MOVING HORIZON CONTROL OF LINEAR SYSTEMS WITH INPUT SATURATION, DISTURBANCES, AND PLANT UNCERTAINTY

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E. Polak* and T. H. Yang*

ABSTRACT

We present a moving horizon feedback system, based on constrained optimal control algorithms, for linear plants with input saturation. The system is a nonconventional sampled-data system: its sampling periods vary from sampling instant to sampling instant, and the control during the sampling time is not constant, but determined by the solution of an open loop optimal control problem. In part I we showed that the proposed moving horizon control system is robustly stable. In this paper we show that it is capable of following a class of reference inputs and suppressing a class of disturbances.

KEY WORDS: Moving horizon control, robust stability, tracking, disturbance rejection.

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1. INTRODUCTION

Model predictive control and moving horizon control are popular in the control of slow dynamical systems, such as chemical process control in the petrochemical and pulp and paper industry, and gas pipeline control (see [Gar.2]). This is the second part of a two part paper, dealing with a new, moving horizon feedback law for linear, time invariant plants, modeled with errors, subject to disturbances, reference inputs, and control constraints, and with the time to solve the optimal control problem accounted for, that was introduced in [Pol.2]. This control algorithm is in the general class of open loop optimal feedback laws, dating back to a 1962 seminal paper by Propoi [Pro.1], that solve open loop optimal control problems with constraints to obtain a feedback law. This class of feedback algorithms includes model predictive control (see [Meh.1, Pre.1, Gar.1, Gar.2]) and moving horizon control (see [Kwo.1, Kwo.2, May.1, May.2, Kee.1]). For a very nice survey of model predictive control, see [Gar.2].

Our algorithm differs from other moving horizon control laws in that it uses a free time, control and state space constrained optimal control problem, and hence it results in a nonconventional sampled-data system: its sampling periods vary from sampling instant to sampling instant, and the control during the sampling time is not constant, but determined by the solution of an open loop optimal control problem.

In part I [Pol.2] we showed that the feedback systems resulting from the use of our control algorithm are robustly stable in the absence of disturbances. In this part, we will examine its disturbance rejection and reference input following characteristics. We note that our results are considerably stronger than those found in the literature dealing with model predictive and moving horizon control, where the state is usually assumed to be measurable, the controls are assumed to be unbounded, and the models are assumed to be exact (see, e.g. [Cla.1, Cla.2, Kwo.2, May.1, Mic.2]).

In Section 2 we introduce our proposed moving horizon feedback control law. In Section 3 we study the effect of disturbances, while in Section 4, we establish a class of reference inputs that can be tracked asymptotically by our system.

2. STRUCTURE OF THE MOVING HORIZON CONTROL LAW

We assume that the plant is a linear-time-invariant (LTI) system, with bounded inputs and an input disturbance, described by the differential equation

$$\dot{\xi}^p(t) = A^p \xi^p(t) + B^p u(t) + B_d^p d(t)\,, \tag{2.1a}$$

$$\eta^p(t) = C^p \xi^p(t)\,, \tag{2.1b}$$

where the state $\xi^p(t) \in \mathbb{R}^n$, the control $u \in U$, with

$$U \triangleq \{\, u \in L_2^m[0,\infty) \mid \|u\|_\infty \le c_u \,\}\,, \tag{2.1c}$$

$c_u \in (0,\infty)$, and the disturbance $d \in L_\infty^{m_d}[0,\infty)$. Consequently, $A^p \in \mathbb{R}^{n\times n}$, $B^p \in \mathbb{R}^{n\times m}$, $B_d^p \in \mathbb{R}^{n\times m_d}$, and $C^p \in \mathbb{R}^{n_o\times n}$. We will denote the solution of (2.1a) at time t, corresponding to the initial state ξ_0^p at time t_0, and the combined input u and d, by $\xi^p(t, t_0, \xi_0^p, u, d)$.

The function of the moving horizon control law is to ensure robust stability and "reasonable" reference signal $r(t)$ tracking, suppress disturbances $d(t)$, while taking into account the fact that the plant inputs are bounded, as in (2.1c), as well as various amplitude constraints on transients.

We assume that the disturbance $d(t)$ cannot be measured and that the matrices A^p, B^p, B_d^p, and C^p are known only to some tolerance. Hence the moving horizon control law must be developed using a plant model, of the same dimension as (2.1a),

$$\dot{\xi}(t) = A\xi(t) + Bu(t) + B_d \hat{d}(t)\,, \tag{2.2a}$$

$$\eta(t) = C\xi(t)\,, \tag{2.2b}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $B_d \in \mathbb{R}^{n\times m_d}$, and $C \in \mathbb{R}^{n_o\times n}$ are approximations to A^p, B^p, B_d^p, and C^p, respectively, and $\hat{d}(t)$ is an estimate of $d(t)$. When $d(t)$ can not be estimated, we set $\hat{d}(t) = 0$. We will denote the solution of (2.2a) at time t, corresponding to the initial state x_0 at time t_0, and the combined input u and $\hat{d}$, by $x(t, t_0, x_0, u, \hat{d})$.

Let Q be a symmetric, positive definite $n\times n$ matrix such that $\langle x, Qx\rangle$ is a Lyapunov function for the linear closed loop system obtained applying state feedback to (2.1a). The reason for this selection will become clear in Section 3. We use this matrix to define the norm $\|x\| \triangleq \langle x, Qx\rangle^{1/2}$. We will denote the usual Eucledean norm on $\mathbb{R}^n$ by $\|\cdot\|_2$.

Assumption 2.1. We will assume that (A, B) and (A, B_d) are a controllable pair, and that (C, A) is an observable pair. $\square$

Let the subspace $S_x \subset \mathbb{R}^n$ be defined by

$$S_x = \{ x \in \mathbb{R}^n \mid x \in R(B), Ax \in R(B) \} , \tag{2.3a}$$

where $R(X)$ denotes the range space of the matrix X. Let H be a matrix whose columns are a basis for S_x. We will show in Section 4 that, when there are no constraints on the control $u(\cdot)$, given any continuously differentiable function $s(t)$, with values in S_x, there exists an input $u_s(t)$ such that for any initial state ξ_0, $\|\xi(t, 0, \xi_0, u_s, 0) - s(t)\| \to 0$ as $t \to \infty$. Let S denote the set of continuously differentiable functions $s : \mathbb{R} \to S_x$. Hence, the reference signals that can be tracked asymptotically, under the best of conditions are those in the set $\mathbf{R} \triangleq C\mathbf{S}$. We will therefore assume that the reference signals to be tracked are in **R**. We will use the following characterization of elements $r \in \mathbf{R}$, because it may help to alleviate the effects of the control constraint. Let $\tilde{C} \triangleq CH$ and let G be a matrix whose columns are a basis for the null space of $\tilde{C}$. Then any reference signal $r \in \mathbf{R}$ can be expressed as follows:

$$r(t) = Cs(t) , \tag{2.3b}$$

where $s(t) \triangleq H(\tilde{C}^T\tilde{C})^{\dagger}\tilde{C}^T r(t)$ ($\dagger$ denotes the Penrose pseudo inverse [Gol.1 pp243]) is continuously differentiable.

We can now define the error dynamics that will be used in defining and analyzing our control law. Suppose that a reference signal $r \in \mathbf{R}$ is given. Let $x^p(t) \triangleq \xi^p(t) - s(t)$, and let $x(t) \triangleq \xi(t) - s(t)$. Then the plant error dynamics are given by

$$\dot{x}^p(t) = A^p x^p(t) + B^p u(t) + B_d^p d(t) + f^p(t) , \tag{2.4a}$$

$$y^p(t) = C^p x^p(t) , \tag{2.4b}$$

where $f^p(t) \triangleq -\dot{s}(t) + A^p s(t)$. Similarly, the model error dynamics become

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d \hat{d}(t) + f(t) , \tag{2.4c}$$

$$y(t) = Cx(t) , \tag{2.4d}$$

where $f(t) \triangleq -\dot{s}(t) + As(t)$.

We will denote the solution of system (2.4c), from the initial state x_0 at time t_0, control u and disturbance $\hat{d}$ by $x(t, t_0, x_0, u, \hat{d})$. Given any time t_k we will let $x_k \triangleq x(t_k, t_0, x_0, u, \hat{d})$.

Assuming that the control law computation takes at most T_C time units, we can now propose a simple, aperiodic sampled-data feedback law, in the form of an algorithm which, during each sampling period, solves an optimal control problem $\mathrm{P}(x_k, t_k, r)$ of the form

$$\mathrm{P}(x_k, t_k, r): \min_{(u,\tau)} \{ g^0(u,\tau) \mid g^i(u,\tau) \le 0,\ i = 1,2,\ldots,l_1,\ \max_{t \in [t_k,\tau]} \phi^j(u,t) \le 0,$$

$$j = 1,\ldots,l_2,\ u \in U,\ \tau \in [t_k + T_C, t_k + \bar{T}] \}, \tag{2.5a}$$

where $0 < T_C < \bar{T} < \infty$, and the constraint functions are defined by

$$g^i(u,\tau) \triangleq h^i(x(\tau, t_k, x_k, u, \hat{d})),\ i = 0,1,\ldots,l_1-1, \tag{2.5b}$$

$$g^{l_1}(u,\tau) = |x(\tau, t_k, x_k, u, \hat{d})|^2 - \alpha^2 |x_k|^2, \tag{2.5c}$$

$$\phi^j(u,t) = h^j(x(t, t_k, x_k, u, \hat{d}), t),\ j = 1,\ldots,l_2-1, \tag{2.5d}$$

$$\phi^{l_2}(u,t) = |x(t, t_k, x_k, u, \hat{d})|^2 - \beta^2 |x_k|^2, \tag{2.5e}$$

where the constraint functions (2.5c,e) with $\alpha \in (0,1)$, $\beta \in [1,\infty)$, are used to ensure robust stability and input tracking, while the other functions, h^i, h^j are convex, locally Lipschitz continuously differentiable functions that can be used to ensure other performance requirements.

We are now ready to state our control algorithm that defines the moving horizon feedback control system. The algorithm uses several parameters: T_C, the time needed to solve the optimal control problem, which must be determined experimentally, and three parameters that are selected partly on the basis of experimentation and partly on judgement, $\bar{T}$, an upper bound on the horizon, and α, β which govern the speed of response of the system.

Control Algorithm 2.2.

Data: $t_0 = 0$, $t_1 = T_C$, $u_{[t_0,t_1]}(t) \equiv 0$, $x_0 \in B_{\hat{\rho}}$. T_C and $\bar{T}$ such that $0 < T_C < \bar{T} < \infty$.

Step 0: Set $k = 0$.

Step 1: At $t = t_k$,

(a) Obtain a measurement or estimate the state $x_k^p = x^p(t_k, t_0, x_0^p, u, d)$ and denote the resulting value by $\bar{x}_k$.

(b) Compute an estimate, $\hat{d}(t)$, of the disturbance $d(t)$ for $t \in [t_k, t_{k+1}]$, if possible; else,

set $\hat{d}(t) = 0$.

(c) Set the plant error dynamics input $u(t) = u_{[t_k, t_{k+1}]}(t) - \hat{d}(t)$ for $t \in [t_k, t_{k+1})$.

(d) Compute an estimate x_{k+1} of the state of the plant error dynamics $x^p(t_{k+1}, t_k, x_k^p, u, d)$ according to the formula

$$x_{k+1} = e^{A(t_{k+1}-t_k)}\tilde{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)}[Bu(t) + B_d\hat{d}(t)]\,dt + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)} f(t)\,dt\,. \tag{2.6}$$

(e) Solve the open loop optimal control problem $P(x_{k+1}, t_{k+1}, r)$ to compute the next sampling time $t_{k+2} \in (t_{k+1}+T_C, t_{k+1}+\bar{T}]$, and the optimal control $u_{[t_{k+1}, t_{k+2}]}(t) \in U$, $t \in [t_{k+1}, t_{k+2})$;

Step 2: Replace k by $k+1$ and go to Step 1. □

In *Step 1 (a)*, the state of the plant has to be estimated if it is not measurable. When the model (2.4c,d) is identical with the actual dynamics (2.4a,b), we can calculate the initial state, x_0^p at $t = 0$, using the standard formula

$$x_0^p = W_o(T_o)^{-1}\int_0^{T_o}(Ce^{At})^T(y^p(t) - \eta(t, 0))dt\,, \tag{2.7a}$$

where $T_o > 0$, the superscript T denotes a transpose, and

$$W_o(T_o) = \int_0^{T_o}(Ce^{At})^T Ce^{At}dt\,, \tag{2.7b}$$

$$\eta(t, s) = C\int_s^t e^{A(t-\tau)}Bu(\tau)d\tau\,. \tag{2.7c}$$

Clearly, $W_o(T_o)^{-1}$ exists because (A, C) is an observable pair. Thus, when there are no modeling errors and no disturbances, for $t \geq T_o$, the state $x^p(t, 0, x_0^p, u, 0)$, can be calculated exactly, and hence this calculated state can be used in Control Algorithm 2.2.

The much more relevant situation occurs when there are modeling errors but no disturbances. In this case formula (2.7a) yields an estimate of the initial state x_0^p. We propose to use it in *Step 1 (a)* of Control Algorithm 2.2, to obtain the estimate $\tilde{x}_k$, with the time T determined by a parameter δ_0, which must be chosen judiciously so as to avoid excessive ill conditioning in the observability grammian $W_o(T_o)$:

Step 1: *(a)* At $t'_k \triangleq t_k + \delta_0(t_{k+1} - t_k)$ with $\delta_0 \in (0, 1)$, estimate the state x_k^p by

$$\tilde{x}_k = W_o(\delta_0(t_{k+1}-t_k))^{-1}\int_{t_k}^{t'_k}(Ce^{A(t-t_k)})^T(y^p(t) - \eta(t, t_k))dt\,. \tag{2.7d}$$

□

Clearly, the fact that the plant inputs are bounded, limits the region of effectiveness of any control law and the class of reference signals that can be tracked. Hence we must assume that the initial states are confined to a ball $B_{\hat{\rho}} \subset \mathbb{R}^n$ and that the reference signals belong to the set $\mathbf{R}_U$, both defined, as follows.

Assumption 2.3. We assume that there exists a $\hat{\rho} \in (0, \infty)$ and $\mathbf{R}_U \subset \mathbf{R}$ such that $0 \in \mathbf{R}_U$ and that $x \in B_{\hat{\rho}} \triangleq \{ x \in \mathbb{R}^n \mid \| x \| \le \hat{\rho} \}$ and for all $r \in \mathbf{R}_U$, the optimal control problem $\mathbf{P}(x, 0, r)$ has a solution. □

The following theorem generalizes a result given in [Pol.1].

Theorem 2.4. Let $B_{\hat{\rho}} \subset \mathbb{R}^n$ and $r \in \mathbf{R}_U$, be defined as in Assumption 2.3. Suppose that *(a)* the systems (2.4a) and (2.4c) are identical, *(b)* $d(t) \equiv 0$, and *(c)* the Control Algorithm 2.2 is used to define the input $u(\cdot)$ for (2.4a). Then the resulting feedback system is asymptotically stable on the set $B_{\hat{\rho}}$, i.e. for any $x_0^p \in B_{\hat{\rho}}$, $x^p(t, 0, x_0^p, u, 0) \to 0$ as $t \to \infty$.

Proof. We begin by showing that for any $r \in \mathbf{R}_U$ and for any $x_0 \in B_{\hat{\rho}}$, the trajectory $x(t_k, 0, x_0, u, 0) = x_k$, $k \in \mathbf{N}$ resulting from the use of the Control Algorithm 2.2 is contained in $B_{\hat{\rho}}$. In turn, this shows that such a trajectory is well defined and that it is bounded.

Suppose that $x_0 \in B_{\hat{\rho}}$ is an arbitrary initial state at $t = 0$. It follows from the form of (2.5c), that for all $k \in \mathbf{N}$,

$$\| x_{k+1} \| = \| x(t_{k+1}, t_k, x_k, u_{[t_k, t_{k+1}]}, 0) \| \le \alpha \| x_k \| \le \alpha^{k+1} \| x_0 \| . \tag{2.8a}$$

Since $\alpha \in (0, 1)$, it follows that $x_k \in B_{\hat{\rho}}$ for all $k \in \mathbf{N}$ and hence that the trajectory $x(t, 0, x_0, u, 0)$ is well defined.

Next, from the form of (2.5e), we see that for all $k \in \mathbf{N}$ and for any $t \in [t_k, t_{k+1}]$,

$$\| x(t, t_k, x_k, u_{[t_k, t_{k+1}]}, 0) \| \le \beta \| x_k \| \le \beta \alpha^k \| x_0 \| \le \beta \| x_0 \| , \tag{2.8b}$$

which implies that since $\beta \alpha^k \to 0$ as $k \to \infty$, we obtain that $x(t, 0, x_0, u, 0) \to 0$ as $t \to \infty$. Hence the feedback system defined by the Control Algorithm 2.2 is asymptotically stable on the set $B_{\hat{\rho}}$. □

We note that Theorem 2.4 did not depend on the form of the cost function $g^0(\cdot, \cdot)$ nor on the form of the constraints defined by (2.5b) and (2.5d). These constraints can be used to shape the transient responses of the closed loop system. We will describe later a procedure for solving problems of the form (2.5a-e).

As stated, Control Algorithm 2.2 only defines a local control law. When the plant is unstable, since the control $u \in U$ is bounded, for some initial state $x_0 \in \mathbb{R}^n$, there is no control which stabilizes the system. In this case, there is not much that one can do about it. However, in the case of stable plants (and models), it is possible to globalize Control Algorithm 2.2 making use of the following observation. First, it should be clear that, in the absence of modeling errors and *disturbances*, for any $r \in \mathbf{R}$ such that for all $t \geq 0$, $\min_{u \in U} \|Bu + f(t)\| = 0$, there exists an admissible control, $u^o(t) \in argmin_{u \in U} \|Bu + f(t)\|$ that results in the error satisfying the equation

$$\dot{x}(t) = Ax(t)\,, \tag{2.9}$$

and hence, if A is a stable matrix, the error goes to zero exponentially, so that for any $x_0 \in \mathbb{R}^n$, $x(t, 0, x_0, u^o, 0) \in B_{\hat{\rho}}$ will occur in finite time. Clearly, in this case, there may be room for a more effective control law, as we will now show. Let M' and Q' be symmetric, positive definite matrices, such that $A^T Q' + Q'A = -M'$, then $V(x(t)) \triangleq \langle x(t), Q'x(t) \rangle$ is a Lyapunov function for (2.9). Let $T_s \in (T_C, \bar{T}]$ and suppose that $x_k \notin B_{\hat{\rho}}$. Then, if we set $t_{k+1} = t_k + T_s$ and we apply the control $u^o(t)$, to (2.4c), for $t \in [t_k, t_{k+1}]$, then we must have that $V(x(t_{k+1}, t_k, x_k, u^o)) \leq e^{-\lambda_{\min}(M')T_s} V(x_k)$. Hence it makes sense to use instead the control defined as the solution of the simple optimal control problem

$$\min_{u \in U} \{ V(x(t_{k+1}, t_k, x_k, u)) \} \ , \tag{2.10}$$

where $x(t_{k+1}, t_k, x_k, u)$ is determined as the solution of (2.4c).

Hence, for stable plants, we propose to modify Control Algorithm 2.2, as follows:

Control Algorithm 2.5.

Data: $t_0 = 0$, t_1, $u_{[t_0, t_1]}(t)$, x_0, T_s, T_C and $\bar{T}$ such that $0 < T_C < T_s \leq \bar{T} < \infty$.

Step 0: Set $k = 0$.

Step 1: At $t = t_k$,

(a) Obtain a measurement or estimate of the state $x_k^p = x^p(t_k, t_0, x_0^p, u, d)$ and denote the resulting value by $\bar{x}_k$.

(b) Compute an estimate, $\hat{d}(t)$, of the disturbance $d(t)$ for $t \in [t_k, t_{k+1}]$, if possible; else, set $\hat{d}(t) = 0$.

(c) Set the plant error dynamics input $u(t) = u_{[t_k, t_{k+1}]}(t) - \hat{d}(t)$ for $t \in [t_k, t_{k+1})$.

(d) Compute an estimate x_{k+1} of the state of the plant error dynamics $x^p(t_{k+1}, t_k, x_k^p, u, d)$ according to the formula (2.6)

$$x_{k+1} = e^{A(t_{k+1}-t_k)}\bar{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)}[Bu(t) + B_d\hat{d}(t)]\,dt + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)} f(t)\,dt\,.$$

(e) If $x_{k+1} \in B_{\hat{\rho}}$, solve the open loop optimal control problem $\mathbf{P}(x_{k+1}, t_{k+1}, r)$ to compute the next sampling time $t_{k+2} \in (t_{k+1} + T_C, t_{k+1} + \bar{T}]$, and the optimal control $u_{[t_{k+1}, t_{k+2}]}(t) \in U$, $t \in [t_{k+1}, t_{k+2})$.

Else set $t_{k+2} = t_{k+1} + T_s$ and $u_{[t_{k+1}, t_{k+2}]}(t) = u^o(t)$, for all $t \in [t_{k+1}, t_{k+2})$.

Step 2: Replace k by $k + 1$ and go to Step 1. □

We will not present a complete analysis of the operation of the closed loop system under Control Algorithm 2.5.

3. DISTURBANCE REJECTION

We will consider two distinct situations. The first is where the disturbance $d(t)$ is a continuous function, such that for some $c_d < \infty$, $\left[\int_t^{t+T} |d(\tau)|^2 d\tau\right]^{1/2} \leq c_d$ for all $t \geq 0$. The second is where the disturbance is the output of a known dynamical system driven by stationary, zero mean, white noise.

We begin with the first case and assume that the disturbance $d(t)$ cannot be estimated. Hence Control Algorithm 2.2 sets $\hat{d}(t) \equiv 0$. Since the more difficult situation occurs when the plant state is estimated, we will assume that this is the case. We begin by defining the error quantities

$$\Delta_1 \triangleq \max_{t \in [0, \bar{T}]} \| e^{A^p t} - e^{At} \|, \tag{3.1a}$$

$$\Delta_2 \triangleq \lambda_{\max}(Q)^{1/2} \sqrt{m}\, c_u \bar{T} \max_{t \in [0, \bar{T}]} \| e^{A^p t} B^p - e^{At} B \|_2, \tag{3.1b}$$

$$K \triangleq \max_{t \in [0, \bar{T}]} \| e^{At} \|, \tag{3.1c}$$

where $\lambda_{\max}(Q)$ denotes the largest singular value of Q.

Lemma 3.1. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with state estimation formula (2.7d). There exist $\Delta_i < \infty$, $i = 3, 4, 5, 6$, such that if Control Algorithm 2.2 constructs the sequences $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{\bar{x}_k\}_{k=0}^{\infty}$ which is the corresponding sequence of the estimates of x_k^p, defined by (2.7d), then for all $k \in \mathbf{N}$,

$$|x_k^p - \bar{x}_k| \leq \Delta_3 |x_k^p| + \Delta_4 , \tag{3.2a}$$

$$|x_{k+1}^p - x_{k+1}| \leq \Delta_5 |x_k^p| + \Delta_6 . \tag{3.2b}$$

Furthermore, when there are no modeling errors and no disturbances, $\Delta_i = 0$, $i = 3, 4, 5, 6$.

Proof. Suppose that $u(\cdot)$ is the control generated by Control Algorithm 2.2 for the plant and model trajectories associated with the sequences $\{ x_k^p \}_{k=0}^{\infty}$, $\{ x_k \}_{k=1}^{\infty}$, and $\{ \bar{x}_k \}_{k=0}^{\infty}$.

We begin with (3.2a). For any $k \in \mathbf{N}$ and any $t \in [t_k , t_{k+1}]$, $y^p(t)$ is given by

$$\begin{aligned} y^p(t) &= C^p e^{A^p(t-t_k)} x_k^p + C^p \int_{t_k}^{t} e^{A^p(t-\tau)}(B^p u(\tau) + B_d^p d(\tau))\, d\tau \\ &= Ce^{A(t-t_k)} x_k^p + \{ C^p e^{A^p(t-t_k)} - Ce^{A(t-t_k)} \}\, x_k^p \\ &\quad + C \int_{t_k}^{t} e^{A(t-\tau)}(Bu(\tau) + B_d d(\tau))\, d\tau + \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B^p - Ce^{A(t-\tau)} B \}\, u(\tau) d\tau \\ &\quad + \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B_d^p - Ce^{A(t-\tau)} B_d \}\, d(\tau) d\tau . \end{aligned} \tag{3.3a}$$

By substituting (3.3a) into (2.7d), we obtain

$$\begin{aligned} \bar{x}_k = x_k^p + W_o(\delta_0(t_{k+1} - t_k))^{-1} \Bigg\{ & \int_{t_k}^{t_k} (Ce^{A(t-t_k)})^T \{ C^p e^{A^p(t-t_k)} - Ce^{A(t-t_k)} \}\, dt\, x_k^p \\ & + \int_{t_k}^{t_k} (Ce^{A(t-t_k)})^T \int_{t_k}^{t} Ce^{A(t-\tau)} B_d d(\tau)\, d\tau dt \\ & + \int_{t_k}^{t_k} (Ce^{A(t-t_k)})^T \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B_d^p - Ce^{A(t-\tau)} B_d \}\, d(\tau))\, d\tau\, dt \\ & + \int_{t_k}^{t_k} (Ce^{A(t-t_k)})^T \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B^p - Ce^{A(t-\tau)} B \}\, u(\tau)\, d\tau\, dt \Bigg\} . \end{aligned} \tag{3.3b}$$

It follows directly from (3.3b) that

$$|x_k^p - \bar{x}_k| \leq \Delta_3 |x_k^p| + \Delta_4 , \tag{3.3c}$$

where

$$\Delta_3 \triangleq C_\Delta \max_{t \in [0,\delta_0\bar{T}]} \| C^p e^{A^p(t-t_k)} - C e^{A(t-t_k)} \|_2 \delta_0 \bar{T} \tag{3.3d}$$

$$\Delta_4 \triangleq C_\Delta \left[\max_{t \in [0,\delta_0\bar{T}]} \| C^p e^{A^p(t-\tau)} B^p - C e^{A(t-\tau)} B \|_2 \sqrt{m}\, c_u \delta_0 \bar{T} \right.$$

$$\left. + \max_{t \in [0,\delta_0\bar{T}]} \{ \| C^p e^{A^p(t-\tau)} B^p - C e^{A(t-\tau)} B \|_2 + \| C e^{At} \|_2 \} \sqrt{m_d}\, \delta_0 \bar{T} c_d \right], \tag{3.3e}$$

with $C_\Delta \triangleq \lambda_{\max}(Q)^{1/2} \max_{t \in [T_c,\bar{T}]} \| W_o(\delta_0 t)^{-1} \|_2 \max_{t \in [0,\delta_0\bar{T}]} \| C e^{At} \|_2$, which proves (3.2a). Clearly, when there are no modeling errors and no disturbances, $\Delta_3 = \Delta_4 = 0$.

Next we will establish (3.2b). Since x_{k+1} is calculated using the estimated initial state $\bar{x}_k$, it follows from the Schwartz inequality in $L_2[0,\bar{T}]$ (i.e., $\int_0^T a(t)b(t)\,dt \le \left[\int_0^T a(t)^2 dt\right]^{1/2} \left[\int_0^T b(t)^2 dt\right]^{1/2}$) that

$$\| x_{k+1}^p - x_{k+1} \| = \| e^{A^p(t_{k+1}-t_k)} x_k^p - e^{A(t_{k+1}-t_k)} \bar{x}_k$$

$$+ \int_{t_k}^{t_{k+1}} \{ e^{A^p(t_{k+1}-\tau)} B^p - e^{A(t_{k+1}-\tau)} B \} u(\tau)\, d\tau + \int_{t_k}^{t_{k+1}} e^{A^p(t_{k+1}-t)} B_d^p d(\tau)\, d\tau \|$$

$$\le K \, \| x_k^p - \bar{x}_k \| + \Delta_1 \| x_k^p \| + \Delta_2$$

$$+ \lambda_{\max}(Q)^{1/2} \int_{t_k}^{t_{k+1}} \| e^{A^p(t_{k+1}-t)} B_d^p - e^{A(t_{k+1}-t)} B_d \|_2 \| d(t) \|_2 dt + \lambda_{\max}(Q)^{1/2} \int_{t_k}^{t_{k+1}} \| e^{A(t_{k+1}-t)} B_d \|_2 \| d(t) \|_2 dt$$

$$\le K \; \{ \Delta_3 \| x_k^p \| + \Delta_4 \} + \Delta_1 \| x_k^p \| + \Delta_2 + \left[\frac{\Delta_2}{c_u \sqrt{\bar{T}}} + K \| B_d \|_2 \sqrt{\bar{T} m} \right] c_d \sqrt{m_d}$$

$$= (K\Delta_3 + \Delta_1) \| x_k^p \| + K\Delta_4 + \Delta_2 + \sqrt{m_d} \left[\frac{\Delta_2}{c_u \sqrt{\bar{T} m}} + K \| B_d \|_2 \sqrt{\bar{T}} \right] c_d \triangleq \Delta_5 \| x_k^p \| + \Delta_6, \tag{3.3f}$$

where K, Δ_1, and Δ_2 were defined in (3.1a,b,c). Hence (3.2b) holds, and our proof is complete. □

Lemma 3.1 leads to the following result that also holds when the state is measured.

Theorem 3.2. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with state estimation as in (2.7d). Suppose that $\varepsilon_1, \varepsilon_2 > 0$ are such that

$$\Delta_5 \le \varepsilon_1 < \frac{1-\alpha}{1+\alpha+K}, \tag{3.4a}$$

$$\Delta_6 \le \varepsilon_2 < \frac{\hat{\rho}}{3+(2+K)/\varepsilon'}, \tag{3.4b}$$

where Δ_5, Δ_6 were defined in (3.3f), and ε' is defined by

$$\varepsilon' \triangleq (1+\alpha)(\alpha+K)/(1+\alpha+K). \tag{3.4c}$$

Then there exists a $\rho_d \in (0, \hat{\rho}]$, such that for all $x_0^p \in B_{\rho_d}$, the trajectory $x^p(t, 0, x_0^p, u, d)$, $t \in [0, \infty)$, is bounded, and there exists an $\varepsilon_3 > 0$ such that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, and $\overline{\lim}_{t \to \infty} \|x^p(t, 0, x_0^p, u, d)\| \le \varepsilon_3$.

Proof. First suppose that the optimal control problem $\mathbf{P}(x_{k+1}, t_{k+1}, 0)$, has a solution for any $x_{k+1} \in \mathbb{R}^n$ and $t_{k+1} \ge 0$. Then, given any initial state x_0^p at time $t_0 = 0$, the dynamics of the moving horizon feedback system, using Control Algorithm 2.2, generate the sequence of states $\{x_k^p\}_{k=0}^{\infty}$, while Control Algorithm 2.2 generates the sequence of estimates $\{x_k\}_{k=1}^{\infty}$, with $x_{k+1} = x(t_{k+1}, t_k, \tilde{x}_k, u, \hat{d})$, $k = 1, 2, \ldots$, according to (2.6), and the sequence $\{x'_k\}_{k=2}^{\infty}$, with $x'_{k+2} = x(t_{k+2}, t_{k+1}, x_{k+1}, u, \hat{d})$, $k = 1, 2, \ldots$, generated in the process of solving the optimal control problem $\mathbf{P}(x_{k+1}, t_{k+1}, 0)$, $k \in \mathbb{N}$.

Now, for any $k \in \mathbb{N}$,

$$\begin{aligned} \|x_{k+2}^p\| &\le \|x_{k+2}^p - x'_{k+2}\| + \|x'_{k+2}\| \le \|x_{k+2}^p - x'_{k+2}\| + \alpha\|x_{k+1}\| \\ &\le K\|x_{k+1}^p - x_{k+1}\| + \Delta_5\|x_{k+1}^p\| + \Delta_6 + \alpha\|x_{k+1}^p - x_{k+1}\| + \alpha\|x_{k+1}^p\| \\ &\le (\Delta_5 + \alpha)\|x_{k+1}^p\| + (K+\alpha)\Delta_5\|x_k^p\| + (1+\alpha+K)\Delta_6. \end{aligned} \tag{3.5a}$$

If we let $a_1 = \Delta_5 + \alpha$, $a_2 = (K+\alpha)\Delta_5$, and $b = (1+\alpha+K)\Delta_6$, then, in view of (3.4) we see that $a_1, a_2, b \ge 0$ and $a_1 + a_2 < 1$, so that the assumptions of Proposition 6.1 satisfied. Hence, if we let $y_0 = \|x_0^p\|$ and $y_1 = \|x_1^p\|$, then it follows from (3.5a) that for y_k defined by (6.1a), $\|x_k^p\| \le y_k$, and that

$$\overline{\lim_{k \to \infty}} \|x_k^p\| \le \frac{(1+\alpha+K)\Delta_6}{\varepsilon'} \triangleq \varepsilon'', \tag{3.5b}$$

and also that for all $k \in \mathbb{N}$,

$$\|x_k^p\| \le y_k \le (K+\alpha)\Delta_5\|x_0^p\| + \|x_1^p\| + \varepsilon''. \tag{3.5c}$$

Since $u(t) = 0$, for all $t \in [0, t_1)$, for $k = 0$, (3.2b) reduces to

$$|x_2^p| \le |x_2^p - x_1| + |x_1| \le \Delta_5 |x_0^p| + \Delta_6 + K\, |x_0^p| = (K + \Delta_5)\, |x_0^p| + \Delta_6 . \tag{3.5d}$$

It then follows from (3.5c,d) that for all $k \ge 2$,

$$|x_k^p| \le ((1+\alpha+K)\Delta_5 + K)\, |x_0^p| + \Delta_6 + \varepsilon'' . \tag{3.5e}$$

Next, making use of (3.5c), we obtain that for all $k \in \mathbb{N}$,

$$|x_{k+1}| \le |x_{k+1}^p - x_{k+1}| + |x_{k+1}^p| \le \Delta_5 |x_k^p| + \Delta_6 + |x_{k+1}^p| \le \Delta_5 y_k + \Delta_6 + y_{k+1}$$

$$\le (1+\Delta_5)(K + (1+\alpha+K)\Delta_5)\, |x_0^p| + (1+\Delta_5)\Delta_6 + \Delta_6$$

$$\triangleq \gamma'_1 |x_0^p| + \gamma'_2 . \tag{3.5f}$$

Since by (3.4), $(1-\alpha)(1+\alpha+K) < 1$, we obtain that $1+\Delta_5 < 2$, and hence it follows that

$$\gamma'_2 = (1+\Delta_5)\Delta_6 + \Delta_6 < 2\Delta_6 + \left[1 + \frac{1-\alpha}{1+\alpha+K}\right] \frac{(1+\alpha+K)}{\varepsilon'} \Delta_6 + \Delta_6$$

$$\le (3 + (2+K)/\varepsilon')\Delta_6 \le (3 + (2+K)/\varepsilon')\varepsilon_2 \triangleq \hat{\gamma}'_2 < \hat{\rho} . \tag{3.5g}$$

Let $B_{\rho_d} \triangleq \{ x \in D \mid |x| \le \rho_d \}$ where, with $\hat{\gamma}_1 \triangleq (1+\varepsilon_1)(K + (1+\alpha+K)\varepsilon_1)$, ρ_d is defined by

$$\rho_d \triangleq (\hat{\rho} - \hat{\gamma}'_2)/\hat{\gamma}_1 . \tag{3.5h}$$

It follows from (3.5g) that $\rho_d > 0$. Furthermore, we conclude that for any $x_0^p \in B_{\rho_d}$, for all $k \in \mathbb{N}$,

$$|x_{k+1}| \le \gamma'_1 |x_0^p| + \gamma'_2 \le \hat{\gamma}'_1 \rho_d + \hat{\gamma}'_2 \le \hat{\rho} .$$

$x_k \in B_{\hat{\rho}}$ for all $k \ge 1$.

It now follows from Proposition 6.1 that $\overline{\lim}_{t\to\infty} |x^p(t, 0, x_0^p, u, d)| \le \varepsilon_3$, where ε_3 is defined by

$$\varepsilon_3 \triangleq \left[\frac{(\beta + (1+K+\beta)\varepsilon_1)(1+\alpha+K)}{\varepsilon'} + 1 + K + \beta\right] \varepsilon_2 . \tag{3.5i}$$

It is again obvious that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, which completes our proof. □

We will now show that when the disturbances are of sufficiently small amplitude, we can still use Control Algorithm 2.2 with slight modification such that when the state is close to the origin, we switch over to LQR feedback control law to obtain the benefit of the disturbance suppression properties of LQR systems (see [Kwa.1]). This depends on the largest real part of the eigenvalues $\lambda_j(A - BK_c)$ of the matrix $A - BK_c$ where K_c is a feedback matrix. Hence a design trade-off is implied: the smaller the largest real part of the eigenvalues, the better is the disturbance suppression. However, to obtain a very negative largest real part may require large elements in K_c, which limits the size of the ball about the origin where the control $u(t) = -K_c x(t)$ will not violate the control constraint.

Thus, suppose that K_c is the gain matrix resulting from the solution of an LQR problem for the model (2.2a) and that K_o is the gain matrix for a corresponding asymptotic state estimator for (2.2a). Since $A - BK_c$ is a stable matrix, there exists a pair of positive definite matrices (Q, M) such that

$$(A - BK_c)^T Q + Q(A - BK_c) = -M \; . \tag{3.6}$$

At this point it becomes clear that for best results, the matrix Q, used to determine the norm $\|\cdot\|$, should also define a Lyapunov function $\langle x, Qx \rangle$ for the system $\dot{x}(t) = (A - BK_c)x(t)$. Assuming that we use the control determined by the gain K_c and the asymptotic state estimator determined by the gain K_o, we get the following augmented dynamics in the well known observer-controller form

$$\dot{x}^p(t) = A^p x^p(t) - B^p K_c x^o(t) + B_d^p d(t), \tag{3.7a}$$

$$\dot{x}^o(t) = K_o C^p x^p(t) + (A - BK_c - K_o C)x^o(t) \, . \tag{3.7b}$$

We will assume that there exists a constant $c'_d < \infty$ such that $\|d(t)\| \le c'_d$ for all $t \ge 0$, and that both c'_d and the modeling errors are sufficiently small to ensure the existence of a ball $B_{LQR} \triangleq \{ x \in \mathbb{R}^n \mid \|x\| \le \rho_{LQR} \}$, $\rho_{LQR} > 0$, such that if for some $t_{\hat{k}}$, $x^o(t_{\hat{k}}) \in B_{LQR}$, then the control given by $u(t) = -K_c x^o(t)$ for all $t \ge t_{\hat{k}}$, with $(x^p(t), x^o(t))$ determined by (3.7a,b), does not violate the bound on the control.

Let $e(t) \triangleq x^p(t) - x^o(t)$ denote the difference between the state of the plant and that of the model. Then

$$\dot{e}(t) = (A^p - K_o C^p)x^p(t) - (A - K_o C)x^o(t) - (B^p K_c - BK_c)x^o(t) + B_d^p d(t) \, . \tag{3.7c}$$

We will assume from now on that the system

$$\dot{\eta}(t) = \tilde{A}\, \eta(t) \, , \tag{3.7d}$$

where $\tilde{A} \triangleq diag((A - K_o C), \hat{A})$, with $\hat{A}$ is defined by

$$\hat{A} \triangleq \begin{bmatrix} A & -BK_c \\ K_oC & A - BK_c - K_oC \end{bmatrix}, \tag{3.7e}$$

corresponding to (3.7a,b,c) when there are no modeling errors and no disturbances, is exponentially stable, and hence that there exists a symmetric, positive definite matrix $\tilde{Q} = diag(Q_o, Q_c)$, with $Q_o \in \mathbb{R}^{n \times n}$ and $Q_c \in \mathbb{R}^{2n \times 2n}$ that defines a Lyapunov function, $\langle \eta, \tilde{Q}\eta \rangle$ for the system (3.7d), so that for some symmetric, positive definite matrix $\tilde{M} = diag(M_o, M_c)$, with $M_o \in \mathbb{R}^{n \times n}$ and $M_c \in \mathbb{R}^{2n \times 2n}$, we have

$$\tilde{A}^T \tilde{Q} + \tilde{Q}\tilde{A} = -\tilde{M}. \tag{3.7f}$$

We will now show for the observer-controller dynamics that when $\|e(0)\|$ and $\|d\|_\infty$ are sufficiently small, $\|e(t)\|$ remains small for all $t \geq 0$.

Lemma 3.3. Suppose that the state $(x^p(t), x^o(t))$ is defined by the observer-controller dynamics described by (3.7a,b), with $(x^p(0), x^o(0))$ arbitrary and let $z(t) \triangleq (e(t), x^p(t), x^o(t))^T$. Let $\Delta\tilde{A}$, $\Delta\tilde{B}$ be defined by

$$\Delta\tilde{A} \triangleq \begin{bmatrix} 0 & \Delta A - K_o\Delta C & \Delta BK_c \\ 0 & \Delta A & -\Delta BK_c \\ 0 & K_o\Delta C & 0 \end{bmatrix}, \tag{3.8a}$$

$$\Delta\tilde{B}_d^T = [\Delta B_d^T, \Delta B_d^T, 0], \tag{3.8b}$$

where $\Delta A = A^p - A$, $\Delta B = B^p - B$, $\Delta B_d = B_d^p - B_d$, and $\Delta C = C^p - C$.

If there exists a $\delta \in (0, 0.5)$ such that *(a)* $\|\Delta\tilde{A}\tilde{Q}\|_2 < \delta\lambda_{\min}(\tilde{M})$, *(b)*

$$\|d\|_\infty < \frac{\lambda_{\min}(M)\lambda_{\min}(\tilde{M})(1-2\delta)^2 \rho_{LQR}\lambda_{\min}(Q)}{4\sqrt{m_d}\,\lambda_{\max}(Q)\lambda_{\max}(\tilde{Q})(\|K_cCQ\|_2 + \delta\lambda_{\min}(M))\|\tilde{B}_d^T\tilde{Q}\|_2 + \delta\lambda_{\min}(\tilde{M}))}, \tag{3.8c}$$

where $\tilde{B}_d^T = [B_d^T, B_d^T, 0]$ and Q and M were defined in (3.6), *(c)* $\|\Delta\tilde{B}_d^T\tilde{Q}\|_2 \leq \delta\lambda_{\min}(\tilde{M})$, and *(d)*

$$\|z(0)\| \leq \frac{\lambda_{\min}(M)(1-2\delta)\rho_{LQR}\lambda_{\min}(Q)^{1/2}}{2\lambda_{\max}(Q)^{1/2}(\|K_cCQ\|_2 + \delta\lambda_{\min}(M))} \triangleq \gamma_e, \tag{3.8d}$$

where $\|z(t)\| \triangleq \|e(t)\| + \|x^p(t)\| + \|x^o(t)\|$ and $\|z(t)\|^2 \triangleq \|e(t)\|^2 + \|x^p(t)\|^2 + \|x^o(t)\|^2$, with $\|x\| = \langle x, Qx \rangle^{1/2}$, then $\|e(t)\|$, $\|x^p(t)\| \leq \gamma_e$ for all $t \geq 0$.

Proof. Referring to (3.7a,b,c) and (3.8a,b), we see that $\dot{z}(t) = [\tilde{A} + \Delta\tilde{A}]z(t) + [\tilde{B}_d + \Delta\tilde{B}_d]d(t)$. Consider the Lyapunov function $V(z)$, for the nominal system (3.7d), defined by $V(\eta) \triangleq \langle \eta, \tilde{Q}\eta \rangle$.

Then,

$$\dot V(z(t)) = \langle \dot z(t), \tilde Q z(t)\rangle + \langle z(t), \tilde Q \dot z(t)\rangle$$

$$= -\langle z(t), \tilde M z(t)\rangle + 2\langle z(t), \Delta\tilde A \tilde Q z(t)\rangle + 2\langle(\tilde B_d + \Delta\tilde B_d)d(t), \tilde Q z(t)\rangle$$

$$\le -\lambda_{\min}(\tilde M)|z(t)|_2^2 + 2|\Delta\tilde A \tilde Q|_2|z(t)|_2^2 + 2|d(t)|_2(|\tilde B_d^T\tilde Q|_2 + |\Delta\tilde B_d^T\tilde Q|_2)|z(t)|_2$$

$$\le (-\lambda_{\min}(\tilde M) + 2\delta\lambda_{\min}(\tilde M))|z(t)|_2^2 + 2\sqrt{m_d}|d|_\infty(|\tilde B_d^T\tilde Q|_2 + \delta\lambda_{\min}(\tilde M))|z(t)|_2$$

$$\le -\frac{\lambda_{\min}(\tilde M)(1-2\delta)V(z(t))^{1/2}|z(t)|_2}{\lambda_{\max}(\tilde Q)^{1/2}} + 2\sqrt{m_d}|d|_\infty(|\tilde B_d^T\tilde Q|_2 + \delta\lambda_{\min}(\tilde M))|z(t)|_2 . \tag{3.9}$$

The last inequality is obtained by $|z(t)|_2 \ge V(z(t))^{1/2}/\lambda_{\max}(\tilde Q)^{1/2}$. Now, it follows from (3.8c) that

$$\dot V(z(t)) \le \left[\frac{-\lambda_{\min}(\tilde M)(1-2\delta)V(z(t))^{1/2}}{\lambda_{\max}(\tilde Q)} + \frac{\lambda_{\min}(M)\lambda_{\min}(\tilde M)(1-2\delta)^2\rho_{LQR}\lambda_{\min}(Q)}{2\lambda_{\max}(Q)^{1/2}\lambda_{\max}(\tilde Q)(|K_c C Q|_2 + \delta\lambda_{\min}(M))}\right]|z(t)|_2 .$$

We can see that if $V(z(t))^{1/2} > \gamma_e\lambda_{\min}(Q)^{1/2}/\lambda_{\max}(\tilde Q)^{1/2}$ then $\dot V(z(t)) < 0$. Since $\gamma_e^2 \ge |z(0)|^2 \ge V(z(0))\lambda_{\min}(Q)/\lambda_{\max}(\tilde Q)$, $V(z(t)) \le \gamma_e^2\lambda_{\min}(Q)/\lambda_{\max}(\tilde Q)$ for all $t \ge 0$. Since $V(z(t)) \ge \lambda_{\min}(Q)|z(t)|^2/\lambda_{\max}(\tilde Q) \ge \lambda_{\min}(Q)|e(t)|^2/\lambda_{\max}(\tilde Q)$, we obtain that $|e(t)| \le \gamma_e$, which establishes the first inequality. Since $|x^p(t)|^2 \le |z(t)|^2$ also holds, we see from the above that the second inequality also holds, which completes our proof. □

It is worth noting that (3.9) implies that $|z(t)| \to 0$ as $|d|_\infty \to 0$, and hence that $|x^o(t)|$, $|x^p(t)| \to 0$ as $|d|_\infty \to 0$.

Now, let

$$\rho_{LQR}^o \triangleq \min\{\gamma_e/4, \rho_{LQR}/4\}, \tag{3.10a}$$

$$\Delta_5 \le \varepsilon_1 < \frac{1-\alpha}{1+\alpha+K}, \tag{3.10b}$$

$$\Delta_6 \le \varepsilon_2 < \min\left[\frac{\hat\rho}{3+(2+K)/\varepsilon'}, \frac{\rho_{LQR}^o(1-\varepsilon_1)}{2}\right], \tag{3.10c}$$

where K and ε' were defined in (3.1c) and (3.4c), respectively. Then, it follows from (3.5b) that

$$\overline{\lim_{k\to\infty}} |x_k| \le \frac{(1+\alpha+K)\Delta_6}{\varepsilon'} \le \frac{(1+\alpha+K)\varepsilon_2}{\varepsilon'} \triangleq \rho_{MH} . \tag{3.10d}$$

Let

$$\rho_{oc} \triangleq (1-\varepsilon_1)\left[\rho \hat{L}^o_{QR} - \varepsilon_2/(1-\varepsilon_1)\right]. \tag{3.10e}$$

Then, it follows from (3.10a,c) that

$$\rho_{oc} = \rho \hat{L}^o_{QR}(1-\varepsilon_1) - \varepsilon_2 > \rho \hat{L}^o_{QR}(1-\varepsilon_1)/2 > \varepsilon_2 . \tag{3.10f}$$

For the case where the state of the plant is not measurable, we propose to incorporate this idea into Control Algorithm 2.2 by modifying *Step 1*, as follows. Let $T_{K_c} \in [T_C, \infty)$ be such that

$$e^{-\lambda_{min}(\tilde{M})(1-2\delta)T_{K_c}/\lambda_{max}(\tilde{Q})} \le \frac{\rho_{oc}^2 \lambda_{min}(Q)}{2\lambda_{max}(\tilde{Q})(\rho_{oc}^2 + (\rho \hat{L}^o_{QR})^2)}, \tag{3.10g}$$

$$\| e^{(A-BK_c)T_{K_c}} \| \le \alpha . \tag{3.10h}$$

Finally, we define the vector valued saturation function $SAT(u) \triangleq (sat(u^1), \ldots, sat(u^m))$, where $sat(y) = y$ if $y \in [-c_u, c_u]$, and $sat(y) = c_u sgn(y)$ otherwise.

Step 1': At $t = t_k$,

(a) If $u(t) = -K_c x^o(t)$ for $t \in [t_{k-1}, t_k)$ and $\max\{\|\tilde{x}_{k-1}\|, \|x_k\|\} \le \rho_{oc}$, set $\tilde{x}_k = x^o(t_k)$; else if $\max\{\|\tilde{x}_{k-1}\|, \|x_k\|\} \le \rho_{oc}$, set $\tilde{x}_k = x_k$ and reinitialize the observer by setting $x^o(t_k) = x_k$, else estimate the state $x^p_k = x^p(t_k, t_0, x^p_0, u, d)$ by (2.7d) and denote the resulting value by $\tilde{x}_k$.

(b) Compute an estimate, $\hat{d}(t)$, of a disturbance $d(t)$ for $t \in [t_k, t_{k+1}]$, if possible; else, set $\hat{d}(t) = 0$.

(c) If $\max\{\|\tilde{x}_{k-1}\|, \|x_k\|\} > \rho_{oc}$, set the plant input $u(t) = u_{[t_k, t_{k+1}]}(t) - \hat{d}(t)$ for $t \in [t_k, t_{k+1})$; else reset t_{k+1} to the new value $t_{k+1} = t_k + T_{K_c}$, and set $u(t) = -SAT(K_c x^o(t) - \hat{d}(t))$ for $t \in [t_k, t_{k+1})$.

(d) Compute an estimate x_{k+1} of the state of the plant $x^p(t_{k+1}, t_k, x^p_k, u, d)$ according (2.6), i.e.,

$$x_{k+1} = e^{A(t_{k+1}-t_k)}\tilde{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)}(B\,u(t) + B_d \hat{d}(t)) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)} f(t)\,dt . \qquad \square$$

Theorem 3.4. Suppose that *(a)* δ, $\Delta\tilde{A}$, $\Delta\tilde{B}$, $\|d\|_\infty$ satisfy the conditions in Lemma 3.3, *(b)* $\|K_c \Delta C Q\| \le \delta\lambda_{min}(M)$, *(c)* that (3.10b,c) holds, *(d)* that $\rho_{MH} < (\rho_{oc} - \varepsilon_2)/(1+\varepsilon_1)$, where ρ_{MH} was defined in (3.10d), *(e)* that $\gamma_e \le \rho_d$, where γ_e and ρ_d were defined in (3.8d) and (3.5h), respectively, and *(f)* that we use *Step 1'* in Control Algorithm 2.2. Then there exists a $\varepsilon_4 \in (0, \infty)$ such that for any

$x_0^p \in B_{\rho_d}$, defined in (3.5h), the trajectory $x^p(t, 0, x_0^p, u, d)$ satisfies that $|x^p(t, 0, x_0^p, u, d)| \le \varepsilon_4$ for all $t \ge 0$ and, furthermore, $\overline{\lim}_{t\to\infty} x^p(t, 0, x_0^p, u, d) \to 0$ as $\|d\|_\infty \to 0$.

Proof. We will prove that for any trajectory $x^p(t, 0, x_0^p, u, d)$, with $x_0^p \in B_{\rho_d}$, there must exist a $\hat{k}$ such that for all $t \in [0, t_{\hat{k}})$, the control $u(t)$ is defined by the solution of the optimal control problem $P(x_k, t_k, 0)$ and $\max\{|\tilde{x}_{\hat{k}-1}|, |x_{\hat{k}}|\} \le \rho_{oc}$, i.e., that the switch will take place in *Step 1' (c)* to the linear feedback control law $u(t) = -K_c x^o(t)$, with $(x^p(t), x^o(t))$ the solution of (3.7a,b), from the initial state $(x^p(t_{\hat{k}}), x^o(t_{\hat{k}}))$ at $t = t_{\hat{k}}$. Then we will show that if the linear feedback control law $u(t) = -K_c x^o(t)$ is used for $t \in [t_{\hat{k}}, T_{oc}]$ with $T_{oc} \ge t_{\hat{k}+1}$, $|x^o(t)| \le \rho_{LQR}$ holds for all $t \in [t_{\hat{k}}, T_{oc}]$, so that the linear feedback control law does not violate the bound on the control. Then, we will consider two possibilities: *(a)* only one switch to the linear feedback control law takes place (at $t_{\hat{k}}$), , i.e., $\max\{|\tilde{x}_{k-1}|, |x_k|\} \le \rho_{oc}$ for all $k \ge \hat{k}$ so that $u(t) = -K_c x^o(t)$ for all $t \ge t_{\hat{k}}$, and *(b)* the condition $\max\{|\tilde{x}_{k-1}|, |x_k|\} \le \rho_{oc}$ fails for some $k \ge \hat{k}$ and the Control Algorithm 2.2 switches back to the solution of the optimal control problem $P(x_k, t_k, 0)$ which implies that the linear feedback control law and the solution of the optimal control problem are used alternatively.

First, we will show that the switch to the linear feedback control law will take place. It follows from (3.2a,b) that if the switch to the linear feedback control law does not take place for any $k \in \mathbb{N}$, then

$$|\tilde{x}_{k-1}| \le |x_{k-1}^p - \tilde{x}_{k-1}| + |x_{k-1}^p| \le (\Delta_3 + 1)|x_{k-1}^p| + \Delta_4 , \tag{3.11a}$$

$$|x_k| \le |x_k^p - x_k| + |x_k^p| \le \Delta_5 |x_k^p| + \Delta_6 + |x_k^p| . \tag{3.11b}$$

Because $\Delta_3 \le \Delta_5 \le \varepsilon_1$, $\Delta_4 \le \Delta_6 \le \varepsilon_2$, and $\rho_{MH} < (\rho_{oc} - \varepsilon_2)/(1 + \varepsilon_1)$, it follows from (3.10d) that there exists a $\hat{k} \in \mathbb{N}$ such that $|\tilde{x}_{\hat{k}-1}| \le \rho_{oc}$ and $|x_{\hat{k}}| \le \rho_{oc}$. Therefore the switch to the linear feedback control law will take place.

Now, it follows from (3.2a) that

$$|x_{\hat{k}-1}^p| \le |x_{\hat{k}-1}^p - \tilde{x}_{\hat{k}-1}| + |\tilde{x}_{\hat{k}-1}| \le \varepsilon_1 |x_{\hat{k}-1}^p| + \varepsilon_2 + |\tilde{x}_{\hat{k}-1}| . \tag{3.11c}$$

From (3.2b), we obtain that

$$|x_{\hat{k}}^P| \le |x_{\hat{k}}^P - x_{\hat{k}}| + |x_{\hat{k}}| \le \varepsilon_1 |x_{\hat{k}-1}^P| + \varepsilon_2 + |x_{\hat{k}}| . \tag{3.11d}$$

From (3.11c,d) and (3.10e) we obtain that

$$|x_{\hat{k}}^P| \le \frac{\varepsilon_1}{1-\varepsilon_1}(\varepsilon_2 + \rho_{oc}) + \varepsilon_2 + \rho_{oc} = \rho_{LQR}^o . \tag{3.11e}$$

Then, it follows from $x^o(t_{\hat{k}}) = x_{\hat{k}}$ that $|x^o(t_{\hat{k}})| \le \rho_{oc} \le \rho_{LQR}^o$.

Next, suppose that the control $u(t) = -K_c x^o(t)$ is used for all $t \in [t_{\hat{k}}, T_{oc}]$, where $t_{\hat{k}}$ is the time when the switch to the linear feedback control law takes place and $T_{oc} \ge t_{\hat{k}+1}$. Let the Lyapunov function $V(\cdot)$ be defined by $V(x^o(t)) \triangleq |x^o(t)|^2 \triangleq \langle x^o(t), Qx^o(t) \rangle$. Then, making use of the matrix M defined by (3.6) and (3.7b), we obtain that for all $\in [t_{\hat{k}}, T_{oc}]$

$$\dot{V}(x^o(t)) \le -\lambda_{\min}(M)|x^o(t)|_2^2 + 2|K_c \Delta CQ|_2 |x^o(t)|_2^2 + 2|e(t)|_2(|K_c CQ|_2 + |K_c \Delta CQ|_2)|x^o(t)|_2$$

$$\le \left[-\frac{\lambda_{\min}(M)(1-2\delta)|x^o(t)|}{\lambda_{\max}(Q)^{1/2}} + \frac{2|e(t)|(|K_c CQ|_2 + \delta\lambda_{\min}(M))}{\lambda_{\min}(Q)^{1/2}} \right] |x^o(t)|_2 . \tag{3.11f}$$

It now follows from (3.11e) that $|x^o(t_{\hat{k}})|, |x_{\hat{k}}^P| \le \rho_{LQR}^o \le \gamma_e/4$ and that $|z(t_{\hat{k}})| \le |e(t_{\hat{k}})| + |x^o(t_{\hat{k}})| + |x^P((t_{\hat{k}})| \le 2(|x^o(t_{\hat{k}})| + |x_{\hat{k}}^P|) \le \gamma_e$, which implies that for all $t \in [t_{\hat{k}}, T_{oc}]$, $|e(t)| \le \gamma_e$, by Lemma 3.3. Now, it follows from (3.8d) that if $|x^o(t)| > \rho_{LQR}$ for any $t \in [t_{\hat{k}}, T_{oc}]$, then $\dot{V}(x^o(t)) < 0$ for $t \in [t_{\hat{k}}, T_{oc}]$. Since $|x^o(t_{\hat{k}})| \le \rho_{LQR}$, we must have that $|x^o(t)| \le \rho_{LQR}$ for all $t \in [t_{\hat{k}}, T_{oc}]$ and therefore $u(t) = -K_c x^o(t)$ satisfies the bound on the control.

Now let us consider the case *(a)*. If we set $T_{oc} = \infty$, then we conclude from the above that $|x^o(t)| \le \rho_{LQR}$ for $t \ge t_{\hat{k}}$. Also, by Lemma 3.3, $|x^P(t)| \le \gamma_e$ for all $t \ge t_{\hat{k}}$, which implies that $x^P(t)$ is bounded. Since by Lemma 3.3, $\lim_{t\to\infty}|z(t)| \to 0$ as $|d|_\infty \to 0$ we must have that $\overline{\lim}_{t\to\infty}|x^P(t)| \to 0$ as $|d|_\infty \to 0$, which completes the proof of *(a)*.

Next, let us consider the case *(b)*. Suppose that there exists a $k' > \hat{k}$ such that $u(t) = -K_c x^o(t)$, for all $t \in [t_{\hat{k}}, t_{k'}]$, and $\max\{|\tilde{x}_{k'-1}|, |x_{k'}|\} > \rho_{oc}$. Since $|x^o(t)| \le \rho_{LQR}$ and $|x^P(t)| \le \gamma_e$ for all $t \in [t_{\hat{k}}, t_{k'}]$, and $\gamma_e \le \rho_d$, we have that $x^P(t_{k'}) \in B_{\rho_d}$, which implies that the optimal control problem has a solution. Hence, by the first part of our proof, there exists a $\hat{k}' > k'$

such that the switch to the linear feedback control law again takes place. We now resort to a continuity argument. If $d(t)=0$ for all $t \in [t_{\hat{k}'}, t_{\hat{k}'+1}]$, we will have that $\max\{\|\bar{x}_{\hat{k}'}\|, \|x_{\hat{k}'+1}\|\} \le \rho_{oc}\max\{\alpha, 1/\sqrt{2}\}$. Hence, by continuity of the solution of (3.7a,b), there must exists a $c''_d > 0$ such that if $\|d(t)\| \le c''_d > 0$ for all $t \in [t_{\hat{k}'}, t_{\hat{k}'+1}]$, then $\max\{\|\bar{x}_{\hat{k}'+1}\|, \|x_{\hat{k}'+2}\|\} < \rho_{oc}$ will hold, and hence the linear control law will be retained for the next interval, $[t_{\hat{k}'+1}, t_{\hat{k}'+2}]$, and similarly, for all the intervals to follow, since c''_d does not depend on t_k. Hence, if $\|d\|_\infty \le c''_d$, then the linear control law will be used for all $t \ge t_{\hat{k}'}$, and therefore, by case *(a)*, we conclude that $\overline{\lim}_{t\to\infty}\|x^p(t)\| \to 0$ and it completes our proof. □

Next we turn to the case where the disturbance is the output of a known dynamical system driven by stationary, zero mean, white noise. To obtain bounds on the disturbance effects, we must assume that there are no modeling errors, i.e., that $A^p = A$, $B^p = B$, $B_d^p = B_d$, and $C^p = C$, and that the state of the plant can be measured. First we will consider the effect of disturbances which are generated by the initial state of an unforced, linear, time invariant system that is described by

$$\dot{x}_d(t) = A_d x_d(t) \tag{3.12a}$$

$$d(t) = C_d x_d(t), \tag{3.12b}$$

where $A_d \in \mathbb{R}^{n_d \times n_d}$, $C_d \in \mathbb{R}^{m_d \times n_d}$. Since the input $u(\cdot)$ is bounded, we can only hope to reduce the effects of bounded disturbances. Therefore, we assume that there exists a $b_d < \infty$ such that $\|e^{A_d t}\| \le b_d$ for all $t \ge 0$.

To estimate the state $x_d(t)$, we can proceed as follows. For all $k \in \mathbb{N}$ and $t \in [t_k, t_{k+1}]$, let $e(t)$ be defined by $e(t) \triangleq x^p(t, t_k, x_k^p, u, d) - x(t, t_k, x_k^p, u, 0)$. Then

$$\dot{e}(t) = Ae(t) + B_d d(t), \tag{3.12c}$$

with $e(t_k) = 0$. Combining (3.12a,b,c), we obtain that

$$\frac{d}{dt}\begin{bmatrix} x_d(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_d & 0 \\ B_d C_d & A \end{bmatrix}\begin{bmatrix} x_d(t) \\ e(t) \end{bmatrix} \triangleq \bar{A}\begin{bmatrix} x_d(t) \\ e(t) \end{bmatrix} \tag{3.12d}$$

$$e(t) = \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} x_d(t) \\ e(t) \end{bmatrix} \triangleq \bar{C}\begin{bmatrix} x_d(t) \\ e(t) \end{bmatrix}. \tag{3.12e}$$

Obviously, when $(\bar{C}, \bar{A})$ is an observable pair, we can use a reduced order estimator to obtain an asymptotically converging estimate of the disturbance state $x_d(t)$. Then, assuming that $\|u(t)-\hat{d}(t)\|_\infty \le c_u$ for all $t \in [t_k, t_{k+1}]$, where $u(t)$ is computed by solving the optimal control problem $P(x_{k-1}, t_{k-1}, 0)$ the use of Control Algorithm 2.2 will result in asymptotically perfect disturbance rejection.

We now give a necessary and sufficient condition for $(\bar{C}, \bar{A})$ to be observable.

Lemma 3.5. Let $\bar{A}$ and $\bar{C}$ be defined as (3.12d,e). Then $(\bar{C}, \bar{A})$ is an observable pair if and only if (B_dC_d, A_d) is an observable pair.

Proof. $\Longrightarrow$ We will give a proof by contraposition. Suppose that (B_dC_d, A_d) is not an observable pair. Then there exists a nonzero vector $z \in \mathbb{R}^{n_d}$ such that

$$B_dC_dA_d^iz = 0, \quad i = 0, 1, \ldots . \tag{3.13a}$$

Now let $\bar{z} \triangleq (z^T, 0)^T \in \mathbb{R}^{n_d+n}$. Then, because of (3.13a), we have that

$$\bar{C}\bar{A}^j\bar{z} = \sum_{i=0}^{j-1} A^{j-i-1}B_dC_dA_d^iz = 0, \; j = 1, 2, \ldots, n_d + n - 1 . \tag{3.13b}$$

Furthermore $\bar{C}\bar{z} = 0$. Hence $(\bar{C}, \bar{A})$ is not an observable pair.

$\Longleftarrow$ Now suppose that $(\bar{C}, \bar{A})$ is not an observable pair. Then there must exist a nonzero $\bar{z} = (z, z') \in \mathbb{R}^{n_d+n}$, such that $\bar{C}\bar{A}^j\bar{z} = 0$ for $j = 0, 1, \ldots, n_d + n - 1$. Since $\bar{C} = (0 \mid I)$, it is clear that $z' = 0$ must hold. Hence (3.13b) must hold, and unraveling this expression, we find that (3.13a) must also hold, which completes our proof. □

As an alternative to using a reduced order observer, at the expense of more computation, we can get an exact estimate of $\hat{d}(t)$ to be used to obtain perfect disturbance rejection, as follows. Let

$$\begin{bmatrix} w_{11}(t) & 0 \\ w_{21}(t) & w_{22}(t) \end{bmatrix} \triangleq \exp(\bar{A}t) = \exp\left\{ \begin{bmatrix} A_d & 0 \\ B_dC_d & A \end{bmatrix} t \right\}, \tag{3.14a}$$

so that $w_{11}(t) = e^{A_dt}$ and $w_{22}(t) = e^{At}$. Hence (3.12e) can be rewritten in the equivalent form

$$e(t) = w_{21}(t)x_d(t_k) + w_{22}(t)e(t_k) = w_{21}(t)x_d(t_k) . \tag{3.14b}$$

Since the state of the plant is measurable, $e(t)$ can be computed for all $t \in [t_k, t_{k+1}]$. Hence, if $\int_{t_k-\delta}^{t_k} w_{21}^T(\tau)w_{21}(\tau)\,d\tau$ is always invertible for some $\delta > 0$, then we can also compute $x_d(t_k - \delta)$ using the formula

$$x_d(t_k-\delta) = \left[\int_{t_k-\delta}^{t_k} w_{21}^T(\tau) w_{21}(\tau)\, d\tau\right]^{-1} \int_{t_k-\delta}^{t_k} w_{21}^T(\tau) e(\tau)\, d\tau \,. \tag{3.15a}$$

We can then use $x_d(t_k-\delta)$ to compute the disturbance $\hat{d}(t)$, for $t \in [t_k, t_{k+1}]$, using the formula:

$$\hat{d}(t) = C_d e^{A_d(t-t_k-\delta)} x_d(t_k-\delta) \triangleq C_d x_d(t) \,. \tag{3.15b}$$

To establish the invertibility of the matrix $\int_{t_k}^{t} w_{21}^T(\tau) w_{21}(\tau)\, d\tau$, for all $t > t_k$, we need the following lemma.

Theorem 3.6. Suppose that $w_{21}(t)$ is defined as in (3.14a). If (C_d, A_d) is an observable pair and B_d has maximum column rank, then, $\int_{t_k}^{t} w_{21}^T(\tau) w_{21}(\tau)\, d\tau$ is invertible for all $t > t_k$.

Proof. To simplify notation, let $\overline{\Phi}(t, \tau) \triangleq \exp((t-\tau)\overline{A})$. Since $(\overline{C}, \overline{A})$ is a observable pair by Lemma 3.5, the observability grammian for the system (3.12d,e), $W(t, t_k)$, defined by

$$W(t, t_k) \triangleq \int_{t_k}^{t} \overline{\Phi}(\tau, t_k)^T \overline{C}^T \overline{C}\, \overline{\Phi}(\tau, t_k)\, d\tau \tag{3.16a}$$

is nonsingular for all $t > t_k$. By substituting the expressions for $\overline{C}$ and $\overline{\Phi}(t, t_k)$ that are given by (3.12e) and (3.14a), respectively, we obtain that

$$W(t, t_k) = \begin{bmatrix} \int_{t_k}^{t} w_{21}^T(\tau) w_{21}(\tau)\, d\tau & \int_{t_k}^{t} w_{21}^T(\tau) w_{22}(\tau)\, d\tau \\ \int_{t_k}^{t} w_{22}^T(\tau) w_{21}(\tau)\, d\tau & \int_{t_k}^{t} w_{22}^T(\tau) w_{22}(\tau)\, d\tau \end{bmatrix} \triangleq \begin{bmatrix} W_{11}(t, t_k) & W_{12}(t, t_k) \\ W_{12}(t, t_k) & W_{22}(t, t_k) \end{bmatrix} \tag{3.16b}$$

Suppose that for some $t > t_k$, $W_{11}(t, t_k)$ is a singular matrix. Then there exists a nonzero vector, $z \in \mathbb{R}^{n_d}$, such that $W_{11}(t, t_k) z = 0$, and hence for $\tilde{z} \triangleq (z^T\ \ 0)^T \in \mathbb{R}^{n_d+n}$,

$$\langle \tilde{z}, W(t, t_k)\tilde{z} \rangle = \langle z, W_{11}(t, t_k) z \rangle = 0 \,, \tag{3.16c}$$

which contradicts to the fact that $W(t, t_k)$ is positive definite matrix for all $t > t_k$. Therefore, $W_{11}(t, t_k)$ is nonsingular for all $t > t_k$, which completes our proof. □

Thus, assuming that $\| u(t) - \hat{d}(t) \|_\infty \le c_u$ for all $t \in [t_k, t_{k+1}]$, where $u(t)$ is computed by solving the optimal control problem $\mathrm{P}(x_{k-1}, t_{k-1}, 0)$ the use of Control Algorithm 2.2 will result in perfect disturbance rejection.

In reality, it is not likely that the disturbance $d(t)$ is the output of a unforced linear time invariant system. It is more realistic to suppose that $d(\cdot)$ is the output of a linear time invariant system driven by stationary zero-mean white noise, with an initial state $x_d(0)$, described by

$$\dot{x}_d(t) = A_d x_d(t) + B_w w(t) \tag{3.17a}$$

$$d(t) = C_d x_d(t)\,. \tag{3.17b}$$

Let $d_1(t) \triangleq C_d e^{A_d t} x_d(0) \triangleq C_d x_{d_1}(t)$ and let $d_2(t) \triangleq C_d \int_0^t e^{A_d(t-\tau)} B_w w(\tau)\, d\tau \triangleq C_d x_{d_2}(t)$ be the contribution of the white noise term in (3.17a). Let $E(\xi)$ denote the expected value of the random variable ξ. Then we see that because $E(w(t)) = 0$ for all $t \geq 0$, $E(x_{d_2}(t)) = 0$ for all $t \geq 0$. Hence (c.f. (3.12d,e) and (3.14a,b)) we obtain that for $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$, $E(e(t)) = w_{21}(t) x_d(t_k)$. Since $\int_{t_k - \delta}^{t_k} w_{21}^T(t) w_{21}(t)\, dt$ is invertible for any $\delta > 0$ by Theorem 3.6, we can compute the estimate of the disturbance $d(t)$, for $t \in [t_k, t_{k+1})$, according to

$$\hat{d}(t) = C_d e^{A_d(t - t_k - \delta)} x_d(t_k - \delta)\,, \tag{3.18}$$

where $x_d(t_k - \delta)$ is defined by (3.14a). Since $E(x_{d_2}(t)) = 0$ for all $t \geq 0$, $E(\hat{d}(t)) = E(d(t))$ for all $t \geq 0$. Therefore, we have perfect estimation of the expected value of the disturbance, which implies that $\| E(d(\cdot) - \hat{d}(\cdot)) \|_\infty = 0$. In conjunction with Theorem 3.4, this fact leads to the following result.

Theorem 3.7. Suppose that *(a)* δ, $\Delta\tilde{A}$, $\Delta\tilde{B}_d$, $\|d\|_\infty$ satisfy the conditions in Lemma 3.3, *(b)* $\|K_c \Delta C\|_2 \leq \delta \lambda_{\min}(M)$, *(c)* that (3.10b) holds, *(d)* that $\rho_{MH} < (\rho_{oc} - \varepsilon_2)/(1 + \varepsilon_1)$, where ρ_{MH} was defined in (3.10d), and *(e)* that we use *Step 1'* in Control Algorithm 2.2. Then there exists an $\varepsilon_5 \in (0, \infty)$ such that for any $x_0^p \in B_{\rho_d}$, defined in (3.5h), the expected value of the trajectory $x^p(t, 0, x_0^p, u, d)$ satisfies that $\| E(x^p(t, 0, x_0^p, u, d)) \| \leq \varepsilon_5$ for all $t \geq 0$ and, furthermore, $\overline{\lim}_{t \to \infty} E(x^p(t, 0, x_0^p, u, d)) = 0$. □

4. TRACKING

We will now examine the reference signal tracking properties of our moving horizon control system, defined by the error dynamics (2.4a,b) and Control Algorithm 2.2. At this point we must assume that the matrix B in (2.4c) has full column rank.

Before we attempt a characterization of inputs which can be tracked asymptotically by our moving horizon control system (with bounded controls), we will extend a result due to Basile and Marro [Bas.1], dealing with asymptotic state tracking of LTI systems without control constraints.

Lemma 4.1. [Bas.1] Consider LTI system (2.2a,b), and let S_x be defined as in (2.3a). Then, S_x is the largest subspace among subspaces $S \subset \mathbb{R}^n$ such that

$$AS + S \subset R(B), \tag{4.1}$$

where $AS + S = \{ x \in \mathbb{R}^n \mid x = x_1 + x_2, \text{ for all } x_1 \in AS, x_2 \in S \}$ and $R(B)$ is a range space of B $\square$

Making use of Lemma 4.1, we obtain the following straightforward generalization of a result in [Bas.1].

Lemma 4.2. Let $r \in R$ and consider the error dynamics (2.4c,d), with $\hat{d}(t) \equiv 0$, and $f(t) = -\dot{s}(t) + As(t)$, where $s(t) \triangleq H(\tilde{C}^T\tilde{C})^{\dagger}\tilde{C}^T r(t)$. Then, there exists a continuous control $u_r(t)$, $t \geq 0$, such that for any initial state $x_0 \in \mathbb{R}^n$, $y(t) \triangleq Cx(t, 0, x_0, u_r, 0) \to 0$ as $t \to \infty$.

Proof. Clearly, if there exists a control $u_r(\cdot)$ such that $x(t, 0, x_0, u_r, 0) \to 0$ as $t \to \infty$, then, since $y(t) = Cx(t, 0, x_0, u_r, 0)$, the desired result must hold.

We recall that by definition $s(t) \in S_x$ for all $t \geq 0$. We will now show that we also have that $\dot{s}(t) \in S_x$. Let z be a nonzero vector in the orthogonal complement of S_x. Then for all $t > 0$,

$$0 = \langle z, (s(t) - s(0)) \rangle = \langle z, \int_0^t \dot{s}(t)\, dt \rangle. \tag{4.2}$$

Since (4.2) holds for all $t \geq 0$, we must have that $\langle z, \dot{s}(t) \rangle = 0$ for all t. Therefore $\dot{s}(t) \in S_x$ for all $t \geq 0$.

Let $u_r(t) \triangleq -Fx(t) + v(t)$ where F is any feedback matrix such that $\sigma(A - BF) \subset C^o_-$ (with $\sigma(A)$ the set of eigenvalues of A and C^o_- the open left half plane of the complex plain), and $v(t)$ is defined by $As(t) - \dot{s}(t) + Bv(t) = 0$ for all $t \geq 0$. The latter is possible because $s(t), \dot{s}(t) \in S_x$ and $AS_x + S_x \subset R(B)$. Then, we have that $x(t, 0, x_0, u_r, 0) = e^{(A-BF)t}x_0$ and obviously, $x(t, 0, x_0, u_r, 0) \to 0$ as $t \to \infty$, which completes our proof. $\square$

So far, we have assumed that there are no constraints on the control. We have assumed in Assumption 2.3 that for all $r \in R_U$ and $x \in B_{\hat{\rho}}$, the optimal control problem $P(x, 0, r)$ has a solution. To show that Control Algorithm 2.2 can be used for input tracking as well as stabilization, we have to prove that for trajectories emanating from the ball $B_{\hat{\rho}}$, the estimated states x_{k+1} defined by (2.6) are in the set $B_{\hat{\rho}}$. To establish this fact, we will follow the pattern set up in Section 3. First, we need the following definition.

Definition 4.3. Let $c_s \in (0, \infty)$. We define $\tilde{R}_U \subset R_U$ by

$$\tilde{R}_U = \{ r \in R_U \mid \max(\|s\|_\infty, \|\dot{s}\|_\infty) \leq c_s \}, \tag{4.3}$$

where $s(t) = H(\tilde{C}^T\tilde{C})^{\dagger}\tilde{C}^T r(t)$.

$\square$

Consider the error dynamics (4a,b) and its model (2.4c,d). We assume that the disturbance $d(t)$ cannot be estimated. Hence Control Algorithm 2.2 sets $\hat{d} \equiv 0$. Since the more difficult situation occurs when the plant state is estimated, we will assume that this is the case. First, we derive a result similar to Lemma 3.1.

Lemma 4.4. Let $r \in \tilde{\mathbf{R}}_U$. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with state estimation formula (2.7d). There exist $\Delta_i < \infty$, $i = 7, 8, 9, 10$, such that if Control Algorithm 2.2 constructs the sequences $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{\bar{x}_k\}_{k=0}^{\infty}$ is the corresponding sequence of the estimates of x_k^p, defined by (2.7d), then for all $k \in \mathbb{N}$,

$$\|x_k^p - \bar{x}_k\| \le \Delta_7 \|x_k^p\| + \Delta_8 , \tag{4.4a}$$

$$\|x_{k+1}^p - x_{k+1}\| \le \Delta_9 \|x_k^p\| + \Delta_{10} . \tag{4.4b}$$

Furthermore, when there are no modeling errors and no disturbances, $\Delta_i = 0$, $i = 7, 8, 9, 10$.

Proof. Suppose that $u(\cdot)$ is the control generated by Control Algorithm 2.2 for the plant and model trajectories associated with the sequences $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{\bar{x}_k\}_{k=0}^{\infty}$. For us to have a similarity with Lemma 3.1, let us modify the error dynamics (2.4a,c) as follows.

For a given $r \in \tilde{\mathbf{R}}_U$, let $s(t) = H(\tilde{C}^T\tilde{C})^{\dagger}\tilde{C}^T r(t)$. Let

$$u(t) = u_1(t) + u_2(t) , \tag{4.5a}$$

where

$$u_2(t) = (B^TB)^{-1}B^T(As(t) - \dot{s}(t)) . \tag{4.5b}$$

Then, since $f^p(t) = -\dot{s}(t) + A^p s(t)$ and $f(t) = -\dot{s}(t) + As(t)$, (2.4a,c) becomes

$$\dot{x}^p(t) = A^p x^p(t) + B^p u_1(t) + B_d^p d(t) + (B^p - B)u_1(t) + (A^p - A)s(t)$$

$$\triangleq A^p x^p(t) + B^p u_1(t) + B_d^p d(t) + d_1(t) , \tag{4.5c}$$

$$\dot{x}(t) = Ax(t) + Bu_1(t) + B_d \hat{d}(t) . \tag{4.5d}$$

Since $\max\{\|s\|_\infty, \|\dot{s}\|_\infty\} \le c_s$, it is clear that $\|u_2\|_\infty$ is bounded. Then,

$$\|u_1\|_\infty \le \|u\|_\infty + \|u_2\|_\infty \triangleq c_r . \tag{4.5e}$$

Next it follows from (4.5c) that

$$\| d_1 \|_\infty \le \| B^p - B \| c_r + \| A^p - A \| c_s \triangleq \Delta_{d_1}. \tag{4.5f}$$

We begin with (4.4a). For any $k \in \mathbb{N}$ and any $t \in [t_k, t_{k+1}]$, $y^p(t)$ is given by

$$\begin{aligned} y^p(t) &= C^p e^{A^p(t-t_k)} x_k^p + C^p \int_{t_k}^{t} e^{A^p(t-\tau)} (B^p u_1(\tau) + B_d^p d(\tau))\, d\tau + C^p \int_{t_k}^{t} e^{A^p(t-\tau)} d_1(\tau)\, d\tau \\ &= C e^{A(t-t_k)} x_k^p + \{ C^p e^{A^p(t-t_k)} - C e^{A(t-t_k)} \}\, x_k^p + C \int_{t_k}^{t} e^{A(t-\tau)} (B u_1(\tau) + B_d d(\tau))\, d\tau \\ &\quad + \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B^p - C e^{A(t-\tau)} B \}\, u_1(\tau) d\tau \\ &\quad + \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B_d^p - C e^{A(t-\tau)} B_d \}\, d(\tau)\, d\tau \\ &\quad + C \int_{t_k}^{t} e^{A(t-\tau)} d_1(\tau)\, d\tau + \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} - C e^{A(t-\tau)} \}\, d_1(\tau)\, d\tau. \end{aligned} \tag{4.5g}$$

By substituting (4.5g) into (2.7d), we obtain

$$\begin{aligned} \tilde{x}_k = x_k^p + W_o(\delta_0(t_{k+1} - t_k))^{-1} \Bigg\{ & \int_{t_k}^{t'_k} (C e^{A(t-t_k)})^T \{ C^p e^{A^p(t-t_k)} - C e^{A(t-t_k)} \}\, dt\, x_k^p \\ & + \int_{t_k}^{t'_k} (C e^{A(t-t_k)})^T \int_{t_k}^{t} C e^{A(t-\tau)} (B_d d(\tau) + d_1(\tau))\, d\tau dt \\ & + \int_{t_k}^{t'_k} (C e^{A(t-t_k)})^T \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B^p - C e^{A(t-\tau)} B \}\, u_1(\tau) d\tau dt \\ & + \int_{t_k}^{t'_k} (C e^{A(t-t_k)})^T \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} B_d^p - C e^{A(t-\tau)} B_d \}\, d(\tau) d\tau dt \\ & + \int_{t_k}^{t'_k} (C e^{A(t-t_k)})^T \int_{t_k}^{t} \{ C^p e^{A^p(t-\tau)} - C e^{A(t-\tau)} \}\, d_1(\tau)\, d\tau\, dt \Bigg\}. \end{aligned} \tag{4.5h}$$

It follows directly from (4.5h) that

$$\| x_k^p - \tilde{x}_k \| \le \Delta_7 \| x_k^p \| + \Delta_8, \tag{4.5i}$$

where $\Delta_7 = \Delta_3$, where Δ_3 was defined in (3.3d) and

$$\Delta_8 \triangleq \Delta'_4 + C_\Delta \left[\max_{t \in [0, \delta_0 \bar{T}]} \{ \| C^p e^{A^p(t-\tau)} - C e^{A(t-\tau)} \|_2 + \| C e^{At} \|_2 \} \delta_0 \bar{T} \Delta_{d_1} \right], \tag{4.5j}$$

with $C_\Delta \triangleq \lambda_{\max}(Q)^{1/2} \max_{t \in [T_c, \bar{T}]} \| W_o(\delta_0 t)^{-1} \|_2 \max_{t \in [0, \delta_0 \bar{T}]} \| C e^{At} \|_2$ and with Δ'_4 replacing c_u of Δ_4

defined in (3.3e) with c_r in (4.5e), which proves (4.4a). Clearly, when there are no modeling errors and no disturbances, $\Delta_7 = \Delta_8 = 0$.

Next we will establish (4.4b). Since x_{k+1} is calculated using the estimated initial state $\bar{x}_k$, it follows from the Schwartz inequality in $L_2[0,\bar{T}]$ (i.e., $\int_0^T a(t)b(t)\,dt \le \left[\int_0^T a(t)^2 dt\right]^{1/2} \left[\int_0^T b(t)^2 dt\right]^{1/2}$) that

$$|x^p_{k+1} - x_{k+1}| = |e^{A^p(t_{k+1}-t_k)}x^p_k - e^{A(t_{k+1}-t_k)}\bar{x}_k + \int_{t_k}^{t_{k+1}} e^{A^p(t_{k+1}-t)} d_1(\tau)\,d\tau$$

$$+ \int_{t_k}^{t_{k+1}} \{ e^{A^p(t_{k+1}-\tau)}B^p - e^{A(t_{k+1}-\tau)}B \}\, u_1(\tau)\,d\tau + \int_{t_k}^{t_{k+1}} e^{A^p(t_{k+1}-t)}B^p_d d(\tau)\,d\tau|$$

$$\le K\, |x^p_k - \bar{x}_k| + \Delta_1 |x^p_k| + \Delta'_2$$

$$+ \lambda_{\max}(Q)^{1/2}\int_{t_k}^{t_{k+1}} |e^{A^p(t_{k+1}-t)} - e^{A(t_{k+1}-t)}|_2 |d_1(t)|_2 dt + \lambda_{\max}(Q)^{1/2}\int_{t_k}^{t_{k+1}} |e^{A(t_{k+1}-t)}|_2 |d_1(t)|_2 dt$$

$$+ \lambda_{\max}(Q)^{1/2}\int_{t_k}^{t_{k+1}} |e^{A^p(t_{k+1}-t)}B^p_d - e^{A(t_{k+1}-t)}B_d|_2 |d(t)|_2 dt + \lambda_{\max}(Q)^{1/2}\int_{t_k}^{t_{k+1}} |e^{A(t_{k+1}-t)}B_d|_2 |d(t)|_2 dt$$

$$\le K\ \{ \Delta_7 |x^p_k| + \Delta_8 \} + \Delta_1 |x^p_k| + \Delta'_2 + \left[\frac{\Delta'_2}{c_r\sqrt{\bar{T}}} + K |B_d| \sqrt{\bar{T}m} \right] c_d \sqrt{m_d}$$

$$+ \left[K + \max_{t \in [0,\bar{T}]} |e^{A^p t} - e^{At}| \right] \sqrt{\bar{T}m_d}\,\Delta_{d_1}$$

$$\triangleq (K\Delta_7 + \Delta_1)\, |x^p_k| + K\Delta_8 + \Delta'_2 + \sqrt{m_d} \left[\frac{\Delta'_2}{c_r\sqrt{\bar{T}m}} + K |B_d| \sqrt{\bar{T}} \right] c_d + \tilde{\Delta}_{d_1}$$

$$\triangleq \Delta_9 |x^p_k| + \Delta_{10}, \qquad (4.5k)$$

where K, Δ_1 were defined in (3.1a,b) and Δ'_2 was obtained by replacing c_u of Δ_2 in (3.1c) with c_r in (4.5e). Hence (4.4b) holds, and our proof is complete. □

In Section 3, Theorem 3.2 was proved by making use of the results in Lemma 3.1 and Proposition 6.1. In the case of tracking, it is clear that if we replace Δ_5 with Δ_9 and Δ_6 with Δ_{10} in the proof of Theorem 3.2 and use Lemma 4.4 instead of Lemma 3.1, still using Proposition 6.1, then the conclusions of Theorem 3.2 assume the following form.

Theorem 4.5. Let $r \in \tilde{\mathbf{R}}_U$. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.2, with state estimation as in (2.7d). Suppose that Δ_9, Δ_{10} satisfy the inequalities

$$\Delta_9 \le \varepsilon_1 < \frac{1-\alpha}{1+\alpha+K}, \tag{4.6a}$$

$$\Delta_{10} \le \varepsilon_2 < \frac{\hat{\rho}}{3+(2+K)/\varepsilon'}, \tag{4.6b}$$

where Δ_9, Δ_{10} were defined in (4.5k), and ε' was defined in (3.4c). Let ρ_d be as in (3.5h). Then, for all $x_0^p \in B_{\rho_d}$, the trajectory $x^p(t, 0, x_0^p, u, d)$, $t \in [0, \infty)$, is bounded, and there exists an $\varepsilon_6 > 0$ such that $\varepsilon_6 \to 0$ as $\varepsilon_2 \to 0$, and $\overline{\lim}_{t \to \infty} |x^p(t, 0, x_0^p, u, d)| \le \varepsilon_6$. □

Since the constants Δ_9, Δ_{10} depend on c_s and the bounds on the modeling errors, we see that there is a trade off involved in chosing a value for c_s, namely, the larger c_s the smaller are the modeling errors under which (4.6a,b) will be satisfied, while the set of admissible inputs $\tilde{\mathbf{R}}_U$ grows with c_s.

In a similar way, the results of Theorem 3.4 can also be extended to the reference signal following case.

5. CONCLUSION.

Moving horizon control is a promising idea for the control of nonlinear systems. In this two part paper we have explored the properties of a moving horizon feedback system, based on constrained optimal control algorithms, with the simplest possible nonlinearity, namely, input saturation. While in the first part, we have shown that moving horizon control results in a robustly stable system, in this part we have shown that it is capable of following a class of reference inputs and suppressing a class of disturbances. A possible issue in the use of the type of moving horizon control system discussed in this paper, is the time needed to solve the optimal control problems. This should cause no difficulties in controlling slow moving plants, such as in process control. For faster plants, it may be necessary to implement the optimal control algorithms in some form of dedicated architecture, so as to reduce to the solution time to acceptable levels.

6. APPENDIX I.

We will now establish two inequalities that form the basis of several of our proofs.

Proposition 6.1 Consider the second order scalar difference equation

$$y_{k+2} = a_1 y_{k+1} + a_2 y_k + b \ , \quad k \in \mathbf{N} \ . \tag{6.1a}$$

If $a_1, a_2 \geq 0$, $b \geq 0$ and $a_1 + a_2 < 1$, then for all $k \geq 1$,

$$y_k \leq a_2 y_0 + y_1 + b/(1 - a_1 + a_2) \ , \tag{6.1b}$$

and

$$\overline{\lim}_{k \to \infty} y_k \leq b/(1 - a_1 + a_2) \ . \tag{6.1c}$$

Proof. We begin by rewriting (6.1a) in first order vector form, as follows. For $k \in \mathbf{N}$, let $z_k = (y_k, y_{k+1})^T$. Then $z_0 = (y_0, y_1)^T$, and

$$z_{k+1} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} z_k + \begin{bmatrix} 0 \\ b \end{bmatrix} \triangleq F z_k + g \ , \tag{6.2a}$$

$$y_k = [1\ 0] z_k \triangleq H z_k \ . \tag{6.2b}$$

The matrix F has two eigenvalues, $\lambda_+, \lambda_- = \frac{1}{2}(a_1 \pm \sqrt{a_1^2 + 4a_2})$, with corresponding eigenvectors, $e_+ = (1, \lambda_+)^T$ and $e_1 = (1, \lambda_-)^T$. We will now show that $-1 < \lambda_- \leq 0 \leq \lambda_+ < 1$, i.e., that (6.2a) is an asymptotically stable system. By assumption

$$0 \leq a_2 < 1 - a_1 \ . \tag{6.2c}$$

If we multiply both sides Of (6.2c) by 4, and add a_1^2 to the both sides, we get that

$$a_1^2 + 4a_2 < (2 - a_1)^2 \ , \tag{6.2d}$$

which implies that $\lambda_- = \frac{1}{2}(a_1 - \sqrt{a_1^2 + 4a_2}) > -1$ and $\lambda_+ = \frac{1}{2}(a_1 + \sqrt{a_1^2 + 4a_2}) < 1$. Thus, we have that $-1 < \lambda_- \leq \lambda_+ < 1$.

We can proceed to establish (6.1b,c). By the Jordan decomposition, we have that

$$F = E^{-1} \Lambda E \ , \tag{6.2e}$$

where $\Lambda = diag(\lambda_+, \lambda_-)$, and $E = (e_+, e_-)$ is a matrix whose columns are the eigenvectors of F. Hence for all $k \geq 2$,

$$y_k = HE^{-1}\Lambda^k E z_0$$

$$= \frac{1}{\lambda_- - \lambda_+}\{\lambda_+\lambda_-(\lambda_+^{k-1} - \lambda_-^{k-1})y_0 + (\lambda_-^k - \lambda_+^k)y_1\} + \frac{b}{\lambda_- - \lambda_+}\sum_{i=0}^{k-1}(\lambda_-^{k-1-i} - \lambda_+^{k-1-i}) . \quad (6.2f)$$

Since $0 < \lambda_+ < 1$ and $-1 < \lambda_- < 0$, it is clear that *(a)* the first term in (6.2f) goes to zero as $k \to \infty$ and *(b)* the last term in (6.2f) satisfies the inequality

$$\frac{b}{\lambda_- - \lambda_+}\sum_{i=0}^{k-1}(\lambda_-^{k-1-i} - \lambda_+^{k-1-i}) \le \frac{b}{\lambda_- - \lambda_+}\left\{\frac{1}{1-\lambda_-} - \frac{1}{1-\lambda_+}\right\} = \frac{b}{1 - a_1 + a_2} , \quad (6.2g)$$

because $(1-\lambda_+)(1-\lambda_-) = 1 - a_1 + a_2$, which proves (6.1c).

Next, for all $k \ge 1$, $\lambda_+^k \le \lambda_+$ and $-\lambda_-^k \le (-\lambda_-)^k \le -\lambda_-$. Hence $\{\lambda_+\lambda_-(\lambda_+^{k-1} - \lambda_-^{k-1})/(\lambda_- - \lambda_+) \le -\lambda_+\lambda_- = a_2$. Also $(\lambda_-^k - \lambda_+^k)/(\lambda_- - \lambda_+) \le 1$, hence (6.1b) hold. □

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