AN ELEMENTARY PROOF OF THE ROUTH-HURWITZ STABILITY CRITERION

by

J. J. Anagnost and C. A. Desoer

Memorandum No. UCB/ERL M90/9

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ABSTRACT

This paper presents an elementary proof of the well-known Routh-Hurwitz stability criterion. The novelty of the proof is that it requires only elementary geometric considerations in the complex plane. This feature makes is useful for use in undergraduate control system courses.

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1. Introduction

The determination of stability of lumped parameter, linear, time invariant systems is one of the most fundamental problems in system theory. According to Gantmacher, [2, pp.172-173] this problem was first solved in essence by Hermite [3] in 1856, but remained unknown. In 1875, E. J. Routh also obtained conditions for stability of such systems [7]. In 1895, A. Hurwitz, unaware of Routh's work, gave another solution based on Hermite's paper. The determinantal inequalities obtained by Hurwitz are known today as the Routh-Hurwitz conditions, taught in virtually every undergraduate course on control theory.

Unfortunately, Hurwitz's proof of the result is very complicated, involving algebraic manipulations. Indeed, the proof is so complicated that most elementary textbooks (for example, [1], [5]) choose not to prove it at all, but rather to state it as a fact.

In a recent paper, Mansour [6] proves the Routh-Hurwitz Theorem in a very simple manner using the Hermite-Biehler Theorem. Motivated by Mansour's proof, this paper presents a proof based on elementary geometric considerations in the complex plane. It thus provides a clear *geometric* insight into what makes the procedure work. It also slightly extends Mansour's work by a providing a proof of the second part of the Routh-Hurwitz criterion: the number of sign changes in the first column of the Routh Table is the number of open right half-plane zeros.

The idea behind the proof of the theorem is simple. It will be shown that at each step the Routh procedure (i) eliminates precisely one zero of the characteristic polynomial (ii) preserves the position of the $j\omega$-axis zeros, and (iii) ensures that the remaining off $j\omega$-axis zeros do not cross the $j\omega$-axis. By observing the sign changes in the first column of the Routh table, it can be determine whether the eliminated zero is a zero in the open right half-plane or the open left half-plane. Thus, in n steps, precisely n zeros have been eliminated and the sign changes indicate the number of right half-plane zeros of the original polynomial.

2. Statement of the Routh-Hurwitz Stability Criterion

Theorem 2.1 (Routh-Hurwitz) - Consider an nth order polynomial in s

$$p(s) = a_0 + a_1 s + a_2 s^2 + \dots a_{n-1} s^{n-1} + a_n s^n$$

where a_i, i=0, 1, ...n $\in \mathbb{R}$ and $a_n > 0$ and $a_0 \neq 0$. (If $a_0 = 0$, simply factor out an appropriate s^k term and proceed.) If possible (i.e., none of the divisors are zero), construct the well-known Routh table, written in the form as shown in Table 1. We have used the notation

$$b_{n-2} = a_{n-2} - \frac{a_n}{a_{n-1}} a_{n-3} \qquad b_{n-4} = a_{n-4} - \frac{a_n}{a_{n-1}} a_{n-5} \quad \dots$$

$$c_{n-4} = a_{n-3} - \frac{a_{n-1}}{b_{n-2}} b_{n-4} \qquad c_{n-6} = a_{n-5} - \frac{a_{n-1}}{b_{n-2}} b_{n-6} \quad \dots$$

$$\vdots$$

$$
\begin{array}{c|ccccccc}
h_n & a_n & a_{n-2} & a_{n-4} & \cdots\cdots & a_4 & a_2 & a_0 \\
g_{n-2} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots\cdots & a_3 & a_1 & \\
h_{n-2} & b_{n-2} & b_{n-4} & b_{n-6} & \cdots\cdots & b_2 & b_0 & \\
g_{n-4} & c_{n-4} & c_{n-6} & c_{n-8} & \cdots\cdots & c_0 & & \\
h_{n-4} & d_{n-4} & d_{n-6} & d_{n-8} & \cdots\cdots & d_0 & & \\
\vdots & \vdots & & & & & & \\
g_2 & k_2 & k_0 & & & & & \\
h_2 & l_2 & l_0 & & & & & \\
g_0 & m_0 & & & & & & \\
h_0 & n_0 & & & & & & \\
g_{-2} & 0 & & & & & &
\end{array}
$$

Table 1 - The Routh Table

Then p(s) is Hurwitz (i.e., p(s) has all its zeros in the open left half-plane) if and only if each element of the first column is positive, i.e., $a_n > 0$, $a_{n-1} > 0$, $b_{n-2} > 0$, ... $m_0 > 0$, $n_0 > 0$.

Remark on Notation - In Table 1 we have assumed (to fix notation) that n is even. The even polynomial p(s) was split into its even and odd part by $p(s) = h_n(s^2) + sg_{n-2}(s^2)$, where $h_n(s^2)$ is even and of degree n, and where $g_{n-2}(s^2)$ is even and of degree n-2. Note that the coefficients of $h_n(s^2)$ are contained in the first row of the Routh table, and the coefficients of $g_{n-2}(s^2)$ are contained in the second row of the Routh table. This explains the presence of the $h_n(s^2)$ and $g_{n-2}(s^2)$ in the column to the left of the Routh table in Table 1. (If we had assumed n was odd there would be a $g_{n-1}(s^2)$ to the left of the first row, and a $h_{n-1}(s^2)$ to the left of the second row.) The remainder of the notation in Table 1 is explained in section 4.

3. Preliminary Lemmas

We first start with a definition which makes precise the notion of net phase change. Let $\mathbb{C}$ denote the complex plane.

Definition 3.1 Consider a polynomial p(s) and a continuous, oriented curve $C \subset \mathbb{C}$ which starts at $s_1 \in \mathbb{C}$ and ends at $s_2 \in \mathbb{C}$. Suppose $p(s) \neq 0$, for all $s \in C$. Let the curve be parametrized by the continuous function $\phi:[0, 1] \to C$. Since $p(s) \neq 0$ for all $s \in C$ this means that $\arg(p(s))$ along C is well-defined mod2π; hence, we choose $\arg(p(\phi(0)))$ arbitrarily and for all $r \in [0, 1]$, we choose $\arg(p(\phi(r)))$ such that $r \to \arg(p(\phi(r)))$ is continuous. Then we define the function

$$\underset{C}{\text{argnet}}(p(\bullet)) := \arg(p(\phi(1))) - \arg(p(\phi(0)))$$
$$= \arg(p(s_2)) - \arg(p(s_1))$$

Roughly speaking, $\underset{C}{\text{argnet}}(p(\bullet))$ is simply the net phase change of p(s) as s traverses C. For example, in Figure 2, if the plotted solid locus is p(C), then $\underset{C}{\text{argnet}}(p(\bullet)) = 2\pi$.

The following lemma gives a relationship between the location of zeros of a polynomial and its net phase change.

Lemma 3.2 - Consider the polynomial $p(s) = a_0 + a_1 s + a_2 s^2 + \ldots a_{n-1}s^{n-1} + a_n s^n$ where a_i, $i=0, 1, \ldots n \in \mathbb{R}$ and $a_n > 0$ and $a_0 \neq 0$ (so p(s) is of degree n and $p(0) \neq 0$). Then p(s) has L zeros in the open left-half plane counting multiplicities, R zeros in open right half-plane counting multiplicities and 2K zeros $j\omega_i$, $\omega_i > 0$, on the $j\omega$ axis with multiplicities m_i, $i=1, \ldots K$ (i.e., there are a total of M $j\omega$-axis zeros) if and only if

(i) $p^{(k)}(j\omega_i)$ $(:= \frac{d^k}{ds^k} p(s) |_{s=j\omega_i}) = 0$ for $k = 0, \ldots, m_{i-1}$, $i = 1, \ldots K$ but $p^{(m_i)}(j\omega_i) \neq 0$, $i=1, \ldots K$, and $p(j\omega) \neq 0$ for all $\omega \in \mathbb{R}^+ \backslash \{\omega_i: i=1, \ldots K\}$;

(ii) $\underset{C}{\text{argnet}}(p(\bullet)) = \frac{\pi}{2}(L - R + M)$

where the oriented curve C is the $j\omega$-axis, except for indentations on the right at each $j\omega$-axis zero $j\omega_i$, i.e., the curve C starts at zero and ends at $+j\infty$, as shown in Figure 1.

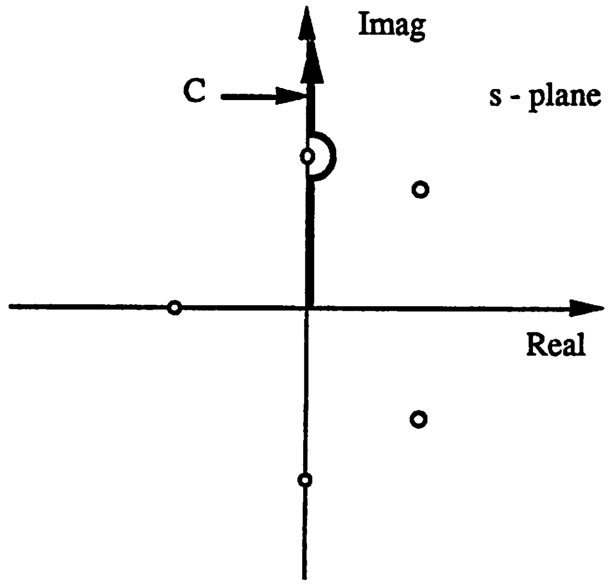

Figure 1 - Plot of the curve C

Proof of Lemma 3.2 - $\Rightarrow$ Since p(s) is an nth degree polynomial, it has precisely n zeros. By assumption, precisely M are on the $j\omega$-axis, while the remaining zeros lie in the open right half-plane, or open left half-plane. In addition, since each zero $j\omega_i$ has multiplicity m_i, this implies that $p^{(k)}(j\omega_i) = 0$ for $k = 0, \ldots m_i-1$ and $p^{(m_i)}(j\omega_i) \neq 0$. Thus, (i) is proved. To prove (ii), note that each simple real open right half-plane zero contributes $-\pi/2$ radians of phase to the net argument as s traverses C, while each simple real open left half-plane zero contributes $\pi/2$ radians of phase. Due to indentations on the right of the $j\omega$-axis zeros, each complex conjugate zero pair contributes either $+\pi$ or $-\pi$ radians of phase depending on whether the pair resides in the *closed* left half-plane or *open* right half-plane, respectively. Since there are a total of L zeros in the open left-half plane counting multiplicities, R zeros in open right half-plane counting multiplicities and M zeros on the $j\omega$ axis counting multiplicities this means

$$\underset{C}{\mathrm{argnet}}(p(\bullet)) = \frac{\pi}{2}(L - R + M)$$

This proves (ii).

$\Leftarrow$ - By assumption p(s) has precisely M/2 pairs of $j\omega$-axis zeros counting multiplicities, so it can be factored as

$$p(s) = \prod_{i=1}^{K}\left(s^2 + \omega_i^2\right)^{m_i} \prod_{i=1}^{n-M}(s - s_{zi})$$

where $\{s_{zi}: i=1, \ldots n-M\}$ denotes the remaining zeros of p(s). Since the curve C is indented to the right at the $j\omega$-axis zeros, we can define $\underset{C}{\mathrm{argnet}}(p(\bullet))$. By computation its value is

$$\underset{C}{\mathrm{argnet}}(p(\bullet)) = \frac{\pi}{2} M + \underset{C}{\mathrm{argnet}}\left(\prod_{i=1}^{n-M}(s - s_{zi})\right)$$

By condition (ii), the second term is equal to $\pi(L-R)/2$. Also, $n-M = L + R$, and M is known by (i); so conditions (i) and (ii) determine uniquely L, R, and M. ■

The following two lemmas are the main results of the section. They characterize the effect of one step of the Routh-Hurwitz procedure when the leading term is even (Lemma 3.3), and when the leading term is odd (Lemma 3.4).

Lemma 3.3 - Consider the polynomial $p(s) = a_0 + a_1 s + a_2 s^2 + \dots a_{n-1}s^{n-1} + a_n s^n$ where a_i, $i=0, 1, \dots n \in \mathbb{R}$ and $a_n > 0$ and $a_0 \neq 0$. Assume in addition that n is even and $a_{n-1} \neq 0$. Let $h_n(s^2)$ and $sg_{n-2}(s^2)$ be the even and odd parts of p(s), respectively, i.e.,

$$h_n(s^2) := a_0 + a_2 s^2 + \dots a_{n-2}s^{n-2} + a_n s^n$$
$$g_{n-2}(s^2) := a_1 + a_3 s^2 + \dots a_{n-3}s^{n-4} + a_{n-1}s^{n-2}$$

Suppose that p(s) has L zeros in the open left half-plane counting multiplicities, M $j\omega$-axis zeros counting multiplicities, and R (=n-L-M) zeros in the open right half-plane counting multiplicities. For any real λ, define

$$\begin{aligned} N(s, \lambda) &:= p(s) + \lambda s^2 g_{n-2}(s^2) \\ &= h_n(s^2) + \lambda s^2 g_{n-2}(s^2) + s g_{n-2}(s^2) \end{aligned}$$

Then,

(i) $j\omega_i$ is a $j\omega$-axis zero of p(s) with multiplicity m_i if and only if $j\omega_i$ is a $j\omega$-axis zero of $N(s, \lambda)$ with multiplicity m_i for all $\lambda \in \mathbb{R}$;

(ii) Given any closed, bounded interval $I \subset \mathbb{R}$, there exists a curve C as in Figure 1 such that $N(s, \lambda) \neq 0$ for all $s \in C$, and for all $\lambda \in I$. Thus, $\underset{C}{\mathrm{argnet}}(N(\bullet, \lambda))$ is well-defined for $\lambda \in I$.

Define the interval I by $I := [-|a_n/a_{n-1}|, |a_n/a_{n-1}|]$. Choose the curve C so that $\underset{C}{\mathrm{argnet}}(N(\bullet, \lambda))$ is well-defined for $\lambda \in I$. (This can be done by part (ii).) Then,

(iii) $\left| \underset{C}{\mathrm{argnet}}(N(\bullet, \lambda)) - \underset{C}{\mathrm{argnet}}(p(\bullet)) \right| \leq \pi$, for all $\lambda \in I$;

(iv) $\{ \underset{C}{\text{argnet}}(p(\bullet)) - \underset{C}{\text{argnet}}(N(\bullet, -a_n/a_{n-1})) \}\text{sign}(a_n/a_{n-1}) = \pi/2$.

(v) $N(s, -a_n/a_{n-1})$ has (L - 1) zeros in the open left half-plane, M zeros on the $j\omega$-axis, and R zeros in the open right half-plane, in each case counting multiplicities if and only if $a_n/a_{n-1} > 0$. In addition, $N(s, -a_n/a_{n-1})$ has L zeros in the open left half-plane, M zeros on the $j\omega$-axis, and R-1 zeros in the open right half-plane, in each case counting multiplicities if and only if $a_n/a_{n-1} < 0$.

Proof of Lemma 3.3 -

Proof of (i) - $\Leftarrow$ Take $\lambda_0 \in \mathbb{R}$, arbitrary. $j\omega_i$ is a $j\omega$-axis zero of $N(s, \lambda_0)$ with multiplicity m_i means that $\frac{d^k}{ds^k}N(s, \lambda_0)\big|_{s=j\omega_i} = 0$ for $k = 0, \dots m_i$-1 and $\frac{d^{m_i}}{ds^{m_i}}N(s, \lambda_0)\big|_{s=j\omega_i} \neq 0$. $\frac{d^k}{ds^k}N(s, \lambda_0)\big|_{s=j\omega_i} = 0$ for $k = 0, \dots m_i$-1 is equivalent to $h_n^{(k)}(-\omega_i^2) + \lambda_0\left(\frac{d^k}{ds^k}\{s^2g_{n-2}(s^2)\}\right)\big|_{s=j\omega_i} + \left(\frac{d^k}{ds^k}\{sg_{n-2}(s^2)\}\right)\big|_{s=j\omega_i} = 0$ for $k = 0, \dots m_i$-1. Since ω_i is real, equating the imaginary and real portions of this expression to zero yields $h_n^{(k)}(-\omega_i^2) + \lambda_0\left(\frac{d^k}{ds^k}\{s^2g_{n-2}(s^2)\}\right)\big|_{s=j\omega_i} = 0$ and $\left(\frac{d^k}{ds^k}\{sg_{n-2}(s^2)\}\right)\big|_{s=j\omega_i} = 0$, for $k = 0, \dots m_i$-1. This latter expression implies that $g_{n-2}^{(k)}(-\omega_i^2) = 0$;using this in the former expression shows that $h_n^{(k)}(-\omega_i^2) = 0$, $k = 0, \dots m_i$-1. Thus, $p^{(k)}(j\omega_i) = h_n^{(k)}(-\omega_i^2) + \left(\frac{d^k}{ds^k}\{sg_{n-2}(s^2)\}\right)\big|_{s=j\omega_i} = 0$, $k = 0, \dots m_i$-1. From $\frac{d^{m_i}}{ds^{m_i}}N(s, \lambda_0)\big|_{s=j\omega_i} \neq 0$ and $g_{n-2}^{(k)}(-\omega_i^2) = 0$, $k = 0, \dots m_i$-1,we further conclude that $\frac{d^{m_i}}{ds^{m_i}}p(s)\big|_{s=j\omega_i} \neq 0$. Hence, $j\omega_i$ is a $j\omega$-axis zero of p(s) with multiplicity m_i.

$\Rightarrow$ $j\omega_i$ is a $j\omega$-axis zero of p(s) with multiplicity m_i means that $p^{(k)}(j\omega_i) = h_n^{(k)}(-\omega_i^2) + \left(\frac{d^k}{ds^k}\{sg_{n-2}(s^2)\}\right)\big|_{s=j\omega_i} = 0$, $k = 0, \dots m_i$-1. Equating the real and imaginary parts of $p^{(k)}(j\omega_i) = 0$ yields $h_n^{(k)}(-\omega_i^2) = 0$ and $g_{n-2}^{(k)}(-\omega_i^2) = 0$ for $k = 0, \dots m_i$-1. This in turn implies that $\lambda_0\left(\frac{d^k}{ds^k}\{s^2g_{n-2}(s^2)\}\right)\big|_{s=j\omega_i} = 0$ for any $\lambda_0 \in \mathbb{R}$, and for $k = 0, \dots m_i$-1. Thus, $\frac{d^k}{ds^k}N(s, \lambda_0)\big|_{s=j\omega_i} = p^{(k)}(j\omega_i) + \lambda_0\left(\frac{d^k}{ds^k}\{s^2g_{n-2}(s^2)\}\right)\big|_{s=j\omega_i} + \left(\frac{d^k}{ds^k}\{sg_{n-2}(s^2)\}\right)\big|_{s=j\omega_i} = 0$ for $k = 0, \dots m_i$-1. It can be shown by similar reasoning that $\frac{d^{m_i}}{ds^{m_i}}N(s, \lambda_0)\big|_{s=j\omega_i} \neq 0$. This proves (i).

Proof of (ii) - This statement merely asserts the existence of a curve C which ensures that $\underset{C}{\text{argnet}}(N(\bullet, \lambda))$ is well-defined for all λ in the closed, bounded interval I. Since the details are not relevant to the rest of the proof, the details are left to the Appendix.

$\left(\frac{d^k}{ds^k}\{sg_{n-2}(s^2)\}\right)\Big|_{s=j\omega_i} = 0$ for $k = 0, \ldots m_i\text{-}1$. Since ω_i is real, equating the imaginary and real portions of this expression to zero yields $h_n^{(k)}(-\omega_i^2) + \lambda_0\left(\frac{d^k}{ds^k}\{s^2g_{n-2}(s^2)\}\right)\Big|_{s=j\omega_i} = 0$ and $\left(\frac{d^k}{ds^k}\{sg_{n-2}(s^2)\}\right)\Big|_{s=j\omega_i} = 0$, for $k = 0, \ldots m_i\text{-}1$. This latter expression implies that $g_{n-2}^{(k)}(-\omega_i^2) =$ 0;using this in the former expression shows that $h_n^{(k)}(-\omega_i^2) = 0$, $k = 0, \ldots m_i\text{-}1$. Thus, $p^{(k)}(j\omega_i) = h_n^{(k)}(-\omega_i^2) + \left(\frac{d^k}{ds^k}\{sg_{n-2}(s^2)\}\right)\Big|_{s=j\omega_i} = 0, k = 0, \ldots m_i\text{-}1$. From $\frac{d^{m_i}}{ds^{m_i}}N(s, \lambda_0)\Big|_{s=j\omega_i} \neq 0$ and $g_{n-2}^{(k)}(-\omega_i^2) = 0$, $k = 0, \ldots m_i\text{-}1$,we further conclude that $\frac{d^{m_i}}{ds^{m_i}}p(s)\Big|_{s=j\omega_i} \neq 0$. Hence, $j\omega_i$ is a $j\omega$-axis zero of $p(s)$ with multiplicity m_i.

$\Rightarrow$ $j\omega_i$ is a $j\omega$-axis zero of $p(s)$ with multiplicity m_i means that $p^{(k)}(j\omega_i) = h_n^{(k)}(-\omega_i^2) + \left(\frac{d^k}{ds^k}\{sg_{n-2}(s^2)\}\right)\Big|_{s=j\omega_i} = 0, k = 0, \ldots m_i\text{-}1$. Equating the real and imaginary parts of $p^{(k)}(j\omega_i) = 0$ yields $h_n^{(k)}(-\omega_i^2) = 0$ and $g_{n-2}^{(k)}(-\omega_i^2) = 0$ for $k = 0, \ldots m_i\text{-}1$. This in turn implies that $\lambda_0\left(\frac{d^k}{ds^k}\{s^2g_{n-2}(s^2)\}\right)\Big|_{s=j\omega_i} = 0$ for any $\lambda_0 \in \mathbb{R}$, and for $k = 0, \ldots m_i\text{-}1$. Thus, $\frac{d^k}{ds^k}N(s, \lambda_0)\Big|_{s=j\omega_i} = p^{(k)}(j\omega_i) + \lambda_0\left(\frac{d^k}{ds^k}\{s^2g_{n-2}(s^2)\}\right)\Big|_{s=j\omega_i} + \left(\frac{d^k}{ds^k}\{sg_{n-2}(s^2)\}\right)\Big|_{s=j\omega_i} = 0$ for $k = 0, \ldots m_i\text{-}1$. It can be shown by similar reasoning that $\frac{d^{m_i}}{ds^{m_i}}N(s, \lambda_0)\Big|_{s=j\omega_i} \neq 0$. This proves (i).

Proof of (ii) - This statement merely asserts the existence of a curve C which ensures that $\underset{C}{\text{argnet}}(N(\bullet, \lambda))$ is well-defined for all λ in the closed, bounded interval I. Since the details are not relevant to the rest of the proof, the details are left to the Appendix.

Proof of (iii) - For simplicity, first assume that $p(s)$ has no $j\omega$-axis zeros. For this case we take the curve C to be the positive $j\omega$-axis. Note that $N(j\omega, \lambda) \neq 0$ on $\mathbb{R} \times I$.

Thus, the only difference in phase occurs for (ω_k, ∞). Since there are no zeros of $\omega g_{n-2}(-\omega^2)$ in this interval, this again implies that sign{Im(N(jω, λ))} is a constant (see Figure 2). This in turn implies that $|\underset{[j\omega_k, j\infty]}{\text{argnet}}(N(\bullet, \lambda)) - \underset{[j\omega_k, j\infty]}{\text{argnet}}(p(\bullet))| \leq \pi$ for all $\lambda \in I$. Since $\underset{[0, j\omega_k]}{\text{argnet}}(N(\bullet, \lambda)) = \underset{[0, j\omega_k]}{\text{argnet}}(p(\bullet))$ we then have

$$|\underset{[0, j\infty]}{\text{argnet}}(N(\bullet, \lambda)) - \underset{[0, j\infty]}{\text{argnet}}(p(\bullet))| \leq \pi$$

for all $\lambda \in I$. This proves (iii) for the case where p(s) has no jω-axis zeros.

Proof of (iv) - Note that by the definition of I that $-a_n/a_{n-1} \in I$. Order the zeros of $\omega g_{n-2}(-\omega^2)$ as before, and use arguments identical to that of part (iii) to obtain

$$\underset{[0, j\infty]}{\text{argnet}}(p(\bullet)) - \underset{[0, j\infty]}{\text{argnet}}(N(\bullet, -a_n/a_{n-1})) = \underset{[j\omega_k, j\infty]}{\text{argnet}}(p(\bullet)) - \underset{[j\omega_k, j\infty]}{\text{argnet}}(N(\bullet, -a_n/a_{n-1}))$$

Since ω_k is a fixed point (i.e., independent of λ), we then obtain

$$= \underset{\omega \to \infty}{\arg}(p(j\omega)) - \underset{\omega \to \infty}{\arg}(N(j\omega, -a_n/a_{n-1}))$$

Since we are taking the limit as $\omega \to \infty$, we only need to consider the leading term of each polynomial. Performing this operation, and using the properties of arg, we obtain in succession

$$= \underset{\omega \to \infty}{\arg}(a_n(j\omega)^n) - \underset{\omega \to \infty}{\arg}(a_{n-1}(j\omega)^{n-1})$$

$$= \underset{\omega \to \infty}{\arg}[a_n(j\omega)^n/a_{n-1}(j\omega)^{n-1}]$$

$$= \underset{\omega \to \infty}{\arg}[a_n j\omega/a_{n-1}]$$

Thus, if $a_n/a_{n-1} > 0$, the net argument difference is π/2, and if $a_n/a_{n-1} < 0$, then the net argument difference is -π/2. This proves (iv) for the case where p(s) has no jω-axis zeros.

If p(s) has jω-axis zeros, then part (i) shows that N(s, λ) has the same jω-axis zeros with the same multiplicities. This means that the only difference in argument can come from the non jω-axis zeros. If we extract the jω-axis zeros by $p_1(s) = p(s)/\prod_{i=1}^{M/2}(s^2 + \omega_i^2)$, then $p_1(s)$ has no jω-axis zeros, so we can apply the arguments above. For example, to prove (iii) we know from above that

$$\left| \underset{[0, j\infty]}{\text{argnet}}(N_1(\bullet, \lambda)) - \underset{[0, j\infty]}{\text{argnet}}(p_1(\bullet)) \right| \leq \pi$$

where $N_1(s, \lambda) := N(s, \lambda) / \prod_{i=1}^{M/2} (s^2 + \omega_i^2)$ for all $\lambda \in I$. If we choose a contour C like Figure 1, indented on the right of the $j\omega$-axis zeros, we then obtain

$$\left| \underset{C}{\text{argnet}}(N(\bullet, \lambda)) - \underset{C}{\text{argnet}}(p(\bullet)) \right| \leq \pi$$

for all $\lambda \in I$, which proves (iii). Statement (iv) is proved similarly.

Proof of (v) - The net argument difference between p(s) and $N(s, -a_n/a_{n-1})$ as s traverses C is $\text{sign}(a_n/a_{n-1})\pi/2$, by applying part (iv) above. Applying both logical implications of Lemma 3.2 shows that $N(s, -a_n/a_{n-1})$ and p(s) have the same number of zeros on the $j\omega$-axis, and a difference of at most one in the number of open right half-plane or open left half-plane zeros, depending on the sign of a_n/a_{n-1}. This proves (v). ■

In the case that n is odd (rather than even as in the statement of Lemma 3.3), we have the corresponding result to Lemma 3.3.

Lemma 3.4 - Consider the polynomial $p(s) = a_0 + a_1 s + a_2 s^2 + \ldots a_{n-1}s^{n-1} + a_n s^n$ where a_i, i=0, 1, ...n $\in \mathbb{R}$ and $a_n > 0$ and $a_0 \neq 0$. Assume n is *odd*. Let $h_{n-1}(s^2)$ and $sg_{n-1}(s^2)$ be the even and odd parts of p(s), respectively, i.e.,

$$g_{n-1}(s^2) := a_1 + a_3 s^2 + \ldots a_{n-2}s^{n-3} + a_n s^{n-1}$$
$$h_{n-1}(s^2) := a_0 + a_2 s^2 + \ldots a_{n-3}s^{n-3} + a_{n-1}s^{n-1}$$

Assume $a_{n-1} \neq 0$. Suppose that p(s) has L zeros in the open left half-plane counting multiplicities, M $j\omega$-axis zeros counting multiplicities, and R (=n-L-M) zeros in the open right half-plane counting multiplicities. For any real λ, define

$$\begin{aligned} N_o(s, \lambda) &:= p(s) + \lambda s h_{n-1}(s^2) \\ &= s g_{n-1}(s^2) + \lambda s h_{n-1}(s^2) + h_{n-1}(s^2) \end{aligned}$$

Then, statements (i)-(v) of Lemma 3.3 hold, with $N_o(s, \lambda)$ replacing $N(s, \lambda)$.

Proof of Lemma 3.4 - The proofs of (i) and (ii) are identical to the analogous results of Lemma

3.3, and are thus omitted. The proofs of (iii)-(v) are also nearly identical to their counterparts in Lemma 3.3. The key point is to note that $j\omega\lambda h_{n-1}(-\omega^2)$ contributes only to the *imaginary* part of $N_o(j\omega, \lambda)$. This means that points a and b of Figure 3 are fixed points, i.e. if ω_1 satisfies $p(j\omega_1) =$ a, then $h_{n-1}(-\omega_1{}^2) = 0$, which means that $N_o(j\omega_1, \lambda) = b$ for all $\lambda \in I$. The details are left to the reader. ■

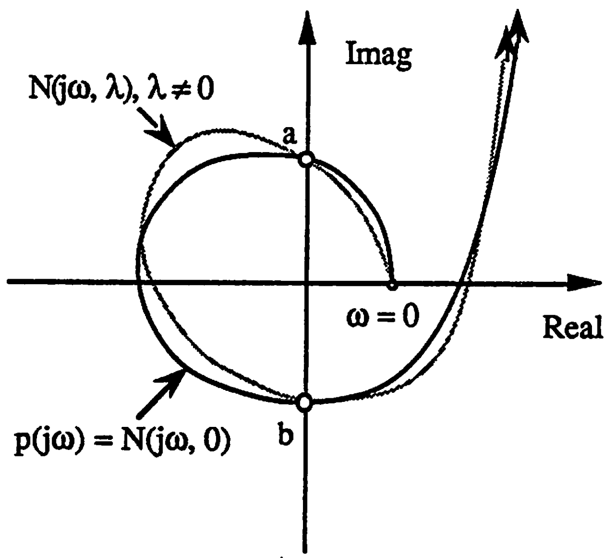

Figure 3 - Graph of $\omega \rightarrow N(j\omega, \lambda)$, $0 \leq \omega < \infty$ (n odd).

4. Proof of Theorem 2.1 (Routh-Hurwitz)

Let us first emphasize some notation. To fix notation, assume n is even.

Let $h_{n-w}(s^2)$ be the even polynomial of degree n-w whose coefficients lie in the row 2w. (See Table 1). Let $g_{n-w}(s^2)$ be the even polynomial of degree n-w whose coefficients lie in the row 2w + 1. (Again see Table 1).

To construct the Routh Table, perform the calculations indicated in section 2. This corresponds to finding a $\lambda_{n-w} \in \mathbb{R}$, or a $\mu_{n-w} \in \mathbb{R}$, such that

$$h_{n-w}(s^2) = h_{n-w+2}(s^2) + \lambda_{n-w} s^2 g_{n-w}(s^2) \tag{4.1}$$

$$g_{n-w}(s^2) = g_{n-w+2}(s^2) + \mu_{n-w} h_{n-w+2}(s^2) \tag{4.2}$$

where the leading terms of $h_{n-w}(s^2)$ and $g_{n-w}(s^2)$, respectively, are canceled. If this procedure cannot be performed (i.e., the leading term of $g_{n-w}(s^2)$ or $h_{n-w}(s^2)$ is zero), then a zero is in the first column of the Routh Table. The standard procedure given in elementary textbooks is to replace the zero by $\varepsilon > 0$, and proceed. See section 5 for some of the implications of this.

Proof of Theorem 2.1 (Routh-Hurwitz) $\Rightarrow$ If p(s) is Hurwitz, then each of its zeros is in the open left half-plane. Consider the first step of the Routh procedure. By Lemma 3.3, part (v), $N(s, -a_n/a_{n-1}) = h_n(s^2) - a_n/a_{n-1}s^2g_{n-2}(s^2) + sg_{n-2}(s^2)$ has the same number of zeros in the open left half-plane as p(s) except for the eliminated zero. Since all the zeros of p(s) are in the open left half-plane, the eliminated zero must also be in the left half-plane. Thus, $N(s, -a_n/a_{n-1}) = sg_{n-2}(s^2) + h_{n-2}(s^2)$ has precisely n-1 zeros in the open left half-plane, and a_n/a_{n-1} is positive. By using Lemma 3.4, in the next step we have that $h_{n-2}(s^2) + sg_{n-4}(s^2)$ has precisely n-2 zeros in the open left-half plane, and a_n/a_{n-1} is positive. After n steps, n zeros have been eliminated and each element in the first column is positive.

$\Leftarrow$ If each element in the first column is positive then use of Lemma 3.3, part (v), and Lemma 3.4, part (v) show that precisely n zeros in the open left half-plane have been eliminated. Thus, p(s) is Hurwitz. ■

Remark 4.1 - In the case that n is odd, the proof of the Routh-Hurwitz theorem is nearly identical. Simply write $p(s) = sg_{n-1}(s^2) + h_{n-1}(s^2)$, where $sg_{n-1}(s^2)$ and $h_{n-1}(s^2)$ are the odd and even parts of p(s), respectively. Apply Lemmas 3.4 and 3.3 appropriately in a manner similar to that in the proof of the even case. The details are left to the reader.

5. The Second Part of the Routh-Hurwitz Theorem

Based on Lemma 3.3 we have the second part of the Routh-Hurwitz criterion.

Theorem 5.1 - Consider an nth order polynomial in s

$$p(s) = a_0 + a_1s + a_2s^2 + \dots a_{n-1}s^{n-1} + a_ns^n$$

where a_i, $i=0, 1, \dots n \in \mathbb{R}$ and $a_n > 0$ and $a_0 \neq 0$. As before, to fix notation assume that n is even. Suppose when calculating the Routh Table that no element in the first column is zero. Then the number of sign changes in the first column of the Routh Table is the number of *open* right half-plane zeros of p(s).

Proof of Theorem 5.1 - At each step the algorithm (i) eliminates precisely one zero of p(s), (ii) preserves the position of the $j\omega$-axis zeros, and (iii) ensures that the remaining off $j\omega$-axis zeros do not cross the $j\omega$-axis. By Lemma 3.3 part(v), and Lemma 3.4 part (v) the eliminated zero is in the open left half -plane if the ratio of the associated leading coefficients is positive, whereas the eliminated zero is in the open right half-plane if the ratio of the associated leading coefficients is negative. Thus, the number of sign changes in the first column indicates the number of open right half-plane zeros of p(s) that were eliminated.

Remark 5.2 - If a zero does appear in the first column during the Routh procedure, care must be exercised in ascertaining the zero positions of the original polynomial. By adding an $\varepsilon > 0$ to a column, the position of the zeros of the original polynomial are being perturbed (since the zeros of a polynomial are continuous functions of their coefficients provided a_n remains bounded away from zero). Attempting to deduce properties of the zeros of the original polynomial based on the properties of the perturbed polynomial can often lead to erroneous conclusions as the following examples show.

Example - Let $p(s) = (s^2 + a)(s^2 + b) = s^4 + (a + b)s^2 + ab$, where $a, b \in \mathbb{R}$. The Routh Table for this example is

$$
\begin{array}{|cccc}
1 & a+b & ab & 0 \\
\varepsilon & 0 & 0 & \\
a+b & ab & 0 & \\
\dfrac{-\varepsilon ab}{a+b} & 0 & & \\
ab & 0 & &
\end{array}
$$

If $a > 0$ and $b > 0$, then there are two sign changes in the first column since $\varepsilon > 0$. This leads to the "conclusion" that there are two zeros in closed right half-plane. Note that adding an $\varepsilon > 0$ merely pushes the $j\omega$-axis zeros of the p(s) off the $j\omega$-axis. Much more insidious examples can be constructed that make it very difficult to tell the position of the zeros of the original polynomial. (See, for example, [2, p. 184, Example 4].) However, we do have the following proposition.

Proposition 5.3 - Suppose that during construction of the Routh table a zero in the first column is encountered. Then

(i) If there are one or more nonzero elements in the same row, then p(s) has at least one zero in the open right half-plane;

(ii) If the row is zero, then (a) p(s) has at least one pair of $j\omega$-axis zeros, or (b) p(s) contains a factor of the form $(s+ \alpha_0)(s- \alpha_0)$ for some $\alpha_0 \in \mathbb{R}$, or (c) p(s) contains a factor of the form $(s+ \alpha_0 + \beta_0 j)(s+ \alpha_0 - \beta_0 j)(s- \alpha_0 + \beta_0 j)(s - \alpha_0 - \beta_0 j)$ for some $\alpha_0, \beta_0 \in \mathbb{R}$.

Proof of Proposition 5.3 - *Proof of (i).* Since there is a zero in the first column in the Routh Table, the Routh-Hurwitz Theorem 2.1 shows that p(s) has at least one zero in the *closed* right half-plane. Without loss of generality, assume that the zero is in the second element of the first column, i.e., $sg_{n-2}(s^2)$ has a leading coefficient of at most order n-3. (Since the row is nonzero by supposition, $sg_{n-2}(s^2)$ is at least of order 1.) Suppose that the only zeros of p(s) in the closed right half-plane are $j\omega$-axis zeros, say M counting multiplicities. Then extract the $j\omega$-axis zero pairs from p(s) by $p_1(s) := p(s)/\prod_{i=1}^{M/2}\left(s^2 + \omega_i^2\right) = h_n(s^2)/\prod_{i=1}^{M/2}\left(s^2 + \omega_i^2\right) + sg_{n-2}(s^2)/\prod_{i=1}^{M/2}\left(s^2 + \omega_i^2\right)$.

(Note that since $sg_{n-2}(s^2)$ is nonzero, $M < n$.) By supposition, $p_1(s)$ is Hurwitz and thus has every coefficient strictly positive. However, $h_n(s^2)/\prod_{i=1}^{M/2}(s^2+\omega_i^2)$ is of order n-M, while $sg_{n-2}(s^2)/\prod_{i=1}^{M/2}(s^2+\omega_i^2)$ is of order at most n - M - 3 (but at least 1). Thus, $p_1(s)$ has its n - M - 1 coefficient equal to zero, which contradicts the fact that $p_1(s)$ is Hurwitz. This proves (i).

Proof of (ii) - Encountering a zero row during construction of the Routh Table means that some step

$$h_{n-w+2}(s^2) = -\lambda_{n-w}s^2g_{n-w}(s^2), \text{ or}$$
$$g_{n-w+2}(s^2) = -\mu_{n-w}h_{n-w+2}(s^2)$$

Hence the polynomial $h_{n-w+2}(s^2) + sg_{n-w}(s^2)$ or $sg_{n-w+2}(s^2) + h_{n-w+2}(s^2)$ equals $(1-\lambda_{n-w}s)sg_{n-w}(s^2)$ or $(1-\mu_{n-w}s)h_{n-w+2}(s^2)$, respectively. Since $g_{n-w}(s^2)$ and $h_{n-w+2}(s^2)$ have real coefficients, this means that $g_{n-w}(s^2)$ or $h_{n-w+2}(s^2)$ have zeros of the type stated in the Proposition. Working our way back *up* the Routh Table, note that p(s) can be written as linear combinations of $g_{n-w}(s^2)$ and $h_{n-w+2}(s^2)$ or $g_{n-w+2}(s^2)$ and $h_{n-w+2}(s^2)$. (Use (4.1)-(4.2) and the Routh Table, Figure 1.) Thus, p(s) also has the stated property. ■

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Appendix - Proof of Lemma 3.3, part (ii)

The idea is to choose a curve C as in Figure 1 with a sufficiently small indentation about each jω-axis zero so that (ii) holds.

Take any bounded interval $I \subset \mathbb{R}$. From part (i) of Lemma 3.3, for any $\lambda \in I$, $N(s, \lambda)$ has the same jω-axis zeros with the same multiplicities as p(s), say a total of M. Therefore, only n-M zeros of $N(s, \lambda)$ depend on λ. Let $\{z_i(\lambda), i=1, \ldots n\text{-}M\}$ be such that $N(z_i(\lambda), \lambda) = 0$. At $\lambda = -a_n/a_{n-1}$, note that the degree of $N(s, \lambda)$ drop by precisely one (since $a_{n-1} \neq 0$). Thus, precisely one member of $\{z_i(\lambda), i=1, \ldots n\text{-}M\}$, say $z_J(\lambda)$, goes to infinity as $\lambda \to -a_n/a_{n-1}$, and it tends to infinity along the real axis, as an asymptotic expansion shows. This means that there is a closed interval $I_J \subset I$ with $|z_J(\lambda)| < \infty$ for all $\lambda \in I_J$, satisfying $\min_{i \in \{1, \ldots M\}} \min_{\lambda \in I}(|z_J(\lambda) - j\omega_i|) =$

$\min_{i \in \{1, \dots M\}} \min_{\lambda \in I_J} (|z_J(\lambda) - j\omega_i|)$. This latter term has an achievable positive minimum, since the locus $z_J(I_J)$ is closed and bounded, and never crosses the imaginary axis. Call this minimum distance R_J.

So now consider $z(I) := \{z_i(\lambda), i=1, \dots n-M, i \neq J, \lambda \in I\} \subset \mathbb{C}$, the locus of all off-imaginary axis zeros of $N(s, \lambda)$ (except for $z_J(\lambda)$) as λ varies over I. Since I is closed and bounded, the locus is a closed and bounded set. Therefore, $\min_{i \in \{1, \dots M\}} (|z(I) - j\omega_i|)$ is a finite, positive constant, say R. Let $R^* = \min(R_J, R) > 0$. Using this value of R^*, we can then make the radius of the indentation about each $j\omega$-axis zero $j\omega_i$ equal to $R^*/2$, say. This allows construction of the desired contour C, which proves (ii). ■