\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}+\sum_{i=1}^{n}\begin{bmatrix}\alpha_{i1}\\ \alpha_{i2}\end{bmatrix}(|x-\gamma_i|+|x+\gamma_i|)+\sum_{i=n+1}^{n+m}\begin{bmatrix}\alpha_{i1}\\ \alpha_{i2}\end{bmatrix}(|y-\gamma_i|+|y+\gamma_i|).$$

EXAMPLE 6.3. Let M be the matrix

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}.$$

Lattice vector fields for which $(\xi\circ \mathrm{M})(\mathbf{x})=\xi(\mathbf{x})$ (i.e. $\xi(-\mathbf{x})=\xi(\mathbf{x})$) are given by

$$\xi\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}+\textstyle\sum_{i=1}^{n}\begin{bmatrix}\alpha_{i1}\\ \alpha_{i2}\\ \alpha_{i3}\end{bmatrix}(|x-\gamma_i|+|x+\gamma_i|)+$$

$$\textstyle\sum_{i=n+1}^{n+m}\begin{bmatrix}\alpha_{i1}\\ \alpha_{i2}\\ \alpha_{i3}\end{bmatrix}(|y-\gamma_i|+|y+\gamma_i|)+\sum_{i=n+m+1}^{n+m+p}\begin{bmatrix}\alpha_{i1}\\ \alpha_{i2}\\ \alpha_{i3}\end{bmatrix}(|z-\gamma_i|+|z+\gamma_i|).$$

§7. Lattice vector fields commuting with matrix multiplication.

A lattice vector field ξ commutes with the matrix $\mathbf{M}$ if $(\mathbf{M}\circ\xi)(\mathbf{x}) = (\xi\circ\mathbf{M})(\mathbf{x})$. Theorem 7.1 proves the basic relationship between the lattice vector field and the matrix. Examples at the end of the section will illustrate applications of the theorem.

Theorem 7.1. *Let* $\mathbf{M}$ *be a matrix and* ξ *be a lattice vector field. The equality* $(\xi\circ\mathbf{M})(\mathbf{x}) = (\mathbf{M}\circ\xi)(\mathbf{x})$ *holds if and only if*

$$\mathbf{M}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (-1)^{l_1} x_{\pi_1} \\ \vdots \\ (-1)^{l_n} x_{\pi_n} \end{bmatrix}$$

for $l_i \in \{0,1\}$, *and* $(\pi_1,\ldots,\pi_n)$ *a permutation of* $(1,\ldots,n)$ *such that*

$$n_i = n_{\pi_i} \qquad 1\le i\le n$$

$$\gamma_{ij} = (-1)^{l_i}\gamma_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)} \qquad 1\le i\le n, 1\le j\le n_i$$

$$\begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} = \begin{bmatrix} (-1)^{l_1}\alpha_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)\ \pi_1} \\ \vdots \\ (-1)^{l_n}\alpha_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)\ \pi_n} \end{bmatrix} \qquad 1\le i\le n, 1\le j\le n_i$$

$$\begin{bmatrix} b_{1i} \\ \vdots \\ b_{ni} \end{bmatrix} = \begin{bmatrix} (-1)^{l_1+l_i} b_{\pi_1\pi_i} \\ \vdots \\ (-1)^{l_n+l_i} b_{\pi_n\pi_i} \end{bmatrix} \qquad 1\le i\le n, 1\le j\le n_i$$

and

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} (-1)^{l_1}\alpha_{\pi_1} \\ \vdots \\ (-1)^{l_n}\alpha_{\pi_n} \end{bmatrix}.$$

Proof. Assume that the stated equality holds, then

$$(\mathbf{M}\circ\xi)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \mathrm{M}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix} + \mathrm{M}\begin{bmatrix}b_{11} & \dots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \dots & b_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\mathrm{M}\begin{bmatrix}\alpha_{ij1}\\ \vdots\\ \alpha_{ijn}\end{bmatrix}|x_i - \gamma_{ij}|$$

$$= \begin{bmatrix}(-1)^{l_1}\alpha_{\pi_1}\\ \vdots\\ (-1)^{l_n}\alpha_{\pi_n}\end{bmatrix} + \begin{bmatrix}(-1)^{l_1}b_{\pi_1 1} & \dots & (-1)^{l_1}b_{\pi_1 n}\\ \vdots & & \vdots\\ (-1)^{l_n}b_{\pi_n 1} & \dots & (-1)^{l_n}b_{\pi_n n}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\begin{bmatrix}(-1)^{l_1}\alpha_{ij\pi_1}\\ \vdots\\ (-1)^{l_n}\alpha_{ij\pi_n}\end{bmatrix}|x_i - \gamma_{ij}|$$

$$= \begin{bmatrix}(-1)^{l_1}\alpha_{\pi_1}\\ \vdots\\ (-1)^{l_n}\alpha_{\pi_n}\end{bmatrix} +$$

$$\begin{bmatrix}(-1)^{l_1+l_1}b_{\pi_1\pi_1} & \dots & (-1)^{l_1+l_n}b_{\pi_1\pi_n}\\ \vdots & & \vdots\\ (-1)^{l_n+l_1}b_{\pi_n\pi_1} & \dots & (-1)^{l_n+l_n}b_{\pi_n\pi_n}\end{bmatrix}\begin{bmatrix}(-1)^{l_1}x_{\pi_1}\\ \vdots\\ (-1)^{l_n}x_{\pi_n}\end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_{\pi_i}}\begin{bmatrix}(-1)^{l_1}\alpha_{\pi_i j\pi_1}\\ \vdots\\ (-1)^{l_n}\alpha_{\pi_i j\pi_n}\end{bmatrix}|x_{\pi_i} - \gamma_{\pi_i j}|$$

$$= \begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix} + \begin{bmatrix}b_{11} & \dots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \dots & b_{nn}\end{bmatrix}\begin{bmatrix}(-1)^{l_1}x_{\pi_1}\\ \vdots\\ (-1)^{l_n}x_{\pi_n}\end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_{\pi_i}}\begin{bmatrix}\alpha_{ij1}\\ \vdots\\ \alpha_{ijn}\end{bmatrix}|(-1)^{l_i}x_{\pi_i} - \gamma_{ij}|$$

$$= (\xi \circ \mathrm{M})\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}.$$

Conversely, assume that $(\mathrm{M} \circ \xi)(\mathrm{x}) = (\xi \circ \mathrm{M})(\mathrm{x})$ holds where

$$(\mathrm{M} \circ \xi)\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \mathrm{M}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix} + \mathrm{M}\begin{bmatrix}b_{11} & \dots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \dots & b_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\mathrm{M}\begin{bmatrix}\alpha_{ij1}\\ \vdots\\ \alpha_{ijn}\end{bmatrix}|x_i - \gamma_{ij}|$$

and

$$(\xi \circ \mathrm{M})\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix} + \begin{bmatrix}b_{11} & \dots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \dots & b_{nn}\end{bmatrix}\mathrm{M}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\begin{bmatrix}\alpha_{ij1}\\ \vdots\\ \alpha_{ijn}\end{bmatrix}|m_{i1}x_1 + \dots + m_{in}x_n - \gamma_{ij}|.$$

As $(\mathrm{M}\circ\xi)(\mathrm{x}) = (\xi\circ\mathrm{M})(\mathrm{x})$, the points at which the two functions $\mathrm{M}\circ\xi, \xi\circ\mathrm{M}$, are not differentiable are the same. Thus, the points along which $\xi \circ \mathrm{M}$ is not differentiable are lines parallel to the coordinate axes. Hence,

$$[m_{i1} \dots m_{in}] = (-1)^{l_i}\mathrm{e}^t_{\pi_i}$$

where $l_i \in \{0,1\}$ and $\pi_i \in \{1,\dots,n\}$. The points at which $\mathrm{M} \circ \xi$ is not differentiable are given by

$$x_1 \equiv \gamma_{11} < \dots < \gamma_{1n_1}$$
$$\vdots$$
$$x_n \equiv \gamma_{n1} < \dots < \gamma_{nn_n}$$

while the points at which $\xi \circ \mathrm{M}$ are not differentiable are given by

$$x_{\pi_1} \equiv \begin{cases}\gamma_{11} < \dots < \gamma_{1n_1} & l_1 = 0\\ -\gamma_{1n_1} < \dots < -\gamma_{11} & l_1 = 1\end{cases}$$
$$\vdots$$
$$x_{\pi_n} \equiv \begin{cases}\gamma_{n1} < \dots < \gamma_{nn_n} & l_n = 0\\ -\gamma_{nn_n} < \dots < -\gamma_{n1} & l_n = 1\end{cases}.$$

If the set of points at which $\mathrm{M} \circ \xi, \xi \circ \mathrm{M}$, are nondifferentiable are the same then $(\pi_1, \ldots, \pi_n)$ is a permutation of $(1, \ldots, n)$. Thus

$$(\mathrm{M} \circ \xi) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^{l_1} \alpha_{\pi_1} \\ \vdots \\ (-1)^{l_n} \alpha_{\pi_n} \end{bmatrix} + \begin{bmatrix} (-1)^{l_1} b_{\pi_1 1} & \ldots & (-1)^{l_1} b_{\pi_1 n} \\ \vdots & & \vdots \\ (-1)^{l_n} b_{\pi_n 1} & \ldots & (-1)^{l_n} b_{\pi_n n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{i=1}^{n} \sum_{j=1}^{n_i} \begin{bmatrix} (-1)^{l_1} \alpha_{ij\pi_1} \\ \vdots \\ (-1)^{l_n} \alpha_{ij\pi_n} \end{bmatrix} |x_i - \gamma_{ij}|$$

$$= \begin{bmatrix} (-1)^{l_1} \alpha_{\pi_1} \\ \vdots \\ (-1)^{l_n} \alpha_{\pi_n} \end{bmatrix} +$$

$$\begin{bmatrix} (-1)^{l_1+l_1} b_{\pi_1 \pi_1} & \ldots & (-1)^{l_1+l_n} b_{\pi_1 \pi_n} \\ \vdots & & \vdots \\ (-1)^{l_n+l_1} b_{\pi_n \pi_1} & \ldots & (-1)^{l_n+l_n} b_{\pi_n \pi_n} \end{bmatrix} \begin{bmatrix} (-1)^{l_1} x_{\pi_1} \\ \vdots \\ (-1)^{l_n} x_{\pi_n} \end{bmatrix} + \sum_{i=1}^{n} \sum_{j=1}^{n_{\pi_i}} \begin{bmatrix} (-1)^{l_1} \alpha_{\pi_i j \pi_1} \\ \vdots \\ (-1)^{l_n} \alpha_{\pi_i j \pi_n} \end{bmatrix} |x_{\pi_i} - \gamma_{\pi_i j}|$$

while

$$(\xi \circ \mathrm{M}) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} \begin{bmatrix} (-1)^{l_1} x_{\pi_1} \\ \vdots \\ (-1)^{l_n} x_{\pi_n} \end{bmatrix} + \sum_{i=1}^{n} \sum_{j=1}^{n_{\pi_i}} \begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} |(-1)^{l_i} x_{\pi_i} - \gamma_{ij}|.$$

Notice that $\mathrm{Part}(\mathrm{M} \circ \xi) = \mathrm{Part}(\xi \circ \mathrm{M})$ and considering the following equalities

$$(\mathrm{M} \circ \xi)_{n_1, \ldots, n_{\pi_i - 1}, j, n_{\pi_i + 1}, \ldots, n_n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

$$(\mathrm{M} \circ \xi)_{n_1, \ldots, n_{\pi_i - 1}, j-1, n_{\pi_i + 1}, \ldots, n_n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + 2 \begin{bmatrix} (-1)^{l_1} \alpha_{\pi_i j \pi_1} \\ \vdots \\ (-1)^{l_n} \alpha_{\pi_i j \pi_n} \end{bmatrix} (x_{\pi_i} - \gamma_{\pi_i j})$$

$$(\xi \circ \mathrm{M})_{n_1, \ldots, n_{\pi_i - 1}, j, n_{\pi_i + 1}, \ldots, n_n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

$$(\xi \circ \mathrm{M})_{n_1, \ldots, n_{\pi_i - 1}, j-1, n_{\pi_i + 1}, \ldots, n_n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{cases} 2 \begin{bmatrix} \alpha_{ij1} \\ \vdots \\ \alpha_{ijn} \end{bmatrix} (x_{\pi_i} - \gamma_{ij}) & l_i = 0 \\ 2 \begin{bmatrix} \alpha_{i\ n_{\pi_i}+1-j\ 1} \\ \vdots \\ \alpha_{i\ n_{\pi_i}+1-j\ n} \end{bmatrix} (x_{\pi_i} - \gamma_{ij}) & l_i = 1 \end{cases}$$

$$(\mathrm{M} \circ \xi)_{n_1, \ldots, n_{\pi_i - 1}, j, n_{\pi_i + 1}, \ldots, n_n}(\mathbf{x}) = (\xi \circ \mathrm{M})_{n_1, \ldots, n_{\pi_i - 1}, j, n_{\pi_i + 1}, \ldots, n_n}(\mathbf{x})$$
$$(\mathrm{M} \circ \xi)_{n_1, \ldots, n_{\pi_i - 1}, j-1, n_{\pi_i + 1}, \ldots, n_n}(\mathbf{x}) = (\xi \circ \mathrm{M})_{n_1, \ldots, n_{\pi_i - 1}, j-1, n_{\pi_i + 1}, \ldots, n_n}(\mathbf{x})$$

it follows that

$$\begin{bmatrix}\alpha_{ij1}\\ \vdots \\ \alpha_{ijn}\end{bmatrix} = \begin{cases}\begin{bmatrix}(-1)^{l_1}\alpha_{\pi_i j \pi_1}\\ \vdots \\ (-1)^{l_n}\alpha_{\pi_i j \pi_n}\end{bmatrix} & l_i = 0, 1 \le i \le n, 1 \le j \le n_{\pi_i} \\ \begin{bmatrix}(-1)^{l_1}\alpha_{\pi_i\ n_{\pi_i}+1-j\ \pi_1}\\ \vdots \\ (-1)^{l_n}\alpha_{\pi_i\ n_{\pi_1}+1-j\ \pi_n}\end{bmatrix} & l_i = 1, 1 \le i \le n, 1 \le j \le n_{\pi_i}\end{cases}$$

$$= \begin{bmatrix}(-1)^{l_1}\alpha_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)\ \pi_1}\\ \vdots \\ (-1)^{l_n}\alpha_{\pi_i\ (1-l_i)j+l_i(n_{\pi_i}+1-j)\ \pi_n}\end{bmatrix}.$$

Now consider the two linear vector fields

$$(\mathrm{M}\circ\xi)_{n_1,\ldots,n_n}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix}(-1)^{l_1}\alpha_{\pi_1}\\ \vdots \\ (-1)^{l_n}\alpha_{\pi_n}\end{bmatrix} +$$

$$\begin{bmatrix}(-1)^{l_1+l_1}b_{\pi_1\pi_1} & \cdots & (-1)^{l_1+l_n}b_{\pi_1\pi_n}\\ \vdots & & \vdots \\ (-1)^{l_n+l_1}b_{\pi_n\pi_1} & \cdots & (-1)^{l_n+l_n}b_{\pi_n\pi_n}\end{bmatrix}\begin{bmatrix}(-1)^{l_1}x_{\pi_1}\\ \vdots \\ (-1)^{l_n}x_{\pi_n}\end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_{\pi_i}}\begin{bmatrix}(-1)^{l_1}\alpha_{\pi_i j\pi_1}\\ \vdots \\ (-1)^{l_n}\alpha_{\pi_i j \pi_n}\end{bmatrix}(x_{\pi_i} - \gamma_{\pi_i j})$$

and

$$(\xi\circ\mathrm{M})_{n_1,\ldots,n_n}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix} + \begin{bmatrix}b_{11} & \cdots & b_{1n}\\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn}\end{bmatrix}\begin{bmatrix}(-1)^{l_1}x_{\pi_1}\\ \vdots \\ (-1)^{l_n}x_{\pi_n}\end{bmatrix} + \sum_{i=1}^{n}\sum_{j=1}^{n_{\pi_i}}\begin{bmatrix}\alpha_{ij1}\\ \vdots \\ \alpha_{ijn}\end{bmatrix}(x_{\pi_i} - (-1)^{l_i}\gamma_{ij})$$

which agree on the set $\mathrm{A}_{n_1,\ldots,n_n}$. Equating the derivatives

$$\frac{\partial}{\partial x_{\pi_i}}(\mathrm{M}\circ\xi)_{n_1,\ldots,n_n}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix}(-1)^{l_1}b_{\pi_1\pi_i}\\ \vdots \\ (-1)^{l_n}b_{\pi_n\pi_i}\end{bmatrix} + \sum_{j=1}^{n_{\pi_i}}\begin{bmatrix}(-1)^{l_1}\alpha_{\pi_i j\pi_1}\\ \vdots \\ (-1)^{l_n}\alpha_{\pi_i j\pi_n}\end{bmatrix}$$

$$\frac{\partial}{\partial x_{\pi_i}}(\xi\circ\mathrm{M})_{n_1,\ldots,n_n}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = (-1)^{l_i}\begin{bmatrix}b_{1i}\\ \vdots \\ b_{ni}\end{bmatrix} + \sum_{j=1}^{n_{\pi_i}}\begin{bmatrix}\alpha_{ij1}\\ \vdots \\ \alpha_{ijn}\end{bmatrix}$$

gives

$$\begin{bmatrix}b_{1i}\\ \vdots \\ b_{ni}\end{bmatrix} = \begin{bmatrix}(-1)^{l_1+l_i}b_{1\pi_i}\\ \vdots \\ (-1)^{l_n+l_i}b_{n\pi_i}\end{bmatrix}.$$

for $1 \le i \le n$.

Equating the constant terms in the linear vector fields requires that

$$\mathrm{d}(\mathrm{M}\circ\xi)_{n_1,\ldots,n_n} = \begin{bmatrix}(-1)^{l_1}\alpha_{\pi_1}\\ \vdots \\ (-1)^{l_n}\alpha_{\pi_n}\end{bmatrix} - \sum_{i=1}^{n}\sum_{j=1}^{n_i}\begin{bmatrix}(-1)^{l_1}\alpha_{\pi_i j\pi_1}\\ \vdots \\ (-1)^{l_n}\alpha_{\pi_i j\pi_n}\end{bmatrix}\gamma_{\pi_i j}$$

$$\mathrm{d}(\xi\circ\mathrm{M})_{n_1,\ldots,n_n} = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix} - \sum_{i=1}^{n}\sum_{j=1}^{n_i}\begin{bmatrix}\alpha_{ij1}\\ \vdots \\ \alpha_{ijn}\end{bmatrix}\gamma_{ij}$$

are identical from which it follows that

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} (-1)^{l_1}\alpha_{\pi_1} \\ \vdots \\ (-1)^{l_n}\alpha_{\pi_n} \end{bmatrix}. \qquad \blacksquare$$

EXAMPLE 7.2. Let $\mathbf{M}$ be the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Lattice vector fields for which $(\xi \circ \mathbf{M})(\mathbf{x}) = (\mathbf{M} \circ \xi)(\mathbf{x})$ (i.e. $\xi(-\mathbf{x}) = -\xi(\mathbf{x})$) are given by

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \end{bmatrix} (|x-\gamma_i| - |x+\gamma_i|) + \sum_{i=n+1}^{n+m} \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \end{bmatrix} (|y-\gamma_i| - |y+\gamma_i|).$$

EXAMPLE 7.3. Let $\mathbf{M}$ be the matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Lattice vector fields for which $(\xi \circ \mathbf{M})(\mathbf{x}) = (\mathbf{M} \circ \xi)(\mathbf{x})$ (i.e. $\xi(-\mathbf{x}) = -\xi(\mathbf{x})$) are given by

$$\xi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \alpha_{i3} \end{bmatrix} (|x-\gamma_i| - |x+\gamma_i|) +$$

$$\sum_{i=n+1}^{n+m} \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \alpha_{i3} \end{bmatrix} (|y-\gamma_i| - |y+\gamma_i|) + \sum_{i=n+m+1}^{n+m+p} \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \alpha_{i3} \end{bmatrix} (|z-\gamma_i| - |z+\gamma_i|).$$

References.

[1] Chua L.O. and Deng A., "Canonical piecewise-linear modeling." *IEEE Transactions on Circuits and Systems.*, vol.33, pp.511-525, May 1986.

[2] Chua L.O. and Deng A., "Canonical piecewise-linear representations." *IEEE Transactions on Circuits and Systems.*, vol.35, pp.101-111, January 1988.

[3] Parker T.S. and Chua L.O., "Practical numerical algorithms for chaotic systems." Springer-Verlag, New York, 1989.

Figure captions.

Figure 1. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}3\\4\end{bmatrix}\left|\begin{bmatrix}4\\-3\end{bmatrix}^t\begin{bmatrix}x\\y\end{bmatrix}-2\right|.$$

Figure 2. Sample orbits corresponding to the vector field given by

$$\xi\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}1\\2\\3\end{bmatrix}\left|\begin{bmatrix}1\\4\\-3\end{bmatrix}^t\begin{bmatrix}x\\y\\z\end{bmatrix}-1\right|+\begin{bmatrix}1\\4\\-3\end{bmatrix}\left|\begin{bmatrix}1\\2\\3\end{bmatrix}^t\begin{bmatrix}x\\y\\z\end{bmatrix}+1\right|.$$

Figure 3. This is the phase portrait corresponding to the lattice vector field given by

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}2&4\\5&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}|x-2|+\begin{bmatrix}1\\0\end{bmatrix}|y-1|.$$

Figure 4. Sample orbits corresponding to the lattice vector field given by

$$\xi\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\1\\2\end{bmatrix}+\begin{bmatrix}2&0&1\\2&3&0\\1&1&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}0\\1\\3\end{bmatrix}|x-1|+\begin{bmatrix}0\\3\\1\end{bmatrix}|x+1|+\begin{bmatrix}1\\0\\-2\end{bmatrix}|y|.$$

Figure 5. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}3\\1\end{bmatrix}\left|\begin{bmatrix}3\\1\end{bmatrix}^t\begin{bmatrix}x\\y\end{bmatrix}+1\right|+\begin{bmatrix}1\\3\end{bmatrix}\left|\begin{bmatrix}-1\\-3\end{bmatrix}^t\begin{bmatrix}x\\y\end{bmatrix}\right|.$$

Figure 6. Sample orbits corresponding to the vector field given by

$$\xi\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}1\\0\\2\end{bmatrix}\left|\begin{bmatrix}-2\\0\\-4\end{bmatrix}^t\begin{bmatrix}x\\y\\z\end{bmatrix}+1\right|+\begin{bmatrix}1\\2\\0\end{bmatrix}\left|\begin{bmatrix}2\\4\\0\end{bmatrix}^t\begin{bmatrix}x\\y\\z\end{bmatrix}-1\right|.$$

Figure 7. This is the phase portrait corresponding to the lattice vector field given by

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}2&1\\1&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}5\\0\end{bmatrix}|x-1|+\begin{bmatrix}4\\0\end{bmatrix}|x|+\begin{bmatrix}0\\2\end{bmatrix}|y+1|.$$

Figure 8. Sample orbits corresponding to the lattice vector field given by

$$\xi\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\1\\-2\end{bmatrix}+\begin{bmatrix}1&4&-1\\4&2&0\\-1&0&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}|x+3|+$$
$$\begin{bmatrix}0\\6\\0\end{bmatrix}|y-3|+\begin{bmatrix}0\\0\\2\end{bmatrix}|z-2|+\begin{bmatrix}0\\0\\3\end{bmatrix}|z+2|.$$

Figure 9. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}+\begin{bmatrix}3&4\\-1&-3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\-3\end{bmatrix}\left|\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-1\right|+\begin{bmatrix}2\\-1\end{bmatrix}\left|\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-1\right|.$$

Figure 10. This is the phase portrait corresponding to the lattice vector field given by

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}3&1\\2&-3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}|x-2|+\begin{bmatrix}0\\3\end{bmatrix}|x+4|+\begin{bmatrix}5\\0\end{bmatrix}|y+1|+\begin{bmatrix}3\\0\end{bmatrix}|y-2|.$$

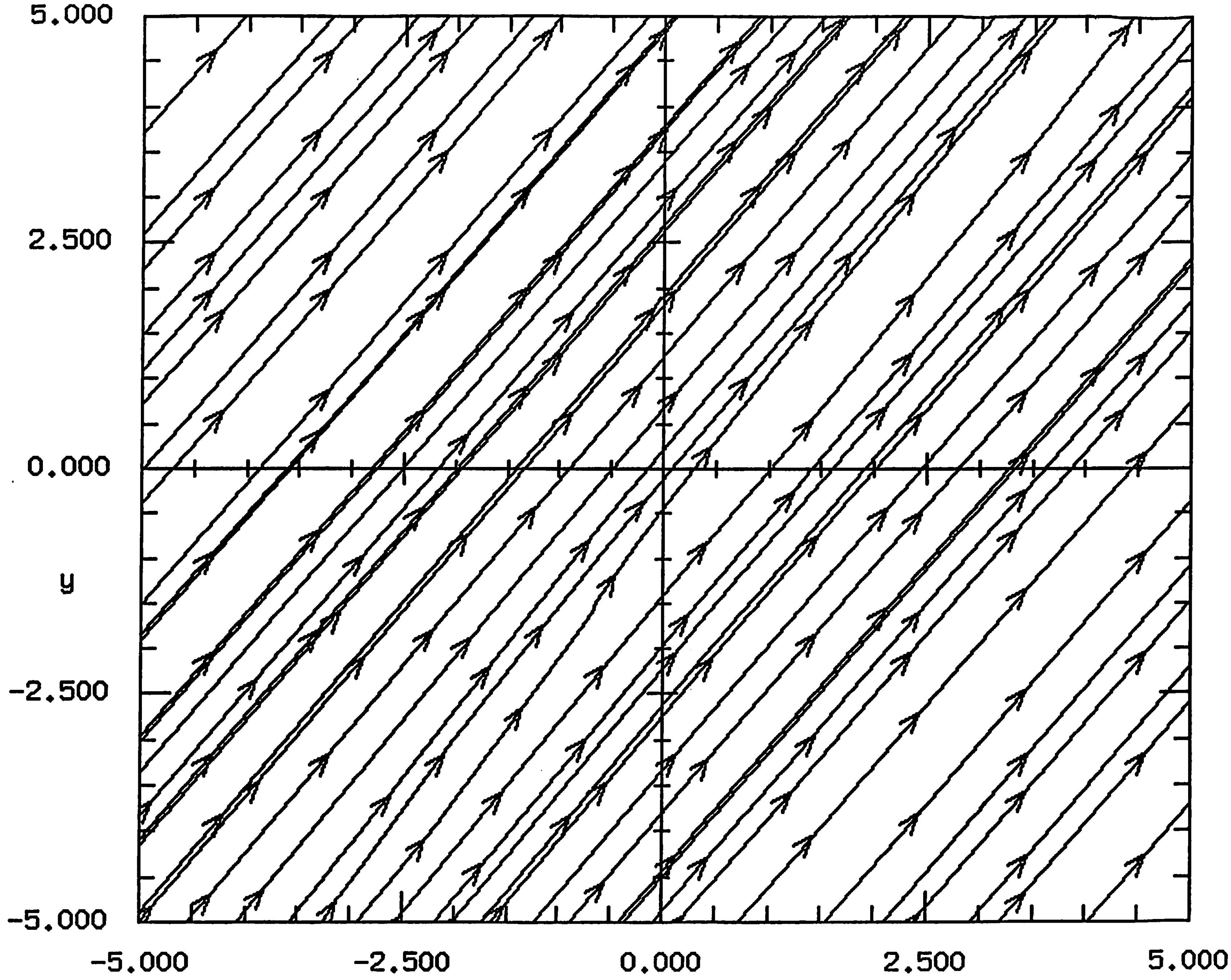

FIGURE: 1

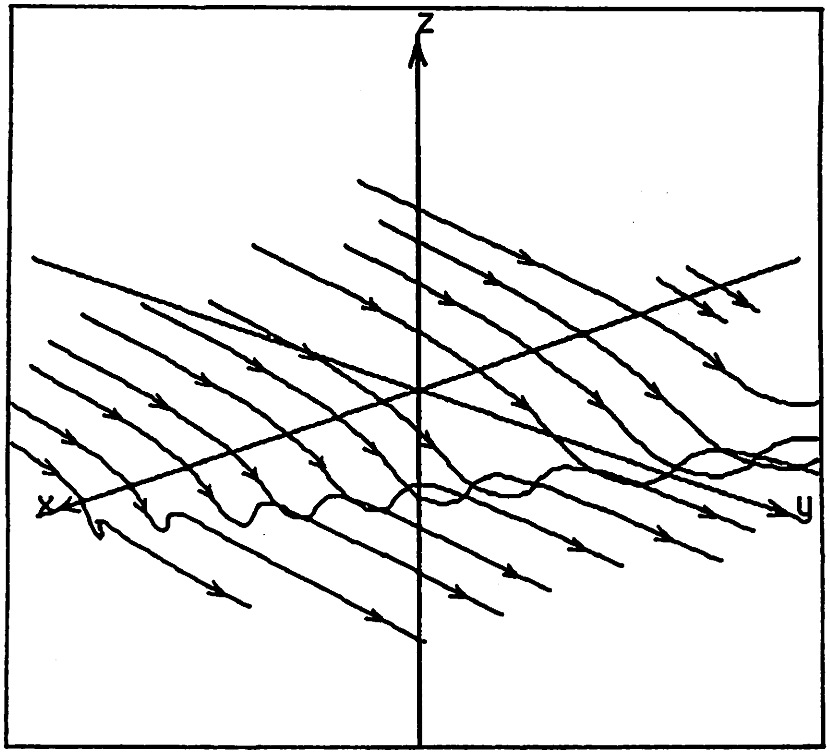

FIGURE: 2

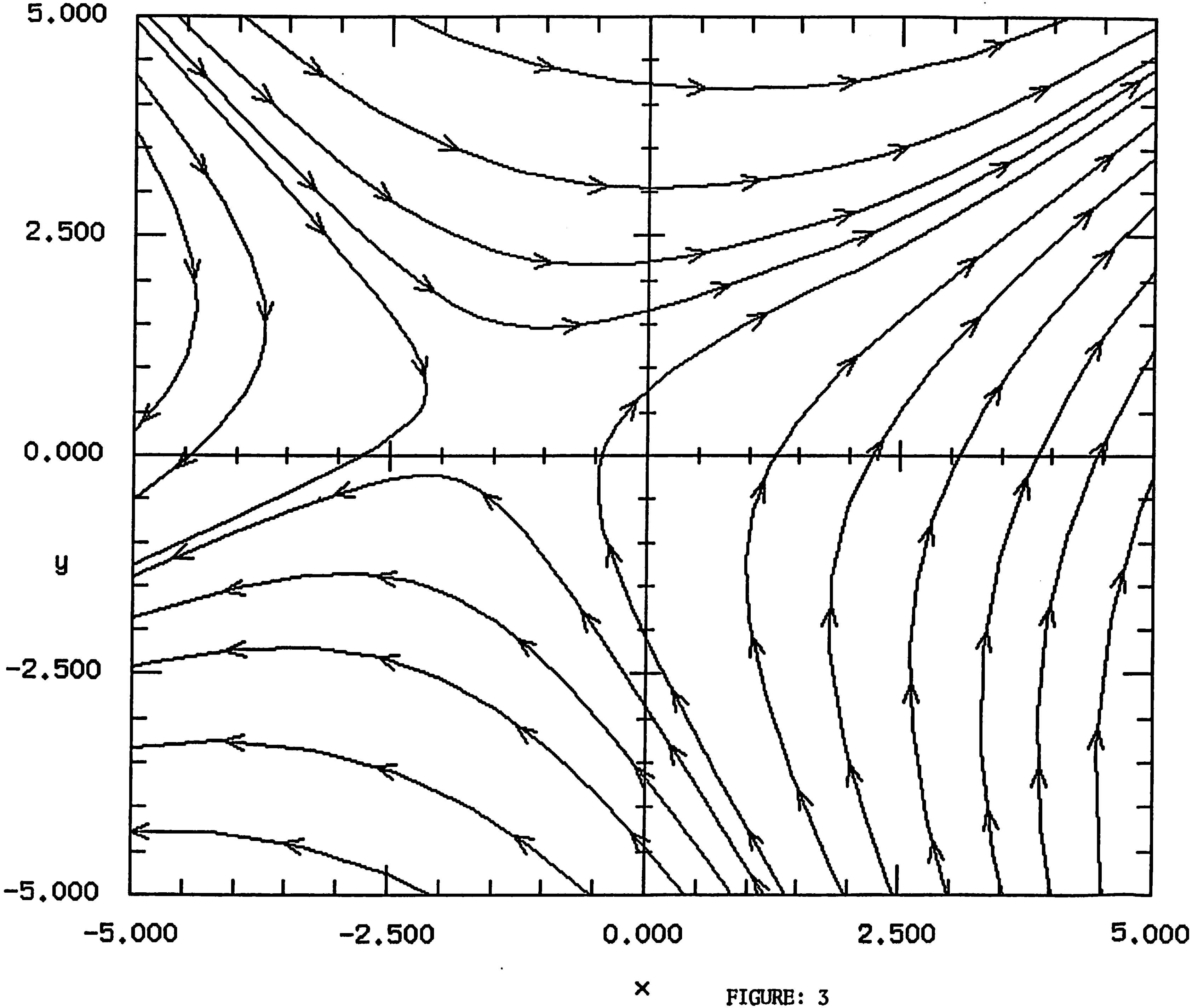

FIGURE: 3

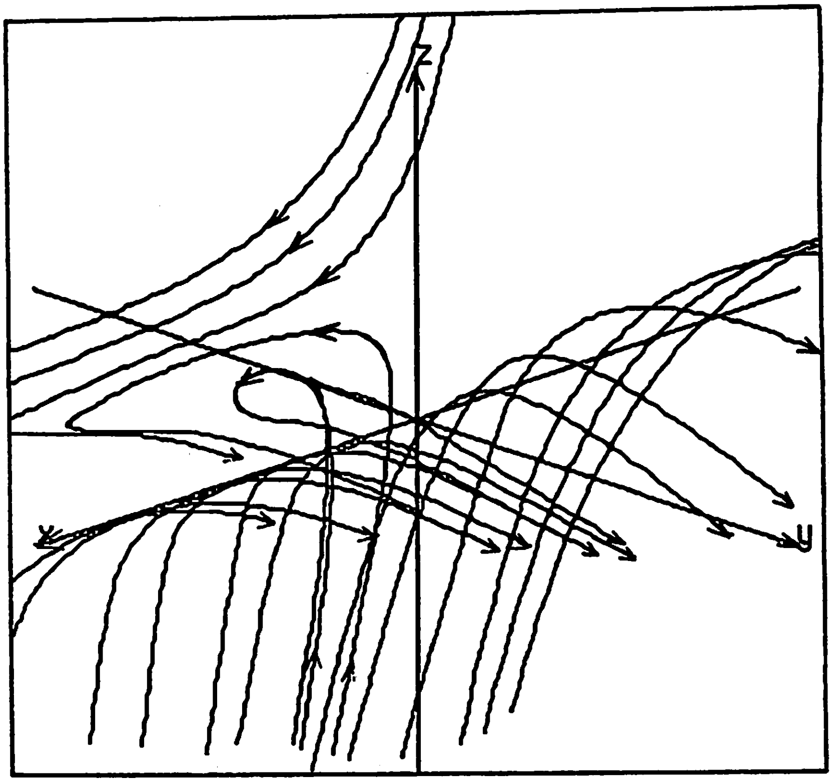

FIGURE: 4

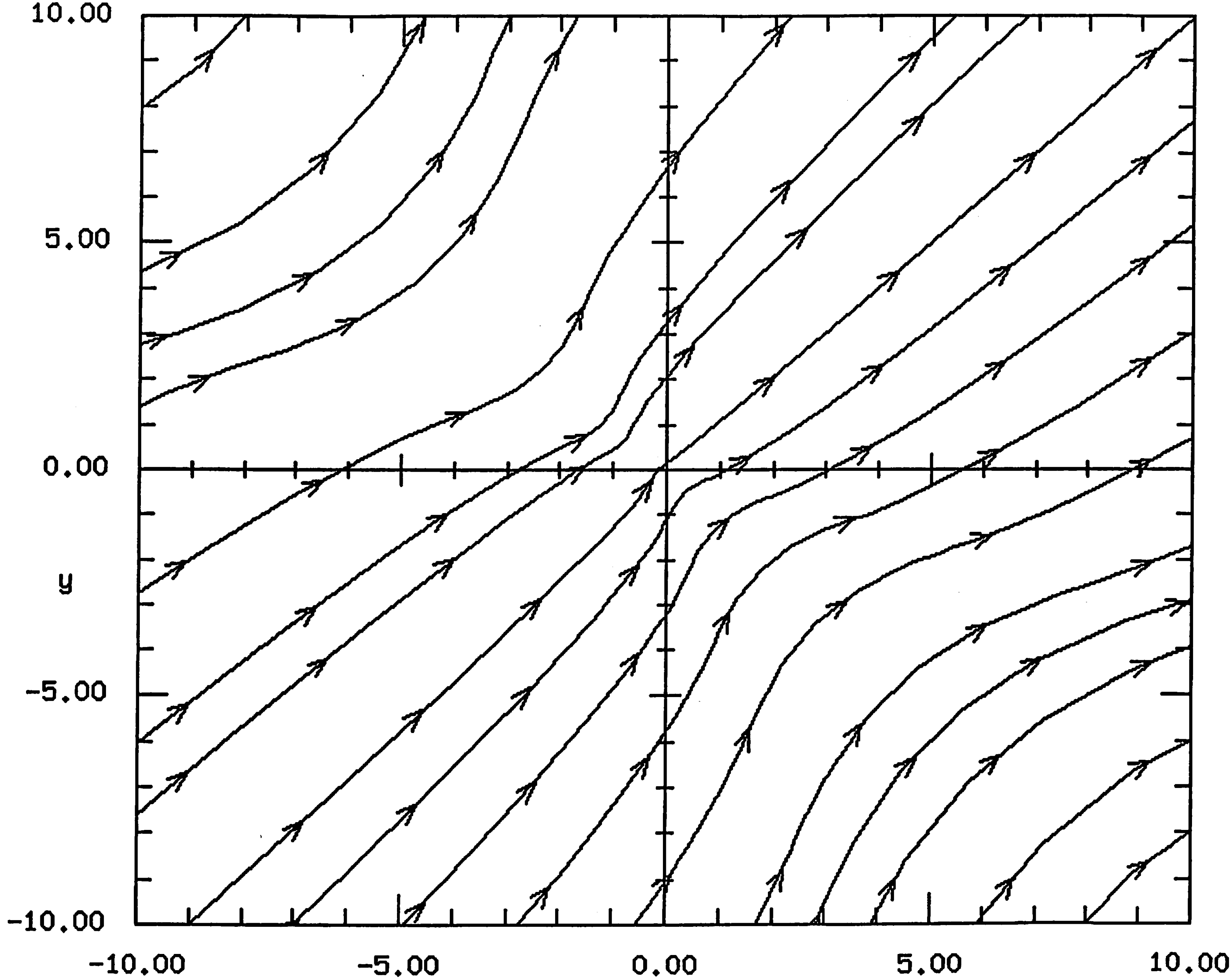

FIGURE: 5

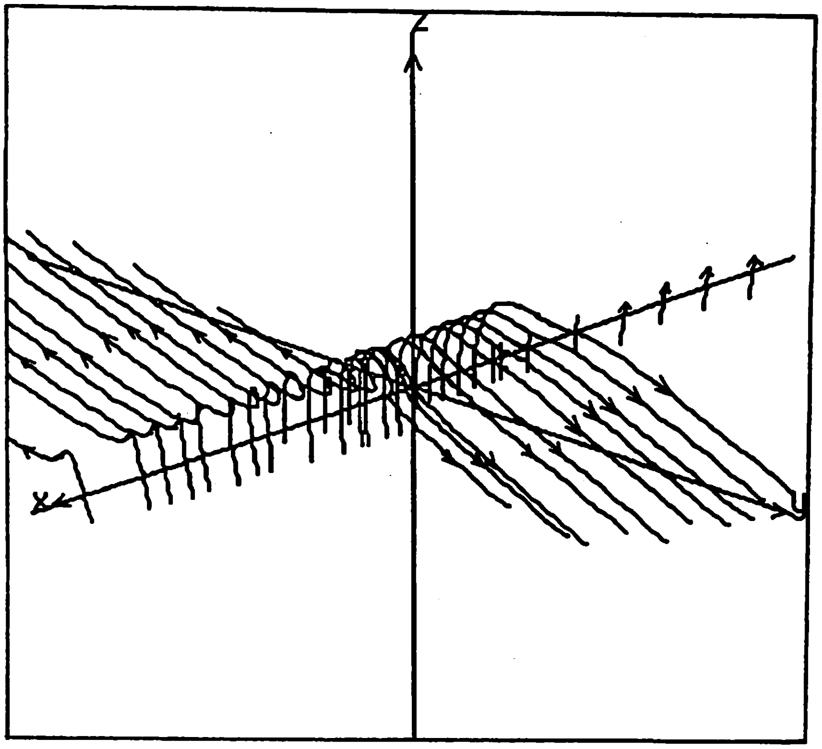

FIGURE: 6

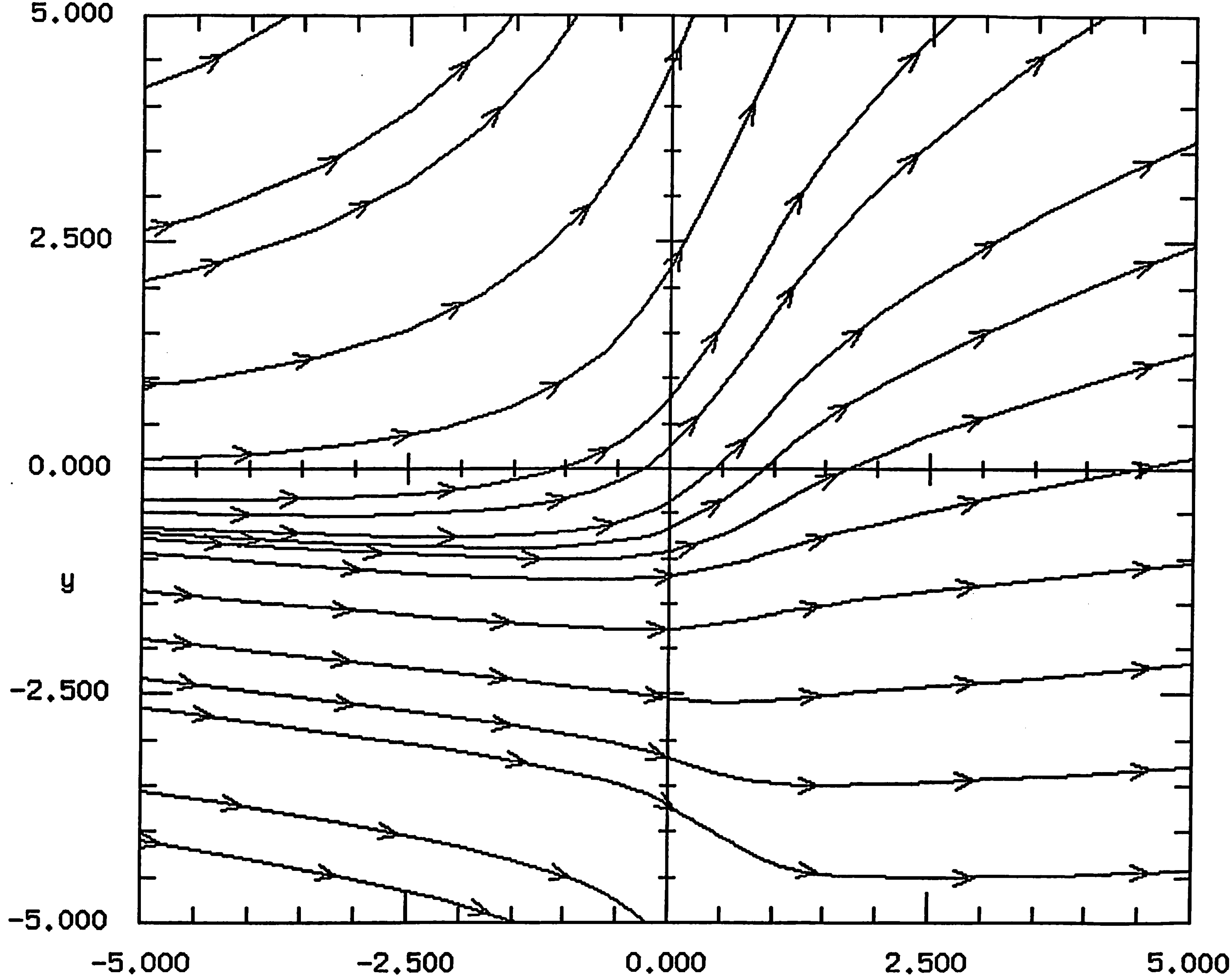

FIGURE: 7

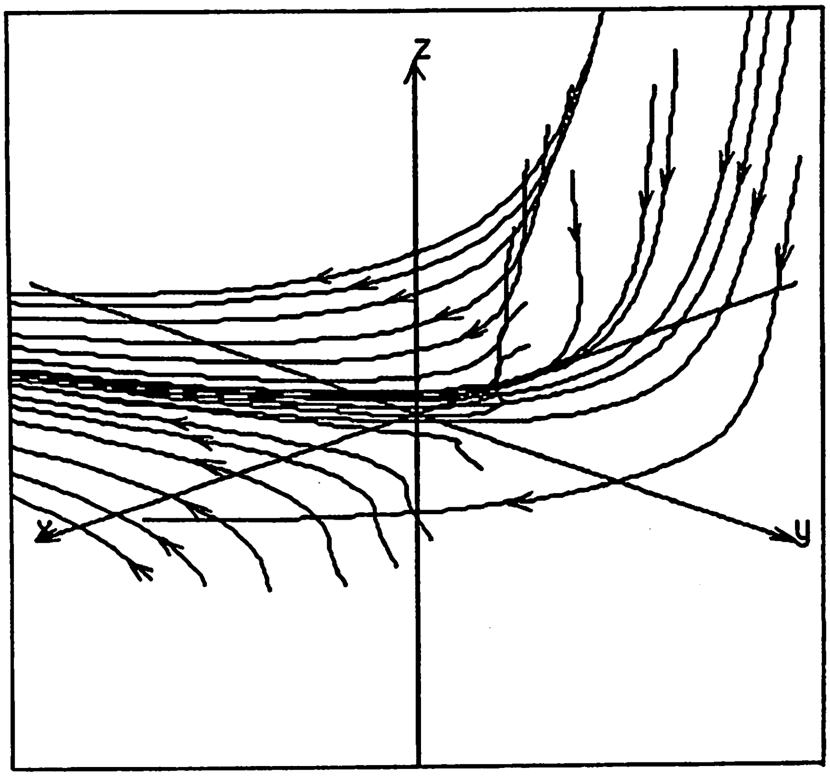

FIGURE: 8

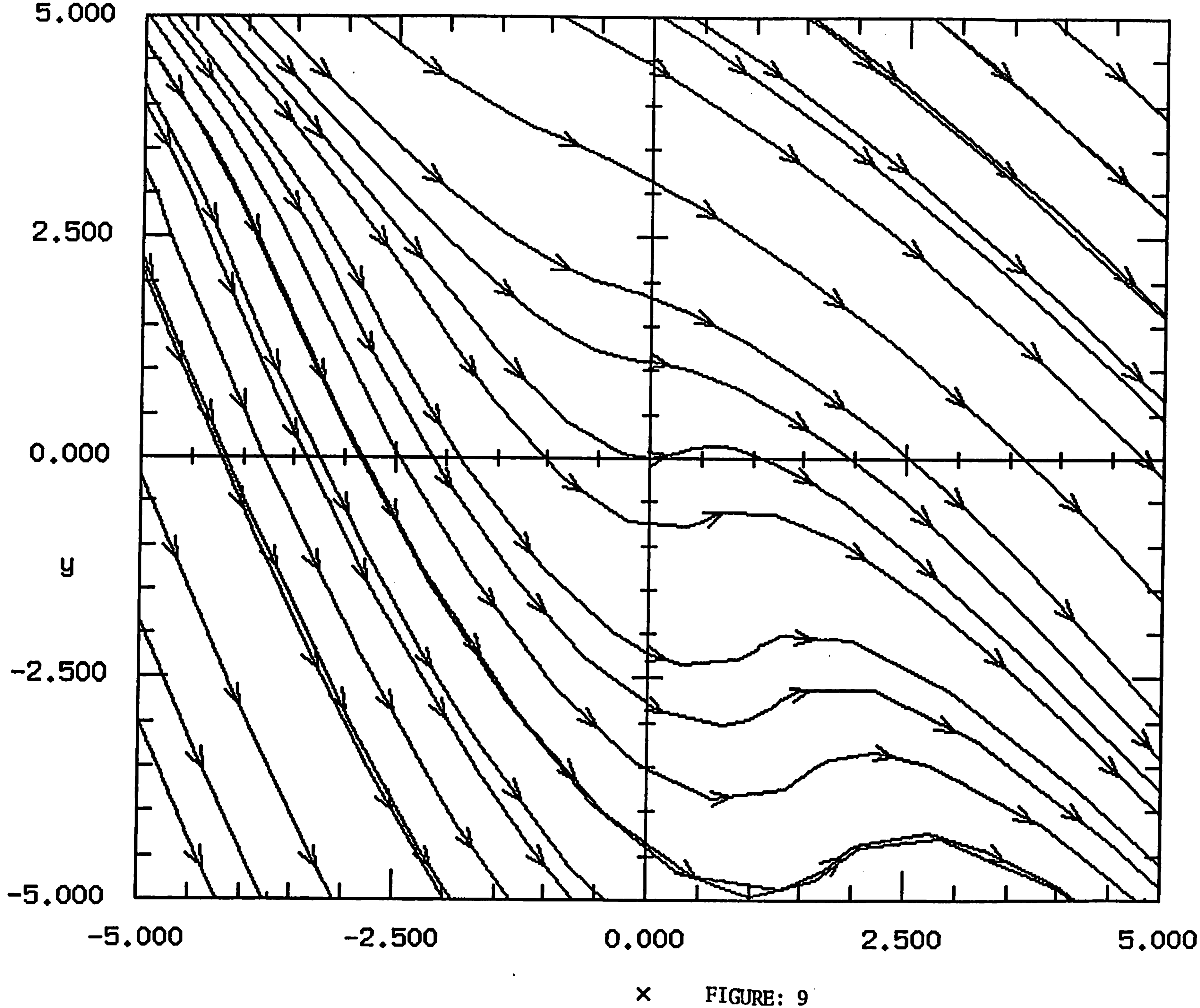

FIGURE: 9

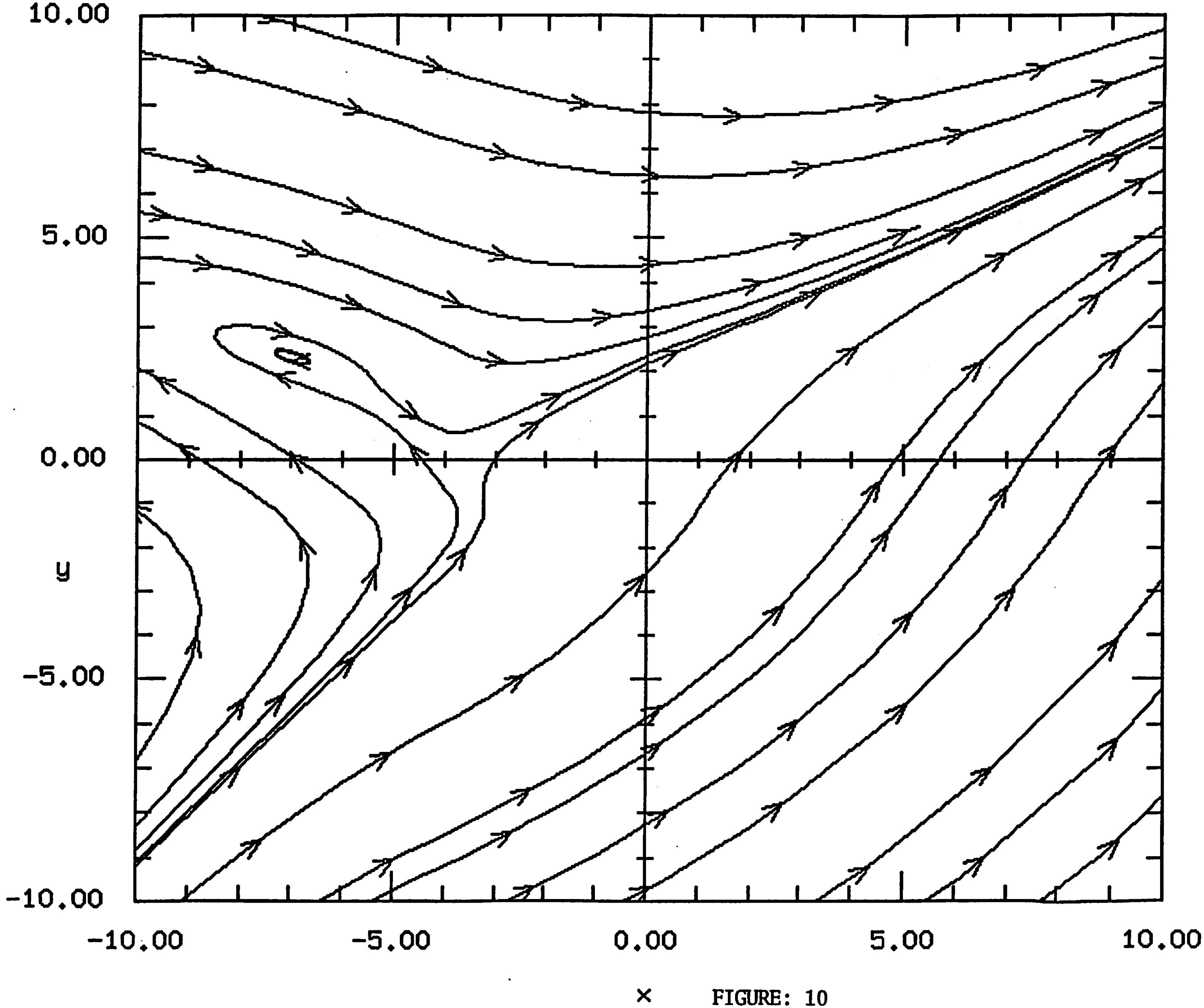

FIGURE: 10