A MODEL FOR PRICING INTERRUPTIBLE ELECTRIC POWER SERVICE

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1 Introduction

The current allocation of electric energy is based on a system of fixed prices. In such a system the gap between marginal cost of energy generation and the marginal value of energy consumption, hence the resulting inefficiency, is quite large [10]. One scheme that closes this gap is that of *spot pricing*, [11], [3], [2].

Spot pricing is impractical today because the necessary communications infrastructure is not yet in place. A more practical scheme might employ *future prices*: the power company announces prices a day (or week) in advance and consumers then have the lead time to adjust their demand. The announced future price would depend on forecasts of some of the determinants of supply (e.g. scheduled generator shutdown times) and demand (e.g. weather).

Future prices can more easily be implemented than spot prices, see [1]. However, since a price is announced in period 1 (now) for energy to be delivered and consumed in period 2 (later), and since significant unanticipated fluctuations in supply and demand can occur in the interim, some consumers will be rationed when the actual period 2 demand exceeds the supply. The model we develop in this paper recognizes the cost of rationing borne by frustrated consumers who have their electricity cut off.

Thus a future pricing scheme must take into account rationing loss, and it must ration on the basis of available information. Also there must be a balance between raising prices to reduce rationing-caused losses and lowering prices to increase welfare gains from increased consumption. The *interruptible service contracts* proposed here incorporate both aspects. These are contingent contracts that condition service on particular events or contingencies. A model for the market operation can be described as

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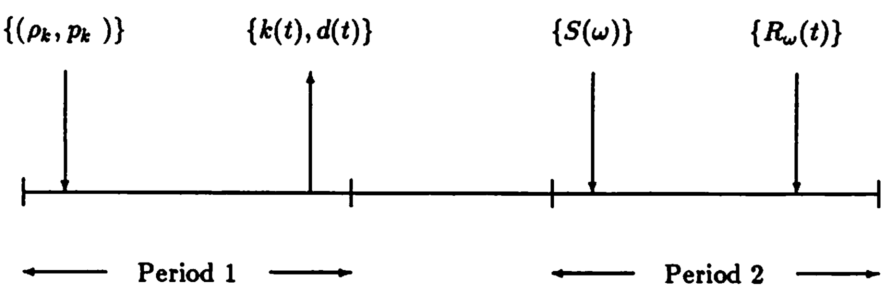

Figure 1: Market operation

a two-step process as depicted in Figure 1. In this paper we assume the supply is random, but the consumer preferences are deterministic.

Step 1. At the beginning of period 1 the power company announces a set of contracts (ρ_k, p_k), $k = 1, \cdots$. Each consumer t chooses one type of contract $k(t)$ and a quantity $d(t)$ kWh of energy for which t pays $p_{k(t)}d(t)$ dollars. In period 2 the company will deliver $d(t)$ kWh of energy to consumer t with probability $\rho_{k(t)}$. With the complementary probability $1 - \rho_{k(t)}$ our consumer will not receive electricity. Thus ρ_k is the guaranteed reliability of service if the kth type of contract is purchased. Note that it is immaterial whether customer t selects the quantity $d(t)$ in period 1 or 2; it is important that the type $k(t)$ is selected in period 1.[1]

Step 2. At the beginning of period 2 the company finds out the actual value of energy supply, $S(\omega)$. Here ω denotes the sample point or *contingency*. The company has to decide which customers to ration so that (i) the total energy delivered does not exceed the available supply for each contingency, and (ii) each customer's contract is fulfilled. The latter decision is represented by the 0-1 valued function $R_\omega(t)$. If $R_\omega(t) = 0$ customer t will not receive service, if it is 1 she will receive $d(t)$ kWh of energy. Hence conditions (i) and (ii) are respectively given by:

$$\sum_t R_\omega(t)d(t) \le S(\omega), \quad \text{for all } \omega \tag{1}$$

$$\text{Prob}\,\{\omega \mid R_\omega(t) = 1\} = \rho_{k(t)}, \quad \text{for all } t \tag{2}$$

[1] If there is randomness in demand, the consumer selects her demand in period 2 after her preference is revealed [9].

The optimal contracts are obtained by first formulating a welfare problem. We then show that the optimum can be sustained by interruptible service contracts. The welfare problem for N types of consumer groups is formulated in §2. In §3 and §4 we consider the consumer and supplier decision problems. In §5 we show that the optimal contract prices are the fixed point of a point-to-set mapping. In §6 we examine several important properties of the fixed point. In §7 we obtain the structure of optimal contracts. The structure turns out to be remarkably simple, and the contract reliabilities are given by an ordering of the contingencies. Some concluding remarks are collected in §8.

The interruptible service contracts discussed here bear a family resemblance to the priority service contracts presented in [4], [7]. However, our consumer model is significantly different, and there are some differences in interpretation.

Part of this work was done while the second author was visiting the International Institute of Applied Systems Analysis in Laxenburg, Austria. He would like to thank Alexander Kurzhanski for the opportunity of working at IIASA.

2 Problem formulation

The structure of optimal contracts is obtained indirectly by first formulating a welfare maximization problem and then by showing that the optimum can be sustained by interruptible service contracts offered to consumers in a decentralized market. We assume that the demand is deterministic and there is no variable supply cost.

We first model the supply side. The total energy supply available (in period 2) is a random variable that takes values $s_i > 0$ with probability $\pi_i > 0$, $i = 1, \cdots, n$. The set of values $\{(s_i, \pi_i)\}$ is known in period 1. However, the actual realization or "contingency" that will occur in period 2 is known only at the beginning of period 2.

Next we model consumer welfare. A consumer is characterized by her preference which consists of a utility and a loss function. We assume there are N distinct types of consumers. The demand of any individual consumer is assumed to be infinitesimal compared with the total demand of all consumers. This permits us to model the set of customers as a continuum indexed by $t \in [0, 1)$.[2] Since there are N types of consumers, the set $t \in [0, 1)$ is partitioned into N sets $G_1, \cdots, G_N$. Let the Lebesgue measure of the set G_j be β_j^*, $j = 1, \cdots, N$; so $\sum_{j=1}^N \beta_j^* = 1$. Suppose a consumer $t \in G_j$ is allocated energy $d(t)$ with reliability $\rho(t)$. The net

[2] With this convention the total number of customers is 1, so the supplies s_i are measured in average kWh per customer.

benefit to this consumer is her welfare. It is given by

$$w(t) = \rho(t)\, U_j(d(t)) - [1 - \rho(t)]\, L_j(d(t)) \tag{3}$$

The interpretation is that if consumer t actually consumes energy $d(t)$ her utility is $U_j(d(t))$, and since this occurs with probability $\rho(t)$, the first term in (3) is the expected utility. But if service is interrupted she suffers a disutility of $L_j(d(t))$, and since that happens with probability $1 - \rho(t)$, the second term in (3) measures the expected rationing loss. The disutility will generally depend on $d(t)$ since the customer planned on using that amount. It is assumed that

$$U_j(0) = L_j(0) = 0;\ U_j'(d) > 0,\ U_j''(d) < 0;\ L_j'(d) \geq 0,\ L_j''(d) \geq 0$$

These are standard assumptions: U_j is strictly concave and L_j is convex, and both are increasing. The total social welfare is the integral

$$W = \int_0^1 w(t)dt = \sum_{j=1}^{N} \int_{t \in G_j} \{\rho(t)\, U_j(d(t)) - [1 - \rho(t)]\, L_j(d(t))\} dt \tag{4}$$

We now consider the allocation problem. In period 1 each t is allocated a pair $(\rho(t), d(t))$. At the beginning of period 2, the contingency is revealed. Suppose it is s_i. The power company now decides which, if any, consumers are to be rationed. This is given by a *rationing function* $R_i : [0,1) \rightarrow \{0,1\}$ defined as

$$R_i(t) = \begin{cases} 0 & \text{if } t \text{ is rationed in contingency } i \\ 1 & \text{otherwise} \end{cases}$$

The rationing function must satisfy the physical constraint

$$\int_0^1 R_i(t)d(t)dt \leq s_i \quad \text{for all } i \tag{5}$$

which simply says that supply meets rationed demand. The rationing functions must also meet the contracts, that is,

$$\sum_{i=1}^{n} \pi_i\, R_i(t) = \rho(t) \quad \text{for all } t \tag{6}$$

The welfare maximization problem is to find functions $d, R_1, \cdots, R_n$ subject to constraints (5) and (6) so as to maximize the total social welfare W. This can be reformulated as an optimal control problem. Introduce the 'state' vector x and the 'control' vector z,

$$x(t) = (x_1(t), \cdots, x_n(t)), \quad z(t) = (d(t), r(t))$$

where

$$r_i(t) := \pi_i R_i(t),\ r(t) = (r_1(t), \cdots, r_n(t)),\ x_i(t) := \int_0^1 \pi_i^{-1} r_i(\tau) d(\tau) d\tau$$

Then the welfare problem can be reformulated as

$$\max\ W = \int_0^1 w(t)dt = \sum_{j=1}^{N} \int_{t \in G_j} \{\rho(t)\ U_j(d(t)) - [1-\rho(t)]\ L_j(d(t))\} dt \tag{7}$$

subject to

$$\dot{x}_i(t) = \frac{1}{\pi_i} d(t) r_i(t),\ t \in [0,1),\ i = 1, ..., n \tag{8}$$

$$x_i(0) = 0,\ x_i(1) \le s_i,\ i = 1, ..., n \tag{9}$$

$$d(t) \ge 0,\ r_i(t) \in \{0, \pi_i\},\ \rho(t) = \sum_{i=1}^{n} r_i(t) \tag{10}$$

This is a standard optimal control problem with state equations (8), state constraints (9), and control constraints (10).

The Maximum Principle [8] gives necessary conditions for a solution of (7)-(10). However, we are interested in sufficiency which will be needed for contract design. For each $j = 1, \cdots, N$, define the *Hamiltonian*

$$H_j(d, r, \mu) = \sum_{i=1}^{n} r_i U_j(d) - [1 - \sum_{i=1}^{n} r_i] L_j(d) - [\sum_{i=1}^{n} \mu_i r_i] d \tag{11}$$

where $d \ge 0$, $r_i \in \{0, \pi_i\}$, and $\mu = (\mu_1, \cdots, \mu_n)$ with $\mu_i \ge 0$. The term $\pi_i \mu_i$ is the *adjoint variable* associated with the supply constraint (9). It is the *scarcity cost* of an additional unit of capacity in contingency i.

Theorem 1 (Sufficiency conditions) *Suppose there exist $\mu^* \ge 0$ and $H_1^*, \cdots, H_N^* \ge 0$ such that for each $j = 1, \cdots, N$, and for all $d \ge 0$ and $r_i \in \{0, \pi_i\}$,*

$$H_j(d, r, \mu^*) \le H_j^* \tag{12}$$

Then the maximum social welfare W^* satisfies

$$W^* = \max\ W \le \sum_{j=1}^{N} \beta_j^* H_j^* + \sum_{i=1}^{n} \pi_i \mu_i^* s_i \tag{13}$$

Moreover, if there is a feasible control $z^* = (d^*, r^*)$ which satisfies

$$H_j(d^*(t), r^*(t), \mu^*) \equiv H_j^*,\ t \in G_j,\ j = 1, \cdots, N \tag{14}$$

and

$$\mu_i^*(s_i - x_i^*(1)) = 0,\ i = 1, \cdots, n \tag{15}$$

then this control is optimal.

Proof. Let z be any feasible control and x the corresponding trajectory. Let W be the welfare attained when control z is applied. From (7), (8), (11), we get

$$\begin{aligned} W &= \sum_{j=1}^{N} \int_{t \in G_j} H_j(d(t), r(t), \mu^*) dt + \sum_{j=1}^{N} \int_{t \in G_j} \sum \mu_i^* r_i(t) d(t) dt \\ &\leq \sum_{j=1}^{N} \beta_j^* H_j^* + \sum_{i=1}^{n} \pi_i \mu_i^* x_i(1) \\ &\leq \sum_{j=1}^{N} \beta_j^* H_j^* + \sum_{i=1}^{n} \pi_i \mu_i^* s_i \end{aligned} \tag{16}$$

where the two inequalities in (16) follow from (12) and (9), respectively. The second part of the assertion follows since (14) and (15) yield equalities in (16). □

Thus an optimal solution z^* maximizes the Hamiltonian $H_j(d, r, \mu^*)$ for each $t \in G_j$, $j = 1, \cdots, N$. Condition (14) means that the net benefit is the same for consumers of the same type. Condition (15) is the complementary slackness condition. It implies, as will be seen later, that at the prevailing prices the power company cannot increase its profit by offering a different set of contracts. Hence (14)-(15) are conditions for consumer equilibrium and supplier equilibrium.

To obtain the optimum, we must find μ^* and $\overline{H}^* = (H_1^*, \cdots, H_N^*)$ that satisfy (14) and (15). We briefly outline the ideas here. First, for $d \geq 0$, $1 \geq \rho \geq 0$, $p \geq 0$, and for each $j = 1, \cdots, N$, define

$$h_j(d, \rho, p) := \rho\, U_j(d) - (1 - \rho)\, L_j(d) - p\, d \tag{17}$$

This is the *net consumer surplus* derived by a consumer in group G_j who purchases d kWh of energy with reliability ρ at unit price p.

Next we order the contingencies in decreasing order of severity so that $s_1 < \cdots < s_n$. The contracted reliability levels are $\rho_1, \cdots, \rho_n$, where $\rho_m := \sum_{i \geq m} \pi_i$. The search for $\overline{H}^*$ can be formulated as a resource allocation problem in which we seek for a fixed point. The fixed point is a vector of prices $\overline{p}^* = (p_1^*, \cdots, p_n^*)$, where p_m^* is the price for the contract with reliability ρ_m. We begin with an arbitrary price system[3] $\overline{p} = (p_1, \cdots, p_n) \geq 0$. The power company offers the n contracts $\{(\rho_m, p_m)\}_{m=1}^{n}$. Each consumer will choose one of these contracts. For this set of prices, the best contract for consumers in group G_j is the one that gives the highest net surplus. However, there may be more than one best choice for consumers in G_j. Hence there is more than one way to allocate contracts to consumers in

[3]The price system in our formulation also includes the price index of all non-electricity commodities. This will be elaborated later.

G_j. Therefore we need to deal with demand correspondences (or point-to-set mappings) rather than demand functions. Note that since every consumer in G_j picks one of her best contracts, they will all end up with the same surplus. Thus (14) is satisfied.

The next step is to derive the supply correspondence, and hence, the excess demand correspondence. We say that a given price system is an equilibrium if the supply meets the demand in all contingencies. Our goal is to show that there is an equilibrium price system for the allocation problem by adjusting the prices appropriately. The idea of the adjustment scheme is to reduce positive excess demands by increasing the prices of those commodities for which the excess demand is the greatest. Suppose the equilibrium prices are $p_1^*, \cdots, p_n^*$. We show that it is possible to construct an optimal solution z^* to the welfare problem from the contracts $\{(\rho_m^*, p_m^*)\}_{m=1}^n$. Hence the contracts offered at the equilibrium prices sustain the optimal solution of the welfare problem. Our desired μ^* and $\overline{H}^*$ can then be computed from these equilibrium prices.

3 Consumer behaviour and allocation of contracts

The main purpose of this section is to describe situations in which the desired actions of the consumers are mutually compatible and can be carried out simultaneously (i.e. in a decentralized market), and for which we can prove that there exists a set of prices that will cause consumers to make mutually compatible decisions.

3.1 A decentralized consumer decision problem

In period 1 the power company offers the n contracts $\{(\rho_m, p_m)\}_{m=1}^n$, where $p_m \geq 0$ is the price of the contract with reliability ρ_m. Each consumer selects one of these contracts. Consider a consumer in group G_j. To decide which contract to pick, she needs to compute the net surplus derived by purchasing a contract (p_m, ρ_m), $m = 1, \cdots, n$. Since each consumer in G_j picks only *one* contract, she will be faced with the following decision problem if she decides to purchase contract (ρ_m, p_m):

$$\max_{0 \leq d \leq M,\ b \in \Re} \rho_m U_j(d) - (1 - \rho_m) L_j(d) + b \tag{18}$$

subject to the constraint

$$p_m d + p_0 d \leq I \tag{19}$$

The constant I, measured in dollars, is the *income* of each consumer. A consumer in G_j spends her income on electricity and other commodities such as food. The price $p_0 \geq 0$ is a *price index* for all commodities other than electricity, and b is the amount of this composite commodity. Thus

(19) is the income constraint. Let the optimal consumption of problem (18)-(19) be denoted by $d_{jm}(p_0, p_m, I)$ and $b(p_0, p_m, I)$. We shall assume $M >> s_n$. Note that if we allow the demand to be a free variable in $[0, \infty)$, it may be unbounded if $p_m = 0$. However, since the supply that is actually available to a consumer is bounded above by s_n, the exclusion of extremely large demand from consideration is justified. Moreover, the limitation of the choice of demand not to exceed the quantity M will not alter the results of our analysis.

Remark 1. The demand $d_{jm}(p_0, p_m, I)$ is homogeneous of degree zero in (p_0, p_m, I). That is, $d_{jm}(\lambda p_0, \lambda p_m, \lambda I) = d_{jm}(p_0, p_m, I)$ for any $\lambda > 0$.

Remark 2. It is not difficult to see that the optimal solution of problem (18)-(19) always satisfies the income constraint with equality. That is, we always have

$$p_m d_{jm}(p_0, p_m, I) + p_0 b(p_0, p_m, I) = I$$

Then problem (18)-(19) is equivalent to

$$\max_{0 \leq d \leq M} \; p_0\{\rho_m U_j(d) - (1 - \rho_m)L_j(d) + I - p_m d\} \tag{20}$$

Since I is a constant, (20) reduces to

$$\max_{0 \leq d \leq M} \; \rho_m U_j(d) - (1 - \rho_m)L_j(d) - p_m d \tag{21}$$

Thus the demand for electricity is independent of income I. From now on, we let $d_{jm} := d_{jm}(p_m) = d_{jm}(p_0, p_m, I)$. Note from (21) that $d_{jm} = \arg\max_{d \geq 0} h_j(d, \rho_m, p_m)$.

3.2 Allocation rule

If consumer $t \in G_j$ purchases contract (ρ_m, p_m), then her surplus is

$$H_{jm} := h_j(d_{jm}, \rho_m, p_m) \geq 0$$

Let $H_j := max_{1 \leq m \leq n} H_{jm}$. Suppose the contracts $\{(\rho_m, p_m)\}_{m \in I_j}$, where $I_j \subseteq \{1, \cdots, n\}$, achieve the surplus H_j. Then each consumer in G_j will pick a contract from $\{(\rho_m, p_m)\}_{m \in I_j}$. Since the contracts indexed by elements in I_j are indifferent to consumers in G_j, we see that in case the cardinality of I_j is greater than one, there is more than one way to allocate the contracts to consumers in G_j. Let β_{jm}, $1 \leq m \leq n$, denote the number of consumers in G_j who are assigned the contract (ρ_m, p_m). The β_{jm} are chosen by the following rule:

<u>Allocation rule:</u>

$$\beta_{jm} \geq 0 \; ; \; \beta_{jm} := 0 \text{ if } H_{jm} < H_j \tag{22}$$

and

$$\sum_{m=1}^{n} \beta_{jm} = \beta_j^* \tag{23}$$

The allocation rule ensures that each consumer in G_j gets the same net surplus. It also implies that each consumer is allocated only one contract. Thus an individual demand for electricity is a point in $\Re^n_+$ (since there are n types of electricity to choose from) with at most one positive component. For a given price system $(p_0, \overline{p}) := (p_0, p_1, \cdots, p_n)$, let

$$\Phi_j(p_0, \overline{p}) := \{\overline{\beta_j} = (\beta_{j1}, \cdots, \beta_{jn}) \mid \overline{\beta_j} \text{ satisfies (22)-(23)}\} \tag{24}$$

be the set of *preferred allocations* for group G_j. Let $d_{jm} := d_{jm}(p_0, \overline{p}) = d_{jm}(p_m)$. The aggregate demand for electricity of type m (i.e. electricity to be delivered at reliability ρ_m) is the set

$$d_m(p_0, \overline{p}) := \{\sum_{j=1}^{N} \beta_{jm} d_{jm} \mid \beta_{jm} \text{ satisfies (22)-(23)}, j = 1, \cdots, N\} \tag{25}$$

The demand for the n types of electricity from group G_j is the set

$$D_j(p_0, \overline{p}) := \{(\beta_{j1} d_{j1}, \cdots, \beta_{jn} d_{jn}) \mid \overline{\beta_j} \in \Phi_j(p_0, \overline{p})\} \tag{26}$$

The *demand correspondence* (or consumption set) for electricity is given by

$$\begin{aligned} D(p_0, \overline{p}) &:= \sum_{j=1}^{N} D_j(p_0, \overline{p}) \\ &= \{(\sum_{j=1}^{N} \beta_{j1} d_{j1}, \cdots, \sum_{j=1}^{N} \beta_{jn} d_{jn}) \mid \overline{\beta_j} \in \Phi_j(p_0, \overline{p}), \\ &\qquad j = 1, 2, \cdots, N\} \end{aligned} \tag{27}$$

Remark 3. By Remark 1 the demand correspondence is homogeneous of degree zero in $(p_0, \overline{p}, I)$.

4 Supplier's production plan

We consider the supplier's plan of action. The production plan of the supplier is constrained to belong to the *technology set*, which represents all feasible supplies of electricity of each type. Let q_m be a supply quantity for electricity of type m. That is, the amount q_m will serve exclusively those consumers who have chosen contract (ρ_m, p_m). Since only those consumers who have chosen one of the contracts in $\{(\rho_m, p_m)\}_{m\leq i}$ will be served in contingency i, we must have $0 \leq q_1 + \cdots + q_i \leq s_i$. So the technology set T is described by

$$T := \{\overline{q} = (q_1, \cdots, q_n) \mid q_m \geq 0 \; ; \; q_1 + \cdots + q_i \leq s_i, 1 \leq i \leq n\} \qquad (28)$$

The supplier chooses a production plan in the technology set T that maximizes profit for given prices $(p_0, \overline{p})$. For a given price system $\overline{p}$ and production plan $\overline{q}$ for electricity, the profit is given by $< \overline{p}, \overline{q} > := \sum_{m=1}^{n} p_m q_m$. The *supply correspondence* is given by

$$S(p_0, \overline{p}) := \{\overline{q} \in T \mid \; < \overline{p}, \overline{q} > \geq < \overline{p}, \overline{q}' > \text{ for all } \overline{q}' \in T\} \qquad (29)$$

Remark 4. The supply correspondence $S(p_0, \overline{p})$ is homogeneous of degree zero in $(p_0, \overline{p})$. Let $\pi(p_0, \overline{p})$ be the maximum profit when the price system is $(p_0, \overline{p})$. Then $\pi(p_0, \overline{p}) = \max_{\overline{q}\in T} < \overline{p}, \overline{q} >$ is homogeneous of degree one in $(p_0, \overline{p})$.

The following lemma characterizes the set $S(p_0, \overline{p})$. By this lemma, if a contract with higher reliability is offered at a lower price than a contract with lower reliability, then it is more profitable for the supplier not to supply electricity of the higher reliability type.

Lemma 1 *Suppose $\overline{q} \in S(p_0, \overline{p})$ and $p_l < p_k$ for some $1 \leq l < k \leq n$. Then $q_l = 0$.*

Proof. Suppose by contradiction that $q_l > 0$. Consider the alternative production plan $\overline{q}' = (q_1', \cdots, q_n')$ given by

$$q_m' = \begin{cases} q_m & , m \neq l, k \\ 0 & , m = l \\ q_l + q_k & , m = k \end{cases}$$

To see that $\overline{q}'$ is technologically feasible, we let $Q_i := \sum_{m\leq i} q_m$ and $Q_i' := \sum_{m\leq i} q_m'$. Then we get $Q_i' = Q_i$ for $1 \leq i < l$ and $i \geq k$, and $Q_i' < Q_i \leq s_i$ for $l \leq i < k$. So $\overline{q}' \in T$. Next

$$< \overline{p}, \overline{q} > - < \overline{p}, \overline{q}' > = q_l(p_l - p_k) < 0$$

which contradicts the hypothesis that $\overline{q} \in S(p_0, \overline{p})$. $\square$

5 Existence of equilibrium prices

To define an equilibrium we must have a "closed" economy. So we must model what happens to the company profits. We do this by assuming a private ownership economy [5]. Let $\theta_j \geq 0$ be the *share* of the supplier's profit for group G_j. The shares $\theta_1, \cdots, \theta_N$ satisfy $\sum_{j=1}^{N} \theta_j = 1$. Suppose $\overline{q} \in S(p_0, \overline{p})$ when the price system is $(p_0, \overline{p})$. Since the profit $\pi(p_0, \overline{p}) = < \overline{p}, \overline{q} >$ is distributed to shareholders, the income of group G_j is given by

$$\beta_j^* I = \int_{t \in G_j} p_0 b_j(t) + \theta_j \pi(p_0, \overline{p}) \tag{30}$$

where $b_j(t)$ is the quantity of the composite commodity that consumer $t \in G_j$ has purchased.

For each $(p_0, \overline{p})$, define the set

$$X(p_0, \overline{p}) := D(p_0, \overline{p}) - S(p_0, \overline{p}) \tag{31}$$

This is a nonempty set of excess demands compatible with the selection by each consumer of a consumption optimal for her income constraint and by the supplier of a supply optimal for that price system. The point-to-set map $(p_0, \overline{p}) \mapsto X(p_0, \overline{p})$ is called the *excess demand correspondence.* We say that $(p_0, \overline{p})$ is an equilibrium price system for electricity if $\overline{x} \leq 0$ for some $\overline{x} \in X(p_0, \overline{p})$. That is, there is an allocation given by the allocation rule (22)-(23) such that the supply meets the demand for each type of electricity. We will show that an equilibrium price system exists and it induces a set of contracts that is optimal for the welfare problem. This is done by showing that a certain correspondence has a fixed point. The fixed point is our desired equilibrium price system. We appeal to the Kakutani Theorem [6] to show the existence of such fixed point.

By Remarks 3 and 4 the excess demand correspondence is homogeneous of degree zero in $(p_0, \overline{p})$. Hence we may fix the level of $(p_0, \overline{p})$ arbitrarily without restricting our analysis in any way. For our purpose, this is most conveniently done by considering those points that belong to the n-dimensional simplex Σ_n, where

$$\Sigma_n := \{(p_0, \overline{p}) \mid \sum_{i=0}^{n} p_i = 1 \, , \; p_i \geq 0\}$$

When $\overline{p} \neq 0$, the set $X(p_0, \overline{p})$ is below the hyperplane through the origin and orthogonal to $\overline{p}$. This is given by the next result.

Lemma 2 *For each $(p_0, \overline{p}) \in \Sigma_n$, $< \overline{p}, \overline{x} > \leq 0$ for all $\overline{x} \in X(p_0, \overline{p})$.*

Proof. Let $\overline{x} \in X(p_0, \overline{p})$. Then $\overline{x} = \overline{d} - \overline{q}$ for some $\overline{d} \in D(p_0, \overline{p})$ and $\overline{q} \in S(p_0, \overline{p})$. From the definition of the demand correspondence there are $\overline{\beta_1}, \cdots, \overline{\beta_N}$ such that $\overline{\beta_j} \in \Phi_j(p_0, \overline{p})$ is a feasible allocation for group G_j and $\overline{d} = (\sum_{j=1}^{N} \beta_{j1} d_{j1}, \cdots, \sum_{j=1}^{N} \beta_{jn} d_{jn})$. Let $\overline{d_j} = (\beta_{j1} d_{j1}, \cdots, \beta_{jn} d_{jn}), j =$

$1, \cdots, N$. The income constraint gives

$$\sum_{m=1}^{n} p_m \beta_{jm} d_{jm} + \int_{t \in G_j} p_0 b_j(t) dt \ \leq \ \beta_j^* I \tag{32}$$

Then (30) and (32) imply $< \overline{p}, \overline{d_j} > \leq \ \theta_j \pi(p_0, \overline{p})$. Summing over j gives $< \overline{p}, \overline{d} > \leq \ \sum_{j=1}^{N} \theta_j \pi(p_0, \overline{p}) = \pi(p_0, \overline{p})$ since $\sum_{j=1}^{N} \theta_j = 1$. Finally, since $\pi(p_0, \overline{p}) =< \overline{p}, \overline{q} >$, we get $< \overline{p}, \overline{d} - \overline{q} > \leq 0$ as desired. □

Lemma 3 *For each $(p_0, \overline{p}) \in \Sigma_n$ and $j = 1, \cdots, N$, the set $\Phi_j(p_0, \overline{p})$ is a nonempty, compact, and convex subset of $\Re_+^n$.*

Proof. The set $I_j = \{1 \leq m \leq n \mid H_{jm} = H_j\}$ is clearly nonempty. Then by the allocation rule the set $\Phi_j(p_0, \overline{p})$ is nonempty. It is also compact since those nonzero β_{jm} are determined by (23). The convexity property is obvious since (23) is a linear constraint. □

The notion of upper semi-continuity was introduced in proving the Kakutani Theorem. We repeat the definition for completeness.

Definition 1 *Let Ψ be a correspondence from X to Y, and $x \in X$. Ψ is* upper semi-continuous *(u.s.c.) at x if for every sequence $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$, the conditions $x_n \to x$, $y_n \to y$, $y_n \in \Psi(x_n)$ imply $y \in \Psi(x)$. Ψ is u.s.c. if it is u.s.c. at every $x \in X$.*

Lemma 4 *For each $j = 1, \cdots, N$, the correspondence $(p_0, \overline{p}) \mapsto \Phi_j(p_0, \overline{p})$ is upper semi-continuous.*

Proof. Let $(p_0, \overline{p}) \in \Sigma_n$ and $\{(p_0^l, \overline{p}^l) = (p_0^l, p_1^l, \cdots, p_n^l)\}_{l=1}^{\infty} \subset \Sigma_n$. Also let $\overline{\beta_j}^l = (\beta_{j1}^l, \cdots, \beta_{jn}^l) \in \Phi_j(p_0^l, \overline{p}^l)$. Suppose $(p_0^l, \overline{p}^l) \to (p_0, \overline{p})$ and $\overline{\beta_j}^l \to \overline{\beta_j}$. We need to show $\overline{\beta_j} \in \Phi_j(p_0, \overline{p})$. Let $I_j = \{m \mid H_{jm} = H_j\}$. Also let $H_{jm}^l = h_j(d_{jm}(p_0^l, \overline{p}^l), \rho_m, p_m^l)$ and $H_j^l = \max_{1 \leq m \leq n} H_{jm}^l$. From (18)-(19) it is easy to see that $h_j(d_{jm}(p_0, \overline{p}), \rho_m, p_m)$ is continuous in $(p_0, \overline{p})$. So there is a positive integer l^* such that $H_{jm}^l < H_j$ for all $m \notin I_j$ and $l \geq l^*$. The allocation rule then gives $\beta_{jm}^l = 0$ for $m \notin I_j$ and $l \geq l^*$. In particular, we get

$$\beta_{jm}^l \xrightarrow{l \to \infty} \beta_{jm}^* := 0 \ , \ m \notin I_j \tag{33}$$

Next consider $m \in I_j$. We have $\beta_{jm}^l = 0$ if $H_{jm}^l < H_j^l$. If $H_{jm}^l = H_j^l$, then $\beta_{jm}^l \geq 0$ is determined by (23) (see allocation rule). Since $\{\beta_{jm}^l\}_{l=1}^{\infty}$ converges and $\beta_{jm}^l = 0$ is also a solution of (23), we must have $\beta_{jm}^l \to \beta_{jm}^* \geq 0$, where β_{jm}^* is a solution of (23). This observation and (33) imply $\overline{\beta_j}^* = (\beta_{j1}^*, \cdots, \beta_{jn}^*) \in \Phi_j(p_0, \overline{p})$. Since $\overline{\beta_j}^l \to \overline{\beta_j}$, we get $\overline{\beta_j} = \overline{\beta_j}^* \in \Phi_j(p_0, \overline{p})$. So the u.s.c. property is established. □

Lemma 5 *For each $(p_0, \overline{p}) \in \Sigma_n$, the set $X(p_0, \overline{p})$ is nonempty, compact, and convex. Moreover, $X(p_0, \overline{p})$ is upper semi-continuous.*

Proof. $S(p_0, \overline{p})$ is clearly nonempty and convex. Its compactness follows from that of the technology set T. Next since $\Phi_j(p_0, \overline{p})$ is nonempty and convex, the set $D(p_0, \overline{p})$ is nonempty and convex. By Lemma 3 $\Phi_j(p_0, \overline{p})$ is compact. Also since $0 \leq d_{jm} \leq M$, and d_{jm} is continuous in $(p_0, \overline{p})$ for all j and m, $D(p_0, \overline{p})$ is compact.

It is immediate that the supply correspondence is u.s.c. So it remains to show that $D(p_0, \overline{p})$ is u.s.c. The demand $d_{jm}(p_0, \overline{p})$ is continuous in $(p_0, \overline{p})$ for each j. This and Lemma 4 imply that $D_j(p_0, \overline{p})$ defined in (26) is u.s.c. The claim follows since $D(p_0, \overline{p}) = \sum_{j=1}^{N} D_j(p_0, \overline{p})$. □

We would like to adjust the current price system so that an equilibrium price system can be obtained. The idea of the adjustment scheme is to increase the prices of those types of electricity for which the excess demand is the greatest. We begin by considering the excess demand correspondence. By Lemma 5 the set $X(p_0, \overline{p})$ is compact for all $(p_0, \overline{p}) \in \Sigma_n$. So there is a compact and convex set $\Sigma \subset \Re^n$ such that $X(p_0, \overline{p}) \subset \Sigma$ for all $(p_0, \overline{p}) \in \Sigma_n$. Let $\overline{x} \in \Sigma$. Define the set $A(\overline{x})$ by

$$A(\overline{x}) := \{(p_0, \overline{p}) \in \Sigma_n \mid \; < \overline{p}, \overline{x} > \; \geq \; < \overline{p}', \overline{x} > \text{ for all } (p_0', \overline{p}') \in \Sigma_n\} \quad (34)$$

If $\overline{x} \in X(p_0, \overline{p})$, then $A(\overline{x})$ adjusts $(p_0, \overline{p})$ to a set of price systems given by (34). Define the correspondence Ψ from $\Sigma_n \times \Sigma$ to itself by

$$((p_0, \overline{p}), \overline{x}) \mapsto \Psi((p_0, \overline{p}), \overline{x}) := A(\overline{x}) \times X(p_0, \overline{p}) \quad (35)$$

Lemma 6 *For each $\overline{x} \in \Sigma$, the set $A(\overline{x})$ is nonempty, convex, and compact. Moreover, the correspondence $\overline{x} \mapsto A(\overline{x})$ is u.s.c.*

Proof. Since Σ_n is nonempty and compact, the set $A(\overline{x})$ is nonempty and bounded. From the definition of $A(\overline{x})$, the set is clearly closed. So $A(\overline{x})$ is compact. The convexity property follows from that of Σ_n and the linear constraint in (34). The u.s.c. property is straightforward to check. □

Proposition 1 *The correspondence Ψ has a fixed point, i.e. there exists a point $((p_0^*, \overline{p}^*), \overline{x}^*) \in \Sigma_n \times \Sigma$ such that $(p_0^*, \overline{p}^*) \in A(\overline{x}^*)$ and $\overline{x}^* \in X(p_0^*, \overline{p}^*)$.*

Proof. The product set $\Sigma_n \times \Sigma$ is nonempty, compact, and convex since Σ_n and Σ are. By Lemmas 5 and 6 the set $A(\overline{x}) \times X(p_0, \overline{p})$ is nonempty, compact, and convex for all $((p_0, \overline{p}), \overline{x}) \in \Sigma_n \times \Sigma$. These two lemmas also imply the correspondence Ψ is u.s.c. Therefore all conditions of the Kakutani Theorem are satisfied, and Ψ has a fixed point. □

Lemma 7 *There exist $(p_0^*, \overline{p}^*) \in \Sigma_n$ and $\overline{x}^* = (x_1^*, \cdots, x_n^*) \in X(p_0^*, \overline{p}^*)$ such that $x_m^* \leq 0$ for all $m = 1, \cdots, n$.*

Proof. By Proposition 1 there exists $((p_0^*, \overline{p}^*), \overline{x}^*) \in \Sigma_n \times \Sigma$ with the property

$$(p_0^*, \overline{p}^*) \in A(\overline{x}^*) \text{ and } \overline{x}^* \in X(p_0^*, \overline{p}^*) \tag{36}$$

The first relation in (36) implies $< \overline{p}^*, \overline{x}^* > \geq < \overline{p}', \overline{x}^* >$ for all $(p_0', \overline{p}') \in \Sigma_n$. By Lemma 2 the second relation in (36) implies $< \overline{p}^*, \overline{x}^* > \leq 0$. Hence

$$< \overline{p}', \overline{x}^* > \leq \; 0 \text{ for all } (p_0', \overline{p}') \in \Sigma_n \tag{37}$$

For each $m = 1, \cdots, n$, consider the point in Σ_n with $p_m = 1$ and $p_i = 0$ for $i \neq m$. Then (37) gives $x_m^* \leq 0$. This completes the proof. □

Hence the fixed point $(p_0^*, \overline{p}^*)$ obtained in Proposition 1 is an equilibrium price system. We obtain further properties of this price system.

Lemma 8 (Walrasian property) *The prices $\overline{p}^* = (p_1^*, \cdots, p_n^*)$ and the excess demand vector $\overline{x}^* = (x_1^*, \cdots, x_n^*)$ satisfy $p_m^* x_m^* = 0$ for all m. Hence $< \overline{p}^*, \overline{x}^* >= 0$.*

Proof. By Lemma 7 we have $x_m^* \leq 0$. So $p_m^* x_m^* \leq 0$, and it suffices to show that for each m, $p_m^* x_m^* < 0$ is not possible. We prove this by contradiction. Suppose $p_l^* > 0$ and $x_l^* < 0$ for some l. Consider the new price system $(p_0', \overline{p}')$ defined by

$$p_m' = \begin{cases} p_m^* & , m \neq 0, l \\ 0 & , m = l \\ p_0^* + p_l^* & , m = 0 \end{cases}$$

Then $(p_0', \overline{p}') \in \Sigma_n$, $(p_0', \overline{p}') \neq (p_0^*, \overline{p}^*)$, and $< \overline{p}^*, \overline{x}^* > - < \overline{p}', \overline{x}^* >= p_l^* x_l^* < 0$. Clearly this contradicts $(p_0^*, \overline{p}^*) \in A(\overline{x}^*)$. □

By Lemma 8, if $x_m^* < 0$ so that there is a positive excess supply of electricity of type m, then the contract (ρ_m^*, p_m^*) is *free* and offered at price $p_m^* = 0$.

6 Properties of an equilibrium price system

In this section we derive some important properties of the equilibrium price system obtained in §5. These results will be used in §7 to construct a set of contracts that sustain the optimal solution of the welfare problem (7)-(10). The next lemma is useful for several of the subsequent results.

Lemma 9 *If $p_m^* \leq p_k^*$, where $k > m$, then no consumer will purchase the contract (ρ_k, p_k^*).*

Proof. Since $\rho_m > \rho_k$, the strict inequality

$$\rho_m U_j(d) - (1 - \rho_m) L_j(d) - p_m^* d > \rho_k U_j(d) - (1 - \rho_k) L_j(d) - p_k^* d$$

holds for all j and $0 < d \leq M$. So $H^*_{jm} := h_j(d_{jm}(p^*_0, \overrightarrow{p}^*), \rho_m, p^*_m) > H^*_{jk}$ for all j. Therefore $\beta_{jk} = 0$ for j, and hence, $d^*_k = 0$. $\square$

The following result gives an ordering on the equilibrium prices.

Lemma 10 *The prices $\overrightarrow{p}^* = (p^*_1, \cdots, p^*_n)$ are monotonically decreasing in the order of decreasing contingency severity, i.e. $p^*_1 \geq p^*_2 \geq \cdots \geq p^*_n$.*

Proof. We prove the assertion by contradiction. Suppose $p^*_i < p^*_{i+1}$ for some $1 \leq i \leq n-1$. Let $\overrightarrow{d}^* \in D(p^*_0, \overrightarrow{p}^*)$ and $\overrightarrow{q}^* \in S(p^*_0, \overrightarrow{p}^*)$ be such $\overrightarrow{x}^* = \overrightarrow{d}^* - \overrightarrow{q}^*$. Then Lemma 9 gives $d^*_{i+1} = 0$. Since $p^*_{i+1} > p^*_i \geq 0$, Lemma 8 implies $q^*_{i+1} = d^*_{i+1} = 0$. We need to consider two cases.

<u>Case 1</u>: $q^*_m = 0$ for all $m > i+1$. (If $i+1 = n$, take $q^*_{n+1} = 0$.) Consider $\overrightarrow{q}' = (q'_1, \cdots, q'_n)$ with $q'_m = q^*_m$ for $m \neq i+1$, and $q'_{i+1} = s_{i+1} - s_i$. Since $\sum_{m \leq i} q^*_m \leq s_i$ and $q^*_m = 0$ for all $m > i+1$, we see that $\overrightarrow{q}' \in T$. Next we obtain

$$< \overrightarrow{p}^*, \overrightarrow{q}' > - < \overrightarrow{p}^*, \overrightarrow{q}^* > = p^*_{i+1}(s_{i+1} - s_i) > 0$$

But this contradicts $\overrightarrow{q}^* \in S(p^*_0, \overrightarrow{p}^*)$.

<u>Case 2</u>: There exists $m > i+1$ such that $q^*_m > 0$. Let $k := \min\{m > i+1 \mid q^*_m > 0\}$. We claim that $p^*_{i+1} > p^*_k$. If $p^*_k \geq p^*_{i+1} > 0$, then by Lemmas 8 and 9 we get $q^*_k = d^*_k = 0$. This contradicts $q^*_k > 0$. So we must have $p^*_{i+1} > p^*_k$. Consider $\overrightarrow{q}'$ defined by $q'_m = q^*_m$ for $m \notin \{i+1, k\}$, $q'_{i+1} = \min\{s_{i+1} - s_i, q^*_k\}$, and $q'_k = \max\{q^*_k - (s_{i+1} - s_i), 0\}$. Then $\overrightarrow{q}' \in T$ since $q'_m = 0$ for $i+1 < m < k$. Next we obtain

$$\begin{aligned} < \overrightarrow{p}^*, \overrightarrow{q}' > - < \overrightarrow{p}^*, \overrightarrow{q}^* > &= p^*_{i+1} q'_{i+1} + p^*_k q'_k - p^*_k q^*_k \\ &> p^*_k (q'_{i+1} + q'_k) - p^*_k q^*_k \\ &= p^*_k \ \max\{q^*_k, s_{i+1} - s_i\} - p^*_k q^*_k \\ &\geq 0 \end{aligned}$$

Again this contradicts $\overrightarrow{q}^* \in S(p^*_0, \overrightarrow{p}^*)$. $\square$

The next result is useful for obtaining the complementary slackness condition (15).

Proposition 2 *Consider the fixed point $(p^*_0, \overrightarrow{p}^*)$. Let $\overrightarrow{d}^* \in D(p^*_0, \overrightarrow{p}^*)$ and $\overrightarrow{q}^* \in S(p^*_0, \overrightarrow{p}^*)$ be such that $\overrightarrow{x}^* = \overrightarrow{d}^* - \overrightarrow{q}^*$. Then for each $1 \leq i \leq n$,*

$$p^*_i > 0 \textit{ implies } s_i = \sum_{m \leq i} q^*_m = \sum_{m \leq i} d^*_m \tag{38}$$

Proof. The proof is carried out by an induction on i.

<u>Step 1</u>: $i = 1$. Suppose $p^*_1 > 0$. We claim that $q^*_1 = s_1$. We show this by contradiction. So suppose $q^*_1 < s_1$. By Lemma 10 the prices are ordered

as $p_1^* \geq \cdots \geq p_n^* \geq 0$. There are two cases to be considered.
(i) If $p_1^* = \cdots = p_n^*$, then by Lemma 9 we get $d_m^* = 0$ for all $m \geq 2$. Now by Lemma 8 we obtain $q_m^* = d_m^* = 0$ for all $m \geq 2$. So the production $(s_1, q_1^*, \cdots, q_n^*)$ is feasible and yields a profit strictly greater than that of $\overline{q}^*$. This contradicts $\overline{q}^* \in S(p_0^*, \overline{p}^*)$.
(ii) If there is $1 \leq k < n$ such that $p_1^* = \cdots = p_k^* > p_{k+1}^* \geq \cdots \geq p_n^*$, then $q_2^* = \cdots = q_k^* = 0$. (If $k = 1$, we simply have $p_1^* > p_2^* \geq \cdots \geq p_n^*$.) Next if $\sum_{m \geq k+1} q_m^* \leq s_1 - q_1^*$, then the production $\overline{q}' := (s_1, 0, \cdots, 0)$ is feasible and

$$
\begin{aligned}
< \overline{p}^*, \overline{q}' > - < \overline{p}^*, \overline{q}^* > &= p_1^*(s_1 - q_1^*) - \sum_{m \geq k+1} p_m^* q_m^* \\
&> p_1^*(s_1 - q_1^*) - p_1^* \sum_{m \geq k+1} q_m^* \\
&\geq 0
\end{aligned}
$$

This contradicts $\overline{q}^* \in S(p_0^*, \overline{p}^*)$. Now if $\sum_{m \geq k+1} q_m^* > s_1 - q_1^*$, then there is a *smallest* integer h in $\{k+1, \cdots, n\}$ such that $\sum_{k+1 \leq m \leq h} q_m^* \geq s_1 - q_1^*$. Let

$$\delta := \sum_{k+1 \leq m \leq h} q_m^* - (s_1 - q_1^*)$$

By our definition of the integer h, we have $0 \leq \delta \leq q_h^*$. Next consider $\overline{q}''$ defined by $q_1'' = s_1$, $q_2'' = \cdots = q_{h-1}'' = 0$, $q_h'' = \delta$, and $q_m'' = q_m^*$ for $m > h$. It is easy to check that $\overline{q}'' \in T$. We also obtain

$$
\begin{aligned}
< \overline{p}^*, \overline{q}'' > - < \overline{p}^*, \overline{q}^* > &= p_1^*(s_1 - q_1^*) + p_h^* \delta - \sum_{k+1 \leq m \leq h} p_m^* q_m^* \\
&= p_1^*(s_1 - q_1^*) - \sum_{k+1 \leq m < h} p_m^* q_m^* - p_h^*(q_h^* - \delta) \\
&> p_1^*(s_1 - q_1^*) - p_1^* \sum_{k+1 \leq m < h} q_m^* - p_1^*(q_h^* - \delta) \\
&= 0
\end{aligned}
$$

where the strict inequality follows from the monotonic property of the prices and the fact that $q_h^* \geq \delta$. Again we get the same contradiction. The equality $q_1^* = d_1^*$ follows from Lemma 8.
Step 2: Induction assumption. Assume (38) holds for all $i = 1, 2, \cdots, l$.
Step 3: $i = l + 1$. Suppose $p_{l+1}^* > 0$. By Lemma 10 we get $p_l^* > 0$. So the induction assumption gives $s_l = \sum_{m \leq l} q_m^*$. Hence $0 \leq q_{l+1}^* \leq s_{l+1} - s_l$. By the same argument used in Step 1, we infer that $q_{l+1}^* = s_{l+1} - s_l$. Consequently, we get $s_{l+1} = \sum_{m \leq l+1} q_m^*$. The other equality in (38) follows from Lemma 8. This completes the proof. □

Corollary 1 *Let $\overrightarrow{q}^*$ be as given in Proposition 2. If $p_i^* > 0$, then $q_m^* = s_m - s_{m-1} > 0$ for all $m \leq i$. (Take $s_0 = 0$.)*

Proof. Since $p_i^* > 0$, Lemma 10 gives $p_m^* > 0$ for all $m = 1, \cdots, i$. The claim then follows from Proposition 2. □

Lemma 11 *If $p_i^* > 0$, then $p_i^* > p_{i+1}^*$.*

Proof. Let $\overrightarrow{d}^*$ and $\overrightarrow{q}^*$ be given as in the proof of Lemma 10. By the same lemma it remains to show that $p_i^* = p_{i+1}^*$ is not possible. Suppose $p_i^* = p_{i+1}^*$. Then by Lemma 9 we get $d_{i+1}^* = 0$. On the other hand, since $p_{i+1}^* = p_i^* > 0$, Corollary 1 implies $q_{i+1}^* > 0$. Hence we get $x_{i+1}^* < 0$. This contradicts Lemma 8 since $p_{i+1}^* > 0$. □

An immediate consequence of Lemma 10 and Lemma 11 is the following corollary.

Corollary 2 *Let k be the smallest integer in $\{1, \cdots, n\}$ such that $p_k^* = 0$. Then $p_1^* > \cdots > p_{k-1}^* > p_k^* = \cdots = p_n^* = 0$. If this integer k does not exist, then $p_1^* > \cdots > p_n^* > 0$.* □

Lemma 12 *Consider the fixed point $((p_0^*, \overline{p}^*), \overline{x}^*)$. If $p_i^* > 0$, then contract (ρ_i, p_i^*) will be purchased by some consumers in $[0, 1)$.*

Proof. Let $\overrightarrow{d}^*$ and $\overrightarrow{q}^*$ be given as in the proof of Lemma 10. Lemma 8 and Corollary 1 imply $d_i^* = q_i^* > 0$. Since $d_i^* = \sum_{j=1}^{N} \beta_{ji} d_{ji}^*$, we must have $\beta_{ji} > 0$ for some j. □

Corollary 3 *(i) If the contract (ρ_i, p_i^*) is purchased by some consumers in $[0, 1)$, then so will be the contracts $(\rho_1, p_1^*), \cdots (\rho_{i-1}, p_{i-1}^*)$.*
(ii) Let $1 \leq k \leq n$ be the integer such that $p_1^* > \cdots > p_{k-1}^* > p_k^* = \cdots = p_n^* = 0$. Then none of the contracts in $\{(\rho_m, p_m^*)\}_{m \geq k+1}$ will be chosen by any consumer in $[0, 1)$.

Proof. (i) Lemma 9 implies $p_{i-1}^* > p_i^* \geq 0$. Next by Corollary 2 we get $p_1^* > \cdots > p_{i-1}^* > 0$. The claim then follows from Lemma 12.
(ii) This is immediate from Lemma 9 since $p_m^* = p_k^*$ for all $m \geq k + 1$. □

7 Optimal contracts and allocation

In this section we construct an optimal allocation for the welfare problem and show that a set of contracts of the form $\{(\rho_m, p_m^*)\}$ sustains this optimum.

Consider the fixed point $((p_0^*, \overline{p}^*), \overline{x}^*)$ obtained in §5. The power company offers the contracts $\{(\rho_m, p_m^*)\}_{m=1}^{n}$. Contract (ρ_m, p_m^*) is a *winning contract* if some consumers in $[0, 1)$ pick this contract. By Lemma 12 and Corollary 3, if $p_1^* > \cdots > p_n^* > 0$, then all the n contracts are winning contracts. If there is $1 \leq l < n$ such that $p_1^* > \cdots > p_l^* = \cdots = p_n^* = 0$, then contracts $\{(\rho_m, p_m^*)\}_{m<l}$ are winning contracts, and contracts $\{(\rho_m, p_m^*)\}_{m>l}$ are not. By Corollary 3 there is $1 \leq k \leq n$ such that $\{(\rho_m, p_m^*)\}_{m=1}^k$ are the *only* winning contracts. The prices are ordered as $p_1^* > \cdots > p_k^* \geq p_{k+1}^* = \cdots = p_n^* = 0$. To construct an optimal allocation we first compute the surplus vector $\overrightarrow{H}^* = (H_1^*, \cdots, H_N^*)$, where $H_j^* = \max_{1\leq m\leq n} H_{jm}^*$ and $H_{jm}^* = h_j(d_{jm}(p_0^*, \overrightarrow{p}^*), \rho_m, p_m)$. Let $\overrightarrow{d}^*$ and $\overrightarrow{q}^*$ be given as in Proposition 2. Then $\overrightarrow{d}^*$ is of the form

$$\begin{aligned}\overrightarrow{d}^* &= (d_1^*, \cdots, d_k^*, 0, \cdots, 0) \\ &= (\sum_{j=1}^N \beta_{j1} d_{j1}^*, \cdots, \sum_{j=1}^N \beta_{jk} d_{jk}^*, 0, \cdots, 0)\end{aligned}$$

where $\overline{\beta_j} = (\beta_{j1}, \cdots, \beta_{jk}, 0, \cdots, 0) \in \Phi_j(p_0^*, \overrightarrow{p}^*)$, $j = 1, \cdots, N$. For each j, let α_j be the cardinality of the set $I_j = \{m \mid H_{jm}^* = H_j^*\}$. Denote the Lebesgue measure of a set $Z \subset \Re$ by $\mathcal{L}(Z)$. We partition the group G_j into α_j sets $G_{j1}, \cdots, G_{j,\alpha_j}$ such that $m \in I_j$ if and only if $\mathcal{L}(G_{jl}) = \beta_{jm}$ for some $1 \leq l \leq \alpha_j$. This is clearly a one-to-one correspondence between I_j and the partition sets $\{G_{jl}\}_{l=1}^{\alpha_j}$. Also, $\sum_{l=1}^{\alpha_j} \mathcal{L}(G_{jl}) = \sum_{m\in I_j} \beta_{jm} = \beta_j^*$. Moreover, a consumer in G_{jl} is assigned to contract (ρ_m, p_m^*), $m \in I_j$, if and only if $\mathcal{L}(G_{jl}) = \beta_{jm}$.

For each $m = 1, \cdots, n$, define the vectors $r^m = (r_1^m, \cdots, r_n^m)$ by

$$r_i^m := \begin{cases} \pi_i & , \text{if } i \geq m \\ 0 & , \text{if } i < m \end{cases} \tag{39}$$

We construct an allocation (or control) $z^* = (d^*, r^*)$ as follows:

$$d^*(t) := d_{jm}^* \ , \ r^*(t) := r^m \ , \ \text{if } t \in G_{jl} \text{ and } \mathcal{L}(G_{jl}) = \beta_{jm} \tag{40}$$

Also let $\mu^* = (\mu_1^*, \cdots, \mu_n^*)$ be defined by

$$\mu_m^* := \frac{p_m^* - p_{m+1}^*}{\pi_m} \ , \ m = 1, \cdots, n \tag{41}$$

where $p_{n+1}^* := 0$. Since $p_1^* > \cdots > p_k^* \geq p_{k+1}^* = \cdots = 0$, we get

$$\mu_m^* > 0, \ 1 \leq m \leq k-1 \ ; \ \mu_k^* \geq 0 \ ; \ \mu_{k+1}^* = \cdots = \mu_n^* = 0 \tag{42}$$

We will show that the μ_m^*'s are ordered as $\mu_1^* \geq \cdots \geq \mu_n^* \geq 0$. To establish this ordering, we need to introduce the concept of bid prices. For $H > 0$, define the *bid price* $p_j(\rho) := p_j(\rho; H)$ for group G_j as

$$p_j(\rho) := \begin{cases} \max\{p \geq 0 \mid \text{there exists } d \geq 0 \text{ with } h_j(d, \rho, p) \geq H\} \\ \text{undefined if there is no } d \geq 0 \text{ with } h_j(d, \rho, p) \geq H \end{cases} \tag{43}$$

This is the maximum price that a consumer in G_j is willing to pay for energy with reliability ρ if she is to attain surplus H.

It follows from the definition of $h_j(d, \rho, p)$ that a bid price curve $\rho \mapsto p_j(\rho)$ is increasing and defined on a set of the form $1 \geq \rho \geq \rho^j_{min}(H)$, where $\rho^j_{min}(H)$ is positive and increasing in H. Since Remark 2 gives $H^*_{jm} = \max_{0 \leq d \leq M} h_j(d, \rho_m, p^*_m) = \max_{d \geq 0} h_j(d, \rho_m, p^*_m)$, it follows easily from (43) that $p^*_m = p_j(\rho_m; H^*_{jm})$ for all m. We have the following useful lemma which says that a bid price curve is convex.

Lemma 13 *The map $\rho \mapsto p_j(\rho)$ is convex over the set $1 \geq \rho \geq \rho^j_{min}$.*

Proof. For each $d > 0$ define the function $\rho \mapsto P_j(\rho; d)$, with d as parameter, by

$$P_j(\rho; d) := \frac{\rho\, U_j(d) - [1 - \rho]\, L_j(d) - H}{d}$$

This is an affine function of ρ, and since by (43), $p_j(\rho) = \sup_d P_j(\rho; d)$, it follows that $p_j(\rho)$ is convex. □

Lemma 14 *The scarcity costs are ordered as $\mu^*_1 \geq \cdots \geq \mu^*_n \geq 0$.*

Proof. From (42) we get $\mu^*_{k-1} > \mu^*_k \geq \mu^*_{k+1} = \cdots = \mu^*_n = 0$. So it remains to show $\mu^*_m \geq \mu^*_{m+1}$ for all $1 \leq m \leq k - 2$. Suppose $\mu^*_l < \mu^*_{l+1}$ for some $1 \leq l \leq k - 2$. By Lemma 13, the bid price curves $p_j(\rho; H^*_{jl})$ and $p_j(\rho; H^*_{j,l+2})$ are convex. So $\mu^*_l < \mu^*_{l+1}$ implies

$$p^*_{l+1} > \min\{p_j(\rho_{l+1}; H^*_{jl})\ ,\ p_j(\rho_{l+1}; H^*_{j,l+2})\} \tag{44}$$

Since the bid price curves are decreasing in surplus, (44) implies $H^*_{j,l+1} < \max\{H^*_{jl}\ ,\ H^*_{j,l+2}\}$. Therefore $l + 1 \notin I_j$. This holds for each j, so no consumer will choose the contract (ρ_l, p^*_l). This contradicts Lemma 12 since $p^*_l > 0$. □

We now show that the allocation z^* is optimal for the welfare problem.

Theorem 2 *The allocation (or control) z^* given by (40) is optimal.*

Proof. Let $t \in G_j$. Then $t \in G_{jl}$ with $\mathcal{L}(G_{jl}) = \beta_{jm}$ for some $m \in I_j$. By (40) $d^*(t) = d^*_{jm}$ and $\rho^*(t) = \rho_m$, and the consumer gets surplus H^*_j. Also since $\sum_{l=1}^{\alpha_j} \mathcal{L}(G_{jl}) = \beta^*_j$, every consumer in G_j gets surplus H^*_j. Condition (14) is thus satisfied.

Next we check the complementary slackness condition (15). By (42) this condition clearly holds for $i > k$. For $i < k$, the total demand in contingency i is $x^*_i(1) = \sum_{m \leq i} d^*_m$. By Proposition 2 we get $x^*_i(1) = s_i$. So condition (15) holds for $i < k$. For $i = k$, we see that $\mu^*_k > 0$ implies $p^*_k > 0$. So it follows from Proposition 2 that condition (15) also holds for

$i = k$.
It remains to show that

$$H_j^* = \max_{d \geq 0,\ r_i \in \{0, \pi_i\}} H_j(d, r, \mu^*)\ ,\ j = 1, \cdots, N \tag{45}$$

Now suppose (d^+, r^+) maximizes

$$H_j(d, r, \mu^*) = \sum_{i=1}^{n} r_i\ U_j(d) - [1 - \sum_{i=1}^{n} r_i]\ L_j(d) - [\sum_{i=1}^{n} \mu_i^* r_i] d \tag{46}$$

Since the term involving r_i in (46) is linear in r_i, it follows that

$$U_j(d^+) + L_j(d^+) - \mu_i^* d^+ \begin{cases} > 0 \Rightarrow r_i^+ = \pi_i \\ < 0 \Rightarrow r_i^+ = 0 \end{cases} \tag{47}$$

Now let $m = \min\{i \mid r_i^+ = \pi_i\}$ and consider the control value $(\hat{d}, \hat{r})$ given by

$$\hat{d} = d^+;\ \ \hat{r}_i = \pi_i,\ i \geq m \text{ and } \hat{r}_i = 0,\ i < m$$

It is straightforward to verify using the ordering of μ_m^* in Lemma 14 and (47) that

$$H_j(d^+, r^+, \mu^*) = H_j(\hat{d}, \hat{r}, \mu^*)$$

But from (40) we see that

$$(\hat{d}, \hat{r}) = (d^*(t), r^*(t)) \text{ for some } t \in G_{jl} \text{ and } 1 \leq l \leq \alpha_j$$

which shows that H_j^* is the maximum value of the Hamiltonian as required in (45). □

8 Concluding remarks

In this paper we have considered a two-period pricing model for an electric power system. The power company offers a set of contracts in period 1. Each customer picks a contract in period 1 and then decides her demand. This is a decentralized decision problem. The supplier, on the other hand, needs to design a rationing scheme so that the demand can be met by the supply available in period 2, and each contract can be fulfilled. We have shown that it is possible to design a set of contracts that induce customers and the supplier to act optimally. The reliability levels of these contracts are given by an ordering of the supply contingencies.

In §7 we have shown that the optimal contracts are $(\rho_1, p_1^*), \cdots, (\rho_n, p_n^*)$, where the reliabilities $\rho_m = \sum_{i \geq m} \pi_i$ are obtained by ordering the supply contingencies such that $s_1 < \cdots < s_n$. That is, the mth contract guarantees delivery under contingencies $m, m+1, \cdots, n$. The price vector

$(p_1^*, \cdots, p_n^*)$ is shown to be a fixed point of a certain correspondence. They are ordered as $p_1^* \geq \cdots \geq p_n^*$, hence a higher price is tagged to a more reliable service. If $p_k^* \geq p_{k+1}^* = 0$, then customers will only demand the first $k-1$ and possibly the kth contracts.

By using the complementary slackness condition (15), it can be shown that the prices $p_1^*, \cdots, p_n^*$ are *market* equilibrium prices, [10]. That is, the contracts $\{(\rho_m, p_m^*)\}_{m=1}^n$ is a feasible allocation of contracts that simultaneously maximizes consumer surplus and company profits.

In this paper we have neglected the variable cost of supply and assumed that consumer preferences are deterministic. Extensions of the present model to include these situations can be found in [9] where it is shown that the structure of optimal contracts is similar to that obtained in this paper. However, the reliabilities are no longer given by orderings that coincide with that of the magnitudes of the contingent supplies, and algorithms that search for these orderings are more complicated.

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