MOVING HORIZON CONTROL OF NONLINEAR SYSTEMS WITH INPUT SATURATION, DISTURBANCES, AND PLANT UNCERTAINTY

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T. H. Yang* and E. Polak*

ABSTRACT

We present a moving horizon feedback system, based on constrained optimal control algorithms, for nonlinear plants with input saturation, disturbances, and plant uncertainty. The system is a nonconventional sampled-data system: its sampling periods vary from sampling instant to sampling instant, and the control during the sampling time is not constant, but determined by the solution of an open loop optimal control problem. We show that the proposed moving horizon control system is robustly stable and is capable of suppressing a class of disturbances.

KEY WORDS: Moving horizon control, robust stability.

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1. INTRODUCTION

There exist several approaches, including classical frequency domain techniques, for designing robust, stabilizing feedback control laws for linear time invariant systems. However, the situation for linear time varying systems or nonlinear systems is quite different. Following some earlier work on model predictive control of linear time invariant systems (see e.g., [Cla.1 ,2], [Gar.1,2]), we find in the literature an exploration of the possibility of determining feedback laws for nonlinear or time varying systems by repeatedly solving open loop, finite horizon optimal control problems ([Kwo.1,2], [May.1,2], [Mic.1]). Such feedback laws are known as *moving horizon control* laws.

In moving horizon control, the control at time t is obtained by setting the current control equal to $\hat{u}(t)$, a solution of on open loop optimal control problem over the interval $[t, t+T]$, where $T > 0$. Since $\hat{u}(t)$ depends on the current state x, repeating this computation continuously yields a feedback control control. The finite horizon open loop optimal control problem has usually a terminal constraint $x(t+T) = 0$ (cf. [Kwo.1,2], [May.1,2], [Mic.1]). This strategy provides a relatively simple conceptual procedure for determining stabilizing feedback control for time varying or nonlinear systems. In [May.2] the authors proposed an implementable version of such a controller which does not require the exact solution of an associated optimal control problem with terminal constraints on the state. Instead, the optimal control problem was solved approximately, with the terminal constraint $x(t+T) = 0$ replaced by the relaxed constraint $x(t+T) \in W$, where W is some neighborhood of the origin.

Although the concept of moving horizon control is not new and has been proposed in conjunction with various applications, process control being one of them (see e.g., [Meh.1], [Pre.1], [Gar.1]), it has not always been realized that a naive application of the strategy, in adaptive control for example, can lead to instability. The literature that provides an analysis of the stabilizing properties of moving horizon control laws for linear time varying and nonlinear systems deals with schemes based on open loop optimal control laws for finite horizon optimal control problems with quadratic criteria and no control constraints. Thus Kwon and Pearson [Kwo.2], and Kwon, Bruckstein and Kailath [Kwo.2] deal with linear time-varying systems, Keerthi and Gilbert [Kee.1] deal with nonlinear discrete-time systems, and, more recently, Mayne and Michalska have established the stability properties of nonlinear, continuous-time systems with moving horizon control [May.1,2], [Mic.1,2]; see also Chen and Shaw [Che.1]. In [May.2], the stability robustness of a moving horizon control was examined, although the analysis is incomplete. In [Mic.2], the nontrivial time needed for the computation of the open loop controls is taken into account, under the assumption that there is no modeling error. In [Pol.1,2], robust stability, disturbance rejection, and reference following properties of a moving horizon control law for linear time invariant systems, with and without a state estimation, were analyzed, taking into account the time needed for the computation of the open loop controls.

In this paper we propose a stabilizing moving horizon feedback law for time invariant nonlinear systems, modeled with errors and subject to control and state space constraints. This feedback law results in a nonconventional sampled-data system: its sampling periods vary from sampling instant to sampling instant, and the control during the sampling time is not constant, but determined by the solution of an open loop optimal control problem. We will see that taking into account the time needed to solve the open loop optimal control problem and the modeling errors, complicates matters considerably, because the computed optimal control is based on the state of a model that is not an exact representation of the plant. In Section 2 we introduce our proposed moving horizon feedback control law. In Section 3 we show that the proposed moving horizon feedback system is robustly stable. In Section 4, we study the effect of disturbances. Finally, in Section 5, we introduce more structure into the nonlinear system and analyze the stability of the system when the state of the plant has to be estimated.

2. STRUCTURE OF THE MOVING HORIZON CONTROL LAW.

We assume that the plant is a non-linear time-invariant system with bounded controls and an input disturbance, described by the differential equation

$$\dot{x}^p(t) = f^p(x^p(t), u(t), d(t)), \tag{2.1a}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m_d} \to \mathbb{R}^n$ is continuously differentiable, $u \in U, d \in D$ with

$$U \triangleq \{ u \in L_\infty^m[0,\infty) \mid u(t) \in G_u, \forall t \in [0,\infty) \}, \tag{2.1b}$$

$$D \triangleq \{ d \in L_\infty^{m_d}[0,\infty) \mid d(t) \in G_d, \forall t \in [0,\infty) \}, \tag{2.1c}$$

where $G_u \triangleq \{ z \in \mathbb{R}^m \mid \|z\|_\infty \le c_u \}$ and $G_d \triangleq \{ z \in \mathbb{R}^{m_d} \mid \|z\|_\infty \le c_d \}$ with $c_u, c_d \in (0,\infty)$. We will denote the solution of (2.1a) at time t, corresponding to the initial state x_0^p at time t_0, the input u, and the disturbance d, by $x^p(t, t_0, x_0^p, u, d)$.

The function of the receding horizon control law that we are going to propose is to ensure robust stability while taking into account the fact that the plant inputs are bounded as in (2.1b), as well as various amplitude constraints on transients. Since the function $f^p(\cdot,\cdot,\cdot)$ is known only to some tolerance, the receding horizon control law must be developed using a plant model of the same dimension as (2.1a),

$$\dot{x}^m(t) = f^m(x^m(t), u(t), 0). \tag{2.2a}$$

Here, we will assume that the disturbance $d(t)$ cannot be estimated. We will denote the solution of (2.2a) at time t, corresponding to the initial state x_0^m at time t_0, and the input u, by $x^m(t, t_0, x_0^m, u, 0)$.

Consider the linearization of the system (2.2a), in the neighborhood of the origin, i.e., the system

$$\dot{x}_L(t) = f^m_x(0,0,0)x_L(t) + f^m_u(0,0,0)u(t). \tag{2.2b}$$

The following assumptions are needed to ensure local stability.

Assumption 2.1. We assume that $f^m(0,0,0) = 0$ and $f^p(0,0,0) = 0$. □

Assumption 2.2. We assume that $(f^m_x(0,0,0), f^m_u(0,0,0))$ is a controllable pair. □

Consider the linear system described by (2.2b). It follows from Assumption 2.2 that there exists a stabilizing linear feedback matrix K, where K is the solution of a linear quadratic regulator problem in terms of (2.2b). Let

$$A \triangleq f^m_x(0,0,0) - f^m_u(0,0,0)K. \tag{2.3a}$$

Hence, since A is an asymptotically stable matrix, there exists a pair of symmetric, positive definite matrices (Q, M) such that

$$A^TQ + QA = -M. \tag{2.3b}$$

Clearly, the matrix Q defines the Lyapunov function $\langle x, Qx\rangle$ for the linear closed loop system $\dot{x}(t) = Ax(t)$. We use the matrix Q to define the norm

$$\|x\| \triangleq \langle x, Qx\rangle^{1/2}, \tag{2.3c}$$

that we will use throughout this paper.

Given any time t_k, we will let $x_k^m \triangleq x^m(t_k, t_0, x_0^m, u, 0)$. The aperiodic sampled-data feedback law which we are about to describe has the form of an algorithm which, during each sampling period, solves a free time, constrained optimal control problem $P(x_k^m, t_k)$ of the form

$$\mathbf{P}(x_k^m, t_k): \min_{(u,\tau)} \{ g^0(u,\tau) \mid g^i(u,\tau) \le 0,\ i = 1,2,\ldots,l_1,$$

$$\max_{t \in [t_k, \tau]} \phi^j(u,t) \le 0,\ j = 1,\ldots,l_2,\ u \in U,\ \tau \in [t_k + T_C, t_k + \bar{T}] \} , \tag{2.4a}$$

where $0 < T_C < \bar{T} < \infty$, with T_C the least time needed to solve the optimal control problem $\mathbf{P}(x_k^m, t_k)$, $\bar{T}$ is an a priori limit on the control horizon, and

$$g^i(u,\tau) \triangleq h^i(x^m(\tau, t_k, x_k^m, u, 0)),\ i = 0,1,\ldots,l_1-1, \tag{2.4b}$$

$$g^{l_1}(u,\tau) = |x^m(\tau, t_k, x_k^m, u, 0)|^2 - \alpha^2 |x_k^m|^2, \tag{2.4c}$$

$$\phi^j(u,t) = h^j(x^m(t, t_k, x_k^m, u, 0), t),\ j = 1,\ldots,l_2-1,\ \ t \in [t_k, \tau] \tag{2.4d}$$

$$\phi^{l_2}(u,t) = |x^m(t, t_k, x_k^m, u, 0)|^2 - \beta^2 |x_k^m|^2,\ \ t \in [t_k, \tau]. \tag{2.4e}$$

The constraint functions (2.4c,e) with $\alpha \in (0,1)$ and $\beta \in [1,\infty)$, are used to ensure robust stability, while the other constraint functions, h^i, h^j are convex, locally Lipschitz continuously differentiable functions that can be used to ensure other performance requirements.

We will denote the optimal solution pair to $\mathbf{P}(x_k^m, t_k)$ by $(u_{[t_k, t_{k+1}]}(\cdot), t_{k+1})$. Clearly, the optimal control is defined only on the interval $[t_k, t_{k+1}]$.

The fact that the plant inputs are bounded limits the region of effectiveness of any control law, particularly for unstable plants. Hence we must assume that the initial states are confined to a ball $B_{\hat{\rho}} \triangleq \{ x \in \mathbb{R}^n \mid |x| \le \hat{\rho} \} \subset \mathbb{R}^n$, postulated as follows.

Assumption 2.3. We assume that there exists a $\hat{\rho} \in (0,\infty)$ such that for all $x \in B_{\hat{\rho}} \triangleq \{ x \in \mathbb{R}^n \mid |x| \le \hat{\rho} \}$, the optimal control problem $\mathbf{P}(x, 0)$ has a solution. □

Assumption 2.4. Let $\hat{\rho} > 0$ be as defined in Assumption 2.3, and let the reachable set $\mathbf{R}$ be defined by

$$\mathbf{R} \triangleq \{ x \in \mathbb{R}^n \mid x = x^m(t, 0, x_0, u, d) \text{ or } x = x^p(t, 0, x_0, u, d),$$

$$t \in [0, \bar{T}], x_0 \in B_{\hat{\rho}}, u \in U, d \in D \} . \tag{2.5a}$$

We assume that there exist a Lipschitz constant $L \in [0,\infty)$ and a modeling bound $K_m \in [0,\infty)$ such that for all $\xi', \xi'' \in \mathbf{R}$, $v', v'' \in G_u$, and $\delta', \delta'' \in G_d$,

$$|f^m(\xi', v', \delta') - f^m(\xi'', v'', \delta'')| \le L(|\xi' - \xi''| + |v' - v''|_\infty + |\delta' - \delta''|_\infty) \tag{2.5b}$$

$$|f^p(\xi', v', \delta') - f^p(\xi'', v'', \delta'')| \le L(|\xi' - \xi''| + |v' - v''|_\infty + |\delta' - \delta''|_\infty) \tag{2.5c}$$

$$|f^p(\xi', v', \delta') - f^m(\xi', v', \delta')| \le K_m(|\xi'| + |v'|_\infty + |\delta'|_\infty)\,. \tag{2.5d}$$

□

We are now ready to state our control algorithm that defines the moving horizon feedback control system.

Control Algorithm 2.5.

Data: $t_0 = 0, t_1 = T_C, x_0^p$, and $u_{[t_0, t_1]}(t) \equiv 0$. T_C and $\bar{T}$ such that $0 < T_C < \bar{T} < \infty$.

Step 0: Set $k = 0$.

Step 1: At $t = t_k$,

(a) Measure the state $x_k^p = x^p(t_k, 0, x_0^p, u, d)$;

(b) Set the plant input $u(t) = u_{[t_k, t_{k+1}]}(t)$, for $t \in [t_k, t_{k+1})$.

(c) Compute the estimate $x_{k+1}^m \triangleq x^m(t_{k+1}, t_k, x_k^p, u_{[t_k, t_{k+1}]}, 0)$ of the state of the plant $x^p(t_{k+1}, t_k, x_k^p, u_{[t_k, t_{k+1}]}, d)$ by solving (2.2a) with state x_k^p at time $t = t_k$, and an input $u_{[t_k, t_{k+1}]}(t)$, $t \in [t_k, t_{k+1}]$.

(d) Solve the open loop optimal control problem $P(x_{k+1}^m, t_{k+1})$ to compute the next sampling time $t_{k+2} \in (t_{k+1} + T_C, t_{k+1} + \bar{T}]$, and the optimal control $u_{[t_{k+1}, t_{k+2}]}(t)$, $t \in [t_{k+1}, t_{k+2})$.

Step 2: Replace k by $k+1$ and go to Step 1. □

The following theorem generalizes a result given in [Pol.3].

Theorem 2.6. Suppose that *(a)* Assumption 2.3 is satisfied, *(b)* $d(t) \equiv 0$, *(c)* the systems (2.1a) and (2.2a) are identical, *(d)* the state of the plant is measurable, and *(e)* the Control Algorithm 2.5 is used to define the input $u(\cdot)$ for (2.1a). Then the resulting feedback system is asymptotically stable on the set $B_{\hat{\rho}}$, i.e., for any $x_0^p \in B_{\hat{\rho}}$, the resulting trajectory $x^p(t, 0, x_0^p, u, 0)$ satisfies that $|x^p(t, 0, x_0^p, u, 0)| \le \beta |x_0^p|$ for all $t \ge 0$ and that $x^p(t, 0, x_0^p, u, 0) \to 0$ as $t \to \infty$.

Proof. We begin by showing that for any $x_0^p \in B_{\hat{\rho}}$, the discrete time trajectory $x_k^p \triangleq x^p(t_k, 0, x_0^p, u, 0)$, $k \in \mathbb{N}$ resulting from the use of the Control Algorithm 2.5 is contained in $B_{\hat{\rho}}$. We note that $x^p(t, 0, x_0^p, u, 0) = x^m(t, 0, x_0^p, u, 0)$ for all $t \ge 0$. It follows from the form of (2.4c), that for all $k \in \mathbb{N}$,

$$|x^p_{k+1}| = |x^p(t_{k+1}, t_k, x^p_k, u_{[t_k, t_{k+1}]}, 0)| = |x^m_{k+1}| \le \alpha |x^m_k| \le \alpha^{k+1}|x^p_0| . \quad (2.6a)$$

Since $\alpha \in (0, 1)$, it follows that $x^m_k \in B_{\hat{\rho}}$ for all $k \in \mathbb{N}$ and hence that the trajectory $x^p(t, 0, x^p_0, u, 0)$ is well defined.

Next, from the form of (2.4e), we see that for all $k \in \mathbb{N}$ and for any $t \in [t_k, t_{k+1}]$,

$$|x^p(t, t_k, x^p_k, u_{[t_k, t_{k+1}]}, 0)| = |x^m(t, t_k, x^p_k, u_{[t_k, t_{k+1}]}, 0)| \le \beta |x^m_k| \le \beta\alpha^k |x^p_0| \le \beta |x^p_0| . \quad (2.6b)$$

Since $x^p_0 \in B_{\hat{\rho}}$, it follows from (2.6b) that $|x^p(t, 0, x^p_0, u, 0)| \le \beta |x^p_0|$ for all $t \ge 0$. Finally, because $\beta\alpha^k \to 0$ as $k \to \infty$, it follows that $x^p(t, 0, x^p_0, u, 0) \to 0$ as $t \to \infty$, and hence that the feedback system defined by the Control Algorithm 2.5 is asymptotically stable on the set $B_{\hat{\rho}}$. □

3. ROBUST STABILITY.

We will now determine under what conditions the local asymptotically stability of the closed loop system resulting from the use of Control Algorithm 2.5 is preserved while there is a difference between the actual plant equation (2.1a) and the model equation (2.2a). We will assume that the state of the plant is measurable and that there is no disturbance, i.e., $d(t) \equiv 0$. Since there is no disturbance, we can simplify our notation by letting $f^m(\xi, v) \triangleq f^m(\xi, v, 0)$, $f^p(\xi, v) \triangleq f^p(\xi, v, 0)$, $x^m(t, 0, x_0, u) \triangleq x^m(t, 0, x_0, u, 0)$, and $x^p(t, 0, x_0, u) \triangleq x^p(t, 0, x_0, u, 0)$. We will consider two strategies: the first is where we use Control Algorithm 2.5 only and the second one where we use a cross-over rule to a linear state feedback law near the origin so that residual errors can be eliminated. (The latter strategy was introduced in [May.2], where the analysis was carried out under the assumption that there are no modeling error.

We observe that Control Algorithm 2.5 generates three sequences. The first sequence is that of measured state of the plant $\{x^p_k\}_{k=0}^{\infty}$, the second sequence is that of state estimates $\{x^m_k\}_{k=1}^{\infty}$, i.e. $x^m_k \triangleq x^m(t_k, t_{k-1}, x^p_{k-1}, u_{[t_{k-1}, t_k]})$, and finally, the sequence $\{x'_k\}_{k=2}^{\infty}$, generated in the process of solving the optimal control problem $P(x^m_{k+1}, t_{k+1})$, $k \in \mathbb{N}$, i.e. $x'_k = x^m(t_k, t_{k-1}, x^m_{k-1}, u_{[t_{k-1}, t_k]})$. By construction, the sequences $\{x^m_k\}_{k=1}^{\infty}$ and $\{x'_k\}_{k=2}^{\infty}$, satisfy the relation $|x'_{k+2}| \le \alpha |x^m_{k+1}|$ for all $k \in \mathbb{N}$.

Lemma 3.1. Consider the moving horizon feedback system resulting from the use of Control Algorithm 2.5. Suppose that Assumptions 2.3 and 2.4 are satisfied. Let $\hat{\rho}$ be as postulated in Assumption

2.3, and let $L, K_m \in [0,\infty)$ satisfy (2.5b-d). If $x_k^p, x_k^m \in B_{\hat\rho}$ for all $k \in \mathbf{N}$, then there exist Δ_1 and $\Delta_2 \geq 0$ such that

$$\|x_{k+1}^p - x_{k+1}^m\| \leq \Delta_1 \|x_k^p\| + \Delta_2, \tag{3.1}$$

and $\Delta_1, \Delta_2 \to 0$ as $K_m \to 0$.

Proof. First, the optimal control problem $\mathbf{P}(x_k^m, t_k)$ has a solution for all $k \geq 1$ since $x_k^m \in B_{\hat\rho}$ for all $k \geq 1$, and as a result, the trajectory $x^p(t, 0, x_0^p, u)$ is well defined. Given $x_k^p \in B_{\hat\rho}$ and $u = u_{[t_k, t_{k+1}]} \in U$, obtained by solving the optimal control problem $\mathbf{P}(x_k^m, t_k)$, let $x_k^p(t) \triangleq x^p(t, t_k, x_k^p, u)$ and $x_k^m(t) \triangleq x^m(t, t_k, x_k^p, u)$. Then for all $t \in [t_k, t_{k+1}]$

$$\|x_k^p(t) - x_k^m(t)\| \leq \int_{t_k}^t \|f^p(x_k^p(\tau), u(\tau)) - f^m(x_k^m(\tau), u(\tau))\| \, d\tau$$

$$\leq \int_{t_k}^t \Big(\|f^p(x_k^p(\tau), u(\tau)) - f^p(x_k^m(\tau), u(\tau))\| + \|f^p(x_k^m(\tau), u(\tau)) - f^m(x_k^m(\tau), u(\tau))\| \Big) \, d\tau$$

$$\leq L \int_{t_k}^t \|x_k^p(\tau) - x_k^m(\tau)\| \, d\tau + K_m \int_{t_k}^t (\|x_k^m(\tau)\| + \|u(\tau)\|_\infty) \, d\tau. \tag{3.2a}$$

We obtain that for all $t \in [t_k, t_{k+1}]$,

$$\|x_k^m(t)\| \leq \|x_k^p\| + \int_{t_k}^t \|f^m(x_k^m(\tau), u(\tau))\| \, d\tau \leq \|x_k^p\| + \int_{t_k}^t L(\|x_k^m(\tau)\| + \|u(\tau)\|_\infty) \, d\tau$$

$$\leq \|x_k^p\| + \int_{t_k}^t L \, \|x_k^m(\tau)\| \, d\tau + L c_u \bar{T}. \tag{3.2b}$$

It follows from the Bellman-Gronwall inequality and from the fact that $t_{k+1} - t_k \leq \bar{T}$ for any $k \in \mathbf{N}$ that

$$\|x_k^m(t)\| \leq e^{L(t-t_k)} \|x_k^p\| + L c_u \bar{T} e^{L(t-t_k)}. \tag{3.2c}$$

Hence, for any $t \in [t_k, t_{k+1}]$

$$\int_{t_k}^t \|x_k^m(\tau)\| d\tau \leq \int_{t_k}^t \{ e^{L(\tau-t_k)} \|x_k^p\| + L c_u \bar{T} e^{L(\tau-t_k)} \} \, d\tau$$

$$= \{ (e^{L(t-t_k)} - 1)/L \} \|x_k^p\| + c_u \bar{T} (e^{L(t-t_k)} - 1)$$

$$\leq \{ (e^{L\bar{T}} - 1)/L \} \|x_k^p\| + c_u \bar{T} (e^{L\bar{T}} - 1). \tag{3.2d}$$

By substituting (3.2d) into (3.2a), we obtain that for all $t \in [t_k, t_{k+1}]$

$$\|x_k^p(t)-x_k^m(t)\| \le K_m\left[((e^{L\bar T}-1)/L)\|x_k^p\|+(e^{L\bar T}-1)c_u\bar T+c_u\bar T\right]+L\int_{t_k}^{t}\|x_k^p(\tau)-x_k^m(\tau)\|\,d\tau. \quad (3.2e)$$

It follows from the Bellman-Gronwall inequality that

$$\|x_{k+1}^p-x_{k+1}^m\| \le \{K_m e^{L\bar T}(e^{L\bar T}-1)/L\}\|x_k^p\|+K_m c_u \bar T e^{2L\bar T} \triangleq \Delta_1\|x_k^p\|+\Delta_2. \quad (3.2f)$$

It is clear that $\Delta_1, \Delta_2 \to 0$ as $K_m \to 0$, which completes our proof. □

To use Control Algorithm 2.5, we must have that $x_k^m \in B_{\hat\rho}$ for all $k \ge 1$ so that the optimal control problem $P(x_k^m, t_k)$ has a solution. In Lemma 3.1, we have assumed that $x_k^m \in B_{\hat\rho}$ for all $k \ge 1$. We will now establish a condition which guarantees that $x_k^m \in B_{\hat\rho}$ for all $k \ge 1$.

Lemma 3.2. Suppose that Assumption 2.3 and 2.4 are satisfied. Let $\hat\rho$, L, K_m be as postulated in Assumption 2.3 and Assumption 2.4, respectively. Then there exist $\bar K_m \in (0,\infty)$ and $\rho_s \in (0,\hat\rho]$ such that if $K_m < \bar K_m$, then for all $x_0^p \in B_{\rho_s}$, the sequences $\{x_k^p\}_{k=0}^{\infty}$ and $\{x_k^m\}_{k=1}^{\infty}$, resulting from the use of Control Algorithm 2.5, are well defined and stay in the set $B_{\hat\rho}$.

Proof. We will prove this lemma by contradiction. Suppose that for every $\bar K_m > 0$, $\rho \in (0,\hat\rho]$, and $K_m \le \bar K_m$, there exist $x_0^p \in B_\rho$ and $\hat k \in \mathbb{N}$ such that $x_k^m, x_k^p \in B_{\hat\rho}$ for all $k < \hat k$ and $x_{\hat k}^p \notin B_{\hat\rho}$ or $x_{\hat k}^m \notin B_{\hat\rho}$. We will consider three cases: *(a)* $\hat k = 1$, *(b)* $\hat k = 2$, and *(c)* $\hat k > 2$. To compress our notation, we let $x_k^p(t) \triangleq x^p(t, t_k, x_k^p, u_{[t_k, t_{k+1}]})$, $x_k^m(t) \triangleq x^m(t, t_k, x_k^p, u_{[t_k, t_{k+1}]})$, and $x'_k(t) \triangleq x^m(t, t_k, x_k^m, u_{[t_k, t_{k+1}]})$ for all $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$. We note that $x_{k+1}^p = x_k^p(t_{k+1})$, $x'_{k+2} = x'_{k+1}(t_{k+2})$, $x_{k+1}^m = x_k^m(t_{k+1})$, and $\|x'_{k+2}\| \le \alpha\|x_k^m(t_{k+1})\|$ for all $k \in \mathbb{N}$.

(a) Suppose that $\hat k = 1$. Since $u(t) = 0$ for all $t \in [0, t_1]$ and $x_0^m = x_0^p$, we have that for all $t \in [0, t_1]$

$$\|x_0^p(t)\| \le \|x_0^p\| + \int_0^t \|f^p(x_0^p(\tau), 0)\|d\tau \le \|x_0^p\| + L\int_0^t \|x_0^p(\tau)\|d\tau \quad (3.3a)$$

$$\|x_0^m(t)\| \le \|x_0^p\| + \int_0^t \|f^m(x_0^m(\tau), 0)\|dt \le \|x_0^p\| + L\int_0^t \|x_0^m(\tau)\|d\tau. \quad (3.3b)$$

It follows from Bellman-Gronwall inequality that for all $t \in [0, t_1]$

$$\|x_0^p(t)\| \le \|x_0^p\|e^{Lt} \quad (3.3c)$$

$$|x_0^m(t)| \le |x_0^p| e^{Lt} . \tag{3.3d}$$

Let $\rho \in (0, \hat{\rho}/e^{L\bar{T}}]$ and $\bar{K}_m > 0$ arbitrary. Then, since $x_0^p \in B_\rho$, it is obvious that $x_1^p, x_1^m \in B_{\hat{\rho}}$ where $x_1^p = x_0^p(t_1)$ and $x_1^m = x_0^m(t_1)$, which contradicts our assumption.

(b) Suppose that $\hat{k} = 2$. Since $x_1^m \in B_{\hat{\rho}}$, there exists a solution to the optimal control problem $P(x_1^m, t_1)$. It follows from the fact that $|x'|_2 \le \alpha |x_1^m|$ that

$$|x_2^p| \le |x_2^p - x'_2| + \alpha |x_1^m| . \tag{3.4a}$$

Next, using the Bellman-Gronwall inequality, it can be easily shown that for any $t \in [t_1, t_2]$,

$$|x_1^p(t)| \le e^{L(t-t_1)} \Big[|x_1^p| + L c_u \bar{T} \Big] . \tag{3.4b}$$

Now it follows from (3.1), (3.3c), and (3.4b) that for all $t \in [t_1, t_2]$

$$|x_1^p(t) - x'_1(t)| \le |x_1^p - x_1^m| + \int_{t_1}^t |f^p(x_1^p(\tau), u(\tau)) - f^m(x'_1(\tau), u(\tau))| d\tau$$

$$\le |x_1^p - x_1^m| + \int_{t_1}^t |f^p(x_1^p(\tau), u(\tau)) - f^m(x_1^p(\tau), u(\tau))| d\tau$$

$$+ \int_{t_1}^t |f^m(x_1^p(\tau), u(\tau)) - f^m(x'_1(\tau), u(\tau))| d\tau$$

$$\le |x_1^p - x_1^m| + K_m \int_{t_1}^t (|x_1^p(\tau)| + |u(\tau)|_\infty) d\tau + L \int_{t_1}^t |x_1^p(\tau) - x'_1(\tau)| d\tau .$$

$$\le \Delta_1 |x_0^p| + \Delta_2 + (K_m/L)(e^{L\bar{T}} - 1)|x_1^p| + K_m c_u \bar{T} e^{L\bar{T}} + L \int_{t_1}^t |x_1^p(\tau) - x'_1(\tau)| d\tau$$

$$\le (\Delta_1 + K_m e^{L\bar{T}}(e^{L\bar{T}} - 1)/L)|x_0^p| + \Delta_2 + K_m c_u \bar{T} e^{L\bar{T}} + L \int_{t_1}^t |x_1^p(\tau) - x'_1(\tau)| d\tau$$

$$= 2\Delta_1 |x_0^p| + (1 + 1/e^{L\bar{T}})\Delta_2 + L \int_{t_1}^t |x_1^p(\tau) - x'_1(\tau)| d\tau , \tag{3.4c}$$

where Δ_1, Δ_2 were defined in (3.2f). Again, making use of the Bellman-Gronwall inequality, we obtain that

$$|x_1^p(t_2) - x'_1(t_2)| = |x_2^p - x'_2| \le 2e^{L\bar{T}}\Delta_1 |x_0^p| + (1 + e^{L\bar{T}})\Delta_2 . \tag{3.4d}$$

By substituting (3.4d) into (3.4a) and using (3.3d), we obtain that

$$|x_2^p| \le (2\Delta_1 + \alpha) e^{L\bar{T}} |x_0^p| + (1 + e^{L\bar{T}})\Delta_2 . \tag{3.4e}$$

Next it follows from (3.1), (3.3c), and (3.4e) that

$$|x_2^m| \le |x_2^m - x_2^p| + |x_2^p| \le \Delta_1 |x_1^p| + \Delta_2 + |x_2^p|$$

$$\le (3\Delta_1 + \alpha) e^{L\bar{T}} |x_0^p| + (2 + e^{L\bar{T}})\Delta_2 . \qquad (3.4f)$$

Since $|x_0^p| \to 0$ as $\rho \to 0$ and $\Delta_1, \Delta_2 \to 0$ as $K_m \to 0$, there exist $\bar{K}_m > 0$ and $\rho_s \in (0, \hat{\rho})$ such that if $K_m < \bar{K}_m$ and $x_0^p \in B_{\rho_s}$, then $x_2^p, x_2^m \in B_{\hat{\rho}}$, which contradicts our assumption.

(c) Suppose that $\hat{k} > 2$. Since $x_{k+1}^p, x_{k+1}^m \in B_{\hat{\rho}}$ for all $k \le \hat{k} - 2$, there exists a solution to the optimal control problem $\mathbf{P}(x_{k+1}^m, t_{k+1})$, $(u_{[t_{k+1}, t_{k+2}]}(\cdot), t_{k+2})$, such that $|x'_{k+2}| \le \alpha |x_{k+1}^m|$. Therefore, $x_{k+1}^p(t)$ is well defined for all $t \in [t_{k+1}, t_{k+2}]$, $k \le \hat{k} - 2$, and

$$|x_{k+2}^p| \le |x_{k+2}^p - x'_{k+2}| + \alpha |x_{k+1}^m| . \qquad (3.5a)$$

Now by comparison with (3.4b), we see that for any $t \in [t_{k+1}, t_{k+2}]$,

$$|x_{k+1}^p(t)| \le e^{L(t - t_{k+1})} \left[|x_{k+1}^p| + L c_u \bar{T} \right] . \qquad (3.5b)$$

By comparison with (3.4c), we obtain that for all $t \in [t_{k+1}, t_{k+2}]$

$$|x_{k+1}^p(t) - x'_{k+1}(t)| \le |x_{k+1}^p - x_{k+1}^m| + L \int_{t_{k+1}}^{t} |x_{k+1}^p(\tau) - x'_{k+1}(\tau)| \, d\tau + K_m \int_{t_{k+1}}^{t} (|x_{k+1}^p(\tau)| + |u(\tau)|_\infty) \, d\tau .$$

$$\le \Delta_1 |x_k^p| + \Delta_2 + ((K_m / L)|x_{k+1}^p| + K_m c_u \bar{T})(e^{L\bar{T}} - 1) + K_m \bar{T} c_u + L \int_{t_{k+1}}^{t} |x_{k+1}^p(\tau) - x'_{k+1}(\tau)| \, d\tau$$

$$= \Delta_1 |x_k^p| + \Delta_2 + \frac{\Delta_1}{e^{L\bar{T}}} |x_{k+1}^p| + \frac{\Delta_2}{e^{L\bar{T}}} + L \int_{t_{k+1}}^{t} |x_{k+1}^p(\tau) - x'_{k+1}(\tau)| \, d\tau , \qquad (3.5c)$$

where Δ_1, Δ_2 were defined in (3.2f). Again making use of the Bellman-Gronwall inequality, we obtain that for all $t \in [t_{k+1}, t_{k+2}]$

$$|x_{k+1}^p(t) - x'_{k+1}(t)| \le \Delta_1 |x_{k+1}^p| + \Delta_1 e^{L\bar{T}} |x_k^p| + \Delta_2 (1 + e^{L\bar{T}}) . \qquad (3.5d)$$

It follows from the fact that $|x_{k+1}^m| \le |x_{k+1}^p - x_{k+1}^m| + |x_{k+1}^p|$, $x'_{k+2} = x'_{k+1}(t_{k+2})$, (3.1), and (3.5d) that for all $k \le \hat{k} - 2$, we obtain that

$$|x_{k+2}^p| \le |x_{k+2}^p - x'_{k+2}| + \alpha |x_{k+1}^m|$$

$$\le \Delta_1 |x_{k+1}^p| + \Delta_1 e^{L\bar{T}} |x_k^p| + \Delta_2 (1 + e^{L\bar{T}}) + \alpha |x_{k+1}^m - x_{k+1}^p| + \alpha |x_{k+1}^p|$$

$$\le (\Delta_1 + \alpha) |x_{k+1}^p| + \Delta_1 (\alpha + e^{L\bar{T}}) |x_k^p| + \Delta_2 (1 + \alpha + e^{L\bar{T}}) . \qquad (3.5e)$$

Next we will show that if

$$\bar{K}_m \triangleq \min\left[\frac{L(1-\alpha)}{e^{L\bar{T}}(e^{L\bar{T}}-1)(1+\alpha+e^{L\bar{T}})}\,,\ \frac{\hat{\rho}}{c_u\bar{T}e^{2L\bar{T}}}\left\{\frac{2+e^{L\bar{T}}}{\bar{\varepsilon}}+1\right\}^{-1}\right], \tag{3.5f}$$

where $\bar{\varepsilon} \triangleq (1-\alpha)(\alpha+e^{L\bar{T}})/(1+\alpha+e^{L\bar{T}})$, then $|x_{\hat{k}}^p|, |x_{\hat{k}}^m| \le \hat{\rho}$ must hold, which contradicts our hypothesis. Thus, suppose that (3.5f) holds. Then, it follows from (3.2f) that

$$\Delta_1 = \frac{K_m e^{L\bar{T}}(e^{L\bar{T}}-1)}{L} \le \frac{\bar{K}_m e^{L\bar{T}}(e^{L\bar{T}}-1)}{L} \triangleq \varepsilon_1 < \frac{1-\alpha}{1+\alpha+e^{L\bar{T}}}\,, \tag{3.5g}$$

and

$$\Delta_2 = K_m c_u\bar{T}e^{2L\bar{T}} \le \bar{K}_m c_u\bar{T}e^{2L\bar{T}} \triangleq \varepsilon_2 < \left[\frac{2+e^{L\bar{T}}}{\bar{\varepsilon}}+1\right]^{-1}\hat{\rho}\,. \tag{3.5h}$$

Let $a_1 = \Delta_1+\alpha$, $a_2 = \Delta_1(\alpha+e^{L\bar{T}})$, and $b = \Delta_2(1+\alpha+e^{L\bar{T}})$. Let $z_k \triangleq (y_k, y_{k+1})^T$ with $y_0 = |x_0^p|$ and $y_1 = |x_1^p|$. Consider the discrete time system

$$z_k = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} z_k + \begin{bmatrix} 0 \\ b \end{bmatrix} \triangleq Fz_k+g\,, \tag{3.5i}$$

$$y_k = [1\ 0]z_k \triangleq Hz_k\,. \tag{3.5j}$$

It is clear that for all $k \le \hat{k}$, $|x_k^p| \le y_k$. Since $a_1, a_2 \ge 0$ and

$$a_1+a_2 = \Delta_1(1+\alpha+e^{L\bar{T}})+\alpha < 1\,, \tag{3.5k}$$

the conditions of Proposition 6.1 (see Appendix) are satisfied. Now,

$$1-a_1+a_2 = 1-\Delta_1-\alpha+(e^{L\bar{T}}+\alpha)\Delta_1 > 1-\alpha-(1-\alpha)/(1+\alpha+e^{L\bar{T}})$$

$$= \frac{(1-\alpha)(e^{L\bar{T}}+\alpha)}{1+\alpha+e^{L\bar{T}}} \triangleq \varepsilon' > 0\,. \tag{3.5l}$$

Hence it follows from Proposition 6.1, (3.5e), and (3.5l) that for all $k \le \hat{k}$,

$$|x_k^p| \le y_k \le a_2|x_0^p| + |x_1^p| + \frac{(1+\alpha+e^{L\bar{T}})\Delta_2}{1-a_1+a_2}$$

$$\leq a_2|x_0^p| + |x_1^p| + \frac{(1+\alpha+e^{L\bar{T}})\varepsilon_2}{\varepsilon'} \triangleq a_2|x_0^p| + |x_1^p| + \varepsilon'' , \tag{3.5m}$$

where ε' was defined in (3.5l). We note that it follows from the proof of Proposition 6.1 that

$$\overline{\lim_{k\to\infty}} |x_k^p| \leq \varepsilon'' , \tag{3.5n}$$

where ε'' was defined in (3.5m). Now,

$$|x_1^p| \leq |x_1^p - x_1^m| + |x_1^m| . \tag{3.5o}$$

Since $x_0^m = x_0^p$ and $u(t) = 0$ for all $t \in [0, t_1]$, it follows from (3.2a-e) that

$$|x_1^p| \leq \Delta_1|x_0^p| + e^{L\bar{T}}|x_0^p| = (\Delta_1 + e^{L\bar{T}})|x_0^p| . \tag{3.5p}$$

Substituting (3.5p) into (3.5m), we obtain that for all $k \leq \hat{k}$,

$$|x_k^p| \leq (e^{L\bar{T}} + (1+\alpha+e^{L\bar{T}})\Delta_1)|x_0^p| + \varepsilon'' . \tag{3.5q}$$

Next, it follows from the fact that $|x_k^m| \leq |x_k^p - x_k^m| + |x_k^p|$, (3.5q), and (3.1) that for all $k \leq \hat{k}$,

$$|x_k^m| \leq (1+\Delta_1)((e^{L\bar{T}} + (1+\alpha+e^{L\bar{T}})\Delta_1)|x_0^p| + \varepsilon'') + \Delta_2$$

$$\triangleq \gamma_1|x_0^p| + \gamma_2 . \tag{3.5r}$$

Now, it follows from (3.5g,i) and (3.5m) that

$$e^{L\bar{T}} \leq \gamma_1 \leq (1+\varepsilon_1)(e^{L\bar{T}} + (1+\alpha+e^{L\bar{T}})\varepsilon_1) \triangleq \hat{\gamma}_1 , \tag{3.5s}$$

$$\gamma_2 = (1+\Delta_1)\varepsilon'' + \Delta_2 \leq \left[1 + \frac{2+e^{L\bar{T}}}{\varepsilon'}\right]\varepsilon_2 \triangleq \hat{\gamma}_2 < \hat{\rho} . \tag{3.5t}$$

Let ρ_s be defined by

$$\rho_s \triangleq (\hat{\rho} - \hat{\gamma}_2)/\hat{\gamma}_1 . \tag{3.5u}$$

Since $\hat{\rho} - \hat{\gamma}_2 > 0$ and $\hat{\gamma}_1 \geq 1$, we conclude that $\rho_s > 0$ and hence that $B_{\rho_s} \subset B_{\hat{\rho}}$ is well defined and its interior is not empty. Next, it follows from (3.5e), (3.5q), and (3.5r) that if $x_0^p \in B_{\rho_s}$ and $K_m \leq \bar{K}_m$, then $|x_{\hat{k}}^p|, |x_{\hat{k}}^m| \leq \hat{\rho}$, which contradicts our assumption and it completes our proof. □

Theorem 3.3. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.5. Suppose that

$$\bar{K}_m \triangleq \min\left[\frac{L(1-\alpha)}{e^{L\bar{T}}(e^{L\bar{T}}-1)(1+\alpha+e^{L\bar{T}})}\,,\ \frac{\hat{\beta}}{c_u\bar{T}\,e^{2L\bar{T}}}\left\{\frac{2+e^{L\bar{T}}}{\bar{\varepsilon}}+1\right\}^{-1}\right], \tag{3.6}$$

where $\bar{\varepsilon} \triangleq (1-\alpha)(\alpha+e^{L\bar{T}})/(1+\alpha+e^{L\bar{T}})$. Let ρ_s be defined as in (3.5u). Suppose that $L \in [0,\infty)$ and $K_m < \bar{K}_m$ satisfy (2.5b-d). Then *(a)* there exists an $\varepsilon_3 < \infty$ such that for any $x_0^p \in B_{\rho_s}$, $\|x^p(t,0,x_0^p,u)\| \le \varepsilon_3$ for all $t \in [0,\infty)$ and *(b)* there exists an $\varepsilon_4 > 0$, depending on K_m, such that $\varepsilon_4 \to 0$ as $K_m \to 0$ and for any $x_0^p \in B_{\rho_s}$, the trajectory $x^p(t,0,x_0^p,u)$, $t \in [0,\infty)$, satisfies $\overline{\lim}_{t\to\infty}\|x^p(t,0,x_0^p,u)\| \le \varepsilon_4$.

Proof. First, we have shown in Lemma 3.2 that for any $x_0^p \in B_{\rho_s}$, the sequences $\{x_k^m\}_{k=1}^{\infty}$ and $\{x_k^p\}_{k=0}^{\infty}$ are in the set $B_{\hat{\rho}}$. We will now prove that for any $x_0^p \in B_{\rho_s}$, $\|x^p(t,0,x_0^p,u)\|$ is bounded. It follows from (3.5d) and (2.4d) that for any $t \in [t_{k+1},t_{k+2}]$, $k \in \mathbb{N}$,

$$\begin{aligned}\|x^p(t,t_{k+1},x_{k+1}^p,u)\| &\le \|x^p(t,t_{k+1},x_{k+1}^p,u)-x^m(t,t_{k+1},x_{k+1}^m,u)\|+\|x^m(t,t_{k+1},x_{k+1}^m,u)\| \\ &\le \Delta_1\|x_{k+1}^p\|+\Delta_1 e^{L\bar{T}}\|x_k^p\|+\Delta_2(1+e^{L\bar{T}})+\beta\|x_{k+1}^m\| \\ &\le (\Delta_1+\beta)\|x_{k+1}^p\|+\Delta_1 e^{L\bar{T}}\|x_k^p\|+\Delta_2(1+e^{L\bar{T}})+\beta(\Delta_1\|x_k^p\|+\Delta_2)\,. \end{aligned} \tag{3.7a}$$

Let $\varepsilon_3 \triangleq (\varepsilon_1\hat{\rho}+\varepsilon_2)(1+\beta+e^{L\bar{T}})+\beta\hat{\rho}$, where $\varepsilon_1, \varepsilon_2$ were defined in (3.5g), (3.5h), respectively. Then, since $x_k^p \in B_{\hat{\rho}}$ for all $k \in \mathbb{N}$, and $\Delta_1 \le \varepsilon_1$ and $\Delta_2 \le \varepsilon_2$, $\|x^p(t,t_{k+1},x_{k+1}^p,u)\| \le \varepsilon_3$ for all $t \in [t_{k+1},t_{k+2}]$, $k \in \mathbb{N}$. Next, it follows from the proof of Lemma 3.2 that $\overline{\lim}_{k\to\infty}\|x_k^p\| \le \varepsilon''$ where ε'' was defined in (3.5m). Hence

$$\begin{aligned}\overline{\lim_{k\to\infty}}\,\|x^p(t,t_{k+1},x_{k+1}^p,u)\| &\le (\beta(\Delta_1+1)+\Delta_1(1+e^{L\bar{T}}))\varepsilon''+\Delta_2(1+\beta+e^{L\bar{T}})\,. \\ &\le (\beta+(1+\beta+e^{L\bar{T}})\varepsilon_1)\varepsilon''+\varepsilon_2(1+\beta+e^{L\bar{T}}) \triangleq \varepsilon_4\,, \end{aligned} \tag{3.7b}$$

where $\varepsilon_1, \varepsilon_2$ were defined in (3.5g), (3.5h), respectively. Since $\varepsilon_1, \varepsilon_2 \to 0$ as $K_m \to 0$, we obtain that $\varepsilon_4 \to 0$ as $K_m \to 0$, which completes our proof. □

So far, we have analyzed the behavior of the closed loop system resulting from the use of Control Algorithm 2.5 and we have obtained a bound on residual errors when there exists a difference between the plant (2.1a) and the model (2.2a). We will now present a strategy for eliminating the residual errors using the suggestion in [May.2], to switch over to a linear quadratic regulator control law when the state is sufficiently close to the origin.

Consider the linear feedback control law $u(t) = -Kx^p(t)$ for all $t \geq 0$ where K was chosen to satisfy (2.3a,b). Suppose that $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$f(x, -Kx) \triangleq Ax + \phi(x), \tag{3.8}$$

for all $x \in \mathbb{R}^n$ and that $|\phi(x)|_2 / |x|_2 \to 0$ as $|x|_2 \to 0$. Then, we have the following local stability result:

Lemma 3.4. Let $\rho_{LQR} \in (0, \infty)$ be such that for all $x \in B_{\rho_{LQR}} \triangleq \{ x \in \mathbb{R}^n \mid |x| \leq \rho_{LQR} \}$,

$$|\phi(x)|_2 / |x|_2 \leq \lambda_{\min}(M) / 4\lambda_{\max}(Q), \tag{3.9}$$

where $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the smallest and the largest eigenvalue of M, respectively. Suppose that $x_0^p \in B_{\rho_{LQR}}$ and that the linear feedback control $u(t) = -Kx^p(t, 0, x_0^p, u)$ is used for all $t \geq 0$. If $K_m < \lambda_{\min}(M)/4\lambda_{\max}(Q)(1 + |K|_2/\lambda_{\max}(Q)^{1/2})$, then *(a)* $x^p(t, 0, x_0^p, u) \in B_{\rho_{LQR}}$ for all $t \geq 0$ and *(b)* $\overline{\lim}_{t \to \infty} |x^p(t, 0, x_0^p, u)| = 0$.

Proof. Let $x^p(t) \triangleq x^p(t, 0, x_0^p, u)$ where u is the linear feedback control given as above. Then, making use of (3.8), we obtain that

$$\dot{x}^p(t) = f^p(x^p(t), -Kx^p(t))$$

$$= f(x^p(t), -Kx^p(t)) + f^p(x^p(t), -Kx^p(t)) - f(x^p(t), -Kx^p(t))$$

$$= Ax^p(t) + \phi(x^p(t)) + f^p(x^p(t), -Kx^p(t)) - f(x^p(t), -Kx^p(t)). \tag{3.10a}$$

Consider the Lyapunov function $V(x^p(t)) = |x^p(t)|^2 \triangleq \langle x^p(t), Qx^p(t) \rangle$. Now, since $|\cdot|_\infty \leq |\cdot|_2$,

$$\langle f^p(x^p(t), -Kx^p(t)) - f(x^p(t), -Kx^p(t)), Qx^p(t) \rangle$$

$$\leq |f^p(x^p(t), -Kx^p(t)) - f(x^p(t), -Kx^p(t))| |Q^{1/2} x^p(t)|_2$$

$$\leq K_m (|x^p(t)| + |K|_2 |x^p(t)|_2) \lambda_{\max}(Q)^{1/2} |x^p(t)|_2$$

$$\leq K_m \lambda_{\max}(Q) (1 + |K|_2 / \lambda_{\max}(Q)^{1/2}) |x^p(t)|_2^2 . \tag{3.10b}$$

It now follows from (3.9), (3.10b), and the condition on K_m that

$$\dot{V}(x^p(t)) = \langle x^p(t), (A^T Q + QA) x^p(t) \rangle + 2 \langle \phi(x^p(t)), Qx^p(t) \rangle$$

$$+ 2 \langle f^p(x^p(t), -Kx^p(t)) - f(x^p(t), -Kx^p(t)), Qx^p(t) \rangle$$

$$\leq (-\lambda_{\min}(M) + 2\lambda_{\max}(Q)\|\phi(x^p(t))\|_2/\|x^p(t)\|_2 + 2K_m(1+\|K\|_2/\lambda_{\max}(Q)^{1/2})\lambda_{\max}(Q))\|x^p(t)\|_2^2 .$$

$$\leq (-\lambda_{\min}(M)/2 + 2K_m(1+\|K\|_2/\lambda_{\max}(Q)^{1/2})\lambda_{\max}(Q))\|x^p(t)\|_2^2 \triangleq -\gamma\|x^p(t)\|_2^2 , \qquad (3.10c)$$

where $\gamma > 0$, which implies that if $x_0^p \in B_{LQR}$, $\|x^p(t)\|^2$ is strictly monotone decreasing. Hence, for any $x_0^p \in B_{\rho_{LQR}}$, we obtain that $x^p(t, 0, x_0^p, -Kx^p(t)) \in B_{\rho_{LQR}}$ for all $t \geq 0$ and that $x^p(t, 0, x_0^p, -Kx^p(t)) \to 0$ as $t \to \infty$, which completes our proof. □

We propose to incorporate the switch over to an LQR feedback law into Control Algorithm 2.5 by modifying *Step 1*, as follows. Let $T_K \geq T_C$ be such that $\|e^{T_K A}\| \leq \alpha$, where A was defined in (2.3a).

Step 1': At $t = t_k$,

(a) Measure the state $x_k^p = x^p(t_k, t_0, x_0^p, u, d)$;

(b) If $x_k^p \notin B_{\rho_{LQR}}$, set the plant input $u(t) = u_{[t_k, t_{k+1}]}(t)$, for $t \in [t_k, t_{k+1})$; else reset t_{k+1} to the new value $t_{k+1} = t_k + T_K$ and set $u(t) = -Kx^p(t, t_k, x_k^p, u, d)$ for $t \in [t_k, t_{k+1})$.

(c) Compute the estimate $x_{k+1}^m \triangleq x^m(t_{k+1}, t_k, x_k^p, u_{[t_k, t_{k+1}]}, 0)$ of the state of the plant $x^p(t_{k+1}, t_k, x_k^p, u_{[t_k, t_{k+1}]}, d)$ by solving (2.2a) with state x_k^p at time $t = t_k$, and an input $u_{[t_k, t_{k+1}]}(t)$ for all $t \in [t_k, t_{k+1}]$. □

Theorem 3.5. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.5 with *Step 1'*. Suppose that

$$\bar{K}_m \triangleq \min\left[\frac{L(1-\alpha)}{e^{L\bar{T}}(e^{L\bar{T}}-1)(1+\alpha+e^{L\bar{T}})}, \frac{\hat{\beta}}{c_u\bar{T}e^{2L\bar{T}}}\left\{\frac{2+e^{L\bar{T}}}{\bar{\varepsilon}}+1\right\}^{-1}, \frac{\lambda_{\min}(M)}{4\lambda_{\max}(Q)(1+\|K\|_2/\lambda_{\max}(Q)^{1/2})}\right], \qquad (3.11)$$

where $\bar{\varepsilon} \triangleq (1-\alpha)(\alpha+e^{L\bar{T}})/(1+\alpha+e^{L\bar{T}})$. Let B_{ρ_*} be defined as (3.5v). Suppose that $L \in [0, \infty)$ and $K_m < \bar{K}_m$ satisfy (2.5b-d), and $\varepsilon'' < \rho_{LQR}$ where ε'' was defined in (3.5m). Then, *(a)* there exists an $\varepsilon_5 < \infty$ such that for any $x_0^p \in B_{\rho_*}$, $\|x^p(t, 0, x_0^p, u)\| \leq \varepsilon_5$ for all $t \in [0, \infty)$ and *(b)* $\overline{\lim}_{t\to\infty}\|x^p(t, 0, x_0^p, u)\| = 0$.

Proof. First, it follows from the proof of Lemma 3.2 and the assumption that $\varepsilon'' < \rho_{LQR}$ that there exists a $k' \in \mathbb{N}$ such that the cross-over to the linear feedback control law $u(t) = -Kx^p(t, 0, x_0^p, u)$ will take place. Then, by (3.11) and Lemma 3.4, we obtain that for all

$t \geq t_{k'}$,

$$\dot{V}(x^p(t\,,t_{k'}\,,x_{k'}^p\,,-Kx^p(t))) \leq -\gamma \| x^p(t\,,t_{k'}\,,x_{k'}^p\,,-Kx^p(t))\|_2^2\,, \tag{3.12}$$

where $V(x^p(t)) = \|x^p(t)\|^2$ and γ was defined in (3.10c). It follows from Lemma 3.4 that $x_k^p \in B_{\rho_{LQR}}$ for all $k \geq k'$ and that $\overline{\lim}_{t\to\infty} \|x^p(t\,,0\,,x_0^p\,,u)\| = 0$, which completes our proof. □

4. DISTURBANCE REJECTION.

We will now determine the joint effect of disturbances and modeling errors on the behavior of the closed loop system resulting from the use of Control Algorithm 2.5. We will assume that the state of the plant is measurable and that the disturbance $d(t)$ cannot be estimated. Since the available controls are bounded, we can only hope to overcome bounded disturbances. Hence we assume that there exists a $c_d \in (0\,,\infty)$ such that $\|d\|_\infty \leq c_d$. We will consider two strategies as we did in Section 3. The first using only Algorithm 2.5 and the second one in which algorithm crosses over to the LQR linear feedback law near the origin. We recall that Control Algorithm 2.5 generates three sequences, which are $\{x_k^p\}_{k=0}^\infty$, $\{x_k^m\}_{k=1}^\infty$, and $\{x'_k\}_{k=2}^\infty$.

Lemma 4.1. Consider the moving horizon feedback system resulting from the use of Control Algorithm 2.5. Suppose that Assumptions 2.3 and 2.4 are satisfied. Let $\hat{\rho}$ be defined as in Assumption 2.3, and suppose that $L\,,K_m \in [0\,,\infty)$ satisfy (2.5b-d). If $x_k^p\,,x_k^m \in B_{\hat{\rho}}$ for all $k \in \mathbf{N}$, then there exist Δ_3 and $\Delta_4 \geq 0$ such that

$$\|x_{k+1}^p - x_{k+1}^m\| \leq \Delta_3 \|x_k^p\| + \Delta_4\,, \tag{4.1}$$

and $\Delta_3, \Delta_4 \to 0$ as $K_m\,,c_d \to 0$.

Proof. First, the optimal control problem $\mathrm{P}(x_k^m\,,t_k)$ has a solution for all $k \geq 1$ since $x_k^p\,,x_k^m \in B_{\hat{\rho}}$ for all $k \geq 1$, and as a result, the trajectory $x^p(t\,,0\,,x_0^p\,,u\,,d)$ is well defined. Let $x_k^p(t) \triangleq x^p(t\,,t_k\,,x_k^p\,,u\,,d)$ and $x_k^m(t) \triangleq x^m(t\,,t_k\,,x_k^p\,,u\,,0)$ for all $t \in [t_k\,,t_{k+1}]$ for all $k \in \mathbf{N}$. Given $x_k^p \in B_{\hat{\rho}}$ and $u = u_{[t_k\,,t_{k+1}]} \in U$, obtained by solving the optimal control problem $\mathrm{P}(x_k^m\,,t_k)$, it follows from Assumption 2.4 that for all $t \in [t_k\,,t_{k+1}]$

$$\|x_k^p(t) - x_k^m(t)\| \leq \int_{t_k}^t \|f^p(x_k^p(\tau)\,,u(\tau)\,,d(\tau)) - f^m(x_k^m(\tau)\,,u(\tau)\,,0)\|\,d\tau$$

$$\leq \int_{t_k}^t \|f^p(x_k^p(\tau)\,,u(\tau)\,,d(\tau)) - f^m(x_k^p(\tau)\,,u(\tau)\,,d(\tau))\|\,d\tau$$

$$+\int_{t_k}^{t} |f^m(x_k^p(\tau), u(\tau), d(\tau)) - f^m(x_k^m(\tau), u(\tau), 0)| \, d\tau$$

$$\leq L \int_{t_k}^{t} |x_k^p(\tau) - x_k^m(\tau)| \, d\tau + Lc_d\bar{T} + K_m \int_{t_k}^{t} (|x_k^p(\tau)| + |u(\tau)|_\infty + |d(\tau)|_\infty) \, d\tau . \quad (4.2a)$$

It follows from (2.1a) and (2.5b) that for all $t \in [t_k , t_{k+1}]$,

$$|x_k^p(t)| \leq |x_k^p| + \int_{t_k}^{t} |f^p(x_k^p(\tau), u(\tau), d(\tau))| \, d\tau$$

$$\leq |x_k^p| + \int_{t_k}^{t} L(|x_k^p(\tau)| + |u(\tau)|_\infty + |d(\tau)|_\infty) \, d\tau$$

$$\leq |x_k^p| + \int_{t_k}^{t} L \, |x_k^p(\tau)| \, d\tau + L(c_u + c_d)\bar{T} . \quad (4.2b)$$

It therefore follows from the Bellman-Gronwall inequality and from the fact that for any $k \in \mathbb{N}$, $t_{k+1} - t_k \leq \bar{T}$, that

$$|x_k^p(t)| \leq e^{L(t-t_k)}|x_k^p| + L(c_u + c_d)\bar{T}e^{L(t-t_k)}. \quad (4.2c)$$

Hence, for any $t \in [t_k , t_{k+1}]$

$$\int_{t_k}^{t} |x_k^p(\tau)| d\tau \leq \int_{t_k}^{t} \{ e^{L(\tau-t_k)}|x_k^p| + L(c_u + c_d)\bar{T}e^{L(\tau-t_k)} \} \, d\tau$$

$$= \{ (e^{L(t-t_k)} - 1)/L \} |x_k^p| + (c_u + c_d)\bar{T}(e^{L(t-t_k)} - 1)$$

$$\leq \{ (e^{L\bar{T}} - 1)/L \} |x_k^p| + (c_u + c_d)\bar{T}(e^{L\bar{T}} - 1) . \quad (4.2d)$$

By substituting (4.2d) into (4.2a), we obtain that for all $t \in [t_k , t_{k+1}]$

$$|x_k^p(t) - x_k^m(t)| \leq K_m \left[(e^{L\bar{T}} - 1)/L \, |x_k^p| + (e^{L\bar{T}} - 1)(c_u + c_d)\bar{T} + (c_u + c_d)\bar{T} \right]$$

$$+ L \int_{t_k}^{t} |x_k^p(\tau) - x_k^m(\tau)| \, d\tau + Lc_d\bar{T} .$$

It therefore follows from the Bellman-Gronwall inequality that

$$|x_{k+1}^p - x_{k+1}^m| \leq \{ K_m e^{L\bar{T}}(e^{L\bar{T}} - 1)/L \} |x_k^p| + K_m(c_u + c_d)\bar{T}e^{2L\bar{T}} + Lc_d\bar{T}e^{L\bar{T}}$$

$$\triangleq \Delta_3|x_k^p| + \Delta_4 . \quad (4.2e)$$

It is clear that $\Delta_3, \Delta_4 \to 0$ as $(K_m , c_d) \to 0$, which completes our proof. □

We note that Δ_3 is equal to Δ_1, defined in (3.2f), and that Δ_4 is the sum of Δ_2 defined in (3.2f) and the effect of the disturbance. Lemma 4.1 leads us to the following result.

Lemma 4.2. Suppose that $\hat{\rho}$ is as postulated in Assumption 2.3, and that $L, K_m \in [0, \infty)$ satisfy (2.5b-d). Then, there exist $\bar{K}_m, \bar{c}_d \in (0, \infty)$ and $\rho_d \in (0, \hat{\rho}]$ such that if $K_m \le \bar{K}_m$ and $c_d \le \bar{c}_d$, then for all $x_0^p \in B_{\rho_d} \triangleq \{ x \in \mathbb{R}^n \mid \|x\| \le \rho_d \}$, the sequences $\{x_k^p\}_{k=0}^{\infty}$ and $\{x_k^m\}_{k=1}^{\infty}$, resulting from the use of Control Algorithm 2.5, are well defined and stay in the set $B_{\hat{\rho}}$.

Proof. We will prove this result by contradiction. Suppose that for any $\bar{K}_m, \bar{c}_d \in [0, \infty)$, $\rho \in (0, \hat{\rho}]$, $K_m \le \bar{K}_m$, and $c_d \le \bar{c}_d$, there exist $x_0^p \in B_\rho$ and $\hat{k} \in \mathbb{N}$ such that $x_k^m, x_k^p \in B_{\hat{\rho}}$ for all $k < \hat{k}$ and $x_{\hat{k}}^p \notin B_{\hat{\rho}}$ or $x_{\hat{k}}^m \notin B_{\hat{\rho}}$. We will consider three cases: *(a)* $\hat{k} = 1$, *(b)* $\hat{k} = 2$, and *(c)* $\hat{k} > 2$. To simplify notation, let $x_k^p(t) \triangleq x^p(t, t_k, x_k^p, u_{[t_k, t_{k+1}]}, d)$, $x'_k(t) \triangleq x^m(t, t_k, x_k^m, u_{[t_k, t_{k+1}]}, 0)$, and $x_k^m(t) \triangleq x^m(t, t_k, x_k^p, u_{[t_k, t_{k+1}]}, 0)$ for all $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$. We note that $x_{k+1}^p = x_k^p(t_{k+1})$, $x_{k+1}^m = x_k^m(t_{k+1})$, $x'_{k+2} = x_{k+1}^m(t_{k+2})$, and $\|x'_{k+1}\| \le \alpha \|x_k^m(t_{k+1})\|$ for all $k \in \mathbb{N}$.

(a) Suppose that $\hat{k} = 1$. Since $u(t) = 0$ for all $t \in [0, t_1]$ and $x_0^m = x_0^p$, we have that for all $t \in [0, t_1]$

$$\|x_0^p(t)\| \le \|x_0^p\| + \int_0^t \|f^p(x_0^p(\tau), 0, d(\tau))\| dt \le \|x_0^p\| + L \int_0^t \|x_0^p(\tau)\| d\tau + L c_d \bar{T} \tag{4.3a}$$

$$\|x_0^m(t)\| \le \|x_0^p\| + \int_0^t \|f^m(x_0^m(\tau), 0, 0)\| dt \le \|x_0^p\| + L \int_0^t \|x_0^m(\tau)\| d\tau . \tag{4.3b}$$

It follows from the Bellman-Gronwall inequality that for all $t \in [0, t_1]$

$$\|x_0^p(t)\| \le \|x_0^p\| e^{Lt} + L c_d \bar{T} e^{Lt} \tag{4.3c}$$

$$\|x_0^m(t)\| \le \|x_0^p\| e^{Lt} . \tag{4.3d}$$

Let $\bar{K}_m > 0$ be arbitrary. Since $\|x_0^p\| \to 0$ as $\rho \to 0$ and $c_d \to 0$ as $\bar{c}_d \to 0$, there exist $\rho, \bar{c}_d > 0$ such that $x_1^p, x_1^m \in B_{\hat{\rho}}$ where $x_1^p = x_0^p(t_1)$ and $x_1^m = x_0^m(t_1)$, which contradicts our assumption.

(b) Suppose that $\hat{k} = 2$. Since $x_1^m \in B_{\hat{\rho}}$, there exists a solution to the optimal control problem $P(x_1^m, t_1)$. It follows from the fact that $\|x'_2\| \le \alpha \|x_1^m\|$ that

$$\|x_2^p\| \le \|x_2^p - x'_2\| + \alpha \|x_1^m\| . \tag{4.4a}$$

We can easily show that using the Bellman-Gronwall inequality, for any $t \in [t_1, t_2]$

$$|x_1^p(t)| \le e^{L(t-t_1)}\left[|x_1^p| + L(c_u + c_d)\bar{T}\right]. \tag{4.4b}$$

It follows from (4.4b) that for all $t \in [t_1, t_2]$,

$$\int_{t_1}^{t} |x_1^p(\tau)| d\tau \le (e^{L\bar{T}} - 1)|x_1^p| + (e^{L\bar{T}} - 1)(c_u + c_d)\bar{T}. \tag{4.4c}$$

Now it follows from (4.1), (4.3c), and (4.4c) that for all $t \in [t_1, t_2]$

$$\begin{aligned}
|x_1^p(t) - x'_1(t)| &\le |x_1^p - x_1^m| + \int_{t_1}^{t} |f^p(x_1^p(\tau), u(\tau), d(\tau)) - f^m(x'_1(\tau), u(\tau), 0)| d\tau \\
&\le |x_1^p - x_1^m| + \int_{t_1}^{t} |f^p(x_1^p(\tau), u(\tau), d(\tau)) - f^m(x_1^p(\tau), u(\tau), d(\tau))| d\tau \\
&\quad + \int_{t_1}^{t} |f^m(x_1^p(\tau), u(\tau), d(\tau)) - f^m(x'_1(\tau), u(\tau), 0)| d\tau \\
&\le |x_1^p - x_1^m| + K_m \int_{t_1}^{t} (|x_1^p(\tau)| + |u(\tau)|_\infty + |d(\tau)|_\infty) d\tau + L \int_{t_1}^{t} (|x_1^p(\tau) - x'_1(\tau)| + |d(\tau)|_\infty) d\tau \\
&\le \Delta_3 |x_0^p| + \Delta_4 + (K_m/L)(e^{L\bar{T}} - 1)|x_1^p| + K_m(c_u + c_d)\bar{T}e^{L\bar{T}} \\
&\quad + Lc_d\bar{T} + L\int_{t_1}^{t} |x_1^p(\tau) - x'_1(\tau)| d\tau \\
&\le \Delta_3 |x_0^p| + (K_m e^{L\bar{T}}(e^{L\bar{T}} - 1)/L)(|x_0^p| + Lc_d\bar{T}) + \Delta_4 \\
&\quad + K_m(c_u + c_d)\bar{T}e^{L\bar{T}} + Lc_d\bar{T} + L\int_{t_1}^{t} |x_1^p(\tau) - x'_1(\tau)| d\tau \\
&= 2\Delta_3 |x_0^p| + (1 + 1/e^{L\bar{T}})\Delta_4 + Lc_d\bar{T}\Delta_3 + L\int_{t_1}^{t} |x_1^p(\tau) - x'_1(\tau)| d\tau,
\end{aligned} \tag{4.4d}$$

where Δ_3, Δ_4 were defined in (4.2e). Again, making use of the Bellman-Gronwall inequality, we obtain that

$$|x_1^p(t_2) - x'_1(t_2)| = |x_2^p - x'_2| \le 2e^{L\bar{T}}\Delta_3|x_0^p| + (1 + e^{L\bar{T}})\Delta_4 + Lc_d\bar{T}e^{L\bar{T}}\Delta_3. \tag{4.4e}$$

By substituting (4.4e) into (4.4a) and using (4.3d), we obtain that

$$|x_2^p| \le (2\Delta_3 + \alpha)e^{L\bar{T}}|x_0^p| + (1 + e^{L\bar{T}})\Delta_4 + Lc_d\bar{T}e^{L\bar{T}}\Delta_3. \tag{4.4f}$$

Next, it follows from (4.1), (4.3c), and (4.4f) that

$$\begin{aligned}
|x_2^m| &\le |x_2^m - x_2^p| + |x_2^p| \le \Delta_3|x_1^p| + \Delta_4 + |x_2^p| \\
&\le (3\Delta_3 + \alpha)e^{L\bar{T}}|x_0^p| + (2 + e^{L\bar{T}})\Delta_4 + 2Lc_d\bar{T}e^{L\bar{T}}\Delta_3.
\end{aligned} \tag{4.4g}$$

Since $|x_0^p| \to 0$ as $\rho \to 0$ and $\Delta_3, \Delta_4 \to 0$ as $K_m, c_d \to 0$, there exist $\bar{K}_m, \bar{c}_d \in [0, \infty)$ and

$\rho_d \in (0, \hat{\rho})$ such that if $K_m \le \bar{K}_m$, $c_d \le \bar{c}_d$, and $x_0^p \in B_{\rho_d}$, then $x_2^p, x_2^m \in B_{\hat{\rho}}$, which contradicts our assumption.

(c) Suppose that $\hat{k} > 2$. Since $x_{k+1}^p, x_{k+1}^m \in B_{\hat{\rho}}$ for all $k \le \hat{k} - 2$, there exists a solution to the optimal control problem $P(x_{k+1}^m, t_{k+1})$, $(u_{[t_{k+1}, t_{k+2}]}(\cdot), t_{k+2})$, such that $|x'_{k+2}| \le \alpha |x_{k+1}^m|$. Therefore, $x_{k+1}^p(t)$ is well defined for all $t \in [t_{k+1}, t_{k+2}]$, $k \le \hat{k} - 2$, and

$$|x_{k+2}^p| \le |x_{k+2} - x'_{k+2}| + \alpha |x_{k+1}^m| . \tag{4.5a}$$

Next it follows from (4.2d) that for all $t \in [t_{k+1}, t_{k+2}]$, $k \le \hat{k} - 2$

$$\int_{t_{k+1}}^{t} |x_{k+1}^p(\tau)| d\tau \le \{ (e^{L\bar{T}} - 1)/L \} |x_{k+1}^p| + (c_u + c_d)\bar{T}(e^{L\bar{T}} - 1) . \tag{4.5b}$$

Hence we obtain that for all $t \in [t_{k+1}, t_{k+2}]$ for all $k \le \hat{k} - 2$,

$$|x_{k+1}^p(t) - x'_{k+1}(t)| \le |x_{k+1}^p - x_{k+1}^m| + \int_{t_{k+1}}^{t} |f^p(x_{k+1}^p(\tau), u(\tau), d(\tau)) - f^m(x'_{k+1}(\tau), u(\tau), 0)| \, d\tau$$

$$\le |x_{k+1}^p - x_{k+1}^m| + \int_{t_{k+1}}^{t} |f^p(x_{k+1}^p(\tau), u(\tau), d(\tau)) - f^m(x_{k+1}^p(\tau), u(\tau), d(\tau))| d\tau$$

$$+ \int_{t_{k+1}}^{t} |f^m(x_{k+1}^p(\tau), u(\tau), d(\tau)) - f^m(x'_{k+1}(\tau), u(\tau), 0)| \, d\tau$$

$$\le |x_{k+1}^p - x_{k+1}^m| + L \int_{t_{k+1}}^{t} |x_{k+1}^p(\tau) - x'_{k+1}(\tau)| \, d\tau$$

$$+ Lc_d\bar{T} + K_m \int_{t_{k+1}}^{t} (|x_{k+1}^p(\tau)| + |u(\tau)|_\infty + |d(\tau)|_\infty) \, d\tau ,$$

$$\le \Delta_3 |x_k^p| + \Delta_4 + ((K_m/L)|x_{k+1}^p| + K_m(c_u + c_d)\bar{T})(e^{L\bar{T}} - 1) + K_m\bar{T}(c_u + c_d)$$

$$+ L \int_{t_{k+1}}^{t} |x_{k+1}^p(\tau) - x'_{k+1}(\tau)| \, d\tau + Lc_d\bar{T}$$

$$= \Delta_3 |x_k^p| + \Delta_4 + \frac{\Delta_3}{e^{L\bar{T}}} |x_{k+1}^p| + \frac{\Delta_4}{e^{L\bar{T}}} + L \int_{t_{k+1}}^{t} |x_{k+1}^p(\tau) - x'_{k+1}(\tau)| \, d\tau \tag{4.5c}$$

where Δ_3, Δ_4 were defined in (4.2e). Making use of the Bellman-Gronwall inequality, we obtain from (4.5c) that for all $t \in [t_{k+1}, t_{k+2}]$,

$$|x_{k+1}^p(t) - x'_{k+1}(t)| \le \Delta_3 |x_{k+1}^p| + \Delta_3 e^{L\bar{T}} |x_k^p| + \Delta_4(1 + e^{L\bar{T}}) . \tag{4.5d}$$

It follows from the fact that $x'_{k+2} = x_{k+1}^m(t_{k+2})$, (4.1), and (4.5d) that for all $k \le \hat{k} - 2$,

$$|x^p_{k+2}| \le |x^p_{k+2} - x'_{k+2}| + \alpha |x^m_{k+1}|$$

$$\le \Delta_3 |x^p_{k+1}| + \Delta_3 e^{L\bar{T}} |x^p_k| + \Delta_4 (1 + e^{L\bar{T}}) + \alpha |x^m_{k+1} - x^p_{k+1}| + \alpha |x^p_{k+1}|$$

$$\le (\Delta_3 + \alpha) |x^p_{k+1}| + \Delta_3 (\alpha + e^{L\bar{T}}) |x^p_k| + \Delta_4 (1 + \alpha + e^{L\bar{T}}) . \tag{4.5e}$$

Now suppose that such that

$$\bar{K}_m < \min \left[\frac{L(1-\alpha)}{e^{L\bar{T}}(e^{L\bar{T}}-1)(1+\alpha+e^{L\bar{T}})} , \frac{\hat{\rho}}{c_u \bar{T} e^{2L\bar{T}}} \left\{ \frac{2+e^{L\bar{T}}}{\bar{\varepsilon}} + 1 \right\}^{-1} \right] , \tag{4.5f}$$

$$\bar{c}_d \triangleq \left[\left\{ \frac{2+e^{L\bar{T}}}{\bar{\varepsilon}} + 1 \right\}^{-1} \hat{\rho} - \bar{K}_m c_u \bar{T} e^{2L\bar{T}} \right] \left[(\bar{K}_m e^{L\bar{T}} + L) \bar{T} e^{L\bar{T}} \right]^{-1} , \tag{4.5g}$$

where $\bar{\varepsilon} \triangleq (1-\alpha)(\alpha + e^{L\bar{T}})/(1+\alpha+e^{L\bar{T}})$. Clearly, $\bar{c}_d > 0$ by the choice of $\bar{K}_m$. We will show that $|x^p_{\hat{k}}|, |x^m_{\hat{k}}| \le \hat{\rho}$ must hold, which contradicts our assumption. It follows from (4.2e), and the fact that $c_d \le \bar{c}_d$, and $K_m \le \bar{K}_m$ that

$$\Delta_3 = \frac{K_m e^{L\bar{T}}(e^{L\bar{T}}-1)}{L} \le \frac{\bar{K}_m e^{L\bar{T}}(e^{L\bar{T}}-1)}{L} \triangleq \varepsilon'_1 < \frac{1-\alpha}{1+\alpha+e^{L\bar{T}}} , \tag{4.5h}$$

and

$$\Delta_4 = K_m (c_u + c_d) \bar{T} e^{2L\bar{T}} + L \bar{T} c_d e^{L\bar{T}}$$

$$\le \bar{K}_m c_u \bar{T} e^{2L\bar{T}} + (\bar{K}_m e^{L\bar{T}} + L) \bar{T} e^{L\bar{T}} \bar{c}_d \triangleq \varepsilon'_2 < \left[\frac{2+e^{L\bar{T}}}{\bar{\varepsilon}} + 1 \right]^{-1} \hat{\rho} . \tag{4.5i}$$

Finally, by comparing (3.5e) with (4.5e) and (3.5g), (3.5h) with (4.5h), (4.5i), respectively, we see that the arguments used in the proof of Lemma 3.2 must hold if we replace Δ_1 by Δ_3, Δ_2 by Δ_4, ε_1 by ε'_1, and ε_2 by ε'_2. Hence we obtain the following results which correspond to those in Lemma 3.2.

$$\overline{\lim}_{k \to \infty} |x^p_k| \le \frac{(1+\alpha+e^{L\bar{T}})\varepsilon'_2}{\varepsilon'} \triangleq \varepsilon''' , \tag{4.6a}$$

where ε' was defined in (3.5l). Let $\hat{\gamma}_3, \hat{\gamma}_4$ be such that

$$\hat{\gamma}_3 \triangleq (1+\varepsilon'_1)(e^{L\bar{T}} + (1+\alpha+e^{L\bar{T}})\varepsilon'_1) \tag{4.6b}$$

and

$$\hat{\gamma}_4 \triangleq \left[1+\frac{2+e^{L\bar{T}}}{\varepsilon'}\right]\varepsilon'_2, \tag{4.6c}$$

where ε', ε'_1, and ε'_2 were defined in (3.5l), (4.5h), and (4.5i), respectively. Let ρ_d be defined as follows:

$$\rho_d \triangleq (\hat{\rho} - \hat{\gamma}_4)/\hat{\gamma}_3. \tag{4.6d}$$

It now follows from (4.5i) and (4.6b,c) that the set $B_{\rho_d} \triangleq \{ x \in \mathbb{R}^n \mid \|x\| \le \rho_d \}$ is well defined and its interior is not empty. Finally, we conclude that $\|x_{\hat{k}}^m\|, \|x_{\hat{k}}^p\| \le \hat{\rho}$, which contradicts our assumption, and it completes our proof. □

Next, we have the following result that corresponds to Theorem 3.3. We will omit the proof since it is exactly same as that of Theorem 3.3 provided that we replace Δ_1 with Δ_3, Δ_2 by Δ_4, ε_1 with ε'_1, and ε_2 by ε'_2.

Theorem 4.3. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.5. Suppose that Assumptions 2.3 and 2.4 are satisfied. Let $\hat{\rho}$ be defined as in Assumption 2.3, and suppose that $L, K_m \in [0, \infty)$ satisfy (2.5b-d). Let $\bar{K}_m$, $\bar{c}_d$ be given by (4.5f), (4.5g), respectively, and suppose that $K_m \le \bar{K}_m$ and $c_d \le \bar{c}_d$. Let ρ_d be defined as (4.6d). Then, *(a)* there exists an $\varepsilon'_3 < \infty$ such that for any $x_0^p \in B_{\rho_d}$, $\|x^p(t, 0, x_0^p, u, d)\| \le \varepsilon'_4$ for all $t \in [0, \infty)$ and *(b)* there exists an $\varepsilon_7 > 0$, depending on K_m and c_d, such that $\varepsilon'_4 \to 0$ as $(K_m, c_d) \to 0$ and for any $x_0^p \in B_{\rho_d}$, the trajectory $x^p(t, 0, x_0^p, u, d)$, $t \in [0, \infty)$, satisfies that $\overline{\lim}_{t\to\infty} \|x^p(t, 0, x_0^p, u, d)\| \le \varepsilon'_4$. □

We will now show that when the disturbances are of sufficiently small amplitude, we can still use Control Algorithm 2.5 with *Step 1'*, to obtain the benefit of the disturbance suppression properties of LQR systems. These depend on the largest real part of the eigenvalues $\lambda_j(A)$ of the matrix A, where A was defined in (2.3a). Hence a design trade-off is implied: the smaller the largest real part of the eigenvalues, the better is the disturbance suppression. However, to obtain a very negative largest real part may require large elements in K, which limits the size of the ball about the origin where the control $u(t) = -Kx^p(t)$ will not violate the control constraint.

Thus, suppose that K is the gain matrix resulting from the solution of an LQR problem for the model (2.2b) satisfying (2.3a,b). In Section 3, due to the absence of disturbances, we obtained that $\overline{\lim}_{t\to\infty} \|x^p(t, 0, x_0, -K x^p(t), 0)\| \to 0$. Here, we have a different situation. Suppose that

$\phi_d : \mathbb{R}^n \times \mathbb{R}^{m_d} \to \mathbb{R}^n$ is defined by

$$f^m(x, -Kx, d) = Ax + f_d^m(0,0,0)d + \phi_d(x, d), \tag{4.7a}$$

where $x \in \mathbb{R}^n$, $\phi_d(0,0) = 0$, and $\|\phi_d(x, d)\|_2 / (\|x\|_2 + \|d\|_\infty) \to 0$ as $(\|x\|_2, \|d\|_\infty) \to 0$. Let $\Phi : \mathbb{R}_+ \to 2^{\mathbb{R}_+^2}$ be defined by

$$\Phi(\nu_\varepsilon) \triangleq \{ (\rho, c) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \frac{\|\phi_d(x, d)\|_2}{\|x\|_2 + \|d\|_\infty} \le \nu_\varepsilon, \ \forall x \in B_\rho, \ \forall d \in G_c \}, \tag{4.7b}$$

where $B_\rho \triangleq \{ x \in \mathbb{R}^n \mid \|x\| \le \rho \}$ and $G_c \triangleq \{ d \in \mathbb{R}^{m_d} \mid \|d\|_\infty \le c \}$. Then we have the following result.

Lemma 4.4. Suppose that Assumption 2.4 is satisfied and $K_m \in [0, \infty)$ satisfies (2.5d). Suppose that

$$(\rho_{LQR}, c'_d) \in \Phi(\lambda_{\min}(M)/4\lambda_{\max}(Q)), \tag{4.8a}$$

where Q, M satisfy (2.3b) and that $K_m < \lambda_{\min}(M)/4\lambda_{\max}(Q)(1 + \|K\|_2/\lambda_{\max}(Q)^{1/2})$ and that γ is defined as (3.10c). Let

$$c''_d \triangleq \min \left[c'_d, \frac{\rho_{LQR}\gamma}{\lambda_{\max}(Q)^{1/2}} \left\{ 2\|f_d^m(0,0,0)\|_2 \lambda_{\max}(Q) + \frac{\lambda_{\min}(M)(2 + \|K\|_2)}{2(1 + \|K\|_2)} \right\}^{-1} \right]. \tag{4.8b}$$

If $x_0^p \in B_{\rho_{LQR}}$, $\|d\|_\infty \le c''_d$, and the linear feedback control $u(t) = -Kx^p(t, 0, x_0^p, u, d)$ is used for all $t \ge 0$, then *(a)* $x^p(t, 0, x_0^p, u, d) \in B_{\rho_{LQR}}$ for all $t \ge 0$ and *(b)* $\overline{\lim}_{t \to \infty} \|x^p(t, 0, x_0^p, u, d)\| \to 0$ as $c'_d \to 0$.

Proof. It follows from (4.7a) that

$$\begin{aligned}
\dot{x}^p(t) &= f^p(x^p(t), -Kx^p(t), d(t)) \\
&= f^m(x^p(t), -Kx^p(t), d(t)) + f^p(x^p(t), -Kx^p(t), d(t)) - f^m(x^p(t), -Kx^p(t), d(t)) \\
&= Ax^p(t) + f_d^m(0,0,0)d(t) + \phi_d(x^p(t), d(t)) \\
&\quad + f^p(x^p(t), -Kx^p(t), d(t)) - f^m(x^p(t), -Kx^p(t), d(t)).
\end{aligned} \tag{4.9a}$$

Consider the Lyapunov function $V(x^p(t)) = \|x^p(t)\|^2 \triangleq \langle x^p(t), Qx^p(t) \rangle$. Then, since $\|d(t)\|_\infty \le \|d(t)\|_2$,

$$\dot{V}(x^p(t)) = \langle x^p(t), (A^TQ + QA)x^p(t) \rangle + 2\langle \phi_d(x^p(t), d(t)), Qx^p(t) \rangle + 2\langle f_d^m(0,0,0)d(t), Qx^p(t) \rangle$$

$$+2\langle f^p(x^p(t), -Kx^p(t), d(t)) - f^m(x^p(t), -Kx^p(t), d(t)), Qx^p(t)\rangle$$

$$\leq -\langle x^p(t), Mx^p(t)\rangle + 2\lambda_{max}(Q)\|\phi_d(x^p(t), d(t))\|_2 \|x^p(t)\|_2 + 2K_m c''_d \lambda_{max}(Q)\|x^p(t)\|_2$$

$$+2K_m(1+\|K\|_2/\lambda_{max}(Q)^{1/2})\lambda_{max}(Q)\|x^p(t)\|_2^2 + 2c''_d \|f_d^m(0,0,0)\|_2 \lambda_{max}(Q)\|x^p(t)\|_2$$

$$\leq \left[-\lambda_{min}(M) + \frac{2\lambda_{max}(Q)\|\phi_d(x^p(t), d(t))\|_2}{\|x^p(t)\|_2 + c''_d} + 2K_m(1+\|K\|_2)\lambda_{max}(Q)\right]\|x^p(t)\|_2^2$$

$$+2\left[K_m + \|f_d^m(0,0,0)\|_2 + \frac{\|\phi_d(x^p(t), d(t))\|_2}{\|x^p(t)\|_2 + c''_d}\right]\lambda_{max}(Q) c''_d \|x^p(t)\|_2 \qquad (4.9b)$$

Hence it follows from (4.8b) and the condition on K_m that

$$\dot{V}(x^p(t)) \leq \left[-\frac{\gamma}{\lambda_{max}(Q)^{1/2}}\|x^p(t)\| + \left\{2\|f_d^m(0,0,0)\|_2 \lambda_{max}(Q) + \frac{\lambda_{min}(M)(2+\|K\|_2)}{2(1+\|K\|_2)}\right\}c''_d\right]\|x^p(t)\|_2$$

$$\leq \left[-\frac{\gamma}{\lambda_{max}(Q)^{1/2}}\|x^p(t)\| + \frac{\rho_{LQR}\gamma}{\lambda_{max}(Q)^{1/2}}\right]\|x^p(t)\|_2, \qquad (4.9c)$$

where γ was defined in (3.10c). (4.9c) implies that if $\|x^p(t)\| > \rho_{LQR}$, then $\dot{V}(x^p(t)) < 0$. Consequently, since $x_0^p \in B_{\rho_{LQR}}$, we must have that $x^p(t, 0, x_0^p, -Kx^p(t), d(t)) \in B_{\rho_{LQR}}$ for all $t \geq 0$. Since $c''_d \to 0$ as $c'_d \to 0$, it follows from (4.9c) that as $c'_d \to 0$, $V(x^p(t)) \to 0$, which completes our proof. □

Lemma 4.4 testifies us to the use of Control Algorithm 2.5 with *Step 1'*, stated in Section 3, when disturbances are present. The corresponding stability result is as follows. Since its proof is immediate from Lemma 4.2, Theorem 4.3, and Lemma 4.4, we will omit it.

Theorem 4.5. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.5 with *Step 1'*. Suppose that Assumptions 2.3 and 2.4 are satisfied and that L, K_m satisfy (2.5b-d). Let $\bar{K}_m$, $\bar{c}_d > 0$ be such that

$$\bar{K}_m < \min\left[\frac{L(1-\alpha)}{e^{L\bar{T}}(e^{L\bar{T}}-1)(1+\alpha+e^{L\bar{T}})}, \frac{\hat{\beta}}{c_u \bar{T} e^{2L\bar{T}}}\left\{\frac{2+e^{L\bar{T}}}{\bar{\varepsilon}}+1\right\}^{-1}\right],$$

$$\left.\frac{\lambda_{\min}(M)}{4\lambda_{\max}(Q)(1+\|K\|_2/\lambda_{\max}(Q))^{1/2})}\right] \tag{4.10a}$$

$$\bar{c}_d \triangleq \min\left[c''_d\,,\ \left[\left\{\frac{2+e^{L\bar{T}}}{\bar{\varepsilon}}+1\right\}^{-1}\hat{\rho}-\bar{K}_m c_u \bar{T} e^{2L\bar{T}}\right]\{(\bar{K}_m e^{L\bar{T}}+L)\bar{T}e^{L\bar{T}}\}^{-1}\right], \tag{4.10b}$$

where c''_d was defined in (4.8b) and $\bar{\varepsilon} \triangleq (1-\alpha)(\alpha+e^{L\bar{T}})/(1+\alpha+e^{L\bar{T}})$. Let B_{ρ_d} be defined as (4.6d). Suppose that $K_m \le \bar{K}_m$, $c_d \le \bar{c}_d$, and $\varepsilon''' < \rho_{LQR}$, where ε''' was defined in (4.6a) and ρ_{LQR} in (4.8a). Then, *(a)* there exists an $\varepsilon_6 < \infty$ such that for any $x_0^p \in B_{\rho_e}$, $\|x^p(t\,,0\,,x_0^p\,,u\,,d)\| \le \varepsilon_6$ for all $t \in [0\,,\infty)$ and *(b)* $\overline{\lim}_{t\to\infty}\|x^p(t\,,0\,,x_0^p\,,u\,,d)\| \to 0$ as $\bar{c}_d \to 0$. □

5. Robust Stability with State Estimation.

Since it is not always possible to measure the state of the plant, $x^p(t\,,0\,,x_0^p\,,u\,,d)$, we will now examine the behavior of our closed loop system resulting from the use of Control Algorithm 2.5, when the state of the plant has to be estimated in the presence of modeling errors. Here, we will assume that there are no disturbances. At this point, we must introduce more structure into the nonlinear functions $f^p(\cdot\,,\cdot\,,\cdot)$ and $f^m(\cdot\,,\cdot\,,\cdot)$ in (2.1a) and (2.2a), respectively, by assuming a parametric structure for the model uncertainty. Hence, consider the nonlinear time invariant system described by

$$\dot{x}(t) = F(x(t)\,,u(t)\,,\theta_x)\,, \tag{5.1a}$$

$$y(t) = h(x(t)\,,\theta_y)\,, \tag{5.1b}$$

with $u \in U$, where U was defined in (2.1b), $\theta_x \in \mathbb{R}^{q_1}$, $\theta_y \in \mathbb{R}^{q_2}$, $F : \mathbb{R}^n \times G_u \times \mathbb{R}^{q_1} \to \mathbb{R}^n$, and $h : \mathbb{R}^n \times \mathbb{R}^{q_2} \to \mathbb{R}^{n_o}$. Let $q \triangleq q_1 + q_2$ and let $\theta \in \mathbb{R}^q$ be defined by $\theta \triangleq ((\theta_x)^T, (\theta_y)^T)^T$. We will denote the solution of (5.1a,b) at time t, corresponding to the initial state x_0 at time t_0, and the input u by $x(t\,,t_0,x_0,u\,,\theta_x)$.

We assume that the actual plant parameter vector θ is θ^p, but due to modeling errors, in the model, the parameter vector θ is set to θ^m. We will denote the solution of (5.1a,b), with $\theta = \theta^p$, by $x^p(t\,,t_0,x_0^p\,,u)$ and $y^p(t)$, and by $x^m(t\,,t_0,x_0,u)$ and $y^m(t)$ when $\theta = \theta^m$.

We propose to obtain an estimate of the initial state by solving the following nonlinear least squares problem:

$$\mathbf{P_E}(\theta, T, u): \min_{x_0 \in \mathbb{R}^n} J(x_0; T, u, \theta), \tag{5.2a}$$

where

$$J(x_0; T, u, \theta) \triangleq \tfrac{1}{2}\int_0^{\gamma T} |y^p(t) - y(t)|^2 \, dt$$

$$= \tfrac{1}{2}\int_0^{\gamma T} |h(x^p(t, 0, x_0^p, u), \theta_y^p) - h(x(t, 0, x_0, u, \theta_x), \theta_y)|^2 \, dt , \tag{5.2b}$$

with $T \in [T_C, \bar{T}]$ and $\gamma \in (0, 1)$. It is obvious that $J(x_0^p; T, u, \theta^p) = 0$ for all $u \in U$ and for all $T \in [T_C, \bar{T}]$.

Assumption 5.1. We assume that for any $\theta_x \in \mathbb{R}^{q_1}$ and $\theta_y \in \mathbb{R}^{q_2}$, $F(0, 0, \theta_x) = 0$ and $h(0, \theta_y) = 0$. □

Assumption 5.2. We assume that for any $u \in U$, and $T \in [T_C, \bar{T}]$, $\mathbf{P_E}(\theta^p, T, u)$ has a unique solution x_0^*, such that $J(x_0^*; T, u, \theta^p) = 0$, i.e., $x_0^* = x_0^p$. □

Assumption 5.3. Let $\hat{\rho}$ be as in Assumption 2.3, let $\rho_x^\theta, \rho_y^\theta \in (0, \infty)$, and let $B^o(\theta_x^p, \rho_x^\theta)$ and $B^o(\theta_y^p, \rho_y^\theta)$ be defined by

$$B^o(\theta_x^p, \rho_x^\theta) \triangleq \{ \theta_x \in \mathbb{R}^{q_1} \mid |\theta_x^p - \theta_x|_2 < \rho_x^\theta \} \tag{5.3a}$$

$$B^o(\theta_y^p, \rho_y^\theta) \triangleq \{ \theta_y \in \mathbb{R}^{q_2} \mid |\theta_y^p - \theta_y|_2 < \rho_y^\theta \} . \tag{5.3b}$$

Let the reachable set $\mathbf{R}(\hat{\rho}, \rho_x^\theta)$ be defined by

$$\mathbf{R}(\hat{\rho}, \rho_x^\theta) \triangleq \{ z \in \mathbb{R}^n \mid z = x^m(t, 0, x_0, u),$$

$$\forall t \in [0, \bar{T}], x_0 \in B_{\hat{\rho}}, u \in U, \theta_x \in B^o(\theta_x^p, \rho_x^\theta) \} . \tag{5.3c}$$

We assume that there exists a Lipschitz constant $L \in [0, \infty)$ such that for all $\xi', \xi'' \in \mathbf{R}(\hat{\rho}, \rho_x^\theta)$, $v', v'' \in U$, $\theta'_x, \theta''_x \in B^o(\theta_x^p, \rho_x^\theta)$, and $\theta'_y, \theta''_y \in B^o(\theta_y^p, \rho_y^\theta)$,

$$|F(\xi', v', \theta'_x) - F(\xi', v', \theta''_x)| \le L |\theta'_x - \theta''_x|_2 (|\xi'| + |v'|_\infty) , \tag{5.3d}$$

$$|F(\xi', v', \theta'_x) - F(\xi'', v'', \theta'_x)| \le L (|\xi' - \xi''| + |v' - v''|_\infty) , \tag{5.3e}$$

$$\left|\frac{\partial}{\partial x} F(\xi', v', \theta'_x) - \frac{\partial}{\partial x} F(\xi', v', \theta''_x)\right|_2 \le L |\theta'_x - \theta''_x|_2 (|\xi'| + |v'|_\infty) , \tag{5.3f}$$

$$\left|\frac{\partial}{\partial x} F(\xi', v', \theta'_x) - \frac{\partial}{\partial x} F(\xi'', v'', \theta'_x)\right|_2 \le L (|\xi' - \xi''| + |v' - v''|_\infty) , \tag{5.3g}$$

$$\left|\frac{\partial}{\partial u}F(\xi',v',\theta'_x)-\frac{\partial}{\partial u}F(\xi',v',\theta''_x)\right|_2 \le L|\theta'_x-\theta''_x|_2(|\xi'|+|v'|_\infty)\,, \tag{5.3h}$$

$$\left|\frac{\partial}{\partial u}F(\xi',v',\theta'_x)-\frac{\partial}{\partial u}F(\xi'',v'',\theta'_x)\right|_2 \le L(|\xi'-\xi''|+|v'-v''|_\infty)\,, \tag{5.3i}$$

$$\left|\frac{\partial}{\partial \theta_x}F(\xi',v',\theta'_x)-\frac{\partial}{\partial \theta_x}F(\xi',v',\theta''_x)\right|_2 \le L|\theta'_x-\theta''_x|_2(|\xi'|+|v'|_\infty)\,, \tag{5.3j}$$

$$\left|\frac{\partial}{\partial \theta_x}F(\xi',v',\theta'_x)-\frac{\partial}{\partial \theta_x}F(\xi'',v'',\theta'_x)\right|_2 \le L(|\xi'-\xi''|+|v'-v''|_\infty)\,, \tag{5.3k}$$

$$\left|\frac{\partial^2}{\partial x^2}F(\xi',v',\theta'_x)-\frac{\partial^2}{\partial x^2}F(\xi',v',\theta''_x)\right|_2 \le L|\theta'_x-\theta''_x|_2(|\xi'|+|v'|_\infty)\,, \tag{5.3l}$$

$$\left|\frac{\partial^2}{\partial x^2}F(\xi',v',\theta'_x)-\frac{\partial^2}{\partial x^2}F(\xi'',v'',\theta'_x)\right|_2 \le L(|\xi'-\xi''|+|v'-v''|_\infty)\,, \tag{5.3m}$$

$$|h(\xi',\theta'_y)-h(\xi',\theta''_y)|_2 \le L|\theta'_y-\theta''_y|_2|\xi'|\,, \tag{5.3n}$$

$$|h(\xi',\theta'_y)-h(\xi'',\theta'_y)|_2 \le L|\xi'-\xi''|\,, \tag{5.3o}$$

$$\left|\frac{\partial}{\partial x}h(\xi',\theta'_y)-\frac{\partial}{\partial x}h(\xi',\theta''_y)\right|_2 \le L|\theta'_y-\theta''_y|_2|\xi'|\,, \tag{5.3p}$$

$$\left|\frac{\partial}{\partial x}h(\xi',\theta'_y)-\frac{\partial}{\partial x}h(\xi'',\theta'_y)\right|_2 \le L|\xi'-\xi''|\,, \tag{5.3q}$$

$$\left|\frac{\partial}{\partial \theta_y}h(\xi',\theta'_y)-\frac{\partial}{\partial \theta_y}h(\xi',\theta''_y)\right|_2 \le L|\theta'_y-\theta''_y|_2|\xi'|\,, \tag{5.3r}$$

$$\left|\frac{\partial}{\partial \theta_y}h(\xi',\theta'_y)-\frac{\partial}{\partial \theta_y}h(\xi'',\theta'_y)\right|_2 \le L|\xi'-\xi''|\,, \tag{5.3s}$$

$$\left|\frac{\partial^2}{\partial x^2}h(\xi',\theta'_y)-\frac{\partial^2}{\partial x^2}h(\xi',\theta''_y)\right|_2 \le L|\theta'_y-\theta''_y|_2|\xi'|\,, \tag{5.3t}$$

$$\left|\frac{\partial^2}{\partial x^2}h(\xi',\theta'_y)-\frac{\partial^2}{\partial x^2}h(\xi'',\theta'_y)\right|_2 \le L|\xi'-\xi''|\,. \tag{5.3u}$$

□

Assumption 5.4. We assume that if x_0^* is the solution of the nonlinear least squares problem $P_E(\theta^p, T, u)$, then the Hessian $(\partial^2/\partial x_0^2)J(x_0^*; T, u, \theta^p)$ exists and there exists a $\delta \in (0, \infty)$ such that for all $z \in \mathbb{R}^n$

$$z^T \frac{\partial^2}{\partial x_0^2} J(x_0^*; T, u, \theta^p) z \geq \delta \|z\|_2^2 . \tag{5.4}$$

□

The following theorem is a direct consequence of the Implicit Function Theorem in [Ale.1].

Theorem 5.5. Suppose that Assumptions 5.1-5.4 are satisfied. Let x_0^* be the solution of $P_E(\theta^p, T, u)$. Then there exist $\rho_\varepsilon > 0$, $\rho_\theta > 0$, and a mapping $\eta : B^o(\theta^p, \rho_\theta) \to \mathbb{R}^n$ of class $C^1(B^o(\theta^p, \rho_\theta))$, where $B^o(\theta, \rho) \triangleq \{ z \in \mathbb{R}^q \mid \|z - \theta\|_2 < \rho \}$, such that

(1) $\eta(\theta^p) = x_0^*$;

(2) for all $\theta \in B^o(\theta^p, \rho_\theta)$, $\|\eta(\theta) - x_0^*\| \leq \rho_\varepsilon$ and the gradient $(\partial/\partial x_0)J(\eta(\theta); T, u, \theta) = 0$;

(3) the equality $(\partial/\partial x_0)J(x_0; T, u, \theta) = 0$ holds in the *rectangle* $B^o(\theta^p, \rho_\theta) \times B^o_{\rho_\varepsilon}(x_0^*)$ only for all $x_0 = \eta(\theta)$, where $B^o_\rho(x) \triangleq \{ z \in \mathbb{R}^n \mid \|z - x\| < \rho \}$ and x_0 is the solution of $P_E(\theta, T, u)$.

Proof. First, it follows from (5.2b) that for any $\theta \triangleq (\theta_x^T, \theta_y^T)^T \in B^o(\theta^p, \rho_\theta)$,

$$\frac{\partial}{\partial x_0} J(x_0; T, u, \theta) = \int_0^{\gamma T} \frac{\partial}{\partial x_0} h(x(t, 0, x_0, u, \theta_x), \theta_y)^T [y^p(t) - y(t)]\, dt$$

$$= \int_0^{\gamma T} \left[\frac{\partial}{\partial x} h(x(t, 0, x_0, u, \theta_x), \theta_y) \frac{\partial}{\partial x_0} x(t, 0, x_0, u) \right]^T$$

$$[h(x^p(t, 0, x_0^p, u), \theta_y^p) - h(x(t, 0, x_0, u, \theta_x), \theta_y)]\, dt . \tag{5.5}$$

It follows from Assumption 5.3 that $(\partial/\partial x_0)J(x_0; T, u, \theta)$ is differentiable in (x_0, θ). Also, it follows from Assumption 5.4 that $(\partial^2/\partial x_0^2)J(x_0^p; T, u, \theta^p)^{-1}$ exists. Hence, the conditions of the Implicit Function Theorem in [Ale.1, pp. 105] are satisfied, which completes our proof. □

Lemma 5.6. Suppose that Assumptions 5.1-5.4 are satisfied. Suppose that $x_0^p, x_0 \in R(\rho, \rho_x^\theta)$ are the solution of $P_E(\theta^p, T, u), P_E(\theta, T, u)$, respectively, where $\theta \in B^o(\theta^p, \hat{\rho}_\theta)$ with $\hat{\rho}_\theta \triangleq \min(\rho_x^\theta, \rho_y^\theta, \rho_\theta)$, $\rho_x^\theta, \rho_y^\theta$ as in Assumption 5.3, and ρ_θ as in Theorem 5.5. Then there exists a $\bar{L} \in [0, \infty)$ such that

$$|\eta(\theta) - x_0^p| = |x_0 - x_0^p| \leq \bar{L}|\theta - \theta^p|_2 , \tag{5.6}$$

for all $\theta \in B^o(\theta^p, \hat{\rho}_\theta)$.

Proof. First, it follows from Assumption 5.2 that $x_0^* = x_0^p$. It follows from [Ale.1, pp.102] that there exists a $K \in [0, \infty)$ such that for all $\theta \triangleq (\theta_x^T, \theta_y^T)^T \in B^o(\theta^p, \hat{\rho}_\theta)$,

$$|\eta(\theta) - x_0^p| \leq K \left| \frac{\partial}{\partial x_0} J(x_0^p ; T, u, \theta) \right| . \tag{5.7a}$$

Now, we have that

$$\frac{\partial}{\partial x_0} J(x_0^p ; T, u, \theta) = \int_0^{\gamma T} \left[\frac{\partial}{\partial x} h(x(t, 0, x_0^p, u, \theta_x), \theta_y) \frac{\partial}{\partial x_0} x(t, 0, x_0^p, u) \right]^T$$

$$[h(x^p(t, 0, x_0^p, u), \theta_y^p) - h(x(t, 0, x_0^p, u, \theta_x), \theta_y)]\, dt . \tag{5.7b}$$

Hence

$$\left| \frac{\partial}{\partial x_0} J(x_0^p ; T, u, \theta) \right|_2 \leq \int_0^{\gamma T} \left| \frac{\partial}{\partial x} h(x(t, 0, x_0^p, u, \theta_x), \theta_y) \right|_2 \left| \frac{\partial}{\partial x_0} x(t, 0, x_0^p, u) \right|_2$$

$$|h(x^p(t, 0, x_0^p, u), \theta_y^p) - h(x(t, 0, x_0^p, u, \theta_x), \theta_y)|_2\, dt .$$

Now, since (5.3o) is satisfied for all $\xi', \xi'' \in R(\hat{\rho}, \rho_x^\theta)$, it follows from direct calculation that $|(\partial h/\partial x) h(x(t, 0, x_0^p, u, \theta_x), \theta_y)|_2 \leq L$. Now

$$\left| \frac{\partial}{\partial x_0} x(t, 0, x_0^p, u) \right|_2 = \left| \frac{\partial}{\partial x_0} \left[x_0 + \int_0^t F(x(\tau, 0, x_0^p, u), u(\tau), \theta_x)\, d\tau \right] \right|_2$$

$$\leq 1 + \int_0^t \left| \frac{\partial}{\partial x} F(x(\tau, 0, x_0^p, u, \theta_x), u(\tau), \theta_x) \right|_2 \left| \frac{\partial}{\partial x_0} x(\tau, 0, x_0^p, u) \right|_2 d\tau . \tag{5.7c}$$

It follows from (5.3e) with $v' = v''$ and by direct calculation that

$$\left| \frac{\partial}{\partial x} F(x(\tau, 0, x_0^p, u, \theta_x), u(\tau), \theta_x) \right|_2 \leq L . \tag{5.7d}$$

Next, it follows from (5.7c,d) and the Bellman-Gronwall inequality that for all $t \in [0, \bar{T}]$,

$$\left| \frac{\partial}{\partial x_0} x(t, 0, x_0^p, u) \right|_2 \leq e^{L\bar{T}} . \tag{5.7e}$$

Next, it follows from (5.3d,e) and the Bellman-Gronwall inequality that for all $t \in [0, \bar{T}]$,

$$|x^p(t,0,x_0^p,u)| \le |x_0^p| + \int_0^t |F(x^p(\tau,0,x_0^p,u),u(\tau),\theta_x^p)|\,d\tau$$

$$\le |x_0^p| + \int_0^t L(|x^p(\tau,0,x_0^p,u)| + |u(\tau)|_\infty)d\tau$$

$$\le (|x_0^p| + Lc_u\bar{T})e^{Lt}\,. \tag{5.7f}$$

Also, it can be easily shown from (5.3d,e) and (5.7f) that

$$|x^p(t,0,x_0^p,u) - x(t,0,x_0^p,u,\theta_x)|$$

$$\le \int_0^t |F(x^p(\tau,0,x_0^p,u),u(\tau),\theta_x^p) - F(x(\tau,0,x_0^p,u,\theta_x),u(\tau),\theta_x)|\,d\tau$$

$$\le L\int_0^t |x^p(\tau,0,x_0^p,u) - x(\tau,0,x_0^p,u,\theta_x)|\,d\tau + L\int_0^t (|x^p(\tau,0,x_0^p,u)| + c_u)d\tau$$

$$\le (|x_0^p| + Lc_u\bar{T})e^{L\bar{T}}|\theta_x^p - \theta_x|_2 + L\int_0^t |x^p(\tau,0,x_0^p,u) - x(\tau,0,x_0^p,u,\theta_x)|\,d\tau\,. \tag{5.7g}$$

Hence, it follows from the Bellman-Gronwall inequality that

$$|x^p(t,0,x_0^p,u) - x(t,0,x_0^p,u,\theta_x)| \le (|x_0^p| + Lc_u\bar{T})e^{L\bar{T}}|\theta_x^p - \theta_x|_2 e^{Lt}\,. \tag{5.7h}$$

It now follows from (5.3n,o) and (5.7f,h) that for all $t \in [0,\bar{T}]$,

$$|h(x^p(t,0,x_0^p,u),\theta_y^p) - h(x(t,0,x_0^p,u,\theta_x),\theta_y)|_2 \le L|\theta^p - \theta|_2\,|x^p(t,0,x_0^p,u)|$$

$$+ L|x^p(t,0,x_0^p,u) - x(t,0,x_0^p,u,\theta_x)|$$

$$\le L(|x_0^p| + Lc_u\bar{T})(1 + e^{L\bar{T}})e^{Lt}\,|\theta^p - \theta|_2\,. \tag{5.7i}$$

Now, it follows from (5.7c-i) that

$$\left|\frac{\partial}{\partial x_0}J(x_0^p;\,T,u,\theta)\right|_2 \le Le^{L\bar{T}}(1 + e^{L\bar{T}})|\theta^p - \theta|_2 \int_0^{\gamma T} L(|x_0^p| + Lc_u\bar{T})e^{Lt}dt$$

$$\le Le^{L\bar{T}}(|x_0^p| + Lc_u\bar{T})(e^{2L\bar{T}} - 1)\,|\theta^p - \theta|_2 \triangleq L'|\theta^p - \theta|_2\,. \tag{5.7j}$$

Setting $\bar{L} \triangleq KL'$, we obtain the desired result. □

We propose to utilize the result of Lemma 5.6 in Control Algorithm 2.5 as follows.

Control Algorithm 5.7.

Data: $t_0 = 0$, $\gamma \in (0,1)$, $t_1 = T_C$, x_0^m, and $u_{[t_0,t_1]}(t) \equiv 0$. T_C and $\bar{T}$ such that $0 < T_C < \bar{T} < \infty$.

Step 0: Set $k = 0$.

Step 1: At $t = t_k$,

(a) Set the plant input $u(t) = u_{[t_k, t_{k+1}]}(t)$, for $t \in [t_k, t_{k+1})$.

(b) At $t = t_k + \gamma(t_{k+1} - t_k)$, estimate the state of the plant $x_k^p \triangleq x^p(t, 0, x_0^p, u)$ by solving the optimal control problem $\mathbf{P_E}(\theta^m, t_{k+1} - t_k, u)$ where θ^m is the parameter of the model, and denote the resulting value by $\bar{x}_k$.

(c) Compute the estimate $x_{k+1}^m \triangleq x^m(t_{k+1}, t_k, \bar{x}_k, u_{[t_k, t_{k+1}]})$ of the state of the plant $x^p(t_{k+1}, t_k, x_k^p, u_{[t_k, t_{k+1}]})$ by solving (5.2a) with state $\bar{x}_k$ at time $t = t_k$, and an input $u_{[t_k, t_{k+1}]}(t)$, $t \in [t_k, t_{k+1}]$.

(d) Solve the open loop optimal control problem $\mathbf{P}(x_{k+1}^m, t_{k+1})$ to compute the next sampling time $t_{k+2} \in (t_{k+1}+T_C, t_{k+1}+\bar{T}]$, and the optimal control $u_{[t_{k+1}, t_{k+2}]}(t)$, $t \in [t_{k+1}, t_{k+2})$.

Step 2: Replace k by $k + 1$ and go to Step 1. □

We will examine the behavior of Control Algorithm 5.7.

Lemma 5.8. Consider the moving horizon feedback system resulting from the use of Control Algorithm 5.7. Suppose that Assumptions 5.1-5.3 are satisfied. Let $\hat{\rho}$ be as in Assumption 2.3, let $L \in [0, \infty)$ be as in Assumption 5.3, and let $\hat{\rho}_\theta$ be as in Lemma 5.6. If $x_k^p, x_k^m \in B_{\hat{\rho}}$ for all $k \in \mathbb{N}$ and $\theta^m \in B^o(\theta^p, \hat{\rho}_\theta)$, then there exist $\Delta_5, \Delta_6 \in [0, \infty)$ such that

$$\| x_{k+1}^p - x_{k+1}^m \| \le \Delta_5 \| x_k^p \| + \Delta_6 , \tag{5.8}$$

and $\Delta_5, \Delta_6 \to 0$ as $\hat{\rho}_\theta \to 0$.

Proof. First, the optimal control problem $\mathbf{P}(x_k^m, t_k)$ has a solution for all $k \ge 1$ since $x_k^m \in B_{\hat{\rho}}$ for all $k \ge 1$, and as a result, the trajectory $x^p(t, 0, x_0^p, u)$ is well defined. Given $x_k^p \in B_{\hat{\rho}}$ and $u = u_{[t_k, t_{k+1}]} \in U$, obtained by solving the optimal control problem $\mathbf{P}(x_k^m, t_k)$, let $x_k^p(t) \triangleq x^p(t, t_k, x_k^p, u_{[t_k, t_{k+1}]})$ and $x_k^m(t) \triangleq x^m(t, t_k, \bar{x}_k, u_{[t_k, t_{k+1}]})$. Then it follows from Lemma 5.6, (5.3d,e), (5.7f), and the fact that $\bar{x}_k$ is the solution of $\mathbf{P_E}(\theta^m, t_{k+1} - t_k, u)$ that for all $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$,

$$\| x_k^p(t) - x_k^m(t) \| \le \| x_k^p - \bar{x}_k \| + \int_{t_k}^{t} \| F(x_k^p(\tau), u_{[t_k, t_{k+1}]}(t), \theta_x^p) - F(x_k^m(\tau), u_{[t_k, t_{k+1}]}(t), \theta_x^m) \| \, d\tau$$

$$\leq \bar{L}|\theta^p - \theta^m|_2 + \int_{t_k}^{t} \Big[L|\theta^p - \theta^m|_2 \, |x_k^p(\tau)| + L \, |x_k^p(\tau) - x_k^m(\tau)| \Big] \, d\tau$$

$$\leq \bar{L}\hat{\rho}_\theta + \hat{\rho}_\theta(|x_k^p| + Lc_u\bar{T})(e^{L\bar{T}} - 1) + L\int_{t_k}^{t} |x_k^p(\tau) - x_k^m(\tau)| \, d\tau \, . \qquad (5.9a)$$

Then, it follows from the Bellman-Gronwall inequality that for all $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$,

$$|x_k^p(t) - x_k^m(t)| \leq \hat{\rho}_\theta e^{L\bar{T}}(e^{L\bar{T}} - 1)|x_k^p| + \hat{\rho}_\theta(\bar{L} + Lc_u\bar{T}(e^{L\bar{T}} - 1))e^{L\bar{T}}$$

$$\triangleq \Delta_5|x_k^p| + \Delta_6 \, . \qquad (5.9b)$$

Then, $\Delta_5, \Delta_6 \to 0$ as $\hat{\rho}_\theta \to 0$, which completes our proof. □

We note that Lemma 3.1 is the basic result that was used to prove other results in Section 3. Now, by comparing Lemma 3.1 and Lemma 5.8, it is clear that similar to those results stated in Section 3 will hold if we replace K_m with $\hat{\rho}_\theta$. Therefore, we omit the proof of the following theorem.

Theorem 5.9. Consider the moving horizon feedback system resulting from the use of Control Algorithm 5.7. Then, there exist $\bar{\rho}_\theta \in (0, \hat{\rho}_\theta]$ and $\rho_s \in (0, \hat{\rho}]$, where $\hat{\rho}_\theta$ is as in Lemma 5.6 and $\hat{\rho}$ is as in Assumption 2.3, such that for any $x_0^p \in B_{\rho_s}$ and for any $\theta^m \in B^o(\rho^p, \bar{\rho}_\theta)$, the trajectory $x^p(t, 0, x_0^p, u)$ is well defined and that there exists a $\varepsilon_7 \in (0, \infty)$ such that $|x^p(t, 0, x_0^p, u)| \leq \varepsilon_7$ for all $t \geq 0$. Furthermore, there exists a $K_x \in (0, \infty)$ such that $\overline{\lim}_{t \to \infty} |x^p(t, 0, x_0^p, u)| \leq K_x\bar{\rho}_\theta$. □

6. APPENDIX

We will now establish two inequalities that form the basis of several of our proofs.

Proposition 6.1 Consider the second order scalar difference equation

$$y_{k+2} = a_1 y_{k+1} + a_2 y_k + b \, , \quad k \in \mathbb{N}. \qquad (6.1a)$$

If $a_1, a_2 \geq 0$, $b \geq 0$ and $a_1 + a_2 < 1$, then for all $k \geq 1$,

$$y_k \leq a_2 y_0 + y_1 + b/(1 - a_1 + a_2) \, , \qquad (6.1b)$$

and

$$\overline{\lim}_{k \to \infty} y_k \leq b/(1 - a_1 + a_2). \qquad (6.1c)$$

Proof. We begin by rewriting (6.1a) in first order vector form, as follows. For $k \in \mathbb{N}$, let $z_k = (y_k, y_{k+1})^T$. Then $z_0 = (y_0, y_1)^T$, and

$$z_{k+1} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} z_k + \begin{bmatrix} 0 \\ b \end{bmatrix} \triangleq F z_k + g \,, \tag{6.2a}$$

$$y_k = [1\,0] z_k \triangleq H z_k . \tag{6.2b}$$

The matrix F has two eigenvalues, $\lambda_+, \lambda_- = ½(a_1 \pm \sqrt{a_1^2 + 4a_2})$, with corresponding eigenvectors, $e_+ = (1, \lambda_+)^T$ and $e_1 = (1, \lambda_-)^T$. We will now show that $-1 < \lambda_- \leq 0 \leq \lambda_+ < 1$, i.e., that (6.2a) is an asymptotically stable system. By assumption

$$0 \leq a_2 < 1 - a_1 . \tag{6.2c}$$

If we multiply both sides 0f (6.2c) by 4, and add a_1^2 to the both sides, we get that

$$a_1^2 + 4a_2 < (2 - a_1)^2 , \tag{6.2d}$$

which implies that $\lambda_- = ½(a_1 - \sqrt{a_1^2 + 4a_2}) > -1$ and $\lambda_+ = ½(a_1 + \sqrt{a_1^2 + 4a_2}) < 1$. Thus, we have that $-1 < \lambda_- \leq \lambda_+ < 1$.

We can proceed to establish (6.1b,c). By the Jordan decomposition, we have that

$$F = E^{-1} \Lambda E \,, \tag{6.2e}$$

where $\Lambda = diag(\lambda_+, \lambda_-)$, and $E = (e_+, e_-)$ is a matrix whose columns are the eigenvectors of F. Hence for all $k \geq 2$,

$$y_k = H E^{-1} \Lambda^k E z_0$$

$$= \frac{1}{\lambda_- - \lambda_+} \{ \lambda_+ \lambda_- (\lambda_+^{k-1} - \lambda_-^{k-1}) y_0 + (\lambda_-^k - \lambda_+^k) y_1 \} + \frac{b}{\lambda_- - \lambda_+} \sum_{i=0}^{k-1} (\lambda_-^{k-1-i} - \lambda_+^{k-1-i}) . \tag{6.2f}$$

Since $0 < \lambda_+ < 1$ and $-1 < \lambda_- < 0$, it is clear that *(a)* the first term in (6.2f) goes to zero as $k \to \infty$ and *(b)* the last term in (6.2f) satisfies the inequality

$$\frac{b}{\lambda_- - \lambda_+} \sum_{i=0}^{k-1} (\lambda_-^{k-1-i} - \lambda_+^{k-1-i}) \leq \frac{b}{\lambda_- - \lambda_+} \left\{ \frac{1}{1 - \lambda_-} - \frac{1}{1 - \lambda_+} \right\} = \frac{b}{1 - a_1 + a_2} , \tag{6.2g}$$

because $(1 - \lambda_+)(1 - \lambda_-) = 1 - a_1 + a_2$, which proves (6.1c).

Next, for all $k \geq 1$, $\lambda_+^k \leq \lambda_+$ and $-\lambda_-^k \leq (-\lambda_-)^k \leq -\lambda_-$. Hence $\{ \lambda_+ \lambda_- (\lambda_+^{k-1} - \lambda_-^{k-1}) / (\lambda_- - \lambda_+) \leq \lambda_+ \lambda_- = a_2$. Also $(\lambda_-^k - \lambda_+^k)/(\lambda_- - \lambda_+) \leq 1$, hence (6.1b) hold. □

7. APPENDIX

We will now establish two inequalities that form the basis of several of our proofs.

Proposition 7.1 Consider the second order scalar difference equation

$$y_{k+2} = a_1 y_{k+1} + a_2 y_k + b \ , \ k \in \mathbb{N} . \tag{7.1a}$$

If $a_1, a_2 \geq 0, b \geq 0$ and $a_1 + a_2 < 1$, then for all $k \geq 1$,

$$y_k \leq a_2 y_0 + y_1 + b/(1 - a_1 + a_2) , \tag{7.1b}$$

and

$$\overline{\lim}_{k \to \infty} y_k \leq b/(1 - a_1 + a_2) . \tag{7.1c}$$

Proof. We begin by rewriting (7.1a) in first order vector form, as follows. For $k \in \mathbb{N}$, let $z_k = (y_k , y_{k+1})^T$. Then $z_0 = (y_0 , y_1)^T$, and

$$z_{k+1} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} z_k + \begin{bmatrix} 0 \\ b \end{bmatrix} \triangleq F z_k + g \ , \tag{7.2a}$$

$$y_k = [1\ 0] z_k \triangleq H z_k . \tag{7.2b}$$

The matrix F has two eigenvalues, $\lambda_+ , \lambda_- = ½(a_1 \pm \sqrt{a_1^2 + 4a_2})$, with corresponding eigenvectors, $e_+ = (1 , \lambda_+)^T$ and $e_1 = (1 , \lambda_-)^T$. We will now show that $-1 < \lambda_- \leq 0 \leq \lambda_+ < 1$, i.e., that (7.2a) is an asymptotically stable system. By assumption

$$0 \leq a_2 < 1 - a_1 . \tag{7.2c}$$

If we multiply both sides Of (7.2c) by 4, and add a_1^2 to the both sides, we get that

$$a_1^2 + 4a_2 < (2 - a_1)^2 , \tag{7.2d}$$

which implies that $\lambda_- = ½(a_1 - \sqrt{a_1^2 + 4a_2}) > -1$ and $\lambda_+ = ½(a_1 + \sqrt{a_1^2 + 4a_2}) < 1$. Thus, we have that $-1 < \lambda_- \leq \lambda_+ < 1$.

We can proceed to establish (7.1b,c). By the Jordan decomposition, we have that

$$F = E^{-1} \Lambda E \ , \tag{7.2e}$$

where $\Lambda = diag(\lambda_+ , \lambda_-)$, and $E = (e_+ , e_-)$ is a matrix whose columns are the eigenvectors of F. Hence for all $k \geq 2$,

$$y_k = H E^{-1} \Lambda^k E z_0$$

$$= \frac{1}{\lambda_- - \lambda_+} \{ \lambda_+\lambda_-(\lambda_+^{k-1} - \lambda_-^{k-1})y_0 + (\lambda_-^k - \lambda_+^k)y_1 \} + \frac{b}{\lambda_- - \lambda_+} \sum_{i=0}^{k-1} (\lambda_-^{k-1-i} - \lambda_+^{k-1-i}) . \tag{7.2f}$$

Since $0 < \lambda_+ < 1$ and $-1 < \lambda_- < 0$, it is clear that *(a)* the first term in (7.2f) goes to zero as $k \to \infty$ and *(b)* the last term in (7.2f) satisfies the inequality

$$\frac{b}{\lambda_- - \lambda_+} \sum_{i=0}^{k-1} (\lambda_-^{k-1-i} - \lambda_+^{k-1-i}) \le \frac{b}{\lambda_- - \lambda_+} \left\{ \frac{1}{1-\lambda_-} - \frac{1}{1-\lambda_+} \right\} = \frac{b}{1-a_1+a_2} , \tag{7.2g}$$

because $(1-\lambda_+)(1-\lambda_-) = 1 - a_1 + a_2$, which proves (7.1c).

Next, for all $k \ge 1$, $\lambda_+^k \le \lambda_+$ and $-\lambda_-^k \le (-\lambda_-)^k \le -\lambda_-$. Hence $\{ \lambda_+\lambda_-(\lambda_+^{k-1} - \lambda_-^{k-1})/(\lambda_- - \lambda_+) \le \lambda_+\lambda_- = a_2$. Also $(\lambda_-^k - \lambda_+^k)/(\lambda_- - \lambda_+) \le 1$, hence (7.1b) hold. □

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