MOVING HORIZON CONTROL OF LINEAR SYSTEMS WITH INPUT SATURATION AND PLANT UNCERTAINTY

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E. Polak* and T. H. Yang*

ABSTRACT

We present a moving horizon feedback system, based on constrained optimal control algorithms, for linear plants with input saturation. The system is a nonconventional sampled-data system: its sampling periods vary from sampling instant to sampling instant, and the control during the sampling time is not constant, but determined by the solution of an open loop optimal control problem. This is in two part paper. In this part, we show that the proposed moving horizon control system is robustly stable, whether the state of the plant is measurable or not. In the second part, we show that the proposed moving horizon control system is capable of following a class of reference inputs and suppressing a class of disturbances. Experimental results show that the behavior of the moving horizon control system is superior to that resulting from some alternative control laws.

KEY WORDS: Moving horizon control, robust stability.

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† The research reported herein was sponsored by the Air Force Office of Scientific Research contract AFOSR-90-0068, the National Science Foundation grant ECS-8916168, and the UK Science and Engineering Research Council.

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1. INTRODUCTION

Open loop optimal feedback, dating back to a 1962 seminal paper by Propoi [Pro.1], is a general approach for the construction of stabilizing feedback laws for systems subject to input constraints and other nonlinearities. Originally, it was based on the idea that in a sample-data system, the control to be applied between sampling times can be determined by solving a fixed time open loop optimal control problem with or without various constraints. Over the years, open loop optimal feedback has been explored under the names of *model predictive control* (see [Meh.1, Pre.1, Gar.1, Gar.2]) and *moving horizon control* (see [Kwo.1, Kwo.2, May.1, May.2, Kee.1]). In model predictive control, the system is operated in sample-data mode, while in moving horizon control, the system is operated either in sample-data or in continuous mode.

The literature dealing with model predictive control presents results dealing with the stability, reference input following and constant disturbance rejection capabilities of the resulting feedback systems, under the assumption that the controls and states are unconstrained (see, e.g., [Cla.1, Cla.2]). No such results are available for systems with bounded control and state space constraints.

As far as moving horizon control is concerned, it has not always been realized that a naive application of the strategy, in adaptive control for example, can lead to instability. The early literature dealt with the stabilizing properties of moving horizon control laws based on open loop optimal control for finite horizon optimal control problems with quadratic criteria and no control constraints. Thus Kwon and Pearson [Kwo.1], and Kwon, Bruckstein and Kailath [Kwo.2] deal with linear time-varying systems, Keerthi and Gilbert [Kee.1] deal with nonlinear discrete-time systems, and, more recently, Mayne and Michalska have established the stability properties of nonlinear, continuous-time systems with moving horizon control [May.1, May.2, Mic.1, Mic.2] with control and state space constraints; see also Chen and Shaw [Che.1]. In [May.2], the robust stability of a moving horizon control was examined although the analysis is incomplete. In [Mic.2], the nontrivial time needed for the computation of the open loop controls is taken into account, under the assumption that there is no modeling error. None of the work cited above deals with the question of state estimation.

To illustrate the range of questions that must be dealt with in designing and analyzing an open loop optimal feedback system, consider the linear, time invariant dynamical system modeled by the finite dimensional ODE:

$$\dot{x}(t) = Ax(t) + Bu(t) \ , \tag{1.1}$$

where $(A \ , \ B)$ is a controllable pair. Suppose that the only requirements are to restore the state of the system to the origin, using bounded controls $u(t)$, and that the resulting trajectory must not violate some state space constraints.

Under ideal conditions, the state of the system can be measured exactly, and there are no modeling errors, and no disturbances. In this case, the above requirements can be used to define the constraints of an open loop optimal control problem, whose instantaneous solution $\hat{u}(t)$ would drive the state to the origin in finite time. Since the zero state is an equilibrium point for (1.1), no further action would be required, and hence, under ideal conditions, there would be no need for feedback laws to drive the system from arbitrary states to the origin.

In practice, only an estimate of the system state may be available, the system model (1.1) is likely to be approximate, and there may be input disturbances. Furthermore, the time needed to solve the optimal control problem is likely to be nonnegligible. Hence the actual terminal state, resulting from the application of an optimal control based on an estimated initial state and the approximate dynamics, need not be the origin, and hence some constant or periodic remedial action is required. Such remedial action invariably results in some sort of a feedback law.

This is the first of a two part paper in which we propose an open loop optimal feedback algorithm for linear plants, modeled with errors, subject to disturbances, reference inputs, and control constraints, and with the time to solve the optimal control problem accounted for. Our control algorithm is closest in concept to those that are classified as moving horizon control laws. It differs from other moving horizon control laws in that it uses a free time, control and state space constrained optimal control problem, and hence it results in a nonconventional sampled-data system: its sampling periods vary from sampling instant to sampling instant, and the control during the sampling time is not constant, but determined by the solution of an open loop optimal control problem.

In this part we will establish the robust stability of the resulting feedback system in the absence of disturbances. In part II, we will examine its disturbance rejection and reference input following characteristics. In Section 2, we introduce our proposed moving horizon feedback control law. In Section 3, we show that the proposed moving horizon feedback system is robustly stable. We will consider cases when the state of the plant is measurable and when it is estimated. Finally, in Section 4, we illustrate the behavior of our moving horizon control law by means of a few simple examples.

2. STRUCTURE OF THE MOVING HORIZON CONTROL LAW

We assume that the plant is a linear-time-invariant (LTI) system, with bounded inputs, described by the differential equation

$$\dot{x}^p(t) = A^p x^p(t) + B^p u(t), \tag{2.1a}$$

$$y^p(t) = C^p x^p(t), \tag{2.1b}$$

where the state $x^p(t) \in \mathbb{R}^n$, the control $u \in U$, with

$$U \triangleq \{ u \in L_2^m[0, \infty) \mid \|u\|_\infty \leq c_u \}, \tag{2.1c}$$

and $c_u \in (0, \infty)$. Consequently, $A^p \in \mathbb{R}^{n \times n}$, $B^p \in \mathbb{R}^{n \times m}$, and $C^p \in \mathbb{R}^{n_o \times n}$. We will denote the solution of (2.1a) at time t, corresponding to the initial state x_0^p at time t_0, and the input u, by $x^p(t, t_0, x_0^p, u)$.

The function of the moving horizon control law is to ensure robust stability while taking into account the fact that the plant inputs are bounded, as in (2.1c), as well as various amplitude constraints on transients.

We assume that the matrices A^p, B^p, and C^p are known only to some tolerance. Hence the moving horizon control law must be developed using a plant model, of the same dimension as (2.1a),

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.2a}$$

$$y(t) = Cx(t), \tag{2.2b}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{n_o \times n}$ are approximations to A^p, B^p, and C^p, respectively. We will denote the solution of (2.2a) at time t, corresponding to the initial state x_0 at time t_0, and the input u by $x(t, t_0, x_0, u)$.

Let Q be a symmetric, positive definite $n \times n$ matrix such that $\langle x, Qx \rangle$ is a Lyapunov function for the linear closed loop system obtained applying state feedback to (2.1a). The reason for this selection will become clear in Section 3. We use this matrix to define the norm $\|x\| \triangleq \langle x, Qx \rangle^{1/2}$. We will denote the usual Eucledean norm on $\mathbb{R}^n$ by $\|\cdot\|_2$.

Let $x_k \triangleq x(t_k, t_0, x_0, u)$. Assuming that the control law computation takes at most T_C time units, we can now propose a simple, aperiodic sampled-data feedback law, in the form of an algorithm which, during each sampling period, solves an optimal control problem $\mathrm{P}(x_k, t_k)$ of the form

$$P(x_k, t_k): \min_{(u,\tau)} \{ g^0(u,\tau) \mid g^i(u,\tau) \le 0,\ i = 1, 2, \ldots, l_1,\ \max_{t \in [t_k, \tau]} \phi^j(u,t) \le 0,$$

$$j = 1, \ldots, l_2,\ u \in U,\ \tau \in [t_k + T_C, t_k + \bar{T}] \}, \tag{2.3a}$$

where $0 < T_C < \bar{T} < \infty$, and the constraint functions are defined by

$$g^i(u,\tau) \triangleq h^i(x(\tau, t_k, x_k, u)),\ i = 0, 1, \ldots, l_1 - 1, \tag{2.3b}$$

$$g^{l_1}(u,\tau) = \|x(\tau, t_k, x_k, u)\|^2 - \alpha^2 \|x_k\|^2, \tag{2.3c}$$

$$\phi^j(u,t) = h^j(x(t, t_k, x_k, u), t),\ j = 1, \ldots, l_2 - 1, \tag{2.3d}$$

$$\phi^{l_2}(u,t) = \|x(t, t_k, x_k, u)\|^2 - \beta^2 \|x_k\|^2, \tag{2.3e}$$

where the constraint functions (2.3c,e) with $\alpha \in (0, 1)$, $\beta \in [1, \infty)$, are used to ensure robust stability and input tracking, while the other functions, h^i, h^j are convex, locally Lipschitz continuously differentiable functions that can be used to ensure other performance requirements.

We are now ready to state our control algorithm that defines the moving horizon feedback control system. The algorithm uses several parameters: T_C, the time needed to solve the optimal control problem, which must be determined experimentally, and three parameters that are selected partly on the basis of experimentation and partly on judgement, $\bar{T}$, an upper bound on the horizon, and α, β which govern the speed of response of the system.

Control Algorithm 2.1.

Data: $t_0 = 0$, $t_1 = T_C$, $u_{[t_0, t_1]}(t) \equiv 0$, $x_0 \in B_{\hat{\rho}}$. T_C and $\bar{T}$ such that $0 < T_C < \bar{T} < \infty$.

Step 0: Set $k = 0$.

Step 1: At $t = t_k$,

(a) Obtain a measurement or estimate of the state $x_k^p = x^p(t_k, t_0, x_0^p, u)$ and denote the resulting value by $\tilde{x}_k$.

(b) Set the plant input $u(t) = u_{[t_k, t_{k+1}]}(t)$ for $t \in [t_k, t_{k+1})$.

(c) Compute an estimate x_{k+1} of the state of the plant $x^p(t_{k+1}, t_k, x_k^p, u)$ according to the formula

$$x_{k+1} = e^{A(t_{k+1} - t_k)} \tilde{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1} - t)} B u(t)\, dt \tag{2.4}$$

(d) Solve the open loop optimal control problem $P(x_{k+1}, t_{k+1})$ to compute the next

sampling time $t_{k+2} \in (t_{k+1}+T_C \, , t_{k+1}+\bar{T}]$, and the optimal control $u_{[t_{k+1}, t_{k+2}]}(t) \in U$, $t \in [t_{k+1}, t_{k+2})$;

Step 2: Replace k by $k+1$ and go to Step 1. □

Clearly, the fact that the plant input is bounded, limits the region of effectiveness of any control law. Hence we must assume that the initial states are confined to a $B_{\hat{\rho}} \subset \mathbb{R}^n$ as follows.

Assumption 2.2. We assume that there exists a $\hat{\rho} \in (0, \infty)$ such that for any $x \in B_{\hat{\rho}} \triangleq \{ z \in \mathbb{R}^n \mid \| z \| \leq \hat{\rho} \}$, the optimal control problem $\mathbf{P}(x, 0)$ has a solution. □

The following theorem generalizes a result given in [Pol.1].

Theorem 2.3. Let $B_{\hat{\rho}} \subset \mathbb{R}^n$ be defined as in Assumption 2.2. Suppose that *(a)* the systems (2.1a) and (2.2a) are identical and *(b)* the Control Algorithm 2.1 is used to define the input $u(\cdot)$ for (2.1a). Then the resulting feedback system is asymptotically stable on the set $B_{\hat{\rho}}$, i.e. for any $x_0^p \in B_{\hat{\rho}}$, $x^p(t, 0, x_0^p, u) \to 0$ as $t \to \infty$.

Proof. We begin by showing that for any $x_0 \in B_{\hat{\rho}}$, the trajectory $x(t_k, 0, x_0, u) = x_k$, $k \in \mathbf{N}$ resulting from the use of the Control Algorithm 2.1 is contained in $B_{\hat{\rho}}$. In turn, this shows that such a trajectory is well defined and that it is bounded.

Suppose that $x_0 \in B_{\hat{\rho}}$ is an arbitrary initial state at $t = 0$. It follows from the form of (2.3c), that for all $k \in \mathbf{N}$,

$$\| x_{k+1} \| = \| x(t_{k+1}, t_k, x_k, u_{[t_k, t_{k+1}]}) \| \leq \alpha \| x_k \| \leq \alpha^{k+1} \| x_0 \| . \tag{2.5a}$$

Since $\alpha \in (0, 1)$, it follows that $x_k \in B_{\hat{\rho}}$ for all $k \in \mathbf{N}$ and hence that the trajectory $x(t, 0, x_0, u)$ is well defined.

Next, from the form of (2.3e), we see that for all $k \in \mathbf{N}$ and for any $t \in [t_k, t_{k+1}]$,

$$\| x(t, t_k, x_k, u_{[t_k, t_{k+1}]}) \| \leq \beta \| x_k \| \leq \beta \alpha^k \| x_0 \| \leq \beta \| x_0 \| , \tag{2.5b}$$

which implies that because $\beta \alpha^k \to 0$ as $k \to \infty$, we obtain that $x(t, 0, x_0, u) \to 0$ as $t \to \infty$. Hence the feedback system defined by the Control Algorithm 2.1 is asymptotically stable on the set $B_{\hat{\rho}}$. □

We note that Theorem 2.3 did not depend on the form of the cost function $g^0(\cdot, \cdot)$ nor on the form of the constraints defined by (2.3b) and (2.3d). These constraints can be used to shape the transient responses of the closed loop system. We will describe later a procedure for solving problems of the form (2.3a-e).

As stated, Control Algorithm 2.1 only defines a local control law. When the plant is unstable, since the control $u \in U$ is bounded, for some initial state $x_0 \in \mathbb{R}^n$, there is no control which stabilizes the system. In this case, there is not much that one can do about it. However, in the case of stable plants (and models), it is possible to globalize Control Algorithm 2.1 since for any $x_0 \in \mathbb{R}^n$, there exists a $t \in [0, \infty)$ such that $x(t, 0, x_0, 0) \in B_{\hat{\rho}}$. Clearly, in this case, there may be room for a more effective control law, as we will now show. Let M' and Q' be symmetric, positive definite matrices, such that $A^T Q' + Q'A = -M'$, then $V(x(t)) \triangleq \langle x(t), Q'x(t) \rangle$ is a Lyapunov function for $\dot{x}(t) = Ax(t)$. Let $T_s \in (T_C, \bar{T}]$ and suppose that $x_k \notin B_{\hat{\rho}}$. Then, if we set $t_{k+1} = t_k + T_s$ and we apply the control $u(t) = 0$, to (2.1a), for $t \in [t_k, t_{k+1}]$, then we must have that $V(x(t_{k+1}, t_k, x_k, u^o)) \le e^{-\lambda_{\min}(M')T_s} V(x_k)$. Hence it makes sense to use the control defined as the solution of the simple optimal control problem

$$\min_{u \in U} \{ V(x(t_{k+1}, t_k, x_k, u)) \} , \tag{2.6}$$

where $x(t_{k+1}, t_k, x_k, u)$ is determined as the solution of (2.2a). Let $u^o(t)$, $t \in [t_k, t_{k+1}]$, be a solution of (2.6).

Hence, for stable plants, we propose to modify Control Algorithm 2.1, as follows:

Control Algorithm 2.4.

Data: $t_0 = 0$, t_1, $u_{[t_0, t_1]}(t)$, x_0, T_s, T_C and $\bar{T}$ such that $0 < T_C < T_s \le \bar{T} < \infty$.

Step 0: Set $k = 0$.

Step 1: At $t = t_k$,

(a) Obtain a measurement or estimate of the state $x_k^p = x^p(t_k, t_0, x_0^p, u)$ and denote the resulting value by $\bar{x}_k$.

(b) Set the plant error dynamics input $u(t) = u_{[t_k, t_{k+1}]}(t)$ for $t \in [t_k, t_{k+1})$.

(c) Compute an estimate x_{k+1} of the state of the plant error dynamics $x^p(t_{k+1}, t_k, x_k^p, u)$ according to the formula (2.6)

$$x_{k+1} = e^{A(t_{k+1}-t_k)} \bar{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)} Bu(t)\, dt .$$

(d) If $x_{k+1} \in B_{\hat{\rho}}$, solve the open loop optimal control problem $\mathrm{P}(x_{k+1}, t_{k+1})$ to compute the

next sampling time $t_{k+2} \in (t_{k+1} + T_C, t_{k+1} + \bar{T}]$, and the optimal control $u_{[t_{k+1}, t_{k+2}]}(t) \in U$, $t \in [t_{k+1}, t_{k+2})$.

Else set $t_{k+2} = t_{k+1} + T_s$ and $u_{[t_{k+1}, t_{k+2}]}(t) = u^o(t)$, for all $t \in [t_{k+1}, t_{k+2})$.

Step 2: Replace k by $k+1$ and go to Step 1. □

We will not present a complete analysis of the operation of the closed loop system under Control Algorithm 2.4.

3. ROBUST STABILITY

In this section, we will analyze the behavior of the closed loop system resulting from the use of Control Algorithm 2.1 under the assumption that there is a difference between the actual plant equations (2.1a) and the model equations (2.2a). We will consider two distinct situations: the first is where we can measure the state, while the second one is where the state has to be estimated. Finally, we will show how a cross over rule to a linear state feedback law, proposed by Michalska and Mayne (see [May.2]) near the origin can be used to eliminate residual errors in both cases.

3.1. Moving horizon control with state measurement.

We begin by defining the error quantities

$$\Delta_1 \triangleq \max_{t \in [0, \bar{T}]} \| e^{A^p t} - e^{At} \| , \tag{3.1a}$$

$$\Delta_2 \triangleq \lambda_{\max}(Q)^{1/2} \sqrt{m}\, c_u \bar{T} \max_{t \in [0, \bar{T}]} \| e^{A^p t} B^p - e^{At} B \| , \tag{3.1b}$$

$$K \triangleq \max_{t \in [0, \bar{T}]} \| e^{At} \| , \tag{3.1c}$$

where $\lambda_{\max}(Q)$ denotes the largest singular value of Q. When either Δ_1 or Δ_2 is not zero, even if $\bar{x}_k \in B_{\hat{\rho}}$, where $B_{\hat{\rho}}$ was defined in Assumption 2.2, the estimated state, x_{k+1} (defined by (2.4)), may not be in $B_{\hat{\rho}}$ and hence there may not exist a solution to the optimal control problem $\mathbf{P}(x_{k+1}, t_{k+1})$. Therefore, we have to specify a set $B_{\rho_*} \subset B_{\hat{\rho}}$, such that for any $x_0^p \in B_{\rho_*}$, Control Algorithm 2.1 is well defined on the emanating trajectory $x(t, 0, x_0^p, u)$. We will obtain a formula for such a set in the process of proving the following result.

Lemma 3.1. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.1, with plant state measurement. There exist $\varepsilon_1, \varepsilon_2 > 0$ such that if $\Delta_1 \le \varepsilon_1$ and $\Delta_2 \le \varepsilon_2$, then there exists a set $B_{\rho_0} \subset B_{\hat{\rho}}$, with nonempty interior, such that for all $x_0^p \in B_{\rho_0}$, the control law defined by Control Algorithm 2.1 is well defined on the resulting trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, i.e., the states, x_{k+1}, $k = 0, 1, 2, \ldots$, computed using (2.4) satisfy that $x_{k+1} \in B_{\hat{\rho}}$ for all $k \geq 0$.

Proof. First suppose that the optimal control problem $\mathbf{P}(x_{k+1}, t_{k+1})$, has a solution for any $x_{k+1} \in \mathbb{R}^n$ and $t_{k+1} \geq 0$. Then, given any initial state x_0^p at time $t_0 = 0$, the Control Algorithm 2.1 generates three sequences of states. The first sequence is that of measured plant states $\{x_k^p\}_{k=0}^{\infty}$, so that $\tilde{x}_k = x_k^p$ for all $k \in \mathbb{N}$, the second sequence is the sequence of estimates $\{x_k\}_{k=1}^{\infty}$, with $x_{k+1} = x(t_{k+1}, t_k, x_k^p, u)$, $k = 1, 2, \ldots$, generated according to (2.4), and finally, the sequence $\{x'_k\}_{k=2}^{\infty}$, with $x'_{k+2} = x(t_{k+2}, t_{k+1}, x_{k+1}, u)$, $k = 1, 2, \ldots$, generated in the process of solving the optimal control problem $\mathbf{P}(x_{k+1}, t_{k+1})$, $k \in \mathbf{N}$.

First we note that it follows form (2.1a), (2.4), (3.1a,b), and the fact that $\|u(t)\|_2 \leq \sqrt{m}\,\|u(t)\|_\infty$ that

$$\|x_{k+1}^p - x_{k+1}\| \leq \Delta_1 \|x_k^p\| + \Delta_2 . \tag{3.2a}$$

Hence, making use of (3.2a), we obtain that

$$\|x_{k+1}\| \leq \|x_{k+1}^p - x_{k+1}\| + \|x_{k+1}^p\| \leq \Delta_1 \|x_k^p\| + \Delta_2 + \|x_{k+1}^p\| . \tag{3.2b}$$

Since by construction, for all $k \in \mathbf{N}$, $\|x'_{k+2}\| \leq \alpha \|x_{k+1}\|$, it follows from (3.1a-c) and (3.2a,b), that

$$\begin{aligned}
\|x_{k+2}^p\| &\leq \|x_{k+2}^p - x'_{k+2}\| + \alpha \|x_{k+1}\| \\
&\leq K \|x_{k+1}^p - x_{k+1}\| + \Delta_1 \|x_{k+1}^p\| + \Delta_2 + \alpha \|x_{k+1}^p - x_{k+1}\| + \alpha \|x_{k+1}^p\| \\
&\leq (K + \alpha)(\Delta_1 \|x_k^p\| + \Delta_2) + (\Delta_1 + \alpha)\|x_{k+1}^p\| + \Delta_2 \\
&= (K + \alpha)\Delta_1 \|x_k^p\| + (\Delta_1 + \alpha)\|x_{k+1}^p\| + (1 + \alpha + K)\Delta_2 .
\end{aligned} \tag{3.2c}$$

Let

$$\hat{\varepsilon}_1 \triangleq (1 - \alpha)/(1 + \alpha + K) . \tag{3.2d}$$

We will now show that if $\Delta_1 \leq \varepsilon_1$ for some $\varepsilon_1 \in (0, \hat{\varepsilon}_1)$, then there exists $\gamma_1, \gamma_2 \in (0, \infty)$ such that for all $k = 1, 2, \ldots$,

$$\|x_k\| \le \gamma_1 \|x_0^e\| + \gamma_2 . \tag{3.2e}$$

We will now make use of Proposition 6.1 (see Appendix I). Hence, let $a_1 = \Delta_1 + \alpha$, $a_2 = (K+\alpha)\Delta_1$, and $b = (1+\alpha+K)\Delta_2$. Because a_1, a_2, and b are positive, if we set $y_0 = \|x_0^e\|$ and $y_1 = \|x_1^e\|$ in (6.1a), then comparing (3.2c) with (6.1a), we see that for all $k \in \mathbb{N}$, $y_k \ge \|x_k^e\|$. Also, because

$$\Delta_1 + \alpha + (K+\alpha)\Delta_1 = a_1 + a_2 < 1 , \tag{3.2f}$$

the assumptions of Proposition 6.1 are satisfied. Since $\Delta_1 \le \varepsilon_1 < \hat{\varepsilon}_1$, and $(K+\alpha)\Delta_1 \ge 0$,

$$1 - a_1 + a_2 = 1 - \Delta_1 - \alpha + (K+\alpha)\Delta_1 > 1 - \alpha - \hat{\varepsilon}_1 = \frac{(1-\alpha)(\alpha+K)}{1+\alpha+K} \triangleq \varepsilon' > 0 . \tag{3.2g}$$

Hence, for all $k \ge 1$,

$$\|x_k^e\| \le y_k \le a_2 \|x_0^e\| + \|x_1^e\| + \frac{b}{1-a_1+a_2} \le a_2 \|x_0^e\| + \|x_1^e\| + \varepsilon'' , \tag{3.2h}$$

$$\overline{\lim_{k\to\infty}} \|x_k^e\| \le \overline{\lim_{k\to\infty}} y_k \le \varepsilon'' , \tag{3.2i}$$

where

$$\varepsilon'' \triangleq \frac{(1+\alpha+K)\Delta_2}{\varepsilon'} . \tag{3.2j}$$

Clearly,

$$\|x_1^e\| \le \|x_1^e - x_1\| + \|x_1\| . \tag{3.2k}$$

Since $x_0 = x_0^e$, and $u(t) = 0$ for $t \in [0, t_1]$ it follows from (3.1a,c), and (3.2k) that

$$\|x_1^e\| \le \Delta_1 \|x_0^e\| + K \|x_0^e\| = (\Delta_1 + K) \|x_0^e\| . \tag{3.2l}$$

Substituting this result into (3.2h), we obtain that for all $k \ge 1$,

$$\|x_k^e\| \le y_k \le (\Delta_1 + K - \lambda_+ \lambda_-) \|x_0^e\| + \varepsilon'' , \tag{3.2m}$$

where ε'' is defined in (3.2j). Since $-\lambda_+\lambda_- = a_2 = (K+\alpha)\Delta_1$, it follows from (3.2m) and (3.2b) that

$$\begin{aligned} \|x_{k+1}\| &\le \Delta_1 y_k + \Delta_2 + y_{k+1} \\ &\le (1+\Delta_1)(\Delta_1 + K + (K+\alpha)\Delta_1) \|x_0^e\| + (1+\Delta_1)\varepsilon'' + \Delta_2 \\ &\triangleq \gamma_1 \|x_0^e\| + \gamma_2 , \end{aligned} \tag{3.2n}$$

which proves (3.2e).

Next we will show that with $\hat{\varepsilon}_2 > 0$ defined by

$$\hat{\varepsilon}_2 \triangleq \hat{\rho} \left[\frac{2+K}{\varepsilon'}+1\right]^{-1} = \hat{\rho} \left[\frac{(2+K)(1+\alpha+K)}{(1-\alpha)(\alpha+K)}+1\right]^{-1} , \tag{3.2o}$$

where $\hat{\rho} > 0$ was used to define the set $B_{\hat{\rho}}$ in Assumption 2.2 and ε' is defined in (3.2g), if $\Delta_2 \le \varepsilon_2$ for some $\varepsilon_2 \in (0, \hat{\varepsilon}_2)$, then there exists a $\rho_s \in (0, \hat{\rho})$, depending on $\varepsilon_1, \varepsilon_2$, such that if $\|x_0^p\| \le \rho_s$, then $\|x_k\| \le \hat{\rho}$ for all $k = 1, 2, \ldots$, i.e., that the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, emanating from x_0^p, constructed under Control Algorithm 2.1 is well defined.

Assuming that $\Delta_1 \le \varepsilon_1$ and that $\Delta_2 \le \varepsilon_2$, we obtain that from (3.2n)

$$\gamma_2 < \left[1+\frac{1-\alpha}{1+\alpha+K}\right]\frac{(1+\alpha+K)}{\varepsilon'}\varepsilon_2+\Delta_2 \le \left[\frac{2+K}{\varepsilon'}+1\right]\varepsilon_2 \triangleq \hat{\gamma}_2 \le \hat{\rho} \ . \tag{3.2p}$$

Let $\hat{\gamma}_1$ and ρ_s be defined as follows:

$$\hat{\gamma}_1 \triangleq (1+\varepsilon_1)(K+(1+\alpha+K)\varepsilon_1) \ , \tag{3.2q}$$

$$\rho_s \triangleq (\hat{\rho} - \hat{\gamma}_2)/\hat{\gamma}_1 \ , \tag{3.2r}$$

Since $\hat{\rho} - \hat{\gamma}_2 > 0$ and $K \le \gamma_1 \le \hat{\gamma}_1$, we conclude that $\rho_s > 0$, and hence that $B_{\rho_s} \subset B_{\hat{\rho}}$, defined by

$$B_{\rho_s} \triangleq \{ x \in B_{\hat{\rho}} \mid \|x\| \le \rho_s \} \ , \tag{3.2s}$$

is well defined and its interior is not empty. Furthermore, for any $x_0^p \in B_{\rho_s}$, the resulting sequence $\{ x_{k+1} \}_{k=0}^{\infty}$ satisfies

$$\|x_{k+1}\| \le \gamma_1 \|x_0^p\| + \gamma_2 \le \gamma_1 (\hat{\rho} - \hat{\gamma}_2)/(\hat{\gamma}_1) + \hat{\gamma}_2 \le \hat{\rho} \ , \ \forall k \in \mathbb{N} , \tag{3.2t}$$

which implies that $x_{k+1} \in B_{\hat{\rho}}$, for all $k \in \mathbb{N}$, and, in turn, that the optimal control problem $P(x_{k+1}, t_{k+1})$ has a solution for all $k \in \mathbb{N}$. Hence the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, emanating from any $x_0^p \in B_{\rho_s}$ is well defined by Control Algorithm 2.1, which completes our proof.□

Theorem 3.2. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.1, with plant state measurement. Suppose that $\Delta_1 \le \varepsilon_1 < \hat{\varepsilon}_1$ and $\Delta_2 \le \varepsilon_2 < \hat{\varepsilon}_2$, where $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$ are defined in (3.2d) and (3.2o), respectively. Let B_{ρ_s} be defined as (3.2s). Then *(a)* for any $x_0^p \in B_{\rho_s}$, the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, is bounded, and *(b)* there exists an $\varepsilon_3 > 0$, depending on $\varepsilon_1, \varepsilon_2$, such that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, and for any $x_0^p \in B_{\rho}$, the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, satisfies $\overline{\lim}_{t \to \infty} \|x^p(t, 0, x_0^p, u)\| \le \varepsilon_3$.

Proof. Let $x_0^p \in B_{\rho}$ be arbitrary and let $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{x'_k\}_{k=2}^{\infty}$ be the sequences constructed by Control Algorithm 2.1, as defined in Lemma 3.1. We recall that by Lemma 3.1, the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, is well defined.

(a) Making use of (2.1a) and (3.1a-c), we obtain that for all $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$,

$$\|x^p(t, t_k, x_k^p, u)\| \le \|x^p(t, t_k, x_k^p, u) - x(t, t_k, x_k, u)\| + \|x(t, t_k, x_k, u)\|$$

$$\le \Delta_1 \|x_k^p\| + K\|x_k^p - x_k\| + \Delta_2 + \|x(t, t_k, x_k, u)\| . \tag{3.3a}$$

Next we note that the form of (2.3e) ensures that $\|x(t, t_k, x_k, u)\| \le \beta \|x_k\|$ for all $t \in [t_k, t_{k+1}]$. Hence, in view of (3.2a), (3.3a) can be replaced by

$$\|x^p(t, t_k, x_k^p, u)\| \le \Delta_1 \|x_k^p\| + K\|x_k^p - x_k\| + \Delta_2 + \beta\|x_k\| = (\Delta_1 + \beta)\|x_k^p\| + (K + \beta)\|x_k^p - x_k\| + \Delta_2$$

$$\le (\Delta_1 + \beta)\|x_k^p\| + (K + \beta)\Delta_1\|x_{k-1}^p\| + (1 + K + \beta)\Delta_2 , \quad t \in [t_k, t_{k+1}] . \tag{3.3b}$$

Clearly, since $u_{[0, t_1]} \equiv 0$, $\|x^p(t, 0, x_0^p, u)\|$ is bounded on $[0, t_1]$. Since, as we have already shown in the proof of Lemma 3.1, $\{\|x_k^p\|\}_{k=0}^{\infty}$ is a bounded sequence, it follows from (3.3b) that $\|x^p(t, t_k, x_k^p, u)\|$ is bounded for all $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$, which completes the proof of *(a)*.

(b) It follows from (3.2i), in the proof of Lemma 3.1, that

$$\overline{\lim}_{k \to \infty} \|x_k^p\| \le \overline{\lim}_{k \to \infty} y_k \le \varepsilon'' \le \frac{(1 + \alpha + K)\varepsilon_2}{\varepsilon'} , \tag{3.3c}$$

where ε' is defined in (3.2g). Let

$$\varepsilon_3 \triangleq \left[\frac{(\beta + (1 + K + \beta)\varepsilon_1)(1 + \alpha + K)}{\varepsilon'} + 1 + K + \beta \right] \varepsilon_2 . \tag{3.3d}$$

Then (3.2i) and (3.3b) lead to the conclusion that $\overline{\lim}_{t \to \infty} \|x^p(t, 0, x_0^p, u)\| \le \varepsilon_3$. It is obvious from (3.2i) and (3.3d) that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, which completes our proof. $\square$

3.2. moving horizon control with state estimation.

Since it is not always possible to measure the plant state x_k^p, we will now examine the behavior of our closed loop system, resulting form the use of Control Algorithm 2.1, when the plant state has to be estimated in the presence of modeling errors, i.e., when the actual dynamics are as in (2.1a,b) and the modeled dynamics as in that (2.2a,b).

When the model (2.2a,b) is identical with the actual dynamics (2.1a,b), we can calculate the initial state, x_0^p at $t = 0$, using the standard formula

$$x_0^p = W_o(T_o)^{-1}\int_0^{T_o} (Ce^{At})^T(y^p(t) - \eta(t\ ,0))dt\ , \tag{3.4a}$$

where $T_o > 0$, the superscript T denotes a transpose, and

$$W_o(T_o) = \int_0^{T_o} (Ce^{At})^T Ce^{At} dt\ , \tag{3.4b}$$

$$\eta(t\ ,s) = C\int_s^t e^{A(t-\tau)}Bu(\tau)d\tau . \tag{3.4c}$$

Clearly, $W_o(T_o)^{-1}$ exists because $(A\ ,C)$ is an observable pair. Thus, when there are no modeling errors, for $t \geq T_o$, the state $x^p(t\ ,0\ ,x_0^p\ ,u)$, can be calculated exactly, and hence this calculated state can be used in Control Algorithm 2.1.

The much more relevant situation occurs when there are modeling errors. In this case formula (3.4a) yields an estimate of the initial state x_0^p. We propose to use it in *Step 1 (a)* of Control Algorithm 2.1, to obtain the estimate $\tilde{x}_k$, with the time T determined by a parameter δ_0, which must be chosen judiciously so as to avoid excessive ill conditioning in the observability grammian $W_o(T_o)$:

Step 1: *(a)* At $t'_k \triangleq t_k + \delta_0(t_{k+1} - t_k)$ with $\delta_0 \in (0\ ,1)$, estimate the state x_k^p by

$$\tilde{x}_k = W_o(\delta_0(t_{k+1} - t_k))^{-1}\int_{t_k}^{t'_k} (Ce^{A(t-t_k)})^T(y^p(t) - \eta(t\ ,t_k))dt\ . \tag{3.5}$$

□

Lemma 3.3. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.1, with state estimation formula (3.5). There exist $\Delta_i < \infty$, $i = 3\ ,\ldots,6$, such that if Control Algorithm 2.1 constructs the sequences $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{\tilde{x}_k\}_{k=0}^{\infty}$ which is the corresponding sequence of the estimates of x_k^p, defined by (3.5), then for all $k \in \mathbb{N}$,

$$|x_k^p - \tilde{x}_k| \leq \Delta_3|x_k^p| + \Delta_4\ , \tag{3.6a}$$

$$|x_{k+1}^p - x_{k+1}| \leq \Delta_5|x_k^p| + \Delta_6 . \tag{3.6b}$$

Furthermore, when there are no modeling errors, $\Delta_i = 0$, $i = 3\ ,\ldots,6$.

Proof. Suppose that $u(\cdot)$ is the control generated by Control Algorithm 2.1 for the plant and model trajectories associated with the sequences $\{x_k^p\}_{k=0}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{\tilde{x}_k\}_{k=0}^{\infty}$.

We begin with (3.6a). For any $k \in \mathbb{N}$ and any $t \in [t_k\ ,t_{k+1}]$, $y^p(t)$ is given by

$$y^p(t) = C^p e^{A^p(t-t_k)} x_k^p + C^p \int_{t_k}^{t} e^{A^p(t-\tau)} B^p u(\tau) d\tau$$

$$= C e^{A(t-t_k)} x_k^p + [C^p e^{A^p(t-t_k)} - C e^{A(t-t_k)}] x_k^p$$

$$+ C \int_{t_k}^{t} e^{A(t-\tau)} B u(\tau) d\tau + \int_{t_k}^{t} [C^p e^{A^p(t-\tau)} B^p - C e^{A(t-\tau)} B] u(\tau) d\tau . \quad (3.7a)$$

By substituting (3.7a) into (3.5), we obtain

$$\bar{x}_k = x_k^p + W_o^{-1}(\delta_0(t_{k+1} - t_k)) \left\{ \int_{t_k}^{t'_k} (C e^{A(t-t_k)})^T [C^p e^{A^p(t-t_k)} - C e^{A(t-t_k)}] \, dt \, x_k^p \right.$$

$$\left. + \int_{t_k}^{t'_k} (C e^{A(t-t_k)})^T \int_{t_k}^{t} [C^p e^{A^p(t-\tau)} B^p - C e^{A(t-\tau)} B] u(\tau) \, d\tau \, dt \right\} . \quad (3.7b)$$

It follows directly from (3.7b) that

$$| x_k^p - \bar{x}_k | \le \Delta_3 | x_k^p | + \Delta_4 , \quad (3.7c)$$

where

$$\Delta_3 = \lambda_{\max}(Q)^{1/2} \max_{t \in [T_c, \bar{T}]} | W_o(\delta_0 t)^{-1} |_2 \max_{t \in [0, \delta_0 \bar{T}]} | C e^{At} |_2 | C^p e^{A^p(t-t_k)} - C e^{A(t-t_k)} |_2 \delta_0 \bar{T} \quad (3.7d)$$

$$\Delta_4 = \lambda_{\max}(Q)^{1/2} \max_{t \in [T_c, \bar{T}]} | W_o(\delta_0 t)^{-1} |_2 \max_{t \in [0, \delta_0 \bar{T}]} | C e^{At} |_2 | C^p e^{A^p(t-\tau)} B^p - C e^{A(t-\tau)} B |_2 \sqrt{m} \, c_u \delta_0 \bar{T} , \quad (3.7e)$$

which proves our first (3.6a). Clearly, when there are no modeling errors, $\Delta_3 = \Delta_4 = 0$.

Next we will establish (3.6b). Since x_{k+1} is calculated using the estimated initial state $\bar{x}_k$, we have that

$$| x_{k+1}^p - x_{k+1} | = | e^{A^p(t_{k+1}-t_k)} x_k^p - e^{A(t_{k+1}-t_k)} \bar{x}_k + \int_{t_k}^{t_{k+1}} \{ e^{A^p(t_{k+1}-\tau)} B^p - e^{A(t_{k+1}-\tau)} B \} u(\tau) d\tau |$$

$$\le K \, | x_k^p - \bar{x}_k | + \Delta_1 | x_k^p | + \Delta_2$$

$$\le K \, \{ \Delta_3 | x_k^p | + \Delta_4 \} + \Delta_1 | x_k^p | + \Delta_2 ,$$

$$= (K \Delta_3 + \Delta_1) | x_k^p | + K \Delta_4 + \Delta_2 \triangleq \Delta_5 | x_k^p | + \Delta_6 , \quad (3.7f)$$

where K, Δ_1, and Δ_2 are defined in (3.1a,b,c), respectively. Hence (3.6b) holds, and our proof is complete. □

Lemma 3.3, leads to the following result.

Theorem 3.4. Consider the moving horizon feedback system resulting from the use of the Control Algorithm 2.1, with state estimation formula (3.5). Let $\varepsilon_1, \varepsilon_2 > 0$ be such that

$$\varepsilon_1 < (1-\alpha)/(1+\alpha+K), \tag{3.8a}$$

$$\varepsilon_2 < \hat{\rho}\left[1+\frac{2+K}{\varepsilon'}\right]^{-1}, \tag{3.8b}$$

where $\hat{\rho}$ was defined in Assumption 2.2 and ε' was defined in (3.2g), in the proof of Lemma 3.1. If $\Delta_5 \leq \varepsilon_1$ and $\Delta_6 \leq \varepsilon_2$, then there exists a set $B_{\rho_s} \subset B_{\hat{\rho}}$ such that *(a)* for any $x_0^p \in B_{\rho_s}$, the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, is well defined and bounded, and *(b)* there exists an $\varepsilon_3 > 0$ such that $\varepsilon_3 \to 0$ as $\varepsilon_2 \to 0$, and for any $x_0^p \in B_{\rho_s}$ the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, satisfies $\overline{\lim}_{t\to\infty} |x^p(t, 0, x_0^p, u)| \leq \varepsilon_3$.

Proof. *(a)* First suppose that the optimal control problem $P(x_{k+1}, t_{k+1})$, has a solution for any $x_{k+1} \in \mathbb{R}^n$ and $t_{k+1} \geq 0$. Then it follows from Lemma 3.3 that

$$\begin{aligned}|x_{k+2}^p| &\leq |x_{k+2}^p - x'_{k+2}| + |x'_{k+2}| \leq K\,|x_{k+1}^p - x_{k+1}| + \Delta_1 |x_{k+1}^p| + \Delta_2 + \alpha |x_{k+1}| \\ &\leq (K+\alpha)\,|x_{k+1}^p - x_{k+1}| + (\Delta_1+\alpha)\,|x_{k+1}^p| + \Delta_2 \\ &\leq (K+\alpha)\Delta_5 |x_k^p| + (K+\alpha)\Delta_6 + (\alpha+\Delta_1)\,|x_{k+1}^p| + \Delta_2 .\end{aligned} \tag{3.9a}$$

Since $\Delta_1 \leq \Delta_5$ and $\Delta_2 \leq \Delta_6$, we have that

$$|x_{k+2}^p| \leq (K+\alpha)\Delta_5 |x_k^p| + (\alpha+\Delta_5)\,|x_{k+1}^p| + (K+\alpha+1)\Delta_6 . \tag{3.9b}$$

Since (3.9b) is of the same form as (3.2c), with Δ_5 replacing Δ_1, and Δ_6 replacing Δ_2, we see that the conclusions of Lemma 3.1 and Theorem 3.2 *(a)* remain valid for the Control Algorithm 2.1 using state estimation formula (3.5).

(b) Referring to (3.2s) (3.3d), we conclude that part *(b)* holds with ε_3 and B_{ρ_s} defined by

$$\varepsilon_3 \triangleq \left[\frac{(\beta+(1+K+\beta)\varepsilon_1)(1+\alpha+K)}{\varepsilon'} + 1 + K + \beta\right]\varepsilon_2 , \tag{3.9c}$$

$$B_{\rho_s} \triangleq \{x \in B_{\hat{\rho}} \mid |x| \leq \rho_s\} , \tag{3.9d}$$

where ε' is defined in (3.2g) and

$$\rho_s \triangleq \frac{\hat{\rho} - ((2+K)/\varepsilon' + 1)\varepsilon_2}{(1+\varepsilon_1)(K + (1+\alpha+K)\varepsilon_1)} . \tag{3.9e}$$

□

3.3. Elimination of residual errors by linear feedback.

Because linear quadratic regulators are robust, when the pair (A, B) is stabilizable and the modeling errors are sufficiently small, we can always find a linear stabilizing state feedback control law $u(t) = -K_c x^p(t, 0, x_0^p, u)$, where K_c is the solution of a linear quadratic regulator problem in terms of the model (2.2a,b), and a ball $B_{LQR} \triangleq \{ x \in \mathbb{R}^n \mid \|x\| \le \rho_{LQR} \}$, $\rho_{LQR} \in (0, \hat{\rho})$, such that if for some $t_{k'}$, $x_{k'}^p \in B_{LQR}$, then the control given by $u(t) = -K_c x(t, 0, x_0, u)$, for $t \ge t_{k'}$ does not violate the bound on the control on the resulting trajectory, i.e, $\|K_c x^p(t, 0, x_0, u)\| \le c_u$ for all $t \ge t_{k'}$. As we will see, a similar, but somewhat more complicated result also holds when $x^p(t, 0, x_0^p, u)$ is estimated using an asymptotic observer. Hence, in both cases, once the plant state is sufficiently near the origin, we can switch over to the LQR control law and thereby eliminate the residual errors resulting from the use of Control Algorithm 2.1.

For the case where the state can be measured, we propose to incorporate this idea into Control Algorithm 2.1 by modifying *Step 1*, as follows. Let $T_{K_c} \ge T_C$ is such that $\|e^{T_{K_c}(A - BK_c)}\| \le \alpha$.

Step 1': At $t = t_k$,

(a) Measure or estimate the state $\tilde{x}_k = x^p(t_k, 0, x_0^p, u)$.

(b) If $\tilde{x}_k \notin B_{LQR}$, set the plant input $u(t) = u_{[t_k, t_{k+1}]}(t)$ for $t \in [t_k, t_{k+1})$; else set $u(t) = -K_c x^p(t, 0, x_0^p, u)$ for $t \in [t_k, t_{k+1})$, where $t_{k+1} = t_k + T_{K_c}$.

(c) Compute an estimate x_{k+1} of the state of the plant $x^p(t_{k+1}, t_k, \tilde{x}_k, u)$ according to the formula (2.4), i.e.,

$$x_{k+1} = e^{A(t_{k+1} - t_k)} \tilde{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1} - t)} B \, u_{[t_k, t_{k+1}]}(t) \, dt.$$

□

At this point it becomes clear that for best results, the matrix Q, used to define the norm $\|\cdot\|$, should also define a Lyapunov function $\langle x, Qx \rangle$ for the system $\dot{x} = (A - BK_c)x$, so that for some positive definite matrix M we have

$$(A - BK_c)^T Q + Q(A - BK_c) = -M . \tag{3.10}$$

Theorem 3.5. Suppose that the matrix Q used to define the norm $\|\cdot\|$ satisfies (3.10) for some positive definite matrix M, and that the state of the plant can be measured. Let $\varepsilon_1 \in (0, \hat{\varepsilon}_1)$, $\varepsilon_2 \in (0, \hat{\varepsilon}_2)$, and $\delta \in (0, \lambda_{\min}(M)/2\lambda_{\max}(Q))$, where $\hat{\varepsilon}_1, \hat{\varepsilon}_2$ were defined in (3.2d), (3.2o), respectively, M is as in (3.10), and let ρ_s be defined by (3.2r), and ρ_{MH} by

$$\rho_{MH} \triangleq \frac{(1+\alpha+K)\varepsilon_2}{\varepsilon'}, \tag{3.11}$$

where ε' was defined in (3.2g). Finally, suppose that $\rho_{MH} < \rho_{LQR}$, with $\rho_{LQR} > 0$, as above. If $\Delta_1 \le \varepsilon_1$, $\Delta_2 \le \varepsilon_2$, and $\|(A^p - A) - (B^p - B)K_c\|_2 \le \delta$, then for any $x_0^p \in B_{\rho_s}$, with B_{ρ_s} defined in (3.2s), the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$ is bounded and, furthermore, $x^p(t, 0, x_0^p, u) \to 0$ as $t \to \infty$.

Proof. Since the conditions imposed in Lemma 3.1 and Theorem 3.2 are satisfied, it follows that for any $x_0^p \in B_{\rho_s}$, the trajectory $x^p(t, 0, x_0^p, u)$, $t \in [0, \infty)$, determined by Algorithm 2.1, using the original *Step 1'*, is well defined, bounded and $\overline{\lim}_{k\to\infty} \|x_k^p\| \le \rho_{MH}$. Since $\rho_{MH} < \rho_{LQR}$, there exists a finite $\hat{k} \in \mathbb{N}$, such that $\|x_{\hat{k}}^p\| \le \rho_{LQR}$, and hence that the cross over to the linear control law, specified in *Step 1' (c)* will take place. Let $V(x) \triangleq \langle x, Qx \rangle$. Hence, for $x^p(t)$ determined by the differential equation $\dot{x}^p(t) = (A^p - B^p K_c)x^p(t)$, $x^p(t_{\hat{k}}) = x_{\hat{k}}^p$, we obtain that for all $t \ge t_{\hat{k}}$,

$$\begin{aligned}
\dot{V}(x^p(t)) &= \langle \dot{x}^p(t), Qx^p(t) \rangle + \langle x^p(t), Q\dot{x}^p(t) \rangle \\
&= \langle x^p(t)T, [(A - BK_c)^T Q + Q(A - BK_c)]x^p(t) \rangle \\
&\quad + \langle x^p(t)[(A^p - A - (B^p - B)K_c]^T Q + Q(A^p - A - (B^p - B)K_c)]x^p(t) \rangle \\
&= -\langle x^p(t), Mx^p(t) \rangle \\
&\quad + \langle x^p(t), ((A^p - A - (B^p - B)K_c)^T Q + Q(A^p - A - (B^p - B)K_c))x^p(t) \rangle.
\end{aligned} \tag{3.12a}$$

Since $\|(A^p - A) - (B^p - B)K_c\|_2 \le \delta$ it follows that for all $t \ge t_{\hat{k}}$,

$$\dot{V}(x^p(t)) \le -\lambda_{\min}(M) \left[1 - \frac{2\delta\lambda_{\max}(Q)}{\lambda_{\min}(M)} \right] \|x^p(t)\|_2^2 < 0, \tag{3.12b}$$

which implies that *(i)* $x^p(t) \in B_{LQR}$ for all $t \ge t_{\hat{k}}$, and *(ii)* that $x^p(t) \to 0$ as $t \to \infty$, completing our proof. □

When the state of the plant cannot be measured, we must augment our control system with an asymptotic state observer that provides the plant state estimate when we switch over to the linear feedback control law. The asymptotic observer must be in operation from time $t = 0$. In this case we get augmented dynamics in the well known observer-controller form

$$\dot{x}^p(t) = A^p x^p(t) - B^p K_c x^o(t), \tag{3.13a}$$

$$\dot{x}^o(t) = K_o C^p x^p(t) + (A - BK_c - K_o C)x^o(t), \tag{3.13b}$$

where K_o is the observer gain matrix. Let $e(t) \triangleq x^p(t) - x^o(t)$ denote the difference between the state of the plant and that of the model in the observer. Then

$$\dot{e}(t) = (A^p - K_o C^p)x^p(t) - (A - K_o C)x^o(t) - (B^p K_c - BK_c)x^o(t). \tag{3.13c}$$

We assume that the system

$$\dot{\eta}(t) = \tilde{A}\,\eta(t), \tag{3.13d}$$

where $\tilde{A} \triangleq diag\,((A - K_o C), \hat{A})$, with $\hat{A}$ is defined by

$$\hat{A} \triangleq \begin{bmatrix} A & -BK_c \\ K_o C & A - BK_c - K_o C \end{bmatrix}, \tag{3.13e}$$

corresponding to (3.13a,b,c) when there are no modeling errors, is exponentially stable, and hence that there exists a symmetric, positive definite matrix $\tilde{Q} = diag\,(Q_o, Q_c)$, with $Q_o \in \mathbb{R}^{n \times n}$ and $Q_c \in \mathbb{R}^{2n \times 2n}$ that defines a Lyapunov function, $\langle \eta, \tilde{Q}\eta \rangle$ for the system (3.13d), so that for some symmetric, positive definite matrix $\tilde{M} = diag\,(M_o, M_c)$, with $M_o \in \mathbb{R}^{n \times n}$ and $M_c \in \mathbb{R}^{2n \times 2n}$, we have

$$\tilde{A}^T\tilde{Q} + \tilde{Q}\tilde{A} = -\tilde{M}. \tag{3.13f}$$

We will now show that the system (3.13d) is robustly stable.

Lemma 3.6 Suppose that the state $(x^p(t), x^o(t))$ is defined by the observer-controller dynamics described by (3.13a,b), with $(x^p(0), x^o(0))$ arbitrary. Let $\delta \in (0, 0.5)$ and $\Delta\tilde{A}$ be defined by

$$\Delta\tilde{A} \triangleq \begin{bmatrix} 0 & \Delta A - K_o \Delta C & \Delta BK_c \\ 0 & \Delta A & -\Delta BK_c \\ 0 & K_o \Delta C & 0 \end{bmatrix}, \tag{3.14}$$

where $\Delta A = A^p - A$, $\Delta B = B^p - B$, and $\Delta C = C^p - C$. If $\|\Delta\tilde{A}\,\tilde{Q}\|_2 < \delta\lambda_{\min}(\tilde{M})$, then $\overline{\lim}_{t \to \infty} \|x^p(t)\| = 0$ and $\overline{\lim}_{t \to \infty} \|x^o(t)\| = 0$.

Proof. Let $z(t) \triangleq (e(t), x^p(t), x^o(t))^T$, where $(x^p(t), x^o(t))$ is a solution of (3.13a,b) and $e(t) \triangleq x^p(t) - x^o(t)$. Then, referring to (3.13a,b,c) and (3.14), we see that $\dot{z}(t) = [\tilde{A} + \Delta\tilde{A}]z(t)$. Consider the Lyapunov function $V(z)$, for the nominal system (3.13d), defined by $V(\eta) \triangleq \langle \eta, \tilde{Q}\eta \rangle$. Then,

$$\dot{V}(z(t)) = \langle \dot{z}(t), \tilde{Q}z(t) \rangle + \langle z(t), \tilde{Q}\dot{z}(t) \rangle$$

$$= -\langle z(t), \tilde{M}z(t) \rangle + 2\langle z(t), \Delta\tilde{A}\tilde{Q}z(t) \rangle$$

$$\leq -\lambda_{\min}(\tilde{M})(1 - 2\delta)\|z(t)\|_2^2. \tag{3.15}$$

It follows immediately from the condition on δ that $\dot{V}(z(t)) < 0$, whenever $z(t) \neq 0$, which completes our proof. □

Lemma 3.7. Suppose that the state $(x^p(t), x^o(t))$ is defined by the observer-controller dynamics (3.13a,b), that $\|x^p(0)\| \leq \varepsilon$, $\|x^o(0)\| \leq \varepsilon$, for some $\varepsilon > 0$, and that $\Delta\tilde{A}$ satisfies the condition in Lemma 3.6. Then for all $t \geq 0$,

$$\|e(t)\| \leq \left[2\lambda_{\max}(Q)\frac{\lambda_{\max}(Q_o) + \lambda_{\max}(\hat{Q})}{\lambda_{\min}(Q)\lambda_{\min}(Q_o)} \right]^{1/2} \varepsilon \triangleq \gamma\varepsilon. \tag{3.16}$$

Proof. First, let $\|x\|_{Q_o} \triangleq \langle x, Q_o x \rangle^{1/2}$. Let the Lyapunov function $V(\cdot)$ be defined as in Lemma 3.6. Then it follows from the definition of $V(\cdot)$ and the fact that by Lemma 3.6, $\dot{V}(z(t)) < 0$, where $z(t) \triangleq (e(t), x^p(t), x^o(t))$, that $\|e(t)\|_{Q_o}^2 \leq V(z(t)) \leq V(z(0))$, for all $t \geq 0$. Hence,

$$\|e(t)\|_{Q_o}^2 \leq \frac{\lambda_{\max}(Q_o)}{\lambda_{\min}(Q)}\|e(0)\|^2 + \frac{2\lambda_{\max}(\hat{Q})}{\lambda_{\min}(Q)}\varepsilon^2$$

$$\leq 2\left[\frac{\lambda_{\max}(Q_o) + \lambda_{\max}(\hat{Q})}{\lambda_{\min}(Q)} \right]\varepsilon^2. \tag{3.17a}$$

It now follows from (3.17a) that

$$\|e(t)\|^2 \leq \frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q_o)}\|e(t)\|_{Q_o}^2 \leq 2\varepsilon^2\lambda_{\max}(Q)\frac{\lambda_{\max}(Q_o) + \lambda_{\max}(\hat{Q})}{\lambda_{\min}(Q)\lambda_{\min}(Q_o)}, \tag{3.17b}$$

which completes the proof. □

To include the use of an observer, we now propose to modify *Step 1* of Control Algorithm 2.1, as follows: Let $\delta \in (0, 0.5)$, let

$$\rho_{LQR}^{o} \in \left[0, \min\{\rho_{LQR}, \frac{\lambda_{\min}(M)\lambda_{\max}(Q)(1-2\delta)\rho_{LQR}}{2\gamma\lambda_{\min}(Q)(\|K_c C Q\|_2+\delta\lambda_{\min}(M))}\}\right], \tag{3.18a}$$

where ρ_{LQR} was defined at the beginning of this subsection and γ was defined in (3.16). Let $\varepsilon_1, \varepsilon_2 > 0$ be such that

$$\varepsilon_1 < \hat{\varepsilon}_1, \tag{3.18b}$$

$$\varepsilon_2 < \min\{\hat{\varepsilon}_2, \frac{\rho_{LQR}^{o}(1-\varepsilon_1)}{2}\}, \tag{3.18c}$$

where K was defined in (3.1c), and $\hat{\varepsilon}_1, \hat{\varepsilon}_2$ were defined in (3.2d), (3.2o), respectively. Finally, let $\rho_{oc} > 0$ be defined by

$$\rho_{oc} \triangleq (1-\varepsilon_1)\left[\rho_{LQR}^{o} - \frac{\varepsilon_2}{1-\varepsilon_1}\right]. \tag{3.18d}$$

Then, it follows from (3.18c) that $\rho_{oc} > (1-\varepsilon_1)(\rho_{LQR}^{o} - \rho_{LQR}^{o}/2) > \varepsilon_2$. Let $T_{K_c} \in [T_C, \infty)$ be such that

$$e^{-\lambda_{\min}(\tilde{M})(1-2\delta)T_{K_c}/\lambda_{\max}(\tilde{Q})} \le \frac{\rho_{oc}^2\lambda_{\min}(Q)}{2\lambda_{\max}(\tilde{Q})(\rho_{oc}^2+(\rho_{LQR}^{o})^2)}, \tag{3.18e}$$

$$\|e^{(A-BK_c)T_{K_c}}\| \le \alpha. \tag{3.18f}$$

Finally, we define the vector valued saturation function $SAT(u) \triangleq (sat(u^1), \ldots, sat(u^m))$, where $sat(y) = y$ if $y \in [-c_u, c_u]$, and $sat(y) = c_u sgn(y)$ otherwise.

Step 1'': At $t = t_k$,

(a) If $u(t) = -K_c x^o(t)$ for $t \in [t_{k-1}, t_k)$ and $\max\{\|\tilde{x}_{k-1}\|, \|x_k\|\} \le \rho_{oc}$, set $\tilde{x}_k = x^o(t_k)$; else if $\max\{\|\tilde{x}_{k-1}\|, \|x_k\|\} \le \rho_{oc}$, set $\tilde{x}_k = x_k$ and reinitialize the observer by setting $x^o(t_k) = x_k$, else estimate the state $x_k^p = x^p(t_k, t_0, x_0^p, u)$ by (3.5) and denote the resulting value by $\tilde{x}_k$.

(b) If $\max\{\|\tilde{x}_{k-1}\|, \|x_k\|\} > \rho_{oc}$, set the plant input $u(t) = u_{[t_k, t_{k+1}]}(t)$ for $t \in [t_k, t_{k+1})$; else reset t_{k+1} to the new value $t_{k+1} = t_k + T_{K_c}$, and set $u(t) = -SAT(K_c x^o(t))$ for $t \in [t_k, t_{k+1})$.

(c) Compute an estimate x_{k+1} of the state of the plant $x^p(t_{k+1}, t_k, \tilde{x}_k, u)$ according (2.4),

i.e.,

$$x_{k+1} = e^{A(t_{k+1}-t_k)}\tilde{x}_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t)} B\, u(t)\, dt\,. \qquad \square$$

Lemmas 3.6 and 3.7 lead us to a following result.

Theorem 3.8. Suppose that *(a)* $\Delta\tilde{A}$ satisfies the condition in Lemma 3.6, *(b)* $\delta \in (0, 0.5)$, $\| K_c \Delta C Q \| \le \delta \lambda_{\min}(M)$ $\Delta_5 \le \varepsilon_1$, $\Delta_6 \le \varepsilon_2$, $\rho_{MH} < (\rho_{oc} - \varepsilon_2)/(1+\varepsilon_1)$, where ε_1 and ε_2 satisfy (3.18b) and (3.19c), respectively, and ρ_{MH} was defined in (3.11), and *(c)* that we use *Step 1''* in Control Algorithm 2.1. Then for any $x_0^p \in B_{\rho_o}$, defined in (3.9d), the trajectory $x^p(t, 0, x_0^p, u)$ is bounded and, furthermore, $x^p(t, 0, x_0^p, u) \to 0$ as $t \to \infty$.

Proof. We will prove that for any trajectory $x^p(t, 0, x_0^p, u)$, with $x_0^p \in B_{\rho_o}$, there must exist a $\hat{k}$ such that the control $u(t)$ is defined by the solution of the optimal control problem $P(x_k, t_k)$ for all $t \in [0, t_{\hat{k}})$ and $\max\{ \|\tilde{x}_{\hat{k}-1}\|, \|x_{\hat{k}}\| \} \le \rho_{oc}$, i.e., that the switch, in *Step 1'' (c)*, to the linear feedback control law $u(t) = -K_c x^o(t)$, with $(x^p(t), x^o(t))$ the solution of (3.13a,b), from the initial state $(x^p(t_{\hat{k}}), x_{\hat{k}})$ at $t = t_{\hat{k}}$, will take place. Then we will show that *(a)* $x^0(t) \in B_{LQR}$ for all $t \ge t_{\hat{k}}$ so that the linear feedback control law does not violate the bound on the control, and *(b)* that $\max\{ \|\tilde{x}_{k-1}\|, \|x_k\| \} \le \rho_{oc}$ must hold for all $k \ge \hat{k}$, so that the linear law is used for all $t \ge t_{\hat{k}}$. It will then follow from Lemma 3.6 that state of the plant will be driven to the origin as $t \to \infty$.

First, it follows from (3.6a,b) that if the control $u(t) = u_{[t_k, t_{k+1})}(t)$ and the times t_k are determined by solving the optimal control problem $P(x_k, t_k)$ for all $k \in \mathbb{N}$, then

$$\|\tilde{x}_{k-1}\| \le \|x_{k-1}^p - \tilde{x}_{k-1}\| + \|x_{k-1}^p\| \le (\Delta_3 + 1)\|x_{k-1}^p\| + \Delta_4\,, \tag{3.19a}$$

$$\|x_k\| \le \|x_k^p - x_k\| + \|x_k^p\| \le \Delta_5 \|x_{k-1}^p\| + \Delta_6 + \|x_k^p\|\,. \tag{3.19b}$$

Next, because $\Delta_3 \le \Delta_5 \le \varepsilon_1$, $\Delta_4 \le \Delta_6 \le \varepsilon_2$, and $\rho_{MH} < (\rho_{oc} - \varepsilon_2)/(1+\varepsilon_1)$, since it follows from (3.2i,j) and (3.18d) that $\varepsilon'' < \rho_{MH}$, we conclude that $\overline{\lim}_{k\to\infty} \|x_k^p\| \le \rho_{MH}$. Hence there exists a $\hat{k} \in \mathbb{N}$ such that $\|\tilde{x}_{\hat{k}-1}\| \le \rho_{oc}$ and $\|x_{\hat{k}}\| \le \rho_{oc}$. Hence a switch to the linear feedback control law will take place at the time $t_{\hat{k}}$.

Next, we will prove that $x^o(t) \in B_{LQR}$ for all $t \ge t_{\hat{k}}$, where $t_{\hat{k}}$ is the time when the switch to the linear control feedback control law takes place. Now, it follows (3.6a) and (3.7d,e,f) that

$$\|x^p_{\hat{k}-1}\| \le \|x^p_{\hat{k}-1} - \tilde{x}_{\hat{k}-1}\| + \|\tilde{x}_{\hat{k}-1}\| \le \varepsilon_1 \|x^p_{\hat{k}-1}\| + \varepsilon_2 + \|\tilde{x}_{\hat{k}-1}\| . \tag{3.19c}$$

From (3.6b), we obtain that

$$\|x^p_{\hat{k}}\| \le \|x^p_{\hat{k}} - x_{\hat{k}}\| + \|x_{\hat{k}}\| \le \varepsilon_1 \|x^p_{\hat{k}-1}\| + \varepsilon_2 + \|x_{\hat{k}}\| . \tag{3.19d}$$

It follows from (3.19c) that $\|x_{\hat{k}-1}\| \le \|\tilde{x}_{\hat{k}-1}\|/(1-\varepsilon_1) + \varepsilon_2/(1-\varepsilon_1)$. Hence it follows from (3.18d), (3.19d), and the fact that $\|\tilde{x}_{\hat{k}-1}\|, \|x_{\hat{k}}\| \le \rho_{oc}$ that

$$\|x^p_{\hat{k}}\| \le \frac{\varepsilon_1}{1-\varepsilon_1}(\varepsilon_2 + \rho_{oc}) + \varepsilon_2 + \rho_{oc} = \rho^o_{LQR} . \tag{3.19e}$$

By *Step 1'' (a)*, we reinitialize the observer by setting $x^o(t_{\hat{k}}) = x_{\hat{k}}$ and hence $\|x^o(t_{\hat{k}})\| \le \rho_{oc} \le \rho^o_{LQR}$.

Now suppose that linear feedback control law is used for all $t \ge t_{\hat{k}}$. Then it follows from (3.19e) and Lemma 3.7 that $\|e(t)\| \le \gamma \rho^o_{LQR}$ for all $t \ge t_{\hat{k}}$. Next, let the Lyapunov function $V(\cdot)$ be defined by $V(x^o(t)) \triangleq \|x^o(t)\|^2 \triangleq \langle x^o(t), Qx^o(t) \rangle$. Then, making use of the matrix M defined by (3.11), we obtain that for all $t \ge t_{\hat{k}}$, with $(x^p(t), x^o(t))$ a solution of (3.13a,b) with initial states $(x^p(t_{\hat{k}}), x_{\hat{k}})$,

$$\begin{aligned}
\dot{V}(x^o(t)) \le & -\lambda_{\min}(M)\|x^o(t)\|_2^2 + 2\|K_c \Delta C Q\|_2 \|x^o(t)\|_2^2 \\
& + \Big[2\|e(t)\|(\|K_c C Q\| \, | \mathit{sub} 2 + \|K_c \Delta C Q\|_2)\|x^o(t)\| \Big] / \lambda_{\max}(Q) \\
\le & -\lambda_{\min}(M)(1-2\delta)\|x^o(t)\|^2 / \lambda_{\min}(Q) \\
& + 2\gamma \rho^o_{LQR}(\|K_c C Q\|_2 + \delta \lambda_{\min}(M))\|x^o(t)\| / \lambda_{\max}(Q) .
\end{aligned} \tag{3.19f}$$

It follows from (3.18a) that if $\|x^o(t)\| > \rho_{LQR}$, then $\dot{V}(x^o(t)) < 0$. Since $\|x^o(t_{\hat{k}})\| \le \rho_{oc} \le \rho_{LQR}$, it follows that $x^o(t) \in B_{LQR}$ for all $t \ge t_{\hat{k}}$. Therefore, if the linear feedback control law is used for all $t \ge t_{\hat{k}}$, then it does not violate the bound on the control.

We will now prove by induction that $\|x_{k+1}\|, \|\tilde{x}_k\| \le \rho_{oc}$, for all $k \ge \hat{k}$, where x_{k+1} is computed by (2.4) and $\tilde{x}_k = x^o(t_k)$, with $(x^p(t), x^o(t))$ a solution of (3.13a,b) from the initial state $(x^p(t_{\hat{k}}), x_{\hat{k}})$. For $(x^p(t), x^o(t), e(t))$ a solution of (3.13a,b,c), let $z(t) \triangleq (x^p(t), x^o(t), e(t))$, and let $\|z(t)\|_Q^2 \triangleq \|e(t)\|^2+\|x^p(t)\|^2+\|x^o(t)\|^2$. Recall that $\|x_{\hat{k}}\|, \|\tilde{x}_{\hat{k}-1}\| \leq \rho_{oc}$, and that $x^o(t_{\hat{k}}) = x_{\hat{k}}$, and that $\|x_{\hat{k}}^p\| \leq \rho\ell_{QR}$ by (3.19e). Now suppose that for some $k \geq \hat{k}+1$, we have that $\|x_k\|, \|\tilde{x}_{k-1}\|, \|x^o(t_k)\| \leq \rho_{oc}$, and $\|x_k^p\| \leq \rho\ell_{QR}$ hold, and that $u(t) = -K_c x^o(t)$ for $t \in [t_{k-1}, t_k)$. We need to prove that $\|x_{k+1}\|, \|\tilde{x}_k\|, \|x^o(t_{k+1})\| \leq \rho_{oc}$ and that $\|x_{k+1}^p\| \leq \rho\ell_{QR}$. Now, since $u(t) = -K_c x^o(t)$ for $t \in [t_{k-1}, t_k)$, we set $\tilde{x}_k = x^o(t_k)$ by *Step 1'' (a)*. Therefore, $\|\tilde{x}_k\| = \|x^o(t_k)\| \leq \rho_{oc}$ by assumption. Next, we must have that $\|x_{k+1}\| \leq \alpha\rho_{oc}$ because $\|e^{(A-BK_c)T_{K_c}}\| \leq \alpha$. We will now prove that the relations $\|x^o(t_{k+1})\| \leq \rho_{oc}$ and $\|x_{k+1}^p\| \leq \rho\ell_{QR}$, both hold.

Let $\tilde{V}(z(t)) = \langle z(t), \tilde{Q}z(t)\rangle$. Then,

$$\tilde{V}(z(t)) \geq \lambda_{\min}(\tilde{Q})\|z(t)\|_2^2 \geq (\lambda_{\min}(\tilde{Q})/\lambda_{\max}(Q))\|z(t)\|_Q^2 \geq (\lambda_{\min}(\tilde{Q})/\lambda_{\max}(Q))\|x^o(t)\|^2. \tag{3.19g}$$

It follows from (3.15) and the fact that $\tilde{V}(z(t)) \leq \lambda_{\max}(\tilde{Q})\|z(t)\|_2^2$ that for $t \in [t_k, t_{k+1})$,

$$\frac{d}{dt}\tilde{V}(z(t)) \leq -\lambda_{\min}(\tilde{M})(1-2\delta)\|z(t)\|_2^2 \leq -(\lambda_{\min}(\tilde{M})/\lambda_{\max}(\tilde{Q}))(1-2\delta)\tilde{V}(z(t)). \tag{3.19h}$$

Clearly, $\tilde{V}(z(t_k)) \leq \lambda_{\max}(\tilde{Q})\|z(t_k)\|_2^2 \leq \lambda_{\max}(\tilde{Q})\|z(t_k)\|_Q^2/\lambda_{\min}(Q)$. Hence, because *(i)* $\|e(t_k)\|^2 \leq \|x^o(t_k)\|^2+\|x^p(t_k)\|^2$, *(ii)* $\|x^o(t_k)\| \leq \rho_{oc}$ and $\|x^p(t_k)\| = \|x_k^p\| \leq \rho\ell_{QR}$ by assumption, and *(iii)* $\|z(t_k)\|_Q^2 \leq 2(\|x^o(t_k)\|^2+\|x^p(t_k)\|^2)$, it follows from (3.19h) that for all $t \in [t_k, t_{k+1})$,

$$\tilde{V}(z(t)) \leq e^{-\lambda_{\min}(\tilde{M})(1-2\delta)(t-t_k)/\lambda_{\max}(\tilde{Q})}\tilde{V}(z(t_k))$$

$$\leq \frac{2\lambda_{\max}(\tilde{Q})}{\lambda_{\min}(Q)} e^{-\lambda_{\min}(\tilde{M})(1-2\delta)(t-t_k)/\lambda_{\max}(\tilde{Q})}\left[\rho_{oc}^2+(\rho\ell_{QR})^2\right]. \tag{3.19i}$$

Since by the triangle inequality, $V(z(t)) \geq 2\|x^o(t)\|^2$ for all $t \in [t_k, t_{k+1})$, it follows from (3.18e) and (3.19g,i) that

$$\|x^o(t_k+T_{K_c})\|^2 \triangleq \|x^o(t_{k+1})\|^2 \leq \rho_{oc}^2/2. \tag{3.19j}$$

Therefore, $\|x^o(t_{k+1})\| \leq \rho_{oc}/\sqrt{2}$. Now, (3.19g) holds when we replace $x^o(t)$ by $x^p(t)$ because $\|z(t)\|_Q^2 \geq \|x^p(t)\|^2$. Then, again it follows from (3.19h,i) that $\|x^p(t_k+T_{K_c})\|^2 = \|x_{k+1}^p\|^2 \leq \rho_{oc}^2 \leq (\rho\ell_{QR})^2$, which completes our proof by induction. It therefore follows that the Control Algorithm 2.1 selects the feedback control law $u(t) = -SAT(K_c x^o(t))$, for the next interval, $t \in [t_{k+1}, t_{k+1}+T_{K_c}]$, where $t_{k+1} = t_k+T_{K_c}$, and since we have already shown that, in this case, the control $u(t) = -K_c x^o(t)$ does not violate the control constraint, it follows that $u(t) = -K_c x^o(t)$, for the next interval, $t \in [t_{k+1}, t_{k+1}+T_{K_c}]$, and hence, by induction, for all $t \geq t_{\hat{k}}$.

It now follows from Lemma 3.6 that $\|x^p(t)\| \to 0$ and $\|x^o(t)\| \to 0$ as $t \to \infty$, which completes our proof. □

4. NUMERICAL RESULTS.

We will now present three examples that illustrate the performance of the moving horizon control system based on Control Algorithm 2.1, for a plant modeled by the state equations

$$\dot{x}(t) = \begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{4.1a}$$

where $u \in U \triangleq \{ u \in L_\infty[0, \infty) \mid \|u\|_\infty \le 1 \}$. Control Algorithm 2.1 used the following optimal control problem:

$$\mathrm{P}(x_k, t_k): \ \min_{u \in U} \tfrac{1}{2} \int_{t_k}^{\tau} (\langle x(t, t_k, x_k, u), Rx(t, t_k, x_k, u) \rangle + \langle u(t), Su(t) \rangle)\, dt \tag{4.1b}$$

subject to

$$\|x(\tau, t_k, x_k, u)\|^2 - 0.01 \|x_k\|^2 \le 0, \tag{4.1c}$$

$$\|x(t, t_k, x_k, u)\|^2 - 100 \|x_k\|^2 \le 0, \ \forall t \in [t_k, \tau], \tag{4.1d}$$

where $\tau \in [t_k + T_C, t_k + \bar{T}]$, $T_C = 5$, $\bar{T} = 40$,

$$R = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.1e}$$

and $S = 2000$.

For comparison, we used the example given in [Gut.1], which has only a control constraint. Since the initial state was known, we solved the optimal control problem $\mathrm{P}(x_0, 0)$ off-line to obtain the initial control $u(t)$, $t \in [0, t_1]$.

Example 4.1. In this example we have assumed that the state can be measured and that there are no modeling errors. This is the case presented in [Gut.1], where a piecewise linear control law was used, defined by

$$u(t) = sat\, [(L - k[0\ 1]P)x(t)], \tag{4.2}$$

where $L = -0.78 \times 10^{-3} \times [4.47\ \ 94.61]$, $k = 0.5 \times 10^{-5}$,

$$P = \begin{bmatrix} 171 & 1433 \\ 1433 & 19435 \end{bmatrix},$$

and $sat(\cdot)$ is the standard saturation function. The matrix L was obtained by solving a Linear Quadratic Regulator problem and P is a correction matrix. Figure 1 shows the resulting trajectories, using both our strategy and the one in [Gut.1] for $t \in [0, 60)$ and $x_0 = [10 \ \ 10]$.

As we can see from the Figure 1, the trajectory generated by Control Algorithm 2.1 converges to the origin faster than the trajectory given by [Gut.1]. The controls for both cases are shown in Figure 2.

Example 4.2. Next, we have again assumed that the state can be measured, but that there are modeling errors, viz. we assumed that the actual plant dynamics were

$$\dot{x}^p(t) = \begin{bmatrix} 0.01 & 1 \\ 0 & 0.01 \end{bmatrix} x(t) + \begin{bmatrix} 0.01 \\ 0.99 \end{bmatrix} u(t), \tag{4.3}$$

while the model was as in (4.1a). For the initial state given in Example 4.1, in Figure 3, we compare the trajectory, $x^p(t, 0, x_0^p, u)$, obtained by applying the control given in [Gut.1] with the trajectory generated by Control Algorithm 2.1. Again, the trajectory generated by Control Algorithm 2.1 converges to the origin faster. The controls for both cases are shown in Figure 4.

Example 4.3. In this example, we consider the case where there are modeling errors and the state has to be estimated. Thus, we assumed that the plant was described by

$$\dot{x}^p(t) = \begin{bmatrix} 0.002 & 1 \\ 0 & 0.003 \end{bmatrix} x(t) + \begin{bmatrix} 0.002 \\ 0.99 \end{bmatrix} u(t), \tag{4.4a}$$

$$y^p(t) = [0.99 \ \ 0.005] x(t), \tag{4.4b}$$

with $x_0^p = [5 \ \ 5]$. The plant was modeled by the equations

$$\dot{x}(t) = \begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{4.4c}$$

$$y(t) = [1 \ \ 0] x(t), \tag{4.4d}$$

with $x_0 = [2 \ \ 2]$.

We applied Control Algorithm 2.1 and the resulting trajectory, $x^p(t, 0, x_0^p, u)$, and control $u(t)$, $t \in [0, 100]$, are shown in Figures 5 and 6, respectively.

5. CONCLUSION.

In this paper we have explored the stability robustness properties of a moving horizon feedback system, based on constrained optimal control algorithms. As a starting point for a more extensive exploration to follow, we have assumed that the plant has the simplest possible nonlinearity, namely, input saturation. We have shown that the particular moving horizon control scheme that we proposed, results in a robustly stable system, both when the state of the plant is measurable and when it must be estimated. Our experimental results show that the behavior of our moving horizon control system is superior to that resulting from one alternative control law. While the time needed for the solution of the optimal control problems defining our control law is nontrivial, it should be acceptable in controlling slow moving plants, such as in process control. For faster plants, it may be necessary to implement the optimal control algorithms in a dedicated architecture, so as to speed up the solution times.

6. APPENDIX I.

We will now establish two inequalities that form the basis of several of our proofs.

Proposition 6.1 Consider the second order scalar difference equation

$$y_{k+2} = a_1 y_{k+1} + a_2 y_k + b \ , \quad k \in \mathbf{N} . \tag{6.1a}$$

If $a_1, a_2 \geq 0$, $b \geq 0$ and $a_1 + a_2 < 1$, then for all $k \geq 1$,

$$y_k \leq a_2 y_0 + y_1 + b/(1 - a_1 + a_2) , \tag{6.1b}$$

and

$$\overline{\lim}_{k \to \infty} y_k \leq b/(1 - a_1 + a_2) . \tag{6.1c}$$

Proof. We begin by rewriting (6.1a) in first order vector form, as follows. For $k \in \mathbf{N}$, let $z_k = (y_k, y_{k+1})^T$. Then $z_0 = (y_0, y_1)^T$, and

$$z_{k+1} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} z_k + \begin{bmatrix} 0 \\ b \end{bmatrix} \triangleq F z_k + g \ , \tag{6.2a}$$

$$y_k = [1\ 0] z_k \triangleq H z_k . \tag{6.2b}$$

The matrix F has two eigenvalues, $\lambda_+, \lambda_- = ½(a_1 \pm \sqrt{a_1^2 + 4a_2})$, with corresponding eigenvectors, $e_+ = (1, \lambda_+)^T$ and $e_1 = (1, \lambda_-)^T$. We will now show that $-1 < \lambda_- \leq 0 \leq \lambda_+ < 1$, i.e., that

(6.2a) is an asymptotically stable system. By assumption

$$0 \le a_2 < 1 - a_1 . \tag{6.2c}$$

If we multiply both sides Of (6.2c) by 4, and add a_1^2 to the both sides, we get that

$$a_1^2 + 4a_2 < (2 - a_1)^2 , \tag{6.2d}$$

which implies that $\lambda_- = \frac{1}{2}(a_1 - \sqrt{a_1^2 + 4a_2}) > -1$ and $\lambda_+ = \frac{1}{2}(a_1 + \sqrt{a_1^2 + 4a_2}) < 1$. Thus, we have that $-1 < \lambda_- \le \lambda_+ < 1$.

We can proceed to establish (6.1b,c). By the Jordan decomposition, we have that

$$F = E^{-1} \Lambda E , \tag{6.2e}$$

where $\Lambda = diag(\lambda_+, \lambda_-)$, and $E = (e_+, e_-)$ is a matrix whose columns are the eigenvectors of F. Hence for all $k \ge 2$,

$$y_k = HE^{-1}\Lambda^k E z_0$$

$$= \frac{1}{\lambda_- - \lambda_+} \{ \lambda_+\lambda_-(\lambda_+^{k-1} - \lambda_-^{k-1}) y_0 + (\lambda_-^k - \lambda_+^k) y_1 \} + \frac{b}{\lambda_- - \lambda_+} \sum_{i=0}^{k-1} (\lambda_-^{k-1-i} - \lambda_+^{k-1-i}) . \tag{6.2f}$$

Since $0 < \lambda_+ < 1$ and $-1 < \lambda_- < 0$, it is clear that *(a)* the first term in (6.2f) goes to zero as $k \to \infty$ and *(b)* the last term in (6.2f) satisfies the inequality

$$\frac{b}{\lambda_- - \lambda_+} \sum_{i=0}^{k-1} (\lambda_-^{k-1-i} - \lambda_+^{k-1-i}) \le \frac{b}{\lambda_- - \lambda_+} \left\{ \frac{1}{1 - \lambda_-} - \frac{1}{1 - \lambda_+} \right\} = \frac{b}{1 - a_1 + a_2} , \tag{6.2g}$$

because $(1 - \lambda_+)(1 - \lambda_-) = 1 - a_1 + a_2$, which proves (6.1c).

Next, for all $k \ge 1$, $\lambda_+^k \le \lambda_+$ and $-\lambda_-^k \le (-\lambda_-)^k \le -\lambda_-$ Hence $\{ \lambda_+\lambda_-(\lambda_+^{k-1} - \lambda_-^{k-1}) / (\lambda_- - \lambda_+) \le -\lambda_+\lambda_- = a_2$. Also $(\lambda_-^k - \lambda_+^k)/(\lambda_- - \lambda_+) \le 1$, hence (6.1b) hold. □

7. APPENDIX II.

The free-time optimal control problem (2.3a-e) has to be solved at every iteration of Control Algorithm 2.1. The major difficulty in solving this problem stems from the fact that functions such as $|x(t, 0, x_0, u)|^2$ that are convex in u, are not convex in t and hence optimal control algorithms, such as the phase I - phase II algorithms described in [Pol.2], can only be counted on to find local minima for this problem. This difficulty can be eliminated by solving a sequence of convex, fixed time optimal control problems, constructed using an interval bisection technique, whose solutions converge to the desired optimal solution of (2.3a-e), as follows. An important aspect of phase I - phase II algorithms, such as those in [Pol.2,3,4], is that when a fixed-time optimal control problem has no solution, then they produce a control which minimizes the maximum constraint violation.

Algorithm 7.1.

Data: $x_k \in B_{\hat{\rho}}$, t_k and $\bar{T}$ such that $\bar{T} - t_k > T_C$ and $\delta \in (0, \bar{T} - T_C - t_k)$.

Step 0: Set $i = 0$, $\tau_0 = \bar{T}$, $T_{\min} = t_k + T_C$, and $T_{\max} = \bar{T}$.

Step 1: Solve the problem (2.5a-e) with τ fixed at the value $\tau = \tau_i$.

Step 2: If the computed control, $u_i(\cdot)$, does not satisfy all the constraints in (2.3a-e),

$$\text{set} \begin{cases} T_{\min} = \tau_i\,,\ T_{\max} = 2\tau_i\,, \text{ and } \tau_{i+1} = T_{\max}\,, & \text{if } \tau_i = T_{\max} \\ T_{\min} = \tau_i \text{ and } \tau_{i+1} = (\tau_i + T_{\max})/2\,, & \text{otherwise.} \end{cases}$$

Else, set $T_{\max} = \tau_i$ and $\tau_{i+1} = (T_{\min} + \tau_i)/2$.

Step 3: If $(T_{\max} - T_{\min}) \le \delta$, set $t_{k+1} = \tau_{i+1} - T_1$, set $u_{[t_k, t_{k+1}]}(t) = u_i(t)$ for $t \in [t_k, t_{k+1}]$, and stop.

Else, set $i = i + 1$ and go to Step 1. □

Since by definition of $\hat{\rho}$, the original free-time optimal control problem has a solution, it is clear that Algorithm 7.1 terminates after a finite number of iterations.

8. ACKNOWLEDGEMENT

The authors are grateful to Prof. David Q. Mayne and Dr. Hannah Michalska for their helpful comments and suggestions.

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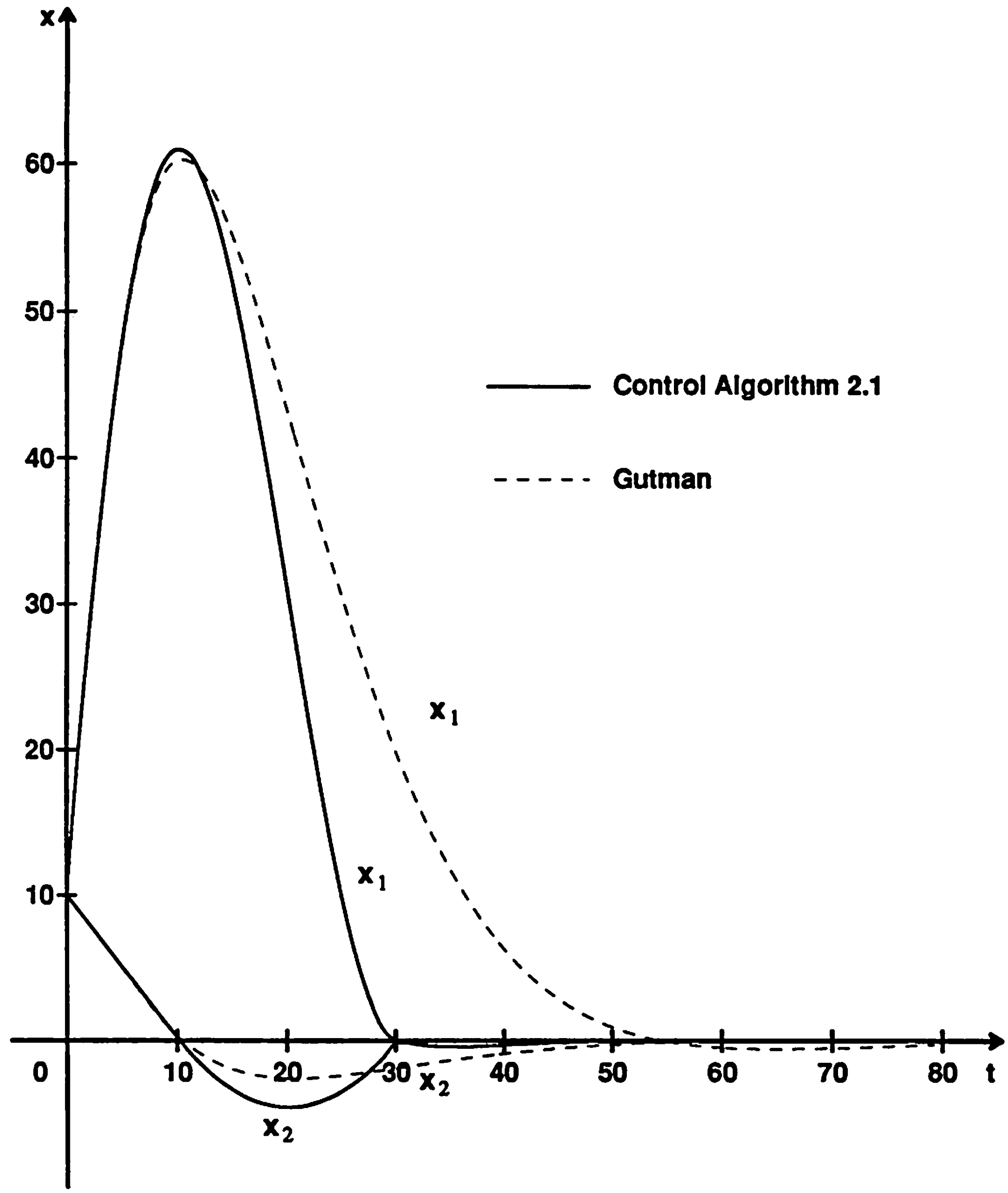

Figure 1. States vs Time for Example 4.1.

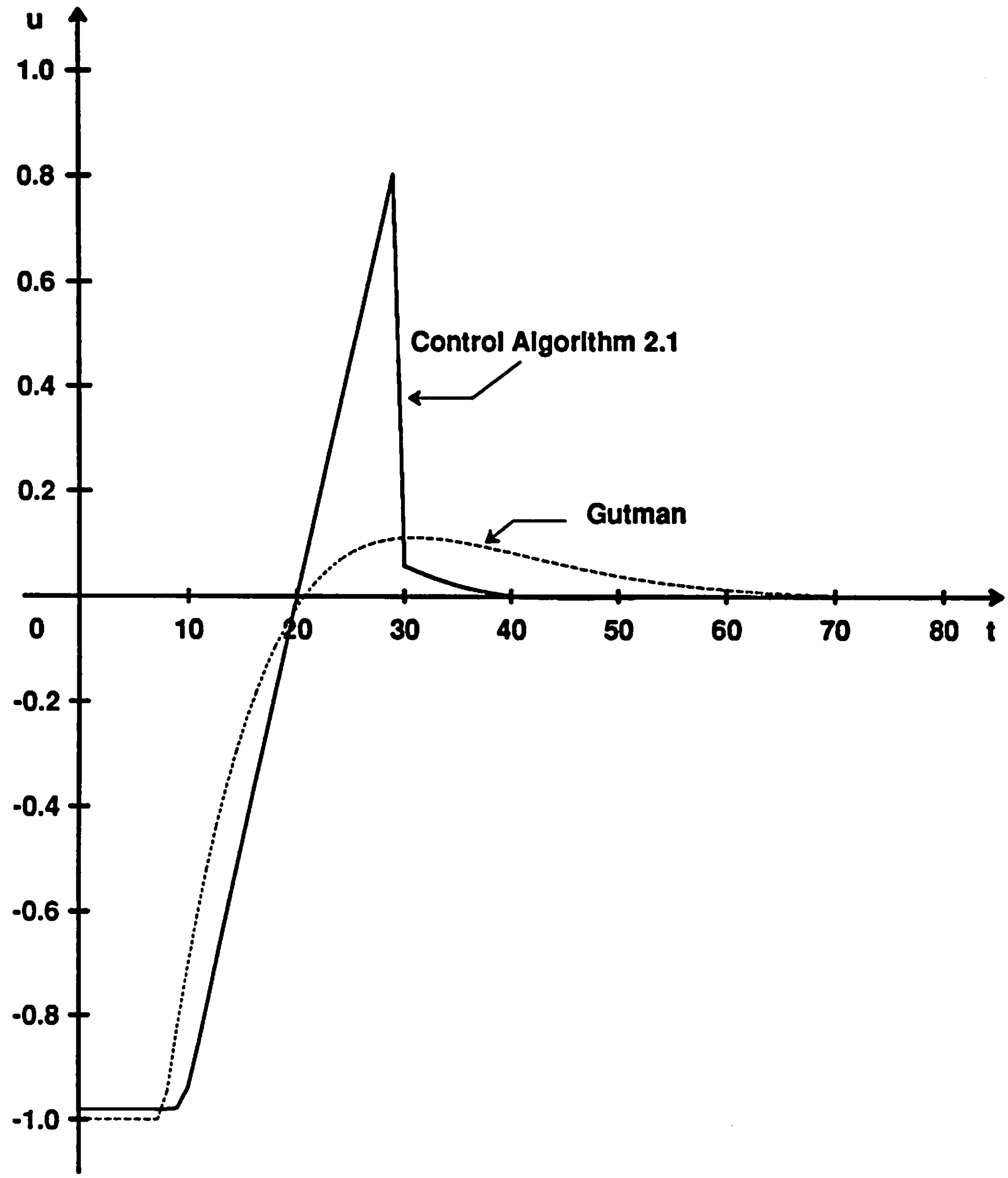

Figure 2. Controls vs Time for Example 4.1.

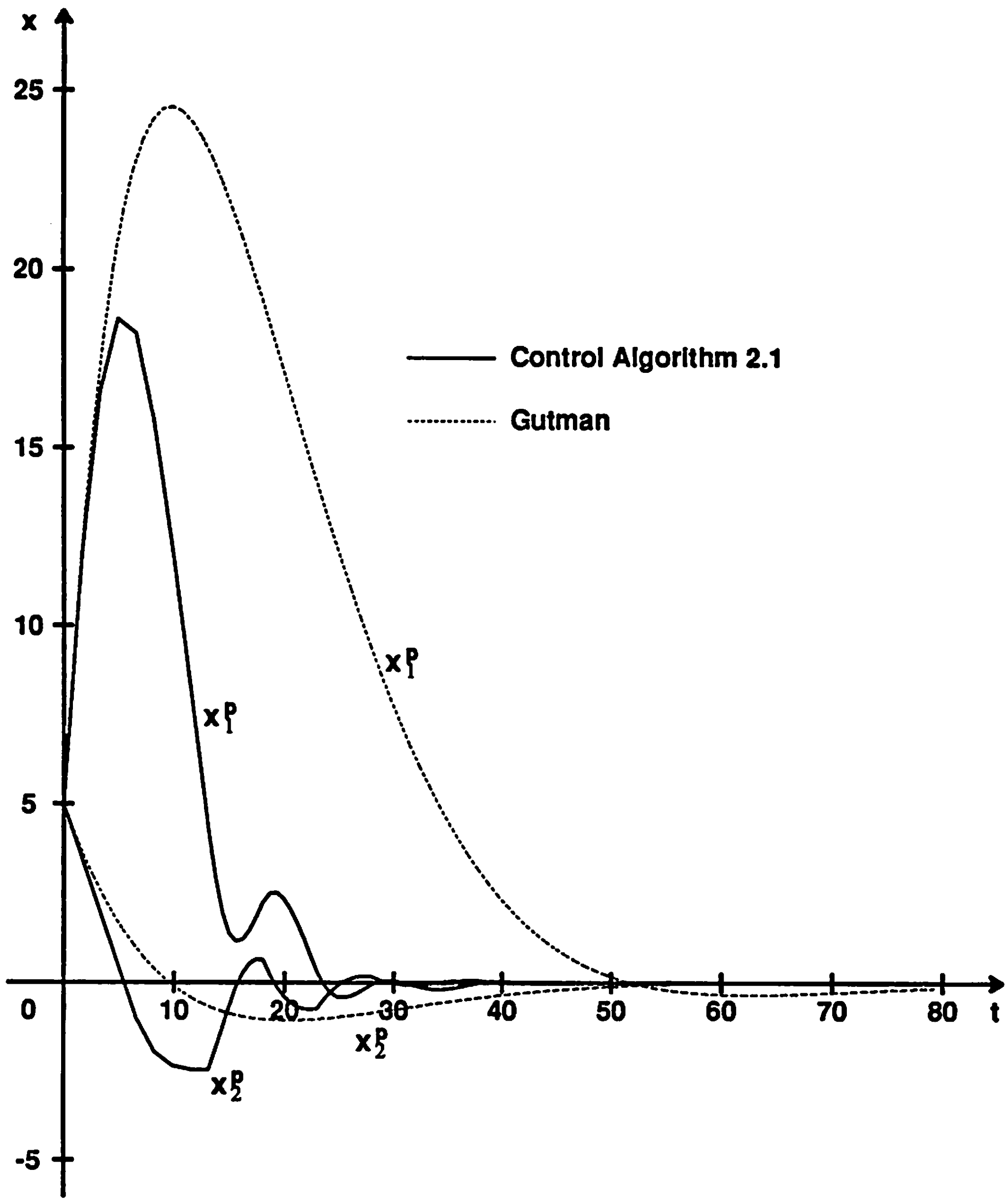

Figure 3. States vs Time with Perturbations for Example 4.2.

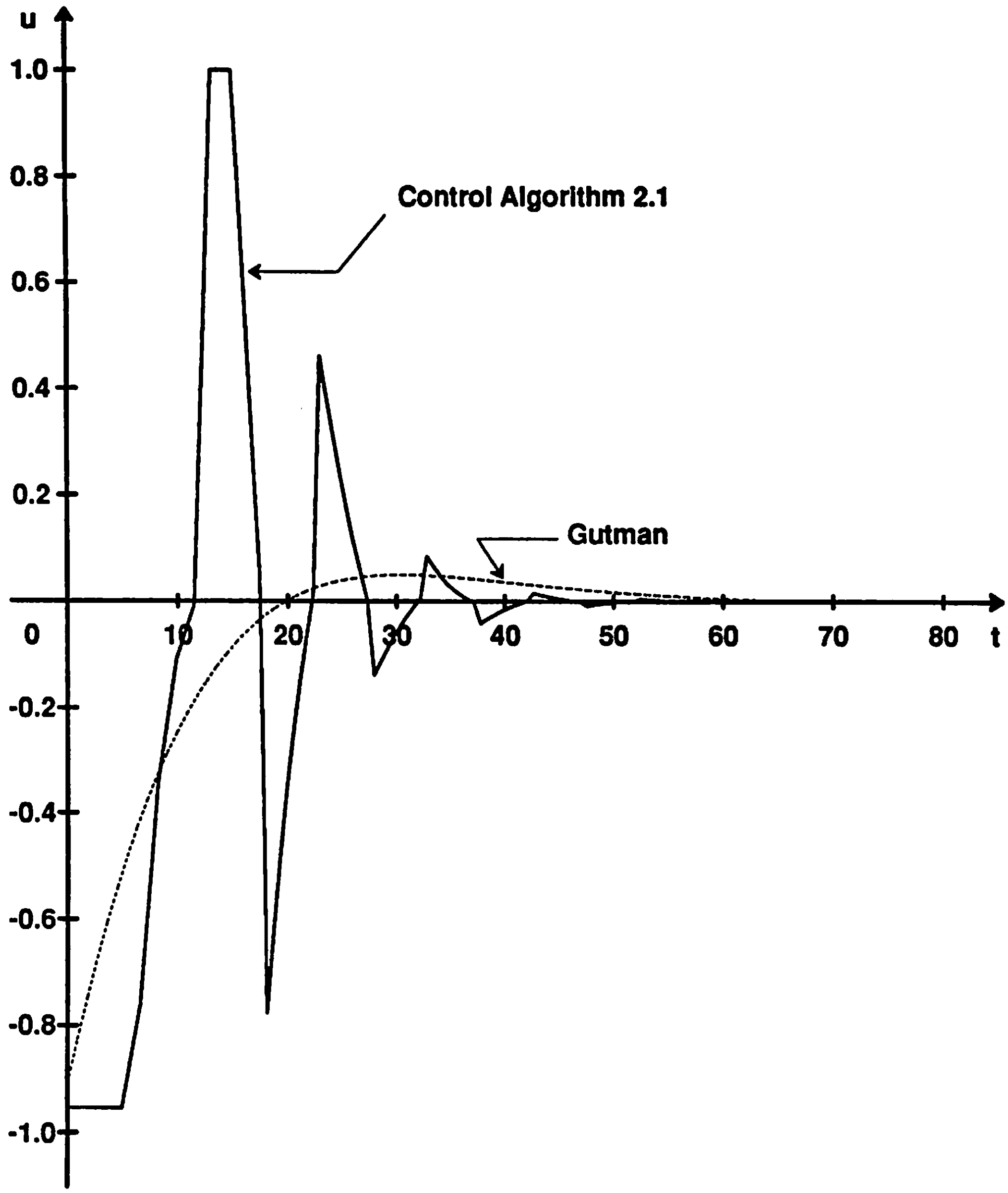

Figure 4. Controls vs Time with Perturbations for Example 4.2.

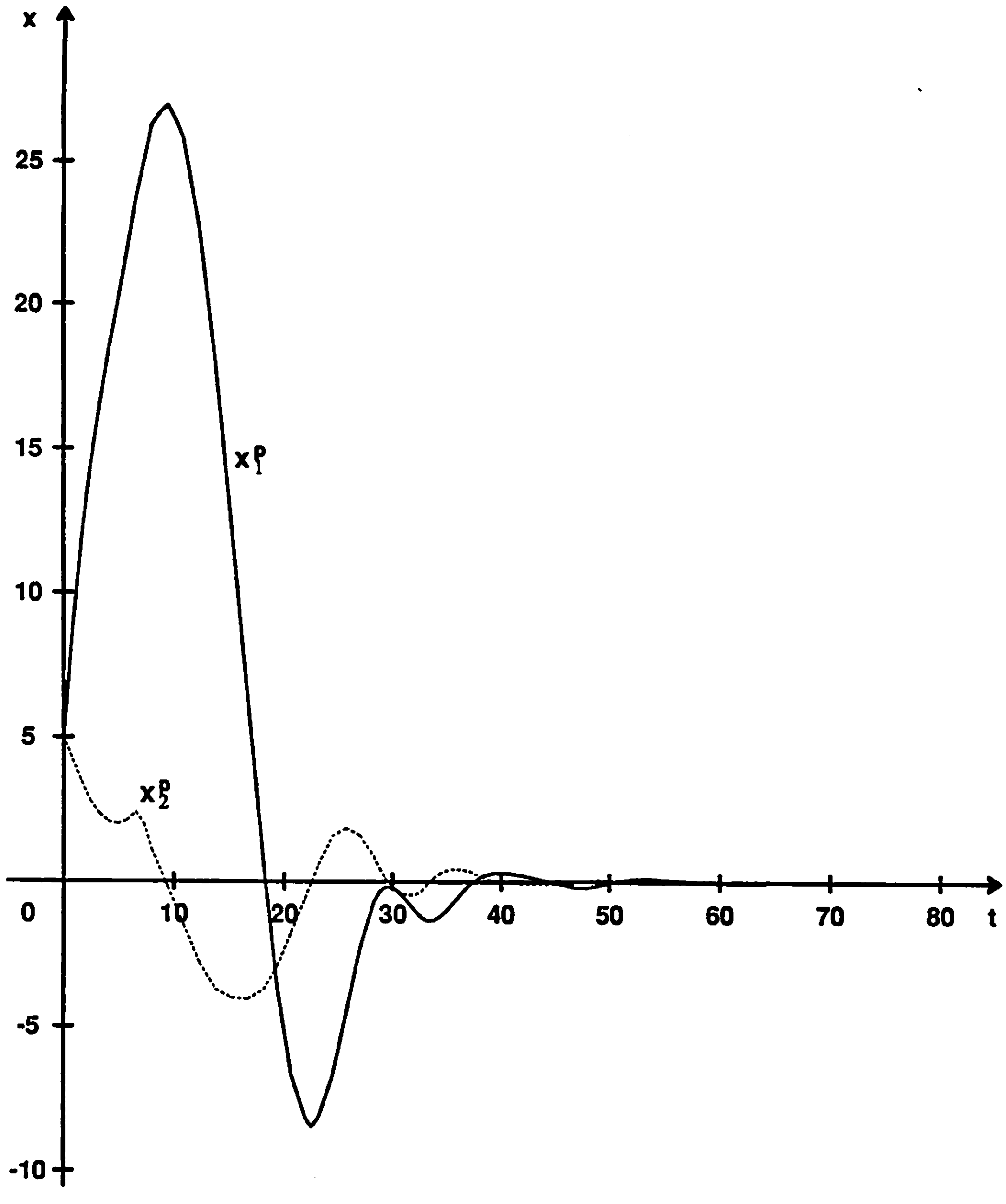

Figure 5. States vs Time with State Estimation for Example 4.3.

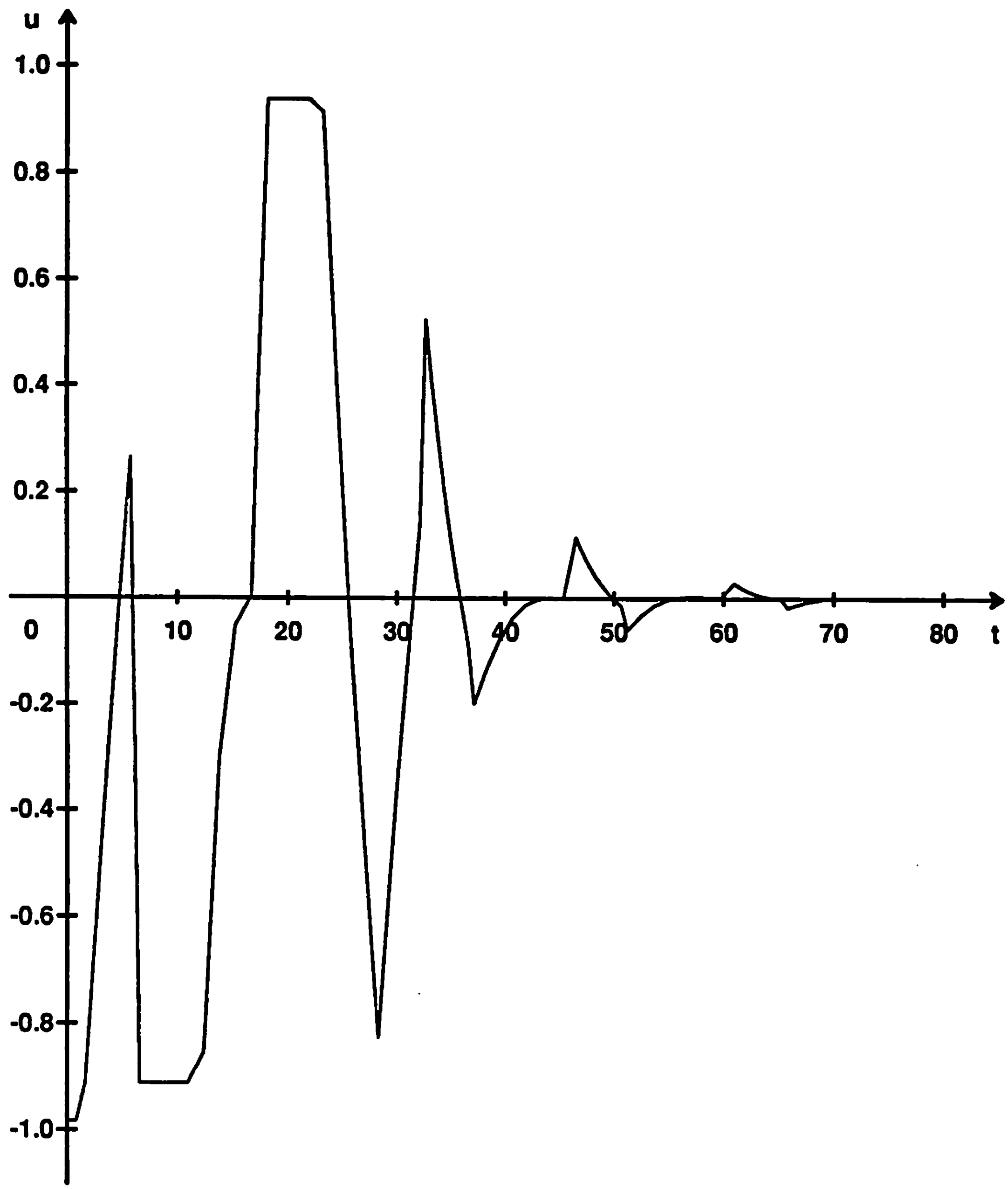

Figure 6. Controls vs Time with State Estimation for Example 4.3.