RECEDING HORIZON CONTROL OF LINEAR SYSTEMS WITH INPUT SATURATION, DISTURBANCES, AND PLANT UNCERTAINTY

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E. Polak* and T. H. Yang*

ABSTRACT

We present a moving horizon feedback system, based on constrained optimal control algorithms, for linear plants with input saturation. The system is a nonconventional sampled-data system: its sampling periods vary from sampling instant to sampling instant, and the control during the sampling time is not constant, but determined by the solution of an open loop optimal control problem. We show that the proposed moving horizon control system is robustly stable, and that it is capable of following a class of reference inputs and suppressing a class of disturbances. Experimental results show that the behavior of the moving horizon control system is superior to that resulting from alternative control laws.

KEY WORDS: Moving horizon control, robust stability, tracking, disturbance rejection.

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1. INTRODUCTION

There is a startling contrast between the considerable difficulty of constructing robust, stabilizing feedback control laws for nonlinear or time varying systems by conventional methods, and the ease with which complex, *open loop* optimal control problems can be solved (see, e.g. [Kwa.2, May.2, May.3]). This observation has led to the suggestion that it might be feasible to determine feedback laws for nonlinear or time varying systems by repeatedly solving open loop, finite horizon optimal control problems. Such feedback laws are known as *receding horizon control* laws.

Although the concept of receding horizon control is not new and has been proposed in conjunction with various applications, process control being one of them, it has not always been realized that a naive application of the strategy, in adaptive control for example, can lead to instability. The literature that provides an analysis of the stabilizing properties of moving horizon control laws deals with schemes based on open loop optimal control laws for finite horizon optimal control problems with quadratic criteria and no control constraints. Thus Kwon and Pearson [Kwo.2], and Kwon, Bruckstein and Kailath [Kwo.2] deal with linear time-varying systems, Keerthi and Gilbert [Kee.1] deal with nonlinear discrete-time systems, and, more recently, Mayne and Michalska have established the stability properties of nonlinear, continuous-time systems with moving horizon control [May.2,May.3]; see also Chen and Shaw [Che.1]. None of this work addresses the questions of robustness (i.e., model errors), input following and disturbance rejection, nor does it take into account the nontrivial computing time associated with the computation of the open loop controls.

Now consider a dynamical system modeled by the finite dimensional ODE:

$$\dot{x}(t) = h(x(t), u(t)) \ , \tag{1.1}$$

where $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies the usual assumptions for optimal control (see, e.g., [Pol.1,Pol.6]), and $h(0, 0) = 0$. We assume that the state of the system can be measured exactly, that the control $u(t)$ is bounded, with $u(t) \in U$, a compact, convex set, and that design requirements may involve some state space constraints on the trajectories of the system. Clearly, if there were no modeling errors, no disturbances and no inputs, there would be no need for feedback laws to drive the system from arbitrary states to the origin. Thus, assuming that the time needed to solve a minimum time optimal control problem is less than T_C seconds, that the state $x(0) \neq 0$ at time $t = 0$ is known, and that one only wishes to take the system to the the zero state as quickly as possible, one could *(i)* use (1.1) to project the state at time T_C with the control $u(t) = 0$ for $t \in [0, T_C]$, *(ii)* solve a constrained minimum time, or free-time quadratic integral cost optimal control problem to compute a control $\hat{u}(\cdot)$ that can steer the state from $x(T_C)$ to the origin in the time $\hat{T}$, and then *(iii)* apply the control $\hat{u}(t)$ over the interval $[T_C, T_C + \hat{T}]$. At the end of this interval the system would be returned to the zero state. To establish a need for supplementing open loop optimal control with a feedback strategy, suppose that the model (1.1) is somewhat in error either in modeling the actual dynamics or in failing to include disturbance effects, or both. Then at time T_C, the actual system state $x^p(T_C + \hat{T})$ will differ from the projected state $x(T_C + \hat{T})$ and hence at time $T_C + \hat{T}$, the system state will not be the zero state. Thus, in the presence of modeling errors or disturbances, some form of closed loop strategy must be used.

In this paper we propose a feedback strategy for the simplest case of a linear plant, modeled with errors, subject to inputs and disturbances, as well as control constraints. This feedback strategy results in a nonconventional sampled-data system: its sampling periods vary from sampling instant to sampling instant, and the control during the sampling time is not constant, but determined by the solution of an open loop optimal control problem. We will see that taking into account the time needed to solve the open loop optimal control problem and modeling errors, complicates matters considerably, because the computed optimal control is based on an *estimated* initial state and corresponds to a model that is not an exact representation of the plant. In Section 2 we introduce our proposed moving horizon feedback control law, based on Control Algorithm 2.2. In Section 3 we show that the proposed moving horizon feedback system is robustly stable. In Section 4 we study the effect of disturbances, while in Section 5, we establish a class of reference inputs that can be tracked asymptotically by our system. Finally, in Section 6, we test illustrate the behavior of our moving horizon control law by means of a few simple examples.

2. STRUCTURE OF THE MOVING HORIZON CONTROL LAW.

We assume that the plant is a linear-time-invariant (LTI) system, with bounded inputs and an input disturbance, described by the differential equation

$$\dot{\xi}^p(t) = A^p \xi^p(t) + B^p(u(t) + d(t)), \tag{2.1a}$$

$$\eta^p(t) = C^p \xi^p(t), \tag{2.1b}$$

where the state $\xi^p(t) \in \mathbb{R}^n$, the control $u \in U$, with

$$U \triangleq \{ u \in L_2^m[0, \infty) \mid \|u\|_\infty \le c_u \} , \tag{2.1c}$$

$c \in (0, \infty)$, and the disturbance $d \in L_2^m[0, \infty)$. Consequently, $A^p \in \mathbb{R}^{n \times n}$ and $B^p \in \mathbb{R}^{n \times m}$. We will denote the solution of (2.1a) at time t, corresponding to the initial state ξ_0^p at time t_0, and the combined input $u + d$, by $\xi^p(t, t_0, \xi_0^p, u + d)$.

The function of the moving horizon control law is to ensure robust stability and "reasonable" reference input $r(t)$ tracking, suppress disturbances $d(t)$, while taking into account the fact that the plant inputs are bounded, as in (2.1c), as well as various amplitude constraints on transients.

We assume that the disturbance $d(t)$ cannot be measured and that the matrices A^p, B^p and C^p are known only to some tolerance. Hence the moving horizon control law must be developed using a plant model, of the same dimension as (2.1a),

$$\dot{\xi}(t) = A\xi(t) + B(u(t) + \hat{d}(t)), \tag{2.2a}$$

$$\eta(t) = C\xi(t), \tag{2.2b}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{n_o \times n}$ are approximations to A^p, B^p and C^p, and $\hat{d}(t)$ is an estimate of $d(t)$. When $d(t)$ can not be estimated, we set $\hat{d}(t) = 0$. We will denote the solution of (2.2a) at time t, corresponding to the initial state x_0 at time t_0, and the combined input $u + \hat{d}$, by $x(t, t_0, x_0, u + \hat{d})$.

Assumption 2.1. We will assume that (A, B) is a controllable pair, and that (C, A) is an observable pair. □

Let the subspace $S_x \subset \mathbb{R}^n$ be defined by

$$S_x = \{ x \in \mathbb{R}^n \mid x \in R(B), Ax \in R(B) \} , \tag{2.3a}$$

where $R(X)$ denotes the range space of the matrix X. Let H be a matrix whose columns are a basis for S_x. We will show in Section 5 that, when there are no constraints on the control $u(\cdot)$, given any continuously differentiable function $s(t)$, with values in S_x, there exists an input $u_S(t)$ such that for any initial state ξ_0, $\|\xi(t, 0, \xi_0, u_S) - s(t)\| \to 0$ as $t \to \infty$. Let S denote the set of continuously differentiable functions $s : \mathbb{R} \to S_x$. Hence, the reference inputs which can be tracked asymptotically, under the best of conditions are those in the set $\mathbf{R} \triangleq C\mathbf{S}$. We will therefore assume that the reference inputs to be tracked are in $\mathbf{R}$. We will use the following characterization of elements $r \in \mathbf{R}$, because it may help to alleviate the effects of the control constraint. Let $\tilde{C} \triangleq CH$ and let G be a matrix whose columns area a basis for the null space of $\tilde{C}$. Then any reference input $r \in \mathbf{R}$ can be expressed as follows:

$$r(t) = Cs(t), \tag{2.3b}$$

where $s(t) \triangleq H(\tilde{C}^T\tilde{C})^{\dagger}\tilde{C}^T r(t)$ († denotes the Penrose pseudo inverse) is continuously differentiable.

We can now define the error dynamics that will be used in defining and analyzing our control law. Suppose that a reference input $r \in \mathbf{R}$ is given. Let $x^p(t) \triangleq \xi^p(t) - s(t)$, and let $x(t) \triangleq \xi(t) - s(t)$. Then the plant error dynamics are given by

$$x^p(t) = A^p x^p(t) + B^p(u(t) + d(t)) + f^p(t), \tag{2.4a}$$

$$y^p(t) = C^p x^p(t), \tag{2.4b}$$

where $f^p(t) \triangleq -\dot{s}(t) + A^p s(t)$. Similarly, the model error dynamics become

$$\dot{x}(t) = Ax(t) + B(u(t) + \hat{d}(t)) + f(t), \tag{2.4c}$$

$$y(t) = Cx(t), \tag{2.4d}$$

where $f(t) \triangleq -\dot{s}(t) + As(t)$.

We will denote the solution of system (2.4b) by $x(t, t_0, x_0, u + \hat{d})$. Given any time t_k we will let $x_k \triangleq x(t_k, t_0, x_0, u + \hat{d})$. Assuming that the control law computation takes at most T_C time units, we can now propose a simple, aperiodic sampled-data feedback law, in the form of an algorithm which, during each sampling period, solves an optimal control problem $\mathrm{P}(x_k, t_k, r)$ of the form

$$\mathrm{P}(x_k, t_k, r): \min_{(u,\tau)} \{ g^0(u,\tau) \mid g^i(u,\tau) \le 0,\ i = 1, 2, \ldots, l_1,$$

$$\max_{t \in [t_k, \tau]} \phi^j(u, t) \le 0,\ j = 1, \ldots, l_2,\ u \in U,\ \tau \in [t_k + T_C, t_k + \bar{T}] \}, \tag{2.5a}$$

where $0 < T_C < \bar{T} < \infty$, and the constraint functions are defined by

$$g^i(u,\tau) \triangleq h^i(x(\tau, t_k, x_k, u)),\ i = 0, 1, \ldots, l_1 - 1, \tag{2.5b}$$

$$g^{l_1}(u,\tau) = \| x(\tau, t_k, x_k, u) \|^2 - \alpha^2 \| x_k \|^2, \tag{2.5c}$$

$$\phi^j(u\,,t) = h^j(x(t\,,t_k\,,x_k\,,u)\,,t)\,,\ j = 1\,,\ldots,l_2-1\,, \tag{2.5d}$$

$$\phi^{l_2}(u\,,t) = \|x(t\,,t_k\,,x_k\,,u)\|^2 - \beta^2\|x_k\|^2\,, \tag{2.5e}$$

where the constraint functions (2.5c,e) with $\alpha \in (0\,,1)$, $\beta \in [1\,,\infty)$, are used to ensure robust stability and input tracking, while the other functions, h^i, $h^j : \mathbb{R}^n \to \mathbb{R}$ are convex, locally Lipschitz continuously differentiable functions that can be used to ensure other performance requirements.

We are now ready to state our control algorithm that defines the moving horizon feedback control system.

Control Algorithm 2.2.

Data: $t_0 = 0, t_1 = T_C, u_{[t_0,t_1]}(t) \equiv 0, x_0 \in B_{\hat{\rho}}$. T_C and $\bar{T}$ such that $0 < \underline{T} < T_C < \bar{T} < \infty$.

Step 0: Set $k = 0$.

Step 1: At $t = t_k$,

(a) Obtain a measurement or estimate of the state $x_k^p = x^p(t_k)$ and denote the resulting value by $\bar{x}_k$.

(b) Compute an estimate, $\hat{d}(t)$, of the disturbance $d(t)$ for $t \in [t_k\,,t_{k+1}]$, if possible; else, set $\hat{d}(t) = 0$.

(c) Set the plant error dynamics input $u(t) = u_{[t_k,t_{k+1}]}(t) - \hat{d}(t)$ for $t \in [t_k\,,t_{k+1})$.

(d) Compute an estimate x_{k+1} of the state of the plant error dynamics $x^p(t_{k+1}\,,t_k\,,\bar{x}_k\,,u)$ according to the formula

$$x_{k+1} = e^{A(t_{k+1}-t_k)}\bar{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)}B\,[u(t)+\hat{d}(t)]\,dt + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)}f(t)\,dt\,. \tag{2.6}$$

(e) Solve the open loop optimal control problem $\mathbf{P}(x_{k+1}\,,t_{k+1}\,,r)$ to compute the next sampling time $t_{k+2} \in (t_{k+1}+T_C\,,t_{k+1}+\bar{T}]$, and the optimal control $u_{[t_{k+1},t_{k+2}]}(t) \in U$, $t \in [t_{k+1}\,,t_{k+2})$;

Step 2: Replace k by $k+1$ and go to Step 1. □

Let Q be a symmetric, positive definite $n \times n$ matrix such that $\langle x\,,Qx\rangle$ is a Lyapunov function for the linear closed loop system obtained applying state feedback to (2.1a). The reason for this selection will become clear in Section 3.3. We use this matrix to define the norm $\|x\| \triangleq \langle x\,,Qx\rangle^{1/2}$.

Clearly, the fact that the plant inputs are bounded, limits the region of effectiveness of any control law and the class of reference inputs that can be tracked. Hence we must assume that the initial states are confined to a Q–ball $B_{\hat{\rho}} \subset \mathbb{R}^n$ and that the reference inputs belong to the set $\mathbf{R}_U$, both defined, as follows.

Assumption 2.3. We assume that there exists a nonempty set $\mathbf{r} \subset (0, \infty)$ and $\mathbf{R}_U \subset \mathbf{R}$ such that $0 \in \mathbf{R}_U$ and that for all $\rho \in \mathbf{r}$, $x \in B_\rho \triangleq \{ x \in \mathbb{R}^n \mid \|x\| \leq \rho \}$ and for all $r \in \mathbf{R}_U$, the optimal control problem $\mathbf{P}(x, 0, r)$ has a solution. Let $\hat{\rho}$ be a relatively large value in $\mathbf{r}$. We define $B_{\hat{\rho}} \triangleq \{ x \in \mathbb{R}^n \mid \|x\| \leq \hat{\rho} \}$. □

The following theorem, which generalizes a result given in [Pol.1].

Theorem 2.4. Let $B_{\hat{\rho}} \subset \mathbf{R}^n$ and $r \in \mathbf{R}_U$, be defined as in Assumption 2.3. Suppose that *(a)* the systems (2.1a) and (2.2a) are identical, *(b)* $d(t) \equiv 0$, and *(c)* the Control Algorithm 2.2 is used to define the input $u(\cdot)$ for (2.1a). Then the resulting feedback system is asymptotically stable in the sense of Lyapunov on the set $B_{\hat{\rho}}$.

Proof. We begin by showing that for any $r \in \mathbf{R}_U$ and for any $x_0 \in B_{\hat{\rho}}$, the trajectory $x(t_k, 0, x_0, u) = x_k$, $k \in \mathbb{N}$ resulting from the use of the Control Algorithm 2.2 is contained in $B_{\hat{\rho}}$. In turn, this shows that such a trajectory is well defined and that it is bounded.

Suppose that $x_0 \in B_{\hat{\rho}}$ is an arbitrary initial state at $t = 0$. It follows from the form of (2.5c), that for all $k \in \mathbb{N}$,

$$\|x_{k+1}\| = \|x(t_{k+1}, t_k, x_k, u_{[t_k, t_{k+1}]})\| \leq \alpha \|x_k\| \leq \alpha^{k+1} \|x_0\| . \tag{2.7a}$$

Since $\alpha \in (0, 1)$, it follows that $x_k \in B_{\hat{\rho}}$ for all $k \in \mathbb{N}$ and hence that the trajectory $x(t, 0, x_0, u)$ is well defined.

Next, from the form of (2.5e), we see that for all $k \in \mathbb{N}$ and for any $t \in [t_k, t_{k+1}]$,

$$\|x(t, t_k, x_k, u_{[t_k, t_{k+1}]})\| \leq \beta \|x_k\| \leq \beta \alpha^k \|x_0\| \leq \beta \|x_0\| , \tag{2.7b}$$

which proves that since $x_0 \in B_{\hat{\rho}}$, this trajectory is bounded.

Finally, because $\beta \alpha^k \to 0$ as $k \to \infty$, it follows that $x(t, 0, x_0, u) \to 0$ as $t \to \infty$, and hence that the feedback system defined by the Control Algorithm 2.2 is asymptotically stable in the sense of Lyapunov on the set $B_{\hat{\rho}}$.

□

We note that Theorem 2.4 did not depend on the form of the cost function $g^0(\cdot\,,\cdot)$ nor on the form of the constraints defined by (2.5b) and (2.5d). These constraints can be used to shape the transient responses of the closed loop system. We will describe later a procedure for solving problems of the form (2.5a-e).

As stated, Control Algorithm 2.2 only defines a local control law. When the plant is unstable, it is not clear that there is much that one can do about it. However, in the case of stable plants (and models), it is possible to globalize Control Algorithm 2.2 making use of the following observation. First, it should be clear that, in the absence of modeling errors and disturbances, for any $r \in \mathbf{R}$ such that for all $t \geq 0$, $\min_{u \in U} \| Bu + f(t) \| = 0$, there exists an admissible control, $u^o(t) \in argmin_{u \in U} \| Bu + f(t) \|$ that results in the error satisfying the equation

$$\dot{x}(t) = Ax(t)\,, \tag{2.8}$$

and hence, if A is a stable matrix, the error goes to zero exponentially, so that $x(t) \in B_{\hat{\rho}}$ will occur in finite time. Clearly, in this case, there may be room for a more effective control law, as we will now show. Let M' and Q' be symmetric, positive definite matrices, such that $A^T Q' + Q'A = -M'$, then $V(x(t)) \triangleq \langle x(t)\,, Q'x(t) \rangle$ is a Lyapunov function for (2.8). Let $T_s \in (T_C\,, \bar{T}]$ and suppose that $x_k \notin B_{\hat{\rho}}$. Then, if we set $t_{k+1} = t_k + T_s$ and we apply the control $u_o(t)$, to (2.4c), for $t \in [t_k\,, t_{k+1}]$, then we must have that $V(x(t_{k+1}\,, t_k\,, x_k\,, u^o)) \leq e^{-\lambda_{min}(M')T_s} V(x_k)$. Hence it makes sense to use instead the control defined as the solution of the simple optimal control problem

$$\min_{u \in U} \{\, V(x(t_{k+1}\,, t_k\,, x_k\,, u)) \,\}\ , \tag{2.9}$$

where $x(t_{k+1}\,, t_k\,, x_k\,, u)$ is determined as the solution of (2.4c).

Hence, for stable plants, we propose to modify Control Algorithm 2.2, as follows:

Control Algorithm 2.5.

Data: $t_0 = 0$, t_1, $u_{[t_0, t_1]}(t)$, x_0, T_s, T_C and $\bar{T}$ such that $0 < T_C < T_s \leq \bar{T} < \infty$.

Step 0: Set $k = 0$.

Step 1: At $t = t_k$,

(a) Obtain a measurement or estimate of the state $x_k^p = x^p(t_k)$ and denote the resulting value by $\bar{x}_k$.

(b) Compute an estimate, $\hat{d}(t)$, of the disturbance $d(t)$ for $t \in [t_k, t_{k+1}]$, if possible; else, set $\hat{d}(t) = 0$.

(c) Set the plant error dynamics input $u(t) = u_{[t_k, t_{k+1}]}(t) - \hat{d}(t)$ for $t \in [t_k, t_{k+1})$.

(d) Compute an estimate x_{k+1} of the state of the plant error dynamics $x^p(t_{k+1}, t_k, \bar{x}_k, u)$ according to the formula (2.6)

$$x_{k+1} = e^{A(t_{k+1}-t_k)}\bar{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)}B[u(t)+\hat{d}(t)]\,dt + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)}f(t)\,dt .$$

(e) If $x_{k+1} \in B_{\hat{\rho}}$, solve the open loop optimal control problem $\mathbf{P}(x_{k+1}, t_{k+1}, r)$ to compute the next sampling time $t_{k+2} \in (t_{k+1} + T_C, t_{k+1} + \bar{T}]$, and the optimal control $u_{[t_{k+1}, t_{k+2}]}(t) \in U, t \in [t_{k+1}, t_{k+2})$.

Else set $t_{k+2} = t_{k+1} + T_s$ and $u_{[t_{k+1}, t_{k+2}]}(t) = u^o(t)$, for all $t \in [t_{k+1}, t_{k+2})$.

Step 2: Replace k by $k + 1$ and go to Step 1. □

We will not present a complete analysis of the operation of the closed loop system under Control Algorithm 2.5.

3. ROBUST STABILITY.

In this section, we will analyze the behavior of the closed loop system resulting from the use of Control Algorithm 2.2 under the assumption that there is a difference between the actual plant equations (2.4a) and the model equations (2.4b), and that $d(t) \equiv 0$ and $r(t) \equiv 0$. We recall that when $r(t) \equiv 0$, we have that $s(t) \equiv 0$, $f^p(t) \equiv 0$, and $f(t) \equiv 0$ in (2.4a,b). Hence we will set $\hat{d}(t) \equiv 0$. We will consider two distinct situations: the first is where we can measure the state, while the second one is where the state has to be estimated. Finally, we will show how a cross over rule to a linear state feedback law near the origin can be used to eliminate residual errors in both cases.

3.1. moving horizon control with state measurement.

We begin by defining the error quantities

$$\Delta_1 \triangleq \max_{t \in [0, \bar{T}]} \| e^{A^p t} - e^{At} \| , \tag{3.1a}$$

$$\Delta_2 \triangleq c_u \bar{T} \max_{t \in [0,\bar{T}]} \| e^{A^p t} B^p - e^{At} B \| , \tag{3.1b}$$

$$K \triangleq \max_{t \in [0,\bar{T}]} \| e^{At} \| . \tag{3.1c}$$

When either Δ_1 or Δ_2 is not zero, even if $\bar{x}_k \in D$, where D was defined in Assumption 2.2, the estimated state, x_{k+1} (defined by (2.6)), may not be in D and hence there may not exist a solution to the optimal control problem $P(x_{k+1}, t_{k+1}, 0)$. Therefore, we have to specify a set $B_{\rho_*} \subset D$, such that for any $x_0^p \in B_{\rho_*}$, Control Algorithm 2.2 is well defined on the emanating trajectory $x(t, 0, x_0^p, u)$. We will obtain a formula for such a set in the process of proving the following result.

Lemma 3.1. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with plant state measurement. There exist $\varepsilon_1, \varepsilon_2 > 0$ such that if $\Delta_1 \le \varepsilon_1$ and $\Delta_2 \le \varepsilon_2$, then there exists a set $B_{\rho_*} \subset D$, with nonempty interior, such that for all $x_0^p \in B_{\rho_*}$, the control law defined by Control Algorithm 2.2 is well defined on the resulting trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, i.e., the states, x_{k+1}, $k = 0, 1, 2, \ldots$, computed using (2.6) satisfy that $x_{k+1} \in D$ for all $k \ge 0$.

Proof. First suppose that the optimal control problem $P(x_{k+1}, t_{k+1}, 0)$, has a solution for any $x_{k+1} \in \mathbb{R}^n$ and $t_{k+1} \ge 0$. Then, given any initial state x_0^p at time $t_0 = 0$, the Control Algorithm 2.2 generates three sequences of states. The first sequence is that of measured plant states $\{ x_k^p \}_{k=0}^{\infty}$, so that $\bar{x}_k = x_k^p$ for all $k \in \mathbb{N}$, the second sequence is the sequence of estimates $\{ x_k \}_{k=1}^{\infty}$, with $x_{k+1} = x(t_{k+1}, t_k, x_k^p, u)$, $k = 1, 2, \ldots$, generated according to (2.6), and finally, the sequence $\{ x'_k \}_{k=2}^{\infty}$, with $x'_{k+2} = x(t_{k+2}, t_{k+1}, x_{k+1}, u)$, $k = 1, 2, \ldots$, generated in the process of solving the optimal control problem $P(x_{k+1}, t_{k+1}, 0)$, $k \in \mathbb{N}$.

First we note that it follows form (2.4a), (2.6) and (3.1a,b) that

$$\| x_{k+1}^p - x_{k+1} \| \le \Delta_1 \| x_k^p \| + \Delta_2 . \tag{3.2a}$$

Hence, making use of (3.2a), we obtain that

$$\| x_{k+1} \| \le \| x_{k+1}^p - x_{k+1} \| + \| x_{k+1}^p \| \le \Delta_1 \| x_k^p \| + \Delta_2 + \| x_{k+1}^p \| . \tag{3.2b}$$

Since by construction, for all $k \in \mathbb{N}$, $\| x'_{k+2} \| \le \alpha \| x_{k+1} \|$, it follows from (3.1a-c) and (3.2a,b), that

$$\| x_{k+2}^p \| \le \| x_{k+2}^p - x'_{k+2} \| + \alpha \| x_{k+1} \|$$

$$\leq K\|x^p_{k+1} - x_{k+1}\| + \Delta_1\|x^p_{k+1}\| + \Delta_2 + \alpha\|x^p_{k+1} - x_{k+1}\| + \alpha\|x^p_{k+1}\|$$

$$\leq (K+\alpha)(\Delta_1\|x^p_k\| + \Delta_2) + (\Delta_1+\alpha)\|x^p_{k+1}\| + \Delta_2$$

$$= (K+\alpha)\Delta_1\|x^p_k\| + (\Delta_1+\alpha)\|x^p_{k+1}\| + (1+\alpha+K)\Delta_2 . \tag{3.2c}$$

Let

$$\hat{\varepsilon}_1 \triangleq (1-\alpha)/(1+\alpha+K) . \tag{3.2d}$$

We will now show that if $\Delta_1 \leq \varepsilon_1 < \hat{\varepsilon}_1$, then there exists $\gamma_1, \gamma_2 \in (0, \infty)$ such that for all $k = 1, 2, \ldots,$

$$\|x_k\| \leq \gamma_1\|x^p_0\| + \gamma_2 . \tag{3.2e}$$

We will now make use of Proposition 8.1. Hence, let $a_1 = \Delta_1 + \alpha$, $a_2 = (K+\alpha)\Delta_1$, and $b = (1+\alpha+K)\Delta_2$. Because a_1, a_2, and b are positive, if we set $y_0 = \|x^p_0\|$ and $y_1 = \|x^p_1\|$ in (8.1a), then comparing (3.2c) with (8.1a), we see that for all $k \in \mathbb{N}$, $y_k \geq \|x^p_k\|$. Also, because

$$\Delta_1 + \alpha + (K+\alpha)\Delta_1 = a_1 + a_2 < 1 , \tag{3.2f}$$

the assumptions of Proposition 8.1 are satisfied. Since $\Delta_1 \leq \varepsilon_1 < \hat{\varepsilon}$, and $(K+\alpha)\Delta_1 \geq 0$,

$$1 - a_1 + a_2 = 1 - \Delta_1 - \alpha + (K+\alpha)\Delta_1 > 1 - \alpha - \varepsilon_1 = \frac{(1-\alpha)(\alpha+K)}{1+\alpha+K} \triangleq \varepsilon' > 0 . \qquad 93.2g)$$

Hence, for all $k \geq 1$,

$$\|x^p_k\| \leq y_k \leq a_2\|x^p_0\| + \|x^p_1\| + \frac{b}{1-a_1+a_2} \leq a_2\|x^p_0\| + \|x^p_1\| + \varepsilon'' , \tag{3.2h}$$

$$\overline{\lim_{k\to\infty}} \|x^p_k\| \leq \overline{\lim_{k\to\infty}} y_k \leq \varepsilon'' , \tag{3.2i}$$

where

$$\varepsilon'' \triangleq \frac{(1+\alpha+K)\Delta_2}{\varepsilon'} . \tag{3.2j}$$

Clearly,

$$\|x^p_1\| \leq \|x^p_1 - x_1\| + \|x_1\| . \tag{3.2k}$$

Since $x_0 = x^p_0$, and $u(t) = 0$ for $t \in [0, t_1]$ it follows from (3.1a,c), and (3.2k) that

$$\|x_1^\ell\| \le \Delta_1 \|x_0^\ell\| + K\|x_0^\ell\| = (\Delta_1 + K)\,\|x_0^\ell\| . \tag{3.2l}$$

Substituting this result into (3.2h), we obtain that for all $k \ge 1$,

$$\|x_k^\ell\| \le y_k \le (\Delta_1 + K - \lambda_+\lambda_-)\|x_0^\ell\| + \varepsilon'' , \tag{3.2m}$$

where ε'' is defined in (3.2j). Since $-\lambda_+ \lambda_- = a_2 = (K+\alpha)\Delta_1$, it follows from (3.2m) and (3.2b) that

$$\begin{aligned} \|x_{k+1}\| &\le \Delta_1 y_k + \Delta_2 + y_{k+1} \\ &\le (1+\Delta_1)(\Delta_1 + K + (K+\alpha)\Delta_1)\|x_0^\ell\| + (1+\Delta_1)\,\varepsilon'' + \Delta_2 \\ &\triangleq \gamma_1 \|x_0^\ell\| + \gamma_2 , \end{aligned} \tag{3.2n}$$

which proves (3.2e).

Next we will show that with $\hat{\varepsilon}_2 > 0$ defined by

$$\hat{\varepsilon}_2 \triangleq \hat{\rho} \left[\frac{2+K}{\varepsilon'} + 1 \right]^{-1} = \hat{\rho} \left[\frac{(2+K)(1+\alpha+K)}{(1-\alpha)(\alpha+K)} + 1 \right]^{-1} , \tag{3.2o}$$

where $\hat{\rho} > 0$ was used to define the set D in Assumption 2.2 and ε' is defined in (3.2g), if $\Delta_2 \le \varepsilon_2 < \hat{\varepsilon}_2$, then there exists a $\rho_\varepsilon \in (0, \hat{\rho})$, depending on $\varepsilon_1, \varepsilon_2$, such that if $\|x_0^\ell\| \le \rho_\varepsilon$, then $\|x_k\| \le \hat{\rho}$ for all $k = 1, 2, \ldots$, i.e., that the trajectory $x(t, 0, x_0^\ell, u)$, $t \in [0, \infty)$, emanating from x_0^ℓ, constructed under Control Algorithm 2.2 is well defined.

Assuming that $\Delta_1 \le \varepsilon_1$ and that $\Delta_2 \le \varepsilon_2$, we obtain that from (3.2n)

$$\gamma_2 < \left[1 + \frac{1-\alpha}{1+\alpha+K} \right] \frac{(1+\alpha+K)}{\varepsilon'}\, \varepsilon_2 + \Delta_2 \le \left[\frac{2+K}{\varepsilon'} + 1 \right] \varepsilon_2 \triangleq \hat{\gamma}_2 \le \hat{\rho} . \tag{3.2p}$$

Let $\hat{\gamma}_1$ and ρ_ε be defined as follows:

$$\hat{\gamma}_1 \triangleq (1+\varepsilon_1)\,(K + (1+\alpha+K)\,\varepsilon_1) , \tag{3.2q}$$

$$\rho_s \triangleq (\hat{\rho} - \hat{\gamma}_2)/\hat{\gamma}_1 , \tag{3.2r}$$

Since $\hat{\rho} - \hat{\gamma}_2 > 0$ and $K \le \gamma_1 \le \hat{\gamma}_1$, we conclude that $\rho_\varepsilon > 0$, and hence that the $B_{\rho_\varepsilon} \subset D$, defined by

$$B_{\rho_s} \triangleq \{ x \in D \mid \|x\| \le \rho_s \} , \tag{3.2s}$$

is well defined and its interior is not empty. Furthermore, for any $x_0^\ell \in B_{\rho_\varepsilon}$, the resulting sequence $\{x_{k+1}\}_{k=0}^{\infty}$ satisfies

$$|x_{k+1}| \le \gamma_1 |x_0^p| + \gamma_2 \le \gamma_1(\hat{\rho} - \hat{\gamma}_2)/(\hat{\gamma}_1) + \hat{\gamma}_2 \le \hat{\rho} \ , \ \ \forall k \in \mathbb{N} \ , \tag{3.2t}$$

which implies that $x_{k+1} \in D$, for all $k \in \mathbb{N}$, and, in turn, that the optimal control problem $\mathrm{P}(x_{k+1}, t_{k+1}, 0)$ has a solution for all $k \in \mathbb{N}$. Hence the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, emanating from any $x_0^p \in B_{\rho}$, is well defined by Control Algorithm 2.2, which completes our proof.□

Theorem 3.2. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with plant state measurement. Suppose that $\Delta_1 \le \varepsilon_1 < \hat{\varepsilon}_1$ and $\Delta_2 \le \varepsilon_2 < \hat{\varepsilon}_2$, where $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$ are defined in (3.2d) and (3.2o), respectively. Let B_{ρ} be defined as (3.2s). Then *(a)* for any $x_0^p \in B_{\rho}$, the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, is bounded, and *(b)* there exists an $\varepsilon_3 > 0$, depending on $\varepsilon_1, \varepsilon_2$, such that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, and for any $x_0^p \in B_{\rho}$ the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, satisfies $\overline{\lim}_{t \to \infty} |x^p(t, 0, x_0^p, u)| \le \varepsilon_3$.

Proof. Let $x_0^p \in B_{\rho}$ be arbitrary and let $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{x'_k\}_{k=2}^{\infty}$ be the sequences constructed by Control Algorithm 2.2, as defined in Lemma 3.1. We recall that by Lemma 3.1, the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, is well defined.

(a) Making use of (2.4a) and (3.1a-c), we obtain that for all $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$,

$$|x^p(t, t_k, x_k^p, u)| \le |x^p(t, t_k, x_k^p, u) - x(t, t_k, x_k, u)| + |x(t, t_k, x_k, u)|$$

$$\le \Delta_1 |x_k^p| + K |x_k^p - x_k| + \Delta_2 + |x(t, t_k, x_k, u)| \ . \tag{3.3a}$$

Next we note that the form of (2.5e) ensures that $|x(t, t_k, x_k, u)| \le \beta |x_k|$ for all $t \in [t_k, t_{k+1}]$. Hence, in view of (3.2a), (3.3a) can be replaced by

$$|x^p(t, t_k, x_k^p, u)| \le \Delta_1 |x_k^p| + K |x_k^p - x_k| + \Delta_2 + \beta |x_k| = (\Delta_1 + \beta)|x_k^p| + (K + \beta)|x_k^p - x_k| + \Delta_2$$

$$\le (\Delta_1 + \beta)|x_k^p| + (K + \beta)\Delta_1 |x_{k-1}^p| + (1 + K + \beta)\Delta_2 \ , \ \ t \in [t_k, t_{k+1}] \ . \tag{3.3b}$$

Clearly, since $u_{[0, t_1]} \equiv 0$, $|x^p(t, 0, x_0^p, u)|$ is bounded on $[0, t_1]$. Since, as we have already shown in the proof of Lemma 3.1, $\{|x_k^p|\}_{k=0}^{\infty}$ is a bounded sequence, it follows from (3.3b) that $|x^p(t, t_k, x_k^p, u)|$ is bounded for all $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$, which completes the proof of *(a)*.

(b) It follows from (3.2i), in the proof of Lemma 3.1, that

$$\overline{\lim}_{k \to \infty} |x_k^p| \le \overline{\lim}_{k \to \infty} y_k \le \varepsilon'' \le \frac{(1 + \alpha + K)\varepsilon_2}{\varepsilon'} \ , \tag{3.3c}$$

where ε' is defined in (3.2n). Let

$$\varepsilon_3 \triangleq \left[\frac{(\beta+(1+K+\beta)\varepsilon_1)(1+\alpha+K)}{\varepsilon'} + 1 + K + \beta\right]\varepsilon_2 . \tag{3.3d}$$

Then (3.2i) and (3.3b) lead to the conclusion that $\overline{\lim}_{t\to\infty} \| x^p(t, 0, x_0^p, u)\| \le \varepsilon_3$. It is obvious from (3.2i) and (3.3d) that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, which completes our proof. □

3.2. moving horizon control with state estimation.

Since it is not always possible to measure the plant state x_k^p, we will now examine the behavior of our closed loop system, resulting form the use of Control Algorithm 2.2, when the plant state has to be estimated in the presence of modeling errors, i.e., when the actual dynamics are as in (2.1a,b) and the modeled dynamics as in that (2.2a,b). We will assume that (A, C) is an observable pair.

When the model (2.2a,b) is identical with the actual dynamics (2.1a,b), we can calculate the initial state, x_0^p at $t = 0$, using the standard formula

$$x_0^p = W_o(T)^{-1}\int_0^T (Ce^{At})^T(y^p(t) - \eta(t, 0))dt , \tag{3.4a}$$

where $T > 0$, the superscript T denotes a transpose, and

$$W_o(T) = \int_0^T (Ce^{At})^T Ce^{At} dt , \tag{3.4b}$$

$$\eta(t, s) = C\int_s^t e^{A(t-\tau)}Bu(\tau)d\tau . \tag{3.4c}$$

Clearly, $W_o(T)^{-1}$ exists because (A, C) is an observable pair. Thus, when there are no modeling errors and no disturbances, for $t \ge T$, the state $x^p(t, 0, x_0^p, u)$, can be calculated exactly, and hence this calculated state can be used in Control Algorithm 2.2.

The much more relevant situation occurs when there are modeling errors but no disturbances. In this case formula (3.4a) yields an estimate of the initial state x_0^p. We propose to use it in in *Step 1 (b)* of Control Algorithm 2.2, to obtain the estimate $\tilde{x}_k$, with the time T determined by a parameter δ_0, which must be chosen judiciously so as to avoid excessive ill conditioning in the observability grammian $W_o(T)$:

Step 1: *(a)* At $t'_k \triangleq t_k + \delta_0(t_{k+1} - t_k)$ with $\delta_0 \in (0, 1)$, estimate the state x_k^p by

$$\tilde{x}_k = W_o(\delta_0(t_{k+1} - t_k))^{-1}\int_{t_k}^{t'_k} (Ce^{A(t-t_k)})^T (y^p(t) - \eta(t, t_k))dt . \tag{3.5}$$

□

Lemma 3.3. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with state estimation formula (3.5). There exist $\Delta_i < \infty$, $i = 3, \ldots, 6$, such that if

Control Algorithm 2.2 constructs the sequences $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{\tilde{x}_k\}_{k=0}^{\infty}$ is the corresponding sequence of the estimates of x_k^p, defined by (3.5), then for all $k \in \mathbb{N}$,

$$\|x_k^p - \tilde{x}_k\| \le \Delta_3 \|x_k^p\| + \Delta_4\,, \tag{3.6a}$$

$$\|x_{k+1}^p - x_{k+1}\| \le \Delta_5 \|x_k^p\| + \Delta_6\,. \tag{3.6b}$$

Furthermore, when there are no modeling errors, $\Delta_i = 0$, $i = 3, \ldots, 6$.

Proof. Suppose that $u(\cdot)$ is the control generated by Control Algorithm 2.2 for the plant and model trajectories associated with the sequences $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{\tilde{x}_k\}_{k=0}^{\infty}$.

We begin with (3.6a). For any $k \in \mathbb{N}$ and any $t \in [t_k, t_{k+1}]$, $y^p(t)$ is given by

$$\begin{aligned} y^p(t) &= C^p e^{A^p(t-t_k)} x_k^p + C^p \int_{t_k}^{t} e^{A^p(t-\tau)} B^p u(\tau) d\tau \\ &= C e^{A(t-t_k)} x_k^p + [C^p e^{A^p(t-t_k)} - C e^{A(t-t_k)}] x_k^p \\ &+ C \int_{t_k}^{t} e^{A(t-\tau)} B u(\tau) d\tau + \int_{t_k}^{t} [C^p e^{A^p(t-\tau)} B^p - C e^{A(t-\tau)} B]\, u(\tau)\, d\tau\,. \end{aligned} \tag{3.7a}$$

By substituting (3.7a) into (3.5), we obtain

$$\begin{aligned} \tilde{x}_k = x_k^p + W_o^{-1}(\delta_0(t_{k+1} - t_k)) \Big\{ &\int_{t_k}^{t_k'} (C e^{A(t-t_k)})^T [C^p e^{A^p(t-t_k)} - C e^{A(t-t_k)}]\, dt\, x_k^p \\ &+ \int_{t_k}^{t_k'} (C e^{A(t-t_k)})^T \int_{t_k}^{t} [C^p e^{A^p(t-\tau)} B^p - C e^{A(t-\tau)} B]\, u(\tau)\, d\tau\, dt \Big\}. \end{aligned} \tag{3.7b}$$

It follows directly from (3.7b) that

$$\|x_k^p - \tilde{x}_k\| \le \Delta_3 \|x_k^p\| + \Delta_4\,, \tag{3.7c}$$

where

$$\Delta_3 = \max_{t \in [T_c, \bar{T}]} \|W_o(\delta_0 t)^{-1}\| \max_{t \in [0, \delta_0 \bar{T}]} \|C e^{At}\| \max_{t \in [0, \delta_0 \bar{T}]} \|C^p e^{A^p(t-t_k)} - C e^{A(t-t_k)}\|\, \delta_0 \bar{T} \tag{3.7d}$$

$$\Delta_4 = \max_{t \in [T_c, \bar{T}]} \|W_o(\delta_0 t)^{-1}\| \max_{t \in [0, \delta_0 \bar{T}]} \|C e^{At}\| \max_{t \in [0, \delta_0 \bar{T}]} \|C^p e^{A^p(t-\tau)} B^p - C e^{A(t-\tau)} B\| c_u \delta_0 \bar{T}\,, \tag{3.7e}$$

which proves our first (3.6a). Clearly, when there are no modeling errors, $\Delta_3 = \Delta_4 = 0$.

Next we will establish (3.6b). Since x_{k+1} is calculated using the estimated initial state $\tilde{x}_k$, we have that

$$|x_{k+1}^p - x_{k+1}| = |e^{A^p(t_{k+1}-t_k)}x_k^p - e^{A(t_{k+1}-t_k)}\bar{x}_k + \int_{t_k}^{t_{k+1}} \{ e^{A^p(t_{k+1}-\tau)}B^p - e^{A(t_{k+1}-\tau)}B \} u(\tau)d\tau|$$

$$\leq K \, |x_k^p - \bar{x}_k| + \Delta_1 |x_k^p| + \Delta_2$$

$$\leq K \, \{ \Delta_3 |x_k^p| + \Delta_4 \} + \Delta_1 |x_k^p| + \Delta_2 ,$$

$$= (K\Delta_3 + \Delta_1) \, |x_k^p| + K\Delta_4 + \Delta_2 \triangleq \Delta_5 |x_k^p| + \Delta_6 , \tag{3.7f}$$

where K, Δ_1, and Δ_2 are defined in (3.1a,b,c). Hence (3.6b) holds, and our proof is complete. □

Lemma 3.3, leads to the following result.

Theorem 3.4. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with state estimation formula (3.5). Let $\varepsilon_1, \varepsilon_2 > 0$ be such that

$$\varepsilon_1 < (1-\alpha)/(1+\alpha+K) , \tag{3.8a}$$

$$\varepsilon_2 < \hat{\rho}/(1+(2+K)/\varepsilon') , \tag{3.8b}$$

where $\hat{\rho}$ was defined in Assumption 2.2 and ε' was defined in (3.2g), in the proof of Lemma 3.1. If $\Delta_5 \leq \varepsilon_1$ and $\Delta_6 \leq \varepsilon_2$, then there exists a set $B_{\rho_*} \subset D$ such that *(a)* for any $x_0^p \in B_{\rho_*}$, the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, is well defined and bounded, and *(b)* there exists an $\varepsilon_3 > 0$ such that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, and for any $x_0^p \in B_{\rho_*}$ the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, satisfies $\overline{\lim}_{t\to\infty} |x^p(t, 0, x_0^p, u)| \leq \varepsilon_3$.

Proof. *(a)* First suppose that the optimal control problem $P(x_{k+1}, t_{k+1}, 0)$, has a solution for any $x_{k+1} \in \mathbb{R}^n$ and $t_{k+1} \geq 0$. Then it follows from Lemma 3.3 that

$$|x_{k+2}^p| \leq |x_{k+2}^p - x'_{k+2}| + |x'_{k+2}| \leq K \, |x_{k+1}^p - x_{k+1}| + \Delta_1 |x_{k+1}^p| + \Delta_2 + \alpha |x_{k+1}|$$

$$\leq (K+\alpha) \, |x_{k+1}^p - x_{k+1}| + (\Delta_1 + \alpha) \, |x_{k+1}^p| + \Delta_2$$

$$\leq (K+\alpha)\Delta_5 |x_k^p| + (K+\alpha)\Delta_6 + (\alpha + \Delta_1) \, |x_{k+1}^p| + \Delta_2 . \tag{3.9a}$$

Since $\Delta_1 \leq \Delta_5$ and $\Delta_2 \leq \Delta_6$, we have that

$$|x_{k+2}^p| \leq (K+\alpha)\Delta_5 |x_k^p| + (\alpha + \Delta_5) \, |x_{k+1}^p| + (K+\alpha+1)\Delta_6 . \tag{3.9b}$$

Since (3.9b) is of the same form as (3.2c), with Δ_5 replacing Δ_1, and Δ_6 replacing Δ_2, we see that the conclusions of Lemma 3.1 and Theorem 3.2 *(a)* remain valid for the Control Algorithm 2.2 using state estimation formula (3.5).

(b) Referring to (3.2s) (3.3d), we conclude that part (b) holds with ε_3 and B_{ρ_s} defined by

$$\varepsilon_3 \triangleq \left[\frac{(\beta+(1+K+\beta)\varepsilon_1)(1+\alpha+K)}{\varepsilon'} + 1 + K + \beta\right]\varepsilon_2, \tag{3.9c}$$

$$B_{\rho_s} \triangleq \{x \in D \mid \|x\| \le \rho_s\}, \tag{3.9d}$$

where ε' is defined in (3.2g) and

$$\rho_s \triangleq \frac{\hat{\rho} - ((2+K)/\varepsilon' + 1)\varepsilon_2}{(1+\varepsilon_1)(K+(1+\alpha+K)\varepsilon_1)}. \tag{3.9e}$$

□

3.3. elimination of residual errors by linear feedback.

Because linear quadratic regulators are robust, when the pair (A, B) is stabilizable and the modeling errors are sufficiently small, we can always find a linear stabilizing state feedback control law $u(t) = -K_c x^p(t, 0, x_0^p, u)$, where K_c is the solution of a linear quadratic regulator problem in terms of the model (2.2a,b), and a ball $B_{LQR} \triangleq \{x \mid \|x\| \le \rho_{LQR}\}$, $\rho_{LQR} \in (0, \hat{\rho})$, such that if for some $t_{k'}$, $x_{k'}^p \in B_{LQR}$, then the control given by $u(t) = -K_c x(t, 0, x_0, u)$, for $t \ge t_{k'}$ does not violate the bound on the control on the resulting trajectory, i.e, $\|K_c x^p(t, 0, x_0, u)\| \le c_u$ for all $t \ge t_{k'}$. As we will see, a similar, but somewhat more complicated result also holds when $x^p(t, 0, x_0^p, u)$ is estimated using an asymptotic observer. Hence, in both cases, once the plant state is sufficiently near the origin, we can switch over to the LQR control law and thereby eliminate the residual errors resulting from the use of Control Algorithm 2.2.

For the case where the state can be measured, we propose to incorporate this idea into Control Algorithm 2.2 by modifying *Step 1*, as follows. Let $T_{K_c} \ge T_C$ is such that $\|e^{T_{K_c}(A-BK_c)}\| \le \alpha$.

Step 1': At $t = t_k$,

(a) measure the state $\tilde{x}_k = x^p(t_k)$.

(b) compute an estimate, $\hat{d}(t)$, of a disturbance $d(t)$ for $t \in [t_k, t_{k+1}]$, if possible; else, set $\hat{d}(t) = 0$.

(c) If $\tilde{x}_k \notin B_{LQR}$, set the plant input $u(t) = u_{[t_k, t_{k+1}]}(t) - \hat{d}(t)$ for $t \in [t_k, t_{k+1})$; else set $u(t) = -K_c x^p(t, 0, x_0^p, u) - \hat{d}(t)$ for $t \in [t_k, t_{k+1})$, where $t_{k+1} = t_k + T_{K_c}$ and $T_{K_c} \ge T_C$ such that $\|e^{(A-BK_c)T_{K_c}}\| \le \alpha$.

(d) compute an estimate x_{k+1} of the state of the plant $x^p(t_{k+1}, t_k, \bar{x}_k, u+d)$ according to the formula (2.6), i.e.,

$$x_{k+1} = e^{A(t_{k+1}-t_k)}\bar{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)} B\, u_{[t_k, t_{k+1}]}(t)\, dt.$$

At this point it becomes clear that for best results, the matrix Q, used to define the norm $\|\cdot\|$, should also define a Lyapunov function $\langle x, Qx\rangle$ for the system $\dot{x} = (A - BK_c)x$, so that for some positive definite matrix M we have

$$(A - BK_c)^T Q + Q(A - BK_c) = -M . \tag{3.10}$$

Theorem 3.5. Suppose that the matrix Q used to define the norm $\|\cdot\|$ satisfies (3.13) for some positive definite matrix M, and that the state of the plant can be measured. Let $\varepsilon_1 \in (0, \hat{\varepsilon}_1)$, $\varepsilon_2 \in (0, \hat{\varepsilon}_2)$, and $\delta \in (0, \lambda_{\min}(Q)/2\|M\|)$, where $\hat{\varepsilon}_1, \hat{\varepsilon}_2$ were defined in (3.2d), (3.2o), respectively, M is as in (3.10), and let ρ_s be defined by (3.2r), and ρ_{MH} by

$$\rho_{MH} \triangleq \frac{(1+\alpha+K)\varepsilon_2}{\varepsilon'} , \tag{3.11}$$

in terms of $\hat{\varepsilon}_1, \hat{\varepsilon}_2$ and ε' that was defined in (3.2g). Finally, suppose that $\rho_{MH} < \rho_{LQR}$, with $\rho_{LQR} > 0$, as above. If $\Delta_1 \le \varepsilon_1$, $\Delta_2 \le \varepsilon_2$, and $\|(A^p - A) - (B^p - B)K_c\| \le \delta$, then for any $x_0^p \in B_{\rho_s}$, with B_{ρ_s} defined in (3.2s), the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$ is bounded and, furthermore, $x^p(t, 0, x_0^p, u) \to 0$ as $t \to \infty$.

Proof. Since the conditions imposed in Lemma 3.1 and Theorem 3.2 are satisfied, it follows that for any $x_0^p \in B_{\rho_s}$, the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, determined by Algorithm 2.2, using the original *Step 1 (c)*, is well defined, bounded and $\overline{\lim}_{k\to\infty}\|x_k^p\| \le \rho_{MH}$. Since $\rho_{MH} < \rho_{LQR}$, there exists a finite $\hat{k} \in \mathbb{N}$, such that $\|x_{\hat{k}}^p\| \le \rho_{LQR}$, and hence that the cross over to the linear control law, specified in *Step 1' (c)* will take place. Let $V(x) \triangleq \langle x, Qx\rangle$. Hence, for $x^p(t)$ determined by the differential equation $\dot{x}^p(t) = (A^p - B^pK_c)x^p(t)$, $x^p(t_{\hat{k}}) = x_{\hat{k}}^p$, we obtain that for all $t \ge t_{\hat{k}}$,

$$\dot{V}(x^p(t)) = \langle \dot{x}^p(t), Qx^p(t)\rangle + \langle x^p(t), Q\dot{x}^p(t)\rangle$$

$$= \langle x^p(t)T, [(A - BK_c)^T Q + Q(A - BK_c)]x^p(t)\rangle$$

$$+ \langle x^p(t)[(A^p - A - (B^p - B)K_c]^T Q + Q(A^p - A - (B^p - B)K_c)]x^p(t)\rangle$$

$$= - \langle x^p(t), Mx^p(t) \rangle$$

$$+ \langle x^p(t), ((A^p - A - (B^p - B)K_c)^T Q + Q(A^p - A - (B^p - B)K_c))x^p(t) \rangle . \quad (3.12a)$$

Since $\|(A^p - A) - (B^p - B)K_c\| \le \delta$ it follows that for all $t \ge t_{\varepsilon}$,

$$\dot{V}(x^p(t)) < 0 , \quad (3.12b)$$

which implies that *(i)* $x^p(t) \in B_{LQR}$ for all $t \ge t_{\varepsilon}$, and *(ii)* that $x^p(t) \to 0$ as $t \to \infty$, completing our proof. □

When the state of the plant cannot be measured, we must augment our control system with an asymptotic state observer that provides the plant state estimate when we switch over to the linear feedback control law. The asymptotic observer must be in operation from time $t = 0$. In this case we get augmented dynamics in the well known observer-controller form

$$\dot{x}^p(t) = A^p x^p(t) - B^p K_c x(t) , \quad (3.13a)$$

$$\dot{x}^o(t) = K_o C^p x^p(t) + (A - BK_c - K_o C)x^o(t) , \quad (3.13b)$$

where K_o is the observer gain matrix. Let $e(t) \triangleq x^p(t) - x^o(t)$ denote the difference between the state of the plant and that of the model in the observer. Then

$$\dot{e}(t) = (A^p - K_o C^p)x^p(t) - (A - K_o C)x^o(t) - (B^p K_c - BK_c)x^o(t) . \quad (3.13c)$$

We assume that the system

$$\dot{\eta}(t) = \tilde{A}\,\eta(t) , \quad (3.13d)$$

where $\tilde{A} \triangleq diag((A - K_o C), \hat{A})$, with $\hat{A}$ is defined by

$$\hat{A} \triangleq \begin{bmatrix} A & -BK_c \\ K_o C & A - BK_c - K_o C \end{bmatrix} , \quad (3.13e)$$

corresponding to (3.13a,b,c) when there are no modeling errors, is exponentially stable, and hence that there exists a symmetric, positive definite matrix $\tilde{Q} = diag(Q_o, Q_c)$, with $Q_o \in \mathbb{R}^{n \times n}$ and $Q_c \in \mathbb{R}^{2n \times 2n}$ that defines a Lyapunov function, $\langle \eta, \tilde{Q}\eta \rangle$ for the system (3.13d), so that for some symmetric, positive definite matrix $\tilde{M} = diag(M_o, M_c)$, with $M_o \in \mathbb{R}^{n \times n}$ and $M_c \in \mathbb{R}^{2n \times 2n}$, we have

$$\tilde{A}^T\tilde{Q} + \tilde{Q}\tilde{A} = -\tilde{M} \ . \tag{3.13f}$$

We will now show that the system (3.13d) is robustly stable.

Lemma 3.6 Suppose that the state $(x^p(t), x^o(t))$ is defined by the observer-controller dynamics described by (3.13a,b), with $(x^p(0), x^o(0))$ arbitrary. Let $\delta \in (0, 0.5)$ and $\Delta\tilde{A}$ be defined by

$$\Delta\tilde{A} \triangleq \begin{bmatrix} 0 & \Delta A - K_o\Delta C & \Delta BK_c \\ 0 & \Delta A & -\Delta BK_c \\ 0 & K_o\Delta C & 0 \end{bmatrix}, \tag{3.14}$$

where $\Delta A = A^p - A$, $\Delta B = B^p - B$, and $\Delta C = C^p - C$. If $\|\Delta\tilde{A}\tilde{Q}\| < \delta\lambda_{\min}(\tilde{M})$, then $\overline{\lim}_{t\to\infty} \|x^p(t)\| = 0$ and $\overline{\lim}_{t\to\infty} \|x^o(t)\| = 0$.

Proof. Let $z(t) \triangleq (e(t), x^p(t), x^o(t))^T$, where $(x^p(t), x^o(t))$ is a solution of (3.13a,b) and $e(t) \triangleq x^p(t) - x^o(t)$. Then, referring to (3.13a,b,c) and (3.14), we see that $\dot{z}(t) = [\tilde{A} + \Delta\tilde{A}]z(t)$. Consider the Lyapunov function $V(z)$, for the nominal system (3.13d), defined by $V(\eta) \triangleq \langle \eta, \tilde{Q}\eta \rangle$. Then,

$$\begin{aligned} \dot{V}(z(t)) &= \langle \dot{z}(t), \tilde{Q}z(t) \rangle + \langle z(t), \tilde{Q}\dot{z}(t) \rangle \\ &= -\langle z(t), \tilde{M}z(t) \rangle + 2\langle z(t), \Delta\tilde{A}\tilde{Q}z(t) \rangle \\ &\le -\lambda_{\min}(\tilde{M})(1-2\delta)\|z(t)\|_2^2 \ . \end{aligned} \tag{3.15}$$

It follows immediately from the condition on δ that $\dot{V}(z(t)) < 0$, whenever $z(t) \ne 0$, which completes our proof. □

Lemma 3.7. Suppose that the state $(x^p(t), x^o(t))$ is defined by the observer-controller dynamics (3.13a,b), that $\|x^p(0)\| \le \varepsilon$, $\|x^o(0)\| \le \varepsilon$, for some $\varepsilon > 0$, and that $\Delta\tilde{A}$ satisfies the condition in Lemma 3.6. Then for all $t \ge 0$,

$$\|e(t)\| \le \left[2\lambda_{\max}(Q_o) \frac{\lambda_{\max}(Q_o) + \lambda_{\max}(\hat{Q})}{\lambda_{\min}(Q)^2} \right]^{1/2} \varepsilon \triangleq \gamma\varepsilon \ . \tag{3.16}$$

Proof. First, let $\|x\|_{Q_o} \triangleq \langle x, Q_o x \rangle^{1/2}$. Let the Lyapunov function $V(\cdot)$ be defined as in Lemma 3.6. Then it follows from the definition of $V(\cdot)$ and the fact that by Lemma 3.6, $\dot{V}(z(t)) < 0$, where $z(t) \triangleq (e(t), x^p(t), x^o(t))$, that $\|e(t)\|_{Q_o}^2 \le V(z(t)) \le V(z(0))$, for all $t \ge 0$. Hence,

$$|e(t)|^2_{\bar{Q}_o} \le \frac{\lambda_{\max}(Q_o)}{\lambda_{\min}(Q)}|e(0)|^2 + \frac{2\lambda_{\max}(\hat{Q})}{\lambda_{\min}(Q)}\varepsilon^2$$

$$\le 2\left[\frac{\lambda_{\max}(Q_o)+\lambda_{\max}(\hat{Q})}{\lambda_{\min}(Q)}\right]\varepsilon^2 . \tag{3.17a}$$

It now follows from (3.17a) that

$$|e(t)|^2 \le \frac{\lambda_{\max}(Q_o)}{\lambda_{\min}(Q)}|e(t)|^2_{\bar{Q}_o} \le 2\varepsilon^2\lambda_{\max}(Q_o)\frac{(\lambda_{\max}(Q_o)+\lambda_{\max}(\hat{Q})}{\lambda_{\min}(Q)^2} , \tag{3.17b}$$

which completes the proof. □

To include the use of an observer, we now propose to modify *Step 1 (c)* of Control Algorithm 2.2, as follows: Let $\delta \in (0, 0.5)$, let

$$\rho_{\hat{L}QR}^{o} \in \left[0 , \min\{\rho_{LQR} , \frac{\lambda_{\min}(M)\lambda_{\max}(Q)(1-2\delta)\rho_{LQR}}{2\gamma\lambda_{\min}(Q)(|K_c CQ| + \delta\lambda_{\min}(M))}\}\right] , \tag{3.18a}$$

where ρ_{LQR} was defined at the beginning of this subsection, and let $\varepsilon_1 , \varepsilon_2 > 0$ be such that

$$\varepsilon_1 < \hat{\varepsilon}_1 , \tag{3.18b}$$

$$\varepsilon_2 < \min\{\hat{\varepsilon}_2 , \frac{\rho_{\hat{L}QR}^{o}(1-\varepsilon_1}{2})\} , \tag{3.18c}$$

where K was defined in (3.1c), and $\hat{\varepsilon}_1 , \hat{\varepsilon}_2$ were defined in (3.2d), (3.2o), respectively. Finally, let $\rho_{oc} > 0$ be defined by

$$\rho_{oc} \triangleq (1-\varepsilon_1)\left[\rho_{\hat{L}QR}^{o} - \frac{\varepsilon_2}{1-\varepsilon_1}\right] . \tag{3.18d}$$

Then, it follows from (3.18c) that $\rho_{oc} > (1-\varepsilon_1)\left(\rho_{\hat{L}QR}^{o} - \rho_{\hat{L}QR}^{o}/2\right) > \varepsilon_2$. Let $T_{K_c} \in [T_C , \infty)$ be such that

$$e^{-\lambda_{\min}(\bar{M})(1-2\delta)T_{K}/\lambda_{\max}(\bar{Q})} \le \frac{\rho_{oc}^2\lambda_{\min}(Q)}{2\lambda_{\max}(\bar{Q})(\rho_{oc}^2 + (\rho_{\hat{L}QR}^{o})^2)} , \tag{3.18e}$$

$$|e^{(A-BK_c)T_K}| \le \alpha . \tag{3.18f}$$

Finally, we define the vector valued saturation function $SAT(u) \triangleq (sat(u^1), \ldots , sat(u^m))$, where

$sat(y) = y$ if $y \in [-c_u, c_u]$, and $sat(y) = c_u sgn(y)$ otherwise.

Step 1'': At $t = t_k$,

(a) If $u(t) = -K_c x^o(t)$ for $t \in [t_{k-1}, t_k)$ and $\max\{\|\tilde{x}_{k-1}\|, \|x_k\|\} \le \rho_{oc}$, set $\tilde{x}_k = x^o(t_k)$; else if $\max\{\|\tilde{x}_{k-1}\|, \|x_k\|\} \le \rho_{oc}$, set $\tilde{x}_k = x_k$ and reinitialize the observer by setting $x^o(t_k) = x_k$, else estimate the state $x_k^p = x^p(t_k)$ by (3.5) and denote the resulting value by $\tilde{x}_k$.

(b) compute an estimate, $\hat{d}(t)$, of a disturbance $d(t)$ for $t \in [t_k, t_{k+1}]$, if possible; else, set $\hat{d}(t) = 0$.

(c) If $\max\{\|\tilde{x}_{k-1}\|, \|x_k\|\} > \rho_{oc}$, set the plant input $u(t) = u_{[t_k, t_{k+1}]}(t) - \hat{d}(t)$ for $t \in [t_k, t_{k+1})$; else reset t_{k+1} to the new value $t_{k+1} = t_k + T_{K_c}$, and set $u(t) = -SAT(K_c x^0(t) - \hat{d}(t))$ for $t \in [t_k, t_{k+1})$.

(d) compute an estimate x_{k+1} of the state of the plant $x^p(t_{k+1}, t_k, \tilde{x}_k, u{+}d)$ according (2.6), i.e.,

$$x_{k+1} = e^{A(t_{k+1}-t_k)}\tilde{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)} B\,(u(t) + \hat{d}(t))\,dt\ .$$

Lemmas 3.6 and 3.7 lead us to a following result.

Theorem 3.8. Suppose that *(a)* $\Delta\tilde{A}$ satisfies the condition in Lemma 3.6, *(b)* $\|K_c \Delta C Q\| \le \delta\lambda_{\min}(M)$ $\Delta_5 \le \varepsilon_1$, $\Delta_6 \le \varepsilon_2$, $\rho_{MH} < (\rho_{oc} - \varepsilon_2)/(1 + \varepsilon_1)$, where ρ_{MH} was defined in (3.11), and *(c)* that we use *Step 1''* in Control Algorithm 2.2. Then for any $x_0^p \in B_{\rho_o}$, defined in (3.9d), the trajectory $x^p(t, 0, x_0^p, u)$ is bounded and, furthermore, $x^p(t, 0, x_0^p, u) \to 0$ *as* $t \to \infty$.

Proof. We will prove that for any trajectory $x^p(t, 0, x_0^p, u)$, with $x_0^p \in B_{\rho_o}$, there must exist a $\hat{k}$ such that the control $u(t)$ is defined by the solution of the optimal control problem $\mathbf{P}(x_k, t_k, 0)$ for all $t \in [0, t_{\hat{k}})$ and $\max\{\|\tilde{x}_{\hat{k}-1}\|, \|x_{\hat{k}}\|\} \le \rho_{oc}$, i.e., that the switch, in *Step 1'' (c)*, to the linear feedback control law $u(t) = -K_c x^o(t)$ (since $\hat{d}(t) \equiv 0$ by assumption), with $(x^p(t), x^o(t))$ the solution of (3.13a,b), from the initial state $(x^p(t_{\hat{k}}), x_{\hat{k}})$ at $t = t_{\hat{k}}$, will take place. Then we will show that *(a)* $x^0(t) \in B_{LQR}$ for all $t \ge t_{\hat{k}}$ so that the linear feedback control law does not violate the bound on the control, and *(b)* that $\max\{\|\tilde{x}_{k-1}\|, \|x_k\|\} \le \rho_{oc}$ must hold for all $k \ge \hat{k}$, so that the linear law is used for all $t \ge t_{\hat{k}}$. It will then follow from Lemma 3.6 that state of the plant will be driven to the origin as

$t \to \infty$.

First, it follows from (3.6a,b) that if the control $u(t) = u_{[t_k, t_{k+1})}(t)$ and the times t_k are determined by solving the optimal control problem $P(x_k, t_k, 0)$ for all $k \in \mathbb{N}$, then

$$\|\tilde{x}_{k-1}\| \le \|x^p_{k-1} - \tilde{x}_{k-1}\| + \|x^p_{k-1}\| \le (\Delta_3 + 1)\|x^p_{k-1}\| + \Delta_4 , \tag{3.19a}$$

$$\|x_k\| \le \|x^p_k - x_k\| + \|x^p_k\| \le \Delta_5\|x^p_{k-1}\| + \Delta_6 + \|x^p_k\| . \tag{3.19b}$$

Next, because $\Delta_3 \le \Delta_5 \le \varepsilon_1$, $\Delta_4 \le \Delta_6 \le \varepsilon_2$, and $\rho_{MH} < (\rho_{oc} - \varepsilon_2)/(1 + \varepsilon_1)$, since it follows from (3.2i,j) that $\varepsilon'' < \rho_{MH}$, we conclude that $\overline{\lim}_{k\to\infty} \|x^p_k\| \le \rho_{MH}$. Hence there exists a $\hat{k} \in \mathbb{N}$ such that $\|\tilde{x}_{\hat{k}-1}\| \le \rho_{oc}$ and $\|x_{\hat{k}}\| \le \rho_{oc}$. Hence a switch to the linear feedback control law will take place at the time $t_{\hat{k}}$.

Next, we will prove that $x^o(t) \in B_{LQR}$ for all $t \ge t_{\hat{k}}$, where $t_{\hat{k}}$ is the time when the switch to the linear control feedback control law takes place. Now, it follows (3.6a) and (3.7d,e,f) that

$$\|x^p_{\hat{k}-1}\| \le \|x^p_{\hat{k}-1} - \tilde{x}_{\hat{k}-1}\| + \|\tilde{x}_{\hat{k}-1}\| \le \varepsilon_1\|x^p_{\hat{k}-1}\| + \varepsilon_2 + \|\tilde{x}_{\hat{k}-1}\| . \tag{3.19c}$$

From (3.6b), we obtain that

$$\|x^p_{\hat{k}}\| \le \|x^p_{\hat{k}} - x_{\hat{k}}\| + \|x_{\hat{k}}\| \le \varepsilon_1\|x^p_{\hat{k}-1}\| + \varepsilon_2 + \|x_{\hat{k}}\| . \tag{3.19d}$$

It follows from (3.19c) that $\|x_{\hat{k}-1}\| \le \|\tilde{x}_{\hat{k}-1}\|/(1 - \varepsilon_1) + \varepsilon_2/(1 - \varepsilon_1)$. Hence it follows from (3.18d), (3.19d), and the fact that $\|\tilde{x}_{\hat{k}-1}\|$, $\|x_{\hat{k}}\| \le \rho_{oc}$ that

$$\|x^p_{\hat{k}}\| \le \frac{\varepsilon_1}{1 - \varepsilon_1}(\varepsilon_2 + \rho_{oc}) + \varepsilon_2 + \rho_{oc} = \rho^o_{LQR} . \tag{3.19e}$$

By *Step 1'' (a)*, we reinitialize the observer by setting $x^o(t_{\hat{k}}) = x_{\hat{k}}$ and hence $\|x^o(t_{\hat{k}})\| \le \rho_{oc} \le \rho^o_{LQR}$.

Now suppose that linear feedback control law is used for all $t \ge t_{\hat{k}}$. Then it follows from (3.19e) and Lemma 3.7 that $\|e(t)\| \le \gamma\rho^o_{LQR}$ for all $t \ge t_{\hat{k}}$. Next, let the Lyapunov function $V(\cdot)$ be defined by $V(x^o(t)) \triangleq \|x^o(t)\|^2 \triangleq \langle x^o(t), Qx^o(t)\rangle$. Then, making use of the matrix M defined by (3.11), we obtain that for all $t \ge t_{\hat{k}}$, with $(x^p(t), x^o(t))$ a solution of (3.13a,b) with initial states $(x^p(t_{\hat{k}}), x_{\hat{k}})$,

$$\dot{V}(x^o(t)) \le -\lambda_{\min}(M)|x^o(t)|_2^2 + 2|K_c \Delta C Q|\, |x^o(t)|_2^2$$

$$+ \Big[2|e(t)|(|K_c C Q| + |K_c \Delta C Q|)|x^o(t)|\Big]/\lambda_{\max}(Q)$$

$$\le -\lambda_{\min}(M)(1-2\delta)|x^o(t)|^2/\lambda_{\min}(Q) + 2\gamma\rho_{LQR}^{\ell}(|K_c C Q| + \delta\lambda_{\min}(M))|x^o(t)|/\lambda_{\max}(Q) \quad (3.19f)$$

It follows from (3.18a) that if $|x^o(t)| > \rho_{LQR}$, then $\dot{V}(x^o(t)) < 0$. Since $|x^o(t_{\hat{k}})| \le \rho_{oc} \le \rho_{LQR}$, it follows that $x^o(t) \in B_{LQR}$ for all $t \ge t_{\hat{k}}$. Therefore, if the linear feedback control law is used for all $t \ge t_{\hat{k}}$, then it does not violate the bound on the control.

We will now prove by induction that $|x_{k+1}|, |\tilde{x}_k| \le \rho_{oc}$, for all $k \ge \hat{k}$, where x_{k+1} is computed by (2.6) and $\tilde{x}_k = x^o(t_k)$, with $(x^p(t), x^o(t))$ a solution of (3.13a,b) from the initial state $(x^p(t_{\hat{k}}), x_{\hat{k}})$. For $(x^p(t), x^o(t), e(t))$ a solution of (3.13a,b,c), let $z(t) \triangleq (x^p(t), x^o(t), e(t))$, and let $|z(t)|_Q^2 \triangleq |e(t)|^2 + |x^p(t)|^2 + |x^o(t)|^2$. Recall that $|x_{\hat{k}}|, |\tilde{x}_{\hat{k}-1}| \le \rho_{oc}$, and that $x^o(t_{\hat{k}}) = x_{\hat{k}}$, and that $|x_{\hat{k}}^p| \le \rho_{LQR}^{\ell}$ by (3.19e). Now suppose that for some $k \ge \hat{k}+1$, we have that $|x_k|, |\tilde{x}_{k-1}|, |x^o(t_k)| \le \rho_{oc}$, and $|x_k^p| \le \rho_{LQR}^{\ell}$ hold, and that $u(t) = -K_c x^o(t)$ for $t \in [t_{k-1}, t_k)$. We need to prove that $|x_{k+1}|, |\tilde{x}_k|, |x^o(t_{k+1})| \le \rho_{oc}$ and that $|x_{k+1}^p| \le \rho_{LQR}^{\ell}$. Now, since $u(t) = -K_c x^o(t)$ for $t \in [t_{k-1}, t_k)$, we set $\tilde{x}_k = x^o(t_k)$ by *Step 1'' (a)*. Therefore, $|\tilde{x}_k| = |x^o(t_k)| \le \rho_{oc}$ by assumption. Next, we must have that $|x_{k+1}| \le \alpha\rho_{oc}$ because $|e^{(A-BK_c)T_k}| \le \alpha$. We will now prove that the relations $|x^o(t_{k+1})| \le \rho_{oc}$ and $|x_{k+1}^p| \le \rho_{LQR}^{\ell}$, both hold.

Let $\tilde{V}(z(t)) = \langle z(t), \tilde{Q} z(t)\rangle$. Then,

$$\tilde{V}(z(t)) \ge \lambda_{\min}(\tilde{Q})|z(t)|_2^2 \ge (\lambda_{\min}(\tilde{Q})/\lambda_{\max}(Q))|z(t)|_Q^2 \ge (\lambda_{\min}(\tilde{Q})/\lambda_{\max}(Q))|x^o(t)|^2 . \quad (3.19g)$$

It follows from (3.15) and the fact that $\tilde{V}(z(t)) \le \lambda_{\max}(\tilde{Q})|z(t)|_2^2$ that for $t \in [t_k, t_{k+1})$,

$$\frac{d}{dt}\tilde{V}(z(t)) \le -\lambda_{\min}(\tilde{M})(1-2\delta)|z(t)|_2^2 \le -(\lambda_{\min}(\tilde{M})/\lambda_{\max}(\tilde{Q}))(1-2\delta)\tilde{V}(z(t)) . \quad (3.19h)$$

Clearly, $\tilde{V}(z(t_k)) \le \lambda_{\max}(\tilde{Q})|z(t_k)|_2^2 \le \lambda_{\max}(\tilde{Q})|z(t_k)|_Q^2/\lambda_{\min}(Q)$. Hence, because *(i)* $|e(t_k)|^2 \le |x^o(t_k)|^2 + |x^p(t_k)|^2$, *(ii)* $|x^o(t_k)| \le \rho_{oc}$ and $|x^p(t_k)| = |x_k^p| \le \rho_{LQR}^{\ell}$ by assumption, and *(iii)* $|z(t_k)|_Q^2 \le 2(|x^o(t_k)|^2 + |x^p(t_k)|^2)$, it follows from (3.19h) that for all $t \in [t_k, t_{k+1})$,

$$\tilde{V}(z(t)) \leq e^{-\lambda_{min}(\tilde{M})(1-2\delta)(t-t_k)/\lambda_{max}(\tilde{Q})} \tilde{V}(z(t_k))$$

$$\leq \frac{2\lambda_{max}(\tilde{Q})}{\lambda_{min}(Q)} e^{-\lambda_{min}(\tilde{M})(1-2\delta)(t-t_k)/\lambda_{max}(\tilde{Q})} \left[\rho_{oc}^2 + (\rho \ell_{QR})^2\right]. \tag{3.19i}$$

Since by the triangle inequality, $V(z(t)) \geq 2\|x^o(t)\|^2$ for all $t \in [t_k, t_{k+1})$, it follows from (3.18e) and (3.19g,i) that

$$\|x^o(t_k + T_{K_c})\|^2 \triangleq \|x^o(t_{k+1})\|^2 \leq \rho_{oc}^2/2. \tag{3.19j}$$

Therefore, $\|x^o(t_{k+1})\| \leq \rho_{oc}/\sqrt{2}$. Now, (3.19g) holds when we replace $x^o(t)$ by $x^p(t)$ because $\|z(t)\|_{\tilde{Q}}^2 \geq \|x^p(t)\|^2$. Then, again it follows from (3.19h,i) that $\|x^p(t_k + T_{K_c})\|^2 = \|x_{k+1}^p\|^2 \leq \rho_{oc}^2 \leq (\rho \ell_{QR}^o)^2$, which completes our proof by induction. It therefore follows that the Control Algorithm 2.2 selects the feedback control law $u(t) = -SAT(K_c x^o(t))$, for the next interval, $t \in [t_{k+1}, t_{k+1} + T_{K_c}]$, where $t_{k+1} = t_k + T_{K_c}$, and since we have already shown that, in this case, the control $u(t) = -K_c x^o(t)$ does not violate the control constraint, it follows that $u(t) = -K_c x^o(t)$, for the next interval, $t \in [t_{k+1}, t_{k+1} + T_{K_c}]$, and hence, by induction, for all $t \geq t_{\hat{k}}$. It now follows from Lemma 3.6 that $\|x^p(t)\| \to 0$ and $\|x^o(t)\| \to 0$ as $t \to \infty$, which completes our proof. □

4. DISTURBANCE REJECTION.

We will consider two distinct situations. The first is where the disturbance $d(t)$ is a continuous function, such that for some $c_d < \infty$, $\left[\int_t^{t+T} \|d(\tau)\|^2 d\tau\right]^{1/2} \leq c_d$ for all $t \geq 0$. The second is where the disturbance is the output of a known dynamical system driven by stationary, zero mean, white noise.

We begin with the first case and assume that the disturbance $d(t)$ cannot be estimated. Hence Control Algorithm 2.2 sets $\hat{d}(t) \equiv 0$. Since the more difficult situation occurs when the plant state is estimated, we will assume that this is the case. First, we derive a result similar to Lemma 3.3.

Lemma 4.1. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with state estimation formula (3.5). There exist $\Delta_i < \infty$, $i = 7, 8, 9, 10$, such that if Control Algorithm 2.2 constructs the sequences $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{\bar{x}_k\}_{k=0}^{\infty}$ is the corresponding sequence of the estimates of x_k^p, defined by (3.5), then for all $k \in \mathbb{N}$,

$$|x_k^p - \bar{x}_k| \le \Delta_7 |x_k^p| + \Delta_8 , \tag{4.1a}$$

$$|x_{k+1}^p - x_{k+1}| \le \Delta_9 |x_k^p| + \Delta_{10} . \tag{4.1b}$$

Furthermore, when there are no modeling errors and no disturbances, $\Delta_i = 0, i = 7, 8, 9, 10$.

Proof. Suppose that $u(\cdot)$ is the control generated by Control Algorithm 2.2 for the plant and model trajectories associated with the sequences $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{\bar{x}_k\}_{k=0}^{\infty}$.

We begin with (4.1a). For any $k \in \mathbb{N}$ and any $t \in [t_k, t_{k+1}]$, $y^p(t)$ is given by

$$y^p(t) = C^p e^{A^p(t-t_k)} x_k^p + C^p \int_{t_k}^{t} e^{A^p(t-\tau)} B^p (u(\tau) + d(\tau))\, d\tau$$

$$= Ce^{A(t-t_k)} x_k^p + \{ C^p e^{A^p(t-t_k)} - Ce^{A(t-t_k)} \}\, x_k^p$$

$$+ C \int_{t_k}^{t} e^{A(t-\tau)} B (u(\tau) + d(\tau))\, d\tau + \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B^p - Ce^{A(t-\tau)} B \}\, (u(\tau) + d(\tau))\, d\tau . \tag{4.2a}$$

By substituting (4.2a) into (3.5), we obtain

$$\bar{x}_k = x_k^p + W_o(\delta_0(t_{k+1} - t_k))^{-1} \Bigg\{ \int_{t_k}^{t_k'} (Ce^{A(t-t_k)})^T \{ C^p e^{A^p(t-t_k)} - Ce^{A(t-t_k)} \}\, dt\, x_k^p$$

$$+ \int_{t_k}^{t_k'} (Ce^{A(t-t_k)})^T \int_{t_k}^{t} Ce^{A(t-\tau)} B d(\tau)\, d\tau$$

$$+ \int_{t_k}^{t_k'} (Ce^{A(t-t_k)})^T \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B^p - Ce^{A(t-\tau)} B \} (u(\tau) + d(\tau))\, d\tau\, dt \Bigg\} . \tag{4.2b}$$

It follows directly from (4.2b) that

$$|x_k^p - \bar{x}_k| \le \Delta_7 |x_k^p| + \Delta_8 , \tag{4.2c}$$

where

$$\Delta_7 \triangleq C_\Delta \max_{t \in [0, \delta_0 \bar{T}]} |C^p e^{A^p(t-t_k)} - Ce^{A(t-t_k)}|\, \delta_0 \bar{T} \tag{4.2d}$$

$$\Delta_8 \triangleq C_\Delta \left[\max_{t \in [0, \delta_0 \bar{T}]} |C^p e^{A^p(t-\tau)} B^p - Ce^{A(t-\tau)} B| (c_u \delta_0 \bar{T} + c_d) + \max_{t \in [0, \delta_0 \bar{T}]} |Ce^{At}| \delta_0 \bar{T} c_d \right] , \tag{4.2e}$$

with $C_\Delta \triangleq \max_{t \in [T_c, \bar{T}]} |W_o(\delta_0 t)^{-1}| \max_{t \in [0, \delta_0 \bar{T}]} |Ce^{At}|$, which proves (4.1a). Clearly, when there are no modeling errors and no disturbances, $\Delta_7 = \Delta_8 = 0$.

Next we will establish (4.1b). Since x_{k+1} is calculated using the estimated initial state $\bar{x}_k$, it follows from the Schwartz inequality in $L_2[0, \bar{T}]$ (i.e., $\int_0^T a(t)b(t)\,dt \le \left[\int_0^T a(t)^2 dt\right]^{1/2} \left[\int_0^T b(t)^2 dt\right]^{1/2}$) that

$$\|x_{k+1}^p - x_{k+1}\| = \| e^{A^p(t_{k+1}-t_k)} x_k^p - e^{A(t_{k+1}-t_k)} \bar{x}_k$$

$$+ \int_{t_k}^{t_{k+1}} \{ e^{A^p(t_{k+1}-\tau)} B^p - e^{A(t_{k+1}-\tau)} B \} u(\tau)\, d\tau + \int_{t_k}^{t_{k+1}} e^{A^p(t_{k+1}-t)} B^p d(\tau)\, d\tau \|$$

$$\le K \, \|x_k^p - \bar{x}_k\| + \Delta_1 \|x_k^p\| + \Delta_2$$

$$+ \int_{t_k}^{t_{k+1}} \| e^{A^p(t_{k+1}-t)} B^p - e^{A(t_{k+1}-t)} B \| \|d(t)\| dt + \int_{t_k}^{t_{k+1}} \| e^{A(t_{k+1}-t)} B \| \|d(t)\| dt$$

$$\le K \; \{ \Delta_7 \|x_k^p\| + \Delta_8 \} + \Delta_1 \|x_k^p\| + \Delta_2 + \left[\frac{\Delta_2}{c_u \sqrt{\bar{T}}} + K \|B\| \sqrt{\bar{T}} \right] c_d$$

$$= (K \Delta_7 + \Delta_1) \|x_k^p\| + K \Delta_8 + \Delta_2 + \left[\frac{\Delta_2}{c_u \sqrt{\bar{T}}} + K \|B\| \sqrt{\bar{T}} \right] c_d \triangleq \Delta_9 \|x_k^p\| + \Delta_{10}, \tag{4.2f}$$

where K, Δ_1, and Δ_2 were defined in (3.1a,b,c). Hence (4.1b) holds, and our proof is complete. □

Lemma 4.1 leads to the following result that also holds when the state is measured.

Theorem 4.2. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with state estimation as in (3.5). Suppose that $\varepsilon_1, \varepsilon_2 > 0$ are such that

$$\Delta_9 \le \varepsilon_1 < \frac{1-\alpha}{1+\alpha+K}, \tag{4.3a}$$

$$\Delta_{10} \le \varepsilon_2 < \frac{\hat{\rho}}{3 + (2+K)/\varepsilon'}, \tag{4.3b}$$

where Δ_9, Δ_{10} were defined in (4.2f), and ε' was defined in (3.2g). Then there exists a $\rho_d \in (0, \hat{\rho}]$, such that for all $x_0^p \in B_{\rho_d}$, the trajectory $x^p(t, 0, x_0^p, u+d)$, $t \in [0, \infty)$, is bounded, and there exists an $\varepsilon_3 > 0$ such that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, and $\overline{\lim}_{t \to \infty} \|x^p(t, 0, x_0^p, u+d)\| \le \varepsilon_3$.

Proof. First suppose that the optimal control problem $\mathrm{P}(x_{k+1}, t_{k+1}, 0)$, has a solution for any $x_{k+1} \in \mathbb{R}^n$ and $t_{k+1} \ge 0$. Then, given any initial state x_0^p at time $t_0 = 0$, the dynamics of the moving horizon feedback system, using Control Algorithm 2.2, generate the sequence of states $\{x_k^p\}_{k=0}^{\infty}$, while Control Algorithm 2.2 generates the sequence of estimates $\{x_k\}_{k=1}^{\infty}$, with $x_{k+1} = x(t_{k+1}, t_k, \bar{x}_k, u)$, $k = 1, 2, \ldots$, according to (2.6), and the sequence $\{x'_k\}_{k=2}^{\infty}$, with

$x'_{k+2} = x(t_{k+2}, t_{k+1}, x_{k+1}, u)$, $k = 1, 2, \ldots$, generated in the process of solving the optimal control problem $\mathbf{P}(x_{k+1}, t_{k+1}, 0)$, $k \in \mathbb{N}$.

Now, for any $k \in \mathbb{N}$,

$$\|x^p_{k+2}\| \le \|x^p_{k+2} - x'_{k+2}\| + \|x'_{k+2}\| \le \|x^p_{k+2} - x'_{k+2}\| + \alpha\|x_{k+1}\|$$

$$\le K\|x^p_{k+1} - x_{k+1}\| + \Delta_9\|x^p_{k+1}\| + \Delta_{10} + \alpha\|x^p_{k+1} - x_{k+1}\| + \alpha\|x^p_{k+1}\|$$

$$\le (\Delta_9 + \alpha)\|x^p_{k+1}\| + (K + \alpha)\Delta_9\|x^p_k\| + (1 + \alpha + K)\Delta_{10}. \tag{4.4a}$$

If we let $a_1 = \Delta_9 + \alpha$, $a_2 = (K + \alpha)\Delta_9$, and $b = (1 + \alpha + K)\Delta_{10}$, then, in view of (4.3) we see that $a_1, a_2, b \ge 0$ and $a_1 + a_2 < 1$, so that the assumptions of Proposition 8.1 are satisfied. Hence, if we let $y_0 = \|x^p_0\|$ and $y_1 = \|x^p_1\|$, then it follows from (4.4a) that for y_k defined by (8.1a), $\|x^p_k\| \le y_k$, and hence (c.f. (3.2i)) that

$$\overline{\lim_{k \to \infty}} \|x^p_k\| \le \frac{(1 + \alpha + K)\Delta_{10}}{\varepsilon'} \triangleq \varepsilon''', \tag{4.4b}$$

and also that for all $k \in \mathbb{N}$,

$$\|x^p_k\| \le y_k \le (K + \alpha)\Delta_9\|x^p_0\| + \|x^p_1\| + \varepsilon'''. \tag{4.4c}$$

Since $u(t) = 0$, for all $t \in [0, t_1)$, for $k = 0$, (4.1b) reduces to

$$\|x^p_1\| \le \|x^p_1 - x_1\| + \|x_1\| \le \Delta_9\|x^p_0\| + \Delta_{10} + K\|x^p_0\| = (K + \Delta_9)\|x^p_0\| + \Delta_{10}. \tag{4.4d}$$

It then follows from (4.4c,d) that for all $k \ge 2$,

$$\|x^p_k\| \le ((1 + \alpha + K)\Delta_9 + K)\|x^p_0\| + \Delta_{10} + \varepsilon'''. \tag{4.4e}$$

Next, making use of (4.4c), we obtain that for all $k \in \mathbb{N}$,

$$\|x_{k+1}\| \le \|x^p_{k+1} - x_{k+1}\| + \|x^p_{k+1}\| \le \Delta_9\|x^p_k\| + \Delta_{10} + \|x^p_{k+1}\| \le \Delta_9 y_k + \Delta_{10} + y_{k+1}$$

$$\le (1 + \Delta_9)(K + (1 + \alpha + K)\Delta_9)\|x^p_0\| + (1 + \Delta_9)\Delta_{10} + \Delta_{10}$$

$$\triangleq \gamma'_1\|x^p_0\| + \gamma'_2. \tag{4.4f}$$

Since by (4.3), $(1 - \alpha)(1 + \alpha + K) < 1$, it follows that $1 + \Delta_9 < 2$, and hence it follows that

$$\gamma'_2 = (1 + \Delta_9)\Delta_{10} + \Delta_{10} < 2\Delta_{10} + \left[1 + \frac{1 - \alpha}{1 + \alpha + K}\right] \frac{(1 + \alpha + K)}{\varepsilon'} \Delta_{10} + \Delta_{10}$$

$$\le (3 + (2 + K)/\varepsilon')\Delta_{10} \le (3 + (2 + K)/\varepsilon')\varepsilon_2 \triangleq \hat{\gamma}'_2 < \hat{\rho} \; . \tag{4.4g}$$

Let $B_{\rho_d} \triangleq \{ x \in D \mid \|x\| \le \rho_d \}$ where, with $\hat{\gamma}_1$ as in (3.2q), ρ_d is defined by

$$\rho_d \triangleq (\hat{\rho} - \hat{\gamma}'_2)/\hat{\gamma}_1 \, . \tag{4.4h}$$

Because ε_2 satisfies (4.3), $\rho_d > 0$. Furthermore, we conclude that for any $x_0^p \in B_{\rho_d}$, for all $k \in \mathbb{N}$,

$$\|x_{k+1}\| \le \gamma'_1 \|x_0^p\| + \gamma'_2 \le \hat{\gamma}'_1 \rho_d + \hat{\gamma}'_2 \le \hat{\rho} \; .$$

$x_k \in D$ for all $k \ge 1$.

It now follows from Proposition 8.1 that $\overline{\lim}_{t \to \infty} \|x^p(t, 0, x_0^p, u)\| \le \varepsilon_3$, where ε_3 is defined by (3.9c) (with ε_1, ε_2 as in this theorem). It is again obvious (3.9c) that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, which completes our proof. □

We will now show that when the disturbances are of sufficiently small amplitude, we can still use Control Algorithm 2.2 with *Step 1'* (with state measurement) or *Step 1''* (with state estimation), to obtain the benefit of the disturbance suppression properties of LQR systems. These depend on the largest real part of the eigenvalues $\lambda_j(A - BK_c)$ of the matrix $A - BK_c$. Hence a design trade-off is implied: the smaller the largest real part of the eigenvalues, the better is the disturbance suppression. However, to obtain a very negative largest real part may require large elements in K_c, which limits the size of the ball about the origin where the control $u(t) = -K_c x(t)$ will not violate the control constraint.

Thus, suppose that K_c is the gain matrix resulting from the solution of an LQR problem for the model (2.2a) and that K_o is the gain matrix for a corresponding asymptotic state estimator for (2.2a). Assuming that we use the control determined by the gain K_c and the asymptotic state estimator determined by the gain K_o, we get the following augmented dynamics in the well known observer-controller form

$$\dot{x}^p(t) = A^p x^p(t) - B^p K_c x(t) + B^p d(t), \tag{4.5a}$$

$$\dot{x}^o(t) = K_o C^p x^p(t) + (A - BK_c - K_o C) x^o(t) \, . \tag{4.5b}$$

We will assume that there exists a constant $c'_d < \infty$ such that $\|d(t)\| \le c'_d$ for all $t \ge 0$, and that both c'_d and the modeling errors are sufficiently small to ensure the existence of a ball $B_{LQR} \triangleq \{ x \in \mathbb{R}^n \mid \|x\| \le \rho_{LQR} \}$, $\rho_{LQR} > 0$, such that if for some $t_{\hat{k}}$, $x^o(t_{\hat{k}}) \in B_{LQR}$, then the control given by $u(t) = -K_c x^o(t)$ for all $t \geq t_{\hat{k}}$, with $(x^p(t), x^o(t))$ determined by (4.5a,b), does not violate the bound on the control.

Let $e(t) \triangleq x^p(t) - x(t)$ denote the difference between the state of the plant and that of the model. Then

$$\dot{e}(t) = (A^p - K_o C^p)x^p(t) - (A - K_o C)x^o(t) - (B^p K_c - BK_c)x^o(t) + B^p d(t). \tag{4.5c}$$

We will assume from now on that the system

$$\dot{\eta}(t) = \tilde{A}\,\eta(t), \tag{4.5d}$$

where $\tilde{A} \triangleq diag((A - K_o C), \hat{A})$, with $\hat{A}$ is defined by

$$\hat{A} \triangleq \begin{bmatrix} A & -BK_c \\ K_o C & A - BK_c - K_o C \end{bmatrix}, \tag{4.5e}$$

corresponding to (4.5a,b,c) when there are no modeling errors and no disturbances, is exponentially stable, and hence that there exists a symmetric, positive definite matrix $\tilde{Q} = diag(Q_o, Q_c)$, with $Q_o \in \mathbb{R}^{n \times n}$ and $Q_c \in \mathbb{R}^{2n \times 2n}$ that defines a Lyapunov function, $\langle \eta, \tilde{Q}\eta \rangle$ for the system (4.5d), so that for some symmetric, positive definite matrix $\tilde{M} = diag(M_o, M_c)$, with $M_o \in \mathbb{R}^{n \times n}$ and $M_c \in \mathbb{R}^{2n \times 2n}$, we have

$$\tilde{A}^T \tilde{Q} + \tilde{Q}\tilde{A} = -\tilde{M}. \tag{4.5f}$$

We will now show for the observer-controller dynamics that when $\|e(0)\|$ and $\|d\|_\infty$ are sufficiently small, $\|e(t)\|$ remains small for all $t \geq 0$.

Lemma 4.3. Suppose that the state $(x^p(t), x^o(t))$ is defined by the observer-controller dynamics described by (4.5a,b), with $(x^p(0), x^o(0))$ arbitrary and let $z(t) \triangleq (e(t), x^p(t), x^o(t))^T$. Let $\Delta\tilde{A}$, $\Delta\tilde{B}$ be defined by

$$\Delta\tilde{A} \triangleq \begin{bmatrix} 0 & \Delta A - K_o \Delta C & \Delta BK_c \\ 0 & \Delta A & -\Delta BK_c \\ 0 & K_o \Delta C & 0 \end{bmatrix}, \tag{4.6a}$$

$$\Delta\tilde{B}^T = [\Delta B^T, \Delta B^T, 0], \tag{4.6b}$$

where $\Delta A = A^p - A$, $\Delta B = B^p - B$, and $\Delta C = C^p - C$.

If there exists a $\delta \in (0, 0.5)$ such that *(a)* $\|\Delta\tilde{A}\,\tilde{Q}\| < \delta\lambda_{\min}(\tilde{M})$, *(b)*

$$\|d\|_\infty < \frac{\lambda_{\min}(M)\lambda_{\min}(\tilde{M})(1-2\delta)^2\rho_{LQR}\lambda_{\min}(Q)}{4\sqrt{m}\,\lambda_{\max}(Q)\lambda_{\max}(\tilde{Q})(\|K_cCQ\| + \delta\lambda_{\min}(M))\|\tilde{B}^T\tilde{Q}\| + \delta\lambda_{\min}(\tilde{M}))}\,, \tag{4.6c}$$

where $\tilde{B}^T = [B^T, B^T, 0]$ and Q and M were defined in (3.11), *(c)* $\|\Delta\tilde{B}^T\tilde{Q}\| \le \delta\lambda_{\min}(\tilde{M})$, and *(d)*

$$\|z(0)\| \le \frac{\lambda_{\min}(M)(1-2\delta)\rho_{LQR}\lambda_{\min}(Q)^{1/2}}{2\lambda_{\max}(Q)^{1/2}(\|K_cCQ\| + \delta\lambda_{\min}(M))} \triangleq \gamma_e\,, \tag{4.6d}$$

where $\|z(t)\| \triangleq \|e(t)\| + \|x^p(t)\| + \|x^o(t)\|$ and $\|z(t)\|^2 \triangleq \|e(t)\|^2 + \|x^p(t)\|^2 + \|x^o(t)\|^2$, with $\|x\| = \langle x, Qx\rangle^{1/2}$, then $\|e(t)\|, \|x^p(t)\| \le \gamma_e$ for all $t \ge 0$.

Proof. Referring to (4.5a,b,c) and (4.6a,b), we see that $\dot{z}(t) = [\tilde{A} + \Delta\tilde{A}]z(t) + [\tilde{B} + \Delta\tilde{B}]d(t)$. Consider the Lyapunov function $V(z)$, for the nominal system (4.5d), defined by $V(\eta) \triangleq \langle\eta, \tilde{Q}\eta\rangle$. Then,

$$\dot{V}(z(t)) = \langle\dot{z}(t), \tilde{Q}z(t)\rangle + \langle z(t), \tilde{Q}\dot{z}(t)\rangle$$

$$= -\langle z(t), \tilde{M}z(t)\rangle + 2\langle z(t), \Delta\tilde{A}\,\tilde{Q}z(t)\rangle + 2\langle(\tilde{B} + \Delta\tilde{B})d(t), \tilde{Q}z(t)\rangle$$

$$\le -\lambda_{\min}(\tilde{M})\|z(t)\|_2^2 + 2\|\Delta\tilde{A}\,\tilde{Q}\|\|z(t)\|_2^2 + 2\|d(t)\|_2(\|\tilde{B}^T\tilde{Q}\| + \|\Delta\tilde{B}^T\tilde{Q}\|)\|z(t)\|_2$$

$$\le (-\lambda_{\min}(\tilde{M}) + 2\delta\lambda_{\min}(\tilde{M}))\|z(t)\|_2^2 + 2\sqrt{m}\,\|d\|_\infty(\|\tilde{B}^T\tilde{Q}\| + \delta\lambda_{\min}(\tilde{M}))\|z(t)\|_2$$

$$\le -\frac{\lambda_{\min}(\tilde{M})(1-2\delta)V(z(t))^{1/2}\|z(t)\|_2}{\lambda_{\max}(\tilde{Q})^{1/2}} + 2\sqrt{m}\,\|d\|_\infty(\|\tilde{B}^T\tilde{Q}\| + \delta\lambda_{\min}(\tilde{M}))\|z(t)\|_2\,. \tag{4.7a}$$

The last inequality is obtained by $\|z(t)\|_2 \ge V(z(t))^{1/2}/\lambda_{\max}(\tilde{Q})^{1/2}$. Now, it follows from (4.6c) that

$$\dot{V}(z(t)) \le \left[\frac{-\lambda_{\min}(\tilde{M})(1-2\delta)V(z(t))^{1/2}}{\lambda_{\max}(\tilde{Q})} + \frac{\lambda_{\min}(M)\lambda_{\min}(\tilde{M})(1-2\delta)^2\rho_{LQR}\lambda_{\min}(Q)}{2\lambda_{\max}(Q)^{1/2}\lambda_{\max}(\tilde{Q})(\|K_cCQ\| + \delta\lambda_{\min}(M))}\right]\|z(t)\|_2\,. \tag{4.7b}$$

We can see from (4.7b) that if $V(z(t))^{1/2} > \gamma_e\lambda_{\min}(Q)^{1/2}/\lambda_{\max}(\tilde{Q})^{1/2}$ then $\dot{V}(z(t)) < 0$. Since $\gamma_e^2 \ge \|z(0)\|^2 \ge V(z(0))\lambda_{\min}(Q)/\lambda_{\max}(\tilde{Q})$, $V(z(t)) \le \gamma_e^2\lambda_{\min}(Q)/\lambda_{\max}(\tilde{Q})$ for all $t \ge 0$. Since $V(z(t)) \ge \lambda_{\min}(Q)\|z(t)\|^2/\lambda_{\max}(\tilde{Q}) \ge \lambda_{\min}(Q)\|e(t)\|^2/\lambda_{\max}(\tilde{Q})$, we obtain that $\|e(t)\| \le \gamma_e$, which establishes the first inequality. Since $\|x^p(t)\|^2 \le \|z(t)\|^2$ also holds, we see from the above that the second inequality also holds, which completes our proof. □

It is worth noting that (4.7a) implies that $\|z(t)\| \to 0$ as $\|d\|_\infty \to 0$, and hence that $\|x^o(t)\|, \|x^p(t)\| \to 0$ as $\|d\|_\infty \to 0$.

Now, let

$$\rho^{o}_{LQR} \triangleq \min\{\gamma_e/4, \rho_{LQR}/4\}, \tag{4.8a}$$

$$\Delta_9 \le \varepsilon_1 < \frac{1-\alpha}{1+\alpha+K} \tag{4.8b}$$

$$\Delta_{10} \le \varepsilon_2 < \min\left[\frac{\hat{\rho}}{3+(2+K)/\varepsilon'}, \frac{\rho^{o}_{LQR}(1-\varepsilon_1)}{2}\right], \tag{4.8c}$$

where K and ε' were defined in (3.1c) and (3.2g), respectively. Then, it follows from (4.4b) that

$$\overline{\lim_{k\to\infty}} \, \|x_k\| \le \frac{(1+\alpha+K)\Delta_{10}}{\varepsilon'} \le \frac{(1+\alpha+K)\varepsilon_2}{\varepsilon'} \triangleq \rho_{MH}. \tag{4.8d}$$

Let

$$\rho_{oc} \triangleq (1-\varepsilon_1)\left[\rho^{o}_{LQR} - \varepsilon_2/(1-\varepsilon_1)\right]. \tag{4.8e}$$

Then, it follows from (4.8a,c) that

$$\rho_{oc} = \rho^{o}_{LQR}(1-\varepsilon_1) - \varepsilon_2 > \rho^{o}_{LQR}(1-\varepsilon_1)/2 > \varepsilon_2. \tag{4.8f}$$

Theorem 4.4. Suppose that *(a)* δ, $\Delta\tilde{A}$, $\Delta\tilde{B}$, $\|d\|_\infty$ satisfy the conditions in Lemma 4.3, *(b)* $\|K_c \Delta C Q\| \le \delta\lambda_{\min}(M)$, *(c)* that (4.8b,c) holds, *(d)* that $\rho_{MH} < (\rho_{oc} - \varepsilon_2)/(1+\varepsilon_1)$, where ρ_{MH} was defined in (4.8d), *(e)* that $\gamma_e \le \rho_d$, where γ_e and ρ_d were defined in (4.6d) and (4.4h), respectively, and *(f)* that we use *Step 1''* in Control Algorithm 2.2. Then for any $x_0^p \in B_{\rho_d}$, defined in (4.4h) (using the formulae (3.2q), (4.4g)), the trajectory $x^p(t, 0, x_0^p, u+d)$ is bounded and, furthermore, $\overline{\lim}_{t\to\infty} x^p(t, 0, x_0^p, u+d) \to 0$ as $\|d\|_\infty \to 0$.

Proof. We will prove that for any trajectory $x^p(t, 0, x_0^p, u)$, with $x_0^p \in B_{\rho_d}$, there must exist a $\hat{k}$ such that for all $t \in [0, t_{\hat{k}})$, the control $u(t)$ is defined by the solution of the optimal control problem $P(x_k, t_k, 0)$ and $\max\{\|\tilde{x}_{\hat{k}-1}\|, \|x_{\hat{k}}\|\} \le \rho_{oc}$, i.e., that the switch will take place in *Step 1'' (c)* to the linear feedback control law $u(t) = -K_c x^o(t)$, with $(x^p(t), x^o(t))$ the solution of (4.5a,b), from the initial state $(x^p(t_{\hat{k}}), x^o(t_{\hat{k}}))$ at $t = t_{\hat{k}}$. Then we will show that if the linear feedback control law $u(t) = -K_c x^o(t)$ is used for $t \in [t_{\hat{k}}, T_{oc}]$ with $T_{oc} \ge t_{\hat{k}+1}$, $\|x^o(t)\| \le \rho_{LQR}$ holds for all $t \in [t_{\hat{k}}, T_{oc}]$, so that the linear feedback control law does not violate the bound on the control. Then, we will consider two possibilities: *(a)* only one switch to the linear feedback control law takes place (at $t_{\hat{k}}$), , i.e., $\max\{\,|\tilde{x}_{k-1}|\,,\,|x_k|\,\} \le \rho_{oc}$ for all $k \ge \hat{k}$ so that $u(t) = -K_c x^o(t)$ for all $t \ge t_{\hat{k}}$,

and *(b)* the condition $\max\{\,|\tilde{x}_{k-1}|\,,\,|x_k|\,\} \le \rho_{oc}$ fails for some $k \ge \hat{k}$ and the Control Algorithm 2.2 switches back to the solution of the optimal control problem $\mathbf{P}(x_k\,,t_k\,,0)$ which implies that the linear feedback control law and the solution of the optimal control problem are used alternatively.

First, we will show that the switch to the linear feedback control law will take place. It follows from (4.1a,b) that if the switch to the linear feedback control law does not take place for any $k \in \mathbb{N}$, then

$$|\tilde{x}_{k-1}| \le |x_{k-1}^p - \tilde{x}_{k-1}| + |x_{k-1}^p| \le (\Delta_7 + 1)|x_{k-1}^p| + \Delta_8 \,, \tag{4.9a}$$

$$|x_k| \le |x_k^p - x_k| + |x_k^p| \le \Delta_9 |x_k^p| + \Delta_{10} + |x_k^p| \,. \tag{4.9b}$$

Because $\Delta_7 \le \Delta_9 \le \varepsilon_1$, $\Delta_8 \le \Delta_{10} \le \varepsilon_2$, and $\rho_{MH} < (\rho_{oc} - \varepsilon_2)/(1 + \varepsilon_1)$, it follows from (4.8d) that there exists a $\hat{k} \in \mathbb{N}$ such that $|\tilde{x}_{\hat{k}-1}| \le \rho_{oc}$ and $|x_{\hat{k}}| \le \rho_{oc}$. Therefore the switch to the linear feedback control law will take place.

Now, it follows from (4.1a) that

$$|x_{\hat{k}-1}^p| \le |x_{\hat{k}-1}^p - \tilde{x}_{\hat{k}-1}| + |\tilde{x}_{\hat{k}-1}| \le \varepsilon_1 |x_{\hat{k}-1}^p| + \varepsilon_2 + |\tilde{x}_{\hat{k}-1}| \,. \tag{4.9c}$$

From (4.1b), we obtain that

$$|x_{\hat{k}}^p| \le |x_{\hat{k}}^p - x_{\hat{k}}| + |x_{\hat{k}}| \le \varepsilon_1 |x_{\hat{k}-1}^p| + \varepsilon_2 + |x_{\hat{k}}| \,. \tag{4.9d}$$

From (4.9c,d) and (4.8e) we obtain that

$$|x_{\hat{k}}^p| \le \frac{\varepsilon_1}{1 - \varepsilon_1}(\varepsilon_2 + \rho_{oc}) + \varepsilon_2 + \rho_{oc} = \rho_{LQR}^o \,. \tag{4.9e}$$

Then, it follows from $x^o(t_{\hat{k}}) = x_{\hat{k}}$ that $|x^o(t_{\hat{k}})| \le \rho_{oc} \le \rho_{LQR}^o$.

Next, suppose that the control $u(t) = -K_c x^o(t)$ is used for all $t \in [t_{\hat{k}}\,, T_{oc}]$, where $t_{\hat{k}}$ is the time when the switch to the linear feedback control law takes place and $T_{oc} \ge t_{\hat{k}+1}$. Let the Lyapunov function $V(\cdot)$ be defined by $V(x^o(t)) \triangleq |x^o(t)|^2 \triangleq \langle x^o(t)\,, Qx^o(t)\rangle$. Then, making use of the matrix M defined by (3.11) and (4.5b), we obtain that for all $\in [t_{\hat{k}}\,, T_{oc}]$

$$\dot V(x^o(t)) \le -\lambda_{\min}(M)\|x^o(t)\|_2^2 + 2\|K_c\Delta CQ\|\,\|x^o(t)\|_2^2 + 2\|e(t)\|_2(\|K_cCQ\| + \|K_c\Delta CQ\|)\|x^o(t)\|_2$$

$$\le \left[-\frac{\lambda_{\min}(M)(1-2\delta)\|x^o(t)\|}{\lambda_{\max}(Q)^{1/2}} + \frac{2\|e(t)\|(\|K_cCQ\| + \delta\lambda_{\min}(M))}{\lambda_{\min}(Q)^{1/2}}\right]\|x^o(t)\|_2 . \qquad (4.9f)$$

It now follows from (4.9e) that $\|x^o(t_{\hat k})\|, \|x^p_{\hat k}\| \le \rho^o_{LQR} \le \gamma_e/4$ and that $\|z(t_{\hat k})\| \le \|e(t_{\hat k})\| + \|x^o(t_{\hat k})\| + \|x^p((t_{\hat k})\| \le 2(\|x^o(t_{\hat k})\| + \|x^p_{\hat k}\|) \le \gamma_e$, which implies that for all $t \in [t_{\hat k}, Tsoboc]$, $\|e(t)\| \le \gamma_e$, by Lemma 4.3. Now, it follows from (4.6d) that if $\|x^o(t)\| > \rho_{LQR}$, then $\dot V(x^o(t)) < 0$ for $t \in [t_{\hat k}, T_{oc}]$. Since $\|x^o(t_{\hat k})\| \le \rho_{LQR}$, we must have that $\|x^o(t)\| \le \rho_{LQR}$ for all $t \in [t_{\hat k}, T_{oc}]$ and therefore $u(t) = -K_c x^o(t)$ satisfies the bound on the control.

Now let us consider the case *(a)*. If we set $T_{oc} = \infty$, then we conclude from the above that $\|x^o(t)\| \le \rho_{LQR}$ for $t \ge t_{\hat k}$. Also, by Lemma 4.3, $\|x^p(t)\| \le \gamma_e$ for all $t \ge t_{\hat k}$, which implies that $x^p(t)$ is bounded. Since by Lemma 4.3, $\lim_{t\to\infty}\|z(t)\| \to 0$ as $\|d\|_\infty \to 0$ we must have that $\overline{\lim}_{t\to\infty}\|x^p(t)\| \to 0$ as $\|d\|_\infty \to 0$, which completes the proof of *(a)*.

Next, let us consider the case *(b)*. Suppose that there exists a $k' > \hat k$ such that $u(t) = -K_c x^o(t)$, for all $t \in [t_{\hat k}, t_{k'}]$, and $\max\{\|\tilde x_{k'-1}\|, \|x_{k'}\|\} > \rho_{oc}$. Since $\|x^o(t)\| \le \rho_{LQR}$ and $\|x^p(t)\| \le \gamma_e$ for all $t \in [t_{\hat k}, t_{k'}]$, and $\gamma_e \le \rho_d$, we have that $x^p(t_{k'}) \in B_{\rho_d}$, which implies that the optimal control problem has a solution. Hence, by the first part of our proof, there exists a $\hat k' > k'$ such that the switch to the linear feedback control law again takes place. We now resort to a continuity argument. If $d(t) = 0$ for all $t \in [t_{\hat k'}, t_{\hat k'+1}]$, then by Theorem 3.8, we will have that $\max\{\|\tilde x_{\hat k'}\|, \|x_{\hat k'+1}\|\} \le \rho_{oc}\max\{\alpha, 1/\sqrt{2}\}$. Hence, by continuity of the solution of (4.5a,b), there must exists a $c''_d > 0$ such that if $\|d(t)\| \le c''_d > 0$ for all $t \in [t_{\hat k'}, t_{\hat k'+1}]$, then $\max\{\|\tilde x_{\hat k'+1}\|, \|x_{\hat k'+2}\|\} < \rho_{oc}$ will hold, and hence the linear control law will be retained for the next interval, $[t_{\hat k'+1}, t_{\hat k'+2}]$, and similarly, for all the intervals to follow, since c''_d does not depend on t_k. Hence, if $\|d\|_\infty \le c''_d$, then the linear control law will be used for all $t \ge t_{\hat k'}$, and therefore, by case *(a)*, we conclude that $\overline{\lim}_{t\to\infty}\|x^p(t)\| \to 0$ and it completes our proof. □

Next we turn to the case where the disturbance is the output of a known dynamical system driven by stationary, zero mean, white noise. To obtain bounds on the disturbance effects, we must assume that there are no modeling errors, i.e., that $A^p = A$, $B_p = B$ and $C^p = C$, and that the the state of the plant can be measured. First we will consider the effect of disturbances which are generated by the initial state of an unforced, linear, time invariant system that is described by

$$\dot{x}_d(t) = A_d x_d(t) \tag{4.10a}$$

$$d(t) = C_d x_d(t), \tag{4.10b}$$

where $A_d \in \mathbb{R}^{n_d \times n_d}$, $C_d \in \mathbb{R}^{m \times n_d}$. Since the input $u(\cdot)$ is bounded, we can only hope to reduce the effects of bounded disturbances. Therefore, we assume that there exists a $b_d < \infty$ such that $\| e^{A_d t} \| \leq b_d$ for all $t \geq 0$.

To estimate the state $x_d(t)$, we can proceed as follows. For all $k \in \mathbb{N}$ and $t \in [t_k, t_{k+1}]$, let $e(t)$ be defined by $e(t) \triangleq x^p(t, t_k, x_k^p, u+d) - x(t, t_k, x_k^p, u)$. Then

$$\dot{e}(t) = Ae(t) + Bd(t), \tag{4.10c}$$

with $e(t_k) = 0$. Combining (4.10a,b,c), we obtain that

$$\frac{d}{dt}\begin{bmatrix} x_d(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_d & 0 \\ BC_d & A \end{bmatrix}\begin{bmatrix} x_d(t) \\ e(t) \end{bmatrix} \triangleq \bar{A}\begin{bmatrix} x_d(t) \\ e(t) \end{bmatrix} \tag{4.10d}$$

$$e(t) = \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} x_d(t) \\ e(t) \end{bmatrix} \triangleq \bar{C}\begin{bmatrix} x_d(t) \\ e(t) \end{bmatrix}. \tag{4.10e}$$

Obviously, when $(\bar{C}, \bar{A})$ is an observable pair, we can use a reduced order estimator to obtain an asymptotically converging estimate of the disturbance state $x_d(t)$. Then, assuming that $\| u(t) - \hat{d}(t) \|_\infty \leq c_u$ for all $t \in [t_k, t_{k+1}]$, where $u(t)$ is computed by solving the optimal control problem $\mathbf{P}(x_{k-1}, t_{k-1}, 0)$ the use of Control Algorithm 2.2 will result in asymptotically perfect disturbance rejection.

We now give a necessary and sufficient condition for $(\bar{C}, \bar{A})$ to be observable.

Lemma 4.5. Let $\bar{A}$ and $\bar{C}$ be defined as (4.10d,e). Then $(\bar{C}, \bar{A})$ is an observable pair if and only if (BC_d, A_d) is an observable pair.

Proof. $\Longrightarrow$ We will give a proof by contraposition. Suppose that (BC_d, A_d) is not an observable pair. Then there exists a nonzero vector $z \in \mathbb{R}^{n_d}$ such that

$$BC_dA_d^iz = 0\,,\ i = 0\,,1\,,\ldots . \tag{4.11a}$$

Now let $\bar{z} \triangleq (z^T\,,0)^T \in \mathbb{R}^{n_d+n}$. Then, because of (4.11a), we have that

$$\bar{C}\bar{A}^j\bar{z} = \sum_{i=0}^{j-1} A^{j-i-1}BC_dA_d^iz = 0\,,\ j = 1\,,2\,,\ldots,n_d+n-1\,. \tag{4.11b}$$

Furthermore $\bar{C}\bar{z} = 0$. Hence $(\bar{C}\,,\bar{A})$ is not an observable pair.

$\Longleftarrow$ Now suppose that $(\bar{C}\,,\bar{A})$ is not an observable pair. Then there must exist a nonzero $\bar{z} = (z\,,z') \in \mathbb{R}^{n_d+n}$, such that $\bar{C}\bar{A}^j\bar{z} = 0$ for $j = 0\,,1\,,\ldots,n_d+n-1$. Since $\bar{C} = (0 \mid I)$, it is clear that $z' = 0$ must hold. Hence (4.11b) must hold, and unraveling this expression, we find that (4.11a) must also hold, which completes our proof. □

As an alternative to using a reduced order observer, at the expense of more computation, we can get an exact estimate of $\hat{d}\,(t)$ to be used to obtain perfect disturbance rejection, as follows. Let

$$\begin{bmatrix} w_{11}(t) & 0 \\ w_{21}(t) & w_{22}(t) \end{bmatrix} \triangleq \exp(\bar{A}t) = \exp\left\{ \begin{bmatrix} A_d & 0 \\ BC_d & A \end{bmatrix} t \right\}, \tag{4.12a}$$

so that $w_{11}(t) = e^{A_dt}$ and $w_{22}(t) = e^{At}$. Hence (4.10e) can be rewritten in the equivalent form

$$e(t) = w_{21}(t)x_d(t_k) + w_{22}(t)e(t_k) = w_{21}(t)x_d(t_k)\,. \tag{4.12b}$$

Since the state of the plant is measurable, $e(t)$ can be computed for all $t \in [t_k\,,t_{k+1}]$. Hence, if $\int_{t_k-\delta}^{t_k} w_{21}^T(\tau)w_{21}(\tau)\,d\tau$ is always invertible for some $\delta > 0$, then we can also compute $x_d(t_k-\delta)$ using the formula

$$x_d(t_k-\delta) = \left[\int_{t_k-\delta}^{t_k} w_{21}^T(\tau)w_{21}(\tau)\,d\tau\right]^{-1} \int_{t_k-\delta}^{t_k} w_{21}^T(\tau)e(\tau)\,d\tau\,. \tag{4.13a}$$

We can then use $x_d(t_k-\delta)$ to compute the disturbance $\hat{d}\,(t)$, for $t \in [t_k\,,t_{k+1}]$, using the formula:

$$\hat{d}\,(t) = C_de^{A_d(t-t_k-\delta)}x_d(t_k-\delta) \triangleq C_dx_d(t)\,. \tag{4.13b}$$

To establish the invertibility of the matrix $\int_{t_k}^{t} w_{21}^T(\tau)w_{21}(\tau)\,d\tau$, for all $t > t_k$, we need the following lemma.

Theorem 4.6. Suppose that $w_{21}(t)$ is defined as in (4.12a). If $(C_d\,,A_d)$ is an observable pair and B has maximum column rank, then, $\int_{t_k}^{t} w_{21}^T(\tau)w_{21}(\tau)\,d\tau$ is invertible for all $t > t_k$.

Proof. To simplify notation, let $\bar{\Phi}(t\,,\tau) \triangleq \exp((t-\tau)\bar{A})$. Since $(\bar{C}\,,\bar{A})$ is a observable pair by Lemma 4.5, the observability grammian for the system (4.10d,e), $W(t\,,t_k)$, defined by

$$W(t\ ,t_k) \triangleq \int_{t_k}^{t} \overline{\Phi}(\tau\ ,t_k)^T \overline{C}^T \overline{C}\,\overline{\Phi}(\tau\ ,t_k)\,d\tau \tag{4.14a}$$

is nonsingular for all $t > t_k$. By substituting the expressions for $\overline{C}$ and $\overline{\Phi}(t\ ,t_k)$ that are given by (4.10e) and (4.12a), respectively, we obtain that

$$W(t\ ,t_k) = \begin{bmatrix} \int_{t_k}^{t} w_{21}^T(\tau) w_{21}(\tau)\,d\tau & \int_{t_k}^{t} w_{21}^T(\tau) w_{22}(\tau)\,d\tau \\ \int_{t_k}^{t} w_{22}^T(\tau) w_{21}(\tau)\,d\tau & \int_{t_k}^{t} w_{22}^T(\tau) w_{22}(\tau)\,d\tau \end{bmatrix} \triangleq \begin{bmatrix} W_{11}(t\ ,t_k) & W_{12}(t\ ,t_k) \\ W_{12}(t\ ,t_k) & W_{22}(t\ ,t_k) \end{bmatrix} \tag{4.14b}$$

Suppose that for some $t > t_k$, $W_{11}(t\ ,t_k)$ is a singular matrix. Then there exists a nonzero vector, $z \in \mathbb{R}^{n_d}$, such that $W_{11}(t\ ,t_k)z = 0$, and hence for $\tilde{z} \triangleq (z^T\ \ 0)^T \in \mathbb{R}^{n_d+n}$,

$$\langle \tilde{z}\ , W(t\ ,t_k)\tilde{z} \rangle = \langle z\ , W_{11}(t\ ,t_k)z \rangle = 0\,, \tag{4.14c}$$

which contradicts to the fact that $W(t\ ,t_k)$ is positive definite matrix for all $t > t_k$. Therefore, $W_{11}(t\ ,t_k)$ is nonsingular for all $t > t_k$, which completes our proof. □

Thus, assuming that $\|u(t) - \hat{d}(t)\|_\infty \le c_u$ for all $t \in [t_k\ ,t_{k+1}]$, where $u(t)$ is computed by solving the optimal control problem $\mathbf{P}(x_{k-1}\ ,t_{k-1}\ ,0)$ the use of Control Algorithm 2.2 will result in perfect disturbance rejection.

In reality, it is not likely that the disturbance $d(t)$ is the output of a unforced linear time invariant system. It is more realistic to suppose that $d(\cdot)$ is the output of a linear time invariant system driven by stationary zero-mean white noise, with an initial state $x_d(0)$, described by

$$\dot{x}_d(t) = A_d x_d(t) + B_d w(t) \tag{4.15a}$$

$$d(t) = C_d x_d(t)\,. \tag{4.15b}$$

Let $d_1(t) \triangleq C_d e^{A_d t} x_d(0) \triangleq C_d x_{d_1}(t)$ and let $d_2(t) \triangleq C_d \int_0^t e^{A_d(t-\tau)} B_d w(\tau)\,d\tau \triangleq C_d x_{d_2}(t)$ be the contribution of the white noise term in (4.15a). Let $E(\xi)$ denote the expected value of the random variable ξ. Then we see that because $E(w(t)) = 0$ for all $t \ge 0$, $E(x_{d_2}(t)) = 0$ for all $t \ge 0$. Hence (c.f. (4.10d,e) and (4.12a,b)) we obtain that for $t \in [t_k\ ,t_{k+1}]$, $k \in \mathbb{N}$, $E(e(t)) = w_{21}(t)x_d(t_k)$. Since $\int_{t_k-\delta}^{t_k} w_{21}^T(t) w_{21}(t)\,dt$ is invertible for any $\delta > 0$ by Theorem 4.6, we can compute the estimate of the disturbance $d(t)$, for $t \in [t_k\ ,t_{k+1})$, according to

$$\hat{d}(t) = C_d e^{A_d(t-t_k-\delta)} x_d(t_k - \delta)\,, \tag{4.16}$$

where $x_d(t_k - \delta)$ is defined by (4.12a). Since $E(x_{d_2}(t)) = 0$ for all $t \ge 0$, $E(\hat{d}(t)) = E(d(t))$ for all

$t \geq 0$. Therefore, we have perfect estimation of the expected value of the disturbance, which implies that $\| E(d(\cdot) - \hat{d}(\cdot)) \|_\infty = 0$. In conjunction with Theorem 4.4, this fact leads to the following result.

Theorem 4.7. Suppose that *(a)* δ, $\Delta\tilde{A}$, $\Delta\tilde{B}$, $\|d\|_\infty$ satisfy the conditions in Lemma 4.3, *(b)* $\|K_c \Delta C\| \leq \delta\lambda_{\min}(M)$, *(c)* that (4.8b) holds, *(d)* that $\rho_{MH} < (\rho_{oc} - \varepsilon_2)/(1 + \varepsilon_1)$, where ρ_{MH} was defined in (4.8d), and *(e)* that we use *Step 1''* in Control Algorithm 2.2. Then for any $x_0^p \in B_{\rho_4}$, defined in (4.4h) (using the formulae (3.2q), (4.4g)), the expected value of the trajectory $x^p(t, 0, x_0^p, u)$ is bounded and, furthermore, $\overline{\lim}_{t \to \infty} E(x^p(t, 0, x_0^p, u + d)) = 0$. □

5. TRACKING.

We will now examine the reference input tracking properties of our moving horizon control system, defined by the error dynamics (2.4a,b) and Control Algorithm 2.2. At this point we must assume that the matrix B in (2.4c) has full column rank.

Before we attempt a characterization of inputs which can be tracked asymptotically by our moving horizon control system (with bounded controls), we will extend a result due to Basile and Marro [Bas.1], dealing with asymptotic state tracking of LTI systems without control constraints.

Lemma 5.1. [Bas.1] Consider LTI system (2.2a,b), and let S_x be defined as in (2.3a). Then, S_x is the largest subspace among subspaces $S \subset \mathbb{R}^n$ such that

$$AS + S \subset R(\mathrm{B}), \tag{5.1}$$

where $AS + S = \{ x \in \mathbb{R}^n \mid x = x_1 + x_2, \text{for all } x_1 \in AS, x_2 \in S \}$. □

Making use of Lemma 5.1, we obtain the following straightforward generalization of a result in [Bas.1].

Lemma 5.2. Let $r \in \mathbf{R}$ and consider the error dynamics (2.4c,d), with $\hat{d}(t) \equiv 0$, and $f(t) = -\dot{s}(t) + As(t)$, where $s(t) \triangleq H(\tilde{C}^T\tilde{C})^{\dagger}\tilde{C}^T r(t)$. Then, there exists a continuous control $u_r(t)$, $t \geq 0$, such that for any initial state $x_0 \in \mathbb{R}^n$, $y(t) \triangleq Cx(t, 0, x_0, u_r) \to 0$ as $t \to \infty$.

Proof. Clearly, if there exists a control $u_r(\cdot)$ such that $x(t, 0, x_0, u_r) \to 0$ as $t \to \infty$, then, since $y(t) = Cx(t, 0, x_0, u_r)$, the desired result must hold.

We recall that by definition $s(t) \in S_x$ for all $t \geq 0$. We will now show that we also have that $\dot{s}(t) \in S_x$. Let z be a nonzero vector in the orthogonal complement of S_x. Then for all $t > 0$,

$$0 = \langle z, (s(t) - s(0)) \rangle = \langle z, \int_0^t \dot{s}(t)\,dt \rangle. \tag{5.2}$$

Since (5.2) holds for all $t \geq 0$, we must have that $\langle z, \dot{s}(t) \rangle = 0$ for all t. Therefore $\dot{s}(t) \in S_x$ for all $t \geq 0$.

Let $u_r(t) \triangleq -Fx(t) + v(t)$ where F is any feedback matrix such that $\sigma(A - BF) \subset C_-^o$ (with $\sigma(A)$ the set of eigenvalues of A and C_-^o the open left half plane of the complex plain), and $v(t)$ is defined by $As(t) - \dot{s}(t) + Bv(t) = 0$ for all $t \geq 0$. The latter is possible because $s(t), \dot{s}(t) \in S_x$ and $AS_x + S_x \subset R(B)$. Then, we have that $x(t, 0, x_0, u_r) = e^{(A-BF)t}x_0$ and obviously, $x(t, 0, x_0, u_r) \to 0$ as $t \to \infty$, which completes our proof. □

So far, we have assumed that there are no constraints on the control. We have assumed in Assumption 2.3 that for all $r \in \mathbf{R}_U$ and $x \in B_{\hat{\rho}}$, the optimal control problem $\mathbf{P}(x, 0, r)$ has a solution. To show that Control Algorithm 2.2 can be used for input tracking as well as stabilization, we have to prove that for trajectories emanating from the ball $B_{\hat{\rho}}$, the estimated states x_{k+1} defined by (2.6) are in the set $B_{\hat{\rho}}$. To establish this fact, we will follow the pattern set up in Section 4. First, we need the following definition.

Definition 5.3. Let $c_s \in (0, \infty)$. We define $\tilde{R}_U \subset \mathbf{R}_U$ by

$$\tilde{\mathbf{R}}_U = \{ r \in \mathbf{R}_U \mid \max(\|s\|_\infty, \|\dot{s}\|_\infty) \leq c_s \}, \tag{5.3}$$

where $s(t) = H(\tilde{C}^T\tilde{C})^\dagger\tilde{C}^T r(t)$. □

Consider the error dynamics (2.4a,b) and its model (2.4c,d). We assume that the disturbance $d(t)$ cannot be estimated. Hence Control Algorithm 2.2 sets $\hat{d} \equiv 0$. Since the more difficult situation occurs when the plant state is estimated, we will assume that this is the case. First, we derive a result similar to Lemma 4.1.

Lemma 5.4. Let $r \in \tilde{\mathbf{R}}_U$. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with state estimation formula (3.5). There exist $\Delta_i < \infty$, $i = 11, 12, 13, 14$, such that if Control Algorithm 2.2 constructs the sequences $\{x_k^p\}_{k=0}^\infty$, $\{x_k\}_{k=1}^\infty$, and $\{\bar{x}_k\}_{k=0}^\infty$ is the corresponding sequence of the estimates of x_k^p, defined by (3.5), then for all $k \in \mathbf{N}$,

$$\|x_k^p - \bar{x}_k\| \leq \Delta_{11}\|x_k^p\| + \Delta_{12}, \tag{5.4a}$$

$$\| x_{k+1}^p - x_{k+1} \| \le \Delta_{13} \| x_k^p \| + \Delta_{14} \,. \tag{5.4b}$$

Furthermore, when there are no modeling errors and no disturbances, $\Delta_i = 0$, $i = 11, 12, 13, 14$.

Proof. Suppose that $u(\cdot)$ is the control generated by Control Algorithm 2.2 for the plant and model trajectories associated with the sequences $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{\tilde{x}_k\}_{k=0}^{\infty}$. For us to have a similarity with Lemma 4.1, let us modify the error dynamics (2.4a,c) as follows.

For a given $r \in \tilde{R}_U$, let $s(t) = H(\tilde{C}^T\tilde{C})^{\dagger}\tilde{C}^T r(t)$. Let

$$u(t) = u_1(t) + u_2(t) \,, \tag{5.5a}$$

where

$$u_2(t) = (B^T B)^{-1} B^T (As(t) - \dot{s}(t)) \,. \tag{5.5b}$$

Then, since $f^p(t) = -\dot{s}(t) + A^p s(t)$ and $f(t) = -\dot{s}(t) + As(t)$, (2.4a,c) becomes

$$\dot{x}^p(t) = A^p x^p(t) + B^p(u_1(t) + d(t)) + (B^p - B)u_1(t) + (A^p - A)s(t)$$

$$\triangleq A^p x^p(t) + B^p(u_1(t) + d(t)) + d_1(t) \,, \tag{5.5c}$$

$$\dot{x}(t) = Ax(t) + B(u_1(t) + \hat{d}(t)) \,. \tag{5.5d}$$

Since $\max\{ \|s\|_\infty , \|\dot{s}\|_\infty \} \le c_s$, it is clear that $\|u_2\|_\infty$ is bounded. Then,

$$\|u_1\|_\infty \le \|u\|_\infty + \|u_2\|_\infty \triangleq c_r \,. \tag{5.5e}$$

Next it follows from (5.5c) that

$$\|d_1\|_\infty \le (\|B^p - B\| + \|A^p - A\|)c_r \triangleq \Delta_{d_1} \,. \tag{5.5f}$$

We begin with (5.4a). For any $k \in \mathbb{N}$ and any $t \in [t_k, t_{k+1}]$, $y^p(t)$ is given by

$$y^p(t) = C^p e^{A^p(t-t_k)} x_k^p + C^p \int_{t_k}^{t} e^{A^p(t-\tau)} B^p (u_1(\tau) + d(\tau))\, d\tau + C^p \int_{t_k}^{t} e^{A^p(t-\tau)} d_1(\tau)\, d\tau$$

$$= Ce^{A(t-t_k)} x_k^p + \{ C^p e^{A^p(t-t_k)} - Ce^{A(t-t_k)} \} \, x_k^p + C\int_{t_k}^{t} e^{A(t-\tau)} B(u_1(\tau) + d(\tau))\, d\tau$$

$$+ \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B^p - Ce^{A(t-\tau)} B \} \, (u_1(\tau) + d(\tau))\, d\tau$$

$$+ C\int_{t_k}^{t} e^{A(t-\tau)} d_1(\tau)\, d\tau + \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} - Ce^{A(t-\tau)} \}\, d_1(\tau)\, d\tau \,. \tag{5.5g}$$

By substituting (5.5g) into (3.5), we obtain

$$\bar{x}_k = x_k^p + W_o(\delta_0(t_{k+1}-t_k))^{-1}\Bigg\{\int_{t_k}^{t'_k}(Ce^{A(t-t_k)})^T\ \{C^pe^{A^p(t-t_k)} - Ce^{A(t-t_k)}\}\ dt\ x_k^p$$

$$+\int_{t_k}^{t'_k}(Ce^{A(t-t_k)})^T\int_{t_k}^{t} Ce^{A(t-\tau)}(Bd(\tau)+d_1(\tau))\,d\tau$$

$$+\int_{t_k}^{t'_k}(Ce^{A(t-t_k)})^T\int_{t_k}^{t}\{C^pe^{A^p(t-\tau)}B^p - Ce^{A(t-\tau)}B\}(u_1(\tau)+d(\tau))\,d\tau\,dt$$

$$+\int_{t_k}^{t'_k}(Ce^{A(t-t_k)})^T\int_{t_k}^{t}\{C^pe^{A^p(t-\tau)} - Ce^{A(t-\tau)}\}\,d_1(\tau)\,d\tau\Bigg\}. \qquad (5.5h)$$

It follows directly from (5.5h) that

$$\|x_k^p - \bar{x}_k\| \le \Delta_{11}\|x_k^p\| + \Delta_{12}, \qquad (5.5i)$$

where $\Delta_{11} = \Delta_7$, which was defined in (4.2d) and

$$\Delta_{12} \triangleq \Delta'_8 + C_\Delta\left[\{\max_{t\in[0,\delta_0\bar{T}]}\|C^pe^{A^p(t-\tau)} - Ce^{A(t-\tau)}\| + \max_{t\in[0,\delta_0\bar{T}]}\|Ce^{At}\|\}\,\delta_0\bar{T}\Delta_{d_1}\right], \qquad (5.5j)$$

with $C_\Delta \triangleq \max_{t\in[T_c,\bar{T}]}\|W_o(\delta_0 t)^{-1}\|\max_{t\in[0,\delta_0\bar{T}]}\|Ce^{At}\|$ and with Δ'_8 replacing c_u of Δ_8 defined in (4.2e) with c_r in (5.5e), which proves (5.4a). Clearly, when there are no modeling errors and no disturbances, $\Delta_{11} = \Delta_{12} = 0$.

Next we will establish (5.4b). Since x_{k+1} is calculated using the estimated initial state $\bar{x}_k$, it follows from the Schwartz inequality in $L_2[0,\bar{T}]$ (i.e., $\int_0^T a(t)b(t)\,dt \le \left[\int_0^T a(t)^2dt\right]^{1/2}\left[\int_0^T b(t)^2dt\right]^{1/2}$) that

$$\|x_{k+1}^p - x_{k+1}\| = \|e^{A^p(t_{k+1}-t_k)}x_k^p - e^{A(t_{k+1}-t_k)}\bar{x}_k + \int_{t_k}^{t_{k+1}} e^{A^p(t_{k+1}-t)}d_1(\tau)\,d\tau$$

$$+\int_{t_k}^{t_{k+1}}\{e^{A^p(t_{k+1}-\tau)}B^p - e^{A(t_{k+1}-\tau)}B\}\,u_1(\tau)\,d\tau + \int_{t_k}^{t_{k+1}} e^{A^p(t_{k+1}-t)}B^pd(\tau)\,d\tau\|$$

$$\le K\,\|x_k^p - \bar{x}_k\| + \Delta_1\|x_k^p\| + \Delta'_2$$

$$+\int_{t_k}^{t_{k+1}}\|e^{A^p(t_{k+1}-t)} - e^{A(t_{k+1}-t)}\|\,\|d_1(t)\|\,dt + \int_{t_k}^{t_{k+1}}\|e^{A(t_{k+1}-t)}\|\|d_1(t)\|\,dt$$

$$+\int_{t_k}^{t_{k+1}}\|e^{A^p(t_{k+1}-t)}B^p - e^{A(t_{k+1}-t)}B\|\,\|d(t)\|dt + \int_{t_k}^{t_{k+1}}\|e^{A(t_{k+1}-t)}B\|\,\|d(t)\|dt$$

$$\leq K\ \{\Delta_{11}\|x_0^p\| + \Delta_{12}\} + \Delta_1\|x_0^p\| + \Delta'_2 + \left[\frac{\Delta'_2}{c_r\sqrt{\bar{T}}} + K\|B\|\sqrt{\bar{T}}\right]c_d$$

$$+ \left[K + \max_{t\in[0,\bar{T}]}\|e^{A^p t} - e^{At}\|\right]\sqrt{\bar{T}}\,\Delta_{d_1}$$

$$\triangleq (K\Delta_{11} + \Delta_1)\|x_0^p\| + K\Delta_{12} + \Delta'_2 + \left[\frac{\Delta'_2}{c_r\sqrt{\bar{T}}} + K\|B\|\sqrt{\bar{T}}\right]c_d + \tilde{\Delta}_{d_1}$$

$$\triangleq \Delta_{13}\|x_0^p\| + \Delta_{14}, \tag{5.5k}$$

where K, Δ_1 were defined in (3.1a,b) and Δ'_2 was obtained by replacing c_u of Δ_2 in (3.1c) with c_r in (5.5e). Hence (5.4b) holds, and our proof is complete. □

In Section 4, Theorem 4.2 was proved by making use of the results in Lemma 4.1 and Proposition 8.1. In the case of tracking, it is clear that if we replace Δ_9 with Δ_{13} and Δ_{10} with Δ_{14} in the proof of Theorem 4.2 and use Lemma 5.4 instead of Lemma 4.1, still using Proposition 8.1, then the conclusions of Theorem 4.2 assume the following form.

Theorem 5.5. Let $r \in \tilde{\mathbf{R}}_U$. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with state estimation as in (3.5). Suppose that Δ_{13}, Δ_{14} satisfy the inequalities

$$\Delta_{13} \leq \varepsilon_1 < \frac{1-\alpha}{1+\alpha+K}, \tag{5.6a}$$

$$\Delta_{14} \leq \varepsilon_2 < \frac{\hat{\rho}}{3 + (2+K)/\varepsilon'}, \tag{5.6b}$$

where Δ_{13}, Δ_{14} were defined in (5.5k), and ε' was defined in (3.2g). Let ρ_d be as in (4.4h). Then, for all $x_0^p \in B_{\rho_d}$, the trajectory $x^p(t, 0, x_0^p, u+d)$, $t \in [0, \infty)$, is bounded, and there exists an $\varepsilon_3 > 0$ such that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, and $\overline{\lim}_{t\to\infty} \|x^p(t, 0, x_0^p, u+d)\| \leq \varepsilon_3$. □

Since the constants Δ_{13}, Δ_{14} depend on c_s and the bounds on the modeling errors, we see that there is a trade off involved in chosing a value for c_s, namely, the larger c_s the smaller are the modeling errors under which (5.6a,b) will be satisfied, while the set of admissible inputs $\tilde{\mathbf{R}}_U$ grows with c_s.

In a similar way, the results of Theorem 4.4 can also be extended to the reference following case.

6. NUMERICAL RESULTS.

We will now present three examples that illustrate the performance of the moving horizon control system based on Control Algorithm 2.2, for a plant modeled by the state equations

$$\dot{x}(t) = \begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{6.1a}$$

where $u \in U \triangleq \{ u \in L_\infty[0,\infty) \mid \|u\|_\infty \leq 1 \}$. Control Algorithm 2.2 used the following optimal control problem:

$$\mathrm{P}(x_k, t_k, 0): \min_{u \in U} \tfrac{1}{2}\int_{t_k}^{\tau} (\langle x(t, t_k, x_k, u), Rx(t, t_k, x_k, u)\rangle + \langle u(t), Su(t)\rangle)\, dt \tag{6.1b}$$

subject to

$$\|x(\tau, t_k, x_k, u)\|^2 - 0.01\|x_k\|^2 \leq 0, \tag{6.1c}$$

$$\|x(t, t_k, x_k, u)\|^2 - 100\|x_k\|^2 \leq 0, \ \forall t \in [t_k, \tau], \tag{6.1d}$$

where $\tau \in [t_k + T_C, t_k + \bar{T}]$, $T_C = 5$, $\bar{T} = 40$,

$$R = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}, \tag{6.1e}$$

and $S = 2000$.

All the computations were performed in double precision on a Sun 3/140 Workstation with a floating point accelerator. For comparison, we used the example given in [Gut.1], which has only a control constraint. Since the initial state was known, we solved the optimal control problem $\mathrm{P}(x_0, 0, r)$ off-line to obtain the initial control $u(t), t \in [0, t_1]$.

Example 6.1. In this example we have assumed that the state can be measured and that there are no modeling errors. Also, we assumed that $r(t) \equiv 0$ and $d(t) \equiv 0$. This is the case presented in [Gut.1], where a piecewise linear control law was used, defined by

$$u(t) = sat\,[(L - k[0\ 1]P)x(t)], \tag{6.2}$$

where $L = -0.78 \times 10^{-3} \times [4.47\ \ 94.61]$, $k = 0.5 \times 10^{-5}$,

$$P = \begin{bmatrix} 171 & 1433 \\ 1433 & 19435 \end{bmatrix},$$

and $sat(\cdot)$ is the standard saturation function. The matrix L was obtained by solving a Linear

Quadratic Regulator problem and P is a correction matrix. Figure 1 shows the resulting trajectories, using both our strategy and the one in [Gut.1] for $t \in [0, 60)$ and $x_0 = [10 \;\; 10]$.

As we can see from the Figure 1, the trajectory generated by Control Algorithm 2.2 converges to the origin faster than the trajectory given by [Gut.1]. The controls for both cases are shown in Figure 2.

Example 6.2. Next, we have again assumed that the state can be measured, that $r(t) \equiv 0$ and $d(t) \equiv 0$, but that there are modeling errors, viz. we assumed that the actual plant dynamics were

$$\dot{x}^p(t) = \begin{bmatrix} 0.01 & 1 \\ 0 & 0.01 \end{bmatrix} x(t) + \begin{bmatrix} 0.01 \\ 0.99 \end{bmatrix} u(t), \tag{6.3}$$

while the model was as in (6.1a). For the initial state given in Example 6.1, in Figure 3, we compare the trajectory, $x^p(t, 0, x_0^p, u)$, obtained by applying the control given in [Gut.1] with the trajectory generated by Control Algorithm 2.2. Again, the trajectory generated by Control Algorithm 2.2 converges to the origin faster. The controls for both cases are shown in Figure 4.

Example 6.3. In this example, we consider the case where there are modeling errors and the state has to be estimated. Thus, we assumed that the plant was described by

$$\dot{x}^p(t) = \begin{bmatrix} 0.002 & 1 \\ 0 & 0.003 \end{bmatrix} x(t) + \begin{bmatrix} 0.002 \\ 0.99 \end{bmatrix} u(t), \tag{6.4a}$$

$$y^p(t) = [0.99 \;\; 0.005] x(t), \tag{6.4b}$$

with $x_0^p = [5 \;\; 5]$. The plant was modeled by the equations

$$\dot{x}(t) = \begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{6.4c}$$

$$y(t) = [1 \;\; 0] x(t), \tag{6.4d}$$

with $x_0 = [2 \;\; 2]$.

We applied Control Algorithm 2.2 and the resulting control $u(t)$ and trajectory, $x^p(t, 0, x_0^p, u)$, $t \in [0, 100]$, are shown in Figures 6 and 5, respective.

7. CONCLUSION.

Moving horizon control is a promising new idea for the control of nonlinear systems. In this paper we have explored the properties of a moving horizon feedback system, based on constrained optimal control algorithms, with the simplest possible nonlinearity, namely, input saturation. We have shown that moving horizon control results in a robustly stable system, capable of following a class of reference inputs and suppressing a class of disturbances. Our experimental results show that the behavior of the moving horizon control system is superior to that resulting from alternative control laws. The main remaining issue in the use of moving horizon control based on the solution of constrained optimal control problems, is the time needed to solve the optimal control problems. This should cause no difficulties in controlling slow moving plants, as in process control. For faster plants, it will be necessary to implement the optimal control algorithms in some form of dedicated architecture, so as to reduce to the solution time to acceptable levels.

8. APPENDIX I.

We will now establish two inequalities that form the basis of several of our proofs.

Proposition 8.1 Consider the second order scalar difference equation

$$y_{k+2} = a_1 y_{k+1} + a_2 y_k + b \ , \quad k \in \mathbf{N} . \tag{8.1a}$$

If $a_1, a_2 \geq 0$, $b \geq 0$ and $a_1 + a_2 < 1$, then for all $k \geq 1$,

$$y_k \leq a_2 y_0 + y_1 + b/(1 - a_1 + a_2) , \tag{8.1b}$$

and

$$\overline{\lim}_{k \to \infty} y_k \leq b/(1 - a_1 + a_2) . \tag{8.1c}$$

Proof. We begin by rewriting (8.1a) in first order vector form, as follows. For $k \in \mathbf{N}$, let $z_k = (y_k, y_{k+1})^T$. Then $z_0 = (y_0, y_1)^T$, and

$$z_{k+1} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} z_k + \begin{bmatrix} 0 \\ b \end{bmatrix} \triangleq F z_k + g \ , \tag{8.2a}$$

$$y_k = [1 \ 0] z_k \triangleq H z_k . \tag{8.2b}$$

The matrix F has two eigenvalues, $\lambda_+, \lambda_- = \frac{1}{2}(a_1 \pm \sqrt{a_1^2 + 4a_2})$, with corresponding eigenvectors, $e_+ = (1, \lambda_+)^T$ and $e_1 = (1, \lambda_-)^T$. We will now show that $-1 < \lambda_- \leq 0 \leq \lambda_+ < 1$, i.e., that

(8.2a) is an asymptotically stable system. By assumption

$$0 \le a_2 < 1 - a_1 . \tag{8.2c}$$

If we multiply both sides Of (8.2c) by 4, and add a_1^2 to the both sides, we get that

$$a_1^2 + 4a_2 < (2 - a_1)^2 , \tag{8.2d}$$

which implies that $\lambda_- = \frac{1}{2}(a_1 - \sqrt{a_1^2 + 4a_2}) > -1$ and $\lambda_+ = \frac{1}{2}(a_1 + \sqrt{a_1^2 + 4a_2}) < 1$. Thus, we have that $-1 < \lambda_- \le \lambda_+ < 1$.

We can proceed to establish (8.1b,c). By the Jordan decomposition, we have that

$$F = E^{-1} \Lambda E , \tag{8.2e}$$

where $\Lambda = diag(\lambda_+ , \lambda_-)$, and $E = (e_+ , e_-)$ is a matrix whose columns are the eigenvectors of F. Hence for all $k \ge 2$,

$$y_k = H E^{-1} \Lambda^k E z_0$$

$$= \frac{1}{\lambda_- - \lambda_+} \{ \lambda_+ \lambda_- (\lambda_+^{k-1} - \lambda_-^{k-1}) y_0 + (\lambda_-^k - \lambda_+^k) y_1 \} + \frac{b}{\lambda_- - \lambda_+} \sum_{i=0}^{k-1} (\lambda_-^{k-1-i} - \lambda_+^{k-1-i}) . \tag{8.2f}$$

Since $0 < \lambda_+ < 1$ and $-1 < \lambda_- < 0$, it is clear that *(a)* the first term in (8.2f) goes to zero as $k \to \infty$ and *(b)* the last term in (8.2f) satisfies the inequality

$$\frac{b}{\lambda_- - \lambda_+} \sum_{i=0}^{k-1} (\lambda_-^{k-1-i} - \lambda_+^{k-1-i}) \le \frac{b}{\lambda_- - \lambda_+} \left\{ \frac{1}{1 - \lambda_-} - \frac{1}{1 - \lambda_+} \right\} = \frac{b}{1 - a_1 + a_2} , \tag{8.2g}$$

because $(1 - \lambda_+)(1 - \lambda_-) = 1 - a_1 + a_2$, which proves (8.1c).

Next, for all $k \ge 1$, $\lambda_+^k \le \lambda_+$ and $-\lambda_-^k \le (-\lambda_-)^k \le -\lambda_-$ Hence $\{ \lambda_+ \lambda_- (\lambda_+^{k-1} - \lambda_-^{k-1}) / (\lambda_- - \lambda_+) \le -\lambda_+ \lambda_- = a_2$. Also $(\lambda_-^k - \lambda_+^k)/(\lambda_- - \lambda_+) \le 1$, hence (8.1b) hold. □

9. APPENDIX II.

The free-time optimal control problem (2.5a-e) has to be solved at every iteration of Control Algorithm 2.2. The major difficulty in solving this problem stems from the fact that functions such as $|x(t, 0, x_0, u)|^2$ that are convex in u, are not convex in t and hence optimal control algorithms, such as the phase I - phase II algorithms described in [Polak-Mayne, Pol.3], can only be counted on to find local minima for this problem. This difficulty can be eliminated by solving a sequence of convex, fixed time optimal control problems, constructed using an interval bisection technique, whose solutions converge to the desired optimal solution of (2.5a-e), as follows. An important aspect of phase I - phase II algorithms, such as those in [Polak-Mayne, Pol.3], is that when a fixed-time optimal control problem has no solution, then they produce a control which minimizes the maximum constraint violation.

Algorithm 9.1.

Data: $x_k \in B_{\hat{\rho}}$, t_k and $\bar{T}$ such that $\bar{T} - t_k > T_C$ and $\delta \in (0, \bar{T} - T_C - t_k)$.

Step 0: Set $i = 0$, $\tau_0 = \bar{T}$, $T_{\min} = t_k + T_C$, and $T_{\max} = \bar{T}$.

Step 1: Solve the problem (2.5a-e) with τ fixed at the value $\tau = \tau_i$.

Step 2: If the computed control, $u_i(\cdot)$, does not satisfy all the constraints in (2.5a-e),

$$\text{set} \begin{cases} T_{\min} = \tau_i \,,\, T_{\max} = 2\tau_i \,,\text{ and } \tau_{i+1} = T_{\max}\,, & \text{if } \tau_i = T_{\max} \\ T_{\min} = \tau_i \text{ and } \tau_{i+1} = (\tau_i + T_{\max})/2\,, & \text{otherwise.} \end{cases}$$

Else, set $T_{\max} = \tau_i$ and $\tau_{i+1} = (T_{\min} + \tau_i)/2$.

Step 3: If $(T_{\max} - T_{\min}) \leq \delta$, set $t_{k+1} = \tau_{i+1} - T_1$, set $u_{[t_k, t_{k+1}]}(t) = u_i(t)$ for $t \in [t_k, t_{k+1}]$, and stop.

Else, set $i = i + 1$ and go to Step 1. □

Since by definition of $\hat{\rho}$, the original free-time optimal control problem has a solution, it is clear that Algorithm 9.1 terminates after a finite number of iterations.

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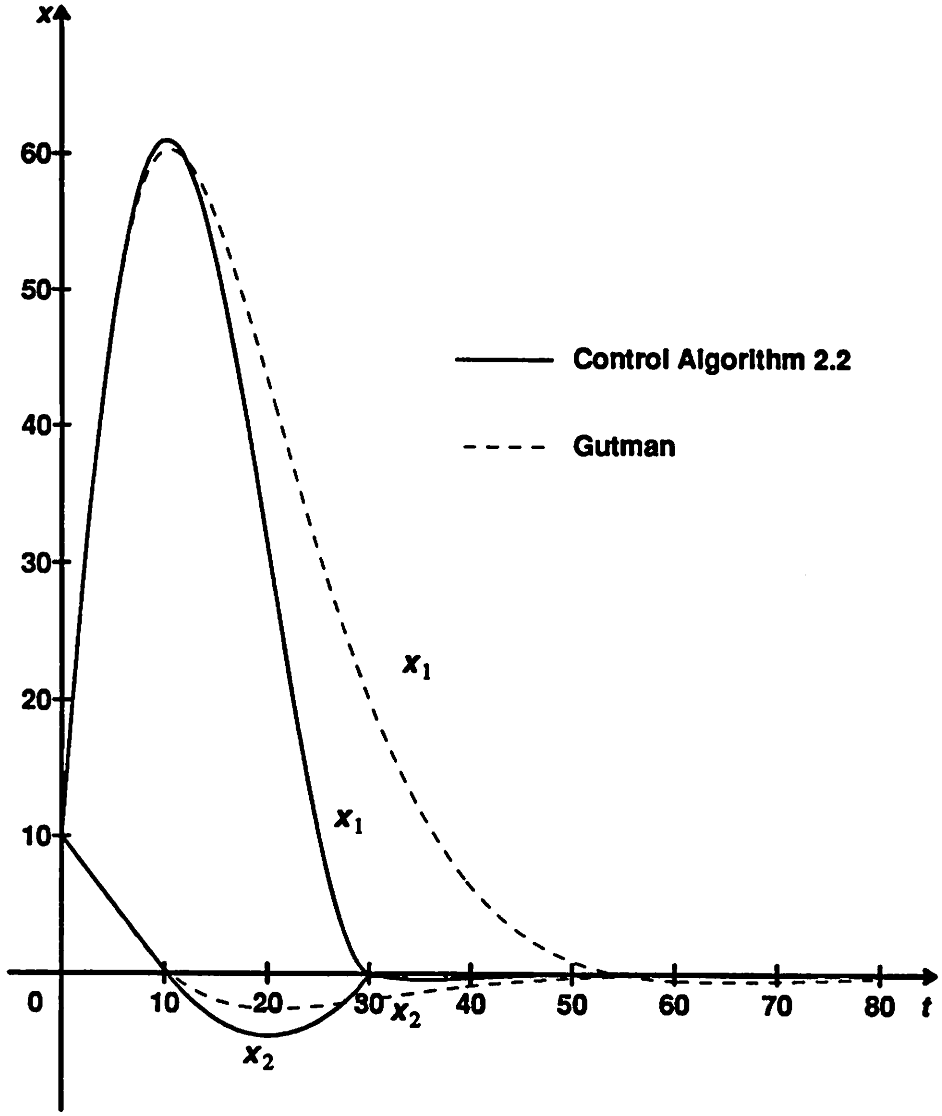

Figure 1. States vs Time

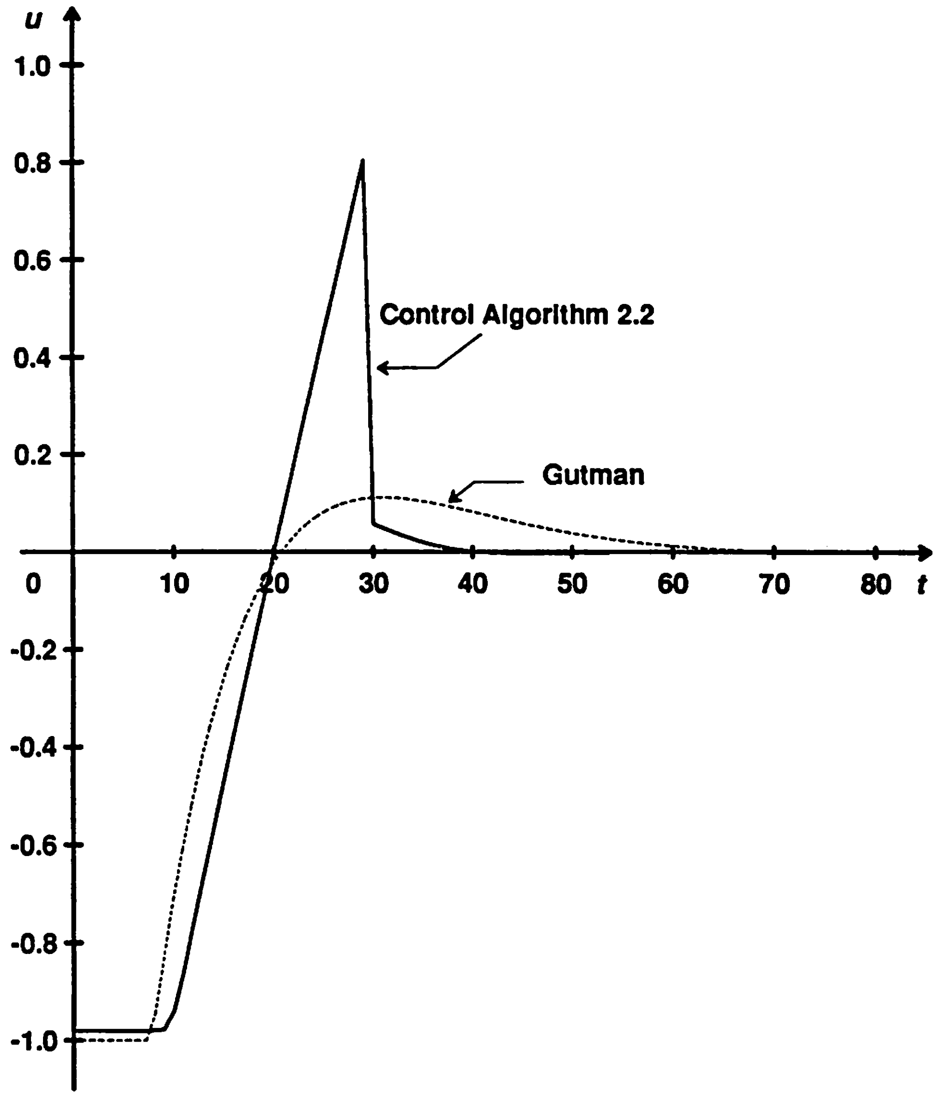

Figure 2. Controls vs Time

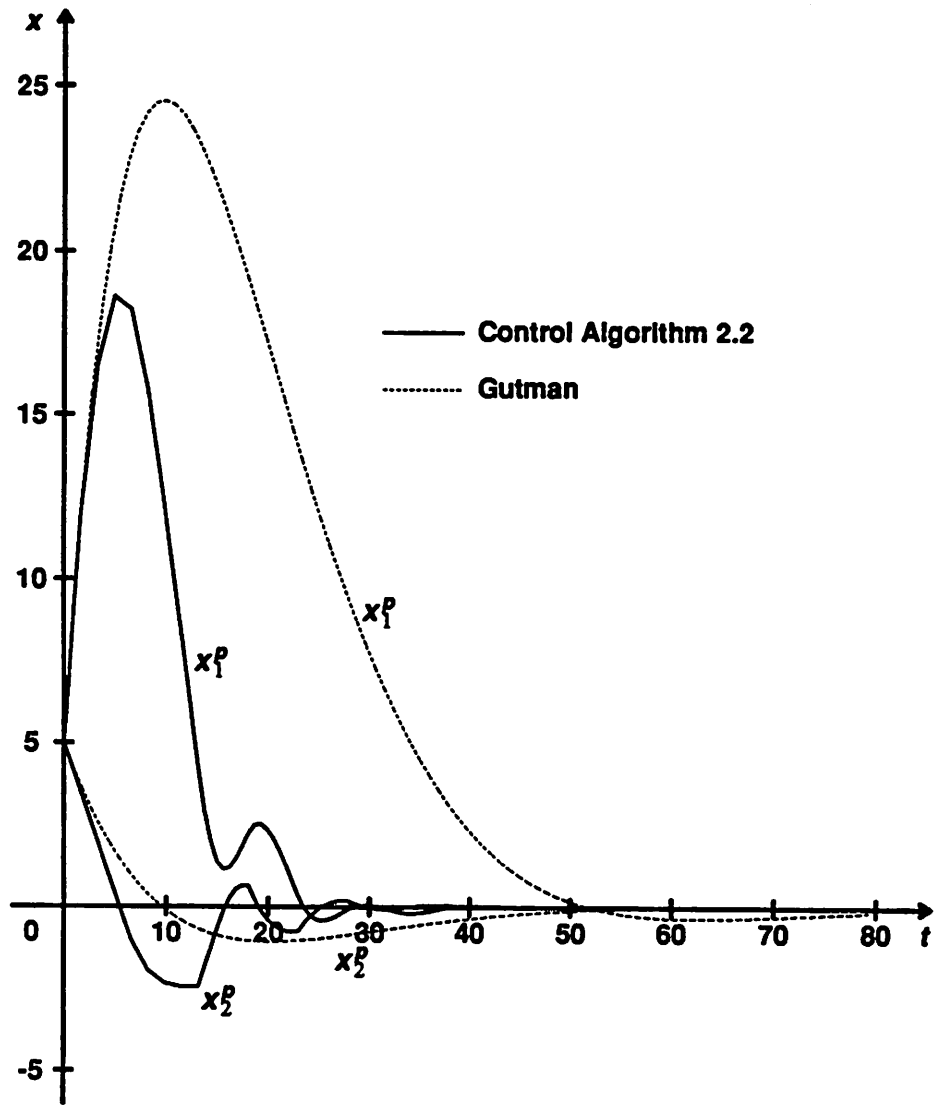

Figure 3. States vs Time with perturbations

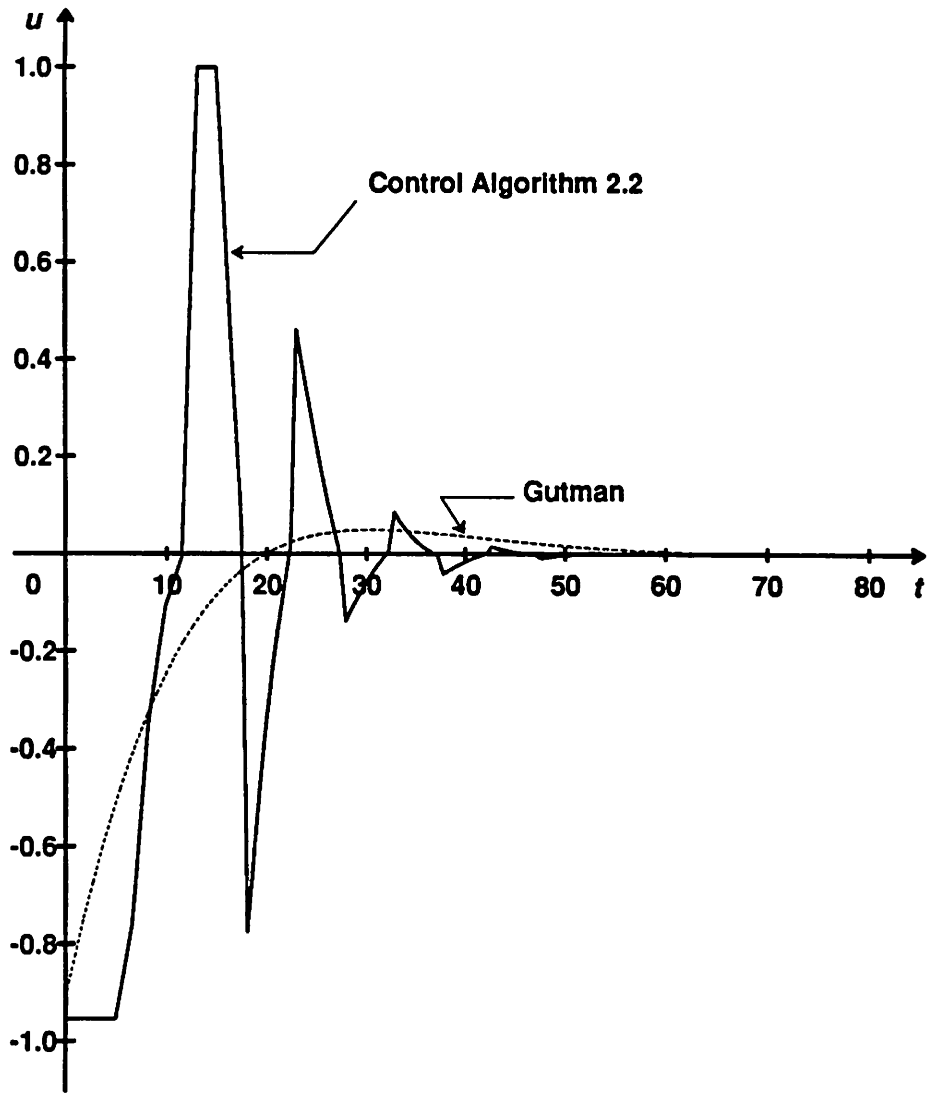

Figure 4. Controls vs Time with Plant Perturbations

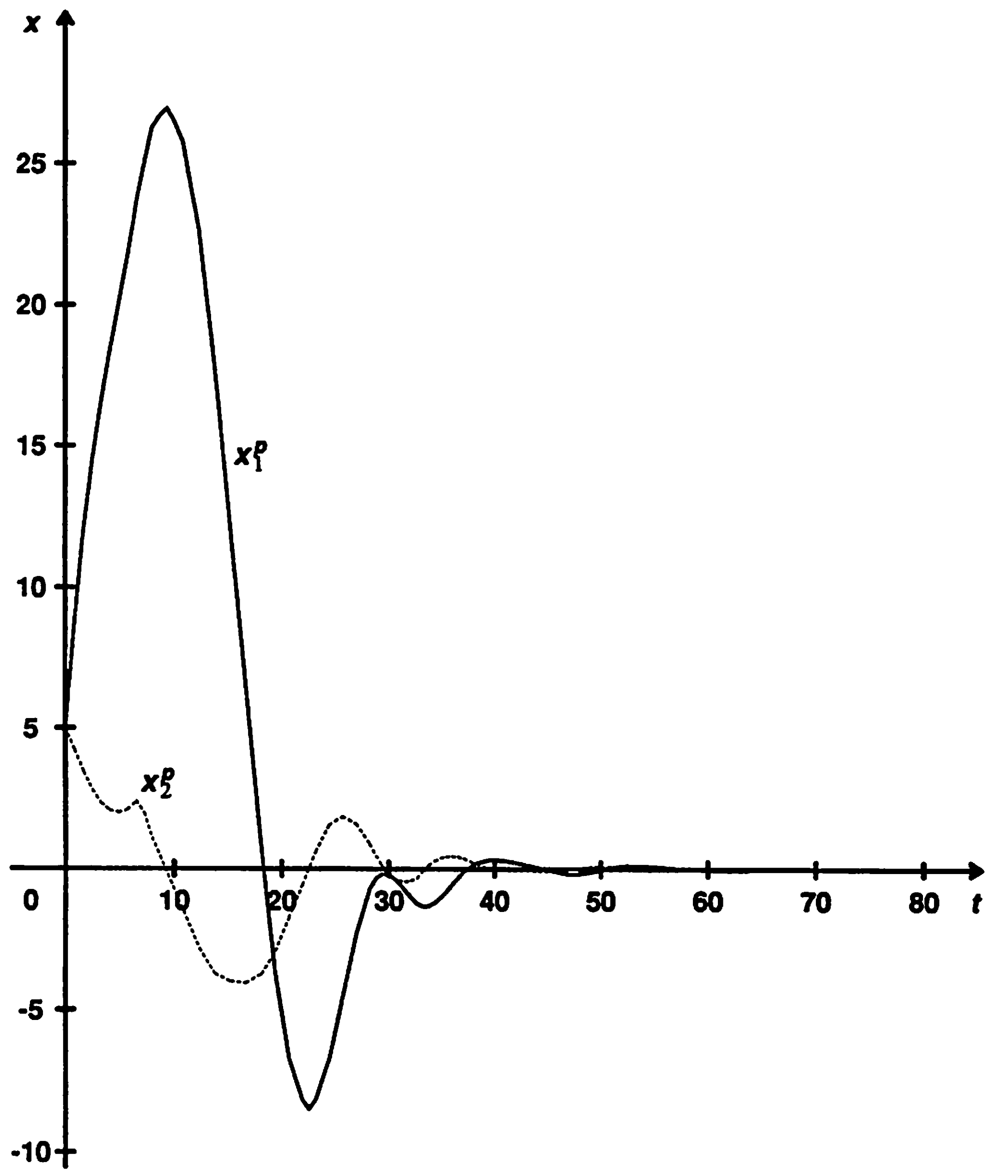

Figure 5. States vs Time with State Estimation

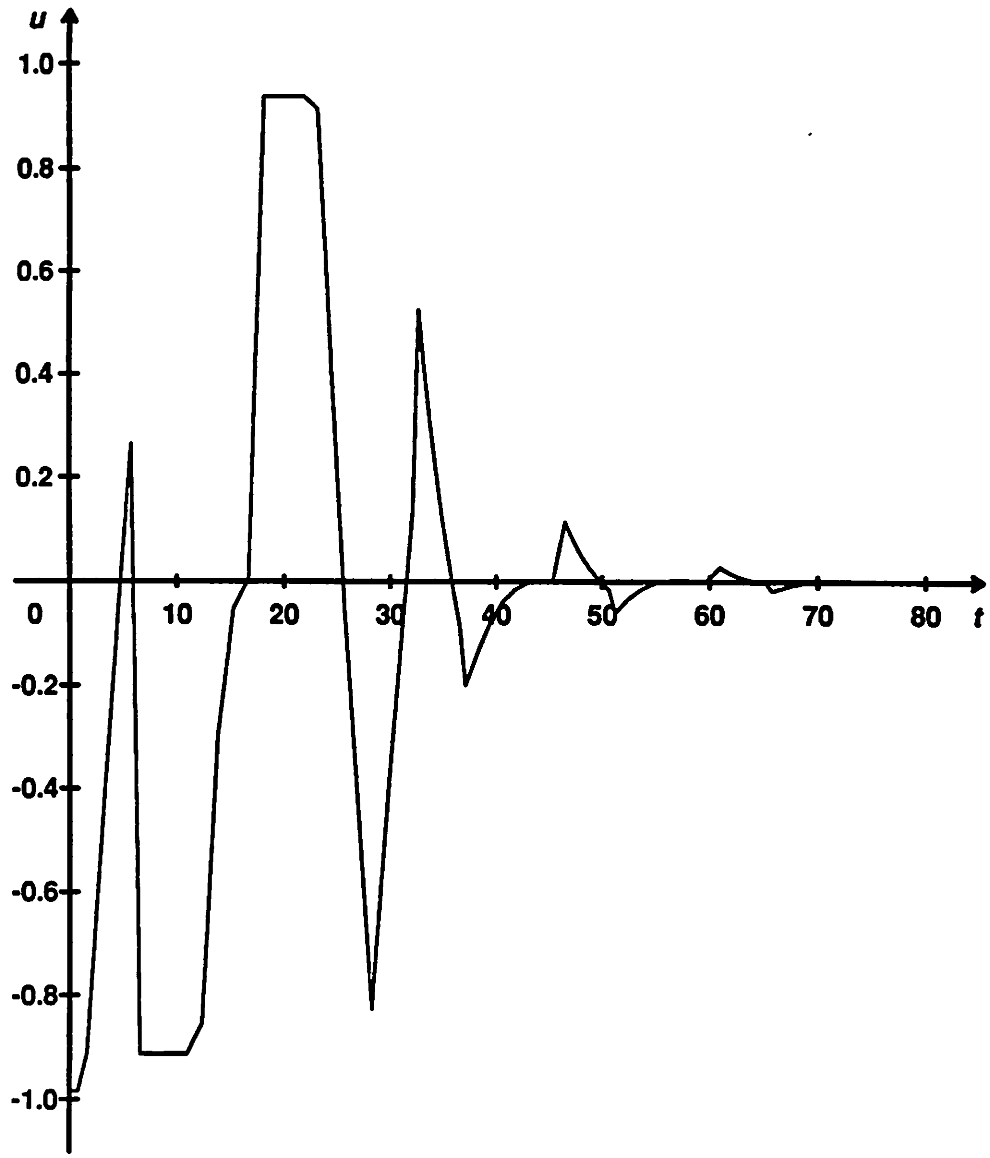

Figure 6. Control vs Time with State Estimation