GENERIC PROPERTIES OF CONTINUOUS PIECEWISE-LINEAR VECTOR FIELDS IN $\Re^2$

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TITLE PAGE

GENERIC PROPERTIES OF CONTINUOUS PIECEWISE-LINEAR VECTOR FIELDS IN $\Re^2$. †

Robert Lum AND Leon O. Chua. ††

Abstract

In recent years there has been a great increase in the use of computer modelling in many fields of science and engineering. The success of such modelling is evident in computer aided design and manufacturing where much of the testing process of the final product can be done without resort to actual construction of intermediate attempts. However, there is another area of computer modelling whose results have been widely studied without recourse to considering the caveats that computer modelling itself may entail. This area is the computer modelling of nonlinear vector fields.

To model a nonlinear vector field by the use of a computer introduces a number of errors intrinsic to computer modelling. This paper considers one such source of computer generated error and its interpretation from the viewpoint of the user who wishes to understand the original vector field.

The approach taken to investigate the pragmatic results of computer modelling in consideration of the properties of the underlying nonlinear vector field will be essentially topological. Generic properties, as will properties of denseness and openess to be discussed will be interpreted from their effects on computer modelling.

As a byproduct of this work the class of continuous piecewise linear vector fields has shown to be very amenable to theorectical analysis. This suggests that the use of piecewise linear vector fields to be the preferred modelling technique from both a practical and theorectical viewpoint. The theorectical results justifying the practical observations.

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§0. Introduction.

The use of computers in the simulation of nonlinear vector fields has grown enormously in recent years with the advent of ever increasingly powerful computational speed in an era of decreasing hardware cost. The computer generated phase portraits of nonlinear vector fields are now commonplace and are indicative of the extent and impact which computers have had on the study of nonlinear vector fields. Despite the pervasiveness with which computers are used in the study of dynamical systems, there still remain questions as to the interpretation of their results, a question that becomes exacerbated when the underlying system is not merely a dynamical system but also possesses chaotic behaviour.

As an example of the problems of computer simulation, consider a parameterised family of nonlinear vector fields for which the vector field corresponding to a fixed set of parameter values has a hyperbolic periodic orbit whose stable and unstable manifold meet nontransverally. This could happen in the case of a researcher interested in particular values of the parameters with a certain significance. Furthermore, the parameter values at which the researcher is interested in are irrational (e.g. π) and as the parameterised vector field passes through this set of parameter values the stable and unstable manifolds of the corresponding periodic orbit change from transversal and nonintersecting (nonchaotic) to transversal and intersecting (chaotic). Since computer storage of numbers is implemented by finite precision approximation, the researcher's parameters would therefore not be stored in their true irrational form but as rational approximations. As a result, the vector field that he or she would be observing would not be the desired vector field but that of a vector field whose parameters are rational. It is then not inconceivable that the researcher may one day observe chaotic behaviour on a microcomputer with 16 bit precision to see such behaviour disappear the next day on a 128 bit precision supercomputer. This is clearly an unacceptable state of affairs.

In this paper the predominant type of vector fields to be studied are continuous piecewise linear vector fields. With respect to these types of vector fields it is possible to give a partial answer of the implications of computer simulation of such nonlinear vector fields. The problem to be addressed is essentially the same as that brought up in the previous paragraph; namely, how does one interpret the results of computer simulation of a piecewise linear vector field whose defining constants may not be accurately stored inside a computer's memory?

A partial answer to this question requires the mathematical concepts of openness and density, leading naturally to the ideas of structural stability and genericity. Beginning with the work of Poincare, Liapunov and Birkhoff, the understanding of these two basic concepts in dynamical systems has seen considerable development as fundamental research questions. It was in 1937 that Andronov and Pontryagin introduced the modern definition of structural stability. Two decades later, Peixoto was able to prove density of structurally stable vector fields on 2-dimensional manifolds. It was at this time that Smale proved a number of fundamental results and set down the main objective of research into dynamical systems as the search for generic and stable properties. Hartman and Grobman, simultaneously and independently, proved that local stability is a generic property. This soon lead to the proof by Kupka and Smale that stable periodic orbits are also a generic property.

It is in this tradition of research into genericity and stability that the current work has been directed. Although it may be tempting to apply previous results from the traditional theory on compact manifolds and smooth vector fields, this is not the case of piecewise linear vector fields. By the nature of their usage, the natural topology to induce on the set of piecewise linear vector fields differs greatly from the traditional C^k topologies traditionally used in differentiable dynamical systems. Because of this departure, a separate analysis is needed to determine the properties of piecewise linear vector fields.

However, by using a topological approach to the problem, the results of the paper have to be reinterpreted from the viewpoint in which they were originally asked. The main result is contingent on two important conjectures (conjecture 3.3 and conjecture 3.9) which serve as perturbation claims on the families of piecewise linear vector fields. The main theorem (theorem 3.13) states that given any continuous piecewise linear vector field in $\Re^2$ that contains saddle connections (including homoclinic orbits) there exist arbitrarily small perturbations that do no contain any saddle connections. Thus, continuous piecewise linear vector fields in $\Re^2$ without saddle connections are dense. Furthermore, there are neighbourhoods about the perturbed vector field that contain vector fields that do not contain any saddle connections at all. Although not strictly an openness property, this result has the same flavour with important implications for computer simulation. The implication of the theorem is that the continuous piecewise linear vector fields without saddle connections are a dense set with nonempty interior. From the standpoint of computer simulation this could be weakly interpreted as saying that in simulating a vector field with a saddle connection there is a possibility of persistently simulating a vector field with no saddle connections at all. A piecewise linear vector field full of saddle connections could be misleadingly simulated as having no saddle connections and attempts to rectify the situation by increasing resolution may not alleviate the problem. Another interpretation is that if a simulation should show the vector field as having a saddle connection then the chances are that the saddle connection is illusory. The density of vector fields without saddle connections throws doubt on the chances of actually simulating a vector field with a saddle connection.

The two previous interpretations of the results of this paper are further complicated by errors that arise from numerical intergration techniques which in themselves are simulating yet another different type of vector field altogether. In conclusion it is remarked that great care needs to be exercised in the interpretation of computer simulation of nonlinear vector fields, with each step of the simulation process introducing its own set of errors the final output has to be interpreted within its proper context.

§1. Definitions.

In this section the definition of a special class of continuous piecewise linear vector fields in $\Re^2$ will be presented. This definition is sometimes called a lattice piecewise linear vector field by other researchers.

Definition 1.1. P(n,m) is the set of order (n,m),$0 \leq$ n,m, continuous piecewise linear vector fields in $\Re^2$ given by

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \end{bmatrix} |x - \gamma_i| + \sum_{i=n+1}^{n+m} \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \end{bmatrix} |y - \gamma_i|$$

where $0 < \alpha_{i1}^2 + \alpha_{i2}^2, \gamma_1 < \ldots < \gamma_n, \gamma_{n+1} < \ldots < \gamma_{n+m}$. Henceforth, this special class of vector fields will be called vector fields in P(n,m). Since only this class of vector fields are considered in this paper, we will sometimes refer to them simply as vector fields to avoid clutter.

The lines $x \equiv \gamma_1, \ldots, \gamma_n$ and $y \equiv \gamma_{n+1}, \ldots, \gamma_{n+m}$ can be considered as boundary lines which lie in $\Re^2$ vertically and horizontally. Together, the vertical and horizontal lines effect a partition of $\Re^2$ into which the vector field ξ is linear in each element of the partition. This observation is elaborated in the next definition of the partition associated with a given vector field ξ.

EXAMPLE 1.2. (Figure 1.) As an example of a continuous piecewise linear vector field consider the vector field given by the equation

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} |x+1| - \begin{bmatrix} 2 \\ -3 \end{bmatrix} |x| - \begin{bmatrix} 5 \\ 1 \end{bmatrix} |x-1| - \begin{bmatrix} 2 \\ 0 \end{bmatrix} |y+1| + \begin{bmatrix} 4 \\ 1 \end{bmatrix} |y-2|.$$

The lines $x \equiv -1, 0, 1$ and $y \equiv -1, 2$ are the horizontal and vertical boundary lines respectively of the vector field. These lines together divide $\Re^2$ into 12 regions in which the vector field is linear in each region.

Definition 1.3. Given a vector field $\xi \in$P(n,m) there is an associated partition of $\Re^2$, Part(ξ) where

$$\text{Part}(\xi) = \{A_{ij} = L_i \times W_j : 0 \leq i \leq n, 0 \leq j \leq m\}$$

with

$$\begin{aligned} L_0 &= (-\infty, \gamma_1], \\ L_i &= [\gamma_i, \gamma_{i+1}], && 1 \leq i < n \\ L_n &= [\gamma_n, \infty), \\ W_0 &= (-\infty, \gamma_{n+1}], \\ W_i &= [\gamma_{n+j}, \gamma_{n+j+1}], && 1 \leq j < m \\ W_m &= [\gamma_{n+m}, \infty). \end{aligned}$$

Some basic topological concepts will be revised. The main topological ideas needed are openness and density, but in order to give a rigorous meaning to these ideas and to understand their intent some auxiliary definitions will be required.

Definition 1.4. Let X be a non-empty set. A metric $d : X \times X \to \Re^+$ is a function that satisfies the following axioms:

(i) : $\forall x, y \in X$, $0 \leq d(x, y)$ with equality holding if and only if $x = y$, .

(ii) : $\forall x, y \in X$, $d(x, y) = d(y, x)$, and

(iii) : $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

EXAMPLE 1.5. In $\Re$ a metric is given by $d(x, y) = |x - y|$. In the natural numbers $\mathbf{N}$, i.e. the set of all integers, a metric can be defined by $d(n, m) = 0$ if $n = m$, and $d(n, m) = \infty$ if $n \neq m$.

A metric enables the concept of distance to be imposed on an arbitratry set X. The first axiom requires distance to be non-negative. Furthermore, two points are zero distance apart if and only if they are the same point. The second axiom states the symmetry property of distance, that the distance between two points is independent of which point the other is measured from. Lastly, the third axiom is the triangle inequality.

Definition 1.6. Let X be a non-empty set with metric $d(x, y)$. The metric $d(x, y)$ defines a collection of open balls as subsets of the set X. Given $x \in X$ and $0 < \epsilon$ the open ball $B(x, \epsilon)$ is the set

$$B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

EXAMPLE 1.7. Using the above metrics, an open ball in $\Re$ is of the form $B(x, \epsilon) = (x - \epsilon, x + \epsilon)$ while an open ball in $\mathbf{N}$ is of the form $B(n, \epsilon) = \{n\}$.

Open balls can be thought of as the collection of points no further than a certain distance from the centre of the ball. Open balls are necessary to define an open set.

Definition 1.8. Let X be a non-empty set with metric $d(x, y)$. An open set $S \subseteq X$ is a set for which every element can be contained inside an open ball lying wholly inside the set S. In other words, for every $x \in S$ there exist $0 < \epsilon$ such that $B(x, \epsilon) \subseteq S$.

EXAMPLE 1.9. An example of an open set in $\Re$ is the set $(0, \infty)$. This can be shown as follows, let $x \in (0, \infty)$ then $B(x, x) = (0, 2x) \subseteq (0, \infty)$. However, the set of rationals $\mathbf{Q}$ is not open in $\Re$. Given a rational point $x \in \mathbf{Q}$, there is no value of $0 < \epsilon$ for which $B(x, \epsilon) = (x - \epsilon, x + \epsilon)$ contains only rationals. All sets in $\mathbf{N}$ are open, this is because each element is an open ball containing itself.

An open set is an extremely useful concept in topology. The set $S \subseteq X$ may be all the elements of X containing a certain property. If the set S is open then elements of S have open balls that lie wholly in S. Thus, elements in close proximity to $x \in S$ also lie in S. The property that x enjoyed is also shared by elements close to x. Under small perturbations of x it is seen that the property that x has is also shared by nearby elements. Thus, the property that defined the set S can be seen to be invariant under small perturbations. In the examples above, the property that a number is positive is invariant under small perturbation of the number, however that a number is rational is not a property that is maintained under small perturbation.

Having defined the open sets is tantamount to defining a topology on the set X. A topology is a preferred collection of subsets called the open sets which satisfy the topological axioms. For the purposes of this paper all topologies will be induced by metrics on the underlying set.

Definition 1.10. Let X be a non-empty set with metric $d(x,y)$. Let $S \subseteq X$. The closure of the set S, written as $\overline{S}$, is the set

$$\overline{S} = \{x \in X : \forall 0 < \epsilon \ B(x,\epsilon) \cap S \neq \{\}\}$$

where $\{\}$ denotes the empty set.

Example 1.11. The closure of the set $(0,\infty)$ is the set $[0,\infty)$ while the closure of the set $\mathbf{Q}$ of rational numbers is $\Re$.

The closure of a set gives those elements whose open balls always meets the given set. Thus, points in the original set can come arbitrarily close to points in the closure of the set. Intuitively, the set $\overline{S}$ consists of the elements of X that can be well approximated by elements of S.

Definition 1.12. Let X be a non-empty set with metric $d(x,y)$. A dense set $S \subseteq X$ is a set for which $\overline{S} = X$.

Example 1.13. The set of rationals is a dense set in $\Re$ because $\overline{\mathbf{Q}} = \Re$. The set $(0,\infty)$ is not a dense set in $\Re$ because $[0,\infty) \neq \Re$. Note that dense sets need not be open (the rationals) and open sets need not be dense (the set $(0,\infty)$).

Because a dense set has closure equal to the whole set, this means that elements of the whole space can be arbitrarily approximated by elements from the dense set. Thus, the dense set gives good approximations to elements in X. The dense set may consist of those elements with a certain property, in which case this property becomes a generic property since elements of the original set can always be approximated by an element with this property. In the examples for $\Re$, that a number is rational is a generic property which many numbers share while that of being a positive number is not as prevalent a property as the former.

In dealing with the topological properties of the set of P(n,m) vector fields there arises the question of the appropiate topology (open sets induced by the different metrics) to place on the set. The underlying manifold $\Re^2$ on which the vector fields are defined is not a compact set. Because of the non-compactness of $\Re^2$ the usual C^k topologies are inapplicable to the current analysis, the C^k topologies only being defined on compact sets. One way to apply a topology to P(n,m) is to consider the vector fields not as vector fields on $\Re^2$ but as vector fields on a compactification of $\Re^2$ and then to apply the usual C^k topologies to the induced vector fields on the compact extension of $\Re^2$. However this approach has several drawbacks. One of the drawbacks of this approach is that the induced vector fields may not necessarily be well-defined on the compact extension of $\Re^2$, this could easily happen with the vector at the image of the point ∞. A second drawback is that the local linearity of the original vector field (away from the boundary lines) will be lost in the induced vector field. Yet another way to induce a topology on P(n,m) is to a priori choose a compact set of $\Re^2$ and apply a C^k topology to the restriction of vector fields to that compact set. The disadvantage to this approach is that it is by no means clear how this compact set is supposed to be chosen. The topology that was eventually chosen is that defined below in definition 1.14.

Definition 1.14. Let $I:P(n,m) \rightarrow \Re^{3(n+m)+6}$ be the isomorphism given by

$$\begin{aligned} I(\xi) &= (I(\xi)_1, \ldots, I(\xi)_{3(n+m)+6}) \\ &= (\alpha_1, \alpha_2, b_{11}, b_{12}, b_{21}, b_{22}, \alpha_{11}, \alpha_{12}, \ldots, \alpha_{n+m1}, \alpha_{n+m2}, \gamma_1, \ldots, \gamma_{n+m}). \end{aligned}$$

The isomorphism induces a metric on P(n,m) by

$$d(\xi, \eta) = \max\{|I(\xi)_i - I(\eta)_i| : 1 \leq i \leq 3(n+m) + 6\}.$$

Thus $B(\xi, \epsilon)$ will denote the open ball $B(\xi, \epsilon) = \{\eta \in \mathrm{P(n, m)} : d(\xi, \eta) < \epsilon\}$.

The topology in definition 1.14 seems to be the most appropiate for computer work with the vector fields to be considered. In the simulation of general vector fields (not necessarily piecewise linear) on a computer there are several potential sources of simulation error. Finiteness of computer arithmetic means that real numbers can be at best approximated by finite decimal representations, thus losing information in the truncated digits of the real number. Under the application of numerical algorithms it becomes an important question as to the accuracy of the algorithm's predictions and the true value of the computer model. This problem becomes increasingly crucial as the time period in which the simulation is performed becomes lengthened. The modelling of the original vector field by a (piecewise linear) vector field introduces questions as to whether the vector field has the same dynamics as the original vector field. Even if the dynamics were known to be faithful to the original vector field, the representation of the vector field in the computer introduces its own set of concerns. Because of the representaion of real numbers in a computer's memory, a vector field with real defining constants would in actuality be represented by a vector field with rational constants approximating the original real defining constants. The possiblilty then arises that the original vector field and the represented vector field in the computer do not have the same dynamics.

Since the vector fields to be considered are stored as a list of defining constants in the computer's memory, a topology on the defining constants would most easily facilitate an analysis of the faithfulness of the modelling of a vector field with real defining constants by its computer representation. Under the topology of definition 1.14, vector fields are considered "close" in the topology if the respective defining constants do not deviate too greatly from each other.

For each element in the partition of $\Re^2$ associated with a given vector field ξ, the vector field restricted to that element is a linear vector field. The linear vector field to which this restriction of ξ induces is given in the lemma following the definition.

Definition 1.15. For each $\mathrm{A}_{ij} \in \mathrm{Part}(\xi)$ there exists a unique linear vector field ξ_{ij} such that $\xi_{ij}|_{\mathrm{A}_{ij}} = \xi|_{\mathrm{A}_{ij}}$. Let $\mathrm{F}(\xi_{ij}) = \{\mathbf{x} \in \Re^2 : \xi_{ij}(\mathbf{x}) = \mathbf{0}\}$ be the set of equilibrium points of ξ_{ij}. An equilibrium point $\mathbf{x}_{ij} \in \mathrm{F}(\xi_{ij})$ is said to be transitional if $\mathbf{x}_{ij} \in \partial\mathrm{A}_{ij}$ and nontransitional if $\mathbf{x}_{ij} \notin \partial\mathrm{A}_{ij}$ †. Nontransitional equilibrium points are classified into those for which $\mathbf{x}_{ij} \in \mathrm{A}_{ij}$ and $\mathbf{x}_{ij} \notin \mathrm{A}_{ij}$, the former being called real equilibrium points and the latter being called virtual equilibrium points. The set of equilibrium point of ξ is the set given by $\mathrm{F}(\xi) = \cup_{i=0}^{n} \cup_{j=0}^{m} \mathrm{F}(\xi_{ij})$.

Lemma 1.16. *ξ_{ij} is given by*

$$\xi_{ij}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1 - \sum_{i'=1}^{i}\alpha_{i'1}\gamma_{i'} + \sum_{i'=i+1}^{n}\alpha_{i'1}\gamma_{i'} - \sum_{i'=n+1}^{n+j}\alpha_{i'1}\gamma_{i'} + \sum_{i'=n+j+1}^{n+m}\alpha_{i'1}\gamma_{i'} \\ \alpha_2 - \sum_{i'=1}^{i}\alpha_{i'2}\gamma_{i'} + \sum_{i'=i+1}^{n}\alpha_{i'2}\gamma_{i'} - \sum_{i'=n+1}^{n+j}\alpha_{i'2}\gamma_{i'} + \sum_{i'=n+j+1}^{n+m}\alpha_{i'2}\gamma_{i'} \end{bmatrix}$$
$$+ \begin{bmatrix} b_{11} + \sum_{i'=1}^{i}\alpha_{i'1} - \sum_{i'=i+1}^{n}\alpha_{i'1} & b_{12} + \sum_{i'=n+1}^{n+j}\alpha_{i'1} - \sum_{i'=n+j+1}^{n+m}\alpha_{i'1} \\ b_{21} + \sum_{i'=1}^{i}\alpha_{i'2} - \sum_{i'=i+1}^{n}\alpha_{i'2} & b_{22} + \sum_{i'=n+1}^{n+j}\alpha_{i'2} - \sum_{i'=n+j+1}^{n+m}\alpha_{i'2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

The linear vector field ξ_{ij} will be written as $\xi_{ij}(\mathbf{x}) = \mathbf{d}_{ij} + \mathbf{M}_{ij}\mathbf{x}$.

PROOF. Since $\gamma_1 < \ldots < \gamma_i \le x \le \gamma_{i+1} < \ldots < \gamma_n, \gamma_{n+1} < \ldots < \gamma_{n+j} \le y \le \gamma_{n+j+1} < \ldots < \gamma_{n+m}$ then $\xi|_{\mathrm{A}_{ij}}$ is given by

$$\xi|_{\mathrm{A}_{ij}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \sum_{i'=1}^{i}\begin{bmatrix} \alpha_{i'1} \\ \alpha_{i'2} \end{bmatrix}(x - \gamma_i') + \sum_{i'=i+1}^{n}\begin{bmatrix} \alpha_{i'1} \\ \alpha_{i'2} \end{bmatrix}(\gamma_i' - x)$$
$$+ \sum_{i'=n+1}^{n+j}\begin{bmatrix} \alpha_{i'1} \\ \alpha_{i'2} \end{bmatrix}(y - \gamma_i') + \sum_{i'=n+j+1}^{n+m}\begin{bmatrix} \alpha_{i'1} \\ \alpha_{i'2} \end{bmatrix}(\gamma_i' - y).$$

Thus,

$$\xi|_{\mathrm{A}_{ij}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1 - \sum_{i'=1}^{i}\alpha_{i'1}\gamma_{i'} + \sum_{i'=i+1}^{n}\alpha_{i'1}\gamma_{i'} - \sum_{i'=n+1}^{n+j}\alpha_{i'1}\gamma_{i'} + \sum_{i'=n+j+1}^{n+m}\alpha_{i'1}\gamma_{i'} \\ \alpha_1 - \sum_{i'=1}^{i}\alpha_{i'2}\gamma_{i'} + \sum_{i'=i+1}^{n}\alpha_{i'2}\gamma_{i'} - \sum_{i'=n+1}^{n+j}\alpha_{i'2}\gamma_{i'} + \sum_{i'=n+j+1}^{n+m}\alpha_{i'2}\gamma_{i'} \end{bmatrix}$$
$$+ \begin{bmatrix} b_{11} + \sum_{i'=1}^{i}\alpha_{i'1} - \sum_{i'=i+1}^{n}\alpha_{i'1} & b_{12} + \sum_{i'=n+1}^{n+j}\alpha_{i'1} - \sum_{i'=n+j+1}^{n+m}\alpha_{i'1} \\ b_{21} + \sum_{i'=1}^{i}\alpha_{i'2} - \sum_{i'=i+1}^{n}\alpha_{i'2} & b_{22} + \sum_{i'=n+1}^{n+j}\alpha_{i'2} - \sum_{i'=n+j+1}^{n+m}\alpha_{i'2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

† ∂S denotes the boundary of a set S, i.e. $\partial S = \bar{S} - S$.

It is clear that there exists a unique linear vector field ξ_{ij} such that $\xi_{ij}|_{A_{ij}} = \xi|_{A_{ij}}$. ∎

§2. The generic equilibrium point structure.

In this section the generic equilibrium point structure of vector fields in P(n,m) will be analysed. This is to establish open and dense subsets of vector fields in P(n,m) whose equilibrium point structure is well understood before considering other phenomema that may be associated with vector fields. It is understood that the term equilibrium point is used in the context of definition 1.4.

The first subset of vector fields to be considered are those with finitely many equilibrium points, all of which are nontransitional. This will be shown to be an open and dense subset of the vector fields in P(n,m). From the stand-point of computer simulation of vector fields it follows by denseness that the generic vector field will have only finitely many equilibrium points. By openness of this property, finite precision arithmetic can also accurately approximate a vector field with finitely many equilibrium points with a vector field with the same number of equilibrium points. That all the equilibrium points are nontransitional means that the number of "real" equilibrium points (corresponding to points for which the vector at that point is the zero vector) is also maintained under small perturbation of the original vector field.

Definition 2.1. $G_0 \subseteq$P(n,m) is the subset consisting of vector fields ξ with (n+1)(m+1) nontransitional equilibrium points.

Theorem 2.2. *G_0 is open in P(n,m).*

Proof. (Figure 2.) Let $\xi \in G_0$. As $F(\xi) = \cup_{i=0}^{n} \cup_{j=0}^{m} F(\xi_{ij})$ the number of equilibrium points $|F(\xi)|$ of ξ is given by

$$|F(\xi)| = \sum_{i=0}^{n} \sum_{j=0}^{m} |F(\xi_{ij})|.$$

The linear vector field ξ_{ij} has the form $\xi_{ij}(\mathbf{x}) = \mathbf{d}_{ij} + M_{ij}\mathbf{x}$. The value of the determinant of the matrix M_{ij} determines the number of equilibrium points of ξ_{ij}. If $\det M_{ij} = 0$ then ξ_{ij} has either 0 or infinitely many equilibrium points. In either case then $|F(\xi)| \neq$(n+1)(m+1). Thus $\det M_{ij} \neq 0$ for $0 \leq i \leq n, 0 \leq j \leq m$. By continuity of the determinant function, there exists $0 < \epsilon_{ij}$ such that if $\overline{\xi} \in B(\xi, \epsilon_{ij})$ then the matrix corresponding to $\overline{\xi}_{ij}$ (the linear vector field for which $\overline{\xi}_{ij}|_{\overline{A}_{ij}} = \overline{\xi}|_{A_{ij}}$) being $\overline{M}_{ij}$ also has nonzero determinant. Thus $|F(\overline{\xi}_{ij})| = 1$. Let $\epsilon_1 = \min\{\epsilon_{ij} : 0 \leq i \leq n, 0 \leq j \leq m\}$. Then for $\overline{\xi} \in B(\xi, \epsilon_1)$ we have $|F(\overline{\xi}_{ij})| = 1$ for $0 \leq i \leq, 0 \leq j \leq m$. Then $|F(\overline{\xi})| =$(n+1)(m+1).

For each $1 \leq i \leq n+m$ we have $0 < \alpha_{i1}^2 + \alpha_{i2}^2$. There exists $0 < \epsilon_{i2}$ such that if $\overline{\xi} \in B(\xi, \epsilon_{i2})$ then $\overline{\alpha}_{i1}, \overline{\alpha}_{i2}$ for the vector field $\overline{\xi}$ also satisfy $0 < \overline{\alpha}_{i1}^2 + \overline{\alpha}_{i2}^2$. Let $\epsilon_2 = \min\{\epsilon_{i2} : i = 1 \leq i \leq n+m\}$. Then for $\overline{\xi} \in B(\xi, \epsilon_2)$ we have $0 < \overline{\alpha}_{i1}^2 + \overline{\alpha}_{i2}^2$ for $1 \leq i \leq n+m$.

Let $H_{ij} : P(n,m) \times \Re^2 \rightarrow \Re^2$ be the function

$$H_{ij}(\overline{\xi}, \overline{\mathbf{x}}) = \mathbf{d}_{ij}(\overline{\xi}) + M_{ij}(\overline{\xi})\overline{\mathbf{x}}.$$

As $\mathbf{D_x H}_{ij}(\xi, \mathbf{x}_{ij}) = \mathbf{M}_{ij}$ and $\det \mathbf{M}_{ij} \neq 0$, by the implicit function theorem for Banach spaces there are neighbourhoods N_{ij}^1, N_{ij}^2 of $\xi, \mathbf{x}_{ij}$ and a function $h_{ij} : N_{ij}^1 \to N_{ij}^2$ assigning to each $\overline{\xi} \in N_{ij}^1$ its unique equilibrium point $\overline{\mathbf{x}}_{ij} = h_{ij}(\overline{\xi})$.

As $\mathbf{x}_{ij}$ is not a transitional equilibrium point then it is either a virtual equilibrium point or a real equilibrium point. The proof will continue in the case that $\mathbf{x}_{ij}$ is a virtual equilibrium point. The proof for a real equilibrium point in analogous. Since $\mathbf{x}_{ij}$ is virtual equilibrium point then $\mathbf{x}_{ij} \notin \mathbf{A}_{ij}$. As disjoint closed sets in $\Re^2$, $\{\mathbf{x}_{ij}\}$ and $\mathbf{A}_{ij}$ may be separated by the existence of $0 < \epsilon_{ij3}, \epsilon_{ij4}$ such that

$$B(\mathbf{x}_{ij}, \epsilon_{ij3}) = \{\mathbf{y} \in \Re^2 : d(\mathbf{y}, \mathbf{x}) < \epsilon_{ij3}\}$$
$$B(\mathbf{A}_{ij}, \epsilon_{ij4}) = \{\mathbf{y} \in \Re^2 : \min_{\mathbf{x} \in \mathbf{A}_{ij}} d(y, x) < \epsilon_{ij4}\}$$

are disjoint. The metric used in $\Re^2$ is that given by $d(\mathbf{y}, \mathbf{x}) = \min\{|y_1 - x_1|, |y_2 - x_2|\}$.

Consider the set $N_{ij}^2 \cap B(\mathbf{x}_{ij}, \epsilon_{ij3}) \subseteq N_{ij}^2$. By continuity of the function h_{ij}, there exists a subset $N_{ij}^0 \subseteq N_{ij}^1$ such that $h_{ij}(N_{ij}^0) = N_{ij}^2 \cap B(\mathbf{x}_{ij}, \epsilon_{ij3})$. Finally, let

$$M_{ij}^1 = N_{ij}^0 \cap B(\xi, \epsilon_{ij4})$$
$$M_{ij}^2 = h_{ij}(M_{ij}^1).$$

Let $\overline{\xi} \in M_{ij}^1$. Then $d(\overline{\xi}, \xi) < \epsilon_{ij4}$. In particular the values $\overline{\gamma}_i, \overline{\gamma}_{i+1}, \overline{\gamma}_j, \overline{\gamma}_{j+1}$ corrresponding to $\overline{\xi}$ satisfy

$$|\overline{\gamma}_i - \gamma_i| < \epsilon_{ij4},$$
$$|\overline{\gamma}_{i+1} - \gamma_{i+1}| < \epsilon_{ij4},$$
$$|\overline{\gamma}_j - \gamma_j| < \epsilon_{ij4},$$
$$|\overline{\gamma}_{j+1} - \gamma_{j+1}| < \epsilon_{ij4}.$$

Thus $\overline{\mathbf{A}}_{ij} = \overline{\mathbf{L}}_i \times \overline{\mathbf{W}}_j$ is a set that satisfies $\overline{\mathbf{A}}_{ij} \subseteq B(\mathbf{A}_{ij}, \epsilon_{ij4})$. Also $\overline{\mathbf{x}}_{ij} = h_{ij}(\overline{\xi})$ satisfies $\overline{\mathbf{x}}_{ij} \in B(\mathbf{x}_{ij}, \epsilon_{ij3})$. As $B(\mathbf{x}_{ij}, \epsilon_{ij3}) \cap B(\mathbf{A}_{ij}, \epsilon_{ij4}) = \{\}$ it follows that $\overline{\mathbf{x}}_{ij} \notin \overline{\mathbf{A}}_{ij}$. Thus, for $\overline{\xi} \in M_{ij}^1, \overline{\mathbf{x}}_{ij}$ is also a virtual equilibrium point. Similarly, if $\mathbf{x}_{ij}$ was initially a real equilibrium point then $\overline{\mathbf{x}}_{ij}$ is also a real equilibrium point for $\overline{\xi} \in M_{ij}^1$ (M_{ij}^1 suitably chosen).

Let $\epsilon_5 = \min\{\min\{\gamma_i - \gamma_{i-1} : 2 \leq i \leq n, n+2 \leq i \leq n+m\}, \infty\}$. For $\overline{\xi} \in B(\xi, \epsilon_5/2)$ the values $\overline{\gamma}_1, \ldots, \overline{\gamma}_{n+m}$ satisfy $|\overline{\gamma}_i - \gamma_i| < \epsilon_5/2$. Thus $\gamma_{i+1} - \epsilon_5/2 < \overline{\gamma}_{i+1}$ and $\overline{\gamma}_i < \gamma_i + \epsilon_5/2$ for $2 \leq i \leq n, n+2 \leq i \leq n+m$. Hence $0 = \gamma_{i+1} - \gamma_i - \epsilon_5 < \overline{\gamma}_{i+1} - \overline{\gamma}_i$. Thus, $\overline{\gamma}_1 < \ldots < \overline{\gamma}_n, \overline{\gamma}_{n+1} < \ldots < \overline{\gamma}_{n+m}$.

Finally, let

$$M = (\cap_{i=0}^n \cap_{j=0}^m M_{ij}^1) \cap B(\xi, \epsilon_1) \cap B(\xi, \epsilon_2) \cap B(\xi, \epsilon_5/2).$$

For $\overline{\xi} \in M, \overline{\xi}$ is a vector field which has (n+1)(m+1) equilibrium points, each of which is nontransitional. Thus G_0 is open. ∎

Theorem 2.3. *G_0 is dense in P(n,m).*

PROOF. (Figure 3.) Let ξ be a vector field in P(n,m). As in the proof that G_0 is open, there exist $0 < \epsilon_1, \epsilon_2$ such that if $\overline{\xi} \in B(\xi, \epsilon_1)$ then $0 < \overline{\alpha}_{i1}^2 + \overline{\alpha}_{i2}^2$ for $i = 1 \ldots, n+m$ and if $\overline{\xi} \in B(\xi, \epsilon_2)$ then $\overline{\gamma}_1 < \ldots < \overline{\gamma}_n, \overline{\gamma}_{n+1} < \ldots < \overline{\gamma}_{n+m}$. It is required to show that for each $0 < \epsilon, B(\xi, \epsilon) \cap G_0 \neq \{\}$. It may be assumed that $0 < \epsilon \leq \min\{\epsilon_1, \epsilon_2\}$. Thus, it remains to show that in $B(\xi, \epsilon)$ there exist vector fields whose equilibrium points are all nontransitional.

For each $\mathbf{A}_{ij} \in \mathrm{Part}(\xi)$ there exists a unique linear vector field ξ_{ij} such that $\xi_{ij}|_{\mathbf{A}_{ij}} = \xi|_{\mathbf{A}_{ij}}$. The linear vector field ξ_{ij} may be writtten as $\xi_{ij}(\mathbf{x}) = \mathbf{d}_{ij} + \mathbf{M}_{ij}\mathbf{x}$. The determinant of $\mathbf{M}_{ij}$ determines the number of equilibrium points of ξ_{ij} and hence the number of equilibrium points of ξ.

If $\det\mathbf{M}_{ij} \neq 0$ then the eigenvalues of $\mathbf{M}_{ij}$ may be written as $\lambda_{ij}^1, \lambda_{ij}^2$ where $0 \neq \lambda_{ij}^1, \lambda_{ij}^2$. Let $||z||$ denote the modulus of the complex number z. It is clear that if $0 < \mu < \min\{||\lambda_{ij}^1||, ||\lambda_{ij}^2||\}$ then the matrix $\mathbf{M}_{ij} + \mu\mathbf{I}$ also has nonzero eigenvalues. If $\det\mathbf{M}_{ij} = 0$ then either one or both of the eigenvalues of $\mathbf{M}_{ij}$ is zero. If one eigenvalue is zero, so that the eigenvalues are $0, \lambda_{ij}^2$, it is possible to take $\lambda_{ij}^1 = \infty$. Then for $0 < \mu < \min\{||\lambda_{ij}^1||, ||\lambda_{ij}^2||\}$ the eigenvalues of the matrix $\mathbf{M}_{ij} + \mu\mathbf{I}$ are nonzero. If both eigenvalues of $\mathbf{M}_{ij}$ are zero then take $\infty = \lambda_{ij}^1, \lambda_{ij}^2$. For $0 < \mu < \min\{||\lambda_{ij}^1||, ||\lambda_{ij}^2||\}$ the matrix $\mathbf{M}_{ij} + \mu\mathbf{I}$ has nonzero eigenvalues.

Choose

$$0 < \mu < \min\{\epsilon, \min\{||\lambda_{ij}^1||, ||\lambda_{ij}^2|| : 0 \leq i \leq n, 0 \leq j \leq m\}\}.$$

Then the vector field $\overline{\xi} = \xi + \mu\mathbf{I}$ lies in $B(\xi, \epsilon)$ and for each $0 \leq i \leq n, 0 \leq j \leq m$ the induced linear vector field for the partition $\overline{\mathbf{A}}_{ij}$ is given by $\overline{\xi}_{ij}(\mathbf{x}) = \mathbf{d}_{ij} + (\mathbf{M}_{ij} + \mu\mathbf{I})\mathbf{x}$. As $\det(\mathbf{M}_{ij} + \mu\mathbf{I}) \neq 0$ then $\overline{\xi}_{ij}$ has a unique equilibrium point. Thus $\overline{\xi}$ has (n+1)(m+1) equilibrium points. It will be shown that if $\overline{\xi}$ has transitional equilibrium points then there is a perturbation of $\overline{\xi}$ to a vector field without any transitional equilibrium points.

As in the proof of openness of G_0 in P(n,m) there is a $0 < \epsilon_3$ such that if $\tilde{\xi} \in B(\overline{\xi}, \epsilon_3)$ then $\tilde{\xi}$ is a vector field with (n+1)(m+1) equilibrium points. Let $\epsilon_4 = \min\{\epsilon_3, \epsilon - \mu\}$. Clearly $B(\overline{\xi}, \epsilon_4) \subseteq B(\xi, \epsilon)$. Without possibility of confusion, let $\xi = \overline{\xi}$.

Say a transitional equilibrium point lies along the vertical line $x \equiv \gamma_{i_1} < \ldots < \gamma_{i_k}, 1 \leq i < \ldots < i_k < n$. Let ξ^{i_1} be a vector field with all the same defining constants as ξ except

$$\begin{bmatrix} \alpha_{i_11}^{i_1} \\ \alpha_{i_12}^{i_2} \end{bmatrix} + \begin{bmatrix} \alpha_{i_1+11}^{i_1} \\ \alpha_{i_1+12}^{i_1} \end{bmatrix} = \begin{bmatrix} \alpha_{i_11} \\ \alpha_{i_12} \end{bmatrix} + \begin{bmatrix} \alpha_{i_1+11} \\ \alpha_{i_1+12} \end{bmatrix},$$

$$\begin{bmatrix} \alpha_{i_11}^{i_1} \\ \alpha_{i_12}^{i_1} \end{bmatrix} \gamma_{i_1}^{i_1} + \begin{bmatrix} \alpha_{i_1+11}^{i_1} \\ \alpha_{i_1+12}^{i_1} \end{bmatrix} \gamma_{i_1+1}^{i_1} = \begin{bmatrix} \alpha_{i_11} \\ \alpha_{i_12} \end{bmatrix} \gamma_{i_1} + \begin{bmatrix} \alpha_{i_1+11} \\ \alpha_{i_1+12} \end{bmatrix} \gamma_{i_1+1}.$$

By considering the formula for $\xi_{ij}^{i_1}, \xi_{ij}$ it happens that $\xi_{ij}^{i_1} = \xi_{ij}$ for $i \neq i_1, 0 \leq j \leq m$. It then happens that $\xi^{i_1} = \xi$ except on the vertical strip $[\gamma_{i_1-1}, \gamma_{i_1+2}] \times \Re$.

If the matrix

$$\begin{bmatrix} \alpha_{i_11} & \alpha_{i_1+11} \\ \alpha_{i_12} & \alpha_{i_1+12} \end{bmatrix}$$

is nonsingular then the implicit function theorem may be applied to the function

$$\mathrm{H}_{i_1}(\alpha_1', \alpha_2', \gamma_1', \gamma_2') = \begin{bmatrix} \alpha_1' \\ \alpha_2' \end{bmatrix} \gamma_1' + \begin{bmatrix} \alpha_{i_1 1} + \alpha_{i_1+11} - \alpha_1' \\ \alpha_{i_1 2} + \alpha_{i_1+12} - \alpha_2' \end{bmatrix} \gamma_2'$$

to determine a function h_{i_1} and neighbourhoods $N_{i_1}^1, N_{i_1}^2$ of $(\alpha_{i_1 1}, \alpha_{i_1 2})$ and (γ_1, γ_2) respectively such that $h_{i_1} : N_{i_1}^1 \to N_{i_1}^2$ assigns to each element in $N_{i_1}^1$ the corresponding vertical lines in $N_{i_1}^2$. In fact the function h_{i_1} is a diffeomorphism. Let

$$\begin{aligned} N_{i_1}^3 &= B((\alpha_{i_1 1}, \alpha_{i_1 2}), \epsilon_4/(n+m)) \cap h_{i_1}^{-1}(N_{i_1}^2 \cap B((\gamma_{i_1}, \gamma_{i_1+1}), \epsilon_4/(n+m))) \\ N_{i_1}^4 &= h_{i_1}(N_{i_1}^3). \end{aligned}$$

Let $(\gamma_{i_1}^{i_1}, \gamma_{i_1+1}^{i_1}) \in N_{i_1}^4 - \{(\gamma_{i_1}, \gamma_{i_1+1})\}$. Then there exist $(\alpha_{i_1 1}^{i_1}, \alpha_{i_1 2}^{i_1})$ such that $h_{i_1}(\alpha_{i_1 1}^{i_1}, \alpha_{i_1 2}^{i_1}) = (\gamma_{i_1}^{i_1}, \gamma_{i_1+1}^{i_1})$. Furthermore, the vector field ξ^{i_1} with the same defining constants as ξ except the values $\alpha_{i_1 1}^{i_1}, \alpha_{i_1 2}^{i_1}, \gamma_{i_1}^{i_1}, \gamma_{i_1+1}^{i_1}$ lies in $B(\xi, \epsilon_4/(n+m))$. Note that, as h_{i_1} is a diffeomorphism , it may be taken that $\gamma_{i_1}^{i_1} \neq \gamma_{i_1}$. As in the proof of openness, ξ^{i_1} can be chosen so that no new nontransitional equilibrium points are created along the lines $x \equiv \gamma_i$ for $1 \le i \le i_1 - 1$.

If the matrix

$$\begin{bmatrix} \alpha_{i_1 1} & \alpha_{i_1+11} \\ \alpha_{i_1 2} & \alpha_{i_1+12} \end{bmatrix}$$

is singular then

$$\begin{bmatrix} \alpha_{i_1+11} \\ \alpha_{i_1+12} \end{bmatrix} \doteq k_{i_1} \begin{bmatrix} \alpha_{i_1 1} \\ \alpha_{i_1 2} \end{bmatrix}$$

for some constant k_{i_1}. As $0 < \alpha_{i_1+11}^2 + \alpha_{i_1+12}^2$ then $k_{i_1} \neq 0$. A solution to the following equalities

$$\begin{bmatrix} \alpha_{i_1 1}^{i_1} \\ \alpha_{i_1 2}^{i_1} \end{bmatrix} + \begin{bmatrix} \alpha_{i_1+11}^{i_1} \\ \alpha_{i_1+12}^{i_1} \end{bmatrix} = \begin{bmatrix} \alpha_{i_1 1} \\ \alpha_{i_1 2} \end{bmatrix} (1 + k_{i_1})$$

$$\begin{bmatrix} \alpha_{i_1 1}^{i_1} \\ \alpha_{i_1 2}^{i_1} \end{bmatrix} \gamma_{i_1}^{i_1} + \begin{bmatrix} \alpha_{i_1+11}^{i_1} \\ \alpha_{i_1+12}^{i_1} \end{bmatrix} \gamma_{i_1+1}^{i_1} = \begin{bmatrix} \alpha_{i_1 1} \\ \alpha_{i_1 2} \end{bmatrix} (\gamma_{i_1} + k_{i_1} \gamma_{i_1+1})$$

is given by

$$\begin{aligned} \begin{bmatrix} \alpha_{i_1 1}^{i_1} \\ \alpha_{i_1 2}^{i_1} \end{bmatrix} &= \begin{bmatrix} \alpha_{i_1 1} \\ \alpha_{i_1 2} \end{bmatrix} \\ \begin{bmatrix} \alpha_{i_1+11}^{i_1} \\ \alpha_{i_1+12}^{i_1} \end{bmatrix} &= k_{i_1} \begin{bmatrix} \alpha_{i_1 1} \\ \alpha_{i_1 2} \end{bmatrix} \\ \gamma_{i_1}^{i_1} &= \gamma_{i_1} + \mu_{i_1} \\ \gamma_{i_1+1}^{i_1} &= \gamma_{i_1+1} - \mu_{i_1}/k_{i_1}. \end{aligned}$$

Choose $0 < \mu_{i_1} < \min\{\epsilon_4/(n+m), \epsilon_4 |k_{i_1}|/(n+m)\}$. The vector field ξ^{i_1} with all the same defining constants as $\overline{\xi}$ except $\gamma_{i_1}^{i_1}, \gamma_{i_1+1}^{i_1}$ lies in $B(\xi, \epsilon_4/(n+m))$. As in the proof of openness, ξ^{i_1} can be chosen so that no new nontransitional equilibrium points are created along the lines $x \equiv \gamma_i$ for $1 \le i \le i_1 - 1$.

Assume a transitional equilibrium point $\mathbf{x}_{i_1 j}$ exists for the linear vector field $\xi_{i_1 j}$ this point lying along the vertical line $x \equiv \gamma_{i_1}$. The point $\mathbf{x}_{i_1 j}$ is also a transitional equilibrium point for $\xi_{i_1 - 1 j}$.

Consider the perturbed vector field ξ^{i_1} with $\gamma_{i_1}^{i_1} \neq \gamma_{i_1}$. Note that for the vector field ξ^{i_1} it happens that $\xi_{i_1-1j}^{i_1} = \xi_{i_1-1j}$ so that $\mathbf{x}_{i_1j}$ is also an equilibrium point for ξ^{i_1}, in particular it is an equilibrium point of $\xi_{i_1-1j}^{i_1}$.

If $\xi_{i_1j}^{i_1}$ also had a transitional equilibrium point for the linear vector field $\mathbf{x}_{i_1j}^{i_1}$ along the line $x \equiv \gamma_{i_1}^{i_1}$ then this would be another equilibrium point for the linear vector field $\xi_{i_1-1j}^{i_1}$. As $\det \mathrm{M}_{i_1j} \neq 0$ then $\xi_{i_1-1j}^{i_1}$ may have at most one equilibrium point. By contradiction, ξ^{i_1} does not have any equilibrium points along the line $x \equiv \gamma_{i_1}^{i_1}$.

By induction, if transitional equilibrium points lie along the lines $\gamma_{i_1} < \ldots < \gamma_{i_k}, 1 \leq i_1 < \ldots < i_k < n$ then there is a sequence of perturbed vector fields $\xi^{i_1}, \ldots, \xi^{i_k}$ with the final vector field ξ^{i_k} having no transitional equilibrium points along $x \equiv \gamma_i, 1 \leq i < n$.

Say a transitional equilibrium point lies along the line $x \equiv \gamma_n$. Consider the vector field ξ^n with all the same defining constants as ξ except that

$$\begin{bmatrix} \alpha_1^n \\ \alpha_2^n \end{bmatrix} = \begin{bmatrix} \alpha_1^{i_k} \\ \alpha_2^{i_k} \end{bmatrix} - \mu_n \begin{bmatrix} \alpha_{n1}^{i_k} \\ \alpha_{n2}^{i_k} \end{bmatrix}$$
$$\gamma_n^n = \gamma_n^{i_k} + \mu_n$$

where $0 < \mu_n < \min\{\epsilon_4/(n+m), \epsilon_4/((n+m)|\alpha_{n1}^{i_k}|), \epsilon_4/((n+m)|\alpha_{n2}^{i_k}|)\}$. By choosing μ_n small enough, it is possible to ensure that no new transitional equilibrium points are created along the lines $x \equiv \gamma_i$ for $1 \leq i \leq n-1$. The vector fields ξ^n, ξ^{i_k} coincide on $(-\infty, \gamma_{n-1}^{i_k}] \times \Re$. Furthermore $\xi_{n-1j}^{i_k}$ and ξ_{n-1j}^n are identical as linear vector fields. Say a transitional equilibrium point existed for the vector field $\xi_{nj}^{i_k}$, it is also an equilibrium point for the vector field $\xi_{n-1j}^{i_k}$, then it is also an equilibrium point for ξ_{n-1j}^n. If ξ_{nj}^n has a transitional equilibrium point along $\gamma_n^n \neq \gamma_n^{i_k}$ then ξ_{n-1j}^n would have two equilibrium points. This is not possible as $\det \mathrm{M}_{n-1j}^n \neq 0$. Thus ξ^n does not have transitional equilibrium points along the line $x \equiv \gamma_i, 1 \leq i \leq n$. Similarly, the vector field ξ^n can be perturbed to a vector field ξ^{n+m} with neither vertical nor horizontal lines $\gamma_1 \ldots, \gamma_{n+m}$ having any transitional equilibrium points. As successive perturbations are at most a distance $\epsilon_4/(n+m)$ apart, and there are at most (n+m) perturbations necessary, then $\xi^{n+m} \in B(\xi, \epsilon_4)$. Thus G_0 is dense in P(n,m). ∎

The next subset of P(n,m) to be considered is an open and dense subset of G_1 consisting of those vector fields all of whose equilibrium points are hyperbolic. The concept of a hyperbolic equilibrium point occurs frequently in differential dynamical systems. Again, it can be shown that hyperbolic equilibrium points are a generic feature of vector fields.

Definition 2.4. The point $\mathbf{x}_{ij}$ is a hyperbolic point if the eigenvalues of the matrix M_{ij} ($\xi_{ij}(\mathbf{x}) = \mathbf{d}_{ij} + \mathrm{M}_{ij}\mathbf{x}$) do not lie on the imaginary axis. The point $\mathbf{x} \in \mathrm{F}(\xi)$ is a hyperbolic if it is hyperbolic for some $\mathrm{F}(\xi_{ij})$.

Definition 2.5. Let $G_1 \subseteq G_0$ be the subset consisting of vector fields $\xi \in G_0$ with (n+1)(m+1) nontransitional hyperbolic equilibrium points.

Theorem 2.6. *G_1 is open in P(n,m).*

PROOF. Let $\xi \in G_1$, there exist $0 < \epsilon_1$ such that if $\bar{\xi} \in B(\xi, \epsilon_1)$ then $\bar{\xi} \in G_0$. Let $\mathbf{x}_{ij}, 0 \le i \le n$ be the equilibrium points for ξ. Each $\mathbf{x}_{ij}$ is the unique equilibrium point for ξ_{ij}. As $\xi_{ij}(\mathbf{x}) = \mathbf{d}_{ij} + \mathbf{M}_{ij}\mathbf{x}$ and $\mathbf{x}_{ij}$ is a hyperbolic equilibrium point for $\mathbf{M}_{ij}$ then the eigenvalues of $\mathbf{M}_{ij}$ do not lie on the imaginary axis. By continuity of the eigenvalues on the defining constants of a vector field, there is an $0 < \epsilon_{ij}$ such that if $\bar{\xi} \in B(\xi, \epsilon_{ij})$ then $\bar{\xi}_{ij}$ has a unique hyperbolic equilibrium point.

Let $\epsilon = \min\{\epsilon_1, \min\{\epsilon_{ij} : 0 \le i \le n, 0 \le j \le m\}\}$. For $\bar{\xi} \in B(\xi, \epsilon)$ the vector field $\bar{\xi}$ has (n+1)(m+1) nontransitional hyperbolic equilibrium points. ∎

Theorem 2.7. *G_1 is dense in P(n.m).*

PROOF. It is suffficient to prove that G_1 is dense in G_0. Let $\xi \in G_0$ and $0 < \epsilon$. Without loss of generality ϵ may be chosen so small that $B(\xi, \epsilon) \subseteq G_0$. Thus, it is needed to show that $B(\xi, \epsilon) \cap G_1 \neq \{\}$. For each linear vector field $\xi_{ij}(\mathbf{x}) = \mathbf{d}_{ij} + \mathbf{M}_{ij}\mathbf{x}$ let $\lambda^1_{ij}, \lambda^2_{ij}$ denote the eigenvalues of the matrix $\mathbf{M}_{ij}$.

If $0 \neq \mathrm{re}\lambda^1_{ij}, \mathrm{re}\lambda^2_{ij}$ then let $\epsilon_{ij} = \min\{|\mathrm{re}\lambda^1_{ij}|, |\mathrm{re}\lambda^2_{ij}|\}$. If $0 = \mathrm{re}\lambda^1_{ij}$ and since ξ has nonzero determinant then λ^1_{ij} is purely imaginary. As the eigenvalues of a 2×2 matrix with real entries occur in complex conjugate pairs then $0 = \mathrm{re}\lambda^2_{ij}$. Let $\epsilon_{ij} = \infty$. If $\bar{\xi} = \xi + \mu\mathrm{I}$ with $0 < \mu < \epsilon_{ij}$ then the vector field $\bar{\xi}_{ij}(\mathbf{x}) = \mathbf{x}_{ij} + (\mathbf{M}_{ij} + \mu\mathrm{I})\mathbf{x}$ has a hyperbolic equilibrium point.

Finally, let $\epsilon_1 = \min\{\epsilon, \min\{\epsilon_{ij} : 0 \le i \le n, 0 \le j \le m\}\}$. For $0 < \mu < \epsilon_1$ the vector field $\bar{\xi}(\mathbf{x}) = (\xi + \mu\mathrm{I})(\mathbf{x})$ has (n+1)(m+1) nontransitional hyperbolic equilibrium points. ∎

Our next generic property is a preparatory result needed for a later proof. The generic property to be presented in definition 2.8 refers to the eigenvalues at a hyperbolic equilibrium point, namely that generically the eigenvalues are distinct. As a result of having distinct eigenvalues, it follows that the eigenvectors associated to the eigenvalues are also distinct and remain so under small perturbations of the original vector field.

Definition 2.8. Let $G_2 \subseteq G_1$ be the subset of vector fields such that if $\xi \in G_2$ then ξ has (n+1)(m+1) nontransitional equilibrium points. For each equilibrium point $\mathbf{x}_{ij}$ the matrix $\mathbf{M}_{ij}$ ($\xi_{ij}(\mathbf{x}) = \mathbf{d}_{ij} + \mathbf{M}_{ij}\mathbf{x}$) has two distinct eigenvalues.

Theorem 2.9. *G_2 is open in P(n,m).*

PROOF. Let $\xi \in G_2$. There exist $0 < \epsilon_1$ such that if $\bar{\xi} \in B(\xi, \epsilon_1)$ then $\bar{\xi} \in G_1$. For each $0 \le i \le n, 0 \le j \le m$ let the eigenvalues of the matrices $\mathbf{M}_{ij}$ be $\lambda^2_{ij}, \lambda^2_{ij}$.

As $0 \neq \lambda^2_{ij} - \lambda^1_{ij}$ then by continuity of the eigenvalues on the defining constants of the vector field ξ there exists $0 < \epsilon_{ij}$ such that if $\bar{\xi} \in B(\xi, \epsilon_{ij})$ then the corresponding eigenvalues $\bar{\lambda}^1_{ij}, \bar{\lambda}^2_{ij}$ are also distinct. Let $\epsilon = \min\{\epsilon_1, \min\{\epsilon_{ij} : 0 \le i \le n, 0 \le j \le m\}\}$. For $\bar{\xi} \in B(\xi, \epsilon)$, the vector field

$\overline{\xi}$ has (n+1)(m+1) nontransitional hyperbolic equilibrium points, the corresponding matrix at each equilibrium point having two distinct eigenvalues. ∎

Theorem 2.10. *G_2 is dense in P(n,m).*

Proof. It is sufficient to show that G_2 is dense in G_1. Let $\xi \in G_1$ and $0 < \epsilon$. Without loss of generality it may be assumed that ϵ is so small that $B(\xi,\epsilon) \subseteq G_1$. It is sufficient to show that $B(\xi,\epsilon) \cap G_1 \neq \{\}$. For $\xi_{ij}(\mathbf{x}) = \mathbf{d}_{ij} + \mathbf{M}_{ij}\mathbf{x}$ let the matrix $\mathbf{M}_{ij}$ be written as

$$\begin{bmatrix} m_{ij}^{11} & m_{ij}^{12} \\ m_{ij}^{21} & m_{ij}^{22} \end{bmatrix}$$

where the entries are polynomial functions in the defining constants of ξ. If the matrix $\mathbf{M}_{ij}$ has two distinct eigenvalues then

$$0 \neq (m_{ij}^{11} + m_{ij}^{22})^2 - 4(m_{ij}^{11}m_{ij}^{22} - m_{ij}^{12}m_{ij}^{21}).$$

By continuity of the above discriminant (being the discriminant of the corresponding characteristic equation of the matrix) there exist $0 < \epsilon_{ij}$ such that if $0 < \mu < \epsilon_{ij}$ then the matrix

$$\begin{bmatrix} m_{ij}^{11} & m_{ij}^{12} + \mu \\ m_{ij}^{21} + \mu & m_{ij}^{22} \end{bmatrix}$$

also has a pair of distince eigenvalues. If the matrix $\mathbf{M}_{ij}$ does not have two distinct eigenvalues then

$$0 = (m_{ij}^{11} + m_{ij}^{22})^2 - 4(m_{ij}^{11}m_{ij}^{22} - m_{ij}^{12}m_{ij}^{21}).$$

If $0 \neq |m_{ij}^{12} + m_{ij}^{21}|$ then let $\epsilon_{ij} = |m_{ij}^{12} + m_{ij}^{21}|$. For $0 < \mu < \epsilon_{ij}$ then the matrix

$$\begin{bmatrix} m_{ij}^{11} & m_{ij}^{12} + \mu \\ m_{ij}^{21} + \mu & m_{ij}^{22} \end{bmatrix}$$

has nonzero discriminant $4\mu(m_{ij}^{12} + m_{ij}^{21} + \mu)$ and thus two distinct eigenvalues. If $0 = |m_{ij}^{12} + m_{ij}^{21}|$ then let $\epsilon_{ij} = \infty$. Again, for $0 < \mu < \epsilon_{ij}$ the matrix

$$\begin{bmatrix} m_{ij}^{11} & m_{ij}^{12} + \mu \\ m_{ij}^{21} + \mu & m_{ij}^{22} \end{bmatrix}$$

will have nonzero discriminant $4\mu^2$ and two distinct eigenvalues.

Finally let $\epsilon_1 = \min\{\epsilon, \min\{\epsilon_{ij} : 0 \leq i \leq n, 0 \leq j \leq m\}\}$. For $0 < \mu < \epsilon_1$ then the vector field

$$\overline{\xi}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} + \mu \\ b_{21} + \mu & b_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \end{bmatrix} |x - \gamma_i| + \sum_{i=n+1}^{n+m} \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \end{bmatrix} |y - \gamma_i|$$

will have (n+1)(m+1) nontransitional hyperbolic equilibrium points, the corresponding matrices having a pair of distinct eigenvalues. ∎

The property of a vector field being properly transversal (to be defined below) will be shown to be a generic property of vector fields in P(n,m), specifically the vector fields that are properly transversal form an open and dense subset of the vector fields in P(n,m). Properly transversal vector fields have importance from the viewpoint of computer simulation of piecewise linear vector fields. In a properly transversal vector field the invariant manifolds of distinct hyperbolic equilibrium points meet transversally, thus any saddle connections between distinct equilibrium points in a properly transversal vector field must contain points of nonzero curvature. It is an immediate observation that saddle connections between manifolds at the same equilibrium point (homoclinic orbits) have points of nonzero curvature along the connection. Any saddle connection must then contain points of nonzero curvature. Along these points of nonzero curvature a numerical algorithm runs the risk of deviating from the original saddle connection. This makes the accurate simulation of saddle connections in the generic vector field more difficult than would otherwise be the case.

Definition 2.11. Let $\xi_{ij}(\mathbf{x}) = \mathbf{d}_{ij} + \mathbf{M}_{ij}\mathbf{x}$ be a linear vector field for which $\mathbf{M}_{ij}$ has a pair of distinct nonzero eigenvalues which do not lie on the imaginary axis. Let the eigenvalues of $\mathbf{M}_{ij}$ be $\lambda^1_{ij}, \lambda^2_{ij}$. If the eigenvalues are complex then let

$$L^1_{ij} = \{\},$$
$$L^2_{ij} = \{\}.$$

If the eigenvalues are real then to each eigenvalue there is a corresponding eigenvector through the equilibrium point $\mathbf{x}_{ij}$; namely $\mathbf{v}^1_{ij}, \mathbf{v}^2_{ij}$. Thus, let

$$L^1_{ij} = \{\mathbf{x}_{ij} + t\mathbf{v}^1_{ij} : t \in \Re\}$$
$$L^2_{ij} = \{\mathbf{x}_{ij} + t\mathbf{v}^2_{ij} : t \in \Re\}.$$

The sets L^1_{ij}, L^2_{ij} are called the lines through $\mathbf{x}_{ij}$.

Definition 2.12. A vector field ξ is called properly transversal whenever $\max\{|i'-i|, |j'-j|\} = 1$ implies that the lines through $\mathbf{x}_{ij}$ intersect each line through $\mathbf{x}_{i'j'}$ transversally.

Definition 2.13. Let $G_3 \subseteq G_2$ be the subset of vector fields that are properly transversal.

Theorem 2.14. *G_3 is open in P(n,m).*

Proof. (Figure 4.) Let $\xi \in G_3$. As $\xi \in G_2$ there exists $0 < \epsilon_1$ such that if $\overline{\xi} \in B(\xi, \epsilon_1)$ then $\overline{\xi} \in G_2$. For the equilibrium points $\mathbf{x}_{ij}, \mathbf{x}_{i'j'}$ with $\max\{|i'-i|, |j'-j|\} = 1$ the lines $L^k_{ij}, L^{k'}_{i'j'}, k, k' = 1, 2$ intersect transversally. By continuity of the lines on the defining constants of the vector field ξ, there exists $0 < \epsilon_{ijki'j'k'}$ such that if $\overline{\xi} \in B(\xi, \epsilon_{ijki'j'k'})$ then the corresponding lines $\overline{L}^k_{ij}, \overline{L}^{k'}_{i'j'}$ also intersect transversally.

Finally, let $\epsilon = \min\{\epsilon_1, \min\{\epsilon_{ijki'j'k'} : 0 \le i \le n, 0 \le j \le m, \max\{|i'-i|, |j'-j|\} = 1, k, k' \in \{1,2\}\}\}$. If $\overline{\xi} \in (\xi, \epsilon)$ then the vector field $\overline{\xi}$ is properly transversal. ∎

Theorem 2.15. *G_3 is dense in P(n,m), 2 ≤n,m.*

PROOF. (Figure 5.) Let $\xi \in G_2$ and $0 < \epsilon$. It is sufficient to show that $B(\xi,\epsilon) \cap G_3 \neq \{\}$. Without loss of generality it may be assumed that ϵ is so small that $B(\xi,\epsilon) \subseteq G_2$.

Consider the matrix

$$\begin{bmatrix} \alpha_{i1} & \alpha_{i+11} \\ \alpha_{i2} & \alpha_{i+12} \end{bmatrix}$$

for $1 \le i \le n-1, n+1 \le i \le n+m-1$. Consider the following sequence of perturbations given by $\xi^1, \ldots, \xi^{n-1}, \xi^{n+1}, \ldots, \xi^{n+m-1}$.

If the matrix

$$\begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix}$$

has nonzero determinant then let $\xi^1 = \xi$. By continuity of the determinant function, there exists $0 < \epsilon_1'$ such that if $\overline{\xi} \in B(\xi^1, \epsilon_1')$ then the corresponding matrix for $\overline{\xi}$ also has nonzero determinant. Let $\epsilon_1 = \min\{\epsilon_1', \epsilon\}$.

If the matrix above has determinant equal to zero then consider the perturbation ξ^1 obtained from ξ as the vector field having all the same defining constants except

$$\alpha_{21}^1 = \alpha_{21} - \mu_1\alpha_{12},$$

$$\alpha_{22}^1 = \alpha_{22} + \mu_1\alpha_{11}$$

for $0 < \mu_1 < \min\{\epsilon/|\alpha_{11}|), \epsilon/|\alpha_{12}|)\}$. Then the matrix

$$\begin{bmatrix} \alpha_{11}^1 & \alpha_{21}^1 \\ \alpha_{12}^1 & \alpha_{22}^1 \end{bmatrix}$$

has nonzero determinant $\mu_1(\alpha_{11}^2 + \alpha_{12}^2)$. There also exists $0 < \epsilon_1'$ such that if $\overline{\xi} \in B(\xi^1, \epsilon_1')$ then the corresponding matrix for $\overline{\xi}$ also has nonzero determinant. Let $\epsilon_1 = \min\{\epsilon_1', \epsilon - d(\xi^1, \xi)\}$.

If the matrix

$$\begin{bmatrix} \alpha_{i1}^{i-1} & \alpha_{i+11}^{i-1} \\ \alpha_{i2}^{i-1} & \alpha_{i+12}^{i-1} \end{bmatrix}$$

has nonzero determinant then let $\xi^i = \xi^{i-1}$. By continuity of the determinant function, there exists $0 < \epsilon_i'$ such that if $\overline{\xi} \in B(\xi^i, \epsilon_i')$ then the corresponding matrix for $\overline{\xi}$ also has nonzero determinant. Let $\epsilon_i = \min\{\epsilon_i', \epsilon_{i-1}\}$.

If the matrix above has determinant equal to zero then consider the perturbation ξ^i obtained from ξ^{i-1} as the vector field having all the same defining constants except

$$\alpha_{i+11}^i = \alpha_{i+11}^{i-1} - \mu_i\alpha_{i2}^{i-1},$$

$$\alpha_{i+12}^i = \alpha_{i+12}^{i-1} + \mu_i\alpha_{i1}^{i-1}$$

for $0 < \mu_i < \min\{\epsilon_{i-1}/|\alpha_{i1}^{i-1}|), \epsilon_{i-1}/|\alpha_{i2}^{i-1}|)\}$. Then the matrix

$$\begin{bmatrix} \alpha_{i1}^{i} & \alpha_{i+11}^{i} \\ \alpha_{i2}^{i} & \alpha_{i+12}^{i} \end{bmatrix}$$

has nonzero determinant $\mu_i((\alpha_{i1}^{i-1})^2+(\alpha_{i2}^{i-1})^2)$. There also exists $0 < \epsilon_i'$ such that if $\overline{\xi} \in B(\xi^i, \epsilon_i')$ then the corresponding matrix for $\overline{\xi}$ also has nonzero determinant. Let $\epsilon_i = \min\{\epsilon_i', \epsilon_{i-1} - d(\xi^i, \xi^{i-1})\}$.

Notice that if $\overline{\xi} \in B(\xi^{n+m-1}, \epsilon_{n+m-1})$ then the matrices

$$\begin{bmatrix} \overline{\alpha}_{i1} & \overline{\alpha}_{i+11} \\ \overline{\alpha}_{i2} & \overline{\alpha}_{i+12} \end{bmatrix}$$

have nonzero determinant for $1 \le i \le n-1, n+1 \le i \le n+m-1$. A vector field that is properly transversal in the ball $B(\xi^{n+m-1}, \epsilon_{n+m-1})$ will also be properly transversal in the original ball $B(\xi, \epsilon)$. Without possibility of confusion let $\xi = \xi^{n+m-1}$ and $\epsilon = \epsilon_{n+m-1}$.

It may also be assumed that the matrices given by

$$\begin{bmatrix} -\alpha_{i1} & \alpha_{i+11} \\ -\alpha_{i2} & \alpha_{i+12} \end{bmatrix} + \begin{bmatrix} \gamma_{i+1} - \gamma_i & 0 \\ 0 & \gamma_{i+1} - \gamma_i \end{bmatrix} \begin{bmatrix} \alpha_{i1} & \alpha_{i+11} \\ \alpha_{i2} & \alpha_{i+12} \end{bmatrix} (2x_{ij} - \gamma_i - \gamma_{i+1})$$

for equilibrium points $\mathbf{x}_{ij} = (x_{ij}, y_{ij})$ and $1 \le i \le n-1, 0 \le j \le m$ have nonzero determinant. Similarly, it may also be taken that the matrices

$$\begin{bmatrix} -\alpha_{j1} & \alpha_{j+11} \\ -\alpha_{j2} & \alpha_{j+12} \end{bmatrix} + \begin{bmatrix} \gamma_{j+1} - \gamma_j & 0 \\ 0 & \gamma_{j+1} - \gamma_j \end{bmatrix} \begin{bmatrix} \alpha_{j1} & \alpha_{j+11} \\ \alpha_{j2} & \alpha_{j+12} \end{bmatrix} (2y_{ij} - \gamma_j - \gamma_{j+1})$$

for equilibrium points $\mathbf{x}_{ij} = (x_{ij}, y_{ij})$ and $0 \le i \le n, n+1 \le j \le n+m-1$ have nonzero determinant. It may also be taken that for $\overline{\xi} \in B(\xi, \epsilon)$ the corresponding matrices for $\overline{\xi}$ also have nonzero determinant. These claims follow by a proof analogous to that for the matrices

$$\begin{bmatrix} \alpha_{i1} & \alpha_{i+11} \\ \alpha_{i2} & \alpha_{i+12} \end{bmatrix}$$

for $1 \le i \le n-1, n+1 \le i \le n-+m-1$ having nonzero determinants.

Consider a perturbation of ξ with all the same defining constants except

$$\begin{bmatrix} \alpha_{11}' \\ \alpha_{12}' \end{bmatrix} + \begin{bmatrix} \alpha_{21}' \\ \alpha_{22}' \end{bmatrix} = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \end{bmatrix} + \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \end{bmatrix},$$
$$\begin{bmatrix} \alpha_{11}' \\ \alpha_{12}' \end{bmatrix} \gamma_1' + \begin{bmatrix} \alpha_{21}' \\ \alpha_{22}' \end{bmatrix} \gamma_2' = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \end{bmatrix} \gamma_1 + \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \end{bmatrix} \gamma_2.$$

Consider the function

$$\mathbf{H}_1(\alpha_1', \alpha_2', \gamma_1', \gamma_2') = \begin{bmatrix} \alpha_1' \\ \alpha_2' \end{bmatrix} \gamma_1' + \begin{bmatrix} \alpha_{11} + \alpha_{21} - \alpha_1' \\ \alpha_{12} + \alpha_{22} - \alpha_2' \end{bmatrix} \gamma_2'.$$

As $\mathbf{D}_\alpha \mathbf{H}_1(\alpha_{11}.\alpha_{12}, \gamma_1, \gamma_2)$,

$$\begin{bmatrix} \gamma_1 - \gamma_2 & 0 \\ 0 & \gamma_1 - \gamma_2 \end{bmatrix},$$

has nonzero determinant, there exist neighbourhoods $N_1^1 \times N_1^2, N_1^3$ of (γ_1, γ_2) and $(\alpha_{11}, \alpha_{12})$ and function $h_1 : N_1^1 \times N_1^2 \to N_1^3$ satisfying

$$\mathbf{H}_1(h_1(\gamma_1', \gamma_2'), \gamma_1', \gamma_2') = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \end{bmatrix} \gamma_1 + \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \end{bmatrix} \gamma_2.$$

As the matrix for $\mathbf{D}_\gamma \mathbf{H}_1(\alpha_{11}, \alpha_{12}, \gamma_1, \gamma_2)$ is

$$\begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix}$$

with nonzero determinant then h_1 is in fact a diffeomorphism. Note that

$$\mathbf{D}_\gamma h_1(\gamma_1, \gamma_2) = \begin{bmatrix} \gamma_2 - \gamma_1 & 0 \\ 0 & \gamma_2 - \gamma_1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix}.$$

Let $\mathrm{X}_1(\gamma_1', \gamma_2')$ be the vector field with the same defining constants as ξ except that

$$\begin{aligned}
\begin{bmatrix} \alpha_{11}^{\mathrm{X}_1} \\ \alpha_{12}^{\mathrm{X}_1} \end{bmatrix} &= h_1(\gamma_1', \gamma_2'), \\
\begin{bmatrix} \alpha_{21}^{\mathrm{X}_1} \\ \alpha_{22}^{\mathrm{X}_1} \end{bmatrix} &= \begin{bmatrix} \alpha_{11} + \alpha_{21} - \alpha_{11}^{\mathrm{X}_1} \\ \alpha_{12} + \alpha_{22} - \alpha_{11}^{\mathrm{X}_1} \end{bmatrix}, \\
\gamma_1^{\mathrm{X}_1} &= \gamma_1', \\
\gamma_2^{\mathrm{X}_1} &= \gamma_2'.
\end{aligned}$$

Let $G_{1j}(\mathrm{X}_1(\gamma_1', \gamma_2'), \mathbf{x}) = \mathbf{d}_{1j}(\mathrm{X}_1(\gamma_1', \gamma_2')) + \mathbf{M}_{1j}(\mathrm{X}_1(\gamma_1', \gamma_2'))\mathbf{x}$. As $\mathbf{D}_x G_{1j}(\mathrm{X}_1(\gamma_1, \gamma_2)(\mathbf{x}_{1j}) = \mathbf{M}_{1j}$, which has nonzero determinant, by the implicit function theorem there exist neighbourhoods $N_{1j}^1 \times N_{1j}^2, N_{1j}^3$ of (γ_1, γ_2) and $\mathbf{x}_{1j}$ and function $g_{1j} : N_{1j}^1 \times N_{1j}^2 \to N_{1j}^3$ assigning to $\mathrm{X}_1(\gamma_1', \gamma_2')$ its unique equilibrium point $\mathbf{x}_{1j}^{\mathrm{X}_1}$.

Now,

$$\begin{aligned}
&G_{1j}(\mathrm{X}_1(\gamma_1', \gamma_2'), \mathbf{x}) \\
&= \begin{bmatrix} \alpha_1 - \alpha_{11}^{\mathrm{X}_1}\gamma_1^{\mathrm{X}_1} + \alpha_{21}^{\mathrm{X}_1}\gamma_2^{\mathrm{X}_1} + \sum_{i'=3}^{n} \alpha_{i'1}\gamma_{i'} - \sum_{i'=n+1}^{n+j} \alpha_{i'1}\gamma_{i'} + \sum_{i'=n+j+1}^{n+m} \alpha_{i'1}\gamma_{i'} \\ \alpha_2 - \alpha_{12}^{\mathrm{X}_1}\gamma_1^{\mathrm{X}_1} + \alpha_{22}^{\mathrm{X}_1}\gamma_2^{\mathrm{X}_1} + \sum_{i'=3}^{n} \alpha_{i'2}\gamma_{i'} - \sum_{i'=n+1}^{n+j} \alpha_{i'2}\gamma_{i'} + \sum_{i'=n+j+1}^{n+m} \alpha_{i'2}\gamma_{i'} \end{bmatrix} \\
&\quad + \begin{bmatrix} b_{11} + \alpha_{11}^{\mathrm{X}_1} - \alpha_{21}^{\mathrm{X}_1} - \sum_{i'=3}^{n} \alpha_{i'1} & b_{12} + \sum_{i'=n+1}^{n+j} \alpha_{i'1} - \sum_{i'=n+j+1}^{n+m} \alpha_{i'1} \\ b_{21} + \alpha_{12}^{\mathrm{X}_1} - \alpha_{22}^{\mathrm{X}_1} - \sum_{i'=3}^{n} \alpha_{i'2} & b_{22} + \sum_{i'=n+1}^{n+j} \alpha_{i'2} - \sum_{i'=n+j+1}^{n+m} \alpha_{i'2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 - \alpha_{11}^{\mathrm{X}_1}\gamma_1^{\mathrm{X}_1} + (\alpha_{11} + \alpha_{21} - \alpha_{11}^{\mathrm{X}_1})\gamma_2^{\mathrm{X}_1} + \sum_{i'=3}^{n} \alpha_{i'1}\gamma_{i'} - \sum_{i'=n+1}^{n+j} \alpha_{i'1}\gamma_{i'} + \sum_{i'=n+j+1}^{n+m} \alpha_{i'1}\gamma_{i'} \\ \alpha_2 - \alpha_{12}^{\mathrm{X}_1}\gamma_1^{\mathrm{X}_1} + (\alpha_{12} + \alpha_{22} - \alpha_{21}^{\mathrm{X}_1})\gamma_2^{\mathrm{X}_1} + \sum_{i'=3}^{n} \alpha_{i'2}\gamma_{i'} - \sum_{i'=n+1}^{n+j} \alpha_{i'2}\gamma_{i'} + \sum_{i'=n+j+1}^{n+m} \alpha_{i'2}\gamma_{i'} \end{bmatrix} \\
&\quad + \begin{bmatrix} b_{11} + \alpha_{11}^{\mathrm{X}_1} - (\alpha_{11} + \alpha_{21} - \alpha_{11}^{\mathrm{X}_1}) - \sum_{i'=3}^{n} \alpha_{i'1} & b_{12} + \sum_{i'=n+1}^{n+j} \alpha_{i'1} - \sum_{i'=n+j+1}^{n+m} \alpha_{i'1} \\ b_{21} + \alpha_{12}^{\mathrm{X}_1} - (\alpha_{12} + \alpha_{22} - \alpha_{12}^{\mathrm{X}_1}) - \sum_{i'=3}^{n} \alpha_{i'2} & b_{22} + \sum_{i'=n+1}^{n+j} \alpha_{i'2} - \sum_{i'=n+j+1}^{n+m} \alpha_{i'2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\end{aligned}$$

Thus, and writing $\mathbf{x}_{1j} = (x_{1j}, y_{1j})$,

$$\mathbf{D}_\gamma G_{1j}(\mathrm{X}_1(\gamma_1, \gamma_1), \mathbf{x}_{1j})$$

$$= \begin{bmatrix} 0 & \alpha_{11}+\alpha_{21} \\ 0 & \alpha_{12}+\alpha_{22} \end{bmatrix} - \begin{bmatrix} \gamma_2-\gamma_1 & 0 \\ 0 & \gamma_2-\gamma_1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} (\gamma_1+\gamma_2) +$$
$$\begin{bmatrix} -\alpha_{11} & -\alpha_{11} \\ -\alpha_{12} & -\alpha_{12} \end{bmatrix} + 2 \begin{bmatrix} \gamma_2-\gamma_1 & 0 \\ 0 & \gamma_2-\gamma_1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} x_{1j}$$

$$= \begin{bmatrix} -\alpha_{11} & \alpha_{21} \\ -\alpha_{12} & \alpha_{22} \end{bmatrix} + \begin{bmatrix} \gamma_2-\gamma_1 & 0 \\ 0 & \gamma_2-\gamma_1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix} (2x_{1j} - \gamma_1 - \gamma_2).$$

As the matrix above has nonzero determinant it follows that $g_{1j} : N_{1j}^1 \times N_{1j}^2 \to N_{1j}^3$ may be considered a diffeomorphism for $0 \le j \le m$. As g_{1j} is a diffeomorphism, the sets $g_{1j}^{-1}(L_{0j'}^k), g_{1j}^{-1}(L_{2j'}^k), k = 1, 2, j' = j-1, j, j+1$ are submanifolds of $N_{1j}^1 \times N_{1j}^2$. Then if

$$(\gamma_1', \gamma_2') \in N_{1j}^1 \times N_{1j}^2 - \cup_{j'=j-1}^{j+1} \cup_{k=1}^2 \left(g_{1j}^{-1}(L_{0j'}^k) \cup g_{1j}^{-1}(L_{2j'}^k)\right)$$

the vector field $\xi' = \mathrm{X}_1(\gamma_1', \gamma_2')$ is identical to ξ except on the region $(-\infty, \gamma_3] \times \Re$.

On the region $(-\infty, \gamma_1'] \times \Re, \xi_{0j}' = \xi_{0j}$ and on the region $[\gamma_2', \gamma_3] \times \Re, \xi_{2j}' = \xi_{2j} 0 \le j \le m$. Thus, $L_{0j'}' = L_{0j'}^1, L_{2j'}' = L_{2j'}$ for $k = 1, 2, j' = j-1, j, j+1$. As the equilibrium point $\mathbf{x}_{1j}'$ does not lie on any of these lines then $L_{1j}'^k, k = 1, 2$ can only intersect the lines $L_{0j'}'^k, L_{2j'}'^k k = 1, 2, j' = j-1, j, j+1$ transversally.

Finally, let

$$(\gamma_1', \gamma_2') \in \cap_{j=0}^m \left(N_{1j}^1 \times N_{1j}^2 - \cup_{j'=j-1}^{j+1} \left(g_{1j}^{-1}(L_{0j'}^k) \cup g_{1j}^{-1}(L_{2j'}^k)\right)\right)$$
$$\cap B((\gamma_1, \gamma_2), \epsilon) \cap h_1^{-1}(N_1^3 \cap B((\alpha_{11}, \alpha_{12}), \epsilon))$$

and $\xi^1 = \mathrm{X}_1(\gamma_1', \gamma_2')$ be the perturbed vector field. Note that the points $\mathbf{x}_{ij}^1, \mathbf{x}_{i'j'}^1, \max\{|i'-i|, |j'-j|\}, i' \ne i$ have lines that only intersect transversally. Note that $d(\xi^1, \xi) < \epsilon$. There exists $0 < \epsilon_1'$ such that if $\overline{\xi} \in B(\epsilon^1, \epsilon_1')$ then the corresponding lines for $\overline{\xi}$ also intersect transversally. Let $\epsilon_1 = \min\{\epsilon_1', \epsilon - d(\xi^1, \xi)\}$. Then $B(\xi^1, \epsilon_1) \subseteq B(\xi, \epsilon)$.

Continuing in this manner, there is a sequence of perturbations $\xi^2, \ldots, \xi^{n-1}$ and suitably chosen $\epsilon_2, \ldots, \epsilon_{n-1}$ such that $B(\xi^{n-1}, \epsilon_{n-1}) \subseteq \cdots \subseteq B(\xi^1, \epsilon_1) \subseteq B(\xi, \epsilon)$. The vector field ξ^{n-1} is such that if $\mathbf{x}_{ij}^{n-1}, \mathbf{x}_{i'j'}^{n-1}$ are equilibrium points with $\max\{|i'-i|, |j'-j|\} = 1, i' \ne i$ then the lines through these points only intersect transversally.

As in the case for the vertical lines $x \equiv \gamma_1, \ldots, \gamma_n$, there are perturbations of the horizontal lines $y \equiv \gamma_{n+1}, \ldots, \gamma_{n+m}$. The corresponding perturbed vector field $\xi^{n+1}, \ldots, \xi^{n+m}$ and suitably chosen $\epsilon_{n+1}, \ldots, \epsilon_{n+m-1}$ satisfy $B(\xi^{n+m-1}, \epsilon_{n+m-1}) \subseteq \cdots \subseteq B(\xi^{n+1}, \epsilon_{n+1}) \subseteq B(\xi^{n-1}, \epsilon_{n-1})$. The final perturbation ξ^{n+m-1} is a vector field such that if $\mathbf{x}_{ij}^{n+m-1}, \mathbf{x}_{i'j'}^{n+m-1}$ are equilibrium points with

$\max\{|i'-i|, |j'-j|\} = 1$ then the lines through these equilibrium points only intersect transversally. The vector field ξ^{n+m} is properly transversal and lies in $B(\xi,\epsilon)$. Thus, G_3 is dense in P(n,m). ∎

Theorem 2.16. *G_3 is dense in P(n,1), $2 \le n$, P(1,m), $2 \le m$.*

Proof. The theorem will be proved for the case P(n,1), $1 \le n$. The proof for the case P(m,1),$1 \le m$ is analogous and will not be repeated.

It is sufficient to prove that G_3 is dense in G_2. Let $\xi \in G_2$ and $0 < \epsilon$. It is required to show that $B(\xi,\epsilon) \cap G_3 \neq \{\}$. Without loss of generality it may be taken that ϵ is so small that $B(\xi,\epsilon) \subseteq G_2$.

As in the proof that G_3 is dense in P(n,m), 2 $\le$n,m there is a sequence of perturbations $\xi^1,\ldots,\xi^{n-1}$such that ξ^j is a perturbation of $\xi^{j-1}(1 \le j < n)$ ending in ξ^{n-1}. The perturbaton ξ^{n-1} is a vector field such that if $\mathbf{x}_{ij}^{n-1}, \mathbf{x}_{i'j'}^{n-1}$ are two equilibrium points with $\max\{|i'-i|, |j'-j|\} = 1, i' \neq i$ then the lines through these points only intersect transversally. The vector field ξ^{n-1} may be constructed with $d(\xi^{n-1},\xi) < \epsilon(n-1)/n$. There also exists $0 < \epsilon_1$ such that if $\overline{\xi} \in B(\xi^{n-1},\epsilon_1)$ then the corresponding lines for $\overline{\xi}$ also intersect transversally.

Let $0 \le i \le n$ and consider the linear vector fields $\xi_{i0}^{n-1}, \xi_{i1}^{n-1}$. Along the vertical strip $[\gamma_i,\gamma_{i+1}] \times \Re$ the vector field ξ^{n-1} is given by

$$\xi^{n-1}|_{[\gamma_i,\gamma_{i+1}]\times\Re}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1^{n-1} \\ \alpha_2^{n-1} \end{bmatrix} + \begin{bmatrix} b_{11}^{n-1} & b_{12}^{n-1} \\ b_{21}^{n-1} & b_{22}^{n-1} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \sum_{i'=1}^{i}\begin{bmatrix} \alpha_{i'1}^{n-1} \\ \alpha_{i'2}^{n-1} \end{bmatrix}(x-\gamma_{i'}^{n-1}) + \sum_{i'=i+1}^{n}\begin{bmatrix} \alpha_{i'1}^{n-1} \\ \alpha_{i'2}^{n-1} \end{bmatrix}(\gamma_{i'}^{n-1}-x) + \begin{bmatrix} \alpha_{n+11}^{n-1} \\ \alpha_{n+12}^{n-1} \end{bmatrix}|y-\gamma_{n+1}^{n-1}|.$$

Consider the perturbation $\mathbf{X}_i(\alpha_1',\alpha_2')$ of $\xi^{n-1}|_{[\gamma_i,\gamma_{i_1}]\times\Re}$ where

$$\mathbf{X}_i(\alpha_1',\alpha_2') = \begin{bmatrix} \alpha_1^{n-1} - (\alpha_1' - \alpha_{n+11}^{n-1})\gamma_{n+1}^{n-1} \\ \alpha_1^{n-1} - (\alpha_2' - \alpha_{n+12}^{n-1})\gamma_{n+1}^{n-1} \end{bmatrix} + \begin{bmatrix} b_{11}^{n-1} & b_{12}^{n-1} + \alpha_1' - \alpha_{n+11}^{n-1} \\ b_{21}^{n-1} & b_{22}^{n-1} + \alpha_2' - \alpha_{n+12}^{n-1} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \sum_{i'=1}^{i}\begin{bmatrix} \alpha_{i'1}^{n-1} \\ \alpha_{i'2}^{n-1} \end{bmatrix}(x-\gamma_{i'}^{n-1}) + \sum_{i'=i+1}^{n}\begin{bmatrix} \alpha_{i'1}^{n-1} \\ \alpha_{i'2}^{n-1} \end{bmatrix}(\gamma_{i'}^{n-1}-x) + \begin{bmatrix} \alpha_1' \\ \alpha_2' \end{bmatrix}|y-\gamma_{n+1}^{n-1}|.$$

Notice that $\mathbf{X}_i(\alpha_1',\alpha_2')_{i0} = \xi_{i0}^{n-1}$ regardless of the values of (α_1',α_2'). Now define the function

$$\mathbf{H}_i(\mathbf{X}_i(\alpha_1',\alpha_2'),\mathbf{x}) = \mathbf{X}_i(\alpha_1',\alpha_2')(\mathbf{x}).$$

As

$$\mathbf{D}_x\mathbf{H}_i(\mathbf{X}_i(\alpha_{n+11}^{n-1},\alpha_{n+12}^{n-1}),\mathbf{x}_{i1}) = \mathbf{M}_{i1}^{n-1}$$

and $\det\mathbf{M}_{i1}^{n-1} \neq 0$ there exist neighbourhoods $N_i^1 \times N_i^2, N_i^3$ of $(\alpha_{n+11}^{n-1},\alpha_{n+11}^{n-1})$ and $\mathbf{x}_{i1}$ respectively and a function $h_i : N_i^1 \times N_i^2 \to N_i^3$ assigning to $\mathbf{X}_i(\alpha_1',\alpha_2')$ its corresponding equilibrium point $\mathbf{x}_{i1}^{\mathbf{X}_i}$. Furthermore, with $\mathbf{x}_{i1} = (x_{i1},y_{i1})$,

$$\begin{aligned}\mathbf{D}_\alpha\mathbf{H}_i(\mathbf{X}_i(\alpha_{n+11}^{n-1},\alpha_{n+12}^{n-1}),\mathbf{x}_{i1}) &= \begin{bmatrix} -\gamma_{n+1}^{n-1} & 0 \\ 0 & -\gamma_{n+1}^{n-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y_{i1} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(y_{i1}-\gamma_{n+1}^{n-1}) \\ &= \begin{bmatrix} 2y_{i1} - 2\gamma_{n+1}^{n-1} & 0 \\ 0 & 2y_{i1} - 2\gamma_{n+1}^{n-1} \end{bmatrix}.\end{aligned}$$

As ξ^{n-1} does not have any transitional equilibrium points then $\mathbf{x}_{i1}$ does not lie on the line $y \equiv \gamma^{n-1}_{n+1}$. Thus, $2y_{i1} - 2\gamma^{n-1}_{n+1} \neq 0$ and

$$\det D_{\alpha} H_i(X_i(\alpha^{n-1}_{n+11}, \alpha^{n-1}_{n+12}), \mathbf{x}_{i1}) \neq 0.$$

Thus, h_i is in fact a diffeomorphism.

Let L^1_{i0}, L^2_{i0} be the lines through the point $\mathbf{x}_{i0}$. By choosing $(\alpha'_1, \alpha'_2) \in N^1_i \times N^i_2 - \cup^2_{k=1} h^{-1}_i(L^k_{i0})$ the vector field $X_i(\alpha'_1, \alpha'_2)$ has a equilibrium point $\mathbf{x}^{X_i}_{i1}$ which does not lie on the lines L^1_{i0}, L^2_{i0}. The lines through $\mathbf{x}^{X_i}_{i1}$ can only intersect the lines through $\mathbf{x}^{X_i}_{i0}$ transversally.

Finally, choose

$$(\alpha'_1, \alpha'_2) \in \cap^n_{i=0} (N^1_i \times N^i_2 - \cup^2_{k=1} h^{-1}_i(L^k_{i0})) \cap B((\alpha^{n-1}_{n+11}, \alpha^{n-1}_{n+12}), \epsilon/n) \cap B((\alpha^{n-1}_{n+11}, \alpha^{n-1}_{n+12}), \epsilon_1).$$

The vector field

$$\xi^n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha^{n-1}_1 - (\alpha'_1 - \alpha^{n-1}_{n+11})\gamma^{n-1}_{n+1} \\ \alpha^{n-1}_1 - (\alpha'_2 - \alpha^{n-1}_{n+12})\gamma^{n-1}_{n+1} \end{bmatrix} + \begin{bmatrix} b^{n-1}_{11} & b^{n-1}_{12} + \alpha'_1 - \alpha^{n-1}_{n+11} \\ b^{n-1}_{21} & b^{n-1}_{12} + \alpha'_2 - \alpha^{n-1}_{n+12} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} +$$
$$\sum_{i'=1}^{n} \begin{bmatrix} \alpha^{n-1}_{i'1} \\ \alpha^{n-1}_{i'2} \end{bmatrix} |x - \gamma^{n-1}_{i'}| + \begin{bmatrix} \alpha'_1 \\ \alpha'_2 \end{bmatrix} |y - \gamma^{n-1}_{n+1}|.$$

is properly transversal. As $d(\xi^n, \xi^{n-1}) < \epsilon/n$ then $d(\xi^n, \xi) < \epsilon$. Thus G_3 is dense in P(n,1). ∎

Theorem 2.17. *G_3 is dense in P(1,1).*

Proof. It is sufficient to prove that G_3 is dense in G_2. Let $\xi \in G_3$ and $0 < \epsilon$. Without loss of generality ϵ may be chosen so small that $B(\xi, \epsilon) \subseteq G_2$. It is required to show that $B(\xi, \epsilon) \cap G_3 \neq \{\}$. The vector field ξ may be written as

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \end{bmatrix} |x - \gamma_1| + \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \end{bmatrix} |y - \gamma_2|.$$

As in the proof that G_3 is dense in P(n,1), there is a perturbation ξ^2 of ξ such that if $\mathbf{x}_{ij}$ is an equilibrium point of ξ^2 then the lines through $\mathbf{x}^2_{ij}$ and $\mathbf{x}^2_{1-ij'}, j' = 0, 1$ only intersect transversally. The vector field ξ^2 has the form

$$\xi^2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1 - (\alpha'_1 - \alpha_{21})\gamma_2 \\ \alpha_2 - (\alpha'_2 - \alpha_{22})\gamma_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} + \alpha'_1 - \alpha_{21} \\ b_{21} & b_{22} + \alpha'_2 - \alpha_{22} \end{bmatrix} + \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \end{bmatrix} |x - \gamma_1| + \begin{bmatrix} \alpha'_1 \\ \alpha'_2 \end{bmatrix} |y - \gamma_2|.$$

for which it is possible to choose $(\alpha'_1, \alpha'_2) \in B((\alpha_{21}, \alpha_{22}), \epsilon/2)$. Thus, $d(\xi^2, \xi) < \epsilon/2$. There exist $0 < \epsilon_1$ such that if $\overline{\xi} \in B(\xi^2, \epsilon_1)$ then the equilibrium point $\overline{\mathbf{x}}_{ij}$ also have nontransversal intersection with lines through the points $\overline{\mathbf{x}}_{1-ij'}, j' = 0, 1$.

As with the proof that G_3 is dense in P(1,m), there is a perturbation $\xi^1 \in B(\xi^2, \min\{\epsilon_1, \epsilon/2\})$. The vector field ξ^1 is such that if $\mathbf{x}^1_{ij}, \mathbf{x}^1_{i'1-j}, i' = 0, 1$ are equilibrium points of ξ^1 then the lines through these points only intersect transversally. The vector field ξ^1 has the form

$$\xi^1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1 - (\alpha'_1 - \alpha_{21})\gamma_2 - (\alpha''_1 - \alpha_{11})\gamma_1 \\ \alpha_2 - (\alpha'_2 - \alpha_{22})\gamma_2 - (\alpha''_2 - \alpha_{12})\gamma_1 \end{bmatrix} + \begin{bmatrix} b_{11} + \alpha''_1 - \alpha_{11} & b_{12} + \alpha'_1 - \alpha_{21} \\ b_{21} + \alpha''_2 - \alpha_{12} & b_{22} + \alpha'_2 - \alpha_{22} \end{bmatrix} +$$
$$\begin{bmatrix} \alpha''_1 \\ \alpha''_2 \end{bmatrix} |x - \gamma_1| + \begin{bmatrix} \alpha'_1 \\ \alpha'_2 \end{bmatrix} |y - \gamma_2|.$$

for $(\alpha_1'', \alpha_2'') \in B((\alpha_{11}, \alpha_{12}), \min\{\epsilon_1, \epsilon/2\})$. The vector field ξ^1 is properly transversal and satisfies $d(\xi^1, \xi) < \epsilon$. Thus G_3 is dense in G_2. ∎

Theorem 2.18. *G_3 is dense in P(0,0), P(n,0) $1 \le n$, and P(0,m) $1 \le m$.*

PROOF. It is sufficient to show that G_3 is dense in G_2. Let $\xi \in G_2$ and $0 < \epsilon$. It is required to show that $B(\xi, \epsilon) \cap G_3 \neq \{\}$. Without loss of generality it may be taken that ϵ is so small that $B(\xi, \epsilon) \subseteq G_2$.

That G_3 is dense in G_2 for P(0,0) is apparent from the fact that P(0,0) only contains linear vector fields. thr proof for P(0,m), $1 \le m$, is analagous to the proof for P(n,0), $1 \le n$.

If $2 \le n$ then by the same technique as used in the proof of denseness of G_3 in P(n,m), $2 \le n, m$, there is a sequence of perturbations $\xi^1, \ldots, \xi^{n-1}$. The final vector field, ξ^{n-1}, has equilibrium points $\mathbf{x}_{i0}^{n-1}$ such that the lines through this equilibrium point only intersects the lines through the equilibrium point $\mathbf{x}_{i\pm 10}^{n-1}$ transversally. The vector field ξ^{n-1} is properly transversal and can be chosen such that $d(\xi^{n-1}, \xi) < \epsilon$.

If $1 = n$ then, as in the proof that G_3 is dense in P(n,1) $2 \le n$, there is a perturbation ξ^1 of ξ of the form

$$\xi^1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1 - (\alpha_1' - \alpha_{11})\gamma_1 \\ \alpha_2 - (\alpha_2' - \alpha_{12})\gamma_1 \end{bmatrix} + \begin{bmatrix} b_{11} + \alpha_1' - \alpha_{11} & b_{12} \\ b_{21} + \alpha_2' - \alpha_{12} & b_{22} \end{bmatrix} + \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \end{bmatrix} |x - \gamma_1|$$

which is properly transversal. Furthermore, ξ^1 can be chosen such that $d(\xi^1, \xi) < \epsilon$. Thus G_3 is dense in G_2. ∎

The final subset of P(n,m) to be considered is a technical result that will aid in some of the proofs in the next section. Again, this subset is an open and dense subset of P(n,m).

Definition 2.19. Let $G_4 \subseteq G_3$ be the subset of vector fields such that if $\xi \in G_4$ and $\xi_{ij}, 0 \le i \le n, 0 \le j \le m$ is the linear vector field that satisfies $\xi_{ij}|_{\mathrm{A}_{ij}} = \xi|_{\mathrm{A}_{ij}}$ with the form

$$\xi_{ij} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_{ij}^1 \\ d_{ij}^2 \end{bmatrix} + \begin{bmatrix} m_{ij}^{11} & m_{ij}^{12} \\ m_{ij}^{21} & m_{ij}^{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

then $0 \neq m_{ij}^{12} m_{ij}^{21}$.

Theorem 2.20. *G_4 is open in P(n,m).*

PROOF. Let $\xi \in G_4$. As $\xi \in G_3$ there exists $0 < \epsilon_1$ such that $B(\xi, \epsilon_1) \subseteq G_3$. For each $0 \le i \le n, 0 \le j \le m$ there exists $0 < \epsilon_{ij}$ such that if $\overline{\xi} \in B(\xi, \epsilon_{ij})$ then $0 \neq \overline{m}_{ij}^{12} \overline{m}_{ij}^{21}$ for the vector field $\overline{\xi}$.

Finally, let $\epsilon = \min\{\epsilon_1, \min\{\epsilon_{ij} : 0 \le i \le n, 0 \le j \le m\}\}$. For $\overline{\xi} \in B(\xi, \epsilon_1)$ the vector field $\overline{\xi}$ satisfies $0 \neq \overline{m}_{ij}^{12} \overline{m}_{ij}^{21}$ for $0 \le i \le n, 0 \le j \le m$. ∎

Theorem 2.21. *G_4 is dense in P(n,m).*

PROOF. It is sufficient to show that G_4 is dense in G_3. Let $\xi \in G_3$ and $0 < \epsilon$. Without loss of generality it may be taken that ϵ is so small that $B(\xi,\epsilon) \subseteq G_3$. It is required to show that $B(\xi,\epsilon) \cap G_4 \neq \{\}$.

For each $0 \leq i \leq n, 0 \leq j \leq m$ consider the linear vector field ξ_{ij} for which $\xi_{ij}|_{A_{ij}} = \xi|_{A_{ij}}$. If $0 \neq m_{ij}^{12}m_{ij}^{21}$ then there exists $0 < \epsilon_{ij}$ such that if $\overline{\xi} \in B(\xi,\epsilon_{ij})$ then $0 \neq \overline{m}_{ij}^{12}\overline{m}_{ij}^{21}$ for the corresponding values of $\overline{\xi}$. In particular, the vector field $\overline{\xi}$ given by

$$\overline{\xi}\begin{bmatrix} x \\ y \end{bmatrix} = \xi\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 & \mu \\ \mu & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

for $0 < \mu < \epsilon_{ij}$ lies in $B(\xi,\epsilon_{ij})$.

If $0 = m_{ij}^{12}m_{ij}^{21}$ then either $0 = m_{ij}^{12}$ or $0 = m_{ij}^{21}$. If $0 \neq m_{ij}^{12}$ then let $\epsilon_{ij} = |m_{ij}^{12}|$. If $0 \neq m_{ij}^{21}$ then let $\epsilon_{ij} = |m_{ij}^{21}|$. If $0 = m_{ij}^{12} = m_{ij}^{21}$ then let $\epsilon_{ij} = \infty$. Clearly the perturbation

$$\overline{\xi}\begin{bmatrix} x \\ y \end{bmatrix} = \xi\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 & \mu \\ \mu & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

for $0 < \mu < \epsilon_{ij}$ satisfies $0 \neq \overline{m}_{ij}^{12}\overline{m}_{ij}^{21}$. Finally let $\epsilon_1 = \min\{\epsilon, \min\{\epsilon_{ij} : 0 \leq i \leq n, 0 \leq j \leq m\}\}$. For $0 < \mu < \epsilon_1$ the vector field given by

$$\overline{\xi}\begin{bmatrix} x \\ y \end{bmatrix} = \xi\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 & \mu \\ \mu & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

lies in G_4. Thus G_4 is dense in G_3 as required. ∎

§3. Saddle connections in the generic vector field.

This section will be concerned with the phenomenon of saddle connections in a vector field. It had been originally hoped to prove that the set of vector fields without saddle connections form an open and dense subset of P(n,m). This goal had not been achieved but in the process results were obtained whose interpretation from the standpoint of computer simulation is identical. The first step in the sequence of the results pertains to the occurence of homoclinic orbits in vector fields.

Definition 3.1. Let $\phi(\mathbf{x},t)$ denote the orbit through the point $\mathbf{x}$ and satisfying the following differential equation

$$\phi'(\mathbf{x},t) = \xi(\phi(\mathbf{x},t))$$
$$\phi(\mathbf{x},0) = \mathbf{x}.$$

Assume that the vector field $\xi \in G_4$ contains a homoclinic orbit through the (necessarily) real equilibrium point $\mathbf{x}_{i_0j_0}$ As linear vector fields do not admit homoclinic orbits, the orbit Γ must cross one of the boundary lines $x \equiv \gamma_1,\ldots,\gamma_n$ or $y \equiv \gamma_{n+1},\ldots,\gamma_{n+m}$. If the vector field Γ should only cross a horizontal line $y \equiv \gamma_{i_1}$ then consider the vector field $\overline{\xi}$ obtained as the rotation of ξ under $\pi/2$ radians. The vector field $\overline{\xi}$ has a homoclinic orbit $\overline{\Gamma}$ crossing one the the vertical lines

$x \equiv -\gamma_{n+m}, \ldots, -\gamma_{n+1}$. It may thus be taken that the original homoclinic orbit crosses a vertical line $x \equiv \gamma_{i_1}$.

If $i_0 < i_1$ then consider the vector field $\bar{\xi}$ obtained from ξ by reflection about the y-axis. The vector field $\bar{\xi}$ has a homoclinic orbit passing through the point $\mathbf{x}_{n-i_0 j_0}$ and crossing the vertical line $x \equiv -\gamma_{i_1} = \bar{\gamma}_{n+1-i_1}$ Thus, for the vector field $\bar{\xi}$ it happens that $n+1-i_1 \leq n - i_0$. It may be assumed that for the original vector field ξ that the homoclinic oribit through $\mathbf{x}_{i_0 j_0}$ crosses a vertical line $x \equiv \gamma_{i_1}$ with $i_1 \leq i_0$.

Theorem 3.2. *Let $\xi \in G_4$ have a homocline orbit Γ through the point $\mathbf{x}_{i_0 j_0}$ crossing the line $x \equiv \gamma_{i_1}$ with $i_1 \leq i_0$. Let*

$$S = \{(\gamma_{i_1}, y) : \Gamma \text{ crosses } x \equiv \gamma_{i_1} \text{ at the point } (\gamma_{i_1}, y)\}.$$

Then S has finitely many elements.

PROOF. Without loss of generality the elements of S may be ordered according the the times of crossing of Γ with the line $x \equiv \gamma_{i_1}$,

$$S = \{(\gamma_{i_1}, y_k) : (\gamma_{i_1}, y_{k+1}) = \phi((\gamma_{i_1}, y_k), t_k),\ t_k = \min\{0 < t : \phi((\gamma_{i_1}, y_k), t) \text{ crosses } x \equiv \gamma_{i_1}\}\}.$$

Thus the y-ordinates $y_1, \ldots, y_k \ldots$ give the sucessive points $(\gamma_{i_1}, y_1), \ldots, (\gamma_{i_1}, y_k), \ldots$ of crossing of Γ with $x \equiv \gamma_{i_1}$.

Assume S has infinitely many elements. If $\lim_{k\to\infty} |y_k| = \infty$ then the points (γ_{i_1}, γ_k) are unbounded. The homoclinic orbit Γ will then be unbounded, as also will be $\Gamma \cup \{\mathbf{x}_{i_0 j_0}\}$. However, $\Gamma \cup \{\mathbf{x}_{i_0 j_0}\}$ is homeomorphic to S^1 (the circle in $\Re^2$). As S^1 is bounded and $\Gamma \cup \{\mathbf{x}_{i_0 j_0}\}$ is unbounded, a contradiction arises. Thus it can be concluded that $\lim_{k\to\infty} |y_k| < \infty$.

By the Bolzano-Weierstrauss theorem, the set $\{y_k : 1 \leq k < \infty\}$ has a convergent subsequence $\{y_{k_l} : 1 \leq l < \infty\}$ with limit y'. Consider the vector $\xi(\gamma_{i_1}, y')$ at the point (γ_{i_1}, y'). There are three cases to consider as to whether $\xi(\gamma_{i_1}, y').[1\ 0]^t$ is less than, equal to, or greater than zero.

Assume $\xi(\gamma_{i_1}, y').[1\ 0]^t < 0$. The vector $\xi(\gamma_{i_1}, y')$ points left into the region $\{(x, y) : x \leq \gamma_{i_1}\}$. By continuity of the vector field there exists $0 < \epsilon$ such that if $y \in (y' - \epsilon, y' + \epsilon)$ then $\xi(\gamma_{i_1}, y).[1\ 0]^t < 0$. As $\lim_{l\to\infty} y_{k_l} = y'$, it is possible to choose $y_{k_{l_0}}, y_{k_{l_1}} \in (y' - \epsilon, y' + \epsilon)$ with $y_{k_{l_0}} < y_{k_{l_1}}$. Let Γ^- be the portion of Γ joining the points $(\gamma_{i_1}, y_{k_{l_0}}), (\gamma_{i_1}, y_{k_{l_1}})$. Consider the closed curve $\Gamma^- \cup (\{\gamma_{i_1}\} \times [y_{k_{l_0}}, y_{k_{l_1}}])$. It is immediate that $\lim_{t\to\infty} \phi((\gamma_{i_1}, y_{k_{l_0}}), t)$ and $\lim_{t\to-\infty} \phi((\gamma_{i_1}, y_{k_{l_1}}), t)$ lie in different components of $\Gamma^- \cup (\{\gamma_{i_1}\} \times [y_{k_{l_0}}, y_{k_{l_1}}])$. Then $\lim_{t\to\infty} \phi((\gamma_{i_1}, y_{k_{l_0}}), t) \neq \lim_{t\to-\infty} \phi((\gamma_{i_1}, y_{k_{l_1}}), t)$ contradicting the assumption of a homoclinic orbit.

Similarly, the assumption that $\xi(\gamma_{i_1}, y').[1\ 0]^t > 0$ will also lead to the same type of contradiction. Thus, consider the case that $\xi(\gamma_{i_1}, y').[1\ 0]^t = 0$.

Consider the two intervals $(y' - \epsilon, y'), (y', y' + \epsilon)$ where ϵ is chosen so small that $(y' - \epsilon, y') \subseteq W_{j_0}, (y', y' + \epsilon) \subseteq W_{j_1}$ for some $0 \leq j_0 \leq j_1 \leq m$. At least one of the two intervals contains infinitely many elements from S. It may be assumed that it is the interval $(y'-\epsilon, y') \subseteq W_{j_0}$ which contains infinitely many elements of S. Consider the linear vector field given by

$$\xi_{i_1 j_0}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d^1_{i_1 j_0} \\ d^2_{i_1 j_0} \end{bmatrix} + \begin{bmatrix} m^{11}_{i_1 j_0} & m^{12}_{i_1 j_0} \\ m^{21}_{i_1 j_0} & m^{22}_{i_1 j_0} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Along the line $x \equiv \gamma_{i_1}$ the linear vector field is given by

$$\xi_{i_1 j_0}\begin{bmatrix} \gamma_{i_1} \\ y \end{bmatrix} = \begin{bmatrix} d^1_{i_1 j_0} \\ d^2_{i_1 j_0} \end{bmatrix} + \begin{bmatrix} m^{11}_{i_1 j_0} & m^{12}_{i_1 j_0} \\ m^{21}_{i_1 j_0} & m^{22}_{i_1 j_0} \end{bmatrix}\begin{bmatrix} \gamma_{i_1} \\ y \end{bmatrix}.$$

If $\xi_{i_1 j_0}(\gamma_{i_1}, y)$ has x-ordinate zero then

$$0 = d^1_{i_1 j_0} + m^{11}_{i_1 j_0}\gamma_{i_1} + m^{12}_{i_1 j_0} y.$$

As $m^{12}_{i_1 j_0} \neq 0$ there is a unique solution to the above equation. Thus, y' is the only value of y for which the above equality holds. This implies that for $y \in (y'-\epsilon, y')$ the vectors $\xi_{i_1 j_0}(\gamma_{i_1}, y)$ are not tangent to the line $x \equiv \gamma_{i_1}$. In other words, $\xi(\gamma_{i_1}, y).[1\ 0]^t < 0$ or $\xi(\gamma_{i_1}, y).[1\ 0]^t > 0$ for $y \in (y'-\epsilon, y')$.

If $\xi(\gamma_{i_1}, y).[1\ 0]^t < 0$ for $y \in (y'-\epsilon, y')$ then choose $y_{k_{l_0}}, y_{k_{l_1}} \in (y'-\epsilon, y')$ with $y_{k_{l_0}} < y_{k_{l_1}}$. Let Γ^- be the portion of Γ joining the points $(\gamma_{i_1}, y_{k_{l_0}}), (\gamma_{i_1}, y_{k_{l_1}})$. Consider the closed curve $\Gamma^- \cup (\{\gamma_{i_1}\} \times [y_{k_{l_0}}, y_{k_{l_1}}])$. It is immediate that $\lim_{t\to\infty}\phi((\gamma_{i_1}, y_{k_{l_0}}), t)$ and $\lim_{t\to-\infty}\phi((\gamma_{i_1}, y_{k_{l_1}}), t)$ lie in different components of $\Gamma^- \cup (\{\gamma_{i_1}\} \times [y_{k_{l_0}}, y_{k_{l_1}}])$. Then $\lim_{t\to\infty}\phi((\gamma_{i_1}, y_{k_{l_0}}), t) \neq \lim_{t\to-\infty}\phi((\gamma_{i_1}, y_{k_{l_1}}), t)$ contradicting the assumption of a homoclinic orbit. Similarly, if $\xi(\gamma_{i_1}, y).[1\ 0]^t > 0$ for $y \in (y'-\epsilon, y')$ then a contradiction also arises. Thus, it cannot happen that $\xi(\gamma_{i_1}, y').[1\ 0]^t = 0$.

Hence, S has finitely many elements. ∎

The following conjecture was found to be necesary to prove the results in this section. Differentiable dynamics is able to utilise local perturbations of vector fields in the C^k topologies, this technique of local perturbation is not available when the vector fields are piecewise linear vector fields. The conjecture is essentially a global perturbation conjecture whose validity has been supported by computer simulation.

Conjecture 3.3. *Let (γ_{i_1}, y) be a point for which $1 \leq i_1 \leq n$. Let the orbit through (γ_{i_1}, y) cross the line $x \equiv \gamma_{i_1}$ at the point (γ_{i_1}, y'). For every $0 < \epsilon$ there exists $\overline{\xi}^1$, $\overline{\xi}^2 \in B(\xi, \epsilon)$ with $\overline{\gamma}^1_{i_1} = \overline{\gamma}^2_{i_1} = \gamma_{i_1}$, and $\overline{\xi}^1_{ij}$, $\overline{\xi}^2_{ij} = \xi_{ij}$ for $i_1 \leq i$ such that the orbit through the point (γ_{i_1}, y) crosses the line $x \equiv \gamma_{i_1}$ at $(\gamma_{i_1}, \overline{y}^1), (\gamma_{i_1}, \overline{y}^2)$ with $\overline{y}^1 < y < \overline{y}^2$.*

Remark: There do exist vector fields that coincide with ξ on the set $[\gamma_{i_1}, \infty) \times \Re$. For example, let

$\overline{\xi}$ be the vector field with all the same defining constants as ξ except

$$\overline{\alpha}_1 - \overline{\alpha}_{i_1 1}\gamma_{i_1} = \alpha_1 - \alpha_{i_1 1}\gamma_{i_1}$$
$$\overline{b}_{11} + \overline{\alpha}_{i_1 1} = b_{11} + \alpha_{i_1 1}$$
$$\overline{\alpha}_2 - \overline{\alpha}_{i_1 2}\gamma_{i_1} = \alpha_2 - \alpha_{i_1 2}\gamma_{i_1}$$
$$\overline{b}_{21} + \overline{\alpha}_{i_1 2} = b_{21} + \alpha_{i_1 2}.$$

Clearly $\overline{\xi}_{ij} = \xi_{ij}$ for $i_1 \leq i$ while

$$\begin{aligned}\overline{\xi}_{ij}\begin{bmatrix} x \\ y \end{bmatrix} = & \begin{bmatrix} \overline{\alpha}_1 - \sum_{i'=1}^{i} \alpha_{i'1}\gamma_{i'} + \sum_{i'=i+1}^{i_1-1} \alpha_{i'1}\gamma_{i'} + \overline{\alpha}_{i_1 1}\gamma_{i_1} + \sum_{i'=i_1+1}^{n} \alpha_{i'1}\gamma_{i'} - \\ \overline{\alpha}_2 - \sum_{i'=1}^{i} \alpha_{i'2}\gamma_{i'} + \sum_{i'=i+1}^{i_1-1} \alpha_{i'2}\gamma_{i'} + \overline{\alpha}_{i_1 2}\gamma_{i_1} + \sum_{i'=i_1+1}^{n} \alpha_{i'2}\gamma_{i'} - \end{bmatrix} \\ & \begin{matrix} \sum_{i'=n+1}^{n+j} \alpha_{i'1}\gamma_{i'} + \sum_{i'=n+j+1}^{n+m} \alpha_{i'1}\gamma_{i'} \\ \sum_{i'=n+1}^{n+j} \alpha_{i'2}\gamma_{i'} + \sum_{i'=n+j+1}^{n+m} \alpha_{i'2}\gamma_{i'} \end{matrix} \Bigg] \\ & + \begin{bmatrix} \overline{b}_{11} + \sum_{i'=1}^{i} \alpha_{i'1} - \sum_{i'=i+1}^{i_1-1} \alpha_{i'1} - \overline{\alpha}_{i_1 1} - \sum_{i'=i_1+1}^{n} \alpha_{i'1} \\ \overline{b}_{21} + \sum_{i'=1}^{i} \alpha_{i'2} - \sum_{i'=i+1}^{i_1-1} \alpha_{i'2} - \overline{\alpha}_{i_1 2} - \sum_{i'=i_1+1}^{n} \alpha_{i'2} \end{bmatrix} \\ & \begin{matrix} b_{12} + \sum_{i'=n+1}^{n+j} \alpha_{i'1} - \sum_{i'=n+j+1}^{n+m} \alpha_{i'1} \\ b_{22} + \sum_{i'=n+1}^{n+j} \alpha_{i'2} - \sum_{i'=n+j+1}^{n+m} \alpha_{i'2} \end{matrix} \Bigg] \begin{bmatrix} x \\ y \end{bmatrix} \\ = & \ \xi_{ij}\begin{bmatrix} x \\ y \end{bmatrix} + 2(\gamma_{i_1} - x)\begin{bmatrix} \overline{\alpha}_{i_1 1} - \alpha_{i_1 1} \\ \overline{\alpha}_{i_1 2} - \alpha_{i_1 2} \end{bmatrix}\end{aligned}$$

for $i < i_1$.

Let $\mathbf{x}$ be a saddle point. To the saddle point can be associated the stable manifold $W^s(\mathbf{x}) = \{\mathbf{y} : \lim_{t\to\infty} \phi(\mathbf{y}, t) = \mathbf{x}\}$ and unstable manifold $W^u(\mathbf{x}) = \{\mathbf{y} : \lim_{t\to-\infty} \phi(\mathbf{y}, t) = \mathbf{x}\}$. Furthermore, $W^s(\mathbf{x}) = W_1^s(\mathbf{x}) \cup \{\mathbf{x}\} \cup W_2^s(\mathbf{x})$ where $W_1^s(\mathbf{x}), W_2^s(\mathbf{x})$ are the connected components of $W^s(\mathbf{x}) - \{\mathbf{x}\}$. Similarly, $W^u(\mathbf{x}) = W_1^u(\mathbf{x}) \cup \{\mathbf{x}\} \cup W_2^u(\mathbf{x})$ where $W_1^u(\mathbf{x}), W_2^u(\mathbf{x})$ are the connected components of $W^u(\mathbf{x}) - \{\mathbf{x}\}$.

Theorem 3.4. *Let $\xi \in G_4$ have a homoclinic orbit Γ through the point $\mathbf{x}_{i_0 j_0}$ joining the manifolds $W_1^u(\mathbf{x}_{i_0 j_0}), W_1^s(\mathbf{x}_{i_0 j_0})$ and crossing the line $x \equiv \gamma_{i_1}$, $i_1 \leq i_0$. For every $0 < \epsilon$ there exist a perturbation ξ^1 of ξ and $0 < \epsilon_1$ such that if $\overline{\xi} \in B(\xi^1, \epsilon_1) \subseteq B(\xi, \epsilon)$ then $\overline{\xi}$ does not have a homoclinic orbit through the point $\overline{\mathbf{x}}_{i_0 j_0}$ joining $W_1^u(\overline{\mathbf{x}}_{i_0 j_0}), W_1^s(\overline{\mathbf{x}}_{i_0 j_0})$.*

Proof. (Figures 6,7.) Let

$$S = \{(\gamma_{i_1}, y_k) : \Gamma \text{ crosses } x \equiv \gamma_{i_1} \text{ at the point } (\gamma_{i_1}, y_k)\}.$$

By theorem 3.2, the set S has finitely many elements. Furthermore, the points (γ_{i_1}, y_k) are ordered by k as to their respective times of crossing of Γ with the line $x \equiv \gamma_{i_1}$.

As S is bounded, there exists values $y_- < y_+$ such that (γ_{i_1}, y_-) and (γ_{i_1}, y_+) are not in the interior of $\Gamma \cup \{\mathbf{x}_{i_0 j_0}\}$ and if $(\gamma_{i_1}, y_k) \in S$ then $y_- < y_k < y_+$. As $(\gamma_{i_1}, y_\pm)$ are not in the interior of $\gamma \cup \{\mathbf{x}_{i_0 j_0}\}$, by the Jordan curve theorem any closed curve joining these two points which crosses $\Gamma \cup \{\mathbf{x}_{i_0 j_0}\}$ finitely does so an even number of times. As γ_{i_1} was chosen so that $i_1 \leq i_0$ and $\mathbf{x}_{i_0 j_0}$ is a nontransitional equilibrium point, the line segment $\{\gamma_{i_1}\} \times [y_-, y_+]$ crosses Γ an even number of times. Thus, S has an even number of elements. Let $S = \{(\gamma_{i_1}, y_k) : 1 \leq k \leq 2p\}$ for some $1 \leq p$. Note that if k is odd then $\xi(\gamma_{i_1}, y_k).[1\ 0]^t < 0$ and if k is even then $\xi(\gamma_{i_1}, y_k).[1\ 0]^t > 0$.

It may be taken that the homoclinic orbit Γ is transversed in a clockwise direction. If this were not the case, then under a reflections about the x-axis the vector field ξ is conjugate to a vector field $\overline{\xi}$ having a homoclinic orbit $\overline{\Gamma}$ through $\overline{\mathbf{x}}_{i_0 j_0}$ transversed in the clockwise direction. Thus $y_1 < y_{2p}$.

There are two cases to consider as to whether $W_2^s(\mathbf{x}_{i_0 j_0})$ lies in the interior or exterior of the region bounded by $\Gamma \cup \{\mathbf{x}_{i_0 j_0}\}$. The proof will proceed in the case that $W_2^s(\mathbf{x}_{i_0 j_0})$ lies in the exterior of the region bounded by $\Gamma \cup \{\mathbf{x}_{i_0 j_0}\}$. The proof in the other case in analagous and will not be repeated.

By conjecture 3.3 there exists $\xi^1 \in B(\xi, \epsilon)$ such that for the vector field ξ^1 the orbit through (γ_{i_1}, y_1) crosses the line $x \equiv \gamma_{i_1}$ at the point (γ_{i_1}, y_{2p}^1) where $y_{2p}^1 < y_{2p}$. Let Γ^1 be the portion of the orbit starting at (γ_{i_1}, y_1) and ending at (γ_{i_1}, y_{2p}^1) under the vector field ξ^1. Consider the region bounded by the five sets: $\Gamma^1, \cup_{t=0}^{-\infty} \phi((\gamma_{i_1}, y_1), t)$, $\{\gamma_{i_1}\} \times [y_{2p}^1, y_{2p}]$, $\cup_{t=0}^{\infty} \phi((\gamma_{i_1}, y_{2p}), t)$, $\{\mathbf{x}_{i_0 j_0}\}$. Note that $W_2^s(\mathbf{x}_{i_0 j_0})$ is not in the region bounded by the above set of points. Note also that the orbit through the point (γ_{i_1}, y_{2p}^1) under ξ^1 cannot exit the aforementioned region, then the orbit of (γ_{i_1}, y_1) under the vector field ξ^1 does not form a homoclinic orbit. The vector field ξ^1 does not have a homoclinic orbit through $\mathbf{x}_{i_0 j_0}^1$ joining $W_1^u(\mathbf{x}_{i_0 j_0}^1)$ and $W_1^s(\mathbf{x}_{i_0 j_0}^1)$. Furthermore, as the the boundary of the above region can be continuously deformed under small perturbations in the defining constants of ξ^1 there exists $0 < \epsilon_1$ such that if $\overline{\xi} \in B(\xi^1, \epsilon_1) \subseteq B(\xi, \epsilon)$ then the vector field $\overline{\xi}$ does not have a homoclinic orbit through $\overline{\mathbf{x}}_{i_0 j_0}$ joining $W_1^u(\overline{\mathbf{x}}_{i_0 j_0})$ and $W_1^s(\overline{\mathbf{x}}_{i_0 j_0})$. ∎

Corollary 3.5. *Let $\xi \in G_4$ and $0 < \epsilon$. There exists ξ' and $0 < \epsilon'$ such that if $\overline{\xi} \in B(\xi', \epsilon') \subseteq B(\xi, \epsilon)$ then $\overline{\xi}$ does not have any homoclinic orbits.*

PROOF. For a given vector field ξ let $\#(\xi)$ denote the number of homoclinic orbits that ξ posseses. For a subset $S \subseteq G_4$ define $N(S) = \max\{\#(\xi) : \xi \in S\}$. Note that since $S \subseteq G_4$ and any $\xi \in G_4$ may have at most $(n+1)(m+1)$ real equilibrium points then ξ may have at most $2(n+1)(m+1)$ homoclinic orbits, thus $N(s) \leq 2(n+1)(m+1)$.

Without loss of generality it may be assumed that $0 < \epsilon$ is so small that $B(\xi, \epsilon) \subseteq G_4$. If $N(B(\xi, \epsilon)) = 0$ then the corollary is true for $\xi' = \xi$ and $\epsilon' = \epsilon$. Thus let $\xi_0^1 \in B(\xi, \epsilon)$ be a vector field for which there is a homoclinic orbit through the point $\mathbf{x}_{i_0 j_0}$ and joining $W_{k_1^1}^u(\mathbf{x}_{i_0 j_0})$, $W_{k_2^1}^s(\mathbf{x}_{i_0 j_0})$ with $k_1^1, k_2^1 \in \{1, 2\}$.

Consider $B(\xi_0^1, \epsilon - d(\xi_0^1, \xi)) \subseteq B(\xi, \epsilon)$. By theorem 3.4 there exists ξ^1 and $0 < \epsilon_1$ such that if $\overline{\xi} \in B(\xi^1, \epsilon_1) \subseteq B(\xi_0^1, \epsilon - d(\xi_0^1, \xi)) \subseteq B(\xi, \epsilon)$ then $\overline{\xi}$ does not have a homoclinic orbit through $\overline{\mathbf{x}}_{i_0 j_0}$

and joining $W^u_{k_1^1}(\overline{\mathbf{x}}_{i_0 j_0}), W^s_{k_2^1}(\overline{\mathbf{x}}_{i_0 j_0})$. Thus $N(B(\xi^1, \epsilon_1)) \leq 2(n+1)(m+1) - 1$. If $N(B(\xi^1, \epsilon_1)) = 0$ then the corollary is true with $\xi' = \xi^1$ and $\epsilon' = \epsilon_1$. Otherwise let $\xi_0^2 \in B(\xi^1, \epsilon_1)$ be a vector field with a homoclinic orbit through the point $\mathbf{x}^2_{i_1 j_1}$ and joining $W^u_{k_1^2}(\mathbf{x}^2_{i_1 j_1}), W^s_{k_2^2}(\mathbf{x}^2_{i_1 j_1})$.

As before, there is a set $B(\xi^2, \epsilon_2) \subseteq B(\xi_0^2, \epsilon_2 - d(\xi_0^2, \xi^1)) \subseteq B(\xi^1, \epsilon_1) \subseteq B(\xi, \epsilon)$ such that if $\overline{\xi} \in B(\xi^2, \epsilon_2)$ then the vector field $\overline{\xi}$ does not have a homoclinic orbit through $\overline{\mathbf{x}}_{i_0 j_0}$ joining $W^u_{k_1^1}(\overline{\mathbf{x}}_{i_0 j_0}), W^s_{k_2^1}(\overline{\mathbf{x}}_{i_0 j_0})$, nor does it have a homoclinic orbit through the point $\overline{\mathbf{x}}_{i_1 j_1}$ joining $W^u_{k_1^2}(\overline{\mathbf{x}}_{i_1 j_1}), W^s_{k_1^2}(\overline{\mathbf{x}}_{i_1 j_1})$. Thus, $N(B(\xi^2, \epsilon_2)) \leq 2(n+1)(m+1) - 2$.

Continuing in this manner there is a sequence of sets $B(\xi^k, \epsilon_k) \subseteq \ldots \subseteq B(\xi^1, \epsilon_1) \subseteq B(\xi, \epsilon)$ such that if $\overline{\xi} \in B(\xi^k, \epsilon_k)$ then $\overline{\xi}$ has at most $2(n+1)(m+1) - k$ homoclinic orbits. If $N(B(\xi^k, \epsilon_k)) = 0$ then the corollary is true with $\xi' = \xi^k$ and $\epsilon' = \epsilon_k$. Since $0 \leq N(B(\xi^k, \epsilon_k)) \leq 2(n+1)(m+1) - k$ then the number of terms in the sequence cannot exceed $k = 2(n+1)(m+1)$. The sequence will thus terminate after a finite number of terms in a set $B(\xi^k, \epsilon_k)$ for which $N(B(\xi^k, \epsilon_k)) = 0$. The corollary is then true for $\xi' = \xi^k$ and $\epsilon' = \epsilon_k$, ∎

Saddle connections between different saddle points will be divided into two types. Saddle connections which are of type I have results amenable under conjecture 3.3 while similar results for saddle connections of type II need conjecture 3.9 for their proof.

Our aim of the proofs about saddle connections is to show that the vector fields without saddle connections are dense in P(n,m). In particular, given a vector field with a saddle connection, there exists a perturbation such that neither the perturbed vector field not any vector field from a sufficiently small neighbourhood of it has the same saddle connection as the original vector field.

Definition 3.6. Let $\xi \in G_4$ have a saddle connection Γ joining the two saddle points $\mathbf{x}_{i_0 j_0}, \mathbf{x}_{i_1 j_1}$ with $i_0 \leq i_1$. If $j_0 \leq j_1$ then define $A = [\gamma_{i_0}, \gamma_{i_1+1}] \times [\gamma_{j_0}, \gamma_{j_1+1}]$. If $j_1 < j_0$ then define $A = [\gamma_{i_0}, \gamma_{i_1+1}] \times [\gamma_{j_1}, \gamma_{j_0+1}]$. The saddle connection Γ is said to be of type I if $\Gamma \cap (\Re^2 - A) \neq \{\}$ and to be of type II if $\Gamma \in \overline{A}$.

Let Γ be a type I saddle connection for a vector field $\xi \in G_4$. As $\Gamma \cap A \neq \{\}$ and the endpoints of Γ lie in the interior of A, then there are points for which Γ crosses ∂A. By rotation and reflection of the original vector field, if necessary, it may be assumed that Γ crosses ∂A at points along the line $x \equiv \gamma_{i_0}$ and that the saddle points which Γ joins are $\mathbf{x}_{i_0 j_0}, \mathbf{x}_{i_1 j_1}$ where $i_0 \leq i_1$. It may further be taken that for $(x, y) \in \Gamma$ it happens that $\lim_{t \to -\infty} \phi((x, y), t) = \mathbf{x}_{i_0 j_0}$ and $\lim_{t \to \infty} \phi((x, y), t) = \mathbf{x}_{i_1 j_1}$.

Theorem 3.7. *Let $\xi \in G_4$ have a saddle connection Γ of type I joining the saddle points $\mathbf{x}_{i_0 j_0}, \mathbf{x}_{i_1 j_1}$ where $i_0 \leq i_1$, and crossing the line $x \equiv \gamma_{i_0}$. Let*

$$S = \{(\gamma_{i_0}, y) : \Gamma \text{ crosses } x \equiv \gamma_{i_0} \text{ at the point } (\gamma_{i_0}, y)\}.$$

Then S has finitely many elements.

PROOF. Order the elements of S according to their times of crossing of Γ with the line $x \equiv \gamma_{i_0}$,

$$S = \{(\gamma_{i_0}, y_k) : (\gamma_{i_0}, y_{k+1}) = \phi((\gamma_{i_0}, y_k), t_k),\ t_k = \min\{0 < t : \phi((\gamma_{i_0}, y_k), t) \text{ crosses } x \equiv \gamma_{i_1}\}\}.$$

Thus the y-ordinates $y_1, \ldots, y_k, \ldots$ give the successive points $(\gamma_{i_0}, y_1), \ldots, (\gamma_{i_0}, y_k), \ldots$ of crossing of Γ with $x \equiv \gamma_{i_0}$.

Assume S has infinitely many elements. If $\lim_{k\to\infty} |y_k| = \infty$ then the points (γ_{i_0}, y_k) are unbounded as a set in $\Re^2$. As the point $\mathbf{x}_{i_0j_0}$ is nontransitional there exists $0 < \epsilon_1$ such that $B(\mathbf{x}_{i_0j_0}, \epsilon_1) \cap \{x \equiv \gamma_{i_0}\} = \{\}$. The point $\mathbf{x}_{i_1j_1}$ is nontransitional, there exists $0 < \epsilon_2$ such that $B(\mathbf{x}_{i_1j_1}) \cap \{x \equiv \gamma_{i_0}\} = \{\}$. Furthermore, there exists $t_1 < t_2$ such that if $t < t_1$ then $\phi((\gamma_{i_1}, y_1), t) \in B(\mathbf{x}_{i_0j_0}, \epsilon_1)$ and if $t_2 < t$ then $\phi((\gamma_{i_1}, y_1), t) \in B(\mathbf{x}_{i_1j_1}, \epsilon_2)$. Consider $\phi((\gamma_{i_0}, y_1), t)$ for $t \in [t_1, t_2]$. For these values of t, $\|\phi'((\gamma_{i_0}, y_1), t)\|$ attains a maximum M for $t \in [t_1, t_2]$. Thus, for any two values $t', t'' \in [t_1, t_2]$ it happens that $\|\phi((\gamma_{i_0}, y_1), t') - \phi((\gamma_{i_0}, y_1), t'')\| \le M|t' - t''| \le M(t_2 - t_1)$. Thus, $|y_1 - y_k| = \|(\gamma_{i_0}, y_1) - (\gamma_{i_0}, y_k)\| \le M(t_2 - t_1)$ from which it follows that $|y_k| \le M(t_2 - t_1) + |y_1|$. Then $\lim_{k\to\infty} |y_k| < \infty$ contradicting $\lim_{k\to\infty} |y_k| = \infty$. Thus it may be assumed that $\lim_{k\to\infty} |y_k| < \infty$.

By the Bolzano-Weierstrauss theorem, the set $\{y_k : 1 \le k < \infty\}$ has a convergent subsequence $\{y_{k_l} : 1 \le l < \infty\}$ with limit y'. Consider the point (γ_{i_0}, y'), as ξ does not have any nontransitional equilibrium points then $\xi(\gamma_{i_0}, y') \neq 0$. Thus there exists $0 < \epsilon_3, \epsilon_4$ such that the flow in $[\gamma_{i_0} - \epsilon_3, \gamma_{i_0} + \epsilon_3] \times [y' - \epsilon_4, y' + \epsilon_4] = V$ conjugate to a a linear flow. Let $1 \le l_0$ be such that if $l_0 \le l$ then $(\gamma_{i_0}, y_{k_l}) \in [\gamma_{i_0} - \epsilon_3/2, \gamma_{i_0} + \epsilon_3/2] \times [y' - \epsilon_4/2, y' + \epsilon_4/2] = U$. Consider the orbit of a point (γ_{i_0}, y_{k_l}) for $l_0 \le l$. As the flow in the regions U, V are conjugate to linear flows then the orbit of the point (γ_{i_0}, y_{k_l}) can only join with the point $(\gamma_{i_0}, y_{k_{l+1}})$ by exiting U, exiting V, re-entering V and re-entering U. As $U \subset V$ there exists a finite time $0 < \tau$ bounding from below the amount of time for the orbit through (γ_{i_0}, y_{k_l}) to travel from ∂S to ∂R. Thus, if $(\gamma_{i_0}, y_{k_{l+1}}) = \phi((\gamma_{i_0}, y_{k_l}), \bar{t}_{k_l})$ then $2\tau \le \bar{t}_{k_l}$. Inductively it follows that $(\gamma_{i_0}, y_{k_l}) = \phi((\gamma_{i_0}, y_{k_{l_0}}), t_{k_l})$ where $2(l - l_0)\tau \le t_{k_l}$. As $l \to \infty$ then $(\gamma_{i_0}, y_{k_l}) \in B(\mathbf{x}_{i_1j_1}, \epsilon_2)$ in contradiction to (γ_{i_0}, y_{k_l}) being on the line $x \equiv \gamma_{i_0}$. Thus S has finitely many elements. ∎

Theorem 3.8. *Let $\xi \in G_4$ have a saddle connection Γ of type I joining the saddle points $\mathbf{x}_{i_0j_0}, \mathbf{x}_{i_1j_1}$ where $i_0 \le i_1$, along $W_1^u(\mathbf{x}_{i_0j_0})$ and $W_1^s(\mathbf{x}_{i_1j_1})$ respectively and crossing the line $x \equiv \gamma_{i_0}$. For every $0 < \epsilon$ there exists ξ' and $0 < \epsilon'$ such that if $\bar{\xi} \in B(\xi', \epsilon') \subseteq B(\xi, \epsilon)$ then the vector field $\bar{\xi}$ does not have a saddle connection joining the saddle points $\bar{\mathbf{x}}_{i_0j_0}, \bar{\mathbf{x}}_{i_1j_1}$ along $W_1^u(\bar{\mathbf{x}}_{i_0j_0})$ and $W_1^s(\bar{\mathbf{x}}_{i_1j_1})$ respectively.*

PROOF. (Figures 8,9,10.) Let

$$S = \{(\gamma_{i_0}, y_k) : \Gamma \text{ crosses } x \equiv \gamma_{i_0} \text{ at the point } (\gamma_{i_0}, y_k)\}.$$

Consider the set $\Re^2 \cup \{\infty\}$ as the one point compactification of $\Re^2$. In this set $\{x \equiv \gamma_{i_0}\} \cup \{\infty\}$ is a closed curve passing through the point ∞. The points $\mathbf{x}_{i_0j_0}, \mathbf{x}_{i_1j_1}$ both lie in the same component of $\Re^2 \cup \{\infty\} - (\{x \equiv \gamma_{i_0}\} \cup \{\infty\})$. As Γ is a curve joining $\mathbf{x}_{i_0 j_0}, \mathbf{x}_{i_1 j_1}$, if it crosses the line $x \equiv \gamma_{i_0}$ a finite number of times then by the Jordan curve theorem, this number of crossings is even. Thus $S = \{(\gamma_{i_0}, y_k) : 1 \leq k \leq 2p\}$ for some $1 \leq p$. Without loss of generality it may be assumed that (γ_{i_0}, y_{2p}) is a point of transversal intersection of Γ and the line $x \equiv \gamma_{i_0}$. The end of the proof will consider the case if this should not happen to be the siutation.

By conjecture 3.3, there exists a vector field $\xi^1 \in B(\xi, \epsilon)$ such that the orbit through (γ_{i_0}, y_1) intersects the line $x \equiv \gamma_{i_0}$ at the point (γ_{i_0}, y^1_{2p}) where $y^1_{2p} < y_{2p}$. Furthermore, y^1_{2p} can be chosen so close to y_{2p} that for $y \in [y^1_{2p}, y_{2p}]$ the vector $\xi^1(\gamma_{i_0}, y)$ is transversal to the line $x \equiv \gamma_{i_0}$.

As (γ_{i_0}, y^1_{2p}) is formed by the transversal intersection of $W^u_1(\mathbf{x}_{i_0 j_0})$ and $x \equiv \gamma_{i_0}$, for sufficiently small $0 < \epsilon^1_1$ the vector field $\overline{\xi} \in B(\xi^1, \epsilon^1_1)$ also has transversal intersection of $W^u_1(\overline{\mathbf{x}}_{i_0 j_0})$ and $x \equiv \overline{\gamma}_{i_0}$ at the point $(\overline{\gamma}_{i_0}, \overline{y}_{2p})$ where $\overline{y}_{2p}$ is close to y^1_{2p}. As (γ_{i_0}, y_{2p}) is formed by the intersection of $W^s_1(\mathbf{x}_{i_1 j_1})$ and $x \equiv \gamma_{i_0}$, for sufficiently small $0 < \epsilon^2_1$ the vector field $\overline{\xi} \in B(\xi^1, \epsilon^2_1)$ has transversal intersection of $W^s_1(\mathbf{x}_{i_1 j_1})$ and $x \equiv \overline{\gamma}_{i_0}$ at $(\overline{\gamma}_{i_0}, \overline{y}_*)$ where $\overline{y}_*$ is close to y_{2p}.

Choose $0 < \epsilon_1 < \min\{\epsilon^1_1, \epsilon^2_1, \epsilon - d(\xi^1, \xi)\}$ so small that for $\overline{\xi} \in B(\xi^1, \epsilon_1) \subseteq B(\xi, \epsilon), \overline{y}_{2p} < \overline{y}_*$ and the vector $\overline{\xi}(\overline{\gamma}_{i_0}, y)$ is transversal to $x \equiv \gamma_{i_0}$ for $y \in [\overline{y}_{2p}, \overline{y}_*]$. Consider the set $B(\xi^1, \epsilon_1)$. If it happens that for every $\overline{\xi} \in B(\xi^1, \epsilon_1)$ the vector field $\overline{\xi}$ does not have a saddle connection joining the saddle points $\overline{\mathbf{x}}_{i_0 j_0}, \overline{\mathbf{x}}_{i_1 j_1}$ along $W^u_1(\overline{\mathbf{x}}_{i_0 j_0}), W^s_1(\overline{\mathbf{x}}_{i_1 j_1})$ then the theorem is true with $\xi' = \xi^1$ and $\epsilon' = \epsilon$.

Let $\xi^2 \in B(\xi^1, \epsilon_1)$ have a saddle connection Γ^2 joining the saddle points $\mathbf{x}^2_{i_0 j_0}, \mathbf{x}^2_{i_1 j_1}$ along $W^u_1(\mathbf{x}^2_{i_0 j_0})$ and $W^s_1(\mathbf{x}^2_{i_1 j_1})$ respectively. Let

$$S^2 = \{(\gamma^2_{i_0}, y_k) : \Gamma^2 \text{ crosses } x \equiv \gamma^2_{i_0} \text{ at the point } (\gamma^2_{i_0}, y_k)\}.$$

As noted before, $S^2 = \{(\gamma^2_{i_0}, y_k) : 1 \leq k \leq 2q\}$ where $p < q$. Observe that $(\gamma^2_{i_0}, y^2_*) \in S^2$ so that $y^2_* = y^2_l$ for some $2p \leq l \leq 2q$. Since there is a saddle connection joining $\mathbf{x}^2_{i_0 j_0}$ and $\mathbf{x}^2_{i_1 j_1}$ the orbit through $(\gamma^2_{i_0}, y^2_{2p})$ meets $x \equiv \gamma^2_{i_0}$ at $(\gamma^2_{i_0}, y^2_*) = (\gamma^2_{i_0}, y^2_l)$. Let $\epsilon_2 = \epsilon - d(\xi, \xi)$ and consider the set $B(\xi^2, \epsilon_2) \subseteq B(\xi^1, \epsilon_1)$.

Say it happens that $l = 2q$. By a second application of conjecture 3.3, there exists $\xi^3 \in B(\xi^2, \epsilon_2)$ with $y^2_{2q} < y^3_{2q}$ and $\xi^3(\gamma^2_{i_0}, y)$ is transversal to $x \equiv \gamma^2_{i_0}$ for $y \in [y^2_{2q}, y^3_{2q}]$. Thus $\xi^3(\gamma^2_{i_0}, y)$ is transversal to $x \equiv \gamma^2_{i_0}$ for $y \in [y^3_{2p}, y^3_{2q}]$. Let Γ^3 be the portion of the orbit that starts at $(\gamma^2_{i_0}, y^3_{2p})$ and ending at $(\gamma^2_{i_0}, y^3_{2q})$ Let $U = \Gamma^3 \cup (\{\gamma^2_{i_0}\} \times [y^3_{2p}, y^3_{2q}])$. There are two possibilities as to whether $\mathbf{x}^3_{i_1 j_1}$ lies in the interior or exterior of the region bounded by U.

Consider the case that $\mathbf{x}^2_{i_1 j_1}$ lies in the exterior of the region bounded by U. Note it happens that $\lim_{t \to -\infty} \phi^3((\gamma^2_{i_0}, y^2_{2q}), t) \in U$ while $\lim_{t \to \infty} \phi^3((\gamma^2_{i_0}, y^3_{2q}), t) \notin U$. It thus follows that the orbit through $(\gamma^2_{i_0}, y^2_{2q})$ cannot join with $(\gamma^2_{i_0}, y^3_{2q})$ to form a saddle connection between $W^u_1(\mathbf{x}^2_{i_0 j_0})$ and $W^s_1(\mathbf{x}^2_{i_1 j_1})$.

In the case that $\mathbf{x}^2_{i_1 j_1}$ lies in the interior of the region bounded by U it is possible to observe that $\lim_{t \to -\infty} \phi^3((\gamma^2_{i_0}, y^2_{2q}), t) \notin U$ while $\lim_{t \to \infty} \phi^3((\gamma^2_{i_0}, y^3_{2q}), t) \in U$. The orbit through $(\gamma^2_{i_0}, y^2_{2q})$

cannot join with $(\gamma^2_{i_0}, y^3_{2q})$ to form a saddle connection between $W^u_1(\mathbf{x}^2_{i_0j_0})$ and $W^s_1(\mathbf{x}^2_{i_1j_1})$.

As the boundary of U is preserved under small perturbations in the defining constants of ξ^3, there exists $0 < \epsilon_3$ such that of $\overline{\xi} \in B(\xi^3, \epsilon_3)$ then the vector field ξ^3 does not have a saddle connection joining $W^u_1(\overline{\mathbf{x}}_{i_0j_0})$ and $W^s_1(\overline{\mathbf{x}}_{i_1j_1})$. The theorem is then true for $\xi' = \xi^3$ and $\epsilon' = \min\{\epsilon_3, \epsilon - d(\xi^3, \xi^2)\}$.

Next, consider the possibility that $l < 2q$. Consider the point $(\gamma^2_{i_0}, y^2_{2q})$ and the orbit through this point. Define the set

$$T^2 = \{(\gamma^2_{i_0}, y_k) : \phi^2((\gamma^2_{i_0}, y_{2q}), t_k) \cap \{x \equiv \gamma^2_{i_0}\} = (\gamma^2_{i_0}, y_k), 2q < k, 0 < t_k\}.$$

Note that if $(\gamma^2_{i_0}, y_k) \in T^2$ then $(\gamma^2_{i_0}, y_k) \notin S$ else $(\gamma^2_{i_0}, y_k)$ would not be the last point of transversal intersection of Γ^2 with $x \equiv \gamma^2_{i_0}$. Hence the orbit through the point $(\gamma^2_{i_0}, y_k)$ is tangent to the line $x \equiv \gamma^2_{i_0}$.

If $T^2 = \{\}$ then $(\gamma^2_{i_0}, y^2_{2q})$ is the last point for which $\phi^2((\gamma^2_{i_0}, y^2_{2q}), t)$ meets $x \equiv \gamma^2_{i_0}$ for $0 \leq t$. As $\mathbf{x}^2_{i_0j_0}$ is a nontransitional equilibrium point there exists $0 < \epsilon_{2q}$ such that $B(\mathbf{x}^2_{i_0j_0}, \epsilon_{2q}) \cap \{x \equiv \gamma^2_{i_0}\} = \{\}$. It may be taken the ϵ_{2q} is so small that for $\overline{\xi} \in B(\xi^2, \epsilon_2)$ it happens that $B(\overline{\mathbf{x}}^2_{i_0j_0}, \epsilon_{2q}) \cap \{x \equiv \overline{\gamma}^2_{i_0}\} = \{\}$. There exist $0 < t_{2q}$ such that if $t_{2q} < t$ then $\phi^2((\gamma^2_{i_0}, y^2_{2q}), t) \in B(\mathbf{x}^2_{i_0j_0}, \epsilon_{2q})$. Let Γ^2_* be the orbit $\phi^2((\gamma^2_{i_0}, y^2_{2q}), t)$ for times $0 \leq t \leq t_{2q}$. Note that Γ^2_* is a compact set for which $\Gamma^2_* \cap \{x \equiv \gamma^2_{i_0}\} = (\gamma^2_{i_0}, y^2_{2q})$. By restricting $0 < \epsilon_2$ it is possible to ensure that for $\overline{\xi} \in B(\xi^2, \epsilon_2)$ it happens that $x \equiv \overline{\gamma}^2_{i_0}$ and $\overline{\Gamma}_*$ only have the point $(\overline{\gamma}^2_{i_0}, \overline{y}_{2q})$ in common. If the vector field $\xi^3 \in B(\xi^2, \epsilon_2)$ has a saddle connection then consider

$$S^3 = \{(\gamma^3_{i_0}, y_k) : \Gamma^3 \text{ crosses } x \equiv \gamma^3_{i_0} \text{ at the point } (\gamma^3_{i_0}, y_k)\}.$$

As before, $S^3 = \{(\gamma^3_{i_0}, y_k) : 1 \leq k \leq 2q'\}$ where $q \leq q'$. Consider the point $(\gamma^2_{i_0}, y^2_{2q})$ with image $(\gamma^3_{i_0}, y^3_l)$ for some $2q \leq l \leq 2q'$. If $l < 2q'$ then the orbit Γ^3_* intersects $x \equiv \gamma^3_{i_0}$ at a point other that the image of $(\gamma^2_{i_0}, y^2_{2q})$. This is in contradiction to the construction of $B(\xi^2, \epsilon_2)$. Thus $l = 2q'$.

If $T^2 \neq \{\}$ then consider the perturbation ξ^μ of ξ^2 as the vector field with all the same defining constants as ξ^2 except

$$\alpha^\mu_1 = \alpha^2_1 - \mu\alpha^2_{i_01}$$
$$\alpha^\mu_2 = \alpha^2_2 - \mu\alpha^2_{i_02}$$
$$\gamma^\mu_{i_0} = \gamma^2_{i_0} - \mu$$

where $0 < \mu < \mu_1 = \min\{\epsilon_2, \epsilon_2/|\alpha^2_{i_01}|, \epsilon_2/|\alpha^2_{i_02}|\}$ is so small that Γ^μ_* has only the point $(\gamma^2_{i_0}, y^\mu_{2q})$ in common with the line $x \equiv \gamma^2_{i_0}$. Thus $T^\mu = \{\}$. If there is a value of μ_0 such that μ_0 has a saddle connection joining $W^u_1(\mathbf{x}^{\mu_0}_{i_0j_0})$ and $W^s_1(\mathbf{x}^{\mu_0}_{i_1j_1})$ then let $\xi^3 = \xi^{\mu_0}$ and $\epsilon_3 < \epsilon_2 - d(\xi^\mu, \xi^2)$ be so small that $\overline{T} = \{\}$ for any vector field $\overline{\xi} \in B(\xi^3, \epsilon_3)$. The proof can then proceed with respect to showing that there is a subset $B(\xi^4, \epsilon_4) \subseteq B(\xi^3, \epsilon_3)$ such that if $\overline{\xi} \in B(\xi^4, \epsilon_4)$ then $\overline{\xi}$ does not have a saddle connection joining $W^u_1(\overline{\mathbf{x}}_{i_0j_0})$ and $W^s_1(\overline{\mathbf{x}}_{i_1j_1})$. If it happens that for every value of μ the vector field ξ^μ does not have a saddle connection joining $W^u_1(\mathbf{x}^\mu_{i_0j_0})$ and $W^s_1(\mathbf{x}^\mu_{i_1j_1})$ then let $\xi^3 = \xi^{\mu_1/2}$ and

$\epsilon_3 < \epsilon_2 - d(\xi^3, \xi^2)$ be so small that $\overline{T} = \{\}$ for vector fields $\overline{\xi} \in B(\xi^3, \epsilon_3)$. If none of the vector fields in $B(\xi^3, \epsilon_3)$ has the required saddle connection then the proof is finished else the proof can proceed with respect to showing that there is a subset $B(\xi^4, \epsilon_4) \subseteq B(\xi^3, \epsilon_3)$ such that if $\overline{\xi} \in B(\xi^4, \epsilon_4)$ then $\overline{\xi}$ does not have a saddle connection joining $W_1^u(\overline{\mathbf{x}}_{i_0j_0})$ and $W_1^s(\overline{\mathbf{x}}_{i_1j_1})$.

Consider the case that the last point of crossing of the saddle connection Γ and the line $x \equiv \gamma_{i_0}$ should not happen to be a point of transversal intersection. For sufficiently small $0 < \mu < \mu_1 \leq \epsilon$ consider a perturbation of the original vector field ξ to ξ^μ where the latter vector field has all the same defining constants as the former vector field except that

$$\alpha_1^\mu = \alpha_1 - \mu\alpha_{i_0 1}$$

$$\alpha_2^\mu = \alpha_2 - \mu\alpha_{i_0 2}\,.$$

$$\gamma_{i_0}^\mu = \gamma_{i_0} - \mu$$

It may be assumed that μ is chosen so small that the last point of crossing of any saddle connection with the line $x \equiv \gamma_{i_0}^\mu$ is transversal. Say there exists a value of μ_0 such that the vector field ξ^{μ_0} has a saddle connection whose last point of crossing with the line $x \equiv \gamma_{i_0}^\mu$ is transversal, then the proof can proceed as before but within the set $B(\xi^{\mu_0}, \epsilon - d(\xi^{\mu_0}, \xi))$. If it should happen that for all values of μ the vector fields ξ^μ do not have the required saddle connection then consider $\xi^{\mu_1/2}$. Choose $0 < \epsilon_{\mu_1/2}$ so small that if $\overline{\xi} \in B(\xi^{\mu_1/2}, \epsilon_{\mu_1/2})$ and the vector field $\overline{\xi}$ has a saddle connection then the last point of crossing of that saddle connection with the line $x \equiv \overline{\gamma}_{i_0}$ is transversal. If none of the vector fields in $B(\xi^{\mu_1/2}, \epsilon_{\mu_1/2})$ have the required saddle connection then the proof is completed, otherwise it reduces to the case at the beginning of the proof. ∎

The following conjecture is to type II saddle connections what conjecture 3.3 is to type I saddle connections. Again, the significance and difficulty of proving the following conjecture lies in the type of perturbations allowable for vector fields. Neither conjecture 3.3 nor conjecture 3.9 are anticipated to have simple proofs, although proofs can be given for restricted cases, the general proof for both these conjectures will have to allow for a wide variety of behaviour for the orbit that these conjectures claim to perturb.

Conjecture 3.9. *Let $x_{i_0j_0}$ be a real equilibrium point for which $1 \leq i_0 \leq n$. Let the orbit through $x_{i_0j_0}$ cross the line $x \equiv \gamma_{i_0+1}$ at the point (γ_{i_0+1}, y'). For every $0 < \epsilon$ there exists $\overline{\xi}^1$, $\overline{\xi}^2 \in B(\xi, \epsilon)$ with $\overline{\gamma}_{i_1}^1 = \overline{\gamma}_{i_1}^2 = \gamma_{i_1}$, and $\overline{\xi}_{ij}^1$, $\overline{\xi}_{ij}^2 = \xi_{ij}$ for $i_0 + 1 \leq i$ such that the orbit through the points $\overline{x}_{i_0j_0}^1, \overline{x}_{i_0j_0}^2$ crosses the line $x \equiv \gamma_{i_0+1}$ at $(\gamma_{i_0+1}, \overline{y}^1), (\gamma_{i_0+1}, \overline{y}^2)$ with $\overline{y}^1 < y < \overline{y}^2$.*

Theorem 3.10. *Let $\xi \in G_4$ have a saddle connection Γ of type II joining the saddle points $\mathbf{x}_{i_0j_0}, \mathbf{x}_{i_1j_1}$ where $i_0 \leq i_1$, and crossing the line $x \equiv \gamma_{i_0+1}$. Let*

$$S = \{(\gamma_{i_0+1}, y) : \Gamma \text{ crosses } x \equiv \gamma_{i_0+1} \text{ at the point } (\gamma_{i_0+1}, y)\}.$$

Then S has finitely many elements.

PROOF. Order the elements of S according to their times of crossing of Γ with the line $x \equiv \gamma_{i_0+1}$,

$$S = \{(\gamma_{i_0+1}, y_k) : (\gamma_{i_0+1}, y_{k+1}) = \phi((\gamma_{i_0+1}, y_k), t_k),$$
$$t_k = \min\{0 < t : \phi((\gamma_{i_0+1}, y_k), t) \text{ crosses } x \equiv \gamma_{i_0+1}\}\}.$$

Thus the y-ordinates $y_1, \ldots, y_k, \ldots$ give the successive points $(\gamma_{i_0+1}, y_1), \ldots, (\gamma_{i_0+1}, y_k), \ldots$ of crossing of Γ with $x \equiv \gamma_{i_0+1}$.

Assume S has infinitely many elements. If $\lim_{k\to\infty} |y_k| = \infty$ then the points (γ_{i_0+1}, y_k) are unbounded as a set in $\Re^2$. As the point $\mathbf{x}_{i_0j_0}$ is nontransitional there exists $0 < \epsilon_1$ such that $B(\mathbf{x}_{i_0j_0}, \epsilon_1) \cap \{x \equiv \gamma_{i_0+1}\} = \{\}$. The point $\mathbf{x}_{i_1j_1}$ is nontransitional, there exists $0 < \epsilon_2$ such that $B(\mathbf{x}_{i_1j_1}) \cap \{x \equiv \gamma_{i_0+1}\} = \{\}$. Furthermore, there exists $t_1 < t_2$ such that if $t < t_1$ then $\phi((\gamma_{i_1}, y_1), t) \in B(\mathbf{x}_{i_0j_0}, \epsilon_1)$ and if $t_2 < t$ then $\phi((\gamma_{i_1}, y_1), t) \in B(\mathbf{x}_{i_1j_1}, \epsilon_2)$. Consider $\phi((\gamma_{i_0+1}, y_1), t)$ for $t \in [t_1, t_2]$. For these values of t, $\|\phi'((\gamma_{i_0+1}, y_1), t)\|$ attains a maximum M for $t \in [t_1, t_2]$. Thus, for any two values $t', t'' \in [t_1, t_2]$ it happens that $\|\phi((\gamma_{i_0+1}, y_1), t') - \phi((\gamma_{i_0+1}, y_1), t'')\| \leq M|t' - t''| \leq M(t_2 - t_1)$. Thus, $|y_1 - y_k| = \|(\gamma_{i_0+1}, y_1) - (\gamma_{i_0+1}, y_k)\| \leq M(t_2 - t_1)$ from which it follows that $|y_k| \leq M(t_2 - t_1) + |y_1|$. Then $\lim_{k\to\infty} |y_k| < \infty$ contradicting $\lim_{k\to\infty} |y_k| = \infty$. Thus it may be assumed that $\lim_{k\to\infty} |y_k| < \infty$.

By the Bolzano-Weierstrauss theorem, the set $\{y_k : 1 \leq k < \infty\}$ has a convergent subsequence $\{y_{k_l} : 1 \leq l < \infty\}$ with limit y'. Consider the point (γ_{i_0+1}, y'), as ξ does not have any nontransitional equilibrium points then $\xi(\gamma_{i_0+1}, y') \neq 0$. Thus there exists $0 < \epsilon_3, \epsilon_4$ such that the flow in $[\gamma_{i_0+1} - \epsilon_3, \gamma_{i_0+1} + \epsilon_3] \times [y' - \epsilon_4, y' + \epsilon_4] = V$ conjugate to a a linear flow. Let $1 \leq l_0$ be such that if $l_0 \leq l$ then $(\gamma_{i_0+1}, y_{k_l}) \in [\gamma_{i_0+1} - \epsilon_3/2, \gamma_{i_0+1} + \epsilon_3/2] \times [y' - \epsilon_4/2, y' + \epsilon_4/2] = U$. Consider the orbit of a point $(\gamma_{i_0+1}, y_{k_l})$ for $l_0 \leq l$. As the flow in the regions U, V are conjugate to linear flows then the orbit of the point $(\gamma_{i_0+1}, y_{k_l})$ can only join with the point $(\gamma_{i_0+1}, y_{k_{l+1}})$ by exiting U, exiting V, re-entering V and re-entering U. As $U \subset V$ there exists a finite time $0 < \tau$ bounding from below the amount of time for the orbit through $(\gamma_{i_0+1}, y_{k_l})$ to travel from ∂S to ∂R. Thus, if $(\gamma_{i_0+1}, y_{k_{l+1}}) = \phi((\gamma_{i_0+1}, y_{k_l}), \bar{t}_{k_l})$ then $2\tau \leq \bar{t}_{k_l}$. Inductively it follows that $(\gamma_{i_0+1}, y_{k_l}) = \phi((\gamma_{i_0+1}, y_{k_{l_0}}), t_{k_l})$ where $2(l - l_0)\tau \leq t_{k_l}$. As $l \to \infty$ then $(\gamma_{i_0+1}, y_{k_l}) \in B(\mathbf{x}_{i_1j_1}, \epsilon_2)$ in contradiction to $(\gamma_{i_0+1}, y_{k_l})$ being on the line $x \equiv \gamma_{i_0+1}$. Thus S has finitely many elements. ∎

Theorem 3.11. *Let $\xi \in G_4$ have a saddle connection Γ of type II joining the saddle points $\mathbf{x}_{i_0j_0}, \mathbf{x}_{i_1j_1}$ where $i_0 \leq i_1$, along $W_1^u(\mathbf{x}_{i_0j_0})$ and $W_1^s(\mathbf{x}_{i_1j_1})$ respectively and crossing the line $x \equiv \gamma_{i_0+1}$. For every $0 < \epsilon$ there exists ξ' and $0 < \epsilon'$ such that if $\bar{\xi} \in B(\xi', \epsilon') \subseteq B(\xi, \epsilon)$ then the vector field $\bar{\xi}$ does not have a saddle connection joining the saddle points $\bar{\mathbf{x}}_{i_0j_0}, \bar{\mathbf{x}}_{i_1j_1}$ along $W_1^u(\bar{\mathbf{x}}_{i_0j_0})$ and $W_1^s(\bar{\mathbf{x}}_{i_1j_1})$ respectively.*

PROOF. (Figures 11,12,13.) Let

$$S = \{(\gamma_{i_0+1}, y_k) : \Gamma \text{ crosses } x \equiv \gamma_{i_0+1} \text{ at the point } (\gamma_{i_0+1}, y_k)\}.$$

Consider the set $\Re^2 \cup \{\infty\}$ as the one point compactification of $\Re^2$. In this set $\{x \equiv \gamma_{i_0+1}\} \cup \{\infty\}$ is a closed curve passing through the point ∞. The points $\mathbf{x}_{i_0 j_0}, \mathbf{x}_{i_1 j_1}$ lie in different components of $\Re^2 \cup \{\infty\} - (\{x \equiv \gamma_{i_0+1}\} \cup \{\infty\})$. As Γ is a curve joining $\mathbf{x}_{i_0 j_0}, \mathbf{x}_{i_1 j_1}$, if it crosses the line $x \equiv \gamma_{i_0+1}$ a finite number of times then by the Jordan curve theorem, this number of crossings is odd. Thus $S = \{(\gamma_{i_0+1}, y_k) : 1 \leq k \leq 2p+1\}$ for some $1 \leq p$. Without loss of generality it may be assumed that $(\gamma_{i_0+1}, y_{2p+1})$ is a point of transversal intersection of Γ and the line $x \equiv \gamma_{i_0+1}$. The end of the proof will consider the case if this should not happen to be the siutation.

By conjecture 3.9, there exists a vector field $\xi^1 \in B(\xi, \epsilon)$ such that the orbit through (γ_{i_0+1}, y_1) intersects the line $x \equiv \gamma_{i_0+1}$ at the point $(\gamma_{i_0+1}, y^1_{2p+1})$ where $y^1_{2p+1} < y_{2p+1}$. Furthermore, y^1_{2p+1} can be chosen so close to y_{2p+1} that for $y \in [y^1_{2p+1}, y_{2p+1}]$ the vector $\xi^1(\gamma_{i_0+1}, y)$ is transversal to the line $x \equiv \gamma_{i_0+1}$.

As $(\gamma_{i_0+1}, y^1_{2p+1})$ is formed by the transversal intersection of $W^u_1(\mathbf{x}_{i_0 j_0})$ and $x \equiv \gamma_{i_0+1}$, for sufficiently small $0 < \epsilon^1_1$ the vector field $\overline{\xi} \in B(\xi^1, \epsilon^1_1)$ also has transversal intersection of $W^u_1(\overline{\mathbf{x}}_{i_0 j_0})$ and $x \equiv \overline{\gamma}_{i_0+1}$ at the point $(\overline{\gamma}_{i_0+1}, \overline{y}_{2p+1})$ where $\overline{y}_{2p+1}$ is close to y^1_{2p+1}. As $(\gamma_{i_0+1}, y_{2p+1})$ is formed by the intersection of $W^s_1(\mathbf{x}_{i_1 j_1})$ and $x \equiv \gamma_{i_0+1}$, for sufficiently small $0 < \epsilon^2_1$ the vector field $\overline{\xi} \in B(\xi^1, \epsilon^2_1)$ has transversal intersection of $W^s_1(\mathbf{x}_{i_1 j_1})$ and $x \equiv \overline{\gamma}_{i_0+1}$ at $(\overline{\gamma}_{i_0+1}, \overline{y}_*)$ where $\overline{y}_*$ is close to y_{2p+1}.

Choose $0 < \epsilon_1 < \min\{\epsilon^1_1, \epsilon^2_1, \epsilon - d(\xi^1, \xi)\}$ so small that for $\overline{\xi} \in B(\xi^1, \epsilon_1) \subseteq B(\xi, \epsilon), \overline{y}_{2p+1} < \overline{y}_*$ and the vector $\overline{\xi}(\overline{\gamma}_{i_0+1}, y)$ is transversal to $x \equiv \gamma_{i_0+1}$ for $y \in [\overline{y}_{2p+1}, \overline{y}_*]$. Consider the set $B(\xi^1, \epsilon_1)$. If it happens that for every $\overline{\xi} \in B(\xi^1, \epsilon_1)$ the vector field $\overline{\xi}$ does not have a saddle connection joining the saddle points $\overline{\mathbf{x}}_{i_0 j_0}, \overline{\mathbf{x}}_{i_1 j_1}$ along $W^u_1(\overline{\mathbf{x}}_{i_0 j_0}), W^s_1(\overline{\mathbf{x}}_{i_1 j_1})$ then the theorem is true with $\xi' = \xi^1$ and $\epsilon' = \epsilon$.

Let $\xi^2 \in B(\xi^1, \epsilon_1)$ have a saddle connection Γ^2 joining the saddle points $\mathbf{x}^2_{i_0 j_0}, \mathbf{x}^2_{i_1 j_1}$ along $W^u_1(\mathbf{x}^2_{i_0 j_0})$ and $W^s_1(\mathbf{x}^2_{i_1 j_1})$ respectively. Let

$$S^2 = \{(\gamma^2_{i_0+1}, y_k) : \Gamma^2 \text{ crosses } x \equiv \gamma^2_{i_0+1} \text{ at the point } (\gamma^2_{i_0+1}, y_k)\}.$$

As noted before, $S^2 = \{(\gamma^2_{i_0+1}, y_k) : 1 \leq k \leq 2q+1\}$ where $p < q$. Observe that $(\gamma^2_{i_0+1}, y^2_*) \in S^2$ so that $y^2_* = y^2_l$ for some $2p+1 \leq l \leq 2q+1$. Since there is a saddle connection joining $\mathbf{x}^2_{i_0 j_0}$ and $\mathbf{x}^2_{i_1 j_1}$ the orbit through $(\gamma^2_{i_0+1}, y^2_{2p+1})$ meets $x \equiv \gamma^2_{i_0+1}$ at $(\gamma^2_{i_0+1}, y^2_*) = (\gamma^2_{i_0+1}, y^2_l)$. Let $\epsilon_2 = \epsilon - d(\xi, \xi)$ and consider the set $B(\xi^2, \epsilon_2) \subseteq B(\xi^1, \epsilon_1)$.

Say it happens that $l = 2q+1$. By a second application of conjecture 3.9, there exists $\xi^3 \in B(\xi^2, \epsilon_2)$ with $y^2_{2q+1} < y^3_{2q+1}$ and $\xi^3(\gamma^2_{i_0+1}, y)$ is transversal to $x \equiv \gamma^2_{i_0+1}$ for $y \in [y^2_{2q+1}, y^3_{2q+1}]$. Thus $\xi^3(\gamma^2_{i_0+1}, y)$ is transversal to $x \equiv \gamma^2_{i_0+1}$ for $y \in [y^3_{2p+1}, y^3_{2q+1}]$. Let Γ^3 be the portion of the orbit that starts at $(\gamma^2_{i_0+1}, y^3_{2p+1})$ and ending at $(\gamma^2_{i_0+1}, y^3_{2q+1})$ Let $U = \Gamma^3 \cup (\{\gamma^2_{i_0+1}\} \times [y^3_{2p+1}, y^3_{2q+1}])$. There are two possibilities as to whether $\mathbf{x}^2_{i_1 j_1}$ lies in the interior or exterior of the region bounded by U.

Consider the case that $\mathbf{x}^3_{i_1 j_1}$ lies in the exterior of the region bounded by U. Note it happens that $\lim_{t \to -\infty} \phi^3((\gamma^2_{i_0+1}, y^2_{2q+1}), t) \in U$ while $\lim_{t \to \infty} \phi^3((\gamma^2_{i_0+1}, y^3_{2q+1}), t) \notin U$. It thus follows that the orbit through $(\gamma^2_{i_0+1}, y^2_{2q+1})$ cannot join with $(\gamma^2_{i_0+1}, y^3_{2q+1})$ to form a saddle connection between $W^u_1(\mathrm{x}^3_{i_0j_0})$ and $W^s_1(\mathrm{x}^2_{i_1j_1})$.

In the case that $\mathrm{x}^2_{i_1j_1}$ lies in the interior of the region bounded by U it is possible to observe that $\lim_{t\to-\infty}\phi^3((\gamma^2_{i_0+1}, y^2_{2q+1}), t) \notin U$ while $\lim_{t\to\infty}\phi^3((\gamma^2_{i_0+1}, y^3_{2q+1}), t) \in U$. The orbit through $(\gamma^2_{i_0+1}, y^2_{2q+1})$ cannot join with $(\gamma^2_{i_0+1}, y^3_{2q+1})$ to form a saddle connection between $W^u_1(\mathrm{x}^3_{i_0j_0})$ and $W^s_1(\mathrm{x}^2_{i_1j_1})$.

As the boundary of U is preserved under small perturbations in the defining constants of ξ^3, there exists $0 < \epsilon_3$ such that of $\overline{\xi} \in B(\xi^3, \epsilon_3)$ then the vector field ξ^3 does not have a saddle connection joining $W^u_1(\overline{\mathrm{x}}_{i_0j_0})$ and $W^s_1(\overline{\mathrm{x}}_{i_1j_1})$. The theorem is then true for $\xi' = \xi^3$ and $\epsilon' = \min\{\epsilon_3, \epsilon - d(\xi^3, \xi^2)\}$.

Next, consider the possibility that $l < 2q + 1$. Consider the point $(\gamma^2_{i_0+1}, y^2_{2q+1})$ and the orbit through this point. Define the set

$$T^2 = \{(\gamma^2_{i_0+1}, y_k) : \phi^2((\gamma^2_{i_0+1}, y_{2q+1}), t_k) \cap \{x \equiv \gamma^2_{i_0+1}\} = (\gamma^2_{i_0+1}, y_k), 2q+1 < k, 0 < t_k\}.$$

Note that if $(\gamma^2_{i_0+1}, y_k) \in T^2$ then $(\gamma^2_{i_0+1}, y_k) \notin S$ else $(\gamma^2_{i_0+1}, y_k)$ would not be the last point of transversal intersection of Γ^2 with $x \equiv \gamma^2_{i_0+1}$. Hence the orbit through the point $(\gamma^2_{i_0+1}, y_k)$ is tangent to the line $x \equiv \gamma^2_{i_0+1}$.

If $T^2 = \{\}$ then $(\gamma^2_{i_0+1}, y^2_{2q+1})$ is the last point for which $\phi^2((\gamma^2_{i_0+1}, y^2_{2q+1}), t)$ meets $x \equiv \gamma^2_{i_0+1}$ for $0 \le t$. As $\mathrm{x}^2_{i_0j_0}$ is a nontransitional equilibrium point there exists $0 < \epsilon_{2q+1}$ such that $B(\mathrm{x}^2_{i_0j_0}, \epsilon_{2q+1}) \cap \{x \equiv \gamma^2_{i_0+1}\} = \{\}$. It may be taken the ϵ_{2q+1} is so small that for $\overline{\xi} \in B(\xi^2, \epsilon_2)$ it happens that $B(\overline{\mathrm{x}}^2_{i_0j_0}, \epsilon_{2q+1}) \cap \{x \equiv \overline{\gamma}^2_{i_0+1}\} = \{\}$. There exist $0 < t_{2q+1}$ such that if $t_{2q+1} < t$ then $\phi^2((\gamma^2_{i_0+1}, y^2_{2q+1}), t) \in B(\mathrm{x}^2_{i_0j_0}, \epsilon_{2q+1})$. Let Γ^2_* be the orbit $\phi^2((\gamma^2_{i_0+1}, y^2_{2q+1}), t)$ for times $0 \le t \le t_{2q+1}$. Note that Γ^2_* is a compact set for which $\Gamma^2_* \cap \{x \equiv \gamma^2_{i_0+1}\} = (\gamma^2_{i_0+1}, y^2_{2q+1})$. By restricting $0 < \epsilon_2$ it is possible to ensure that for $\overline{\xi} \in B(\xi^2, \epsilon_2)$ it happens that $x \equiv \overline{\gamma}^2_{i_0+1}$ and $\overline{\Gamma}_*$ only have the point $(\overline{\gamma}^2_{i_0+1}, \overline{y}_{2q+1})$ in common. If the vector field $\xi^3 \in B(\xi^2, \epsilon_2)$ has a saddle connection then consider

$$S^3 = \{(\gamma^3_{i_0+1}, y_k) : \Gamma^3 \text{ crosses } x \equiv \gamma^3_{i_0+1} \text{ at the point } (\gamma^3_{i_0+1}, y_k)\}.$$

As before, $S^3 = \{(\gamma^3_{i_0+1}, y_k) : 1 \le k \le 2q' + 1\}$ where $q \le q'$. Consider the point $(\gamma^2_{i_0+1}, y^2_{2q})$ with image $(\gamma^3_{i_0+1}, y^3_l)$ for some $2q + 1 \le l \le 2q' + 1$. If $l < 2q' + 1$ then the orbit Γ^3_* intersects $x \equiv \gamma^3_{i_0+1}$ at a point other that the image of $(\gamma^2_{i_0+1}, y^2_{2q+1})$. This is in contradiction to the construction of $B(\xi^2, \epsilon_2)$. Thus $l = 2q' + 1$.

If $T^2 \neq \{\}$ then consider the perturbation ξ^μ of ξ^2 as the vector field with all the same defining constants as ξ^2 except

$$\alpha^\mu_1 = \alpha^2_1 - \mu\alpha^2_{i_0+11}$$

$$\alpha^\mu_2 = \alpha^2_2 - \mu\alpha^2_{i_0+12}$$

$$\gamma^\mu_{i_0+1} = \gamma^2_{i_0+1} - \mu$$

where $0 < \mu < \mu_1 = \min\{\epsilon_2, \epsilon_2/|\alpha^2_{i_0+11}|, \epsilon_2/|\alpha^2_{i_0+12}|\}$ is so small that Γ^μ_* has only the point $(\gamma^2_{i_0+1}, y^\mu_{2q})$ in common with the line $x \equiv \gamma^2_{i_0+1}$. Thus $T^\mu = \{\}$. If there is a value of μ_0 such that μ_0 has a saddle connection joining $W^u_1(x^{\mu_0}_{i_0j_0})$ and $W^s_1(x^{\mu_0}_{i_1j_1})$ then let $\xi^3 = \xi^{\mu_0}$ and $\epsilon_3 < \epsilon_2 - d(\xi^\mu, \xi^2)$ be so small that for any vector field $\xi \in B(\xi^3, \epsilon_3)$ it happens that $\overline{T} = \{\}$. The proof can then proceed with respect to showing that there is a subset $B(\xi^4, \epsilon_4) \subseteq B(\xi^3, \epsilon_3)$ such that if $\overline{\xi} \in B(\xi^4, \epsilon_4)$ then $\overline{\xi}$ does not have a saddle connection joining $W^u_1(\overline{x}_{i_0j_0})$ and $W^s_1(\overline{x}_{i_1j_1})$. If it happens that for every value of μ the vector field ξ^μ does not have a saddle connection joining $W^u_1(x^\mu_{i_0j_0})$ and $W^s_1(x^\mu_{i_1j_1})$ then let $\xi^3 = \xi^{\mu_1/2}$ and $\epsilon_3 < \epsilon_2 - d(\xi^3, \xi^2)$ be so small that $\overline{T} = \{\}$ for vector fields $\overline{\xi} \in B(\xi^3, \epsilon_3)$. If none of the vector fields in $B(\xi^3, \epsilon_3)$ has the required saddle connection then the proof is finished else the proof can proceed with respect to showing that there is a subset $B(\xi^4, \epsilon_4) \subseteq B(\xi^3, \epsilon_3)$ such that if $\overline{\xi} \in B(\xi^4, \epsilon_4)$ then $\overline{\xi}$ does not have a saddle connection joining $W^u_1(\overline{x}_{i_0j_0})$ and $W^s_1(\overline{x}_{i_1j_1})$.

Consider the case that the last point of crossing of the saddle connection Γ and the line $x \equiv \gamma_{i_0+1}$ should not happen to be a point of transversal intersection. For sufficiently small $0 < \mu < \mu_1 \leq \epsilon$ consider a perturbation of the original vector field ξ to ξ^μ where the latter vector field has all the same defining constants as the former vector field except that

$$\alpha^\mu_1 = \alpha_1 - \mu\alpha_{i_0+11}$$

$$\alpha^\mu_2 = \alpha_2 - \mu\alpha_{i_0+12}\,.$$

$$\gamma^\mu_{i_0+1} = \gamma_{i_0+1} - \mu$$

It may be assumed that μ is chosen so small that the last point of crossing of any saddle connection with the line $x \equiv \gamma^\mu_{i_0+1}$ is transversal. Say there exists a value of μ_0 such that the vector field ξ^{μ_0} has a saddle connection whose last point of crossing with the line $x \equiv \gamma^\mu_{i_0+1}$ is transversal, then the proof can proceed as before but within the set $B(\xi^{\mu_0}, \epsilon - d(\xi^{\mu_0}, \xi))$. If it should happen that for all values of μ the vector fields ξ^μ do not have the required saddle connection then consider $\xi^{\mu_1/2}$. Choose $0 < \epsilon_{\mu_1/2}$ so small that if $\overline{\xi} \in B(\xi^{\mu_1/2}, \epsilon_{\mu_1/2})$ and the vector field $\overline{\xi}$ has a saddle connection then the last point of crossing of that saddle connection with the line $x \equiv \overline{\gamma}_{i_0+1}$ is transversal. If none of the vector fields in $B(\xi^{\mu_1/2}, \epsilon_{\mu_1/2})$ have the required saddle connection then the proof is completed, otherwise it reduces to the case at the beginuing of the proof. ∎

Corollary 3.12. *Let $\xi \in G_4$ and $0 < \epsilon$. There exists ξ' and $0 < \epsilon'$ such that if $\overline{\xi} \in B(\xi', \epsilon') \subseteq B(\xi, \epsilon)$ then $\overline{\xi}$ does not have any saddle connections between distinct saddle points.*

Proof. For a given vector field ξ let $\#(\xi)$ denote the number of saddle connections between distict saddle points that ξ posseses. For a subset $S \subseteq G_4$ define $N(S) = \max\{\#(\xi) : \xi \in S\}$. Note that since $S \subseteq G_4$ and any $\xi \in G_4$ may have at most $(n+1)(m+1)$ real equilibrium points then ξ may have at most $(n+1)(m+1)(2(n+1)(m+1)-1)$ saddle connections between distinct saddle points, thus $N(s) \leq (n+1)(m+1)(2(n+1)(m+1-1)$.

Without loss of generality it may be assumed that $0 < \epsilon$ is so small that $B(\xi,\epsilon) \subseteq G_4$. If $N(B(\xi,\epsilon)) = 0$ then the corollary is true for $\xi' = \xi$ and $\epsilon' = \epsilon$. Thus let $\xi_0^1 \in B(\xi,\epsilon)$ be a vector field for which there is a saddle connection joining the points $\mathbf{x}_{i_0j_0}, \mathbf{x}_{i_1j_1}$ and joining $W^u_{k_1^1}(\mathbf{x}_{i_0j_0}), W^s_{k_2^1}(\mathbf{x}_{i_1j_1})$ with $k_1^1, k_2^1 \in \{1,2\}$.

Consider $B(\xi_0^1, \epsilon - d(\xi_0^1,\xi)) \subseteq B(\xi,\epsilon)$. By either theorem 3.8 or theorem 3.11 there exists ξ^1 and $0 < \epsilon_1$ such that if $\overline{\xi} \in B(\xi^1,\epsilon_1) \subseteq B(\xi_0^1, \epsilon - d(\xi_0^1,\xi)) \subseteq B(\xi,\epsilon)$ then $\overline{\xi}$ does not have a saddle connection through $\overline{\mathbf{x}}_{i_0j_0}, \mathbf{x}_{i_1j_1}$ and joining $W^u_{k_1^1}(\overline{\mathbf{x}}_{i_0j_0}), W^s_{k_2^1}(\overline{\mathbf{x}}_{i_1j_1})$. Thus $N(B(\xi^1,\epsilon_1)) \leq (n+1)(m+1)(2(n+1)(m+1)-1)-1$. If $N(B(\xi^1,\epsilon_1)) = 0$ then the corollary is true with $\xi' = \xi^1$ and $\epsilon' = \epsilon_1$. Otherwise let $\xi_0^2 \in B(\xi^1,\epsilon_1)$ be a vector field with a saddle connection through the points $\mathbf{x}^2_{i_2j_2}, \mathbf{x}^2_{i_3j_3}$ and joining $W^u_{k_1^2}(\mathbf{x}^2_{i_2j_2}), W^s_{k_2^2}(\mathbf{x}^2_{i_3j_3})$.

As before, there is a set $B(\xi^2,\epsilon_2) \subseteq B(\xi_0^2, \epsilon_2 - d(\xi_0^2,\xi^1)) \subseteq B(\xi^1,\epsilon_1) \subseteq B(\xi,\epsilon)$ such that if $\overline{\xi} \in B(\xi^2,\epsilon_2)$ then the vector field $\overline{\xi}$ does not have a saddle connection through $\overline{\mathbf{x}}_{i_0j_0}, \overline{\mathbf{x}}_{i_1j_1}$ joining $W^u_{k_1^1}(\overline{\mathbf{x}}_{i_0j_0}), W^s_{k_2^1}(\overline{\mathbf{x}}_{i_1j_1})$, nor does it have a saddle connection through the points $\overline{\mathbf{x}}_{i_2j_2}, \overline{\mathbf{x}}_{i_3j_3}$ joining $W^u_{k_1^2}(\overline{\mathbf{x}}_{i_2j_2}), W^s_{k_1^2}(\overline{\mathbf{x}}_{i_3j_3})$. Thus, $N(B(\xi^2,\epsilon_2)) \leq (n+1)(m+1)(2(n+1)(m+1)-1)-2$.

Continuing in this manner there is a sequence of sets $B(\xi^k,\epsilon_k) \subseteq \ldots \subseteq B(\xi^1,\epsilon_1) \subseteq B(\xi,\epsilon)$ such that if $\overline{\xi} \in B(\xi^k,\epsilon_k)$ then $\overline{\xi}$ has at most $(n+1)(m+1)(2(n+1)(m+1)-1)-k$ saddle connections. If $N(B(\xi^k,\epsilon_k)) = 0$ then the corollary is true with $\xi' = \xi^k$ and $\epsilon' = \epsilon_k$. Since $0 \leq N(B(\xi^k,\epsilon_k)) \leq (n+1)(m+1)(2(n+1)(m+1)-1)-k$ then the number of terms in the sequence cannot exceed $k = (n+1)(m+1)(2(n+1)(m+1)-1)$. The sequence will thus terminate after a finite number of terms in a set $B(\xi^k,\epsilon_k)$ for which $N(B(\xi^k,\epsilon_k)) = 0$. The corollary is then true for $\xi' = \xi^k$ and $\epsilon' = \epsilon_k$, ∎

The following is the main theorem of this paper as alluded to in the introduction. The proof is an immediate consequence of corollaries 3.5 and 3.12.

Theorem 3.13. *Let $\xi \in G_4$ and $0 < \epsilon$. There exists ξ' and $0 < \epsilon'$ such that if $\overline{\xi} \in B(\xi',\epsilon') \subseteq B(\xi,\epsilon)$ then $\overline{\xi}$ does not have any saddle connections.*

References.

[1] Chua L.O. and Deng A., "Canonical piecewise-linear modeling." *IEEE Transactions on Circuits and Systems.*, vol.33, pp.511-525, May 1986.

[2] Chua L.O. and Deng A., "Canonical piecewise-linear representations." *IEEE Transactions on Circuits and Systems.*, vol.35, pp.101-111, January 1988.

[3] Hirsch M. and Smale S., "Differential equations, dynamical systems and linear algebra." Academic Press, 1974.

[4] Palis J. and de Melo W., "Geometric Theory of Dynamical Systems. " Springer-Verlag, 1982.

[5] Peixoto M., "Structural stability on two-dimensional manifolds." Topology 1, 1962.

Figure captions.

Figure 1. This is the phase portrait corresponding to the vector field given by

$$\xi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} |x+1| - \begin{bmatrix} 2 \\ -3 \end{bmatrix} |x| - \begin{bmatrix} 5 \\ 1 \end{bmatrix} |x-1| - \begin{bmatrix} 2 \\ 0 \end{bmatrix} |y+1| + \begin{bmatrix} 4 \\ 1 \end{bmatrix} |y-2|.$$

Figure 2. The equilibrium point $\mathbf{x}_{ij}$ and the set $\mathbf{A}_{ij}$ can both be perturbed so that the corresponding images $\overline{\mathbf{x}}_{ij}$ and $\overline{\mathbf{A}}_{ij}$ do not intersect, thus preserving the nontransitional nature of the original point $\mathbf{x}_{ij}$ and the line $x \equiv \gamma_i$. This means that the property of being a nontransitional point is persistent under small perturbations of the original vector field.

By limiting the amount of perturbation of the equilibrium point and the set $\mathbf{A}_{ij}$ with constraints on the defining constants of the vector field, the nontransitional nature of an equilibrium point can be preserved.

Figure 3. If it should happen that the point $\mathbf{x}_{i_1 j_1}$ is a transitional equilibrium point then a perturbation of the line $x \equiv \gamma_{i_1}$ to $x \equiv \gamma_{i_1}^{i_1}$ changes the nature of $\mathbf{x}_{i_1 j_1}$ to that of a nontransitional equilibrium point. In contrast to nontransitional points, the transitional nature of a point is not necessarily preserved under perturbation of the original vector field. Being transitional is not a persistent property of equilibrium points. Furthermore, since transitional points can always be perturbed to nontransitional points, the latter property is a denseness property. That is, being a nontransitional point is generic property.

Given a vector field with transitional points, by carefully manipulating the defining constants of the vector field, the formerly transitional equlibruium point can be made nontransitional. By repeating the process to all the equilibrium points results in a vector field that consists only of nontransitional points. It should be noted that in perturbing a point to be nontransitional that care must be exercise not to create new transitional equilibrium points.

Figure 4. In this example, the equilibrium points $\mathbf{x}_{00}, \mathbf{x}_{01}, \mathbf{x}_{10}, \mathbf{x}_{11}$ have transversal intersection of their lines, for sufficiently small perturbations to $\overline{\mathbf{x}}_{00}, \overline{\mathbf{x}}_{01}, \overline{\mathbf{x}}_{10}, \overline{\mathbf{x}}_{11}$ the properly transversal nature of the original vector field is still preserved. A properly transversal vector field is thus preserved under small perturbations of the original vector field. Being properly transversal is a persistent property of a vector field.

By contraints on the original defining constants of the vector field, the properly transversal nature of the vector field is maintained.

Figure 5. The equilibrium point $\mathbf{x}_{ij-1}$ has lines that do not meet transversally with the lines of $\mathbf{x}_{i-1j-1}$ and $\mathbf{x}_{i+1j-1}$. By perturbing the equilibrium point $\mathbf{x}_{ij-1}$ to $\mathbf{x}^1_{ij-1}$, the new equilibrium point has only transversal intersection of its lines with the points $\mathbf{x}_{i-1j-1}$ and $\mathbf{x}_{i+1j-1}$. In contrast to properly transversal vector fields, those which are not properly transversal can always be perturbed to a vector field that is properly transversal. Being properly transversal is thus a generic property.

A pair of lines which intersect along their entire lengths can be pertubed to intersect transversally. By perturbing all pairs of lines which do not intersect transversally to transversally intersecting lines, the original vector field becomes one which is properly transversal. It should be noted that in perturbing a pair of lines to intersect transversally, that care must be taken so that previously transversal intersection of lines are still preserved.

Figure 6. The vector field ξ has a homoclinic orbit at the equilibrium point $\mathbf{x}_{i_0j_0}$ joining the invariant manifolds $W_1^u\mathbf{x}_{i_0j_0}$ and $W_1^s\mathbf{x}_{i_0j_0}$. This is the assumed set-up of the vector field for the proof of the theorem.

Figure 7. Because of the set formed by Γ^1 and portions of the invariant manifolds $W_1^u\mathbf{x}_{i_0j_0}$ and $W_1^s\mathbf{x}_{i_0j_0}$, the vector field ξ^1 does not have a homoclinic orbit joining $W_1^u\mathbf{x}_{i_0j_0}$ and $W_1^s\mathbf{x}_{i_0j_0}$.

By perturbing the vector field so that the homoclinic orbit enters a region that it cannot escape, the orignal homoclinic orbit has been destroyed. This process can be applied repeatedly to the original vector field to create a vector field without any homoclinic orbits. However, in repeated applications of the process, case must be taken that new homoclinic orbits are not created.

Figure 8. The vector field ξ has a type I saddle connection between the saddle points $\mathbf{x}_{i_0j_0}$ and $\mathbf{x}_{i_1j_1}$. This saddle connection joins the invariant manifolds $W_1^u(\mathbf{x}_{i_0j_0})$ and $W_1^s(\mathbf{x}_{i_1j_1})$. This is the assumed set-up of the vector field for the proof of the theorem.

Figure 9. The vector field ξ^2, after perturbation of the vector field ξ still has a saddle connection joining the saddle points $\mathbf{x}^2_{i_0j_0}$ and $\mathbf{x}^2_{i_1j_1}$ along the invariant manifolds $W_1^u(\mathbf{x}^2_{i_0j_0})$ and $W_1^s(\mathbf{x}^2_{i_1j_1})$.

It is possible that after small perturbation of the vector field that a saddle connection still exists, as illustrated by this figure. By perturbing a second time it is possible to remove the possibility of a saddle connection between the original manifolds by forcing the orbit to enter a region from which it cannot escape nor form the connection between the original manifolds. This process can be repeatedly applied to ensure that the ensuing vector field does not have any type I saddle connections. However care must be exercised, that in the process new saddle connections are not formed.

Figure 10. The vector field ξ^3, being a perturbation of the original vector field ξ, does not have a saddle connection joining the saddle points $\mathbf{x}^2_{i_0j_0}$ and $\mathbf{x}^2_{i_1j_1}$ along the invariant manifolds $W_1^u(\mathbf{x}^2_{i_0j_0})$ and $W_1^s(\mathbf{x}^2_{i_1j_1})$. This is because of the set formed by Γ^3 and part of the line $x \equiv \gamma^3_{i_0}$. The orbit cannot form a saddle connection along the given manifolds as it cannot escaped the formed region.

Figure 11. The vector field ξ has a type II saddle connection between the saddle points $\mathbf{x}_{i_0j_0}$ and $\mathbf{x}_{i_1j_1}$. This saddle connection joins the invariant manifolds $W_1^u(\mathbf{x}_{i_0j_0})$ and $W_1^s(\mathbf{x}_{i_1j_1})$. This is the assumed set-up of the vector field for the proof of the theorem.

Figure 12. The vector field ξ^2, after perturbation of the vector field ξ still has a saddle connection joining the saddle points $\mathbf{x}^2_{i_0j_0}$ and $\mathbf{x}^2_{i_1j_1}$ along the invariant manifolds $W_1^u(\mathbf{x}^2_{i_0j_0})$ and $W_1^s(\mathbf{x}^2_{i_1j_1})$.

It is possible that after small perturbation of the vector field that a saddle connection still exists, as illustrated by this figure. This is a situation analagous to that for a type I saddle connection of persistence of a saddle connection under a particular choice of perturbation. By perturbing a second time it is possible to remove the possibility of a saddle connection between the original manifolds by forcing the orbit to enter a region from which it cannot escape nor form the connection between the original manifolds. This process can be repeatedly applied to ensure that the ensuing vector field does not have any type II saddle connections. However care must be exercised, that in the process new saddle connections are not formed.

Figure 13. The vector field ξ^3, being a perturbation of the original vector field ξ, does not have a saddle connection joining the saddle points $\mathbf{x}^3_{i_0j_0}$ and $\mathbf{x}^2_{i_1j_1}$ along the invariant manifolds $W^u_1(\mathbf{x}^3_{i_0j_0})$ and $W^s_1(\mathbf{x}^2_{i_1j_1})$. This is because of the set formed by Γ^3 and part of the line $x \equiv \gamma^3_{i_0}$. Unlike the case in figure 11, the equilibrium point $\mathbf{x}^2_{i_0j_0}$ may be perturbed to $\mathbf{x}^3_{i_0j_0}$ due to side effects of corollary 3.9. The orbit cannot form a saddle connection along the given manifolds as it cannot escaped the formed region.

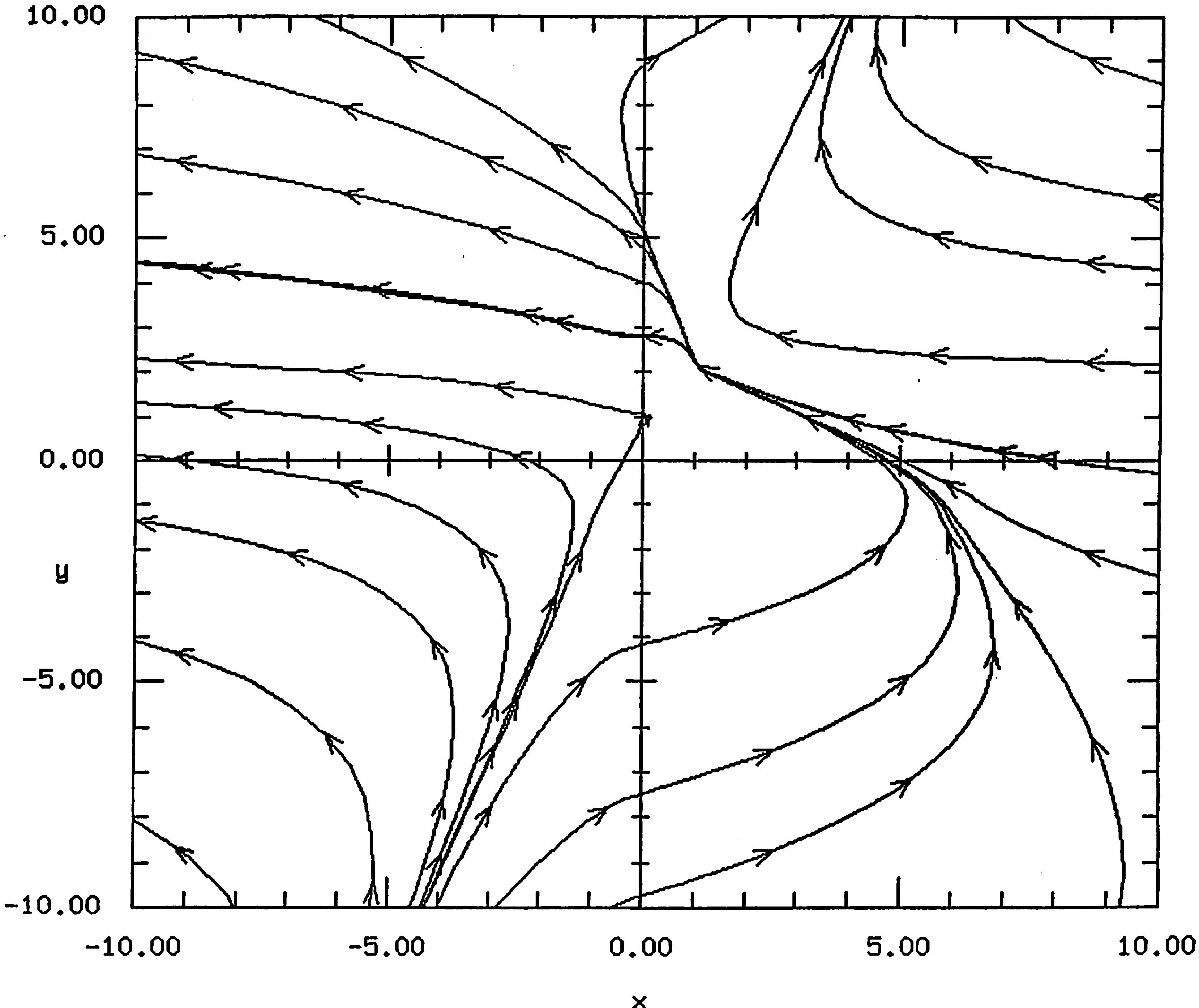
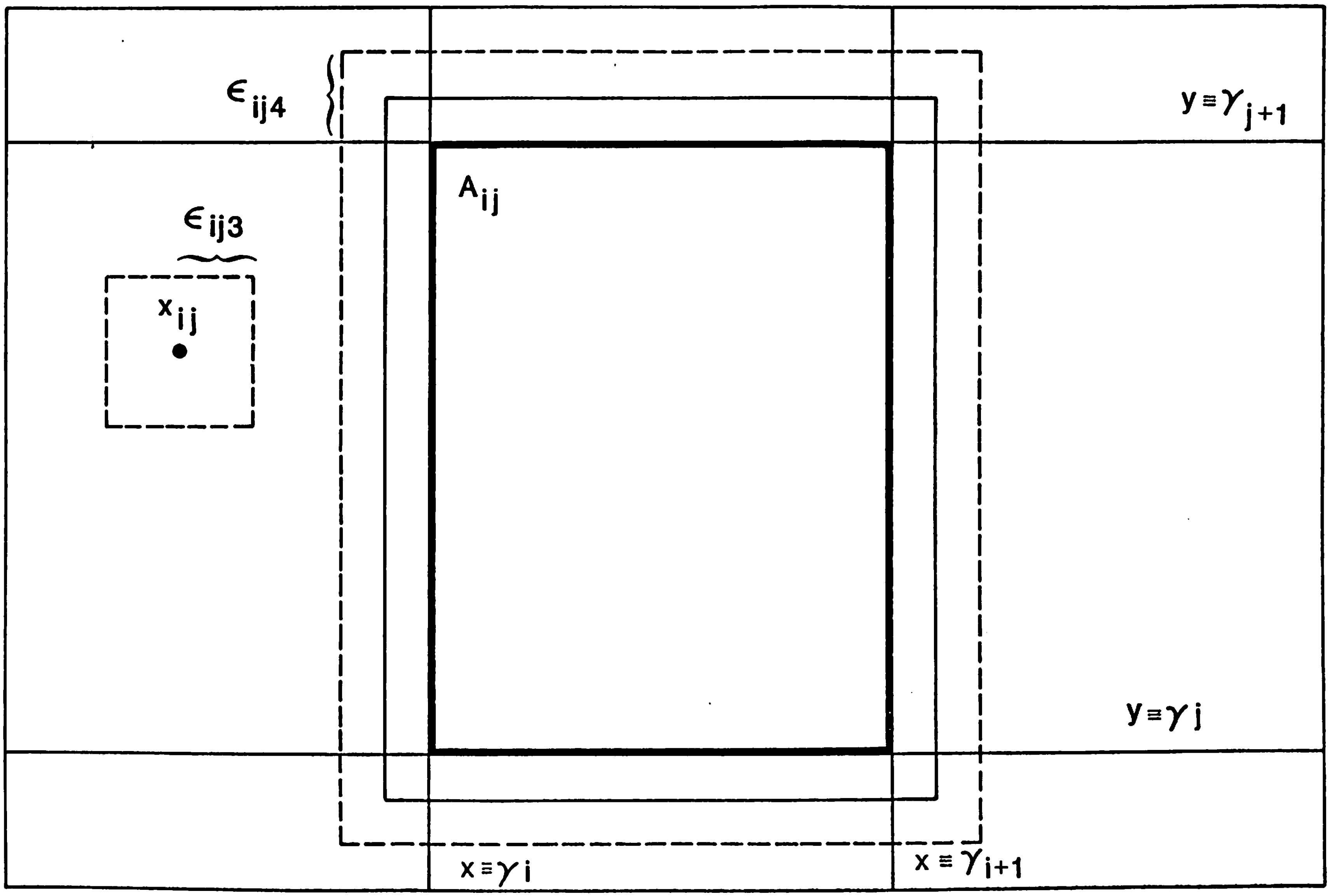


Figure 1.



ϵ_{ij4}
$y \equiv \gamma_{j+1}$
A_{ij}
ϵ_{ij3}
x_{ij}
$y \equiv \gamma_j$
$x \equiv \gamma_i$
$x \equiv \gamma_{i+1}$

Figure 2

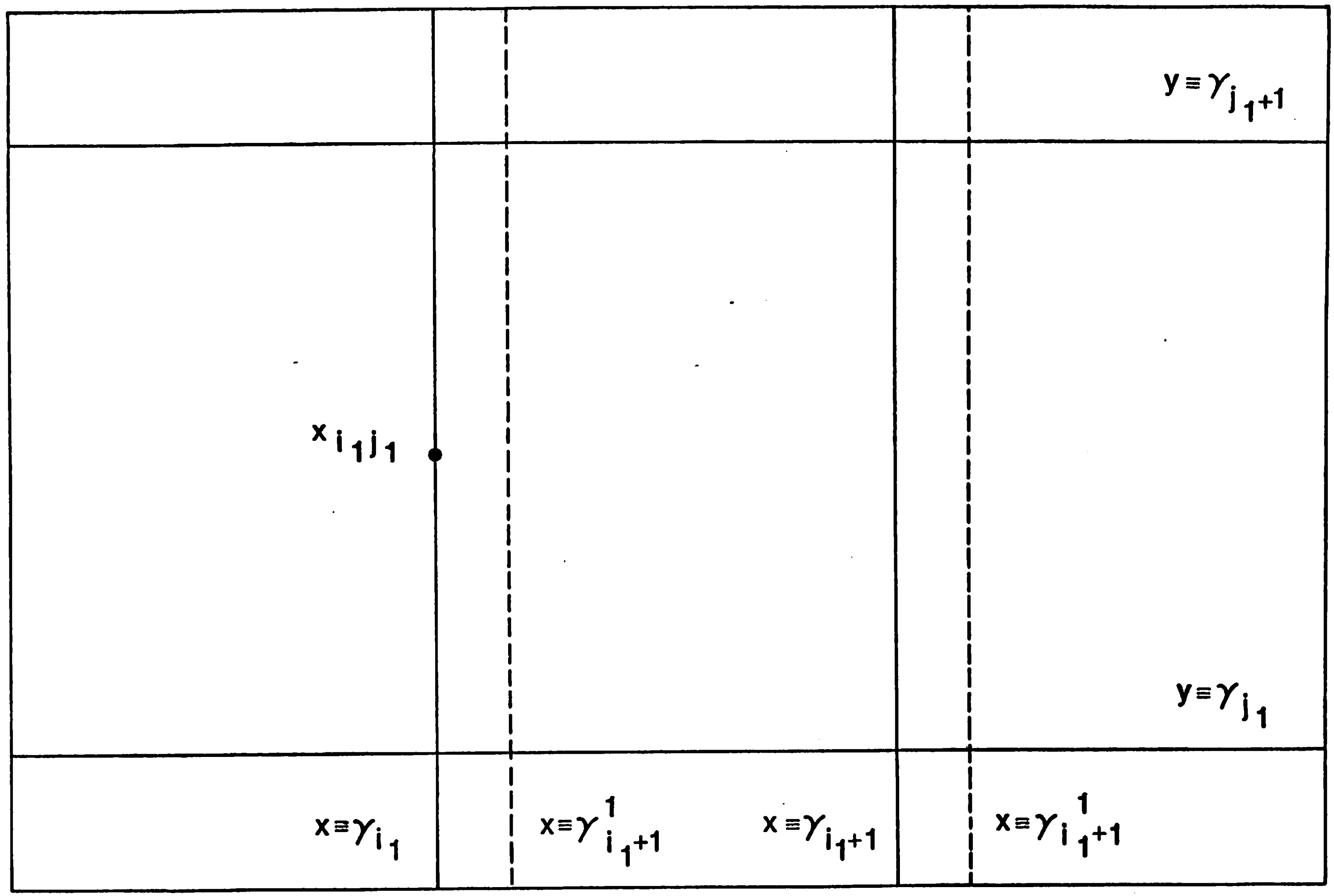
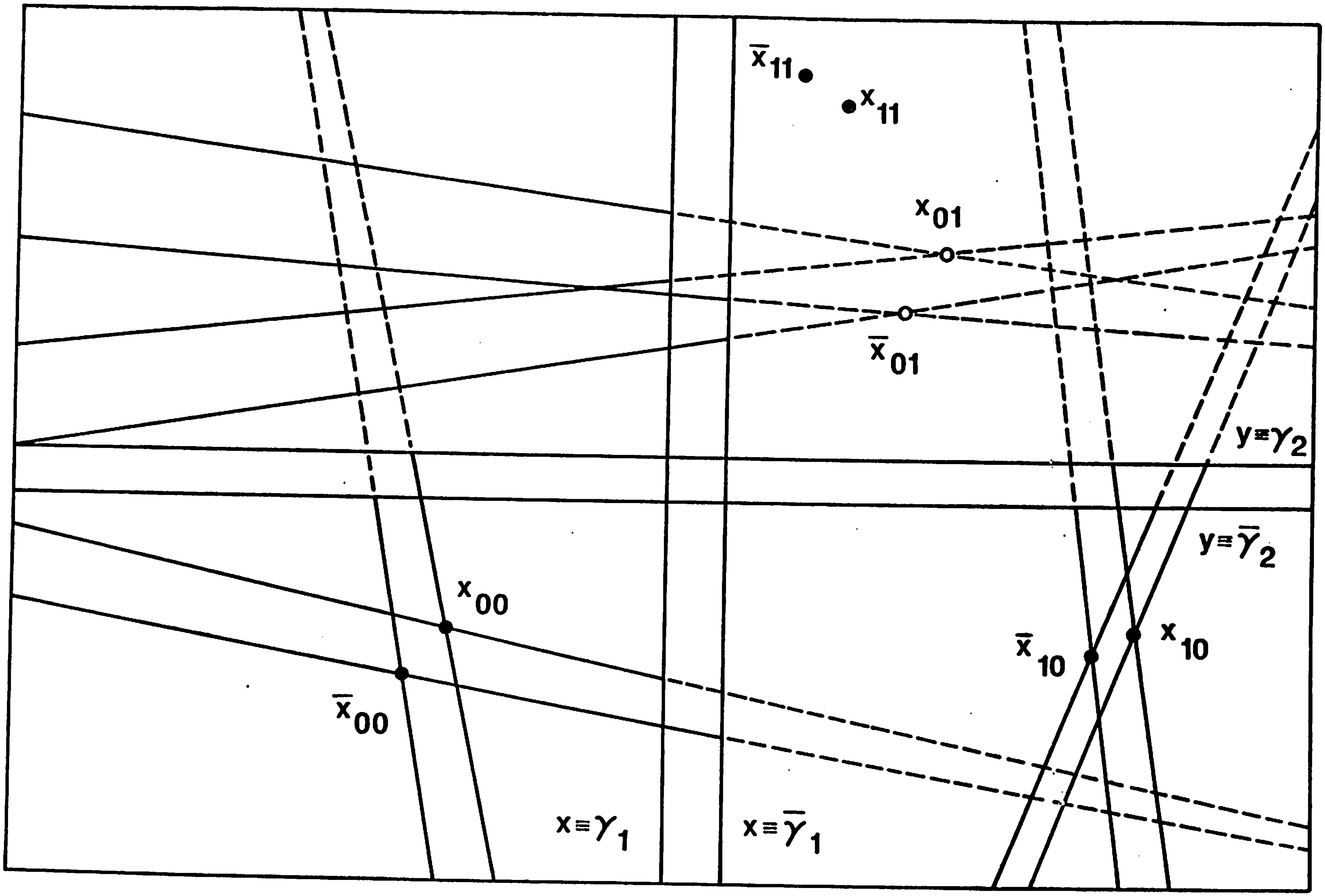
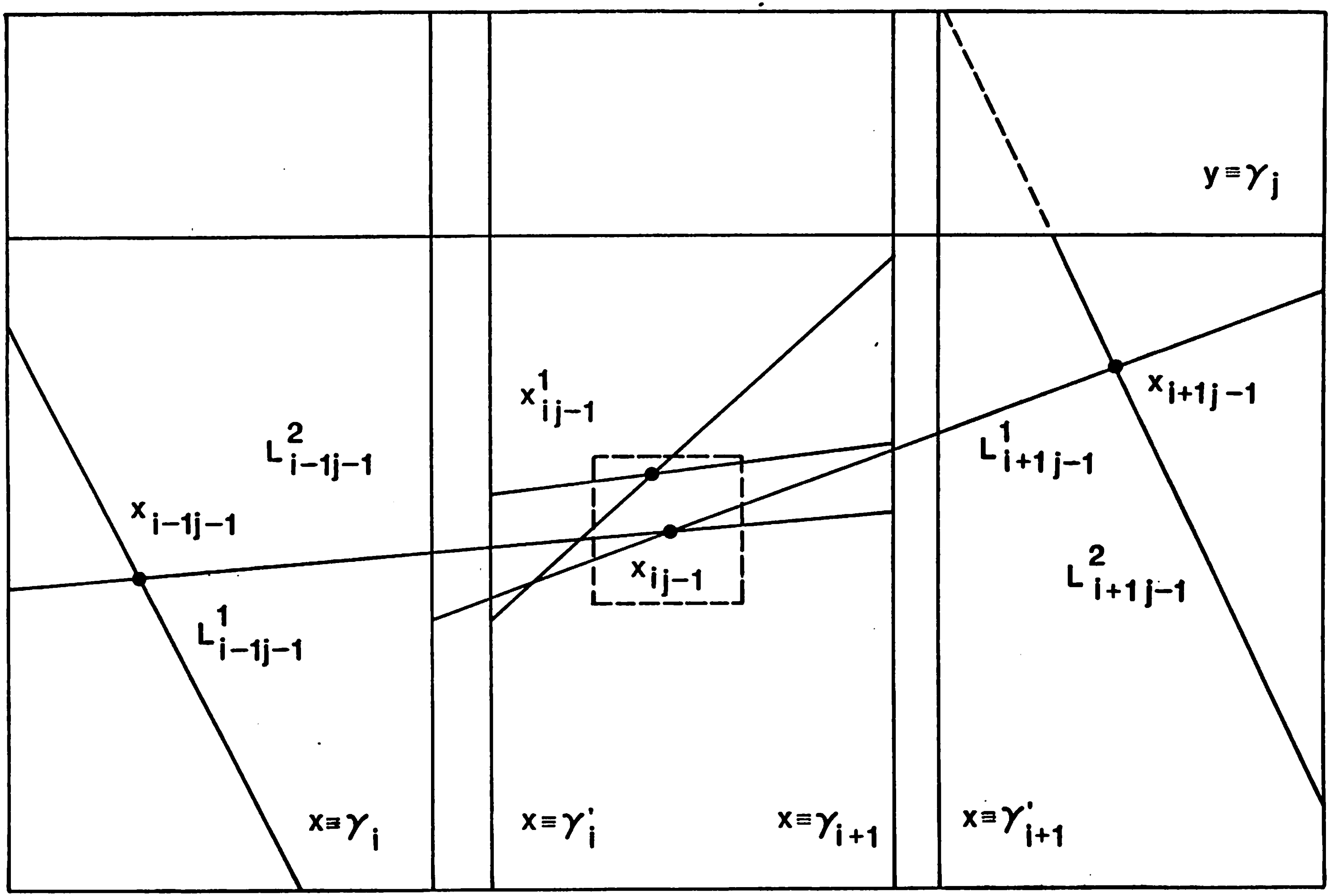

$y \equiv \gamma_{j_1+1}$
$x_{i_1 j_1}$
$y \equiv \gamma_{j_1}$
$x \equiv \gamma_{i_1}$
$x \equiv \gamma^1_{i_1+1}$
$x \equiv \gamma_{i_1+1}$
$x \equiv \gamma^1_{i_1+1}$

Figure 3



$\bar{x}_{11}$
x_{11}
x_{01}
$\bar{x}_{01}$
$y \equiv \gamma_2$
$y \equiv \bar{\gamma}_2$
x_{00}
$\bar{x}_{00}$
$\bar{x}_{10}$
x_{10}
$x \equiv \gamma_1$
$x \equiv \bar{\gamma}_1$

Figure 4



$y = \gamma_j$
x^1_{ij-1}
x_{i+1j-1}
L^2_{i-1j-1}
L^1_{i+1j-1}
x_{i-1j-1}
x_{ij-1}
L^2_{i+1j-1}
L^1_{i-1j-1}
$x = \gamma_i$
$x = \gamma'_i$
$x = \gamma_{i+1}$
$x = \gamma'_{i+1}$

Figure 5

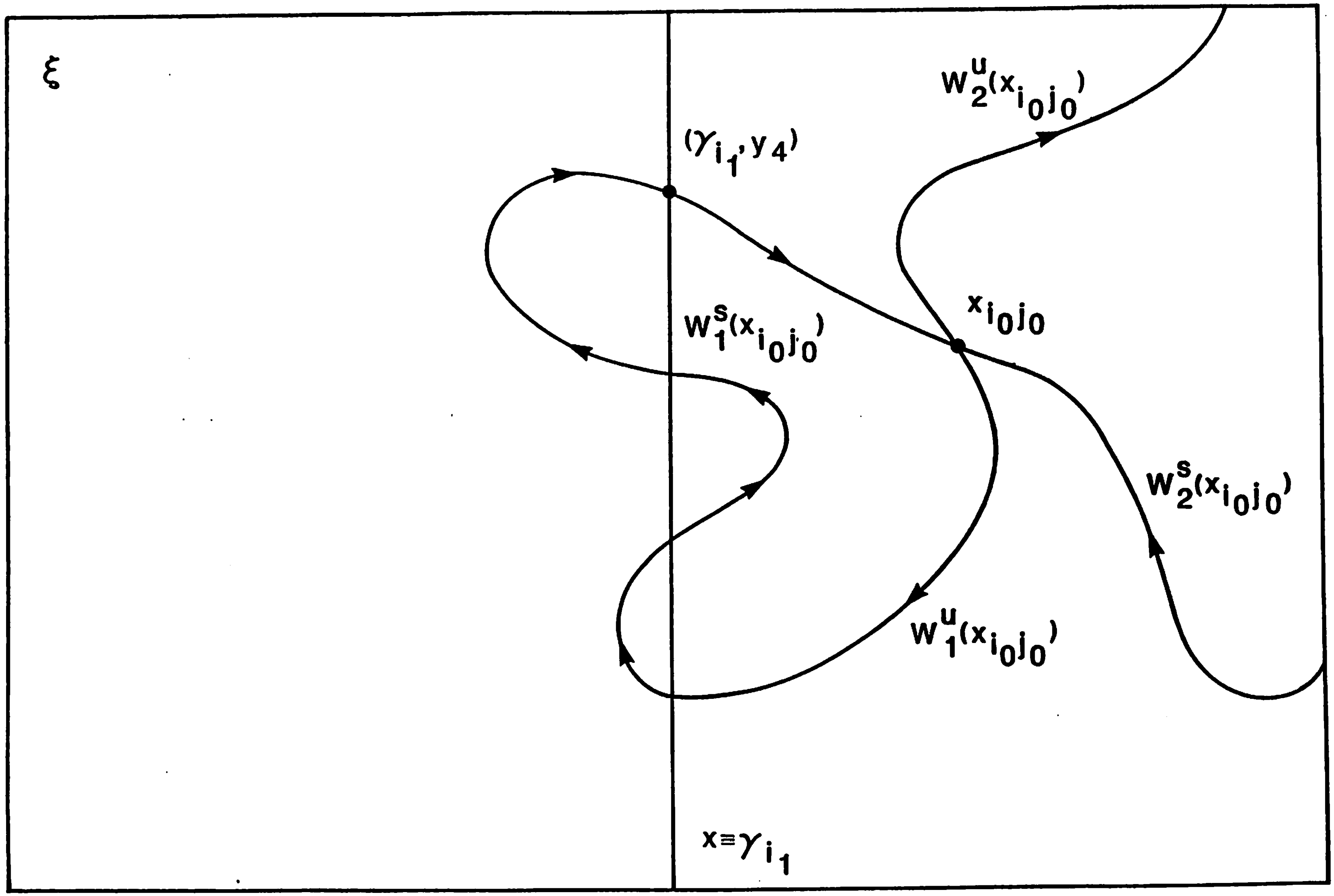

ξ
$W_2^u(x_{i_0j_0})$
(γ_{i_1}, y_4)
$x_{i_0j_0}$
$W_1^s(x_{i_0j_0})$
$W_2^s(x_{i_0j_0})$
$W_1^u(x_{i_0j_0})$
$x \equiv \gamma_{i_1}$

Figure 6

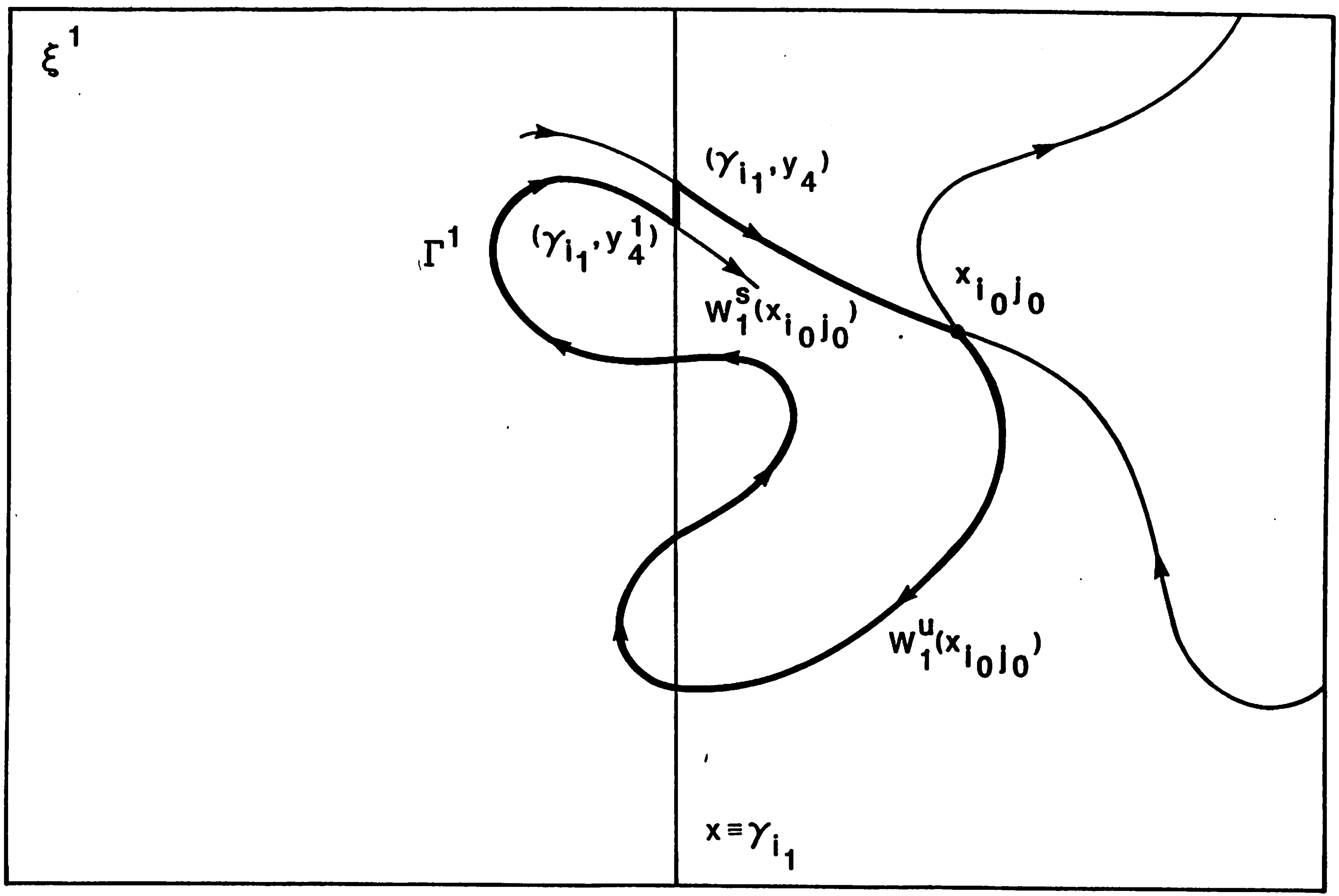

ξ^1
(γ_{i_1}, y_4)
Γ^1
(γ_{i_1}, y_4^1)
$x_{i_0 j_0}$
$W_1^s(x_{i_0 j_0})$
$W_1^u(x_{i_0 j_0})$
$x \equiv \gamma_{i_1}$

Figure 7

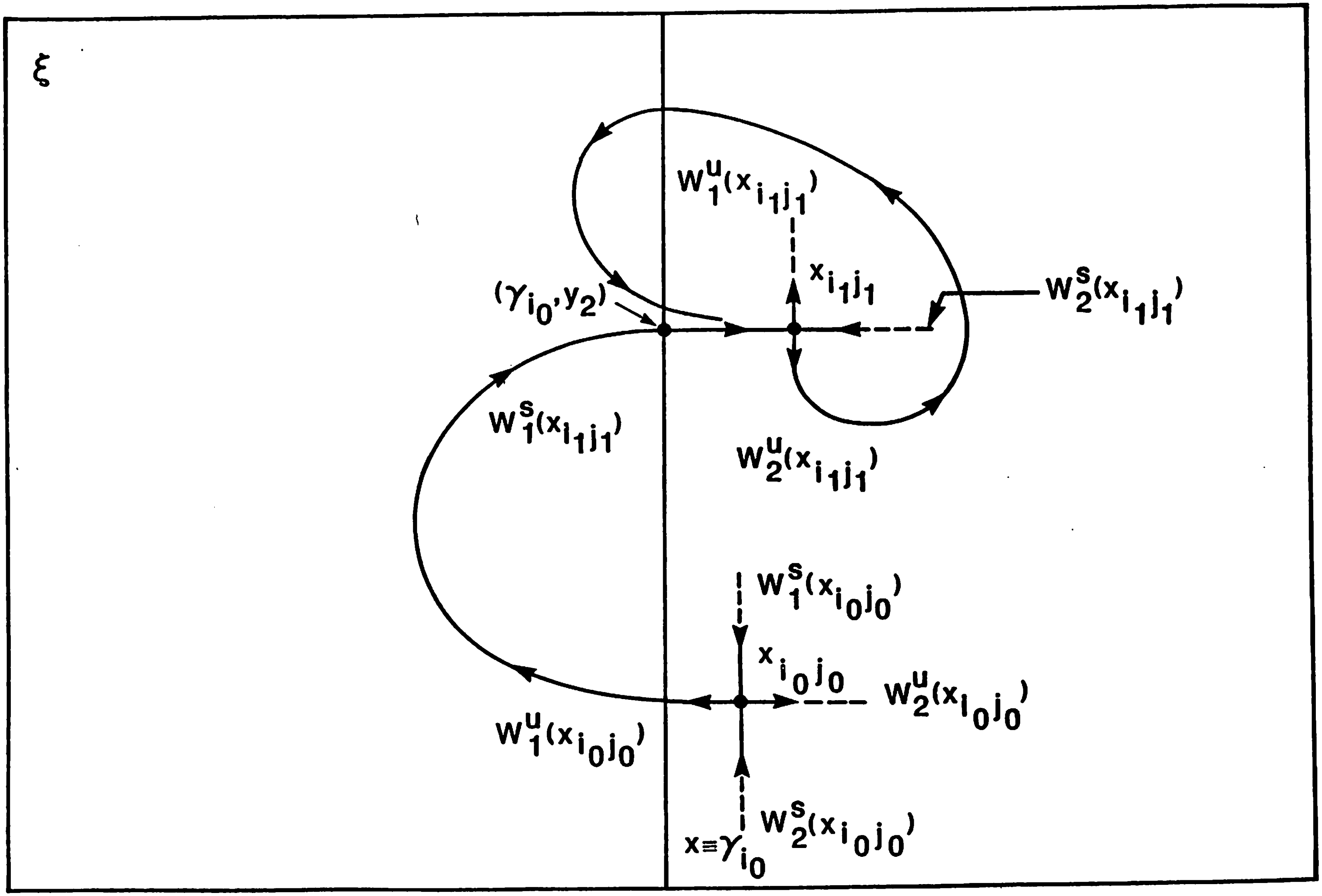
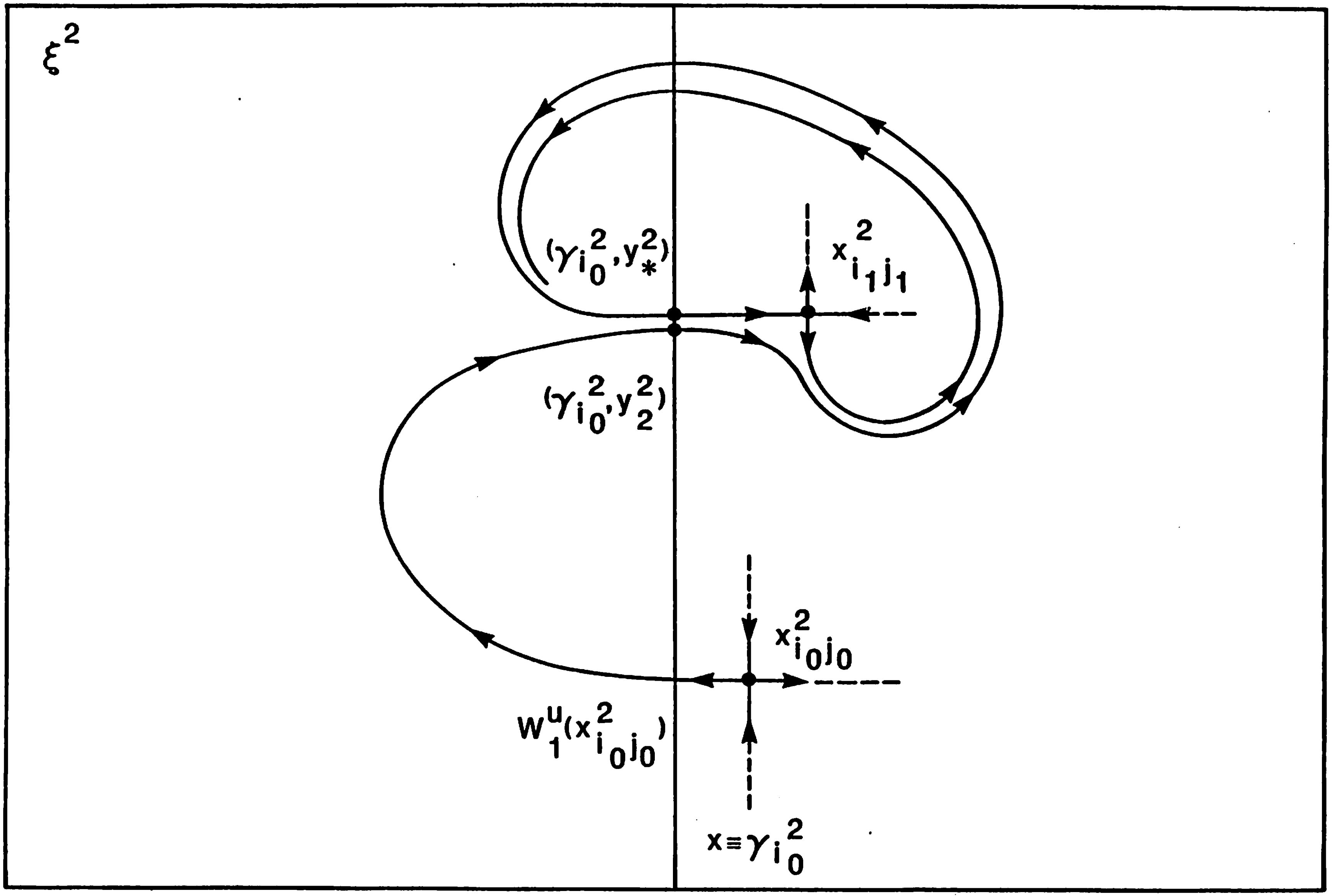
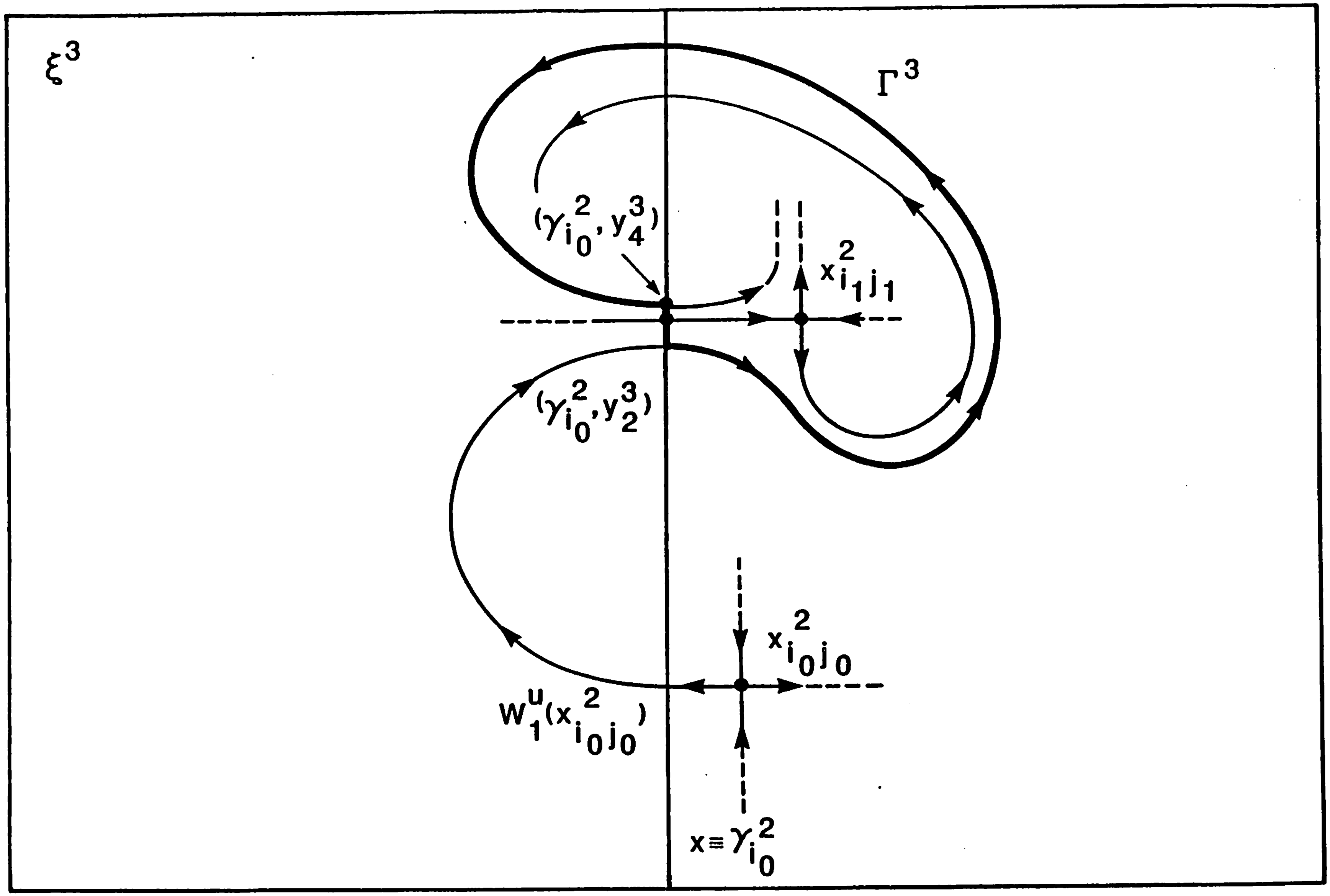

ξ
$W_1^u(x_{i_1j_1})$
$x_{i_1j_1}$
$W_2^s(x_{i_1j_1})$
(γ_{i_0}, y_2)
$W_1^s(x_{i_1j_1})$
$W_2^u(x_{i_1j_1})$
$W_1^s(x_{i_0j_0})$
$x_{i_0j_0}$
$W_2^u(x_{i_0j_0})$
$W_1^u(x_{i_0j_0})$
$W_2^s(x_{i_0j_0})$
$x \equiv \gamma_{i_0}$

Figure 8



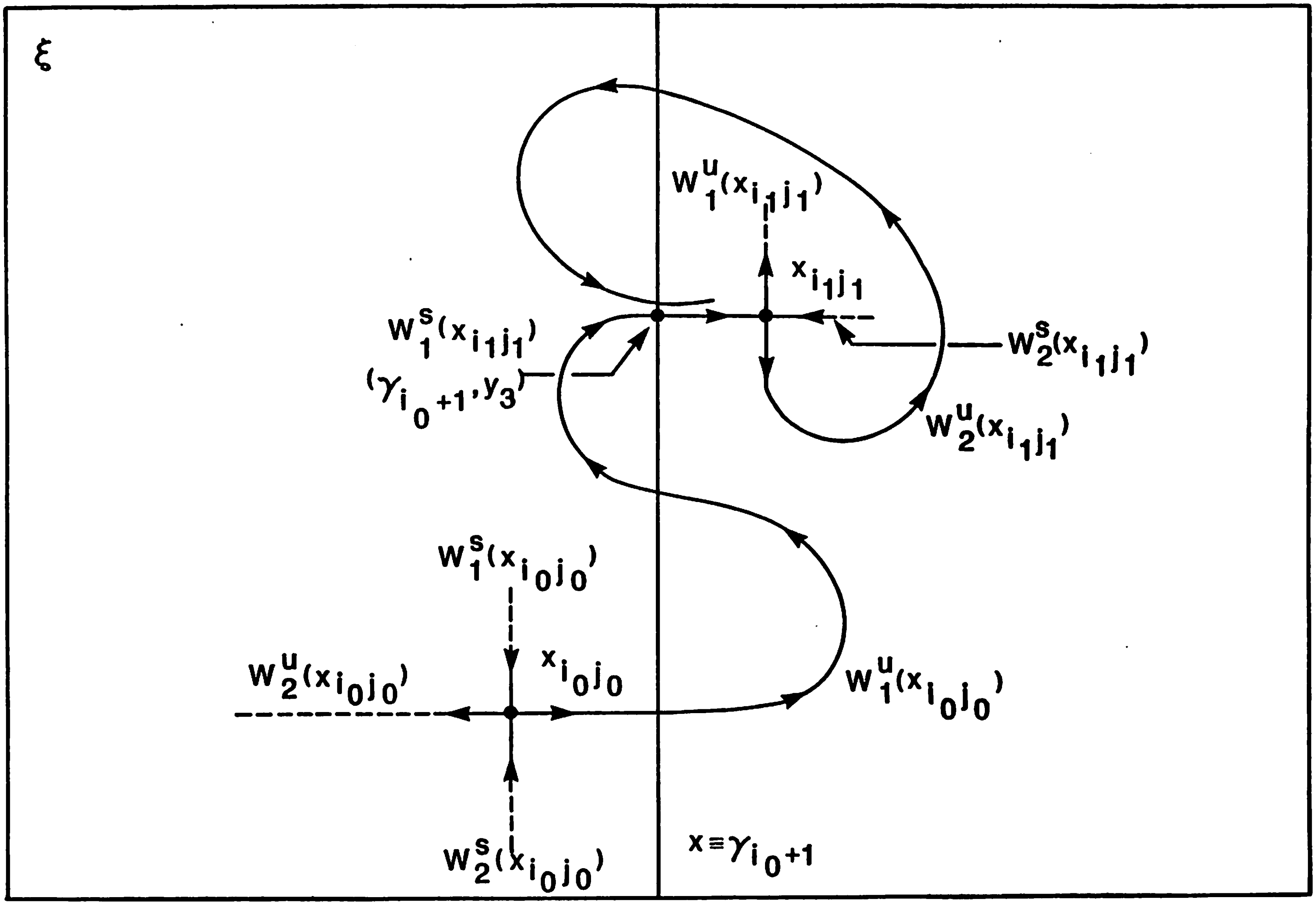
ξ^2
$(\gamma_{i_0}^2, y_*^2)$
$x_{i_1 j_1}^2$
$(\gamma_{i_0}^2, y_2^2)$
$x_{i_0 j_0}^2$
$W_1^u(x_{i_0 j_0}^2)$
$x \equiv \gamma_{i_0}^2$

Figure 9



ξ^3
Γ^3
$(\gamma^2_{i_0}, y^3_4)$
$x^2_{i_1 j_1}$
$(\gamma^2_{i_0}, y^3_2)$
$x^2_{i_0 j_0}$
$W^u_1(x^2_{i_0 j_0})$
$x \equiv \gamma^2_{i_0}$

Figure 10



ξ
$W_1^u(x_{i_1j_1})$
$x_{i_1j_1}$
$W_1^s(x_{i_1j_1})$
(γ_{i_0+1}, y_3)
$W_2^s(x_{i_1j_1})$
$W_2^u(x_{i_1j_1})$
$W_1^s(x_{i_0j_0})$
$x_{i_0j_0}$
$W_2^u(x_{i_0j_0})$
$W_1^u(x_{i_0j_0})$
$W_2^s(x_{i_0j_0})$
$x = \gamma_{i_0+1}$

Figure 11

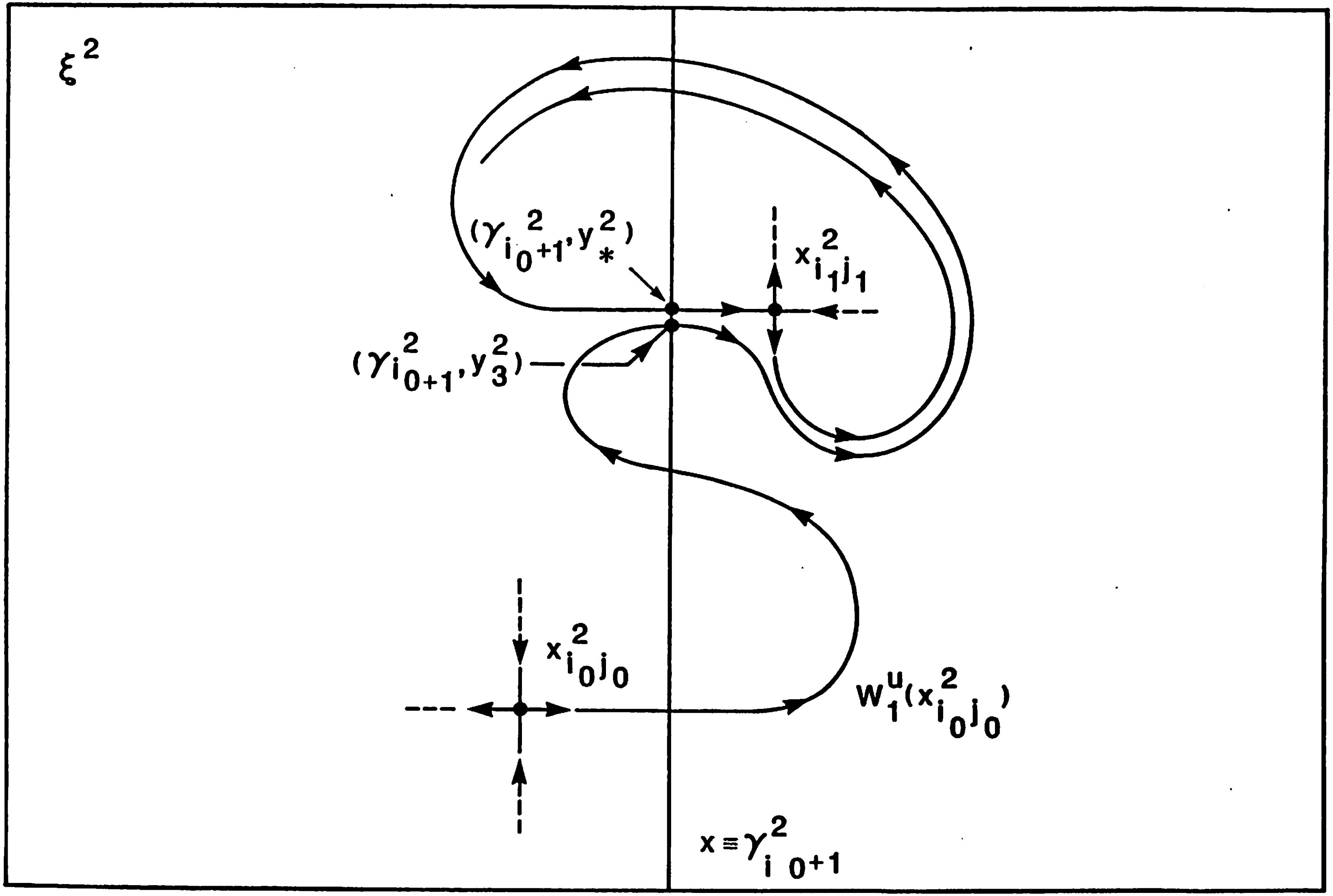
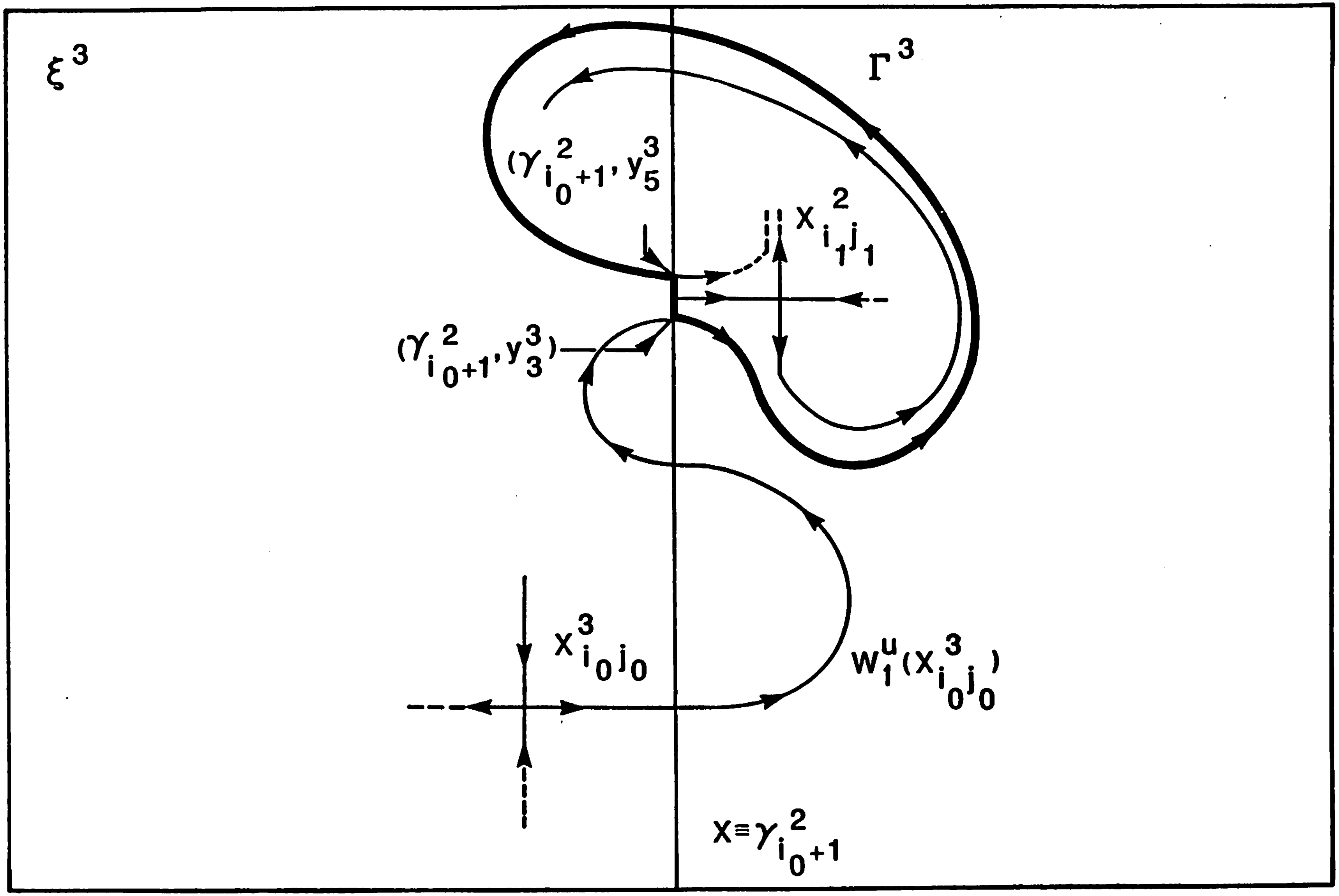

ξ^2
$(\gamma^2_{i_0+1}, y^2_*)$
$x^2_{i_1 j_1}$
$(\gamma^2_{i_0+1}, y^2_3)$
$x^2_{i_0 j_0}$
$W^u_1(x^2_{i_0 j_0})$
$x \equiv \gamma^2_{i\,0+1}$

Figure 12



ξ^3
Γ^3
$(\gamma^2_{i_0+1}, y^3_5$
$x^2_{i_1 j_1}$
$(\gamma^2_{i_0+1}, y^3_3)$
$x^3_{i_0 j_0}$
$W^u_1(x^3_{i_0 j_0})$
$x \equiv \gamma^2_{i_0+1}$

Figure 13