SEMI-GLOBAL STABILIZATION OF MINIMUM PHASE NONLINEAR SYSTEMS IN SPECIAL NORMAL FORMS

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Semi-Global Stabilization of Minimum Phase Nonlinear Systems in Special Normal Forms *

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Abstract

We semi-globally stabilize certain minimum phase nonlinear systems which are in a normal form where the nonlinear subsystem is driven by an output of a linear system that possesses (possibly) nonzero peaking exponents. We eliminate the peaking phenomenon by stabilizing part of the linear system with a high-gain linear control and part of the linear system with a small, bounded control. The interpretation of this approach will be that we are redefining the outputs to add asymptotically stable *nonlinear* zeros to the system in a manner that allows the new composite zero dynamics to be asymptotically stable on arbitrarily large compact sets.

Keywords. Semi-global stabilization, normal forms, bounded controls, nonlinear zeros, peaking.

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1 Introduction

This paper is an extension of the semi-global stabilizability results of [1] and [6] for multi-input minimum phase nonlinear systems in the normal form:

$$\begin{array}{rcl} \dot{\eta} & = & f(\eta, \xi, u) \\ \dot{\xi}_1^i & = & \xi_2^i \\ & \vdots & \\ \dot{\xi}_{r_i}^i & = & u_i \\ y_i & = & \xi_1^i \qquad\qquad for \quad i = 1, \ldots, m \end{array} \tag{1}$$

where the state $x = (\eta, \xi) \in \mathbf{R}^n$ and f is smooth with $f(0,0,0) = 0$. By minimum phase it is meant that the equilibrium point $\eta = 0$ of

$$\dot{\eta} = f(\eta, 0, 0)$$

is globally asymptotically stable.

In the works of [1] and [6], the standard semi-global stabilization problem is to find a family of *linear* feedbacks (of the states ξ only) with tunable gain parameters that allows for local asymptotic stability and regulation to the origin for any initial condition in some (arbitrarily large) a priori bounded set. As described in [6], in general such a family of general feedbacks can fail to exists due to peaking in the linear variables. Loosely speaking, the linear variables can get large before they get small, inducing instability in the nonlinear dynamics. In [1] the problem is seen as an undesirable reduction of the domain of asymptotic stability of the nonlinear dynamics as a result of redefining new outputs to add *linear* zeros in the left-half plane and by employing high gain output feedback to the new output.

We will be able to achieve our extension by allowing our family of feedbacks to be possibly nonlinear, again as a function of ξ only. Our primary tool will be the bounded controls of [9] to eliminate peaking when possible. As motivating examples, we consider two very similar examples in [1] and [6] that serve as warnings that simple high gain *linear* feedbacks will not always be able to solve the nonlinear semi-global stabilizability problem:

Example 1.1 (Example 8.2 of [1]) *Consider the single-input system (in normal form (1))*

$$\begin{array}{rcl} \dot{\eta} & = & -(1 - \eta\xi_2)\eta \\ \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & u \end{array} \tag{2}$$

Example 1.2 (Example 1.1 of [6]) *Consider the single-input system (in normal form (1))*

$$\begin{aligned} \dot{\eta} &= -0.5(1+\xi_2)\eta^3 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \tag{3}$$

Both examples are globally minimum phase. Our philosophy for semi-globally stabilizing these systems can be summed up in the following heuristic argument that we make more precise in the sequel. In each case, if the state ξ_1 were not a part of the system we would choose a linear high-gain feedback function of ξ_2 alone to drive ξ_2 exponentially to the origin. The necessary rate of decay would be determined by the initial state of η. In both cases, if asymptotic regulation of ξ_2 were not crucial, it would actually be sufficient to drive ξ_2 exponentially to an arbitrarily small neighborhood of the origin. The rate of decay and size of the neighborhood would be chosen based on the initial state of η. But this allows us to reintroduce the state ξ_1 since it can be steered to the origin with an arbitrarily small bounded "control" ξ_2. In summary, for both examples, we will choose to drive ξ_2 arbitrarily fast to an arbitrarily small control that will (slowly) drive ξ_1 to zero without destabilizing the dynamics of η. The problem with the fully high gain approach of [1] and [6] is that they drive both ξ_1, ξ_2 rapidly to the origin. To drive ξ_1 fast requires peaking in ξ_2. The peaking in ξ_2 destabilizes the original zero dynamics. However, in the examples, the rate of convergence of ξ_1 is unimportant.

One interpretation of our approach is that we are adding a (slow) nonlinear zero to the system by reducing the order of the linear subsystem by one. Most importantly, the addition of this nonlinear zero still allows for asymptotic stability of the new composite zero dynamics on arbitrarily large compact sets.

2 Problem Statement

We make the following definition to clarify the problem at hand:

Definition 1 *The system (1) is semi-globally stabilizable by state feedback if for any compact set of initial conditions X there exists a smooth state feedback*

$$u = \alpha(\xi, \eta) \tag{4}$$

such that the equilibrium $(0,0)$ *of the closed-loop system (1),(4) is locally asymptotically stable and* X *is contained in the domain of attraction of* $(0,0)$.

We will focus on generating feedbacks that depend only on the linear states ξ. (i.e. $u = \alpha(\xi)$.)

We will be able to achieve semi-global stabilization for multi-input systems in the following special normal form:

$$\begin{array}{rcll}
\dot{\eta} & = & f(\eta, \xi^1_{j_1}, \ldots, \xi^m_{j_m}) & j_i \in \{1, \ldots, r_i + 1\} \\
\dot{\xi}^i_1 & = & \xi^i_2 & \\
 & \cdots & & \\
\dot{\xi}_{r_i} & = & u & \\
y_i & = & \xi^i_1 & for \quad i = 1, \ldots, m
\end{array} \tag{5}$$

where $\xi^i_{r_i+1} \equiv u_i$. With respect to the outputs y_i the system (5) is said to have *vector relative degree* $\{r_1, \ldots, r_m\}$. We define $r = r_1 + \ldots + r_m$. We then have $\xi \in \mathbf{R}^r$ and $\eta \in \mathbf{R}^{n-r}$.

We make the following standard assumption:

Assumption 1 *The equilibrium point* $\eta = 0$ *of the dynamics*

$$\dot{\eta} = f(\eta, 0, \ldots, 0) \tag{6}$$

ie. the zero dynamics of (5), are globally asymptotically stable.

The distinguishing feature of the systems in the special normal form of (5) is that no more than one state in each of the m chains of integrators appears in the η dynamics. Systems of the form (5) are more general than those in [1] in that the one state is *not* required to be the first state of the chain associated with y_i, namely ξ^i_1. In the terminology of [6], $(0, \ldots, 0)$ is not necessarily an achievable sequence of peaking exponents.

3 Main Results

Our general approach for nonlinear systems in the form (5) is to redefine the nonlinear subsystem to include the dynamics of $\xi^i_1, \ldots, \xi^i_{j_i-1}$ for $i = 1, \ldots, m$ and redefine the ith output to be $\tilde{y}_i = \xi^i_{j_i}$. We also define the nonnegative constants $\tilde{r}_i = r_i - j_i + 1$. We then have the following nonlinear system:

$$\begin{array}{rcl} \dot{\eta} & = & f(\eta, \tilde{y}_1, \ldots, \tilde{y}_m) \\ \dot{z}_1^i & = & z_2^i \\ & \vdots & \\ \dot{z}_{j_i-1}^i & = & \tilde{y}_i \\ \dot{\tilde{\xi}}_1^i & = & \tilde{\xi}_2^i \\ & \vdots & \\ \dot{\tilde{\xi}}_{\tilde{r}_i} & = & u_i \\ \tilde{y}_i & = & \tilde{\xi}_1^i \qquad\qquad for \quad i = 1, \ldots, m \end{array} \tag{7}$$

The *vector relative degree* with respect to the new outputs $\tilde{y}_i$ is given by $\{\tilde{r}_1, \ldots, \tilde{r}_m\}$. Observe that some entries of the vector relative degree may in fact be zero. Define $\tilde{r} = \tilde{r}_1 + \ldots + \tilde{r}_m$. We now have $\tilde{\xi} \in \mathbf{R}^{\tilde{r}}$ and $z \in \mathbf{R}^{r-\tilde{r}}$. With respect to the new outputs, it is straightforward to see that the system is not minimum phase. We are now interested in some further output redefinition that makes the system (7) minimum phase at least on sets $U = V \times \mathbf{R}^{r-\tilde{r}}$ where $V \subset \mathbf{R}^{n-r}$ is any arbitrarily large compact set. This will be sufficient since we are only interested in semi-global stabilizability.

In preparation for our choice of output redefinition we establish the following result for the system

$$\begin{array}{rcl} \dot{\eta} & = & f(\eta, \varphi_1(v_1, t), \ldots, \varphi_m(v_m, t)) \\ \dot{z}_1^i & = & z_2^i \\ & \vdots & \\ \dot{z}_{j_i-1}^i & = & v_i(z^i) \qquad\qquad for \quad i = 1, \ldots, m \end{array} \tag{8}$$

Proposition 3.1 *Assume the system (8) satisfies assumption 1. Then, given the compact set $V \subset \mathbf{R}^{n-r}$, there exists a positive constant ν_0 such that for any set of controls $\{v_i\}_{i=1}^m$ that globally stabilizes z and is such that $|v_i| < \nu_0$ and $v_i(0) = 0$, and for any functions $\varphi_i(\cdot, t)$ such that*

$$\begin{array}{rcl} \lim_{t\to\infty} v_i(t) = 0 & \Rightarrow & \lim_{t\to\infty} \varphi_i(v_i, t) = 0 \\ |v_i| < \nu_0 & \Rightarrow & |\varphi_i(v_i, t)| < M\nu_0 \end{array}$$

for some $M > 0$, the dynamics of (8) are asymptotically stable with basin of attraction containing $V \times \mathbf{R}^{r-\tilde{r}}$.

Proof. The local asymptotic stability is straightforward (see, for example, [1, Lemma 4.2].) To determine the basin of attraction note that any state $z \in \mathbf{R}^{r-\bar{r}}$ is driven to the origin by the assumption on $v(z)$. Now consider initial conditions $\eta \in V$. We will demonstrate that $\exists \nu_0$ such that if $|v_i| < \nu_0$ then the trajectories of η remain bounded for all $t \geq 0$. Regulation to the origin then follows from the main theorem of [3] since $z \to 0$ as $t \to \infty$ by assumption, $v_i(\cdot)$ is smooth with $v_i(0) = 0$ and

$$\lim_{t\to\infty} v_i(t) = 0 \Rightarrow \lim_{t\to\infty} \varphi(v_i, t) = 0$$

To this end consider a smooth positive definite and proper Lyapunov function

$$W : \mathbf{R}^{n-r} \to \mathbf{R}$$

such that

$$dW(\eta) \cdot f(\eta, 0) < 0 \tag{9}$$

for all nonzero η. The existence of such a Lyapunov function follows from assumption 1. It then follows that

$$dW(\eta) \cdot f(\eta, \varphi(v, t)) < 0 \tag{10}$$

for all $||\varphi(v,t)|| < \nu(||\eta||)$ for some continuous function ν that is decreasing on $[1, +\infty)$. (See [4, Lemmas 3.1,3.2].) Now let c be the largest value of W on the compact set V and let $||\eta|| \leq R \; \forall \eta \in \{\eta : W(\eta) \leq c\}$. Such an R exists because W is proper. Then R and the function ν together with the constant M determine a bound ν_0 and an additional constant $L < R$ such that

$$dW(\eta) \cdot f(\eta, \varphi(v, t)) < 0 \tag{11}$$

for all $L \leq ||\eta|| \leq R$ and all $||v|| < \nu_0$. Now, by assumption, $\eta(0) \in V$ and hence $W(0) \leq c$. Finally, since W is decreasing whenever $W = c$ it follows that $W(t) \leq c$ for all $t \geq 0$. This in turn implies $||\eta(t)|| \leq R$ for all $t \geq 0$. □

Remark. It is well known that such bounded controls v_i exist since any finite length chain of integrators can be globally stabilized with an arbitrarily small control. (see [2], [5], [9].)

We could now proceed with a standard output definition procedure choosing new outputs as

$$\bar{y}_i = \tilde{y}_i - v_i(z^i)$$

In this case, the zero output dynamics would be given by (8) with $\varphi_i(v_i, t) = v_i$. The drawback to this choice is that the procedure to generate the closed loop control involves repeated differentiation of the necessarily complicated (see [7]) bounded controls v_i. Instead we choose a procedure that avoids this repeated differentiation. (This type of procedure has also been used with success in [8] for certain stabilization problems when the system is not initially minimum phase.) The outputs chosen will depend on the feedback gains used and so will be saved for the last step.

We begin by choosing a high-gain feedback law to stabilize the dynamics of $\tilde{\xi}$ in (7). We also include the small bounded control v which will be instrumental in stabilizing the zero dynamics. We choose

$$u_i = -K^{\bar{r}_i} c_{i,\bar{r}_i} \tilde{\xi}_1^i - \cdots - K c_{i,1} \tilde{\xi}_{\bar{r}_i}^i + K^{\bar{r}_i} v_i \tag{12}$$

where $v_i(\cdot)$ will be specified and $K > 0$.

Next, we make a linear coordinate change to move v_i so that it directly controls the z^i states. To do so, we begin by defining

$$\zeta_k^i = \frac{1}{K^{k-1}} \tilde{\xi}_k^i \tag{13}$$

Then the dynamics for ζ^i are

$$\begin{aligned} \dot{\zeta}_1^i &= K \zeta_2^i \\ &\vdots \\ \dot{\zeta}_{\bar{r}}^i &= K(-c_{i,\bar{r}} \zeta_1^i - \ldots - c_{i,1} \zeta_{\bar{r}}^i + v) \end{aligned} \tag{14}$$

Recalling that

$$\frac{d}{dt} z_{j_i-1}^i = \tilde{\xi}_1^i = \zeta_1^i \tag{15}$$

we define

$$\bar{z}_{j_i-1}^i = c_{i,\bar{r}} z_{j_i-1} + \frac{1}{K}(c_{i,\bar{r}-1} \zeta_1^i + \ldots + c_{i,1} \zeta_{\bar{r}-1}^i + \zeta_{\bar{r}}^i) \tag{16}$$

It can then be shown that

$$\frac{d}{dt} \bar{z}_{j_i-1}^i = v_i \tag{17}$$

Likewise, we define $\bar{z}_k^i$ for $k = 1, \ldots, j_i - 2$ such that

$$\frac{d}{dt} \bar{z}_k^i = \bar{z}_{k+1}^i \tag{18}$$

It is straightforward to show that this can be done in a way such that the transformation between (ζ^i, z^i) and $(\zeta^i, \bar{z}^i)$ is invertible.

We are now ready to define the appropriate outputs. To do so, we denote by A_i the controllable canonical form matrix associated with the Hurwitz polynomial

$$s^{\bar{r}_i} + c_{i,1}s^{\bar{r}_i-1} + \cdots + c_{i,\bar{r}_i} \tag{19}$$

We also let $C_i \in \mathbf{R}^{1\times\bar{r}_i}$ and $B_i \in \mathbf{R}^{\bar{r}_i\times 1}$ be such that

$$\begin{array}{rcl} C_i & = & \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \\ B_i & = & \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T \end{array} \tag{20}$$

Then we define the ith output to be

$$\bar{y}_i = C_i e^{KA_it}\zeta^i(0) \tag{21}$$

Observe that

$$\begin{array}{rcl} \tilde{y}_i(t) = \tilde{\xi}^i_1 & = & \zeta^i_1 \\ & = & \bar{y}_i(t) + \int_0^t C_i e^{KA_i(t-\tau)} K B_i v_i(\tau) d\tau \\ & = & \bar{y}_i(t) + \varphi_i(v_i, t) \end{array} \tag{22}$$

Finally, we have the nonlinear system

$$\begin{array}{rcl} \dot{\eta} & = & f(\eta, \bar{y}_i + \varphi_i(v_i, t)) \\ \dot{\bar{z}}^i_1 & = & \bar{z}^i_2 \\ & \vdots & \\ \dot{\bar{z}}^i_{j_i-1} & = & v_i \\ \dot{\zeta}^i_1 & = & K\zeta^i_2 \\ & \vdots & \\ \dot{\zeta}_{\bar{r}} & = & K(-c_{i,\bar{r}}\zeta^i_1 - \ldots - c_{i,1}\zeta^i_{\bar{r}} + v_i) \\ \bar{y}_i & = & C_i e^{KA_it}\zeta^i(0) \qquad\qquad for \quad i = 1, \ldots, m \end{array} \tag{23}$$

To check the minimum phase property we must examine the system

$$\begin{array}{rcl} \dot{\eta} & = & f(\eta, \varphi_i(v_i, t)) \\ \dot{\bar{z}}^i_1 & = & \bar{z}^i_2 \\ & \vdots & \\ \dot{\bar{z}}^i_{j_i-1} & = & v_i \end{array} \tag{24}$$

It is easy to show from (22) that the functions φ_i satisfy the requirements of proposition 3.1 with the constant M independent of the choice of K. Further, it is important to note that K can be chosen to drive the outputs $\bar{y}_i$ to zero exponentially with an arbitrarily fast rate of decay without exhibiting peaking. We then have the following results.

Theorem 3.1 *Assume the system (5) satisfies assumption 1. Then the system (5) is semi-globally stabilized by the family of feedbacks (12). That is, (12) locally asymptotically stabilizes (5) and for any compact set X of the state space (η, ξ) there exists a $K_X > 0$ and $\nu_X > 0$ such that, for all $K > K_X$ and all globally asymptotically stabilizing $v(\bar{z})$ such that $||v(\bar{z})|| < \nu_X$, the basin of attraction for the closed-loop system (5),(12) contains X.*

Proof. The proof of this theorem follows from the proof of [1, Theorem 7.2] together with proposition 3.1. Following the proof of Theorem 7.2 in [1], we can show that it is possible to choose K in (23) large enough such that the trajectories of $\eta, \bar{z}$ with exponentially decaying inputs converge to trajectories of the undriven $\eta, \bar{z}$ dynamics that take initial conditions in some compact set $\bar{X}$ determined by X. Then applying proposition 3.1, given $\bar{X}$, there exists ν_0 sufficiently small such that if v is chosen with $||v(z)|| < \nu_0$, all trajectories of $\eta, \bar{z}$ that originate in the compact set X are driven to zero. Finally, the states ζ converge to zero since they are the states of a linear system driven by bounded inputs that converge to zero. □

It is possible to slightly weaken the compact set restriction since the dynamics of $\bar{z}$ are autonomous and globally asymptotically stable.

Corollary 3.1 *Assume the system (5) satisfies assumption 1. Then the feedbacks (12) locally asymptotically stabilizes the origin of (5) and, for all initial conditions in the set $Y = X_\eta \times \mathbf{R}^{r-\bar{r}} \times X_\xi$ where $X_\eta \subset \mathbf{R}^{n-r}$ is compact and $X_\xi \subset \mathbf{R}^{\bar{r}}$ is compact, there exists $K_Y > 0$ and $\nu_Y > 0$ such that, for all $K > K_Y$ and all $v(z)$ such that $||v(z)|| < \nu_Y$, the basin of attraction for the closed-loop system (5),(12) contains Y.*

4 Examples

We return now to examples 1.1 and 1.2. Both examples have essentially the same structure and are solved by the same family of feedbacks. To describe this class of feedbacks we define the smooth function $\sigma : \mathbf{R} \to \mathbf{R}$ by

$$\begin{array}{rcl} s\sigma(s) & > & 0 \quad for \quad s \neq 0 \\ |\sigma(s)| & \leq & \nu \end{array} \tag{25}$$

The semi-global stabilization problem for examples 1.1 and 1.2 are then solved by the family of feedbacks

$$u = -K[\xi_2 + \sigma(\xi_1 + \frac{1}{K}\xi_2)] \tag{26}$$

parametrized by $K, \nu > 0$.

We give one further example to demonstrate the methods when $j = r+1$.

Example 4.1 *Consider the single-input system (in normal form (1))*

$$\begin{aligned} \dot{\eta} &= -\eta + \eta^2 u \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \tag{27}$$

Given $\eta \in V$ where $V \subset \mathbf{R}$ compact, the family of feedbacks is specified by

$$u = -\sigma(c_1\xi_1 + c_2\xi_2) \tag{28}$$

where c_1 and c_2 are chosen such that the dynamics of ξ_1, ξ_2 are globally asymptotically stable and where ν is chosen such that

$$\eta > \eta^2 \nu \tag{29}$$

for all $\eta \in V$.

5 Conclusion

We have solved the semi-global stabilization problem for a slightly more general class of systems than in [1] by combining small, bounded controls with high-gain feedback to eliminate the peaking phenomenon. This was systematically done by redefining the outputs to add *nonlinear* zeros to the system in a manner that allows the new composite zero dynamics to be asymptotically stable on arbitrarily large compact sets.

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