CONTROL OF INTERCONNECTED NONLINEAR DYNAMICAL SYSTEMS: THE PLATOON PROBLEM

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Control of Interconnected Nonlinear Dynamical Systems:the platoon problem*

Shahab Sheikholeslam and Charles A. Desoer†

Abstract The problem in this paper is motivated by a highway automation project [2]. The overall system consists of N vehicles,(the platoon); each vehicle is driven by the same input u and the state of the k-th vehicle affects the dynamics of the $(k+1)$-th vehicle; furthermore, the dynamics of each vehicle is affected by its (local) state-feedback controller. Under very general conditions, it is shown that for sufficiently slowly varying inputs, the local feedbacks can be designed so that the platoon maintains its cohesion.

1 Introduction

The study of interconnections of dynamical systems has a long history usually under the heading of "Large Scale Systems". Some of the main results are to be found in the monograph of Michel and Miller [10], in that of Vidyasagar [11]; their long lists of references are particularly useful. The treatise of Kokotovic, Khalil and O'Reilly [12] on singular perturbations is an excellent reference on the concepts and techniques associated with the notions of slow and fast dynamics. Key features of any such interconnection are a) the graph of the interconnection [11] and b) the time-scale separation of dynamics [12].

The system under study here has a special interconnection scheme which is dictated by the application: a platoon of N vehicles follows, under automatic control, a lead vehicle.

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†Authors are with the Department of Electrical Engineering and Computer Science, University of California at Berkeley, Berkeley, CA 94720, tel:(415)643-8847.

(For background information, see [2],[13] and references therein) To maintain the cohesion of the platoon, the lead vehicle's velocity and acceleration is transmitted to each vehicle of the platoon, and vehicle k measures the distance Δ_k between it and the preceding vehicle. As an approximation we may view the dynamics of the sensors and actuators and that of the engine as fast with respect to that of the vehicle. We show that by suitable design of each controller in each vehicle it is possible to achieve the following: given that the platoon is operating in the steady state at constant velocity, v, at $t = t_0$, and that the lead vehicle accelerates to reach a constant velocity v_1 at some later time T, the control laws can be designed so that for all $k \geq 1$, $\Delta_k(t)$ is bounded on $[t_0, \infty)$ and, for some $\alpha < 1$, $||\Delta_k(.)||_\infty \leq \alpha ||\Delta_{k-1}(.)||_\infty + ||\phi_k(.)||_\infty$ where $\phi_k(t) \to 0$ exponentially as $t \to \infty$ at a rate controlled by the choice of the control laws; here $||.||_\infty$ denotes the sup norm over $[T, \infty)$.

2 Problem motivation

Consider a "platoon" of $N+1$ vehicles traveling in the same lane of a straight stretch of highway and following closely one another. Initially, all vehicles travel at the same constant velocity, say v. The lead vehicle is labelled "l", the next one is labelled "1", and the last one "N": x_k denotes the abscissa of the rear bumper of the k-th vehicle; each vehicle is alloted a slot of length L; let Δ_k be defined by

$$\Delta_k := x_{k-1} - (x_k + L)$$

Δ_k measures the deviation in the assigned distance between vehicle $k-1$ and vehicle k.

Each vehicle is equipped with sensors that measure $\dot{x}_k, \ddot{x}_k, \Delta_k, \dot{\Delta}_k$, and $\ddot{\Delta}_k$ as well as $\dot{x}_l$ and $\ddot{x}_l$ (the last two measurements are obtained by a communication link). Using a nonlinear first order model of the engine, for the k-th vehicle we obtain the following dynamical model in terms of the state $\zeta_k := (\Delta_k, \dot{x}_k - v, \ddot{x}_k)$ and the engine input u_k, (say, the throttle input)

$$\dot{\zeta}_k = f_k(\zeta_k, \zeta_{k-1}) + g_k(\zeta_k)u_k \tag{1}$$

for $k \geq 2$, [2],[13].

It turns out that these equations have such a form that a suitable nonlinear control will lead to the following equation for the k-th vehicle

$$\dot{\zeta}_k = f(\zeta_k, \zeta_{k-1}, u) \tag{2}$$

for $k = 2, 3, \ldots, N$, where $u(t) = (\dot{x}_l(t), \ddot{x}_l(t))$, the velocity and acceleration of the lead vehicle.

Note that in (2), the function $f(.,.,.)$ depends only on the state of the k-th and $(k-1)$-th vehicle and the "input" u: the dependence on the vehicle characteristics (mass m_k, cross section A_k, aerodynamic coefficient C_k, and engine characteristic $\tau(\dot{x}_k)$ have been eliminated by the nonlinear feedback law[2]); hence, $f(.,.,.)$ does not depend on k for $k \geq 2$. For the first vehicle, the control law leads to an equation of the form

$$\dot{\zeta}_1 = f_1(\zeta_1, u) \tag{3}$$

The above discussion suggests the following problem: suppose the platoon of $N+1$ vehicles is initialized as above and suppose that at $t = t_0$ the lead vehicle accelerates from the velocity v to some other constant velocity, say v_1, which it reaches at some time T. Is it possible to choose the vehicle control such that, for such increases in velocity, for $k = 1, 2, \ldots, N$, $\Delta_k(.)$ is bounded, $\Delta_k(t) \to 0$ as $t \to \infty$ and, for t' sufficiently large, $max_{t \geq T+t'} |\Delta_k(t)|$ is a decreasing function of k?

3 Problem formulation

With the above considerations in mind, we state the problem as follows: we are given an interconnection of nonlinear time-invariant dynamical systems described by the following differential equations:

$$\begin{aligned} \dot{\zeta}_1 &= f_1(\zeta_1, u) \\ \dot{\zeta}_2 &= f(\zeta_2, \zeta_1, u) \end{aligned}$$

$$
\begin{aligned}
\dot{\zeta}_3 &= f(\zeta_3, \zeta_2, u) \\
&\vdots \\
\dot{\zeta}_N &= f(\zeta_N, \zeta_{N-1}, u)
\end{aligned} \tag{4}
$$

where the exogeneous control u belongs to an open set $U \subset R^m$ and for $k = 1, 2, \ldots, N$, ζ_k belongs to an open set $P_{u0} \subset R^n$; f_1 and f are C^2 functions of their arguments; ζ_k includes x_k,$\dot{x}_k$, and $\ddot{x}_k$ as components.

Consider the situation where all vehicles travel at the same constant velocity, say v (i.e., $\dot{x}_k = v$) and are at their assigned positions (i.e., $\Delta_k = 0$ for $k = 1, 2, \ldots, N$). Call u_0 the corresponding input $u_0 = (v, 0)$: then by the nature of the vehicle dynamics we have $f_1(\zeta_e, u_0) = 0$, and $f(\zeta_e, \zeta_e, u_0) = 0$, where the *equilibrium state* ζ_e is a function of u_0. We assume that, by clever design of the control law within each vehicle, the dynamical system (4) has a whole set of such equilibrium points for appropriate values of u_0 and that about each such equilibrium of (4) there is a suitable basin of attraction.

Theorem 1 considers a special case of (4) and gives precise conditions under which a slowly varying input u will cause ζ to vary slowly and remain within the basin of attraction of the corresponding equilibrium point. Theorem 2 considers the interconnection of nonlinear dynamical systems described by (4) and gives precise conditions under which the deviations of ζ_k ($k = 1, 2, \ldots, N$) from the equilibrium state ζ_e remain bounded for a slowly varying input u; furthermore, if after some time T, $u(t)$ becomes constant, then the peak value of these deviations decreases as k increases.

Consider the dynamical system described as follows:

$$\dot{\zeta} = f(\zeta, \zeta_p, u) \tag{5}$$

where ζ and ζ_p belong to P_{u0}, an open set of R^n, and u belongs to U, an open set of R^m, with $u(t_0) = u_0$.

Definition A point ζ_0 in P_{u0} is called a *sink* of (5) corresponding to the constant state-input ζ_{p0} in P_{u0} and constant input w_0 in U if $f(\zeta_0, \zeta_{p0}, w_0) = 0$ and $Re\sigma[D_1 f(\zeta_0, \zeta_{p0}, w_0)] < 0$;

where $D_1 f(.,.,.)$ denotes the Jacobian matrix of $f(.,.,.)$ with respect to the first variable and $\sigma[.]$ denotes the spectrum of a matrix.

It is well known that if ζ_0 is a sink corresponding to (ζ_{p0}, w_0), then there is a ball $B(\zeta_0, r)$, centered on ζ_0, such that for all $\zeta(t_0) \in B(\zeta_0, r)$, the solution of (5) is bounded and decays exponentially to ζ_0 (see e.g. [8],[9]).

We also assume that by suitable design of the control law in (5) we may move the spectrum of $D_1 f(\zeta_e, \zeta_e, u)$ further into the left half plane.

Theorem 1 Suppose that $P_{u0} \subset R^n$ is open and convex, and $U \subset R^m$ is open; let $f : P_{u0} \times P_{u0} \times U \to R^n$ be a C^2 function such that

$$M_{u0} := \{(\zeta_e, \zeta_p, u) \in P_{u0} \times P_{u0} \times U |\ \zeta_e \text{ is a sink of (5) corresponding to } (\zeta_p, u)\}$$

has a non-empty interior (relative to topology of M_{u0}). Let $\overline{Q}_{u0}$ be a compact, connected subset of M_{u0}, with a non-empty interior Q_{u0}. Let $u : [t_0, \infty) \to U$, with $u(t_0) = u_0$, $\zeta_p : [t_0, \infty) \to P_{u0}$, and $\zeta_e : [t_0, \infty) \to P_{u0}$ be three given C^1 functions such that $(\zeta_e(t), \zeta_p(t), u(t)) \in Q_{u0}$ for all $t \geq t_0$. Let $\zeta(.)$ be the solution of (5) with the $(\zeta_p(.), u(.))$ defined above and with initial condition $\zeta(t_0)$.

Then, for any $\rho > 0$, there exist $\delta_0 > 0$, $\delta_u > 0$, $\delta_\zeta > 0$ independent of t_0, such that for all $u(.)$,$\zeta_p(.)$, and $\zeta_e(.)$ as defined above and satisfying $|\zeta(t_0) - \zeta_e(t_0)| \leq \delta_0$, $\max_{t \geq t_0} |\dot{u}(t)| \leq \delta_u$ and $\max_{t \geq t_0} |\dot{\zeta}_p(t)| \leq \delta_\zeta$ we have:
i) $|\zeta(t) - \zeta_e(t)| < \rho$ for all $t \geq t_0$,
ii) if, in addition, ρ is sufficiently small, then for all $t \geq t_0$, $\zeta(t)$ belongs to the basin of attraction of the sink $\zeta_e(t)$ with respect to $(\zeta_p(t), u(t))$.

There are two methods for proving this theorem: 1) estimation in the time domain (see Kelemen[1], with improvements [3]); 2) using Lyapunov functions (the existence follows from lemma 2 of Hoppensteadt [4], the technique is detailed by Khalil, et al.[5]).

Since $\overline{Q}_{u0}$ is compact, from i) of theorem 1, we have for all $t \geq t_0$, $\zeta(t) \in Z_{u0}$ where Z_{u0} is a compact set.

4 Main result

We consider now the composite dynamical system described by (4). Let f satisfy the assumptions of theorem 1; consider some slowly varying $u(t)$ and the corresponding $\zeta_e(t)$.

With respect to the first equation of (4):

$$\dot{\zeta}_1 = f_1(\zeta_1, u), \tag{6}$$

we assume that $f_1 : P_{u0} \times U \to R^n$ is a C^2 function such that

$$M^1_{u0} := \{(\zeta_e, u) \in P_{u0} \times U | \ \zeta_e \text{ is a sink of (5) corresponding to } u\}$$

has a non-empty interior (relative to topology of M^1_{u0}). Let $\overline{Q}^1_{u0}$ be a compact, connected subset of M^1_{u0}, with a non-empty interior Q^1_{u0}. Let $u : [t_0, \infty) \to U$, with $u(t_0) = u_0$ and $\zeta_e : [t_0, \infty) \to P_{u0}$ be two given C^1 functions such that $(\zeta_e(t), u(t)) \in Q^1_{u0}$ for all $t \geq t_0$.

Consider (6). It is a well known result (e.g. [1],[5]) that given these assumptions on f_1, for any $\rho > 0$, there exist $\delta^1_0 > 0$ and $\delta^1_u > 0$ such that if $|\zeta_1(t_0) - \zeta_e(t_0)| \leq \delta^1_0$ and $max_{t \geq t_0} |\dot{u}(t)| < \delta^1_u$ then for all $t \geq t_0$, $\zeta_1(t) \in Z_{u0}$ and $|\zeta_1(t) - \zeta_e(t)| < \rho$.

Lemma 1 Consider the nonlinear dynamical system described by (4) keeping in mind the above considerations. Under the conditions stated above, by suitable design of the control laws, if ρ is chosen sufficiently small, then for $k = 1, 2, \ldots, N$: 1) for all $t \geq t_0$, $\zeta_k(t) \in Z_{u0}$, and 2) for all $t \geq t_0$, $max_{t \geq t_0} |\dot{\zeta}_k(t)| \leq \delta_\zeta$.

Proof We use induction.

Writing the Taylor expansion of (6) about (ζ_e, u) and noting that $f_1(\zeta_e, u) = 0$ we obtain

$$\dot{\zeta}_1 = H_1(\zeta_e, u, \zeta_1)(\zeta_1 - \zeta_e) \tag{7}$$

where $H_1(\zeta_e, u, \zeta_1) := \int_0^1 D_1 f_1[\zeta_e + \lambda(\zeta_1 - \zeta_e), u] d\lambda$; note that $H_1(.,.,.)$ is continuous.

Since for all $t \geq t_0$, $(\zeta_e(t), u(t), \zeta_1(t)) \in \overline{Q}^1_{u0} \times Z_{u0}$, a compact set, and $H_1(.,.,.)$ is continuous, there exists a constant, $h_1 \geq 0$, such that

$$h_1 = max_{t \geq t_0} |H_1(\zeta_e(t), u(t), \zeta_1(t))|. \tag{8}$$

From (7) and (8) we obtain

$$max_{t \geq t_0} |\dot{\zeta}_1(t)| \leq h_1 \rho \tag{9}$$

hence,

$$\text{If } \rho \leq \frac{\delta_\zeta}{h_1} \text{ then } max_{t \geq t_0} |\dot{\zeta}_1(t)| \leq \delta_\zeta. \tag{10}$$

Induction step We use the notations of theorem 1. Assume that for some $k \geq 1$, $|\zeta_{k+1}(t_0) - \zeta_e(t_0)| \leq \delta_0$, $max_{t \geq t_0} |\dot{u}(t)| \leq \delta_u$, and for all $t \geq t_0$, $\zeta_k(t) \in Z_{u0}$ and $max_{t \geq t_0} |\dot{\zeta}_k(t)| \leq \delta_\zeta$; we will show that for all $t \geq t_0$, $\zeta_{k+1}(t) \in Z_{u0}$ and $max_{t \geq t_0} |\dot{\zeta}_{k+1}(t)| \leq \delta_\zeta$.

Consider the following dynamical system

$$\dot{\zeta}_{k+1} = f(\zeta_{k+1}, \zeta_k, u) \tag{11}$$

Since the assumptions of theorem 1 are satisfied for (11), we have for all $t \geq t_0$, $\zeta_{k+1}(t) \in Z_{u0}$ and $|\zeta_{k+1}(t) - \zeta_e(t)| < \rho$.

Writing the Taylor expansion of (11) about (ζ_e, ζ_e, u) and noting that $f(\zeta_e, \zeta_e, u) = 0$ we obtain

$$\begin{aligned} \dot{\zeta}_{k+1} &= \int_0^1 D_1 f[\zeta_e + \lambda(\zeta_{k+1} - \zeta_e), \zeta_e + \lambda(\zeta_k - \zeta_e), u] d\lambda \; (\zeta_{k+1} - \zeta_e) \\ &+ \int_0^1 D_2 f[\zeta_e + \lambda(\zeta_{k+1} - \zeta_e), \zeta_e + \lambda(\zeta_k - \zeta_e), u] d\lambda \; (\zeta_k - \zeta_e). \end{aligned} \tag{12}$$

Here $D_k f(.,.,.)$ denotes the partial derivative of $f(.,.,.)$ with respect to its k-th argument.

We can write (12) as follows

$$\dot{\zeta}_k = G_1(\zeta_e, u, \zeta_k, \zeta_{k+1})(\zeta_{k+1} - \zeta_e) + G_2(\zeta_e, u, \zeta_k, \zeta_{k+1})(\zeta_k - \zeta_e) \tag{13}$$

where $G_1(\zeta_e, u, \zeta_k, \zeta_{k+1})$ and $G_2(\zeta_e, u, \zeta_k, \zeta_{k+1})$ denote the first and the second integrals in the right hand side of (12), respectively.

Let $P_{Q_{u0}, u0} := \{(\zeta_e, u) | (\zeta_e, \zeta_p, u) \in Q_{u0}\}$. Now, for all $t \geq t_0$, $(\zeta_e(t), u(t), \zeta_k(t), \zeta_{k+1}(t)) \in \overline{P}_{Q_{u0}, u0} \times Z_{u0} \times Z_{u0} := Y_{u0}$, a compact set; $G_1(.,.,.,.) \in C^1$, and $G_2(.,.,.,.) \in C^1$. Hence $G_1(.,.,.,.)$ and $G_2(.,.,.,.)$ are bounded on Y_{u0} by $g_1 \geq 0$ and $g_2 \geq 0$ respectively.

Using these bounds and (13), and noting that by the induction hypothesis for all $t \geq t_0$, $|\zeta_k(t) - \zeta_e(t)| < \rho$ we obtain

$$max_{t \geq t_0} |\dot{\zeta}_{k+1}(t)| \leq (g_1 + g_2)\rho \tag{14}$$

hence,

$$\text{If } \rho \leq \frac{\delta_\zeta}{g_1 + g_2} \text{ then } max_{t \geq t_0} |\dot{\zeta}_{k+1}(t)| \leq \delta_\zeta. \tag{15}$$

From (10) and (15) we note that for $k = 1, 2, \ldots, N$, if $\rho \leq min\left\{\frac{\delta_\zeta}{h_1}, \frac{\delta_\zeta}{g_1+g_2}\right\}$ then $max_{t\geq t_0}|\dot{\zeta}_k(t)| \leq \delta_\zeta$.

Again, let f satisfy the assumptions of theorem 1; consider some slowly varying $u(t)$ and the corresponding $\zeta_e(t)$. Let for $k \geq 2$, $d_k(t) + \zeta_e(t) = \zeta_k(t)$ and assume $d_k(t_0) = 0$ for all k.

Theorem 2 Under these conditions,

1. if (a) ρ is sufficiently small so that lemma 1 holds, (b) for some sufficiently small $\delta_u > 0$ as in the statement of theorem 1, $max_{t\geq t_0}|\dot{u}(t)| < \delta_u$, and (c) for some sufficiently small $\delta_\zeta > 0$ as in the statement of theorem 1, $max_{t\geq t_0}|\dot{\zeta}_e(t)| < \delta_\zeta$, then, by suitable design of control laws, there exist some constants α and β such that $0 \leq \alpha < 1$, $0 < \beta < \infty$, and for $k \geq 2$,

$$||d_{k+1}||_\infty \leq \alpha||d_k||_\infty + \beta||\dot{\zeta}_e||_\infty \tag{16}$$

   hence, for large k,

$$||d_k||_\infty \leq \frac{\beta}{1-\alpha}||\dot{\zeta}_e||_\infty + O(\alpha^{k-1}); \tag{17}$$

   i.e., there is a uniform bound on $||\zeta_k - \zeta_e||_\infty$;

2. if, in addition, after some time T, $u(t)$ and (consequently) $\zeta_e(t)$ become constant, then by local control law design, $\eta > \tilde{k}\gamma$ can be increased; hence, we can obtain

$$||d_{k+1}||_\infty \leq \alpha||d_k||_\infty + ||\phi_k||_\infty. \tag{18}$$

   where, as in (16), $\alpha < 1$; here $\phi_k(t) \to 0$ exponentially as $t \to \infty$, and $||.||_\infty$ denotes the sup norm on $[T, \infty)$.

   In other words, in case of a change in u in the lead vehicle from the initial steady-state value u_0 to the final steady-state value u_f, the peak disturbances down the platoon, i.e., $d_2(.), d_3(.), \ldots$ decrease as k increases.

   Proof(theorem 2, part 1) Adding and subtracting $D_1 f(\zeta_e, \zeta_e, u)d_{k+1}$ to the right hand side of (12) and noting that $\dot{\zeta}_{k+1} = \dot{\zeta}_e + \dot{d}_{k+1}$ we obtain

$$\dot{d}_{k+1} = A(t)d_{k+1} + R(t)d_{k+1} + B(t)d_k - \dot{\zeta}_e \tag{19}$$

where

$$A(t) := D_1 f(\zeta_e(t), \zeta_e(t), u(t)) \tag{20}$$

$$R(t) := \int_0^1 \{D_1 f[\zeta_e(t) + \lambda d_{k+1}(t), \zeta_e(t) + \lambda d_k(t), u(t)] - D_1 f(\zeta_e(t), \zeta_e(t), u(t))\}\, d\lambda \tag{21}$$

and

$$B(t) := \int_0^1 D_2 f[\zeta_e(t) + \lambda d_{k+1}(t), \zeta_e(t) + \lambda d_k(t), u(t)] d\lambda. \tag{22}$$

Let $\Phi(t,\tau)$ be the state transition matrix of $\dot{z} = A(t)z$. Then from (19) we obtain

$$\begin{aligned} d_{k+1}(t) &= \Phi(t,t_0) d_{k+1}(t_0) \\ &+ \int_{t_0}^{t} \Phi(t,\tau) R(\tau) d_{k+1}(\tau) d\tau \\ &+ \int_{t_0}^{t} \Phi(t,\tau)[B(\tau) d_k(\tau) - \dot{\zeta}_e(\tau)] d\tau \end{aligned} \tag{23}$$

here the first term in the right hand side of (23) is zero since $d_{k+1}(t_0) = 0$.

Note that for all $t \geq t_0$, $(\zeta_e(t), u(t)) \in \overline{P}_{Q_{u0},u0}$, a compact set, and $D_1 f(.,.,.)$ is continuous; hence,

$$A(.) \text{ is bounded on } [t_0, \infty). \tag{24}$$

Since $\sigma[A(t)] = \sigma[D_1 f(\zeta_e(t), \zeta_e(t), u(t))]$ is a continuous function of its entries and for all $t \geq t_0$, $(\zeta_e(t), \zeta_e(t), u(t)) \in \overline{Q}_{u0}$ with $\zeta_e(t)$ being a sink of (11) corresponding to $(\zeta_e(t), u(t))$, there exists a constant $\mu < 0$ such that

$$\text{for all } t \geq t_0 \text{ , } Re\sigma[A(t)] \leq \mu. \tag{25}$$

From (24) and (25) we note [6,Theorem 2, sec.32] that there exists a constant $\epsilon > 0$ such that if $|\dot{A}(t)| \leq \epsilon$ then

$$\text{for some } \tilde{k} \geq 1 \text{ and some } \eta > 0 \text{ and for all } t \geq s \geq t_0 \text{ , } |\Phi(t,s)| \leq \tilde{k} exp[-\eta(t-s)]. \tag{26}$$

Differentiating the right hand side of (20) with respect to t and using the chain rule we obtain

$$\begin{aligned} \dot{A}(t) &= \{D_1 D_1 f(\zeta_e(t), \zeta_e(t), u(t)) + D_2 D_1 f(\zeta_e(t), \zeta_e(t), u(t))\}\, \dot{\zeta}_e(t) \\ &+ D_3 D_1 f(\zeta_e(t), \zeta_e(t), u(t)) \dot{u}(t) \end{aligned} \tag{27}$$

Since $D_1D_1f(.,.,.)$,$D_2D_1f(.,.,.)$, and $D_3D_1f(.,.,.)$ are continuous and for all $t \geq t_0$, $(\zeta_e(t), \zeta_e(t), u(t)) \in \overline{Q}_{u0}$, a compact set, $D_1D_1f(.,.,.)$,$D_2D_1f(.,.,.)$, and $D_3D_1f(.,.,.)$ are bounded on $\overline{Q}_{u0}$. Let

$$a_1 := max\left\{|D_1D_1f(\zeta_e, \zeta_e, u) + D_2D_1f(\zeta_e, \zeta_e, u)| \; : \; (\zeta_e, \zeta_e, u) \in \overline{Q}_{u0}\right\}$$

and

$$a_2 := max\left\{|D_3D_1f(\zeta_e, \zeta_e, u)| \; : \; (\zeta_e, \zeta_e, u) \in \overline{Q}_{u0}\right\}.$$

If $max_{t\geq t_0}|\dot{u}(t)| < \delta_u \leq \frac{\epsilon}{2a_2}$ and $max_{t\geq t_0}|\dot{\zeta}_e(t)| < \delta_\zeta \leq \frac{\epsilon}{2a_1}$ then from (27) we obtain $|\dot{A}(t)| \leq \epsilon$ and (26) is satisfied.

Now, from lemma 1, for all $t \geq t_0$, $(\zeta_e(t), u(t), \zeta_k(t), \zeta_{k+1}(t)) \in Y_{u0}$, a compact set, and $B(.,.,.,.)$ is continuous; hence, there exists a constant $b \geq 0$ such that

$$b = max_{t\geq t_0}|B(t)|. \tag{28}$$

Similarly, $R(.,.,.,.)$ is continuous and bounded on Y_{u0}; hence, by compactness, for some constant $\gamma \geq 0$ we have

$$\gamma = max_{t\geq t_0}|R(t)| \tag{29}$$

From (23),(26),(28), and (29) we obtain

$$\begin{aligned} |d_{k+1}(t)| \;\; &\leq \;\; \int_{t_0}^{t} \tilde{k}\gamma \; exp[-\eta(t-\tau)]|d_{k+1}(\tau)|d\tau \\ &+ \;\; \int_{t_0}^{t} \tilde{k} \; exp[-\eta(t-\tau)][b|d_k(\tau)| + |\dot{\zeta}_e(\tau)|]d\tau \end{aligned} \tag{30}$$

Applying a form of Gronwall lemma to (30), [7, Corollary 1.9.1], we obtain

$$\begin{aligned} |d_{k+1}(t)| \;\; &\leq \;\; \int_{t_0}^{t} \tilde{k} \; exp[(-\eta + \tilde{k}\gamma)(t-\tau)][b|d_k(\tau)| + |\dot{\zeta}_e(\tau)|]d\tau \\ &\leq \;\; \int_{t_0}^{t} \tilde{k} \; exp[(-\eta + \tilde{k}\gamma)(t-\tau)]d\tau[b||d_k||_\infty + ||\dot{\zeta}_e||_\infty] \end{aligned} \tag{31}$$

By suitable design of the control law, we can increase η sufficiently beyond $\tilde{k}\gamma$ so that $0 \leq \alpha := \frac{b\tilde{k}}{\eta - \tilde{k}\gamma} < 1$; let $\beta := \frac{\tilde{k}}{\eta - \tilde{k}\gamma}$. Then, from (31), for all $t \geq t_0$,

$$|d_{k+1}(t)| \leq \alpha||d_k||_\infty + \beta||\dot{\zeta}_e||_\infty \tag{32}$$

hence,

$$||d_{k+1}||_\infty \leq \alpha||d_k||_\infty + \beta||\dot{\zeta}_e||_\infty \tag{33}$$

By recurrence, noting that $\alpha < 1$, we see that for all $k \geq 2$,

$$||d_k||_\infty \leq \frac{\beta}{1-\alpha}||\dot{\zeta}_e||_\infty + O(\alpha^{k-1}). \tag{34}$$

Proof(theorem 2, part 2) From (19) we note that for $t \geq T$,

$$\begin{aligned} d_{k+1}(t) &= \Phi(t,T)d_{k+1}(T) \\ &+ \int_T^t \Phi(t,\tau)R(\tau)d_{k+1}(\tau)d\tau \\ &+ \int_T^t \Phi(t,\tau)[B(\tau)d_k(\tau) - \dot{\zeta}_e(\tau)]d\tau \end{aligned} \tag{35}$$

Hence, noting that for $t \geq T, \dot{\zeta}_e(t) = 0$, from (35), (26),(28), and (29) we obtain

$$\begin{aligned} |d_{k+1}(t)| &\leq \tilde{k}\ exp[-\eta(t-T)]|d_{k+1}(T)| \\ &+ \int_T^t \tilde{k}\gamma\ exp[-\eta(t-\tau)]|d_{k+1}(\tau)|d\tau \\ &+ \int_T^t \tilde{k}b\ exp[-\eta(t-\tau)]|d_k(\tau)|d\tau \end{aligned} \tag{36}$$

Applying a form of Gronwall lemma [7, Corollary 1.9.1] to (36) and using the previously defined α, we obtain for all $t \geq T$,

$$\begin{aligned} |d_{k+1}(t)| &\leq \tilde{k}\ exp[(-\eta+\tilde{k}\gamma)(t-T)]|d_{k+1}(T)| \\ &+ \int_T^t \tilde{k}b\ exp[(-\eta+\tilde{k}\gamma)(t-\tau)]|d_k(\tau)|d\tau \qquad (37) \\ &\leq \tilde{k}\ exp[(-\eta+\tilde{k}\gamma)(t-T)]|d_{k+1}(T)| \\ &+ \alpha||d_k||_\infty \qquad (38) \end{aligned}$$

By design, $\eta > \tilde{k}\gamma$ can be increased so that $\alpha := \frac{\tilde{k}b}{\eta-\tilde{k}\gamma} < 1$ and we have $|d_{k+1}(t)| \leq \alpha||d_k||_\infty + ||\phi_k||_\infty$ where $\phi_k(t) \to 0$ exponentially as $t \to \infty$, and $||.||_\infty$ denotes the sup norm on $[T,\infty)$.

These theorems, together with simulations using realistic vehicle models [2],[13], establish that it is reasonable to contemplate platoons of vehicles traveling down the highway at high speed and maintaining a tight formation by automatic control.

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