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Stochastic Cycle Time Optimization of Sequential Systems

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Abstract

The cycle time of a sequential digital system is the key determinant of the performance of the system. Most approaches to cycle time optimization assume fixed delays through logic blocks. In reality, the delays are random variables whose distribution can be characterized by historic data obtained from previously manufactured systems.

In this paper, we describe a mathematical model and a method for optimizing the cycle time of a sequential digital system in the presence of randomly varying logic delays. First, we give an efficient polynomial-time algorithm for the case in which the delays are deterministic and then solve the related stochastic problem. We obtain a polynomial time bound on the solution of the stochastic problem under practical conditions. Also, we derive results that can be used to devise efficient algorithms for obtaining the optimal clock period, where optimality is defined as the most reliable clock period which yields the minimum manufacturing cost. Preliminary tests on industrial data indicate that a practical implementation of the technique is efficient and useful for obtaining reliability-cycle time tradeoff curves. Results from some real examples are shown in the paper.

1 Introduction

The cycle time of a sequential digital system is the key determinant of the performance of the system. Given a fixed computation to be performed on the system, reducing the cycle time will result in a proportional decrease in the time to perform the computation. Several techniques that do not modify the logic circuitry have been proposed to optimize the cycle time of sequential systems. *Skew optimization* [Fis81] and *retiming* [LRS83] are two such techniques.

However, these approaches assume fixed delays through logic blocks. In reality, the delays are random variables whose distributions can be characterized using historic data. The commonly used approach to deal with variable delays is to increase the cycle time by some fixed percentage according to some rules of thumb or perform worst-case analysis. However, this may not be the best approach in terms of the reliability-performance tradeoff and in some sense, may be overly conservative. For a competitive design that pushes technology to the limits, an improvement in the performance can be optimal often at the risk of system failure. In this paper, we present a systematic approach for obtaining reliability-tradeoff curves for a given sequential digital system. Our optimization technique is a mathematical approach for setting the appropriate design point. It allows the designer to obtain the smallest clock period such that a given reliability level for the system is achieved. Typical systems that are amenable to such optimizations are multi-chip modules, high-performance packages and printed circuit boards.

The outline of the paper is as follows: First, we describe the physical and timing models of the digital system under consideration. A formulation of the problem is presented, followed by an efficient algorithm for solving it when the delays are deterministic. We then generalize the formulation to the case of stochastic delays and discuss computational strategies. Our approach for solving the stochastic problem is described, along with some key theorems and propositions. Finally, we present some graphs of reliability-cycle time tradeoff curves for an industrial example for various distribution functions.

2 Skew Optimization

2.1 Physical Model

For explanation purposes, consider the physical model of a multi-chip module (MCM); the techniques can be applied to other hierarchical systems as well. The system is assumed to be composed of *chips*. Chips are composed of *macros* containing primitive elements such as cells or books, some of which may be clocked or synchronizing elements. The input specification consists of *chip description netlists* that specify for each chip, the netlist internal to the chip, delays though the macros, delays in the interconnect between macros, synchronizing element setup and hold times and wiring delays between chip clock pins and each synchronizing element. In addition, a *chip-to-chip netlist* specifies the interconnectivity between chips and wire delays. Wire delays that exist from the MCM clock sources to the chips may also be specified. For the purposes of simplicity, we assume a single phase clock.

2.2 Timing Model

A model for detecting clock hazards and optimizing skew was presented in [Fis81]. The synchronous digital system used in this model consists of blocks of combinational elements separated by edge-triggered latches. The set of latches is denoted by $L = \{l_1, \ldots, l_n\}$. Let T denote the clock period. The circuit can be modeled as a *synchronous communication graph* G containing n vertices, one for each latch and m edges. There is a directed edge between vertex i and j if at some time during the clock cycle, there is a combinational logic path from latch l_i to l_j. Each edge has weights D_{ij} and d_{ij} which denote the largest and smallest combinational logic delays between latch l_i and l_j respectively. The setup and hold times of a latch, l_i, are denoted by $t_{SETUP\,i}$ and $t_{HOLD\,i}$ respectively. The delay from the clock source to the latch l_i is denoted by δ_i.

Such an abstract communication graph may be easily constructed by a *timing analysis* on the physical model presented in the previous section.

2.3 Formulation

We are now ready to formulate the skew optimization problem as defined in [Fis81]. The first constraint concerns *double clocking*. It states that the signal from latch l_i to latch l_j through the fastest path should not race through the circuit before the end of one clock period. The equation can be written as follows:

$$\delta_i + d_{ij} \geq \delta_j + t_{HOLD\,j}, \ \forall (i,j) \in G. \tag{1}$$

The second equation concerns *zero clocking*, i.e., the slowest signal from latch l_i to latch l_j should not arrive at latch l_j later than the end of one period. The corresponding constraint is:

$$\delta_i + t_{SETUP\,j} + D_{ij} \leq \delta_j + T, \ \forall (i,j) \in G. \tag{2}$$

Frequently, we may have constraints on the maximum achievable clock offset between any pair of nodes.

$$0 \leq \delta_i \leq \Delta_{max} \tag{3}$$

The optimization problem may be stated as follows:

Find the smallest clock period T such that equations 1, 2 and 3 are satisfied.

$$
\begin{aligned}
\mathcal{P}: \quad & \text{minimize}\, T \\
& \text{subject to} \\
& \delta_i + d_{ij} \geq \delta_j + t_{HOLD\,j}, \;\; \forall (i,j) \in \mathrm{G} \\
& \delta_i + t_{SETUP\,j} + D_{ij} \leq \delta_j + T, \;\; \forall (i,j) \in \mathrm{G} \\
& 0 \leq \delta_i \leq \Delta_{max}
\end{aligned} \tag{4}
$$

2.4 An Efficient Algorithm for Solving $\mathcal{P}$ when Delays are Deterministic

$\mathcal{P}$ may be solved efficiently using graph-based algorithms as follows. A lower bound on T is easily obtained as:

$$T_{min} = \max\{0, \min_{ij}(D_{ij} - \Delta_{max})\}$$

An upper bound on T is:

$$T_{max} = \max_{ij}\{D_{ij}\}$$

The algorithm is:

1. Perform a bisection search on the values of T between T_{max} and T_{min} to find the minimum feasible period. This is possible because if a clock period T_1 is feasible (infeasible) then any clock period $T_2 > T_1$ ($T_2 < T_1$) is also feasible (infeasible).

2. Test each T_i obtained during the bisection search for feasibility using the Bellman-Ford algorithm [Dre69]

Proposition 1 *The problem $\mathcal{P}$ can be solved in using $O(\log(T_{max} - T_{min})n^3)$ operations*

Let $\mathcal{F}$ denote the following problem:

For a given fixed constant T_0, find a feasible solution to

$$
\begin{aligned}
& \delta_i + d_{ij} \geq \delta_j + t_{HOLD\,j}, \;\; \forall (i,j) \in \mathrm{G} \\
& \delta_i + t_{SETUP\,j} + D_{ij} \leq \delta_j + T_0, \;\; \forall (i,j) \in \mathrm{G} \\
& 0 \leq \delta_i \leq \Delta_{max}
\end{aligned} \tag{5}
$$

Let $\mathcal{F}'$ be the following problem: Find a feasible solution to

$$
\begin{aligned}
& \delta_i + d_{ij} \geq \delta_j + t_{HOLD\,j}, \;\; \forall (i,j) \in \mathrm{G} \\
& \delta_i + t_{SETUP\,j} + D_{ij} \leq \delta_j + T_0, \;\; \forall (i,j) \in \mathrm{G} \\
& \delta_i - \delta_j \leq \Delta_{max}, \forall i,j
\end{aligned} \tag{6}
$$

Let $S' = \{\delta_1, \ldots, \delta_n\}$ be any feasible solution to $\mathcal{F}'$. Let

$$\delta_{min} = \min_i \{\delta_i\}$$

Clearly, $\delta_j - \delta_{min} \leq \Delta_{max}, \forall j$. Define $S = \{\delta_1 - \delta_{min}, \ldots, \delta_n - \delta_{min}\}$. It can be easily verified that S' satisfies all the constraints of $\mathcal{F}$. Also, any solution to $\mathcal{F}$ trivially satisfies $\mathcal{F}'$. Therefore, $\mathcal{F}$ and $\mathcal{F}'$ are equivalent.

Note that for a given fixed clock period T_0, all the inequalities in $\mathcal{F}$ have the form:

$$\delta_i - \delta_j \leq c_{ij}$$

where c_{ij} are real numbers.

A system of inequalities in which all the equations have this form can be checked for feasibility and solved very efficiently in $O(mn)$ time, where m is the number of inequalities and n is the number of variables, using the Bellman-Ford algorithm. Thus, the complexity of the above algorithm is $O(\log(T_{max} - T_{min})n^3)$. [LS88] have shown that even the mixed-integer version of the above form can be solved efficiently. ∎

2.5 Stochastic Delays

For simplicity of explanation, in this and following sections, we will assume that the hold and setup times t_{HOLD} and t_{SETUP} are included in the delays D_{ij}. In a practical system, the delays D_{ij} are not fixed values for that particular system. There may be variations in D_{ij} and d_{ij} due to processing, temperature, signal variations and other sources. We assume that D_{ij} and d_{ij} are random variables with some known probability distribution functions.

The stochastic optimization problem may be stated as:

$$\begin{aligned} \mathcal{P}1: \quad & \text{minimize}\, T \\ & \text{subject to} \\ & Prob[\textstyle\bigcap_{(i,j)\in G}\{(\delta_i + d_{ij} \geq \delta_j) \bigcap (\delta_i + D_{ij} \leq \delta_j + T)\}] \geq \alpha \\ & 0 \leq \delta_i \leq \Delta_{max} \text{ for } i = 1, \ldots, n \end{aligned} \tag{7}$$

where $\alpha \in (0, 1)$ is the desired reliability level for the system.

For the purposes of discussion, let us introduce additional variables in the problem as follows:

$$\begin{aligned}
\mathcal{P}1: \quad & \text{minimize}\, T \\
& \text{subject to} \\
& Prob[\textstyle\bigcap_{(i,j)\in G}\{(x_{ij} + d_{ij} \geq 0) \bigcap (x_{ij} + D_{ij} \leq T)\}] \geq \alpha \qquad (8)\\
& x_{ij} = \delta_i - \delta_j \;\; \forall (i,j) \in \text{G} \\
& 0 \leq \delta_i \leq \Delta_{max} \text{ for } i = 1, \ldots, n
\end{aligned}$$

3 Computational Strategies

There are several options available for solving the problem $\mathcal{P}1$. The first is to impose penalties on the violation of the probabilistic constraints. For example, instead of minimizing T subject to the timing constraints, one can minimize the following objective function:

$$T + \{\sum_{i,j} \log[1 - F^1_{ij}(x_{ij})] + \log[F^2_{ij}(T - x_{ij})]\}^2$$

subject to

$$x_{ij} = \delta_i - \delta_j, \;\; \forall (i,j) \in G$$

$$0 \leq \delta_i \leq \Delta_{max} \text{ for } i = 1, \ldots, n$$

where F^1_{ij} is the probability distribution function associated with d_{ij} and F^2_{ij} is the distribution function associated with D_{ij}. d_{ij} and D_{ij} are assumed to be independent random variables. This objective function is nonlinear and may not be easy to optimize, since it is not separable.

Converting the probabilistic constraints to deterministic ones is the second option considered. Since D_{ij} and d_{ij} involve random variables on the same arc, we make the following simplifying assumption:

$$D_{ij} - d_{ij} = \gamma_{ij} + \mu_{ij} D_{ij}$$

where γ_{ij} and μ_{ij} are non-negative constants. As suggested by Prekopa [Ee88], one technique is to convert the chance constraints into equivalent deterministic ones. The deterministic constraints would have the form:

$$\prod Prob[D_{ij} \leq T - x_{ij} \bigcap D_{ij} \leq (T - \gamma_{ij})/\mu_{ij}] \geq \alpha$$

Prekopa shows that the left hand side of this constraint is logconcave under the assumption A: D_{ij} has a logarithmic concave distribution. This can be used to demonstrate that the constraint set is convex. We could replace the above constraint by:

$$\sum_{(i,j)\in G} \min\{\log(Prob[D_{ij} \leq T - x_{ij}]), \log(Prob[D_{ij} \leq (T - \gamma_{ij})/\mu_{ij}])\} \geq \log \alpha$$

The well known sufficient condition for the concavity of a function of the form:

$$\sum_{i\in I} \log F_i(z_i)$$

where F_i is a distribution function is that there exists $0 \leq p \leq 1$ such that $f_i(x)$, the density function, is nonincreasing for $x \geq x(p_i)$. $x(p_i)$ is the p_i'th fractile. Define the set of functions

$$\wp = \{F(x) | \exists\ 0 \leq p \leq 1, F(x) \text{is nonincreasing for } x \geq x(p)\}$$

Let

$$p_0 = \max_{i\in I}(p_i)$$

Then, $\sum_{i\in I} \log F_i(z_i)$ is concave for all z_i when

$$\prod_{i\in I} F_i(z_i) \geq p_0$$

For all symmetric unimodal distribution functions $F_i \in \wp$, $p_0 = 0.5$ [Kam84]. However, we have found no suitable linearization strategy for dealing with such a constraint.

The actual strategy chosen to efficiently solve the stochastic optimization problem involves maximizing the reliability and is discussed in the next section.

3.1 The Solution Approach

The strategy we use is to maximize the probability of satisfying all the constraints for a given fixed clock period T_0. After conversion, the optimization problem takes form:

$$\begin{aligned}
&\mathcal{P}2 : \text{maximize } \textstyle\sum_{(i,j)\in G} \log(\min(Prob[D_{ij} \leq (T_0 - \gamma_{ij})/\mu_{ij}], Prob[D_{ij} \leq T_0 - x_{ij}])) \\
&\text{subject to} \\
&x_{ij} = \delta_i - \delta_j \ \forall (i,j) \in \text{G} \\
&0 \leq \delta_i \leq \Delta_{max} \text{ for } i = 1, \ldots, n
\end{aligned} \tag{9}$$

We prove that the problem $\mathcal{P}2$ is equivalent to $\mathcal{P}1$. We also show that because of the simple structure of the problem, the maximization problem turns out to be a separable convex optimization problem subject to totally unimodular constraints under the assumption A. Thus, efficient polynomial time algorithms exist for solving such a problem [HS89].

Proposition 2 *$\mathcal{P}1 \Leftrightarrow \mathcal{P}2$ when the distribution functions are increasing in T*

We will write instances of $\mathcal{P}1$ as $P_1(\Delta_{max}, \alpha)$ and of $\mathcal{P}2$ as $P_2(\Delta_{max}, T)$. For simplicity of presentation we will represent the optimal solutions as:

$$P_1(\Delta_{max}, \alpha) = T^*$$

$$P_2(\Delta_{max}, T) = \alpha^*$$

For a given fixed value of Δ_{max}, we can drop that parameter from the notation above. Let

$$P_1(\alpha_1) = T^*$$

$$P_2(T^*) = \alpha^*$$

Let the realized probability of satisfying the constraints in $\mathcal{P}1$ at T^* be α_1. If $\alpha_1 > \alpha^*$, we can reduce T^* in $\mathcal{P}1$ under the assumption that the distribution functions are continuous and increasing in T, thus obtaining a contradiction. α_1 cannot be less than α^* since otherwise, α^* cannot be optimal for $\mathcal{P}2$. So, $\alpha_1 = \alpha^*$.

Can

$$P_1(\alpha^*) = T^*$$

$$P_2(T) = \alpha^*, \text{ and } T \neq T^*?$$

If $T < T^*$ we have a contradiction. If $T > T^*$, let (x^1_{ij}) and (x^2_{ij}) be the solutions to $\mathcal{P}1$ and $\mathcal{P}2$. Let us assume that the distribution functions are continuous and increasing in T. Then, if we substitute the values (x^1_{ij}) into the objective function of $\mathcal{P}2$, we get

$$\prod_{(i,j)\in G} Prob[D_{ij} \leq \min(T - x^1_{ij}, (T - \gamma_{ij})/\mu_{ij})] > \alpha^*$$

since the distribution functions are non-decreasing in T. This leads to a contradiction, unless the reliability achievable has reached its upper limit. So in the region of interest $T = T^*$. ■

Proposition 3 *The constraint set of $\mathcal{P}2$ is totally unimodular*

Consider

$$x_{ij} - \delta_i + \delta_j = 0, \ \forall (i,j) \in G$$

The submatrix of the constraints corresponding to all the x_{ij}'s is an identity matrix. The columns corresponding to δ_i are that of a node-arc incidence matrix. So the constraint set is totally unimodular (see [Law76]). ■

In reality, a given clock period has associated with it some cost of implementing the digital system to run at T. The trade-offs are as follows:

1. Increasing T reduces the rejection rate and thus decreases costs

2. Decreasing T increases the value of the digital system

3. Increasing Δ_{max} decreases T but increases manufacturing costs

Finding the best tradeoff between T and Δ_{max} could be tedious. However, we obtained the following results that ensure that not many iterations are required to search for the optimal values of T and Δ_{max} that minimize the cost. We first show that the optimal objective function value is a concave function of T.

Proposition 4 *The optimal objective function value of $\mathcal{P}2$ is concave increasing in T*

By the equivalence of $\mathcal{P}1$ and $\mathcal{P}2$, we analyse $\mathcal{P}1$. Let

$$P_1(\Delta^1_{max}, \alpha^1) = T_1$$

$$P_1(\Delta^2_{max}, \alpha^2) = T_2$$

Let T_1 be less than T_2. Let the solutions be $(\delta^1_i), (x^1_{ij})$ and $(\delta^1_i), (x^2_{ij})$ respectively. For any $0 \leq \lambda \leq 1$ let

$$\Delta_{max} = \lambda\Delta^1_{max} + (1-\lambda)\Delta^2_{max}$$

$$x_{ij} = \lambda x^1_{ij} + (1-\lambda)x^2_{ij}$$

$$\delta_i = \lambda\delta^1_i + (1-\lambda)\delta^2_i$$

Then,

$$x_{ij} = \delta_i - \delta_j, \ \forall(i,j) \in G$$

$$0 \leq \delta_i \leq \Delta_{max}, i = 1, \ldots, n$$

And for $\alpha \geq p_0$, the reliabilty constraint is satisfied when we set $T = \alpha T_1 + (1-\alpha)T_2$ since the constraint set is concave. This implies $P_1(\Delta_{max}, \lambda\alpha^1 + (1-\lambda)\alpha^2) \leq T$. Also, T^* is nondecreasing in α. Therefore, if the distribution functions are continuous and increasing in T, T^* is an increasing convex function of α, for $\alpha \geq p_0$. For the distribution functions chosen for analysis these conditions are satisfied. Under the monotonicity assumption we can therefore invert this relationship and obtain α to be concave increasing function of T. (Note also that the convexity of T in Δ_{max} follows).

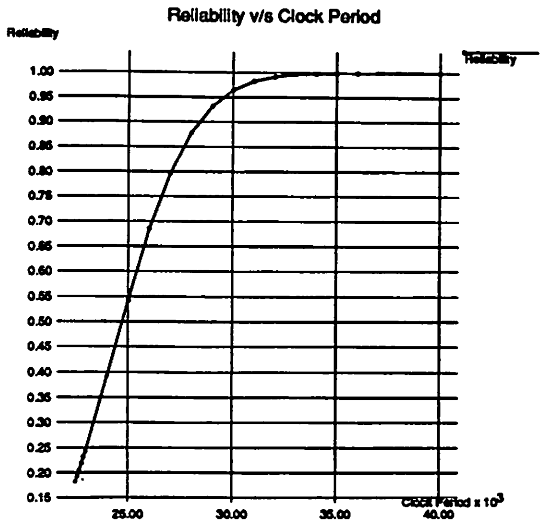

Figure 1: Reliability v/s Clock Period, normal distribution

Proposition 5 *The optimal objective function value of $\mathcal{P}2$ is an increasing concave function of Δ_{max}*

This result is obtained by observing that Δ_{max} does not appear in the objective function. Therefore, for any given granularity in the solution vector to $\mathcal{P}2$, the optimal objective function is a piecewse linear concave function of Δ_{max} [Mur83]. Since it can be easily shown that reliability increases with increasing Δ_{max}, the optimal objective function value is an increasing concave function of Δ_{max}. ∎

4 Results

We have implemented a prototype program that obtains tradeoff curves between P and reliability and Δ_{max} and reliability. Typical analysis of a digital system with about 100 nodes and 375 edges takes a few seconds of CPU time. We tested the program on an industrial example and obtained tradeoff curves for some sample distribution functions for D_{ij}. For a normal distribution, the curves are shown in Figures 1 and 2. The curves for a triangular distribution are shown in Figures 3 and 4. (Note: there may be small deviations from concavity in the plots due to the limited numerical resolution of the plotter). We are currently investigating the best overall sequence of obtaining the cost tradeoffs given two degrees of freedom: Δ_{max} and T.

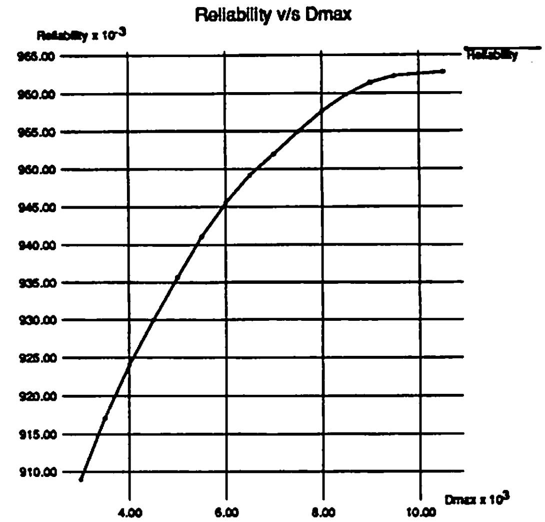

Figure 2: Reliability v/s Δ_{max}, normal distribution, $T_0 = 30000$

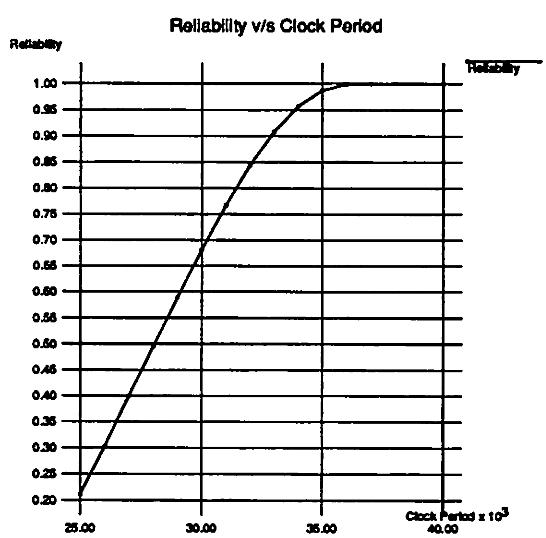

Figure 3: Reliability v/s Clock Period, triangular distribution

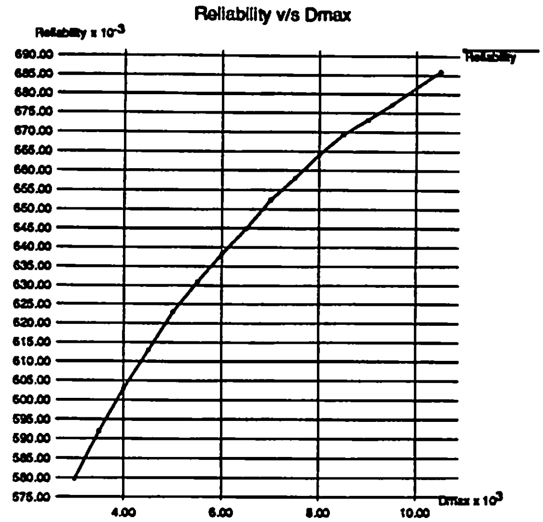

Figure 4: Reliability v/s Δ_{max}, triangular distribution, $T_0 = 30000$

References

[Dre69] S.E. Dreyfus. An appraisal of some shortest path algorithms. *Operations Research*, 17, 1969.

[Ee88] Yu Ermoliev and R.J-B Wets (editors). *Numerical Techniques for Stochastic Optimization.* Springer-Verlag, New York, 1988.

[Fis81] J.P. Fishburn. Clock skew optimization. *AT&T Bell Laboratories. Murray Hill, NJ 07974*, 1981.

[HS89] Dorit S. Hochbaum and J. George Shanthikumar. Convex separable optimization is not nuch harder than linear optimization. *Manuscript, University of California, Berkeley*, 1989.

[Kam84] N. S. Kambo. *Mathematical Programming Techniques.* Affiliated East-West Press, Madras, 1984.

[Law76] Eugene L. Lawler. *Combinatorial Optimization: Networks and Matroids.* Holt, Rinehart and Winston, 1976.

[LRS83] C. Leiserson, F. Rose, and J. Saxe. Optimizing synchronous circuitry by retiming. *Third Caltech Conf. on VLSI*, 1983.

[LS88] Charles E. Leiserson and James B. Saxe. A mixed-integer linear programming problem which is efficiently solvable. *Journal of Algorithms*, 9:114–128, 1988.

[Mur83] Katta G. Murty. *Linear Programming.* John Wiley and Sons, 1983.