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Generalizing the Twist and Flip Paradigm

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Abstract

In this paper we generalize the horseshoe twist theorem of [Brown & Chua, 1991] and derive a wide class of ODEs, with and without dissipation terms, for which the Poincaré can be expressed in closed form as FTFT where T is a generalized twist. We show how to approximate the Poincaré maps of nonlinear ODEs with continuous periodic forcing by Poincaré maps which have a closed form expression of the form $FT_1T_2 \ldots T_n$ where the T_i are twists. We extend the twist and flip to three dimensions with and without damping. Further, we demonstrate how to use the square-wave analysis to argue the existence of a twist and flip paradigm for the Poincaré map of the van der Pol equation with square-wave forcing. We apply this analysis to the cavitation bubble oscillator that appears in [Parlitz, et. al., 1991] and prove a variation of the horseshoe twist theorem for the twist and translate used by Parlitz. We present illustrations of the diversity of the dynamics that can be found in the generalized twist and flip map, and we use a pair of twist maps to provide a specific and very simple illustration of the Smale horseshoe. Finally, we use the twist and translate of [Parlitz, et. al., 1991] to demonstrate that the addition of sufficient linear damping to a dynamical system having PBS chaos may cause the chaos to become *visible*.

1 Introduction

In [Brown & Chua, 1991] three conjectures were stated about the presence of horseshoes in a very general class of ordinary differential equations (ODEs). For the case when the Poincaré map is of the form FTFT where T is the simple twist [Brown & Chua, 1991], all three of these conjectures have now been proven. The proofs of these conjectures depend on an important intermediate result. The conjectures and this intermediate result will all be stated in Sec. 2, with a brief indication of the proofs which may be found in detail in [Brown,1990].

Encouraged by our success in proving the existence of horseshoes for the simple twist and flip map, it is reasonable to ask if this result is simply a special case, or is it part of a broader theory. We argue that it is indeed part of a broader theory through the following steps: In Sec. 3 we show that the class of ODEs having Poincaré maps of the form FTFT, where T is a *generalized* twist is very large, and we derived a lemma that gives us some idea of how large this class is. This lemma provides a mechanism for obtaining the Poincaré map in closed form when a square-wave force is present, and we demonstrate this by deriving five broad classes of non-dissipative, nonlinear, square-wave forced ordinary differential equations and their closed form Poincaré maps.

In Sec. 4 we go on to show that what we have done for non-dissipative equations of Sec. 3.2 can also be done for equations having a damping or dissipative force, and thus we are able to obtain a closed form solution for the Poincaré map of a large class of damped ODEs.

In Sec. 5 we outline the proof of a generalization of the horseshoe twist theorem.

In Sec. 6 we demonstrate how to extend the twist and flip paradigm from square-wave forcing to the case of continuous periodic forcing.

In Sec. 7 we show how the twist and flip can be generalized to three dimensional systems.

In Sec. 8 we extend the concept of the twist map by defining the concept of a two dimensional *shear* map.

In Sec. 9 we indicate how to use the square-wave analysis to find a twist and flip map in the van der Pol equation and suggest that the complexity of the van der Pol equation is explained by the twist and flip paradigm.

In Sec. 10 we will show that in place of the twist and flip map, some equations have a twist and translate paradigm. This fact, found in the cavitation bubble oscillator of [Parlitz, et. al., 1991], suggests an extension of our results to include Poincaré maps of the form LT where L denotes a translation. In fact we may extend the twist and flip paradigm to include maps of the form T_2T_1 where T_1 is a twist map, and T_2 is any combination of twist maps, flip maps, translate maps, or diffeomorphism defined by any linear second order ODE. In this section we prove the horseshoe twist and translate theorem for the non-dissipative twist and translate map.

In Sec. 11 we illustrate the diversity of the dynamics of the twist and flip map by presenting four figures consisting of unstable manifolds and elliptic regions generated by generalized twist and flip maps. In this section we also illustrate the Smale horseshoe using a map of the form FT_2T_1, where T_i are both simple twist maps. The computer code for this illustration is included.

In Sec. 12, we illustrate a basic mechanisn of the onset of chaos in a broad class of dynamical systems using the damped twist and translate map of Parlitz. In particular we show that in the case of linear damping, the complexity of a dynamical system can be found in the non-dissipative form of the equations describing the system and that this complexity may become *visible* with the addition of sufficient damping even though linear damping cannot create horseshoes.

2 Proof of Conjectures for the Horseshoe Twist Theorem

We have stated the horseshoe twist theorem for the Poincaré map of the first order non-linear, square-wave forced system of ODEs in [Brown & Chua, 1991]. We restate this ODE here for convenience of reference:

$$\dot{x} = -y\sqrt{\left(x - a\,\mathrm{sgn}(\sin(\omega t))\right)^2 + y^2} \tag{1}$$

$$\dot{y} = \left(x - a\,\mathrm{sgn}(\sin(\omega t))\right) \sqrt{\left(x - a\;\mathrm{sgn}(\sin(\omega t))\right)^2 + y^2}$$

where $a > 0$. In the following discussion we will adopt the same terminology used in [Brown & Chua, 1991].

Let M be the unstable manifold of FT at a hyperbolic fixed point, where M consists of two branches that are joined at the fixed point. It was proven in [Brown & Chua, 1991] that the local unstable manifold of one of these branches lies on the right side of the vertical axis. Recall that we labeled this branch as M_{rhs}.

Our starting point is the horseshoe twist theorem of [Brown & Chua, 1991], hereafter referred to as the *horseshoe twist theorem* I. The key to the proof of the generalized horseshoe twist theorem, hereafter referred to as the *horseshoe twist theorem* II, can be found in the following theorem,[1] theorem 2, which states that the right hand branch, M_{rhs}, of every homoclinic manifold (i.e., M=S, where S is the stable manifold) from a fixed point of FT lying on the positive vertical axis lies in an annulus bounded away from the fixed points of T. From this we conclude that if M contains a fixed point of T it cannot be homoclinic.

Before indicating the proof of these results we recall some definitions from [Brown & Chua, 1991].

We have defined $\mathbf{C}_r$ to be the circle of radius r centered at $(a, 0)$ and define $\mathbf{D}_r$ to be the circle of radius r centered at $(-a, 0)$.

Using these definitions we define $\mathbf{G}_r$ to be the intersection of $\mathbf{D}_r$ and the right half plane $\{(x, y)|x > 0\}$ and let $\mathbf{H}_r$ denote the intersection of $\mathbf{C}_r$ and the left half plane. We will use S_{lhs} to denote the branch of the stable manifold that originates on the LHS of the plane. This must exist by lemma 7 of [Brown & Chua, 1991]

Lemma 1 *If* $M = S$ *then* $M_{rhs} = S_{lhs}$.

Proof: Suppose that M=S and let $\mathbf{D}_r$ be the circle centered at $(-a, 0)$ having radius r.

Assume that the right hand branch of the stable and unstable manifolds meet, and thus coincide. By lemma 7 of [Brown & Chua,1991] M_{rhs} begins in the interior of $\mathbf{D}_r$. By the same lemma S_{rhs} lies on the RHS of the vertical axis and lies outside of $\mathbf{D}_r$. To meet S_{rhs} on the RHS of the vertical axis, M_{rhs} must cross this circle by continuity. But this cannot happen unless M_{rhs} has already intersected the vertical axis.

But if M_{rhs} intersects the vertical axis, so does S and at the same point, since by lemma 5 [Brown & Chua, 1991], **R**(M)=S, where **R** denotes reflec-

[1] A complete proof of the horseshoe twist theorem II can be found in a recent Ph.D. dissertation [Brown, 1990].

tion about the vertical axis. So if S=M, $\mathbf{R}(\mathrm{M_{rhs}})$ meets $\mathrm{S_{lhs}}$ and so must coincide. ∎

We recall without proof the following proposition [Brown & Chua, 1991]

Proposition 1 *Let* $(0, y)$ *be a hyperbolic fixed point of* FT *on the positive vertical axis. Let* M *be the unstable manifold at* $(0, y)$*. Then there is a point where* M *crosses the vertical axis other than the fixed point* $(0, y)$.

Before we state and prove theorem 1 we need the following 2 lemmas.

Lemma 2 *Let* $(0, y)$, $y > 0$ *be a hyperbolic fixed point for* FT *and let* $\mathrm{M_{rhs}}$ *be as in lemma 1. Let* p *be the point on the vertical axis that is in* M*, and has shortest arc length measured along the unstable manifold to the point* $(0, y)$*. Let* $\mathbf{z}_\alpha$ *be* $(\mathrm{FT})^{-1}(p)$*. Let* $r = \sqrt{y^2 + a^2}$*, and let* $r_\alpha = \|\mathbf{z}_\alpha - \mathbf{a}\|$*. Define the annulus* A_α *by the equation*

$$\mathrm{A}_\alpha = \{\mathbf{z} | r_\alpha \leq \|\mathbf{z} - \mathbf{a}\| \leq r\}$$

Let $\mathrm{V} = (\mathrm{M_{rhs}} \cap \mathrm{A}_\alpha)$ *Then* $\mathrm{T}(\mathrm{V} - \{p\}) \subset \mathbf{H}_r$.

Proof: We have the following facts: $\mathrm{T}(\mathrm{V}) \subset \mathbf{C}_r$, $\mathrm{T}(0, y) = (0, -y)$, and $\mathrm{T}(\mathbf{z}_\alpha) = -p$. By the choice of $\mathbf{z}_\alpha$, there can be no other points of T(V) on the vertical axis other than $(0, -y)$ and $-p$, so that T(V) must lie entirely in $\mathbf{H}_r$. ∎

We need the following lemma from [Brown, 1990] in the proof of theorem 1

Lemma 3 *Given a hyperbolic fixed point of* FT*, let* S *be the stable manifold, and* M *the unstable manifold at this fixed point. If* S = M*, then*

$$\mathbf{P}\mathrm{T}(\mathrm{M}) = \mathrm{M}$$

where **P** *is reflection about the horizontal axis.*

Proof: FT(M)=**RP**T(M)=M=**R**(M). Therefore **RP**T(M)=**R**(M), so that **P**T(M)=M. ∎

Theorem 1 (Homoclinic Manifold Theorem) *If* M *is a homoclinic manifold of* FT*, then* $\mathrm{M_{rhs}}$ *lies in an annulus bounded away from the fixed points of* T.

Proof: Since M is homoclinic we have $M = S = \mathbf{R}(M)$. Let M_{rhs} be the branch of M lying in the RHS of the plane. $\mathbf{R}(M_{rhs}) = S_{lhs}$ (where "lhs" stands for left hand side) and so

$$\mathbf{R}(M_{rhs} \cup S_{lhs}) = (M_{rhs} \cup S_{lhs})$$

forms a connected curve in $\mathbf{R}^2$. We will label this curve as N and note that it contains the branch of the unstable manifold M lying in the RHS of the vertical axis.

Let q be the point other than $(0, y)$ where N meets and crosses the vertical axis and let $\mathbf{z}_\alpha = (FT)^{-1}(p)$. By lemma 7 [Brown & Chua, 1991] and lemma 3 above, $PT(N) = N$.

For any annulus A, we have $\mathbf{PT}(A) = A$. So, $\mathbf{PT}(A \cap M) = A \cap M$ and also, $\mathbf{PT}(A \cap N) = A \cap N$. This relation must hold for A_α. By the choice of $\mathbf{z}_\alpha$ we have $T(A_\alpha \cap M_{rhs})$ must lie entirely on the LHS of the vertical axis (lemma 2 above), except for the point q which must lie on the vertical axis. For convenience we define $M_\alpha = (A_\alpha \cap M_{rhs})$. Therefore $\mathbf{PT}(M_\alpha) = S_{lhs}$. Since S_{rhs} lies entirely in A_α, we conclude that N is contained entirely in $A_\alpha \cup \mathbf{R}(A_\alpha)$.

Now $r_0 \neq r_\alpha$. If $\mathbf{w}$ is a fixed point of T that is in M_{rhs}, then $FT(\mathbf{w}) = F(\mathbf{w})$ which is on the LHS of the vertical axis, i.e., $r_0 < r_\alpha$, hence M_α cannot intersect $\mathbf{C}_{r_0}$ and so N cannot intersect $\mathbf{C}_{r_0} \cup \mathbf{D}_{r_0}$.

We conclude that M is bounded away from the circle of fixed points of T; namely, $\mathbf{C}_{r_0}$. Thus N lies inside the circle $\mathbf{C}_r$ and outside the circle $\mathbf{C}_s$ where $s > r_0$. ∎

Corollary 1 *Assume the definitions of the above lemma. If* M_{rhs} *is a homoclinic manifold, it must meet and cross the negative vertical axis.*

Proof: Follows directly from lemma 2 above. ∎

Corollary 2 *Assume the definitions of the above lemma. If* M_{rhs} *crosses the positive vertical axis at a point other than a fixed point, then it is a c-manifold.*

Proof: Follows directly from corollary 1 above. ∎

Theorem 2 (Horseshoe Twist Theorem II) *Let* $(0, y)$, $y > 0$, *be a hyperbolic fixed point of* FT *and let* M *be the unstable manifold of* FT *at this hyperbolic fixed point. If* M_{rhs} *contains a fixed point of* T, *then* FT *has a horseshoe, and hence the Poincaré map* FTFT *also has a horseshoe.*

Proof: By theorem 1 if M_{rhs} contains a fixed point of T then it cannot be homoclinic. M_{rhs} must meet the vertical axis by proposition 1, at p and by lemma 5 [Brown & Chua, 1991] this is a homoclinic point. If M meets the stable manifold at this point there is a horseshoe by [Smale, 1965]. We assert that M cannot be tangent to the stable manifold at this point. Thus there exists a horseshoe. ∎

We may now restate conjecture 3 of [Brown & Chua, 1991] as a corollary.

Corollary 3 *Given a hyperbolic fixed point* $(0, y)$, *with* $r = \sqrt{y^2 + a^2}$ *and* $r_0 = 2\omega[r/2\omega]$, *a sufficient condition for the unstable manifold* M *of this fixed point to be a c-manifold is that the two circles,*

$$(x - a)^2 + y^2 = r_0^2$$

and

$$(x + a)^2 + y^2 = r^2$$

intersect.

Proof: Recall that by the definitions in [Brown & Chua, 1991] the first circle above is denoted $\mathbf{C}_{r_0}$ and the second is denoted $\mathbf{D}_r$. Assume these two circles intersect. Note that the circle $\mathbf{C}_{r_0}$ is a circle of fixed points of T.

If M_{rhs} crosses the positive vertical axis at a point other than the fixed point we are done by corollary 2. Thus assume that M_{rhs} crosses the negative vertical axis. By lemma 2 and the fact that M_{rhs} crosses the negative vertical axis, this crossing must lie below the circle of fixed points of T. Hence there are points in M_{rhs} that lie above the circle $\mathbf{C}_{r_0}$ and below $\mathbf{C}_{r_0}$ and so because $\mathbf{D}_r$ intersects $\mathbf{C}_{r_0}$ and M_{rhs} is connected, it must intersect $\mathbf{C}_{r_0}$ and thus contain a fixed point of T. By theorem 2, M must be a c-manifold. ∎

Remark 1: We recall that another form of this condition is given by the following inequality:

$$r \leq 2a + r_0$$

or

$$r \bmod (2\omega) \leq 2a.$$

Remark 2: The horseshoe twist theorem I of [Brown & Chua, 1991] is now a special case of corollary 3. In particular the horseshoe twist theorem of [Brown & Chua, 1991] required that two conditions be satisfied. The second condition of the two, namely, $r_0^2 \geq (c-a)^2$, is equivalent to the condition of corollary 3. Therefore, corollary 3 is more general than the horseshoe twist theorem I and, by inspection, can be seen to be much easier to verify.

3 Differential Equations Having Closed Form Poincaré Maps

Our success in deriving a formula relating the parameters of an ODE to chaos depended on the existence of a closed form expression for the Poincaré map. This suggests the following question: How common are nonlinear, periodically forced ODEs having closed form Poincaré maps? To aid in answering this question we have the following factorization lemma.

3.1 Factorization Lemma for the Poincaré Map of Certain ODEs

Lemma 4 *Let* $\mathbf{x}(t)$ *be a solution of*

$$\dot{\mathbf{x}} = \mathrm{G}(\mathbf{x}, t), \ \ \textit{and}\ \mathbf{x}(0) = \mathbf{x}_0 \tag{2}$$

where $(\mathbf{x}_0, t) \in \mathbf{R}^n \times \mathbf{R}$. *Also, suppose that there exists a constant* p *such that for all* $(\mathbf{x}, t) \in \mathbf{R}^n \times \mathbf{R}$ *the function* G *satisfies the relation*

$$-\mathrm{G}(\mathbf{x}, t + p/2) = \mathrm{G}(-\mathbf{x}, t) \tag{3}$$

Let

$$\mathbf{y}(t) = -\mathbf{x}(t + p/2)$$

for $0 \leq t < \infty$. *Then* $\mathbf{y}(t)$ *is a solution of the equation*

$$\dot{\mathbf{y}} = \mathrm{G}(\mathbf{y}, t), \ \ \textit{and}\ \mathbf{y}(0) = -\mathbf{x}(p/2).$$

Proof: Let $\mathbf{u}(t) = -\mathbf{x}(t+p/2)$. Then $\mathbf{u}(0) = -\mathbf{x}(p/2)$ and

$$\dot{\mathbf{u}} = -\dot{\mathbf{x}}(t+p/2) =$$

$$-\mathrm{G}(\mathbf{x}(t+p/2), t+p/2) = \mathrm{G}(-\mathbf{x}(t+p/2), t) =$$

$$\mathrm{G}(\mathbf{u}, t)$$

Therefore, $\mathbf{u}(t) = \mathbf{y}(t) = -\mathbf{x}(t+p/2)$. ∎

Remark We note that G in Eq.(2) is periodic as a function of t with period p. If we let T_t be the one parameter family of phase plane diffeomorphisms in $\mathbf{R}^2$ defined by Eq.(2), then the Poincaré map is T_p. We use this fact in the following corollary.

Corollary 4 *Let* $\mathbf{F}$ *be a 180 degree rotation about the origin in* $\mathbf{R}^2$ *and let* T_t *be the one parameter family of phase plane diffeomorphisms defined by Eq.(2). Then the Poincaré map for Eq.(2), i.e.* T_p, *is given by the diffeomorphism,* Φ *of* $\mathbf{R}^2$ *defined by*

$$\Phi(\mathbf{x}_0) = \mathrm{F}\mathrm{T}_{p/2}\mathrm{F}\mathrm{T}_{p/2}(\mathbf{x}_0)$$

Proof: Let $\mathbf{x}(t, \mathbf{x}_0)$ be the solution of Eq.(2) over the interval $[0, p/2]$ with $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$, and let $\mathbf{y}(t, \mathbf{y}_0)$ be the solution of Eq.(2), for $0 \le t \le p/2$, having the initial condition

$$\mathbf{y}_0 = \mathbf{x}_1 = -\mathbf{x}(p/2, \mathbf{x}_0)$$

Then, by lemma 4, for $0 \le t \le p/2$ we have

$$\mathbf{x}(t+p/2, \mathbf{x}_0) = -\mathbf{y}(t, \mathbf{x}_1)$$

If we take $t = p/2$ in corollary 4 we have the relation

$$\mathbf{x}(p, \mathbf{x}_0) = -\mathbf{y}(p/2, \mathbf{x}_1)$$

where

$$\mathbf{x}_1 = -\mathbf{x}(p/2, \mathbf{x}_0)$$

We conclude that

$$\mathbf{x}(p, \mathbf{x}_0) = \mathrm{FT}_{p/2}\mathrm{FT}_{p/2}(\mathbf{x}_0)$$

∎

Remark: This corollary is also true for $\mathbf{R}^n$.

Since we will always assume $p = 2\pi/\omega$ we will henceforth drop the subscripts and write the Poincaré map simply as FTFT.

3.2 Some Differential Equations for Which there are Closed Form Poincaré Maps

With the above factorization lemma in hand we now determine several classes of ODEs for which the Poincaré map can be obtained in closed form.

It may be readily verified that the equations of this section satisfy the hypothesis of lemma 4, where $p = 2\pi/\omega$, and therefore by corollary 4 the Poincaré map is of the form FTFT.

We have the following convenient definition used extensively in the following sections:

Definition $\mathrm{sg}(t) = \mathrm{sgn}(\sin(t))$.

Class 1 is determined by the pair of equations:

$$\begin{aligned} \dot{x} &= -yf(\lambda) \\ \dot{y} &= (x - a\,\mathrm{sg}(\omega t))f(\lambda) \end{aligned}$$

where $\lambda = \sqrt{(x - a\,\mathrm{sg}(\omega t))^2 + y^2}$.

For $f(u) = u$ this equation is the same as in [Brown & Chua, 1991].

The Poincaré map is FTFT where T is the solution of the equations

$$\begin{aligned} \dot{x} &= -yf(\lambda) \\ \dot{y} &= (x - a)f(\lambda) \end{aligned}$$

at $t = \pi/\omega$, where $\lambda = \sqrt{(x-a)^2 + y^2}$, and F is the 180 degree rotation.

Note that λ is a constant defined by the initial conditions and represents a constraint on the variables x and y. Also note that since

$$\frac{dy}{dx} = -(x - a)/y$$

in the last pair of equations above, the phase portrait of the solutions consists of a continuum of circles centered at $(a,0)$. In particular, $(x(t), y(t))$ must lie on a circle of radius λ centered at $(a,0)$.

The matrix form of the twist map for class 1 is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\left(f(\lambda)t\right) & -\sin\left(f(\lambda)t\right) \\ \sin\left(f(\lambda)t\right) & \cos\left(f(\lambda)t\right) \end{pmatrix} \begin{pmatrix} x_0 - a \\ y_0 \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix}$$

Class 2 is determined by the following equations:

$$\dot{x} = -yf\left(\sqrt{(x - a\,\mathrm{sg}(\omega t))^2 + (1-k^2)y^2}\,\right)H(x,y) \tag{4}$$

$$\dot{y} = (x - a\,\mathrm{sg}(\omega t))f\left(\sqrt{(x - a\,\mathrm{sg}(\omega t))^2 + (1-k^2)y^2}\,\right)H(x,y) \tag{5}$$

Where k is the elliptic modulus and

$$H(x,y) = \frac{\sqrt{(x - a\,\mathrm{sg}(\omega t))^2 + (1-k^2)y^2}}{\sqrt{(x - a\,\mathrm{sg}(\omega t))^2 + y^2}}$$

where $f(r)$ is any C^1 function of r.

The Poincaré map is given by FTFT where T is the solution of the equation

$$\dot{x} = -yf\left(\sqrt{(x-a)^2 + (1-k^2)y^2}\,\right)h(x,y) \tag{6}$$

$$\dot{y} = (x-a)f\left(\sqrt{(x-a)^2 + (1-k^2)y^2}\,\right)h(x,y) \tag{7}$$

where,

$$h(x,y) = \frac{\sqrt{(x-a)^2 + (1-k^2)y^2}}{\sqrt{(x-a)^2 + y^2}}.$$

evaluated at the time $t = \pi/\omega$, and, as above, F is a 180 degree rotation about the origin. Note that as for class 1 equations the phase portrait of Eqs. 6 and 7 is also a continuum of circles centered at $(a,0)$ since,

$$\frac{dy}{dx} = -(x-a)/y$$

We may write out the solution of the latter autonomous vector ODEs explicitly using the addition formulae for the Jacobi elliptic functions:

$$x(t) = \frac{(x_0 - a)\mathrm{cn}\big(f(r)t, k\big) - Ky_0\mathrm{sn}\big(f(r)t, k\big)\mathrm{dn}\big(f(r)t, k\big)}{\mathrm{cn}^2\big(f(r)t, k\big) + K^2\mathrm{sn}^2\big(f(r)t, k\big)} + a$$

$$y(t) = \frac{K(x_0 - a)\mathrm{sn}\big(f(r)t, k\big) + y_0\mathrm{cn}\big(f(r)t, k\big)\mathrm{dn}\big(f(r)t, k\big)}{\mathrm{cn}^2\big(f(r)t, k\big) + K^2\mathrm{sn}^2\big(f(r)t, k\big)}$$

where $x(0) = x_0$, $y(0) = y_0$, $r = \sqrt{(x_0 - a)^2 + y_0^2}$, $K = \sqrt{1 - (ky_0/r)^2}$, and sn, cn, and dn are the Jacobi elliptic functions having elliptic modulus k. From this point on we will drop reference to the elliptic modulus k in the elliptic functions unless it is needed for clarification.

The matrix form of the twist map for class 2 is given by:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c(t) \begin{pmatrix} \mathrm{cn}\big(f(r)t\big) & -K\mathrm{sn}\big(f(r)t\big)\ \mathrm{dn}\big(f(r)t\big) \\ K\mathrm{sn}\big(f(r)t\big) & \mathrm{cn}\big(f(r)t\big)\ \mathrm{dn}\big(f(r)t\big) \end{pmatrix} \begin{pmatrix} x_0 - a \\ y_0 \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix}$$

where

$$c(t) = 1/\big[\ \mathrm{cn}^2\big(f(r)t\big) + K^2\mathrm{sn}^2\big(f(r)t\big)\big]$$

For $k = 0$ class 2 reduces to class 1.

Class 3 equations are obtained from class 2 equations by letting $k = g(\lambda)$ in Eq. 4 and 5, where $g(\lambda)$ is any C^1 function. Since $\lambda = \sqrt{(x - a)^2 + y^2}$ is determined by the initial conditions we see that the elliptic modulus is also determined by the initial conditions. In this case, the matrix form of the twist map is the same as in class 2, except that $k = g(\lambda)$.

Class 4 equations are those motivated by the function

$$x(t) = \lambda\mathrm{cn}\big(\lambda t + \theta, k\big) + a$$

where $\lambda^2 = 0.5\big[\ (1-2k^2)(x_0-a)^2 + \sqrt{(1 - 2k^2)^2(x_0 - a)^4 + 4\big(\dot{x}_0^2 + k^2(x_0 - a)^4\big)}\ \big]$

This function solves the ODE

$$\ddot{x} + p(x, y)(1 - 2k^2)(x - a) + 2k^2(x - a)^3 = 0$$

where, $p(x,y) = 0.5\left[\ (1-2k^2)(x-a)^2+\sqrt{(1-2k^2)^2(x-a)^4+4\left(\dot{x}^2+k^2(x-a)^4\right)}\ \right]$.

It should be noted here that the appearance of a term containing $\dot{x}$, such as occurs above, does not necessarily imply the presence of damping in the ODE.[2] On the contrary, the solution of this equation consists of concentric closed curves parameterized by λ defined above. If we replace a in the above equations by $a\,\mathrm{sg}(\omega t)$ we obtain an ODE having a square-wave forcing term whose Poincaré map, is FTFT, where

$$\mathrm{T}(x,y) = \left(\ \lambda \mathrm{cn}(\lambda t+\theta)+a, -\lambda^2 \mathrm{sn}(\lambda t+\theta)\mathrm{dn}(\lambda t+\theta)\right)$$

and where $t = \pi/\omega$.

When $k^2 = 0.5$, the equation for T reduces to a variation of Duffing's equation, namely,

$$\ddot{x}+(x-a)^3 = 0$$

In this case $\lambda^4/4 = 4\dot{x}_0^2+(x_0-a)^4$, If we replace a by $a\,\mathrm{sg}(\omega t)$ we get a square-wave forced variation of Duffing's equation:

$$\ddot{x}+(x-a\,\mathrm{sg}(\omega t))^3 = 0$$

The Poincaré map of this equation is given by FTFT where the two components of T are

$$\begin{aligned} x(t) &= \lambda \mathrm{cn}\left(\lambda t+\theta, \sqrt{0.5}\ \right)+a \\ \dot{x}(t) &= -\lambda^2 \mathrm{sn}\left(\lambda t+\theta, \sqrt{0.5}\ \right)\mathrm{dn}\left(\lambda t+\theta, \sqrt{0.5}\ \right) \end{aligned}$$

Also note that by taking $k = 0$ we obtain a special case where T is expressible in terms of sines and cosines:

$$\begin{aligned} x(t) &= \lambda \cos\left(\lambda t+\theta\right)+a \\ \dot{x}(t) &= -\lambda^2 \sin\left(\lambda t+\theta\right) \end{aligned}$$

These two functions define a family of ellipses centered at $(a,0)$. If these equations are expanded using the addition formula for sine and cosine and we note that $x_0 = \lambda\cos(\theta)+a$ and $\dot{x}_0 = -\lambda^2\sin(\theta)$ then

[2]To see this in a simpler case consider the equation

$$\ddot{x}+x/(\dot{x}^2+1) = 0$$

which is solvable in terms of elliptic functions.

$$\lambda^2 = 0.5[\ (x_0 - a)^2 + \sqrt{(x_0 - a)^4 + 4\dot{x}_0^2}\,]$$

and we have the following matrix form for the twist T:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(\lambda t) & -\sin(\lambda t)/\lambda \\ \lambda \sin(\lambda t) & \cos(\lambda t) \end{pmatrix} \begin{pmatrix} x_0 - a \\ y_0 \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix}$$

In this case, the ODE associated with T is given by:

$$\ddot{x} + 0.5[\ (x - a)^2 + \sqrt{(x - a)^4 + 4\dot{x}^2}\,]\ (x - a) = 0$$

In this map, the twisting takes place around an ellipse rather than a circle.

When $k \neq 0$ in class 4, it is possible to carry this generalization one step further by assuming that $k = g(\lambda)$, where as before $g(\lambda)$ is any C^1 function. In this case the equation for λ can be very complex. For example, if $k^2 = \lambda$, then this equation becomes

$$\lambda^2 = 0.5[\ (1 - 2\lambda)(x_0 - a)^2 + \sqrt{(1 - 2\lambda)^2(x_0 - a)^4 + 4\big(\dot{x}_0^2 + \lambda(x_0 - a)^4\big)}\,]\ .$$

Class 5 equations for T are given by general elliptic differential equations, See [Davis, 1960], page 209:

$$\ddot{y} = A + By + Cy^2 + Dy^3 \tag{8}$$

This equation when considered as a complex ODE is solved in terms of elliptic functions, [Davis,1960]. Unlike the case for linear equations, the real and imaginary parts of the complex solution are not solutions of Eq. 8. To obtain real solutions in terms of elliptic functions we consider only special cases, for example:

$$\ddot{y} = -Ay - By^3 \tag{9}$$

where $A \geq 0$ and $B > 0$. This equation has the following real solution:

$$y(t) = C\mathrm{dn}(\lambda t + \theta, k)$$

where $k^2 = (2\lambda + A)/\lambda^2$ and $C^2 = 2\lambda^2/B$.

As in the above examples, the square-wave forced equation is obtained by replacing y by $\big(y - a\,\mathrm{sg}(\omega t)\big)$ in the equation for T.

From the form of this solution we see that the initial conditions can affect the amplitude, frequency, and the elliptic constant, k, of the solutions of Eq.(9).

Remark: The above equations may be divided into two groups. Group one arises from a first order vector system in $\mathbf{R}^2$. Group two arises from a second order scalar ODE. Thus we see that Classes 1,2,3 are of increasing complexity from group 1, whereas classes 4 and 5 are from group two and also of increasing complexity.

4 Differential Equations with Damping which have Closed Form Poincaré Maps

Corresponding to the above described five classes of undamped ODEs there are five classes of damped ODEs having Poincaré maps in a closed form. We illustrate how to introduce damping into these equations in a specific example and a general example only, without deriving the five classes.

4.1 The Damped Twist

To obtain ODEs with damping having a Poincaré map in closed form let us begin with the following example which we will show how to generalize in the following section. Consider:

$$x(t) = \lambda \exp(-\alpha t)\mathrm{cn}(u) \tag{10}$$

where $u = \lambda\big(\exp(-\alpha t) - 1\big) + \theta$; λ and θ are arbitrary constants.

Taking the first derivative we get

$$\dot{x} = -\alpha x + \alpha\lambda^2 \exp(-\alpha t)\mathrm{sn}(u)\mathrm{dn}(u) \tag{11}$$

The ODE for which $x(t)$ is the solution is given by

$$\ddot{x} + 3\alpha\dot{x} + 2\alpha^2 x + 2\alpha^2 k^2 x^3 = -\alpha^2(1 - 2k^2)\mu x \tag{12}$$

where $\mu = \lambda^2 \exp(-2\alpha t)$. The factor μ contains both the arbitrary constant of integration, λ, and time, t, and thus it appears at first that Eq. 12 is not autonomous. However, by using identities for the Jacobi elliptic functions the factor μ may be eliminated. In particular we may do so by solving the quadratic equation:

$$(1-k^2)\mu^2 + (2k^2-1)x^2\mu - (k^2x^4 + (\dot{x}/\alpha + x)^2) = 0 \qquad (13)$$

This is a quadratic equation for μ^2 and so the positive root must be chosen. The origin of this quadratic equation is as follows. From [Bowman, 1961] we know that the Jacobi cn satisfies the ODE:

$$\ddot{x} = -(1-2k^2)x - 2k^2x^3$$

where k is a parameter of the ODE which turns out to be the elliptic modulus. This equation has a first integral

$$\dot{x}/2 = -(1-2k^2)x^2/2 - k^2x^4/2 + \mathrm{H}/2 \qquad (14)$$

where H is determined by evaluating the functions sn, cn, dn at $t = 0$. Specifically $\mathrm{H} = (1-k^2)$. By noting from Eq. 10 that

$$\lambda x = \lambda^2 \exp(-\alpha t)\mathrm{cn}(u)$$

and

$$(\dot{x} + \alpha x)/\alpha = \lambda^2 \exp(-\alpha t)\mathrm{sn}(u)\,\mathrm{dn}(u)$$

we obtain Eq. 13 from Eq. 14.

The square-wave forced differential equation may be obtained as before by replacing x by $(x - a\,\mathrm{sg}(\omega t))$ in Eq. 12.

More complex relationships for μ may be obtained by allowing k to be a function of μ. For example, if $k = \mu$, then the equation for μ becomes

$$\mu^4 - 2x^2\mu^3 - (1-x^4)\mu^2 + x^2\mu + (\dot{x}/\alpha + x)^2 = 0$$

Doing this makes the *elliptic modulus*, k, a function of the initial conditions of the ODE.

4.2 Generalizations

We may generalize the idea presented in the preceding section by the following lemma:

Lemma 5 *Assume the solutions of the following the differential equations are unique for each set of initial conditions:*

$$\dot{x}(t) = G(u)\big(x - a\,\mathrm{sg}(\omega t)\big) - H(u)y\sqrt{u^2 - k^2y^2}$$

$$\dot{y}(t) = G(u)y + H(u)\big(x - a\,\mathrm{sg}(\omega t)\big)\sqrt{u^2 - k^2y^2}$$

where

$$u^2 = (x - a\,\mathrm{sg}(\omega t))^2 + y^2,$$

$x(0) = \lambda\mathrm{cn}(\theta)$ *and* $y(0) = \lambda\mathrm{sn}(\theta)$, *and* G, H *are continuous.*

Define two functions g *and* f *by the differential equations*

$$g'(t,\lambda) = g(t,\lambda)G\big(g(t,\lambda)\big) \text{ and } g(0,\lambda) = \lambda$$

$$f'(t,\lambda) = g(t,\lambda)H\big(g(t,\lambda)\big) \text{ and } f(0,\lambda) = 0$$

and then define

$$x(\pi/\omega) = g(\pi/\omega,\lambda)\mathrm{cn}\big(f(\pi/\omega,\lambda) + \theta\big) + a$$

$$y(\pi/\omega) = g(\pi/\omega,\lambda)\mathrm{sn}\big(f(\pi/\omega,\lambda) + \theta\big)$$

If the map T *is defined by the equation*

$$\mathrm{T}(x_0, y_0) = \big(x(\pi/\omega), y(\pi/\omega)\big)$$

then the Poincaré map of the square-wave forced system above is FTFT.

Proof: Apply the factorization lemma to exchange the square-wave forced system for an autonomous equation which determines T. Differentiate the functions x, y to obtain the autonomous ODEs and use the relation $g^2 = (x-a)^2 + y^2$ to eliminate the arbitrary constants from these equations. Then by the uniqueness assumption we are done. ■

For the case where $k = 0$, the matrix form of the twist map given by this lemma is:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c(t,\lambda)\begin{pmatrix} \cos(f(t)) & -\sin(f(t)) \\ \sin(f(t)) & \cos(f(t)) \end{pmatrix}\begin{pmatrix} x_0 - a \\ y_0 \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix}$$

where $c(t,\lambda) = g(t,\lambda)/\lambda$, $g(0,\lambda) = \lambda$, and $f(0,\lambda) = 0$, $\lambda^2 = (x_0 - a)^2 + y_0^2$.

Remark: Because we have above a pair of first order equations there must be two arbitrary constants in the solutions. These constants are λ and θ. Note, however, that θ makes no contribution to the twisting action.

Remark: It would appear that if we add damping to a twist map that appears in a twist and flip map the combined damped twist and flip would converge to a fixed point since the effect of damping is to decrease the distance between any point and the center of the twist. However, lemma 13 of [Brown & Chua, 1991] implies that if the damped twist maps a point to the right hand side of the vertical axis near the center of the twist $(a, 0)$, then the flip must map it to the left hand side of the vertical axis, away from the center of the twist. At best, the damped twist and flip map can settle down to a period-two point.

4.3 Generalizations of the Twist Map and the Damped Twist Map

The preceding considerations lead us to a generalization of a twist map and a damped twist map beyond that found in [Brown & Chua, 1991].

Definition: The following functions define a one parameter family of *generalized damped twist maps* in $\mathbf{R}^2$, with parameter t:

$$x(t) = g(t, \lambda)\text{cn}\big(f(t, \lambda) + \theta\big) + a$$

$$y(t) = g(t, \lambda)\text{sn}\big(f(t, \lambda) + \theta\big)$$

where λ and θ are determined by $x(0)$, and $y(0)$, and

$$\frac{\partial g}{\partial \lambda} \neq 0$$

and

$$\frac{\partial f}{\partial \lambda} \neq 0$$

In the event that

$$\frac{\partial g}{\partial t} = 0$$

we have a *generalized undamped twist map* and the differential equations will not have a damping factor.

Remark: These equations define a twist map because both f and g depend on λ which is determined by the initial conditions. This generalization is consistent with our original definition in [Brown & Chua, 1991].

5 Horseshoe Twist Theorem III for Class 1 ODEs

We have the following extension of the horseshoe twist theorem for the equations of class 1:

Theorem 3 *Let $(0, y)$, $y > 0$, be a hyperbolic fixed point of* FT *where* T *is a generalized twist having rotation $f(r)$. Assume that $f'(\sqrt{a^2 + y^2}) > 0$. Let* M *be the unstable manifold of* FT *at this hyperbolic fixed point. If* $\mathrm{M}_{\mathbf{rhs}}$ *contains a fixed point of* T, *or if* $\mathrm{M}_{\mathbf{rhs}}$ *contains a point of the circle $\|\mathbf{z} - \mathbf{a}\| = a$, then* M *is a c-manifold and therefore* FT *has a horseshoe.*

Outline of Proof: It is sufficient to show that the key lemmas of [Brown, 1990] carry over under these assumptions. In the fixed point lemmas simply replace r by $f(r)$. We assume that f is such that there are hyperbolic fixed points. Note that all symmetry lemmas are independent of f. The generalized twist is area preserving and the trace of D(FT) is given by:

$$\mathrm{tr} = 2(1 + \frac{a\pi y f'(r)}{\omega r}) - 4(a/r)^2$$

The lemma for the slope of the expanding eigenvector is unchanged by f. Further the "energy" lemmas are unchanged by f. Given these facts, the theorem follows. ■

We have the following corollary,

Corollary 5 *Given a hyperbolic fixed point $(0, y)$, with $r = \sqrt{y^2 + a^2}$ and $r_0 < r$ such that $f(r_0) = 2\omega n$, a sufficient condition for the unstable manifold* M *of this fixed point to be a c-manifold is that the two circles,*

$$(x - a)^2 + y^2 = r_0^2$$

and

$$(x + a)^2 + y^2 = r^2$$

intersect, or the two circles,

$$(x-a)^2+y^2=a^2$$

and

$$(x+a)^2+y^2=r^2$$

intersect.

A simpler form of this condition is

$$r-r_0 \leq 2a$$

When $f(r)=r$ this reduces to

$$r \bmod (2\omega) \leq 2a.$$

We have the following conjecture:
Recall that the square-wave forced Duffing equation

$$\ddot{x}+x^3=a\,\mathrm{sg}(\omega t)$$

has the Poincaré map FTFT, where T is defined by the map $\mathrm{T}(x_0,\dot{x}_0)=(x(\pi/\omega),\dot{x}(\pi/\omega))$ where $x(t)$ is the solution of

$$\ddot{x}+x^3=a$$

having initial conditions $(x_0,\dot{x}_0)$.

Conjecture 1 *Given a hyperbolic fixed point of* FT *on the negative vertical axis,* FT *has a horseshoe if the unstable manifold associated to this fixed point contains a fixed point of* T.

Almost surely a similar theorem is true for classes 2 through 5, but at this time it has not been proven.

It should be noted that by structural stability of the horseshoe, if a horseshoe twist theorem is proven for classes 1-5, it will also be true for the damped equations corresponding to classes 1-5 for some sufficiently small damping factor.

6 Extending the Square-wave Analysis

6.1 Generalizing Square-wave Forcing

The square-wave forcing term is an example of a piecewise constant, periodic function. Another example is as follows:

$$f(t) = a_1 \text{ for } 0 \le t \le \pi/2$$

$$f(t) = a_2 \text{ for } \pi/2 < t \le \pi$$

$$f(t) = -a_2 \text{ for } \pi < t \le 3\pi/2$$

$$f(t) = -a_1 \text{ for } 3\pi/2 < t \le 2\pi$$

where f is extended to all real t to have period 2π. If this piecewise constant, periodic function is used in place of the square-wave as a forcing term in the equations of the previous sections, we obtain Poincaré maps of the form $\mathrm{FT_1T_2FT_1T_2}$, where the T_i come from the same equations as T. The only difference is that t is now evaluated at $\pi/2\omega$. Thus the T_i are given by

$$\begin{pmatrix} x(\pi/2\omega) \\ y(\pi/2\omega) \end{pmatrix} = \begin{pmatrix} \cos\left(f(\lambda)\pi/2\omega\right) & -\sin\left(f(\lambda)\pi/2\omega\right) \\ \sin\left(f(\lambda)\pi/2\omega\right) & \cos\left(f(\lambda)\pi/2\omega\right) \end{pmatrix} \begin{pmatrix} x_0 - a_i \\ y_0 \end{pmatrix} + \begin{pmatrix} a_i \\ 0 \end{pmatrix}$$

Note that the center of the rotation of T_i is a_i.

The above observations lead to a generalization that is suggested in [Brown & Chua, 1991]. The forcing function of the equations of classes 1 through 5 may be any periodic function which is piecewise constant, and the Poincaré map is still obtained in closed form. As mentioned in [Brown & Chua, 1991] we may approximate the sine function uniformly by a piecewise constant function and thus obtain an approximation of the Poincaré map for sinusoidal forcing by a Poincaré map of the form FTFT where T is a composition of twists around centers a_i, determined by the sine function. In this regard we have the following lemma suggested by Prof. Morris Hirsch at U. C. Berkeley.

Lemma 6 *Let*

$$\dot{\mathrm{x}} = \mathrm{F}(\mathrm{x}, g(t))$$

and

$$\dot{\mathbf{y}} = \mathrm{F}(\mathbf{y}, h(t))$$

for $0 \leq t \leq \tau$*, where* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in* $\mathbf{R}^n$*,* g, h *are real valued (not necessarily continuous) functions of* t*, and* F *is any* C^1 *function in* $\mathbf{R}^{n+1}$*. Then for any compact region of* R^n*, we can find a number* δ *small enough such that if*

$$\int_0^t |h(s) - g(s)| ds < \delta$$

for $0 \leq t \leq \tau$*, then (1)*

$$\|x(t) - y(t)\| < \epsilon$$

for all $0 \leq t \leq \tau$*.*

Further, (2) the derivatives of x, y *with respect to their initial conditions also satisfies this relationship.*

Proof: Since F is C^1 we may find a compact neighborhood of N, call it D_τ, such that if the solution of the first ODE starts in N it remains in D_τ for all $0 \leq t \leq \tau$. Likewise for the second equation we can find a compact neighborhood of N, call it D_{τ_1} such that the solution of the second equation remains in D_{τ_1} for $0 \leq t \leq \tau$. We find a Lipschitz constant, K, for F on the union of these two neighborhoods and apply theorem 3.3.1 of [Hille, 1969], using the integral assumption to obtain the estimate

$$\|x(t) - y(t)\| < \delta K \exp(Kt)$$

For K fixed, we may choose δ such that $\delta K \exp(K\tau) < \epsilon$.

To prove (2) we invoke Peano's theorem, theorem 3.1 from [Hartman, 1964] and repeat the argument. ∎

This lemma states that the Poincaré map of a periodically forced ODE from any of the above described classes of ODEs may be approximated uniformly on any compact subset of $\mathbf{R}^2$ by a map of the form FTFT, where T is a finite composition of twist maps T_i, each of which can be obtained in closed form. The conclusion we reach is that on any compact set, the Poincaré map of a periodically forced ODE from any of the above described classes can be uniformly approximated on compact subsets by a closed form Poincaré map.

We have the following theorem:

Theorem 4 (Twist and Flip Approximation Theorem I) *Given the conditions of lemma 6 above, if the approximating Poincaré map has a horseshoe, then so does the periodically forced Poincaré map.*

Proof: If the approximating Poincaré map has a horseshoe, it exists on a compact set by [Smale, 1965]. Since the approximating Poincaré map is C^1 close to the periodically forced Poincaré map, the horseshoe is preserved by structural stability. ∎

Remark: By the same argument we have the following result given the conditions of lemma 6 above. Let the first system be a square-wave forced system and the second system be a C^∞ approximation of the square-wave obtained by "rounding-off" the corners of the square-wave forcing function. If the square-wave Poincaré map has a horseshoe, then there exists a "rounded-off" C^∞ forced Poincaré map having a horseshoe also.

Remark Numerical experiments indicate that the above generalizations allow us to integrate many square-wave forced ODEs by the use of a composition of twist maps followed by a flip. Doing this results in an integration technique using about six lines of code that is about ten times faster than conventional methods. The following is a computer program in QuickBASIC for the integration of the following class 1 ODE with a sine forcing term:

$$\dot{x} = -r(t)y$$

$$\dot{y} = r(t)(x - a\ \sin(\omega t))$$

where $r(t) = \sqrt{(x - a\ \sin(\omega t))^2 + y^2}$.

FOR i=1 TO 1000
FOR j=1 TO M

$$aa = a \sin(j\pi/\mathrm{M})$$

$$r = \sqrt{(x - aa)^2 + y^2}$$

$$u = (x - aa)\cos(r/\mathrm{M}) - y\sin(r/\mathrm{M}) + aa$$

$$v = (x - aa)\sin(r/\mathrm{M}) + y\cos(r/\mathrm{M})$$

$$x = u$$

$$y = v$$

```
PSET (x,y),13
   NEXT j
   FOR j=1 TO M
```

$$aa = -a\sin(j\pi/\mathrm{M})$$

$$r = \sqrt{(x-aa)^2 + y^2}$$

$$u = (x-aa)\cos(r/\mathrm{M}) - y\sin(r/\mathrm{M}) + aa$$

$$v = (x-aa)\sin(r/\mathrm{M}) + y\cos(r/\mathrm{M})$$

$$x = u$$

$$y = v$$

```
PSET (x,y),10
   NEXT j
   NEXT i
   END
```

Remark: The symbol *aa* in the above code denotes a programming variable.

7 Extensions to Three Dimensions

The Duffing equation,

$$\ddot{x} + \alpha\dot{x} + x^3 = a\ \cos(\omega t)$$

can be viewed as an autonomous equation of degree 2 in three dimensions. This three dimensional system is <u>not</u> the usual autonomous system obtained by considering time as a new variable but rather is the following:

$$\begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & z \\ ((\dot{z} + \alpha z + 3x^2 y)/\omega)^2 & = & a^2 - (z + \alpha y + x^3)^2 \end{array}$$

For $\dot{x} = \dot{y} = \dot{z} = 0$ we have $x = \pm a^{1/3}, y = 0, z = 0$. This equation is related to Duffing's equation by the correct choice of initial conditions. If the original Duffing equation has the initial conditions $x(0) = x_0, y(0) = y_0$ then the third order equation must have the additional condition $z(0) = a - \alpha y_0 - x_0^3$. Therefore, the Poincaré map has a natural extension to these equations as the map associated with the period of an *implicit* forcing term that is not immediately visible in the three dimensional ODE.

This extension suggests that we should seek to expand our investigations to three dimensions. In particular, we seek an extension of the twist and flip map to three dimensions in such a way that FTFT is a three dimensional Poincaré map for a square-wave forced, nonlinear, first order system of ODEs. We find that this can be done easily, and the three dimensional system is considerably larger than the class 1, two dimensional system described earlier.

We now illustrate this generalization to three dimensions for the twist and flip map. In particular, the following three dimensional system of ODEs has a closed form solution for the Poincaré map which is the analogue of the class 1 system.

Class 1, three dimensions

$$\dot{x} = \frac{\big(x - a\,\mathrm{sg}(\omega t)\big) z f(r)}{\sqrt{\big(x - a\,\mathrm{sg}(\omega t)\big)^2 + y^2}} - y g(r)$$

$$\dot{y} = \frac{z y f(r)}{\sqrt{\big(x - a\,\mathrm{sg}(\omega t)\big)^2 + y^2}} - \big(x - a\,\mathrm{sg}(\omega t)\big) g(r)$$

$$\dot{z} = -\sqrt{\big(x - a\,\mathrm{sg}(\omega t)\big)^2 + y^2}\, f(r)$$

where $r = \sqrt{\big(x - a\,\mathrm{sg}(\omega t)\big)^2 + y^2 + z^2}$ and the functions f, g are any C^1 functions. We note that for $f = 0$ then this system reduces to the two dimensional, class 1 system. The three dimensional system defines a larger class of equations than the two dimensional system due to the fact that in three dimensions we are able to twist in two directions with rotation functions f and g.

These equations can be derived directly from the following polar equations for a twist:

$$(r, \phi, \theta) \to \big(r, \phi + g(r)\tau, \theta + f(r)\tau\big)$$

and the polar equations for the associated three dimensional ODE:

$$(\dot{r}, \dot{\phi}, \dot{\theta}) = \big(0, g(r), f(r)\big)$$

The rectangular-coordinate matrix form of the twist map in three dimensions about the point (a, b, c) is given by the following equations: Given a point in three space, (x, y, z), define $r^2 = (x-a)^2 + (y-a)^2 + (z-c)^2$ and $s^2 = r^2 - (z-c)^2$, then $\mathrm{T}(x, y, z) = (u+a, v+b, w+c)$

where,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \zeta(r,\tau) \begin{pmatrix} \cos\big(g(\lambda)\tau\big) & -\sin\big(g(\lambda)\tau\big) \\ \sin\big(g(\lambda)\tau\big) & \cos(g(\lambda)\tau) \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

$$\zeta(r,\tau) = \big[\cos\big(f(r)\tau\big) + (z-c)\sin\big(f(r)\tau/s\big)\big]$$

and,

$$w = \cos\big(f(r)\tau\big)(z-c) - s\sin\big(f(r)\tau\big)$$

Of course we may introduce damping as we did in the two dimensional case and obtain three dimensional damped ODEs with square-wave forcing.

Class 2, three dimensions

The three dimensional system of first order, nonlinear ODEs

$$\begin{aligned} \dot{x} &= -yz \\ \dot{y} &= xz \\ \dot{z} &= -k^2 xy \end{aligned}$$

where,

$$x(0) = \lambda \mathrm{cn}(\theta),\ y(0) = \lambda \mathrm{sn}(\theta),\ z(0) = \lambda \mathrm{dn}(\theta)$$

has the Jacobi elliptic functions as a two parameter set of solutions:

$$\begin{aligned} x(t) &= \lambda \mathrm{cn}(\lambda t + \theta, k) \\ y(t) &= \lambda \mathrm{sn}(\lambda t + \theta, k) \\ z(t) &= \lambda \mathrm{dn}(\lambda t + \theta, k) \end{aligned}$$

By the form of the above solutions, we see that these equations define a twist.

The following set of equations have, as a two parameter set of solutions, a twist about the point $(a, 0, 0)$

$$\begin{aligned}
\dot{x} &= -yz \\
\dot{y} &= (x-a)z \\
\dot{z} &= -k^2(x-a)y
\end{aligned}$$

By replacing the constant a by $a\,\mathrm{sg}(\omega t)$ we produce a square-wave forced equation whose Poincaré map is of the form FTFT, where T is a twist map and F is the flip in three dimensions, and T is defined by the Jacobi elliptic functions. By combining the solution of the above equations with a 180 degree rotation we obtain a twist and flip map in three space in which the twisting takes place on a set of concentric elliptic surfaces. The map FTFT determined by the twist and flip map is a Poincaré map for a three dimensional system with square-wave forcing. The matrix form of this map may be obtained by using the addition formulae for the Jacobi elliptic functions and identifying the terms $\lambda\mathrm{cn}(\theta)$, $\lambda\mathrm{sn}(\theta)$, $\lambda\mathrm{dn}(\theta)$ with the initial conditions.

Another generalization is suggested by the following set of functions:

$$\begin{aligned}
x(t) &= \lambda\mathrm{cn}(f(\lambda)g(\lambda)t+\theta, k) \\
y(t) &= \lambda\mathrm{sn}(f(\lambda)h(\lambda)t+\theta, k) \\
z(t) &= \lambda\mathrm{dn}(g(\lambda)h(\lambda)t+\theta, k)
\end{aligned}$$

The above equations define a general three dimensional twist which solve the following system of equations, where $\lambda^2 = x^2 + y^2$

$$\begin{aligned}
\dot{x} &= -yz\,f(\lambda)\,g(\lambda)/\lambda \\
\dot{y} &= xz\,f(\lambda)\,h(\lambda)/\lambda \\
\dot{z} &= -k^2xy\,g(\lambda)\,h(\lambda)/\lambda
\end{aligned}$$

We may add square-wave forcing as before and generate nonlinear, square-wave forced ODEs in three dimensions whose Poincaré maps are of the form FTFT, and where T has a closed form expression of a generalized twist.

This analysis can be extended to obtain further examples of twist on surfaces. In particular, given any family of concentric, closed surfaces in three spaces we may obtain a generalization of the twist and flip map. Such a surface can be defined by a first order partial differential equation (PDE) in three variables, [Sneddon, 1957]. Such a surface is made up of the integral curves of a system of three first order ODEs derived from the defining PDE.

8 Generalizing the Twist: Shearing

In this section we will describe a generalization of the twist map when considered in isolation from its role as a Poincaré map. Thus we now consider the twist map as a transformation on $\mathbf{R}^2$ and ask how it may be generalized. The following example will provide the motivation for this generalization.

The function $x(t) = \lambda \tan(\lambda t + \theta)$ solves the nonlinear ODE

$$\ddot{x} - 2x\dot{x} = 0$$

where $x(0) = x_0 = \lambda \tan(\theta)$ and $\dot{x}(0) = x_0 = \lambda^2 \sec^2(\theta)$.

Similarly, the function $x(t) = \lambda a \exp(\lambda t)$ where $a = \dot{x}_0/x_0^2$ and $\lambda = \dot{x}_0/x_0$ solves the ODE

$$\ddot{x} - \dot{x}^2/x = 0$$

where $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$.

In these examples, the initial conditions affect both the "amplitude" and "frequency" of the solution as in the case of the twist, except in this case there is no "bending" action taking place around closed curves because the solution of this ODE is not a set of closed curves.

The presence of a stretching action without the bending action of a twist indicates that the above examples are more general than the twist map. In recognition of this fact we will define such maps as *two dimensional shears.*

Another example of a shear that is not associated with an ODE is given by the formula:

$$\mathrm{T}(x, y) = (\lambda x, y/\lambda)$$

where, $\lambda = (xy)^2 + 1$. This map is modeled after the linear hyperbolic map $(x, y) \rightarrow (\lambda x, y/\lambda)$, where λ is any real number greater than 1, and is determined by a partial differential equation.

The primary reason for defining the shear is that it can be used to demonstrate indirectly that the Poincaré map of some nonlinear ODEs have a twist and flip action. In the following two sections, we proceed to follow this program and we note that, at present, our methods are somewhat closer to art than science.

9 The van der Pol Equation

In this section we indicate that we can find a twist and flip paradigm in the van der Pol equation with square-wave forcing; namely:

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + \lambda x = a\,\mathrm{sg}(\omega t)$$

Since it is not possible to drop the damping term in this equation at this point without losing all of the interesting dynamics, let us retain it for now. We first note that the vector form of this equation in $\mathbf{R}^2$ does satisfy the conditions of the factorization lemma, hence the Poincaré map can be factored as FTFT. We now determine the nature of T from the following autonomous equation:

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + \lambda x = a$$

For $|x| < 1$, the coefficient of the $\dot{x}$ term is negative and therefore the solution of this equation is spiraling outward from the point $(a, 0)$. For $|x| > 1$, the solution is spiraling inward. Clearly something must give in this autonomous equation and so a periodic solution appears. We know from the solution of the autonomous van der Pol equation that the Poincaré map contracts inward when $|x| > 1$ and expands outward when $|x| < 1$. In order to find the twisting action in these two cases we consider them separately.

Case 1, $|x| << 1$. In this case we drop the x^2 term due to its size and obtain the linear autonomous equation

$$\ddot{x} - \epsilon\dot{x} + \lambda x = a$$

which contains no twisting action. An approximate T map for $|x| << 1$ can be obtained in closed form by solving the above linear equation by conventional methods. For $|x|$ close to 1 we have an expanding version of the case 2 analysis which follows.

Case 2, $|x|$ near 1 or $|x| > 1$. In this case we rewrite the equation as follows:

$$\ddot{x} - \epsilon\dot{x} + (\lambda + \epsilon x\dot{x})x = a$$

The twisting action of this equation, we conjecture, can be found by dropping the linear terms and taking $a = 0$ to obtain the equation:

$$\ddot{x} + (\epsilon x\dot{x})x = 0$$

Changing to $(x, \dot{x})$ coordinates and letting $\dot{x} = p$ the above equation reduces to

$$\dot{p} + \epsilon x^2 = 0$$

where $\dot{p}$ is the derivative of p with respect to x. We find a shearing action in this equation as follows. Integrate the above equation with respect to x to obtain:

$$\dot{x} = (\mathrm{H}^3 - \epsilon x^3)/3$$

or

$$\dot{x}/\mathrm{H} = \mathrm{H}^2(1 - \epsilon(x/\mathrm{H})^3)/3$$

where the constant of integration, H, is given by $\mathrm{H} = (\epsilon x_0^3 + 3\dot{x}_0)^{1/3}$ Letting $u = x/\mathrm{H}$ and $s = \mathrm{H}^2 t/3$ we have

$$\frac{du}{ds} = (1 - \epsilon u^3)$$

which has a closed form, nonperiodic, solution $u(s + \theta, \epsilon)$, see [Bois, 1961]. Changing coordinates back to x we have the solution

$$x(t) = \mathrm{H}u(\mathrm{H}^2 t/3 + \theta, \epsilon)$$

As we now know, the key feature of a solution of an ODE that implies shearing is the *simultaneous presence of the constant of integration defined by the initial conditions in both amplitude and frequency of the solution.* For class 1 recall that the twist map was defined by the equations

$$\begin{aligned} x(t) &= \lambda \cos\left(f(\lambda)t + \theta\right) + a \\ y(t) &= \lambda \sin\left(f(\lambda)t + \theta\right) \end{aligned}$$

Hence, the appearance of the constant H^2 as a coefficient on the time variable and H as a coefficient on u assures us of a shearing action. Although we cannot find the Poincaré map exactly, we know that it consists of components FTFT, where T spirals outward for $|x| < 1$, and T spirals inward for other values of x. We indirectly conclude the presence of twisting in the map T from the presence of shearing combined with the above described inward and outward spiraling actions.

10 The Cavitation Bubble Oscillator

We now consider the following equation in [Parlitz, et. al., 1991]

$$\ddot{x} + c\dot{x} + 1 - \exp(-x) = a\,\mathrm{sg}(\omega t)$$

where we have introduced a square-wave forcing function in place of $a\ \cos(\omega t)$. We decompose this non-autonomous equation into the following two autonomous equations:

$$\ddot{x} + c\dot{x} + 1 - \exp(-x) = a$$

and

$$\ddot{x} + c\dot{x} + 1 - \exp(-x) = -a$$

Our above factorization lemma does not apply to these equations because the exponential function is not an odd function of x but we can still proceed without it. First, as in [Brown & Chua, 1991] we drop the damping terms and consider the pair of equations

$$\ddot{x} + 1 - \exp(-x) = a$$

$$\ddot{x} + 1 - \exp(-x) = -a$$

where $a > 0$. A first integral can be obtained to get the equations

$$\dot{x}^2/2 = (a-1)x + \mathrm{H} - \exp(-x)$$

$$\dot{x}^2/2 = -(a+1)x + \mathrm{H} - \exp(-x)$$

where H is the constant of integration. Inspection of this equation reveals two separate cases. Case 1 is when $|a| < 1$. In this case both equations have periodic solutions which are generalized twists about the point $(\log(1/(1 \mp a)), 0)$. In place of a twist and flip map we have the Poincaré map as a composition of two generalized twists, $\mathrm{T}_1\mathrm{T}_2$, which cannot be written out explicitly at this time.

For the second case where $|a| \geq 1$ we see that one component of the Poincaré map comes from the equation:

$$\ddot{x} + 1 - \exp(-x) = a$$

This equation has the first integral:

$$\dot{x}^2/2 + \exp(-x) = (a-1)x + \mathrm{H}$$

where H is the constant of integration. This first integral shows that the solutions are unbounded. Clearly if $x(t) > 5$ the exponential term is small and this first integral takes the form

$$\dot{x}^2/2 = \mathrm{H} + (a-1)x$$

so that $x(t) \approx \beta_1 t^2 + \beta_2 t + \beta_3$ which, as a function of time, is a translation in the phase plane.

The second component of the Poincaré map is determined by

$$\ddot{x} + 1 - \exp(-x) = -a$$

which has first integrals which are closed curves (periodic solutions) and looks like a twist. These two cases suggest further generalizations of the horseshoe twist theorem. In the case of the twist and translate we have the following theorem where T_a is a twist centered at $(a, 0)$, and $\mathrm{L}_\tau(\mathbf{z}) = \mathbf{z} + \tau\mathbf{e}_1$, were $\mathbf{e}_1 = (1, 0)$

Theorem 5 *Let (c, y), $y > 0$ be a hyperbolic fixed point of the twist and translate map $\mathrm{L}_\tau\mathrm{T}_a$, where $a > 0$ and $\tau < 0$ and let* M *be the unstable manifold of $\mathrm{L}_\tau\mathrm{T}_a$ at this hyperbolic fixed point. If $\mathrm{M}_{\mathrm{lhs}}$ contains a fixed point of T_a^2, then* M *is a c-manifold and therefore the map $\mathrm{L}_\tau\mathrm{T}_a$ has a horseshoe.*

From this we get the following corollary which assumes the existence of a hyperbolic fixed point (c, y), for $\mathrm{L}_\tau\mathrm{T}_a$ with $r = \sqrt{y^2 + (0.5\tau)^2}$. Define $r_0 = (\omega[r/\omega])$, where $[x]$ is the integer part of x. Since $r_0 = (2n+1)\omega$, the circle $r_0^2 = (x-a)^2 + y^2$ is made up entirely of fixed points of T_a^2.

Corollary 6 *Given a hyperbolic fixed point (c, y), the unstable manifold* M *of this fixed point is a c-manifold if the two circles,*

$$(x-a)^2 + y^2 = r_0^2$$

and

$$(x-a-\tau)^2 + y^2 = r^2$$

intersect.

The above intersection may be restated as follows:

$$r \leq \tau + r_0$$

or

$$r \bmod (\omega) \leq \tau.$$

We explain the connection of theorem 5 to the non-dissipative twist and translate of [Parlitz et. al., 1991] at the conclusion of the proof of theorem 5.

The following subsections provide the proof of theorem 5. The figures used in [Brown & Chua, 1991] can be adapted to this proof simply by translating the coordinate origin to the point $(a + 0.5\tau, 0)$, Thus we refer the reader to those figures.

10.1 Definitions

We will use T_a to denote the twist centered at the point $(a, 0)$ with $a > 0$ and $\omega > 0$.

Let $\mathbf{z}$ denote a vector in $\mathbf{R}^2$ and let $\mathbf{a}$ denote the vector $(a, 0)$ then,

$$\mathrm{T}_a(\mathbf{z}) \quad = \quad \mathbf{A}(\pi r/\omega)(\mathbf{z} - \mathbf{a}) \quad + \quad \mathbf{a}$$

where

$$\mathbf{A}(\pi r/\omega) = \begin{bmatrix} \cos(\pi r/\omega) & -\sin(\pi r/\omega) \\ \sin(\pi r/\omega) & \cos(\pi r/\omega) \end{bmatrix}$$

and $r = \|\mathbf{z} - \mathbf{a}\|$.

For any vector $\mathbf{z} \in \mathbf{R}^2$ define the "energy" function as $\rho(\mathbf{z}) = \|\mathbf{z} - \mathbf{a}\|$.

L_τ will denote a translation of τ units horizontally from the origin. The equation for L_τ is

$$\mathrm{L}_\tau(\mathbf{z}) = \mathbf{z} + \tau \mathbf{e}_1$$

where $\mathbf{e}_1 = (1, 0)$.

K_τ will denote a translation of τ units vertically from the origin. The equation for K_τ is

$$K_\tau(\mathbf{z}) = \mathbf{z} + \tau \mathbf{e}_2$$

where $\mathbf{e}_2 = (0,1)$.

Define the reflection operator about the horizontal axis by the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Define the reflection operator about any vertical line $x = \alpha$ by the rule

$$\mathbf{R}_\alpha(\mathbf{z}) = 2\alpha - \mathbf{z}$$

We will define c by the equation $c = a + 0.5\tau$, and define c' by the equation $c' = a - 0.5\tau$. The reflection operator about the vertical line, $x = c$ is, therefore, given by the rule

$$\mathbf{R}_c(\mathbf{z}) = 2c - \mathbf{z}$$

We will use $\mathbf{F}_\alpha$ to denote a flip about the point $(\alpha, 0)$ and therefore, $\mathbf{F}_\alpha = \mathbf{P}\mathbf{R}_\alpha$.

We define $\mathbf{C}_r$ to be the circle of radius r centered at $(a, 0)$ and define $\mathbf{D}_r$ to be the circle of radius r centered at $(a + \tau, 0)$.

Using these definitions we define $\mathbf{G}_r$ to be the intersection of $\mathbf{D}_r$ and the half plane $\{(x,y)|x > c\}$ and let $\mathbf{H}_r$ denote the intersection of $\mathbf{C}_r$ and the half plane $\{(x,y)|x < c\}$

We will use $[x]$ to denote the integer part of x. We will use r_0 to denote $\omega[r/\omega]$.

For any transformation Φ we will use $\mathrm{D}(\Phi)$ to denote the derivative of Φ.

We use the abbreviations RHS and LHS to denote the right-hand side and left-hand side, respectively.

10.2 Lemmas

Lemma 7 *(1) Let* $\mathbf{z}_0 \in \mathbf{R}^2$. *If* $\mathbf{F}_c(\mathbf{z}_0)$ *and* $\mathbf{z}_0$ *are on the same energy curve* $\mathbf{C}_r$, *then* $\mathbf{z}_0 = (c, y_0)$ *for some* $y_0 \in \mathbf{R}$. *i.e.,* $\mathbf{z}_0$ *is on the line* $x = c$.

(2) If $\mathrm{T}_a^2(c,y) = (c,y)$ *then* $\mathrm{L}_\tau \mathrm{T}_a(c,y) = \mathbf{F}_c(c,y)$

Proof: Direct computation. ∎

Lemma 8 *If* $L_\tau T_a(c, y) = (c, y)$ *then* $T_a(c, y) = (c', y)$.

Proof: Direct computation. ∎

Lemma 9 $L_\tau T_a$ *has an infinite number of fixed points. Moreover, they all lie on the line* $x = c$ *and approach infinity in both directions.*

Proof: At a fixed point, $\mathbf{z}$, we have the equation:

$$T_a(\mathbf{z}) = L_\tau^{-1}(\mathbf{z})$$

consequently the following equation holds:

$$\mathbf{A}(\pi r/\omega)(\mathbf{z} - \mathbf{a}) = (\mathbf{z} - \mathbf{a} - \tau).$$

From this matrix equation follow three scalar equations

$$\|\mathbf{z} - \mathbf{a}\| = \|\mathbf{z} - \mathbf{a} - \tau\|$$

$$\cos(\pi r/\omega) = 1 - 0.5(\tau/r)^2$$

$$\sin(\pi r/\omega) = \tau y/r^2$$

The first of these equations shows that all fixed points lie on the vertical line as stated.

The last two equations show where the fixed points lie exactly. From the last two equations for the sine and cosine we may derive the following equation:

$$\tan(\pi r/\omega) = \tau y/(2r^2 - \tau^2)$$

The existence of an infinite number of fixed points follows directly from this functional equation. ∎

Lemma 10 $\det(D(L_\tau T_a)) = 1$ *everywhere.*

Proof: Clearly L_τ is area preserving for any τ and the same is true of the twist thus the determinant is 1. ∎

Lemma 11 *The trace of* $\mathrm{D}(\mathrm{L}_\tau \mathrm{T}_a)$ *at a fixed point is given by*

$$\mathrm{tr} = 2 - (\frac{\tau}{r})^2 - (\frac{\tau y \pi}{r\omega})$$

and $\mathrm{tr} > 2$ *for* $-y > \tau\omega/r\pi$. *Consequently, all fixed points for which* $-y > \tau y \omega / r\pi$ *along the positive vertical line* $x = c$ *are hyperbolic, and the eigenvalues are also positive. For* tr *between -2 and 2, the fixed points are elliptic.*

Proof: Note that as a function of u, the rotation matrix $\mathbf{A}$ satisfies the first order ODE,

$$\mathbf{A}'(u) = \mathbf{B}\mathbf{A}(u)$$

where,

$$\mathbf{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

In this proof we will use the abbreviations, $r_x = \partial r/\partial x$ and $r_y = \partial r/\partial y$

The derivative of $\mathrm{L}_\tau \mathrm{T}_a$, i.e., the Jacobian matrix of $\mathrm{L}_\tau \mathrm{T}_a$ with respect to (x, y) is as follows: Let $\mu = \pi/\omega$. Note that

$$\mathrm{D}(\mathrm{L}_\tau)(\mathbf{z}) = \mathbf{I}$$

and

$$\mathrm{D}(\mathrm{L}_\tau \mathrm{T}_a)(\mathbf{z}) = \mathbf{I}\left[\frac{\partial \mathbf{A}(\mu r)(\mathbf{z}-\mathbf{a})}{\partial x}, \frac{\partial \mathbf{A}(\mu r)(\mathbf{z}-\mathbf{a})}{\partial y}\right]$$

(Note that in the above expression

$$\frac{\partial \mathbf{A}(\mu r)(\mathbf{z}-\mathbf{a})}{\partial x}$$

and

$$\frac{\partial \mathbf{A}(\mu r)(\mathbf{z}-\mathbf{a})}{\partial y}$$

are two dimensional column vectors.)

This is equal to

$$\mu[r_x\mathbf{A}'(\mu r)(\mathbf{z}-\mathbf{a}), r_y\mathbf{A}'(\mu r)(\mathbf{z}-\mathbf{a})] + \mathbf{A}(\mu r)$$

Since $\mathbf{A}'(u) = \mathbf{BA}(u)$, we have

$$\mathrm{D}(\mathrm{L}_\tau\mathrm{T}_a)(\mathbf{z}) = \mu[r_x\mathbf{BA}(\mu r)(\mathbf{z}-\mathbf{a}), r_y\mathbf{BA}(\mu r)(\mathbf{z}-\mathbf{a})] + \mathbf{A}(\mu r)$$

Using the equation from lemma 9, i.e.

$$\mathbf{A}(\pi r/\omega)(\mathbf{z}-\mathbf{a}) = (\mathbf{z}-\mathbf{a}-\tau)$$

in the equation for $\mathrm{D}(\mathrm{L}_\tau\mathrm{T}_a)$ we have, at a fixed point of $\mathrm{L}_\tau\mathrm{T}_a$:

$$\mathrm{D}(\mathrm{L}_\tau\mathrm{T}_a)(\mathbf{z}) = \mu[r_x\mathbf{B}(\mathbf{z}-\mathbf{a}-\tau), r_y\mathbf{B}(\mathbf{z}-\mathbf{a}-\tau)] + \mathbf{A}(\mu r)$$

Therefore at a fixed point of $\mathrm{L}_\tau\mathrm{T}_a$, the derivative of $\mathrm{L}_\tau\mathrm{T}_a$ is given by the matrix

$$\mathrm{D}(\mathrm{L}_\tau\mathrm{T}_a)(\mathbf{z}) = \begin{bmatrix} -\mu y\tau/2r + \cos(\mu r) & -\sin(\mu r) - \mu y^2/r \\ -(\mu\tau^2/4r + \sin(\mu r)) & -\mu\tau y/2r + \cos(\mu r) \end{bmatrix}$$

Using this matrix equation we can compute the trace of $\mathrm{D}(\mathrm{L}_\tau\mathrm{T}_a)$:

$$\mathrm{trace}(\mathrm{D}(\mathrm{L}_\tau\mathrm{T}_a))(c,y) = 2 - (\tau/r)^2 - (\tau\pi y/r\omega)$$

∎

Lemma 12 *For any hyperbolic fixed point of* $\mathrm{L}_\tau\mathrm{T}_a$ *on the positive line* $x = c$*, each branch of the unstable manifold is mapped onto itself by* $\mathrm{L}_\tau\mathrm{T}_a$*.*

Proof: For a hyperbolic fixed point the eigenvalues are given by

$$\lambda = \mathrm{tr}/2 \pm \sqrt{(\mathrm{tr}/2)^2 - 1}$$

For $\mathrm{tr} > 2$, $\mathrm{tr}/2 > \sqrt{(\mathrm{tr}/2)^2 - 1}$ and so both eigenvalues are positive. ∎

Lemma 13 *Let (c, y) be a hyperbolic fixed point of* $\mathrm{L}_\tau \mathrm{T}_a$ *on the vertical line, $x = c$. The expanding eigenvector of the unstable manifold has a slope given by*

$$\text{slope} = \frac{\tau}{2y}\sqrt{\frac{(\mathrm{tr}/2)+1}{(\mathrm{tr}/2)-1}}.$$

Consequently, the unstable manifold meets the vertical line transversely at (c, y).

Proof:

The eigenvalues of $\mathrm{D}(\mathrm{L}_\tau \mathrm{T}_a)$ at a fixed point are given by the formula:

$$\lambda = \mathrm{tr}/2 \pm \sqrt{(\mathrm{tr}/2)^2 - 1}$$

where tr is the trace of $\mathrm{D}(\mathrm{L}_\tau \mathrm{T}_a)$. The slope of the expanding eigenvector is given by

$$\text{slope} = -(\lambda - \mathrm{tr}/2)/(\sin(\mu r) + \mu y^2/r)$$

and this is equal to

$$-\sqrt{(\mathrm{tr}/2)^2 - 1}/(\sin(\mu r) + \mu y^2/r).$$

Now, $\sin(\mu r) = \tau y/r^2$ so that we conclude from this that the slope is given by

$$\text{slope} = \frac{\tau}{2y}\sqrt{\frac{(\mathrm{tr}/2)+1}{(\mathrm{tr}/2)-1}}$$

∎

Lemma 14 *Let z_0 be a point on the line $x = c$. If $\mathrm{L}_\tau \mathrm{T}_a(\mathrm{z}_0)$ is on this line, then $\mathrm{T}_a(\mathrm{z}_0)$ is also on this line and z_0 is either a fixed point of T_a^2 or a fixed point of $\mathrm{L}_\tau \mathrm{T}_a$.*

Proof: If $\mathrm{L}_\tau \mathrm{T}_a(\mathrm{z}_0)$ lies on the line $x = c$ then $\mathrm{L}_\tau^{-1} \mathrm{L}_\tau \mathrm{T}_a(\mathrm{z}_0)$ must lie on the line $x = c'$ and hence the same energy curve as z_0. $\mathrm{T}_a \mathrm{z}_0)$ must also be on the line $x = c'$, hence either 180 degrees from z_0 or a fixed point of $\mathrm{L}_\tau \mathrm{T}_a$. ∎

Lemma 15 *(1) For any a,* $\mathbf{P}\mathrm{T}_a\mathbf{P}\mathrm{T}_a = \mathbf{I}$

(2) For any τ, $\mathrm{L}_\tau\mathbf{P} = \mathbf{P}\mathrm{L}_\tau$

(3) For any α, $\mathbf{R}_\alpha = \mathrm{L}_{2\alpha}\mathbf{R}_0$ *and therefore* $\mathbf{R}_\alpha\mathrm{L}_\beta = \mathrm{L}_{-\beta}\mathbf{R}_\alpha$

(4) For any α, $\mathbf{R}_\alpha\mathrm{T}_\alpha = \mathrm{T}_\alpha^{-1}\mathbf{R}_\alpha$.

(5) For any α, $\mathrm{L}_\alpha^{-1} = \mathrm{L}_{-\alpha}$.

Proof: All are direct computations. ■

Lemma 16 *For any* a, τ $\mathbf{R}_c(\mathrm{L}_\tau\mathrm{T}_a) = (\mathrm{L}_\tau\mathrm{T}_a)^{-1}\mathbf{R}_c$.

Proof: Recall that $c = 2a - \tau$. From (4) of lemma 15, $\mathbf{R}_a\mathrm{T}_a(\mathbf{z}) = \mathrm{T}_a^{-1}\mathbf{R}_a$. The result follows from the observation that $\mathbf{R}_c\mathrm{L}_\tau = \mathbf{R}_a$. ■

Lemma 17 *Given a hyperbolic fixed point of* $\mathrm{L}_\tau\mathrm{T}_a$, *let* S *be the stable manifold, and let* M *be the unstable manifold at this fixed point. Then* $\mathbf{R}_c(\mathrm{M}) = \mathrm{S}$

Proof: Lemma 16. ■

Lemma 18 *Given a hyperbolic fixed point of* $\mathrm{L}_\tau\mathrm{T}_a$, *let* S *be the stable manifold, and let* M *be the unstable manifold at this fixed point. If* $\mathrm{M} = \mathrm{S}$, *then*

$$\mathbf{R}_c\mathrm{L}_\tau\mathrm{T}_a(\mathrm{M}) = \mathrm{M}$$

or equivalently,

$$\mathbf{R}_a\mathrm{T}_a(\mathrm{M}) = \mathrm{M}$$

Proof: $\mathrm{L}_\tau\mathrm{T}_a(\mathrm{M}) = \mathbf{R}_a(\mathrm{M})$. ■

We state without proof the following fact:

Lemma 19 *Let* A *be a* 2×2 *real matrix with positive eigenvalues of the form* $\lambda, 1/\lambda$, *where* $\lambda > 1$. *Let* $\mathbf{u}, \mathbf{v}$ *be real eigenvectors for* λ *and* $1/\lambda$ *respectively. Let* $\mathbf{w}$ *be any vector lying between* $\mathbf{u}$ *and* $\mathbf{v}$. *Then* $\mathrm{A}(\mathbf{w})$ *lies between* $\mathbf{u}$ *and* $\mathbf{w}$.

Lemma 20 *Let* $p_1 = (c, y_1)$ *and* $p_2 = (c, y_2)$, $y_1 < y_2$ *be two hyperbolic fixed points of* $\mathrm{L}_\tau\mathrm{T}_a$ *on the positive line* $x = c$ *having no other fixed point of* $\mathrm{L}_\tau\mathrm{T}_a$ *between them. Then there exists a fixed point of* T_a^2 *on the positive line* $x = c$ *between* p_1 *and* p_2.

Proof: By lemma 13 a branch of the local unstable manifold of $L_\tau T_a$ at the fixed point p_1 lies on the LHS of the plane. By the same lemma a branch of the local unstable manifold at the fixed point p_2 of $L_\tau T_a$ lies on the RHS of the plane. Let LL be the vertical line from p_1 to p_2 on the line $x = c$. By lemma 12 (positive eigenvalues) and lemma 19 $L_\tau T_a$ maps a small segment of LL, near p_1, into the left half line $x = c$. Also, $L_\tau T_a$ maps a small segment of LL, near p_2, into the right side of $x = c$. By connectedness of the line LL and the continuity of the diffeomorphism $L_\tau T_a$, there must be a point between p_1 and p_2 which is mapped onto the line $x = c$. We will call this point $\mathbf{z}_0$. By lemma 14 this must be a fixed point, or a period-two point for T_a. By hypothesis $\mathbf{z}_0$ cannot be a fixed point and so it must be a period-two point for T_a on the line $x = c$, i.e., a fixed point of T_a^2. ∎

Lemma 21 *If* $\mathbf{z}_0$ *lies on the* RHS *of the line* $x = c$ *and if the* $L_\tau T_a(\mathbf{z}_0)$*, lies on the* RHS *of the line* $x = c$*, then* $\rho(L_\tau T_a(\mathbf{z}_0))$ *is strictly less than* $\rho(\mathbf{z})$*.*

Proof: If $\mathbf{z}_0$ in on the RHS of the line $x = c$ and $L_\tau T_a(\mathbf{z}_0)$ in on the RHS of this line, then $T_a(\mathbf{z}_0) = \mathbf{z} = (x, y)$ is on the RHS of the line $x = c'$. Since $a < c'$, the result follows. ∎

Lemma 22 *Let* (c, y_0) *be a hyperbolic fixed point of* $L_\tau T_a$ *on the positive line* $x = c$*. Let* $r = \sqrt{(\tau/2)^2 + y_0^2}$*. If* M_{rhs} *intersects the circle* $(x-a-\tau)^2+y^2 = r^2$ *on the* RHS *of the vertical axis then it also intersects the line* $x = c$*.*

Proof: If M_{rhs} intersect this circle on the RHS of the vertical, then there is a first intersection (minimum arc length from the fixed point). Call this point p. Since we assume that M_{rhs} has not intersected the line $x = c$, it must be true that $(L_\tau T_a)^{-1}(p)$ must be in the interior of the circle $(x-a-\tau)^2+y^2 = r^2$ and must lie entirely on the RHS of this line. This is because the slope of the unstable manifold at the fixed point is less than $\tau/2y$ by lemma 13 and $\tau/2y$ is the slope of the circle $(x-a-\tau)^2+y^2 = r^2$ at the fixed point, (c, y_0). But $(L_\tau T_a)^{-1}(p) = T^{-1}(p-\tau\mathbf{e}_1)$ which must lie on the circle $(x-a)^2+y^2 = r^2$ and thus is not in the interior of the intersection of the circle $(x-a-\tau)^2+y^2 = r^2$ and the RHS of the vertical axis. ∎

10.3 The Horseshoe Twist and Translate Theorem

As in the twist and flip map, theorem 5 is an easy consequence of two facts about the unstable manifolds of $L_\tau T_a$. The first fact, proposition 2, states that every unstable manifold of $L_\tau T_a$ from a hyperbolic fixed point on the positive vertical axis, meets and crosses the vertical axis at a point other than the fixed point. The second fact is theorem 6, which states that the right hand branch, M_{rhs}, of every homoclinic manifold (i.e. M=S) from a fixed point of $L_\tau T_a$ lying on the positive vertical axis lies in an annulus bounded away from the fixed points of T_a^2. From this we conclude that if M contains a fixed point of T_a it cannot be homoclinic.

Lemma 23 *If* $M = S$ *then* $M_{rhs} = S_{lhs}$.

Proof: Suppose that M=S.

Assume that the right hand branch of the stable and unstable manifolds meet, and thus coincide. By lemma 13 M_{rhs} begins in the interior of $\mathbf{D}_r$. By the same lemma S_{rhs} lies on the RHS of the line $x = c$ and lies outside of $\mathbf{D}_r$. To meet S_{rhs} on the RHS of the vertical, M_{rhs} must cross this circle by continuity. But this cannot happen by lemma 22 unless M_{rhs} has already intersected the line $x = c$.

But if M_{rhs} intersects this line, so does S and at the same point, since By lemma 17, $\mathbf{R}_c$(M)=S. So if S=M, $\mathbf{R}_c(M_{rhs})$ meets S_{lhs} and so must coincide. This contradicts our assumption that $M_{rhs} = S_{rhs}$. ∎

Proposition 2 *Let* (c, y) *be a hyperbolic fixed point of* $L_\tau T_a$ *on the positive line* $x = c$. *Let* $r = \sqrt{(\tau/2)^2 + y^2}$. *Let* M *be the unstable manifold at* (c, y) *and let* S *be the stable manifold at* (c, y). *Then there is a point where* M *meets and crosses the line* $x = c$ *other than the fixed point* (c, y).

Proof: If M=S we are done by lemma 23.

Assume that $S \neq M$ and that there is no point on the line $x = c$ where M meets and crosses other than the fixed point. $\mathbf{M} - \{(c, y)\}$ has two branches (which, by lemma 12 are mapped onto themselves) and since the slope at the fixed point is not vertical (lemma 13), one branch must lie entirely on the RHS of the vertical. Call this branch M_{rhs}.

If p is any point in M_{rhs} the iterates of p by $L_\tau T_a$ define an infinite sequence of points all on the RHS of the line $x = c$. The energy curves of this sequence of iterates must have a limit (lemma 21).

Consequently, the ω-limit set of the iterates consists of a fixed point or a period-two orbit. But the ω-limit set cannot contain a period-two orbit since the unstable manifold of a fixed point cannot terminate at a period-two point.

Thus assume that the ω-limit set is a fixed point other than (c, y).

If the unstable manifold is to reach another hyperbolic fixed point on the positive axis it must first intersect a circle of fixed points of T_a^2 (lemma 20). Let $T_a^2(p) = p$ be a fixed point of T_a^2 which is on the RHS of the line $x = c$. Then $L_\tau T_a(p) = L_\tau(\mathbf{F}_c(p))$ which is on the LHS and hence M must intersect the vertical axis.

The unstable manifold M cannot terminate at the stable manifold of a hyperbolic fixed point on the negative axis without intersecting and crossing the line $x = c$ because of such fixed points have negative eigenvalues.

The only remaining possibility is that the ω-limit set is an elliptic fixed point of $L_\tau T_a$. But this is also impossible. Therefore every unstable manifold of a positive fixed point of $L_\tau T_a$ must meet the line $x = c$. ∎

Lemma 24 *Let (c, y), $y > 0$ be a hyperbolic fixed point for $L_\tau T_a$ and let M_{rhs} be as in proposition 2. Let p be the point on the vertical axis that is in M, and has shortest arc length measured along the unstable manifold to the point (c, y). Let $\mathbf{z}_\alpha$ be $(L_\tau T_a)^{-1}(p)$. Let $r = \sqrt{y^2 + (\tau/2)^2}$ and $r_\alpha = \|\mathbf{z}_\alpha - \mathbf{a}\|$ and define the annulus A_α by the equation*

$$A_\alpha = \{\mathbf{z} | r_\alpha \leq \|\mathbf{z} - \mathbf{a}\| \leq r\}$$

Let $V = (M_{rhs} \cap A_\alpha)$ Then $T_a(V - \{p\}) \subset L_\tau^{-1}(\mathbf{H}_\tau)$.

Proof: We have the following facts $T_a(V) \subset \mathbf{C}_\tau$, $T_a(c, y) = (c', y)$, and $T_a(\mathbf{z}_\alpha) = L_\tau^{-1}(p)$. By the choice of $\mathbf{z}_\alpha$, there can be no other points on the line $x = c'$ other than (c', y) and $L_\tau^{-1}(p)$, so that $T_a(V)$ must lie entirely in $L_\tau^{-1}(\mathbf{H}_\tau)$. ∎

Theorem 6 (Homoclinic Manifold Theorem) *If M is a homoclinic manifold of $L_\tau T_a$, then M_{rhs} lies in an annulus bounded away from the fixed points of T_a^2.*

Proof: Since M is homoclinic we have $M = S = \mathbf{R}_c(M)$. Let M_{rhs} be the branch of M lying in the RHS of the plane. $\mathbf{R}_c(M_{rhs}) = S_{lhs}$ (where "lhs" stands for left hand side) and so

$$\mathbf{R}_c(\mathrm{M_{rhs}} \cup \mathrm{S_{lhs}}) = (\mathrm{M_{rhs}} \cup \mathrm{S_{lhs}})$$

form a connected curve in $\mathbf{R}^2$. We will label this curve N and note that it contains the branch of the unstable manifold M lying in the RHS of the line $x = c$.

Let q be the point other than (c, y) where N meets and crosses the line $x = c$ and let $\mathbf{z}_\alpha = (\mathrm{L}_\tau \mathrm{T}_a)^{-1}(p)$. By lemmas 13 and 18, $\mathbf{R}_c\mathrm{LT(N)} = \mathrm{N}$.

For any annulus A centered at $(a, 0)$, we have $\mathbf{R}_c\mathrm{L}_\tau\mathrm{T}_a(\mathrm{A}) = \mathrm{A}$. So, $\mathbf{R}_c\mathrm{L}_\tau\mathrm{T}_a(\mathrm{A} \cap \mathrm{M}) = \mathrm{A} \cap \mathrm{M}$ and also, $\mathbf{R}_c\mathrm{L}_\tau\mathrm{T}_a(\mathrm{A} \cap \mathrm{N}) = \mathrm{A} \cap \mathrm{N}$. This relation must hold for A_α. By the choice of $\mathbf{z}_\alpha$ we have $\mathrm{T}_a(\mathrm{A}_\alpha \cap \mathrm{M_{rhs}})$ must lie entirely on the RHS of the line $x = c'$ (lemma 24), except for the point q which must lie on this line. For convenience we define $\mathrm{M}_\alpha = (\mathrm{A}_\alpha \cap \mathrm{M_{rhs}})$. Therefore $\mathbf{R}_c\mathrm{L}_\tau\mathrm{T}_a(\mathrm{M}_\alpha) = \mathrm{S_{lhs}}$. Since $\mathrm{S_{rhs}}$ lies entirely in A_α, we conclude that N is contained entirely in $\mathrm{A}_\alpha \cup \mathbf{R}_c(\mathrm{A}_\alpha)$.

Now $r_0 \neq r_\alpha$. If $\mathbf{w}$ is the fixed point of T_a^2 that is in $\mathrm{M_{rhs}}$, then $\mathrm{L}_\tau\mathrm{T}_a(\mathbf{w}) = \mathrm{L}_\tau\mathbf{F}_c(\mathbf{w})$ which is on the LHS of the line $x = c$, i.e. $r_0 < r_\alpha$ hence M_α cannot intersect $\mathbf{C}_{r_0}$ and so N cannot intersect $\mathbf{C}_{r_0} \cup \mathbf{D}_{r_0}$.

We conclude that M is bounded away from the circle of fixed points of T_a^2, $\mathbf{C}_{r_0}$. Thus N lies inside the circle $\mathbf{C}_r$ and outside the circle $\mathbf{C}_s$ where $s > r_0$. ∎

Corollary 7 *Assume the definitions of the above lemma. If* $\mathrm{M_{rhs}}$ *is a homoclinic manifold, it must meet and cross the negative line* $x = c$.

Corollary 8 *Assume the definitions of the above lemma. If* $\mathrm{M_{rhs}}$ *crosses the positive line* $x = c$ *at a point other than the fixed point, then it is a c-manifold.*

Theorem 5 (Horseshoe Twist and Translate Theorem) *Let* (c, y), $y > 0$, *be a hyperbolic fixed point of* $\mathrm{L}_\tau\mathrm{T}_a$ *and let* M *be the unstable manifold of* $\mathrm{L}_\tau\mathrm{T}_a$ *at this hyperbolic fixed point. If* $\mathrm{M_{rhs}}$ *contains a fixed point of* T_a^2, *then* M *is a c-manifold.*

Proof: By theorem 6 if $\mathrm{M_{rhs}}$ contains a fixed point of T_a^2 then it cannot be homoclinic. $\mathrm{M_{rhs}}$ must meet the line $x = c$ by proposition 2, at p and by lemma 17 this is a homoclinic point. If M meets the stable manifold at this point there is a horseshoe by[16]. If M is tangent to the stable manifold at this point then there is still a horseshoe. Thus in any case there exist a horseshoe, and M is a c-manifold. ∎

Corollary 2 *Given a hyperbolic fixed point* (c, y)*, with* $r = \sqrt{y^2 + (\tau/2)^2}$ *and* $r_0 = \omega[r/\omega]$*, a sufficient condition for the unstable manifold* M *of this fixed point to be a c-manifold is that the two circles,*

$$(x-a)^2 + y^2 = r_0^2$$

and

$$(x-a-\tau)^2 + y^2 = r^2$$

intersect.

Proof: Note that by the definitions in section 10.1, the first circle above is denoted $\mathbf{C}_{r_0}$ and the second is the circle is denoted $\mathbf{D}_r$. Assume these two circles intersect. Note that the circle $\mathbf{C}_{r_0}$ is a circle of fixed points of T_a^2.

If $\mathrm{M}_{\mathrm{rhs}}$ crosses the positive line $x = c$ at a point other than the fixed point we are done by corollary 8. Thus assume that $\mathrm{M}_{\mathrm{rhs}}$ crosses the negative line $x = c$. By lemma 24 and the fact that $\mathrm{M}_{\mathrm{rhs}}$ crosses the negative line $x = c$, this crossing must lie below the circle of fixed points of T_a^2. Hence there are points in $\mathrm{M}_{\mathrm{rhs}}$ that lie above the circle $\mathbf{C}_{r_0}$ and below $\mathbf{C}_{r_0}$ and so because $\mathbf{D}_r$ intersects $\mathbf{C}_{r_0}$ and $\mathrm{M}_{\mathrm{rhs}}$ is connected, it must intersect $\mathbf{C}_{r_0}$ and thus contain a fixed point of T_a^2. By theorem 5, M must be a c-manifold. ∎

10.4 Connection of the Twist and Translate to the Map of Parlitz

We note that the map of Parlitz is a dissipative twist centered at $(0, 0)$ with a translate of the form $\mathrm{K}_\tau(\mathbf{z}) = \mathbf{z} + \tau \mathbf{e}_2$ where in the notation of their paper $\tau = a$. If we define the matrix C by the equation

$$\mathrm{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then for any τ,

$$\mathrm{K}_\tau = \mathrm{C}\mathrm{L}_\tau\mathrm{C}$$

and so the map of [Parlitz, et. al.,1991] is $K_\tau T_0 = CL_\tau T_0 C$. This states that the map of Parlitz is topologically conjugate to the map of theorem 5 and the specific conjugacy is given by the matrix C. (Note that $C^2 = I$).

The addition of dissipation to a twist can remove all hyperbolic fixed points and thus eliminate chaos [Brown & Chua, 1991], but this requires a very large damping term. In a subsequent paper we will provide a direct theorem on the dissipative twist and flip or translate map that will avoid the use of structural stability to establish chaos in a dissipative equation. In the present case however, chaos is established in the dissipative twist and translate by applying theorem 5 and then appealing to the structural stability of the twist and translate map.

It is clear that a theorem is needed that places a specific limit on the amount of damping that can be added to any of the maps in this paper before an existing horseshoe is removed. This would provide the first of two theorems needed to make chaos a design tool in the development of dynamical systems. The second theorem needed is one describing when an existing horseshoe in a dissipative twist and flip or translate map creates a strange attractor.

We have two conjectures along these lines.

Conjecture 2 *If a horseshoe exist for a non-dissipative twist and flip or translate map, then it will continue to exist in the associated dissapitive twist and flip or translate map until enough dissipition is added to remove the associated hyperbolic fixed point.*

Conjecture 3 *If a horseshoe exist in a dynamical system defined by a dissapitive twist and flip or translate map due to a hyperbolic fixed point of minimum energy, then a strange attractor may be created from the associated hyperbolic period-two point, if it exist, by the addition of sufficient damping to remove the hyperbolic fixed point.*

This last conjecture is illustrated in section 12 using the map of Parlitz.

The above considerations suggest the following additional conjecture:

Conjecture 4 *Let*

$$\begin{aligned} \dot{x} &= F(x, y, g(t)) \\ \dot{y} &= G(x, y, g(t)) \end{aligned}$$

where $g(t)$ is periodic, F, G are nonlinear functions of x, y, and the above equations admit a unique solution for each set of initial conditions in the plane. Then, at any periodic point of the Poincaré map of the above equations, the Poincaré map can be C^1-approximated on a compact set which contains the periodic point by maps of the form $\mathrm{T}_1\mathrm{T}_2$ where T_2 is a finite composition of maps at least one of which is a twist map, and T_1 is some finite combination of twist maps, flip maps, translate maps, or diffeomorphism defined by some linear second order ODE.

A sort of converse to the above conjecture is the following lemma which is true in n-dimensions but which we state in two dimensions for simplicity:

Lemma 25 *Let* C *be a two dimensional constant, real matrix and let the following linear matrix ODE be given:*

$$\dot{\mathbf{x}} = \mathrm{C}\mathbf{x}$$

Then for any time $t = \tau$, the solution of this equation defines a factor of a Poincaré map for a square-wave forced, two dimensional, non-linear ODE. The non-linear ODE may be chosen so that the Poincaré map is of the form LTLT, *where* T *is a simple twist.*

Proof: We construct the required equation explicitly:

$$\begin{aligned} \dot{\mathbf{y}} &= 0.5(1 + \mathrm{sg}(\omega t))\mathrm{F}(\mathbf{y}) + \\ &\quad 0.5(1 - \mathrm{sg}(\omega t))\exp(\mathrm{C}\tau)\mathrm{F}(\exp(-\mathrm{C}\tau)\mathbf{y}) \end{aligned}$$

where $\mathbf{y}$ is a two dimensional vector, and F is a two dimensional vector valued function of a two dimensional vector, and F is such that this defines an ODE having a unique solution for each vector initial condition. T is the diffeomorphism defined by the solution of

$$\dot{\mathbf{y}} = \mathrm{F}(\mathbf{y})$$

evaluated at the time $t = \pi/\omega$, and so the Poincaré map of the square-wave equation above is given by LTLT where

$$\mathrm{L} = \exp(\mathrm{C}\pi/\omega)$$

∎

In the case where C is the flip map, and $F(x, \dot{x}) = a - x^3$ The square-wave forced ODE is given by $\ddot{x} + x^3 = a\,\mathrm{sg}(\omega t)$.

The same procedure shows how to construct square-wave forced ODEs having Poincaré maps of the form contained in conjecture 4. We may use a variation of this procedure to construct an ODE having a twist and translate as a Poincaré map. This equation is:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathrm{sg}_1(\omega t) \begin{pmatrix} y \\ a - x^3 \end{pmatrix} + \mathrm{sg}_2(\omega t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $\mathrm{sg}_1(u) = 0.5(1 + \mathrm{sg}(u))$ and $\mathrm{sg}_2(u) = 0.5(1 - \mathrm{sg}(u))$.

11 Modeling and Simulation Using the Non-Dissipative Twist and Flip Maps

The work of Parlitz [Parlitz, et. al, 1991] demonstrates the value of maps as a tool for modeling and simulation of important dynamical systems. We would like to provide support to their position by way of some examples.

The first fact which has been graphically revealed by a study of the twist and flip (twist and translate may also be used) map is that a non-dissipative twist and flip dynamical system is made up of elliptic and hyperbolic periodic points. Around these periodic points are elliptic and hyperbolic regions. Some of these hyperbolic regions are chaotic and some are not. Within the chaotic regions can be found islands of elliptic regions. Of course, all of this is known to be true for Hamiltonian systems as was mentioned in [Brown & Chua, 1991], but these facts may be easily studied and analyzed very quickly by the use of the twist and flip map.

For example, if one were to try to produce the portrait of the elliptic and the hyperbolic regions shown in the following figures by conventional numerical integration, it would take approximately 100 times longer than using the twist and flip map. Moreover, it is possible to build dedicated hardware made of off-the-shelf electronic components (e.g., DSP IC chips) which implements the twist and flip map in real-time. Further, what can now be revealed by the twist and flip map is the remarkable complexity of these islands in terms of size and number. This fact has implications for the study of dynamical systems in general by experts in signal processing, encryption, control theory and the life sciences: The shape of the unstable manifolds and their relation to the elliptic regions may provide a *classification* of dynamical systems since it is likely to be unique for each Poincaré map.[3]

The complexity of the unstable manifold and the elliptic regions that appear with it can be seen in the examples in Figs. 1, 2, 3, and 4. Figs. 1, 2, and 3 are from class 1 above; Fig. 4 is from class 3. We include detailed data on these figures that will permit their reproduction. The first three figures from class 1 differ only in the location of their periodic or fixed points and the rotation function f. In Fig. 1, the unstable manifold is shown in green; in Fig. 2 there are two, one is shown in red, the other in pink; in Fig. 3 it is shown in orange. In Fig. 4 it is shown in pink. The other colors define elliptic regions.[4]

The unstable manifolds in these figures are typically produced by iterating 5000 points from a small line segment of length *eps*. The elliptic regions are obtained by iterating a single point or several points 1000 times. The number of points iterated is denoted below as M. As mentioned, for an unstable manifold M=5000. To produce the various elliptic regions, M ranges from 1 to 7. When producing an unstable manifold, the length of the line segment ranges from 0.02 to 0.0002. When the parameter *eps* is used in the generation of elliptic points, it denotes the maximum spacing between the x−ordinate of each initial point used.

To describe each figure we must only specify the class of twist from among the five classes described above from which each twist comes, specify the rotation function used within that class, and then specify the following six parameters: amplitude, a, frequency, ω, the initial conditions of the fixed or periodic points, x_0, y_0, the slope of the unstable manifold, and the number of iterations, N, of FT. The generic code used in Figs. 1- 3 is reproduced below. It is adapted to each figure by changing the rotation function.

FOR i=1 to M+1

$$x = x_0 + (-1 + (2(i-1)/\mathrm{M}))eps$$

[3]This only provides a classification and not a unique signature for a dynamical system since the Poincaré map is not unique to a dynamical system.

[4]The method of forming new twist maps by variation of the rotation function was suggested by Morris Hirsch.

$$y = y_0 + slope(x - x_0)$$

FOR j=1 to N

$$r = \sqrt{(x-a)^2 + y^2}$$
$$r = f(r)$$
$$u = (x-a)\cos(r) - y\sin(r) + a$$
$$v = y\cos(r) + (x-a)\sin(r)$$
$$x = -u$$
$$y = -v$$

PSET (x, y)
NEXT j
NEXT i

The code used for computing the elliptic regions is the same except that the parameters are different.

We have collected this data in the following four subsections.

11.1 Data for Reproducing Fig. 1

Fig. 1 is produced by a class 1 equation.

Rotation Function: $f(r) = 10r/((1 + r\log(r))$

Unstable Manifold Data (Fixed Point):

$a = 0.2$	$\omega = \pi$	$x_0 = 0$	$y_0 = -0.21808$	N=16	$slope = .59$	M=5000

$eps = .0001$

The parameters for computing the elliptic regions are as follows:

Elliptic Region 1 data:

$a = 0.2$	$\omega = \pi$	$x_0 = -0.6$	$y_0 = 1.1$	N=1000	$slope = 1.1$	M=7

$eps = 1.0$

Elliptic Region 2 data:

$a = 0.2$	$\omega = \pi$	$x_0 = -0.1$	$y_0 = -2$	N=1000	$slope = 1.1$	M=7

$eps = 0.6$

Elliptic Region 3 data:

$a = 0.2$	$\omega = \pi$	$x_0 = -0.1$	$y_0 = -.93$	N=1000	$slope = 4$	M=4

$eps = 0.04$

11.2 Data for Reproducing Fig. 2

Fig. 2 is a class 1 equation.

Rotation Function: $f(r) = -\sin(4\log(r))$

Unstable Manifold 1 Data (Fixed Point):

$a = 1.0$	$\omega = \pi$	$x_0 = .950$	$y_0 = -1.002$	N=10	$slope = 1.24$	M=5000

Unstable Manifold 2 Data (Period 5 Point):

$a = 1.0$	$\omega = \pi$	$x_0 = 0.0$	$y_0 = -6.17665$	N=35	$slope = 0.09$	M=1000

$eps = .01$

The parameters for computing the elliptic regions are as follows:

Elliptic Region 1 Data:

$a = 1.0$	$\omega = \pi$	$x_0 = 1.0$	$y_0 = 6$	N=1000	$slope = -1.0$	M=4

$eps = 0.5$

Elliptic Region 2 Data:

$a = 1.0$	$\omega = \pi$	$x_0 = 5.0$	$y_0 = 3.3$	N=1000	$slope = 1.0$	M=2

$eps = 0.5$

Elliptic Region 3 Data:

$a = 1.0$	$\omega = \pi$	$x_0 = 0.0$	$y_0 = -4.5$	N=1000	$slope = 1.0$	M=2

$eps = 0.3$

Elliptic Region 4 Data:

$a = 1.0$	$\omega = \pi$	$x_0 = 5.6$	$y_0 = -0.0$	N=1000	$slope = 0.0$	M=3

$eps = 0.15$

11.3 Data for Reproducing Fig. 3

Fig. 3 is a class 1.

Rotation Function: $f(r) = (1 + \log(r))/r$

Unstable Manifold Data (Period 4 Point):

$a = 0.2$	$\omega = \pi$	$x_0 = 0.0$	$y_0 = -0.21065$	N=12	$slope = 0.0$	M=5000

$eps = .0001$

The parameters for computing the elliptic regions are as follows:

Elliptic Region 1 Data:

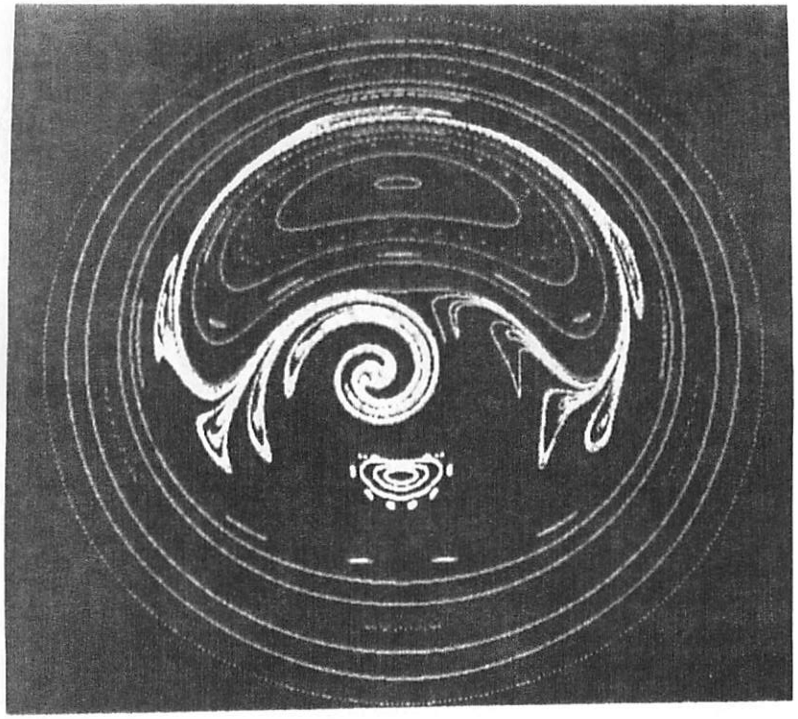


Figure 1

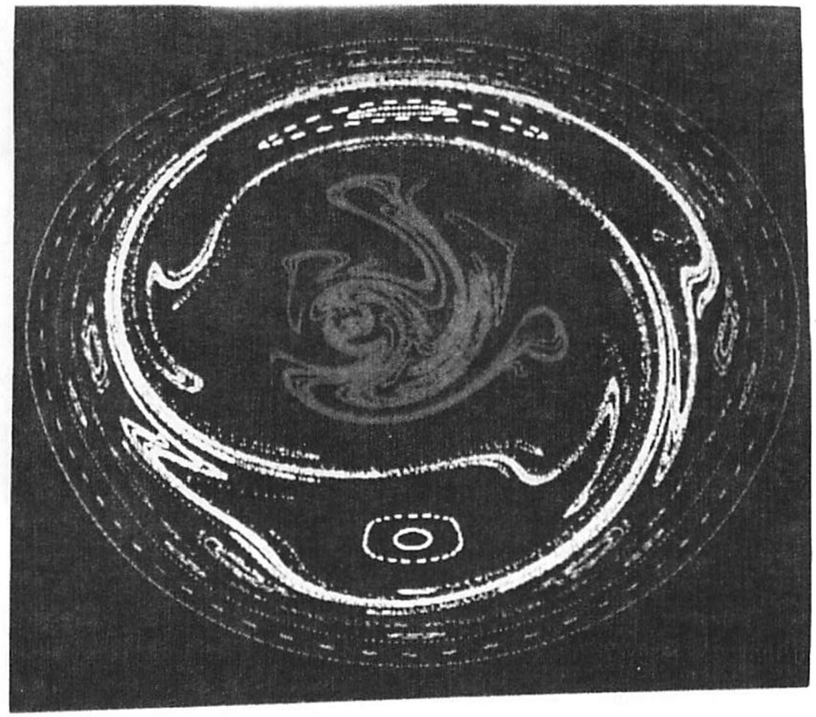


Figure 2

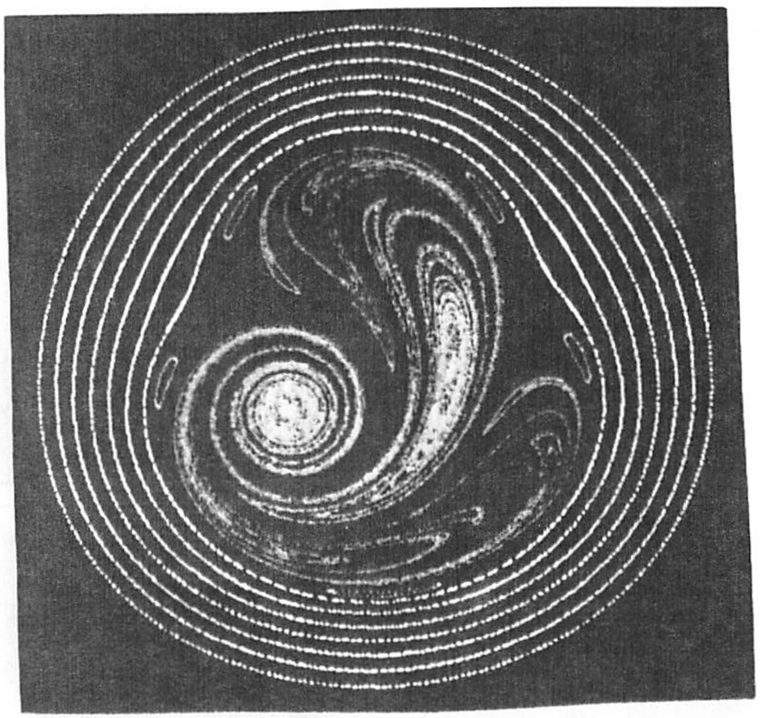


Figure 3

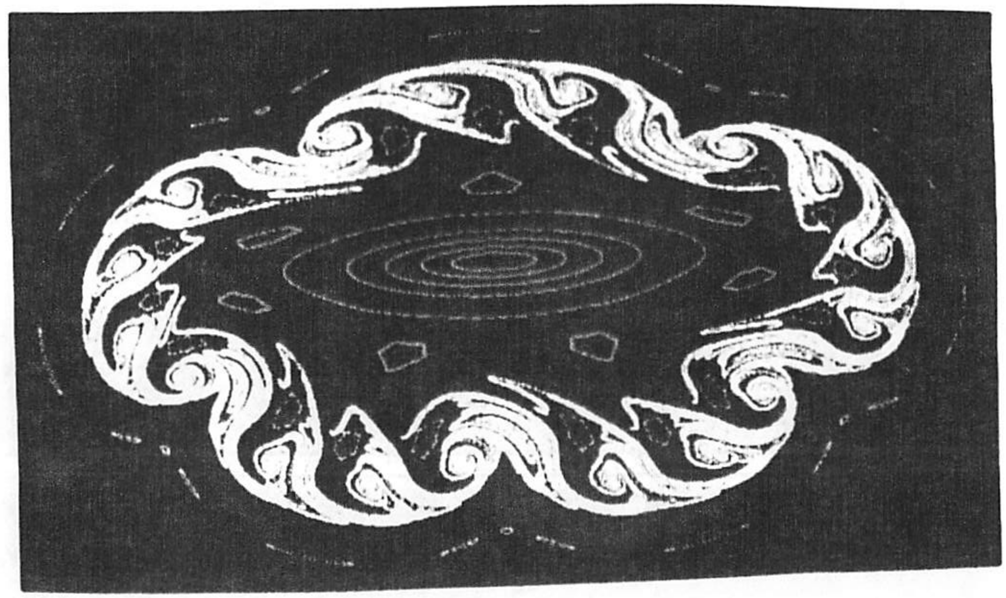


Figure 4

$a = 0.2$	$\omega = \pi$	$x_0 = -0.8$	$y_0 = -0.0$	N=1000	$slope = 0.1$	M=7

$eps = .3$

Elliptic Region 2 Data:

$a = 0.2$	$\omega = \pi$	$x_0 = 0.5$	$y_0 = 0.1$	N=1000	$slope = 1.0$	M=1

$eps = .1$

11.4 Data for Reproducing Fig. 4

We note that Fig 4 is produced by use of the Jacobi elliptic functions in place of the sine and cosine.

Fig. 4 is a class 3, period 22 point.

Rotation Function: $f(r) = 1/r; k^2 = 1/(r^2 + 1)$

Unstable Manifold Data (Period 22 Point):

$a = 1.0$	$\omega = \pi$	$x_0 = 0.0$	$y_0 = -0.36$	N=88	$slope = -1.2$	M=5000

$eps = 0.02$

The parameters for computing the elliptic regions are as follows:

Elliptic Region 1 Data:

$a = 1.0$	$\omega = \pi$	$x_0 = 0.0$	$y_0 = 0.8$	N=1000	$slope = 1.0$	M=1

$eps = 0.5$

Elliptic Region 2 Data:

$a = 1.0$	$\omega = \pi$	$x_0 = 0.0$	$y_0 = 0.8$	N=1000	$slope = 1.0$	M=1

$eps = 0.2$

Elliptic Region 3 Data:

$a = 1.0$	$\omega = \pi$	$x_0 = 0.0$	$y_0 = 0.7$	N=1000	$slope = -2.0$	M=5

$eps = 0.2$

Elliptic Region 4 Data:

$a = 1.0$	$\omega = \pi$	$x_0 = 0.0$	$y_0 = 1.2$	N=1000	$slope = 100$	M=1

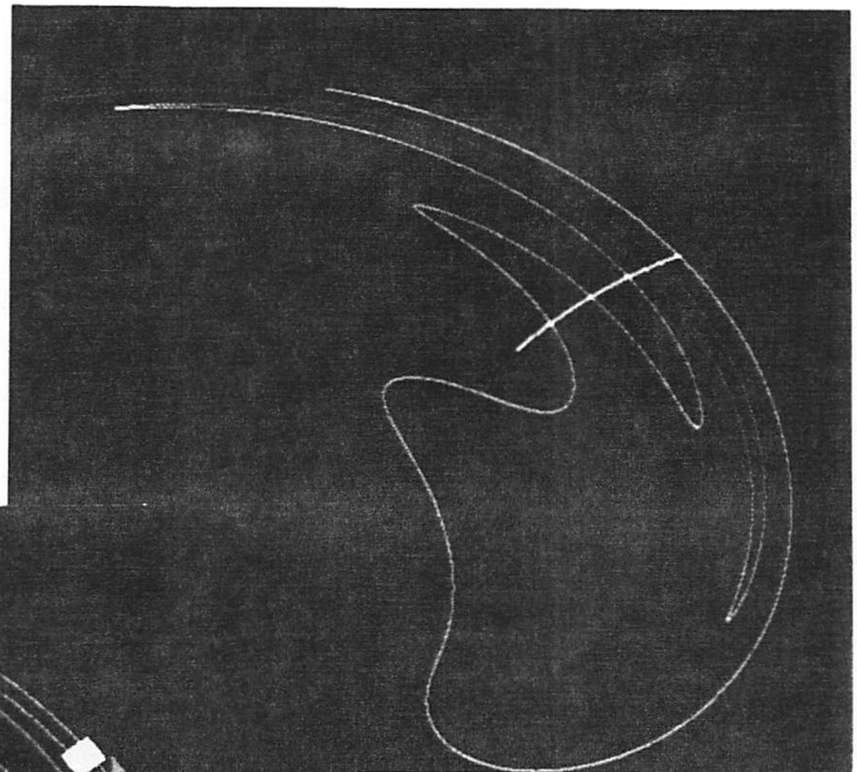
$eps = 0.2$

11.5 Illustrating the Smale Horseshoe Using the Twist and Flip Map

There are numerous accounts of the Smale horseshoe paradigm, [Guckenheimer & Holmes, 1983] being a very good one. However, all of these explanations have one thing in common: none of them utilizes a concrete example as an illustration, but instead all depend on defining the horseshoe map by a description rather than a function. This abstract approach can be explained by the general absence of simple, intuitive examples on which to base an illustration. In [Brown & Chua, 1991], we provided a simple, concrete illustration of homoclinic tangles that can easily be reproduced and studied. In this section we provide a similarly simple, concrete illustration of the horseshoe. In particular, the horseshoe is produced by using a map of the form FT_2T_1 where the T_i are each a simple twist map (i.e., rotation function $f(r) = r$). The only difference in these two twists is the center of the twist. In particular, for T_1 the center of the twist is $(0.3, 0.0)$ and for T_2 the center of the twist is $(0.5, 0.0)$.

In Fig. 5 we show the unstable manifold in green and a small arc of the stable manifold in pink as a line of reference. The fixed point is approximately $(0.1887, 1.12)$ and is seen in the figure as the highest point of intersection of the two manifolds. In Fig. 6 we have placed a white square whose far left corner is at $(0.139, 0.935)$. As seen there the square is placed in just such a way as to be intersected by the unstable manifold in green. This is essential if a horseshoe is to be produced. In this figure we show the successive iterates of the square by FT_2T_1 in different colors. For example the first iterate is in light blue and is to the right and partly below the white square. Note that is has been pulled toward the unstable manifold, and conforms to the shape of the part of the unstable manifold of Fig. 5 that it is located near and that it has been stretched or "sheared". The second iterate is in red and is closer still to the unstable manifold as can be judged by referring back to Fig. 5 again. The third iterate of the square is in pink, the fourth is in orange, the fifth is in grey, the sixth is in blue. The seventh iterate is in green and seems to duplicate an arc of the unstable manifold from Fig. 5. This iterate cuts through the original white square in two places, producing the familiar image of a horseshoe found in texts on nonlinear dynamics. In Fig. 7 we have enlarged the area around the square and have suppressed the interior points, leaving only the outline of the square. We have included the seventh forward iterate of the square in green and the seventh backward iterate in pink. As can be seen in the figure, both of these iterates intersect the square and each other forming the first set of intersections that will produce the Cantor set on which the horseshoe map exists. If now we were to form the seven forward and backward iterates of the green and pink intersections of

Figure 5



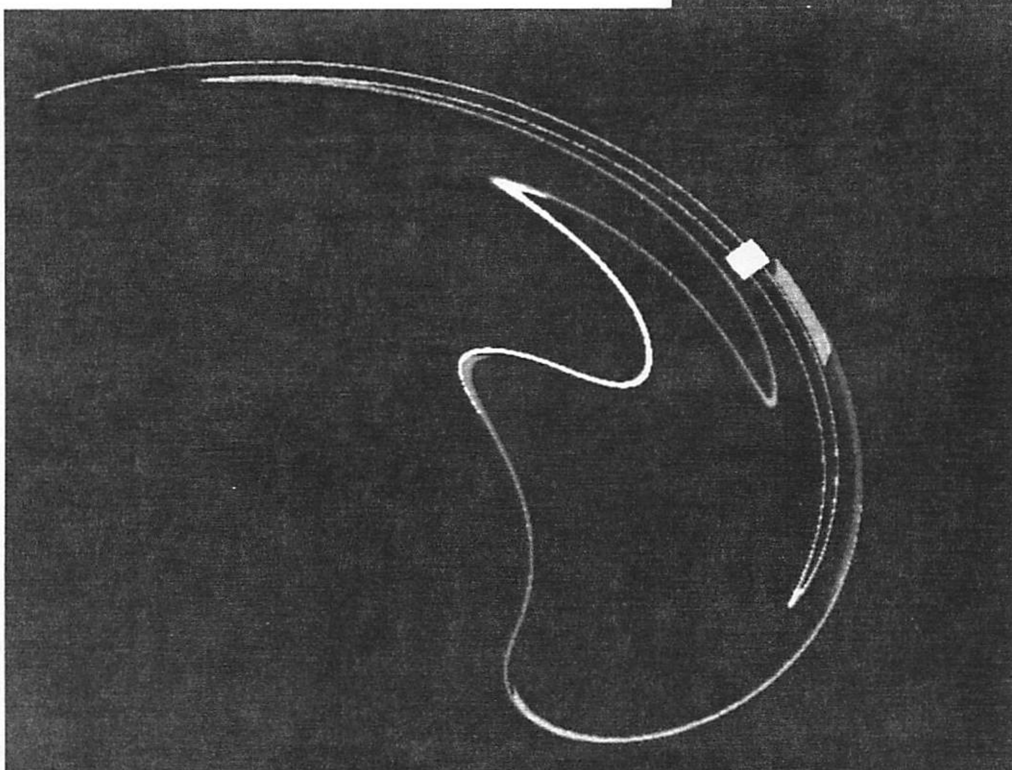


Figure 6

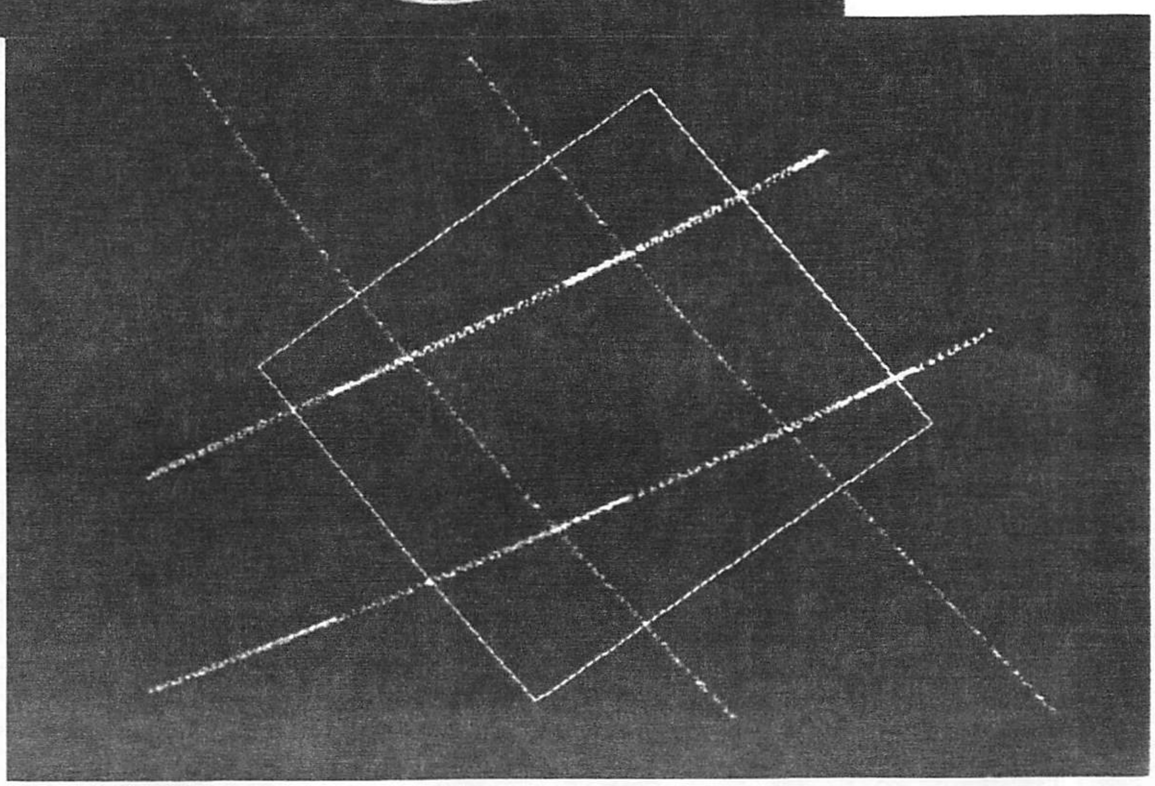


Figure 7

the square, more components of the Cantor set would be created.

The essential data for reproducing these three figures is given in the following tables:

Data for The Unstable Manifold Computation:

$a_1 = 0.3$	$a_2 = 0.5$	$\omega = \pi$	$x_0 = 0.1887$	$y_0 = 1.12$	$slope = -1.2$	N=11

Length of line segment is $eps = .002$ and the number of points in the line segment is M=2000.

The QuickBASIC code used for these computations is as follows:

FOR i=1 to M

$$x = x_0 + ((i+1)/M)eps$$

$$y = y_0 + slope(x - x_0)$$

FOR j=1 to N

$$r = \sqrt{(x-a_1)^2 + y^2}$$

$$u = (x-a_1)\cos(r) - y\sin(r) + a_1$$

$$v = y\cos(r) + (x-a_1)\sin(r)$$

$$r = \sqrt{(u-a_2)^2 + v^2}$$

$$u1 = (u-a_2)\cos(r) - v\sin(r) + a_2$$

$$v1 = v\cos(r) + (u-a_2)\sin(r)$$

$$x = -u1$$

$$y = -v1$$

PSET (x, y)

NEXT j

NEXT i

12 Illustration of Conjectures 2 and 3 Using the Map of Parlitz

In this section we illustrate conjectures 2 and 3 using the map of [Parlitz et. al.,1991]. The illustration consists of six figures described in the following four subsections. The map we use is given by

$$\mathrm{L}_\tau \mathrm{T}_a(\mathbf{z}) \quad = \quad \mathbf{A}(\pi r/\omega)(\mathbf{z} - \mathbf{a}) \quad + \quad \mathbf{a} + \tau \mathbf{e}_1$$

where

$$\mathbf{A}(\pi r/\omega) = \exp(-d\pi/2\omega) \begin{bmatrix} \cos(\pi r/\omega) & -\sin(\pi r/\omega) \\ \sin(\pi r/\omega) & \cos(\pi r/\omega) \end{bmatrix}$$

We now choose $a = 0$ so as to obtain a map that is C^∞ conjugate to the map of Parlitz by a simple conjugacy C, where

$$\mathrm{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For simplicity of illustration we take $\omega = \pi$ and relabel τ as a since we have already set the original a in our definition above equal to 0. The resulting map we will use for illustration is given by

$$\mathrm{L}_a \mathrm{T}(\mathbf{z}) \quad = \quad \mathbf{A}(r)(\mathbf{z}) \quad + \quad a\mathbf{e}_1$$

where **A** is as given above. For convenience we define

$$\alpha = \exp(-d/2)$$

Recall that the map of Parlitz is given by $\mathrm{K}_a\mathrm{T}$ where $\mathrm{K}_a(\mathbf{z}) = \mathbf{z} + a\mathbf{e}_2$ and that $\mathrm{K}_a\mathrm{T} = \mathrm{C}\mathrm{L}_a\mathrm{T}\mathrm{C}$. Thus there is no loss of generality in using L_a in place of K_a. For convenience, since a is fixed, we will drop the subscript of a in the following discussion and refer to our map from now on as LT.

We are going to illustrate what happens to the unstable manifold of a fixed point of LT and a period-two point of LT as we increase $d = -2\log(\alpha)$ the damping factor from 0.0 to approximately 0.657. This is the same as decreasing α from 1 to 0.72.

What we are going to see first is a portrait of the two unstable manifolds as they appear in the non-dissipative system. The fixed-point manifold is in blue, the period-two point manifold is in green. Then we are going to illustrate what happens to these two manifolds as α takes on the values of 0.9, 0.8, and 0.72. This will correspond to damping factors of 0.21, 0.45, and 0.657, respectively.

We will see that as long as the fixed-point manifold exists, the attractor of LT is a fixed point sink, and therefore only transient chaos is possible except for initial conditions that start on a horseshoe. Once the fixed-point manifold is destroyed by the addition of enough damping, which we may prove is always possible to do, the period-two manifold becomes a strange attractor and remains so until enough damping is added to destroy it also. It will be seen that the period-two manifold will still exist when $\alpha = 0.72$ or $d = 0.65$, a very large damping factor.

These figures illustrate that some "small" damping of a map having a horseshoe can produce transient chaos and that true chaos has a zero probability of being found when the damping is small enough, although the horseshoe does exist and so chaos is theoretically possible. What is interesting is that sufficient damping must be added to destroy the fixed-point manifold before a strange attractor can appear associated with the period-two unstable manifold and thus produce chaos. These facts provide the illusion that damping is the source of chaos. But damping does not produce horseshoes, it can only destroy them and so this is only an illusion. The true chaos can only arise in the presence of horseshoes or perhaps homoclinic tangencies.

In the following subsections the data for reproduction of the figures is organized parallel to the previous figures.

12.1 Unstable Manifold for Fixed and Period-Two Points for $\alpha = 1.0,\ d = 0.0$

Fig. 8 is the unstable manifold for a fixed point in blue, and the unstable manifold for a period-two point is green.

Fixed-Point Unstable Manifold Data (Blue):

$a = -3.0$	$d = 0$	$x_0 = -1.5$	$y_0 = 5.556$	N=8	$slope = -0.43$	M=1000

$eps = .01$

Period-Two Unstable Manifold (Green):

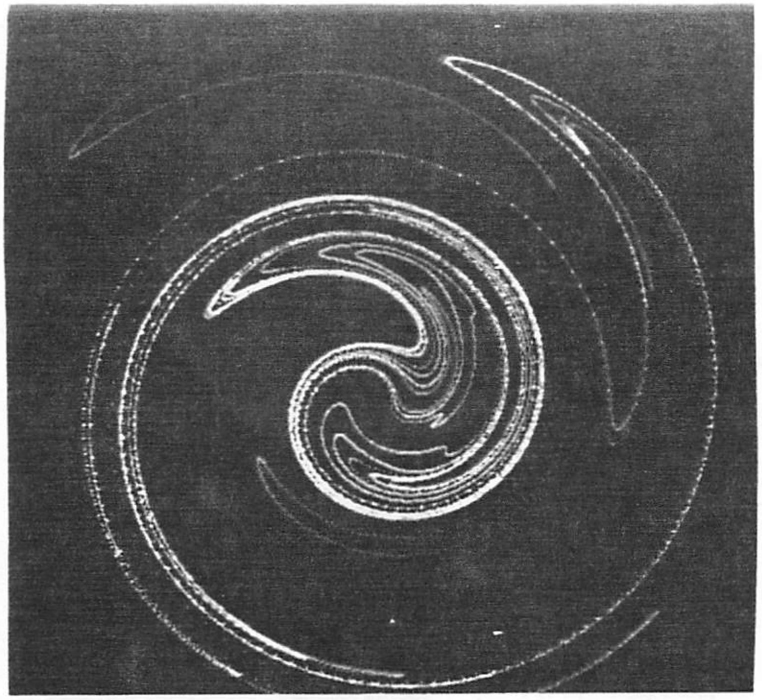


Figure 8

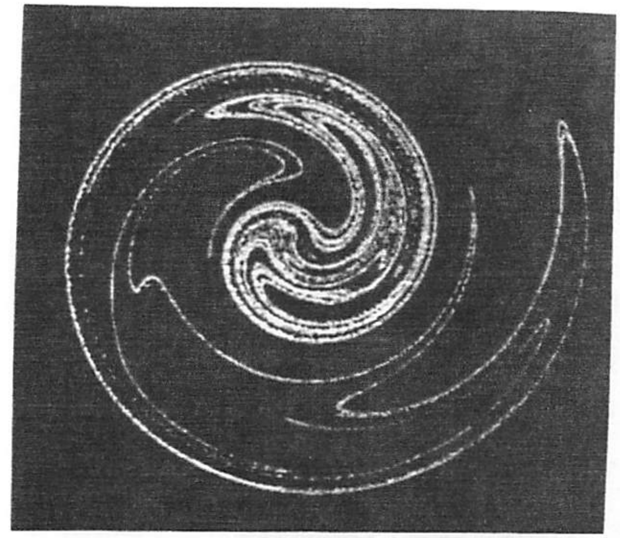


Figure 9

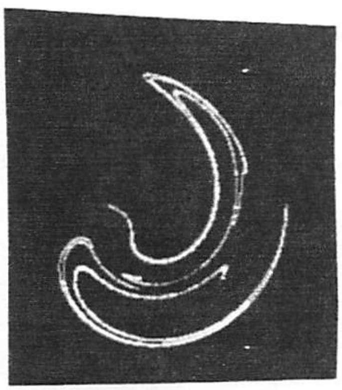


Figure 10

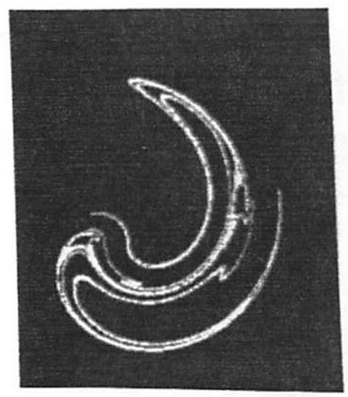


Figure 11

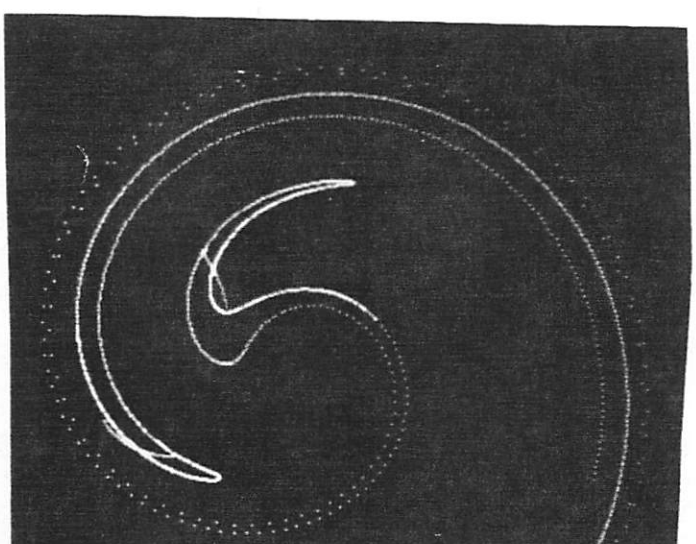


Figure 12

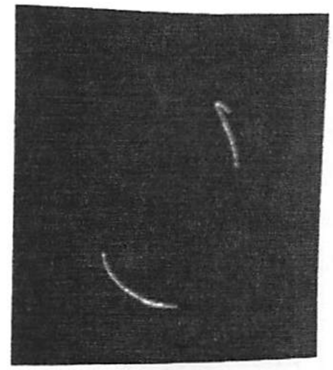


Figure 13

$a = -3.0$	$d = 0.0$	$x_0 = -1.5$	$y_0 = 2.76$	N=12	$slope = -1.46$	M=4000

$eps = .01$

12.2 Figure 8 with the Addition of Damping Factor = 0.21

Fig. 9 is the unstable manifold for a fixed point in blue, and the unstable manifold for a period-two point is green. The data is as follows:

Fixed-Point Unstable Manifold Data (Blue):

$a = -3.0$	$d = 0.21$	$x_0 = -2.535$	$y_0 = 5.14$	N=8	$slope = -0.43$	M=1000

$eps = .1$

Period-Two Unstable Manifold (Green):

$a = -3.0$	$d = 0.21$	$x_0 = -1.465$	$y_0 = 2.25$	N=16	$slope = -1.485$	M=2000

$eps = .05$

The attractor in this case is approximately the point $(-2.94, -6.069)$.

12.3 Figure 8 with the Addition of Damping Factor = 0.45

In Fig. 10 the unstable manifold for the fixed point is gone thus this figure is the unstable manifold for a period-two point in green. The data for this figure is as follows:

Period-Two Unstable Manifold (Green):

$a = -3.0$	$d = 0.45$	$x_0 = -1.46$	$y_0 = 2.285$	N=16	$slope = -1.66$	M=2000

$eps = .03$

Fig. 11 is the strange attractor for LT when $d = 0.45$. The data used in generating this figure is as follows:

Strange Attractor (Red):

$a = -3.0$	$d = 0.45$	$x_0 = -2.535$	$y_0 = 5.14$	N=5000	$slope = -1.66$	M=1

12.4 Figure 8 with the Addition of Damping Factor = 0.656

Fig. 12 shows both the unstable manifold (green) for the period-two point of LT when $d = 0.656$ and the stable manifold (violet) for this period-two point.

The data used in generating this figure is as follows:

Period-Two Point Unstable Manifold Data (Green):

$a = -3.0$	$d = 0.656$	$x_0 = -1.4625$	$y_0 = 2.022$	N=16	$slope = -2.0$	M=1000

$eps = .005$

Period-Two Stable Manifold Data (Violet):

$a = -3.0$	$d = 0.656$	$x_0 = -1.4625$	$y_0 = 2.022$	N=8	$slope = 2.0$	M=1000

$eps = .005$

The transverse crossing of the stable and unstable manifolds for LTLT assures us of a horseshoe and thus chaos still persists at this point. We see in the next figure, the strange attractor is clearly in two parts.

Fig. 13 is the strange attractor for LT when $d = 0.656$. The data used in generating this figure is as follows:

Strange Attractor (Red):

$a = -3.0$	$d = 0.655$	$x_0 = -1.4625$	$y_0 = 2.022$	N=5000	$slope = -2.0$	M=1

Fig. 13 suggests that it may be possible, through the careful addition of damping, to produce homoclinic tangencies in a strange attractor.

12.5 Summary of Section 12

We have seen by the Figures that the fixed-point unstable manifold *encases* the period-two unstable manifold with both manifolds having horseshoes. With damping added, the fixed-point unstable manifold also has between its folds areas of the basin of attraction of a fixed-point sink. Prior to the addition of damping, this sink was an elliptic fixed point of FT. The fact that the fixed-point unstable manifold prior to the addition of damping encircled the elliptic region around the elliptic fixed point is significant. This meant that upon the addition of damping, initial conditions near the fixed-point unstable manifold could wind up in the basin of attraction of the sink that developed from this elliptic fixed point. In some sense, the outer unstable manifold prevented chaos from appearing even though it was always possible for some choices of initial conditions. Once enough damping was added to strip away the fixed-point unstable manifold the period-two unstable manifold became the attractor. It is likely that this happened because the period-two fixed points are on a lower energy curve of the undamped twist than the elliptic fixed point.

We conjecture that this genesis is common to second order square-wave forced nonlinear dynamical systems of the Duffing class described in the following section.

13 Concluding Remarks

Crucial to the advancement of any area of scientific research is the presence of simple examples on which hypothesis may be tested quickly and efficiently. In this paper we have shown how to construct countless such examples of simple transformations that produce chaos. These simple examples differ from the few predecessors such as the Henon map and the gingerbread map [Devaney, 1984] which existed prior to the twist and flip, in that they are Poincaré maps. We consider this "road map" for construction of examples to be one of our main contributions to the study of chaos.

However, the analysis of the van der Pol and cavitation oscillator equations reveal something more of the genesis of chaos: There appear to be two principal mechanisms for generating a twisting action in autonomous, nonlinear, second order differential equations that can lead to the formation of chaotic solutions of the equation when a periodic forced is added. We shall refer to these two classes as the class of Duffing and the class of van der Pol/Rayleigh.

Class of Duffing This class is defined by an equation of the form

$$\ddot{x} + \mathrm{V}(x) = 0$$

and

$$\ddot{x} + \alpha\dot{x} + \mathrm{V}(x) = 0$$

where V is a nonlinear function of x, and the first integral of the undamped equation defines a one parameter system of closed curves in the phase plane.

Such a system produces a twist or a damped twist and by adding a periodically forcing term to the equation, we may produce a twist or a damped twist combined with a flip or some other factor in the Poincaré map.

Class of van der Pol/ Rayleigh This class is defined in two steps: First we consider an equation of the form

$$\ddot{x} + \mathrm{W}(\dot{x}) = 0$$

where W is a nonlinear function of $\dot{x}$. The second step in the definition of this class is to add a simple linear function of x. We require that the final equation, which in general looks like

$$\ddot{x} + \mathrm{W}(\dot{x}) + ax = 0$$

have a periodic solution. This equation will have a shear action arising from the term $\mathrm{W}(\dot{x})$. The requirement that the solutions be periodic with the addition of the linear potential term changes this shear into a twist. The addition of a periodic forcing term produces a twist factor in the Poincaré map.

These two classes are very different physically and the twist factors in their Poincaré maps are also different. We note that in the van der Pol/Rayleigh class the presence of a damping term is essential to producing the chaos, which is not the case for the Duffing class.

Clearly we may form hybrid equations by combining these two classes of equations[5]

We conjecture that a horseshoe twist theorems is true for the Duffing class equations when the periodic forcing term is a square wave if we replace the circles that play an important role in the horseshoe twist theorem by the closed curves defined by the first integral. A special case of this conjecture was presented in [Brown & Chua, 1991].

As for the van der Pol/Rayleigh class, we believe a variation of the horseshoe twist theorem is true for square wave forces, but we do not conjecture the form of the variation at this time.

The results of this paper suggest numerous questions, but of considerable interest among all questions that we could ask are two sets.

The first set is concerned with the ergodic properties of the twist and flip, or more generally, the ergodic properties of Poincaré maps that are of

[5]Ueda's form of the van der Pol equation is an example, [Moon, 1987].

the form $\Phi \circ T$, where Φ is not a twist. Among the questions of interest to engineers and scientist are:

(1) Given the existence of a horseshoe for such a Poincaré map, what are its asymptotic properties when restricted to the closure of the unstable manifold, $\bar{M}$, which produced the horseshoe?

(2) What are the ergodic factors of such Poincaré maps on $\bar{M}$.

(3) Compute the topological entropy for these maps on $\bar{M}$.

(4) Classify these maps on $\bar{M}$ in the categories of weak mixing, strong mixing, Kolmogorov, or Bernoulli.

(5) Since it appears that $\bar{M}$ could be unbounded, extend the ergodic concepts to include this case.

It should be clear that, in addition to questions of ergodic theory we have posed, we may also ask many questions about the statistical properties of the time series generated by Poincaré maps of the form FT or ΦT.

The second set of questions pertain to the harmonic analysis of the chaotic solutions of the classes of ODEs presented in this paper. In particular, for square-wave forcing we can determine the form of the most extreme case of a chaotic solution of these classes of ODEs by the following considerations.

By the Smale-Birkhoff theorem, some power of the Poincaré map is topologically conjugate to a bilateral shift on n-symbols on some Cantor set. Therefore, there exist initial conditions in this Cantor set such that the successive iterates of this power of the Poincaré map defines a pseudo-random sequence of points in the plane. Since the forcing is a square wave, the behavior of the solution between successive points is determined by an autonomous equation whose solutions are periodic functions or spirals. Hence between two points of the Poincaré map is an arc of such a curve. In the case of the simple twist this is the arc of a circle. In the case of the undamped Duffing equation it is the arc of a curve defined by the elliptic functions. In any case, the solution of the ODE between two successive iterates of the power of the Poincaré map determined by the Smale-Birkhoff theorem is a finite number of arcs of closed curves or spirals determined by an autonomous ODE.

Given this description of the generic chaotic solution of a square-wave forced nonlinear ODE from the classes we have presented we may now construct the simplest example of such a curve and pose some questions about its harmonic analysis.

Starting with the one sided shift defined by the equation

$$x_{n+1} = 2x_n \bmod 1$$

on the interval [0,1] we choose an initial condition which generates a dense subset of [0,1][6]. This sequence is the analogue of the successive iterates of the Poincaré map. Next we consider this sequence x_n to define the values of a function, f, by the rule $f(n) = x_n$. We now extend the definition of this function, $f(t)$, $t \geq 0$, to all positive real numbers by requiring that between two points the function is linear and that the function is continuous for all $t \geq 0$. Essentially we are using straight lines in place of the arcs of the closed curves that appeared in the solution of the ODEs.

If we were to graph such a function we would start by plotting the points (n, x_n), or equivalently, $(n, f(n))$ in the plane and then we would connect two successive points by a straight line. This process produces a bounded continuous function. We now ask the following questions about this function:

(1) Does such a function, $f(t)$, have a Fourier transform?

(2) What is the power spectral density of f as a function of the initial condition of the iteration sequence and is the power spectral density continuous?

(3) What questions of spectral synthesis can be answered for such a function?

(4) What does wavelet theory tell us about such a function?

Our ability to answer these questions about the function f will determine to a great extent our ability to deal with chaos through the classical methods of harmonic analysis.

The twist and flip map has suggested numerous directions of research into the mechanism of chaos. We now suggest that all of the points we have been considering so far can be formulated in terms of Lie Groups, and that the twist and flip paradigm can be extended to this setting.

From what has been presented we conclude that the twist and flip or translate map is fundamental in producing PBS (Poincaré-Birkhoff-Smale) chaos in second order nonlinear periodically forced ODEs and is likely to be the fundamental paradigm found in many second order ODEs having chaos.

[6]This sequence should have positive complexity in the sense of Kolmogorov, see [Ford, 1986].

The strange attractor, as was illustrated in [Brown & Chua, 1991], is also accounted for with this paradigm. The twist and flip map can be extended to any number of dimensions and may be the key process at work in the equations of Lorenz, Rossler, and in the Chua circuit. We believe the reason for the pervasiveness of the twist and flip map as a source of PBS chaos is that the twist is actually a special case of shearing and the flip is a form of folding. These two actions can be found in the formation of vortices in turbulent flows, a form of chaos, and thus it is not surprising that they produce chaos in ODEs.

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List of Figure Captions

Fig. 1 Selected Trajectories from Generalized Twist and Flip Maps when the Twist is Defined by a Rotation Function $f(r) = 10r/(1 + r\log(r))$.

Fig. 2 Selected Trajectories from Generalized Twist and Flip Maps when the Twist is Defined by a Rotation Function $f(r) = -\sin(4\log(r))$.

Fig. 3 Selected Trajectories from Generalized Twist and Flip Maps when the Twist is Defined by a Rotation Function $f(r) = (1 + \log(r))/r$.

Fig. 4 Selected Trajectories from Generalized Twist and Flip Maps when the Twist is Defined by a Rotation Function $f(r) = -1/r; k^2 = 1/(r^2 + 1)$

Fig. 5 Unstable Manifold Used in the Formation of the Smale Horseshoe.

Fig. 6 Seven Forward Iterates of the Square Used in the Formation of the Smale Horseshoe.

Fig. 7 Close-up of the Seventh Forward and Backward Iterate Used in the Formation of the Smale Horseshoe.

Fig. 8 The Fixed-Point Unstable Manifold is in Blue and the Period-Two Unstable Manifold is in Green. The Two Unstable Manifolds are Wrapped so Tightly Together that they Give the Illusion of Being Only One Unstable Manifold.

Fig. 9 The Fixed-Point Unstable Manifold is in Blue and the Period-Two Unstable Manifold is in Green. The Addition of Damping Has Contracted the Unstable Manifolds. The Damping Factor is 0.21 and the Attractor is the Single Point $(-2.94, -6.069)$.

Fig. 10 The Fixed-Point Unstable Manifold is Gone. The Period-Two Unstable Manifold Remains and is in Green. Sufficient Damping Has Eliminated the Fixed-Point Unstable Manifold and Further Contracted the Period-Two Unstable Manifold. The Damping Factor is 0.45.

Fig. 11 The Attractor for the Damping Factor of 0.45 *Appears* to be the Period-Two Unstable Manifold. The Addition of Damping Has Eliminated the Fixed Point as the Attractor in Favor of the Period-Two Unstable Manifold. Increased Damping Appears to have Caused the Onset of Chaos.

Fig. 12 The Period-Two Unstable Manifold is Seen in Green and has been Further Contracted by Increasing the Damping Factor to 0.656. The Stable Manifold in Violet Empirically Verifies the Presence of Horseshoes and thus Chaos.

Fig. 13 The Period-Two Unstable Manifold Still Appears to be the Attractor Which is Shown in Red. The Damping Factor is 0.656.

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