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1 Introduction

The current allocation of electric energy is based on a system of fixed prices. In such a system the gap between marginal cost of energy generation and the marginal value of energy consumption, hence the resulting inefficiency, is quite large [6]. One scheme that closes this gap is that of *spot pricing*, [7], [3], [2].

Spot pricing is impractical today because the necessary communications infrastructure is not yet in place. A more practical scheme might employ *future prices*: the power company announces prices a day (or week) in advance and consumers then have the lead time to adjust their demand. The announced future price would depend on forecasts of some of the determinants of supply (e.g. scheduled generator shutdown times) and demand (e.g. weather).

Future prices can more easily be implemented than spot prices, see [1]. However, since a price is announced in period 1 (now) for energy to be delivered and consumed in period 2 (later), and since significant unanticipated fluctuations in supply and demand can occur in the interim, some consumers will be rationed when the actual period 2 demand exceeds the supply. The model we develop in this paper recognizes the cost of rationing borne by frustrated consumers who have their electricity cut off.

Thus a future pricing scheme must take into account rationing loss, and it must ration on the basis of available information. Also there must be a balance between raising prices to reduce rationing-caused losses and lowering prices to increase welfare gains from increased consumption. The *interruptible service contracts* proposed here incorporate both aspects. These are contingent contracts that condition service on particular events or contingencies. A model for the market operation can be described as a two-step process as depicted in Figure 1. In this paper we assume the consumers are

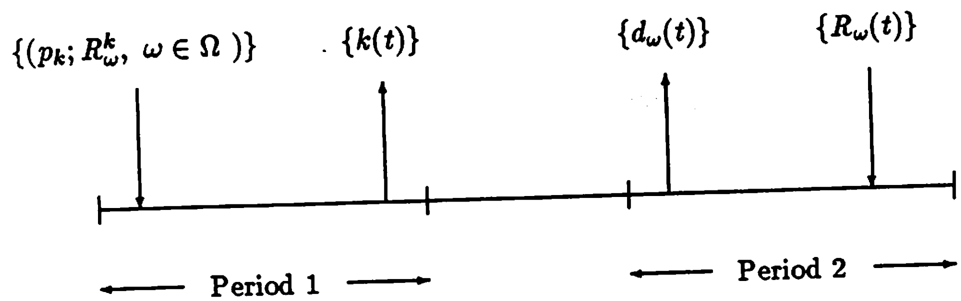

Figure 1: Market operation

identical, i.e. they have the same demand preferences. We also assume the supply is a fixed constant $s > 0$, but the consumer demand preferences are random with n possible sample points or *contingencies* $\omega \in \Omega$.

Step 1. At the beginning of period 1 the power company announces a set of service contracts $(p_k; R_\omega^k,\ \omega \in \Omega)$, $k = 1, 2, \cdots$. Each consumer t chooses one type of contract $(p_{k(t)}; R_\omega^{k(t)},\ \omega \in \Omega)$, where $p_{k(t)}$ is the price per kWh of energy consumption for consumer t. The function $R_\omega^{k(t)}$ is a 0-1 valued function of ω which specifies the contingencies under which the service will be interrupted. When contingency ω occurs the company will deliver the service if $R_\omega^{k(t)} = 1$, and the service will be interrupted if $R_\omega^{k(t)} = 0$. For example, we may have a contract which ensures no interruption of service if the outdoor temperature falls in the range of 70F to 90F, where the event that the outdoor temperature is in the range of 70F to 90F is one of the contingencies in Ω.

Step 2. At the beginning of period 2 consumer t observes the occurrence of a contingency, say ω. Consumer t then selects a quantity $d_\omega(t)$ kWh of energy. (We will see how this quantity is obtained in §2.) So consumer t pays $p_{k(t)}d_\omega(t)$ if $R_\omega^{k(t)} = 1$. Lastly, the power company has to decide which consumers to ration, if any, so that (i) the total energy delivered does not exceed the supply for each demand contingency, and (ii) each consumer's contract is fulfilled. The latter decision is represented by the 0-1 valued function $R_\omega(t)$. The company will deliver $d_\omega(t)$ to consumer t if $R_\omega(t) = 1$. If $R_\omega(t) = 0$ consumer t will not receive the service. Hence conditions (i) and (ii) are respectively given by:

$$\sum_t R_\omega(t) d_\omega(t) \leq s \quad \text{for all } \omega \tag{1}$$

$$R_\omega(t) = R_\omega^{k(t)} \quad \text{for all } t \text{ and for all } \omega \tag{2}$$

The outline of other sections is as follows. The optimal contracts are obtained by first formulating a welfare problem. We then show that the optimum of the welfare problem can be sustained by interruptible service contracts of the type described earlier. It turns out that the optimal solution requires a proper ordering of the demand contingencies. The welfare problem is formulated in §2. We obtain the structure of an optimal ordering of demand contingencies in §3. The optimal allocation is obtained for the special case when demand preferences are additive. This will be worked out in §4. It turns out that the prices for the contracts are conditional expectations of scarcity costs. Some concluding remarks are collected in §5.

2 Problem formulation

The structure of optimal contracts is obtained indirectly by first formulating a welfare maximization problem and then by showing that the optimum can be *sustained* by interruptible service contracts offered to consumers in a decentralized market. We assume the supply is a constant $s > 0$, the demand is random, and there is no variable supply cost.

We first model consumer welfare. The set of demand contingencies is denoted by Ω, and the cardinality of Ω is n. A consumer is characterized by her preference which consists of contingency dependent utility functions U_ω, $\omega \in \Omega$. It is standard to assume that U_ω are strictly concave functions with $U_\omega(0) = 0$. The demand of any individual consumer is assumed to be infinitesimal compared with the total demand of all consumers. This permits us to model the set of customers as a continuum indexed by $t \in [0, 1)$.[1] Consider a consumer who has chosen contract $(p; \hat{R}_\omega, \omega \in \Omega)$, where $\hat{R}_\omega$ is 0-1 valued. Suppose contingency ω is realized at the beginning of period 2. The consumer needs to decide her demand. If there is no service interruption,

[1] With this convention the total number of customers is 1, so the supplies s_i are measured in average kWh per customer.

i.e. $\hat{R}_\omega = 1$, the consumer's demand in period 2, upon the realization of contingency ω, is given by:

$$\max_{d \geq 0} U_\omega(d) - pd \tag{3}$$

Let $\phi_\omega(p) := \text{argmax}_{d \geq 0} U_\omega(d) - pd$ be the solution to problem (3). This is the consumer's demand curve if contingency ω occurs. It is a decreasing function since $\phi_\omega(p) = (U'_\omega)^{-1}(p)$, where U'_ω denotes the derivative of U_ω. Since the consumer is engaged in an interruptible service contract, her load demand $\phi_\omega(p)$ will be met only if $\hat{R}_\omega = 1$. If $\hat{R}_\omega = 0$, the consumer will not plan on making any demand since she knows she will not receive the service. Hence the consumer's decision on her demand depends on both the realization of random elements and the type of service interruption specified on her contract. Moreover, since the consumer decides her demand after the realization is observed, she suffers no loss when she knows she will not get the service because, in that circumstance, her optimal demand is zero. This is unlike the model studied in [6] where each contract specifies a reliability level but does not specify when the service will be interrupted. Therefore, in that model, a consumer suffers a loss term if her demand is interrupted.

Suppose consumer t is allocated the pair $(p(t); \hat{R}_\omega(t), \omega \in \Omega)$. Her welfare is given by

$$w(t) = \sum_{\omega \in \Omega} \pi_\omega \hat{R}_\omega(t) U_\omega(\phi_\omega(p(t))) \tag{4}$$

The total social welfare is the integral

$$W = \int_0^1 \sum_{\omega \in \Omega} \pi_\omega \hat{R}_\omega(t) U_\omega(\phi_\omega(p(t))) dt \tag{5}$$

Next we consider the allocation problem. In period 1 each t is allocated a pair $(p(t); \hat{R}_\omega(t), \omega \in \Omega)$. At the beginning of period 2 a contingency is revealed. Suppose it is ω. The power company now decides which, if any, consumers are to be rationed. This is given by a *rationing function* $R_\omega : [0, 1) \to \{0, 1\}$ defined as

$$R_\omega(t) = \begin{cases} 0 & \text{if } t \text{ is rationed in contingency } \omega \\ 1 & \text{otherwise} \end{cases}$$

The rationing function must satisfy the physical constraint

$$\int_0^1 R_\omega(t) \phi_\omega(p(t)) dt \leq s \quad \text{for all } \omega \tag{6}$$

which simply says that supply meets rationed demand. The rationing functions must also meet contracts, that is,

$$R_\omega(t) = \hat{R}_\omega(t) \quad \text{for all } t \text{ and for all } \omega \tag{7}$$

The welfare maximization problem is to find functions $p, R_\omega, \omega \in \Omega$ subject to constraints (6) and (7) so as to maximize the total social welfare W. It turns out that we need a two-part tariff so that the optimal solution can be sustained as an equilibrium in a decentralized market by no more than n types of interruptible service contracts. The welfare problem can be reformulated as an optimal control problem. To do this, we first define an *ordering* of the set of contingencies Ω. An ordering on Ω is an one-to-one correspondence $f : \Omega \to \{1, 2, \cdots, n\}$. Suppose Ω has been ordered such that the contingencies are indexed by $\{1, 2, \cdots, n\}$. Introduce the 'state' vector x and the 'control' vector z,

$$x(t) = (x_1(t), \cdots, x_n(t)), \quad z(t) = (p(t), R(t))$$

where

$$R(t) = (R_1(t), \cdots, R_n(t)), \; x_i(t) := \int_0^1 R_i(\tau)\phi_i(p(\tau))d\tau$$

Then the welfare problem can be reformulated as

$$\max \; W = \int_0^1 w(t)dt = \int_0^1 \sum_{i=1}^n \pi_i R_i(t) U_i(\phi_i(p(t)))dt \tag{8}$$

subject to

$$\dot{x}_i(t) = R_i(t)\phi_i(p(t)), \; t \in [0,1), \; i = 1, ..., n \tag{9}$$

$$x_i(0) = 0, \; x_i(1) \leq s, \; i = 1, ..., n \tag{10}$$

$$p(t) \geq 0 \;, \; R_i(t) \in \{0,1\} \tag{11}$$

This is a standard optimal control problem with state equations (9), state constraints (10), and control constraints (11). The Maximum Principle [4] gives necessary conditions for a solution of (8)-(11). However, we are interested in sufficiency which will be needed for contract design. For each $p \geq 0$, $R = (R_1, \cdots, R_n)$ with $R_i \in \{0,1\}$, and $\lambda = (\lambda_1, \cdots, \lambda_n)$ with $\lambda_i \geq 0$, define the *Hamiltonian*

$$H(p, R, \lambda) := \sum_{i=1}^n \pi_i R_i \{U_i(\phi_i(p)) - \lambda_i \phi_i(p)\} \tag{12}$$

The term $\pi_i\lambda_i$ is the *adjoint variable* associated with the supply constraint (10). It is the *scarcity cost* of an additional unit of capacity in contingency i. The following sufficiency theorem for optimality serves as the backbone for designing the optimal contracts.

Theorem 1 (Sufficiency conditions) *Suppose there exist $\lambda^* \in \Re^n_+$ and $H^* > 0$ such that for all $p \geq 0$ and $R_i \in \{0,1\}$,*

$$H(p, R, \lambda^*) \leq H^* \tag{13}$$

Then the maximum social welfare W^* satisfies

$$W^* = \max\ W \leq H^* + s(\sum_{i=1}^{n} \pi_i\lambda_i^*) \tag{14}$$

Moreover, if there is a feasible control $z^* = (p^*, R^*)$ which satisfies

$$H(p^*(t), R^*(t), \lambda^*) \equiv H^*\ ,\ t \in [0,1) \tag{15}$$

and

$$\lambda_i^*(s - \int_0^1 R_i^*(t)\phi_i(p^*(t))dt) = 0\ ,\ i = 1, \cdots, n \tag{16}$$

then this control is optimal.

Proof. Let z be any feasible control and x the corresponding trajectory. Let W be the welfare attained when control z is applied. From (8), (9), (12), we get

$$\begin{aligned} W &= \int_0^1 H(p(t), R(t), \lambda^*)dt + \int_0^1 \sum_{i=1}^{n} \pi_i\lambda_i^* R_i(t)\phi_i(p(t))dt \\ &\leq H^* + \sum_{i=1}^{n} \pi_i\lambda_i^* x_i(1) \\ &\leq H^* + s(\sum_{i=1}^{n} \pi_i\lambda_i^*) \end{aligned} \tag{17}$$

where the two inequalities in (17) follow from (13) and (10), respectively. The second part of the assertion follows since (15) and (16) yield equalities in (17). □

Thus an optimal solution z^* maximizes the Hamiltonian $H(p, R, \lambda^*)$ for each t. Condition (15) means that the net benefit is the same for all consumers. Condition (16) is the complementary slackness condition. It implies that at the prevailing prices the power company cannot increase its profit by offering a different set of contracts. Hence (15)-(16) are conditions for consumer equilibrium and supplier equilibrium.

To find the optimum of the welfare problem (8)-(11), we need to find λ^* and H^* that satisfy (15) and (16). This requires a proper ordering of the contingencies. We examine this question in §3. In §4 we proceed to show that the optimal solution of the welfare problem can be sustained by contracts of the form (p_i, R^i), where $R^i = (R^i_1, \cdots, R^i_n) \in \{0,1\}^n$, $i = 1, 2, \cdots, n$, is defined by

$$R^i_m = \begin{cases} 0 & \text{if } m < i \\ 1 & \text{if } m \geq i \end{cases} \tag{18}$$

Note that contract (p_i, R^i) guarantees a service reliability of $\sum_{m \geq i} \pi_m$.

3 Optimal ordering of demand contingencies

For each $\omega \in \Omega$ and $x \geq 0$, let

$$h_\omega(p, x) := U_\omega(\phi_\omega(p)) - x\phi_\omega(p) \tag{19}$$

Given a set of values $\{\lambda_\omega \geq 0; \omega \in \Omega\}$, we let p^0_ω be the solution to the algebraic equation

$$h_\omega(p, \lambda_\omega) = 0 \tag{20}$$

We also let

$$p^0_\omega := \begin{cases} 0 & \text{if } h_\omega(p, \lambda_\omega) > 0 \text{ for all } p > 0 \\ \infty & \text{if } h_\omega(p, \lambda_\omega) \leq 0 \text{ for all } p \geq 0 \end{cases} \tag{21}$$

Now if f is an ordering of Ω and $i = f(\omega)$, then $U_i := U_\omega$, $\phi_i := \phi_\omega$ and $p^0_i := p^0_\omega$. It follows from the strict concavity of U_ω that p^0_ω is well-defined. We have the following useful lemma.

Lemma 1 *Suppose* $0 < p^0_i < \infty$. *Then for any* $p \geq 0$,

$$p \leq p^0_i \Rightarrow \phi_i(p) \geq \phi_i(p^0_i) \Rightarrow h_i(p, \lambda_i) \leq 0 \tag{22}$$

$$p > p^0_i \Rightarrow \phi_i(p) < \phi_i(p^0_i) \Rightarrow h_i(p, \lambda_i) > 0 \tag{23}$$

Proof. The two implications in (22) follow from the decreasing property of ϕ_i and the strict concavity of U_i, respectively. The same argument applies to (23). □

The following lemma suggests a way to order the contingencies such that rationing functions of the form given by (18) maximize the Hamiltonian. It also implies that no more than n such contracts are needed at optimum.

Lemma 2 *Let $\{\lambda_\omega \geq 0; \omega \in \Omega\}$ be given. Obtain $\{p^0_\omega; \omega \in \Omega\}$ and order the contingencies such that*

$$p^0_1 \geq p^0_2 \geq \cdots \geq p^0_n \tag{24}$$

For the given $\lambda = (\lambda_1, \cdots, \lambda_n)$, let p^* be a maximizer of $\max_{p\geq 0} H(p, R, \lambda)$. Assume $H := \max_{R\in\{0,1\}^n} H(p^*, R, \lambda) > 0$. Then $\max_{R\in\{0,1\}^n} H(p^*, R, \lambda)$ is attained by an R^i.

Proof. We first note that $H(p^*, R, \lambda) = \sum_{i=1}^n \pi_i R_i h_i(p^*, \lambda_i)$ is linear in each R_i. So optimal R_i is 1 (respectively 0) if $h_i(p^*, \lambda_i)$ is positive (respectively negative). Let $n_1 := \min\{i \mid p^0_i < \infty\}$ and $n_2 := \max\{i \mid p^0_i > 0\}$. By Lemma 1 and since $p^0_1 \geq \cdots \geq p^0_n$, we get the following conclusions:
(i) If $p^* > p^0_{n_1}$, then $h_i(p^*, \lambda_i) > 0$ for all $i \geq n_1$ and R^{n_1} maximizes $H(p^*, R, \lambda)$.
(ii) If $p^0_j \geq p^* > p^0_{j+1}$ for some $n_1 \geq j \geq n_2 - 1$, then $h_i(p^*, \lambda_i) > 0$ for all $i \geq j+1$ and $h_i(p^*, \lambda_i) \leq 0$ for all $i < j+1$. So the maximum is achieved by R^{j+1}.
(iii) If $p^* \leq p^0_{n_2}$, then $h_i(p^*, \lambda_i) \leq 0$ for all $i \leq n_2$ and $h_i(p^*, \lambda_i) > 0$ for all $i > n_2$. The maximum is achieved by R^{n_2+1}. Note that we cannot have $n_2 = n$ and $p^* \leq p^0_n$ since $H > 0$. □

The next result shows that a contract which guarantees less frequent interruption should be sold at a higher price.

Lemma 3 *Let $\{\lambda_\omega \geq 0; \omega \in \Omega\}$ be given. Order the contingencies such that (24) holds. Obtain $\lambda = (\lambda_1, \cdots, \lambda_n)$ and let $p_i = \mathrm{argmax}_{p\geq 0} H(p, R^i, \lambda)$. Also let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ be such that*

$$H(p_{i_1}, R^{i_1}, \lambda) = \cdots = H(p_{i_k}, R^{i_k}, \lambda) =: H$$

and

$$H(p_i, R^i, \lambda) < H \text{ for } i \notin \{i_1, \cdots, i_k\}$$

Then

$$p_i = \frac{\sum_{j\geq i} \pi_j \phi'_j(p_i)\lambda_j}{\sum_{j\geq i} \pi_j \phi'_j(p_i)} \tag{25}$$

and $p_{i_1} \geq p_{i_2} \geq \cdots \geq p_{i_k}$.

Proof. Since $\max_{p\geq 0} H(p, R^i, \lambda) = \max_{p\geq 0} \sum_{j\geq i} \pi_j[U_j(\phi_j(p)) - \lambda_j\phi_j(p)]$ is maximized at p_i, we get

$$\sum_{j\geq i} \pi_j[U_j'(\phi_j(p_i))\phi_j'(p_i) - \lambda_j\phi_j'(p_i)] = 0 \tag{26}$$

by evaluating $\frac{\partial}{\partial p}H(p, R^i, \lambda) = 0$ at p_i. Now (25) follows from (26) since $U_j'(\phi_j(p_i)) = p_i$.

To see the second assertion we need to show that $p_{i_l} \geq p_{i_m}$ for $1 \leq l < m \leq k$. Suppose $p_{i_l} < p_{i_m}$. By Lemma 2 we see that

$$h_j(p_{i_l}, \lambda_j)\begin{cases} > 0 & \text{if } j \geq i_l \\ \leq 0 & \text{if } j < i_l \end{cases} \tag{27}$$

and

$$h_j(p_{i_m}, \lambda_j)\begin{cases} > 0 & \text{if } j \geq i_m \\ \leq 0 & \text{if } j < i_m \end{cases} \tag{28}$$

Our assumption $p_{i_l} < p_{i_m}$ gives $\phi_j(p_{i_l}) > \phi_j(p_{i_m})$ for all j. Hence by Lemma 1 and (28) we see that $h_j(p_{i_l}, \lambda_j) \leq 0$ for all $j < i_m$. This contradicts (27) since $i_l < i_m$. □

The specific ordering exhibited in (24) is not very useful since it depends on the given set of values $\{\lambda_\omega; \omega \in \Omega\}$. The following lemma provides a sufficient condition which generates the ordering given in (24) and is independent of the set of values $\{\lambda_\omega; \omega \in \Omega\}$.

Lemma 4 *Suppose the contingencies are ordered arbitrarily. If $\lambda \in \Re^n_+$ satisfies*

$$H(p_i, R^i, \lambda) = H > 0 \textit{ for all } i \tag{29}$$

where $p_i = argmax_{p\geq 0} H(p, R^i, \lambda)$. Then we get $p_1^0 \geq p_2^0 \geq \cdots \geq p_n^0$.

Proof. Since $\max_{p\geq 0} H(p, R^i, \lambda)$ is maximized at p_i, we must have

$$H \geq H(p_{i+1}, R^i, \lambda) = \pi_i h_i(p_{i+1}, \lambda_i) + H(p_{i+1}, R^{i+1}, \lambda) \tag{30}$$

Since $H(p_{i+1}, R^{i+1}, \lambda) = H$, (30) gives

$$h_i(p_{i+1}, \lambda_i) \leq 0 \tag{31}$$

On the other hand, we have

$$H = H(p_i, R^i, \lambda) = \pi_i h_i(p_i, \lambda_i) + H(p_i, R^{i+1}, \lambda) \tag{32}$$

Since $H(p_i, R^{i+1}, \lambda) \le H$, (32) gives

$$h_i(p_i, \lambda_i) \ge 0 \tag{33}$$

Then by Lemma 1, (31) and (33) imply $p_i \ge p_i^0 \ge p_{i+1}$. Hence we obtain $p_1 \ge p_1^0 \ge p_2 \ge p_2^0 \ge \cdots \ge p_n \ge p_n^0$. □

4 Optimal Allocation for a Special Case: Additive Demand Preferences

The results in Lemma 4 can be used to construct an algorithm that gives optimal λ^* and H^* as required in Theorem 1. We first try an arbitrary $H > 0$. Suppose there is a way to find $\lambda \in \Re_+^n$ such that (29) holds. Then by Lemma 2 each (p_i, R^i) is a maximizer of $\max_{(p,R)} H(p, R, \lambda)$, where p_i is given by (25). Moreover, the contracts $\{(p_i, R^i)\}_{i=1}^n$ are optimal for consumers. Finally, optimality for the supplier is achieved by adjusting H such that the complementary slackness condition (16) is satisfied. In general, it is difficult to show the existence of a vector λ for which (29) holds. However, a special situation for which this is true is to assume that the demand functions ϕ_i differ by constants. This assumption implies that the demand functions do not intersect. Such non-intersecting property is essential since it implies that a consumer who consumes more in contingency i than in contingency j at one price p will do the same at other prices.

4.1 A two-part tariff and optimal prices

We assume there is a strictly concave function U_0 with $U_0(d) \to \infty$ as $d \to \infty$ (i.e. U_0 does not saturate), and constants $0 \le c_1 < c_2 < \cdots < c_n$ such that the demand curves are ordered in the following manner:

$$\phi_i(p) = \phi_0(p) - c_i \ , \ \phi_0 = (U_0')^{-1} \tag{34}$$

Since we have assumed $U_i(0) = 0$, the following identities are immediate from (34).

$$U_i'(d) = U_0'(d + c_i) \ , \ 1 \le i \le n \tag{35}$$

$$U_i(d) = U_0(d + c_i) - U_0(c_i) \ , \ 1 \le i \le n \tag{36}$$

Also since U_0 is strictly concave and $c_i < c_{i+1}$, we get $U_0(d + c_i) - U_0(c_i) > U_0(d + c_{i+1}) - U_0(c_{i+1})$ for all $d > 0$. By (36) this is equivalent to $U_i(d) >$

$U_{i+1}(d)$ for all $d > 0$. Formula (36) also gives $U_i(d + c_{i+1} - c_i) = U_0(d + c_i) - U_0(c_i) = U_{i+1}(d) + U_0(c_{i+1}) - U_0(c_i)$. By applying (36) again the previous equation becomes

$$U_{i+1}(d) = U_i(d + c_{i+1} - c_i) - U_i(c_{i+1} - c_i) \text{ , for all } d > 0 \tag{37}$$

In other words, the curve $d \mapsto U_{i+1}(d)$ is the truncated curve $d \mapsto U_i(d + c_{i+1} - c_i)$ with the origin shifted to the point $(c_{i+1} - c_i, U_i(c_{i+1} - c_i))$. By the concavity of U_i and (37) we infer that

$$\frac{U_i(c_{i+1} - c_i)}{c_{i+1} - c_i} \geq \frac{U_{i+1}(d)}{d} \text{ , for all } d > 0 \tag{38}$$

Lemma 5 *For $p \geq 0$ and $1 \leq i \leq n-1$, suppose $\phi_{i+1}(p) > 0$. Then*

$$\frac{U_i(\phi_i(p))}{\phi_i(p)} \geq \frac{U_{i+1}(\phi_{i+1}(p))}{\phi_{i+1}(p)} \tag{39}$$

Proof. By (34) and (37) we get

$$\begin{aligned} \frac{U_i(\phi_i(p))}{\phi_i(p)} &= \frac{U_{i+1}(\phi_{i+1}(p)) + U_i(c_{i+1} - c_i)}{\phi_{i+1}(p) + c_{i+1} - c_i} \\ &\geq \frac{U_{i+1}(\phi_{i+1}(p))}{\phi_{i+1}(p)} \text{ , by (38)} \end{aligned}$$

□

Proposition 1 *For any $H > 0$, there exists $\lambda = (\lambda_1, \cdots, \lambda_n)$ such that*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0 \tag{40}$$

and

$$H(p_i, R^i, \lambda) = H \text{ , for all } i \tag{41}$$

where $p_i = argmax_{p \geq 0} H(p, R^i, \lambda) = \sum_{j \geq i} \pi_j \lambda_j / \sum_{j \geq i} \pi_j$. Moreover, $p_1 \geq p_2 \geq \cdots \geq p_n$ and each (p_i, R^i) maximizes $H(p, R, \lambda)$.

Proof. By Lemma 4 and (25) in Lemma 3 we see that $p_n = \lambda_n$. So we solve the equation

$$\pi_n\{U_n(\phi_n(\lambda_n)) - \lambda_n \phi_n(\lambda_n)\} = H$$

and use the assumption $U_0(d) \to \infty$ as $d \to \infty$ to obtain $\lambda_n > 0$. For each $1 \leq i \leq n$, let $\lambda^i := (0, \cdots, 0, \lambda_i, \cdots, \lambda_n) \in \Re^n_+$. Also for each $1 \leq i \leq n-1$, let $(x, \lambda^{i+1}) := (0, ..., 0, x, \lambda_{i+1}, ..., \lambda_n) \in \Re^n_+$. Assume $\lambda_{i+1} \geq \lambda_{i+2} \geq \cdots \geq \lambda_n$ have been obtained and they satisfy $H(p_j, R^j, \lambda^j) = H$ for all $j \geq i+1$. We will find λ_i that satisfies (40), (41). By Lemma 2 and since $p_{i+1} = \text{argmax}_{p \geq 0} H(p, R^{i+1}, \lambda^{i+1})$, we must have

$$h_{i+1}(p_{i+1}, \lambda_{i+1}) = U_{i+1}(\phi_{i+1}(p_{i+1})) - \lambda_{i+1}\phi_{i+1}(p_{i+1}) \geq 0 \qquad (42)$$

Next we get

$$\begin{aligned}
\max_{p \geq 0} H(p, R^i, (\lambda_{i+1}, \lambda^{i+1})) &\geq H(p_{i+1}, R^i, (\lambda_{i+1}, \lambda^{i+1})) \\
&= H(p_{i+1}, R^{i+1}, \lambda^{i+1}) \\
&\quad + \pi_i \{U_i(\phi_i(p_{i+1})) - \lambda_{i+1}\phi_i(p_{i+1})\} \\
&= H + \pi_i h_i(p_{i+1}, \lambda_{i+1}) \\
&\geq H
\end{aligned}$$

The last inequality follows from (42), (39). Since $\max_{p \geq 0} H(p, R^i, (x, \lambda^{i+1}))$ is decreasing in x, there exists $\lambda_i \geq \lambda_{i+1}$ such that $\max_{p \geq 0} H(p, R^i, (\lambda_i, \lambda^{i+1})) = H$. This proves the first assertion.

The formula $p_i = \sum_{j \geq i} \pi_j \lambda_j / \sum_{j \geq i} \pi_j$ follows from (25) and the fact that $\phi_i'(p) = \phi_0'(p)$ for all i. The remaining assertions follow directly from Lemma 2 and Lemma 3. □

The number $\pi_i \lambda_i$ is interpreted as the scarcity cost or dual variable associated with the demand-supply constraint for contingency i. Hence $p_i = \sum_{j \geq i} \pi_j \lambda_j / \sum_{j \geq i} \pi_j$ is the conditional expectation of scarcity costs given that the service is uninterrupted only in contingencies $i, i+1, \cdots, n$. This is different from the bid prices obtained in [6]. The expected demand for a consumer who picks contract (p_i, R^i) is $\sum_{j \geq i} \pi_j \phi_j(p_i)$, so her electric bill is $p_i \sum_{j \geq i} \pi_j \phi_j(p_i)$. Moreover, her net surplus (welfare minus electric bill) is $\sum_{j \geq i} \pi_j \{U_j(\phi_j(p_i)) - p_i \phi_j(p_i)\}$. Since $p_i = \sum_{j \geq i} \pi_j \lambda_j / \sum_{j \geq i} \pi_j$, it is easy to see that the net surplus is *different* from the Hamiltonian $H(p_i, R^i, \lambda) = H$. By (15) in Theorem 1 every consumer must end up with the same net surplus at the optimum. Hence the set of contracts $\{(p_i, R^i)\}$ cannot sustain the optimal solution of the welfare problem (8)-(11). To overcome this difficulty we consider a two-part tariff. The two-part tariff consists of a price p_i for contract (p_i, R^i) and a cost $C_i(\lambda)$ such that the net surplus for consumers who have chosen this contract is now $\sum_{j \geq i} \pi_j \{U_j(\phi_j(p_i)) - p_i \phi_j(p_i)\} - C_i(\lambda)$ and equal to H. If $C_i(\lambda)$ is negative, it is a reimbursement for consumers who have picked contract (p_i, R^i). By Proposition 1 the scarcity costs

$\lambda_1, \lambda_2, \cdots, \lambda_n$ satisfy $H = H(p_i, R^i, \lambda) = \sum_{j \geq i} \pi_j \{U_j(\phi_j(p_i)) - \lambda_j \phi_j(p_i)\}$ for each i. Hence $C_i(\lambda)$ are given by

$$\begin{aligned} C_i(\lambda) &\equiv C_i(\lambda_i, \lambda_{i+1}, \cdots, \lambda_n) \\ &:= \sum_{j \geq i} \pi_j(\lambda_j - p_i)\phi_j(p_i) \ , \ i = 1, \cdots, n \end{aligned} \tag{43}$$

Thus the contracts are of the form $((p_i, R^i), C_i(\lambda))$, and all consumers have the same net surplus. Note that since $p_n = \lambda_n$, we get $C_n(\lambda_n) = 0$. Now by Theorem 1 and since each consumer has surplus H, the contracts are optimal for consumers. However, the prices $\{p_i\}_{i=1}^n$ may not be optimal for the supplier. To obtain the equilibrium prices we need to adjust H appropriately. This requires an examination of the dependence of λ_i and p_i on H. We will show that both λ_i and p_i are decreasing in H. To simplify the notation, let $\sigma_i := \sum_{j \geq i} \pi_j$, $1 \leq i \leq n$. We first obtain some useful lemmas.

Lemma 6 *Consider the vectors $\lambda = (\lambda_1, \cdots, \lambda_n)$ and $p = (p_1, \cdots, p_n)$ obtained in Proposition 1.*

$$\frac{\partial p_i}{\partial \lambda_j} = \begin{cases} \frac{\pi_j}{\sigma_i} & \text{for } j \geq i \\ 0 & \text{for } j < i \end{cases} \tag{44}$$

Proof. The partial derivatives in (44) are immediate since $p_i = \frac{1}{\sigma_i} \sum_{j \geq i} \pi_j \lambda_j$, $1 \leq i \leq n$. □

Lemma 7 *(i) For each $1 \leq i \leq n$, we get*

$$\frac{\partial}{\partial \lambda_i} h_i(p_i, \lambda_i) = \frac{\pi_i}{\sigma_i^2} \left(\sum_{j > i} \pi_j(\lambda_j - \lambda_i)\right)\phi_0'(p_i) - \phi_i(p_i) \tag{45}$$

(ii) For each $1 \leq i \leq n-1$ and $j > i$, we get

$$\frac{\partial}{\partial \lambda_i} h_j(p_i, \lambda_j) = \frac{\pi_i}{\sigma_i^2} \left(\sum_{l \geq i,\ l \neq j} \pi_l(\lambda_l - \lambda_j)\right)\phi_0'(p_i) \tag{46}$$

Proof. (i) By using (44) we get

$$\frac{\partial}{\partial \lambda_i} h_i(p_i, \lambda_i) = \frac{\partial}{\partial \lambda_i} \{U_i(\phi_i(p_i)) - \lambda_i \phi_i(p_i)\}$$

$$\begin{aligned}
&= U_i'(\phi_i(p_i))\phi_i'(p_i)\frac{\pi_i}{\sigma_i} - \phi_i(p_i) - \lambda_i\phi_i'(p_i)\frac{\pi_i}{\sigma_i} \\
&= (p_i - \lambda_i)\frac{\pi_i}{\sigma_i}\phi_0'(p_i) - \phi_i(p_i) \text{ , since } \phi_i'(p) = \phi_0'(p) \text{ for all } i \\
&= \frac{\pi_i}{\sigma_i^2}(\sum_{j>i} \pi_j(\lambda_j - \lambda_i))\phi_0'(p_i) - \phi_i(p_i)
\end{aligned}$$

(ii) The calculations are similar to those in (i). We have

$$\begin{aligned}
\frac{\partial}{\partial \lambda_i} h_j(p_i, \lambda_j) &= \frac{\partial}{\partial \lambda_i}\{U_j(\phi_j(p_i)) - \lambda_j\phi_j(p_i)\} \\
&= U_j'(\phi_j(p_i))\phi_j'(p_i)\frac{\pi_i}{\sigma_i} - \lambda_j\phi_j'(p_i)\frac{\pi_i}{\sigma_i} \\
&= (p_i - \lambda_j)\frac{\pi_i}{\sigma_i}\phi_0'(p_i) \\
&= \frac{\pi_i}{\sigma_i^2}(\sum_{l\geq i,\ l\neq j} \pi_l(\lambda_l - \lambda_j))\phi_0'(p_i)
\end{aligned}$$

□

Lemma 8 *For each $1 \leq i \leq n-1$, the number A_i defined below is zero.*

$$A_i := \pi_i \sum_{j>i} \pi_j(\lambda_j - \lambda_i) + \sum_{j>i} \pi_j \sum_{l\geq i,\ l\neq j} \pi_l(\lambda_l - \lambda_j) \tag{47}$$

Proof. The second term in the right hand side of (47) for $l = i$ is $\sum_{j>i} \pi_j\pi_i(\lambda_i - \lambda_j)$, which is the negative of the first summation in (47). Thus (47) becomes

$$A_i = \sum_{j>i} \sum_{l>i,\ l\neq j} \pi_j\pi_l(\lambda_l - \lambda_j) \tag{48}$$

It is now clear that $A_i = 0$ since the indices l and j appear symmetrically in (48). □

Proposition 2 *(i) For each $1 \leq i \leq n$, $\frac{\partial H}{\partial \lambda_i} = -\pi_i\phi_i(p_i)$. Hence $\frac{\partial H}{\partial \lambda_i} < 0$ for all i.*
(ii) $\frac{\partial H}{\partial p_i} < 0$ for each $1 \leq i \leq n$.

Proof. (i) We first show $\frac{\partial H}{\partial \lambda_n} = -\pi_n\phi_n(p_n) < 0$. By (41) $H = \pi_n\{U_n(p_n) - \lambda_n\phi_n(p_n)\} = \pi_n h_n(p_n, \lambda_n)$. Then (45) gives $\frac{\partial H}{\partial \lambda_n} = -\pi_n\phi_n(p_n)$. Next we show the claim is also true for $1 \leq i \leq n-1$. By (41) we have

$$\begin{aligned}
H &= \sum_{j\geq i} \pi_j\{U_j(\phi_j(p_j)) - \lambda_j\phi_j(p_j)\} \\
&= \pi_i h_i(p_i, \lambda_i) + \sum_{j>i} \pi_j h_j(p_j, \lambda_j)
\end{aligned}$$

By using the derivatives (45) and (46), we get

$$\begin{aligned}\frac{\partial H}{\partial \lambda_i} &= \pi_i\{\frac{\pi_i}{\sigma_i^2}(\sum_{j>i}\pi_j(\lambda_j-\lambda_i))\phi_0'(p_i)-\phi_i(p_i)\} \\ &\quad +\sum_{j>i}\pi_j\{\frac{\pi_i}{\sigma_i^2}(\sum_{l\geq i,\ l\neq j}\pi_l(\lambda_l-\lambda_j))\phi_0'(p_i)\} \\ &= -\pi_i\phi_i(p_i)+\frac{\pi_i}{\sigma_i^2}A_i\phi_0'(p_i)\end{aligned}$$

where A_i is the number defined in (47). The claim is proved since $A_i = 0$ by Lemma 8.
(ii) We have $\frac{\partial H}{\partial \lambda_j} = \frac{\partial H}{\partial p_i}\cdot\frac{\partial p_i}{\partial \lambda_j}$ for $j \geq i$. The claim then follows from part (i) above and Lemma 6. □

4.2 Optimal allocation and contracts

We are now ready to construct optimal λ^*, H^*, and contracts $\{(p_i^*, R^i)\}_{i=1}^n$. We begin with a trial surplus H and find the numbers of consumers that can be assigned to the n contracts. The sum of these numbers are shown to be decreasing in H, hence we can tune the parameter H until this sum equals one. When this occurs the resultant prices are optimal. The algorithm can be described in two steps.
Step 1: Begin with an arbitrary $H > 0$. Obtain the numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ as given in Proposition 1. Calculate the prices

$$p_i = \frac{\sum_{j\geq i}\pi_j\lambda_j}{\sum_{j\geq i}\pi_j}\ ,\ i = 1,\cdots,n$$

Consider the contracts $\{(p_i, R^i)\}_{i=1}^n$.
Step 2: Let β_i denote the number of consumers who are assigned to contract (p_i, R^i). These quantities are obtained by solving the following n equations.

$$\sum_{j\leq i}\beta_j\phi_i(p_j) = s\ ,\ i = 1,2,\cdots,n \tag{49}$$

The next lemma gives an useful expression of β_i in terms of $\{\beta_j\ ;\ j \leq i-1\}$ for $i \geq 2$.

Lemma 9

$$\beta_1 = \frac{s}{\phi_1(p_1)} \tag{50}$$

$$\beta_i = \frac{(c_i - c_{i-1}) \sum_{j \leq i-1} \beta_j}{\phi_i(p_i)} \ , \ i \geq 2 \tag{51}$$

Proof. (50) is exactly (49) when $i = 1$. For $i \geq 2$, we have $\phi_i(p) = \phi_{i-1}(p) - c_i - c_{i-1}$ if $\phi_i(p) > 0$. Then

$$\begin{aligned} s &= \sum_{j \leq i} \beta_j \phi_i(p_j) \\ &= \sum_{j \leq i-1} \beta_j(\phi_{i-1}(p_j) - c_i - c_{i-1}) + \beta_i \phi_i(p_i) \\ &= \sum_{j \leq i-1} \beta_j(\phi_{i-1}(p_j)) - (c_i - c_{i-1}) \sum_{j \leq i-1} \beta_j + \beta_i \phi_i(p_i) \\ &= s - (c_i - c_{i-1}) \sum_{j \leq i-1} \beta_j + \beta_i \phi_i(p_i) \ , \ \text{by (49)} \end{aligned} \tag{52}$$

It is now clear that (51) follows from (52). □

Proposition 3 *Each β_i, $i = 1, \cdots, n$, is monotonically decreasing in H.*

Proof. We show this by an induction on i. By Proposition 2 an increase in H will decrease all p_1 and therefore increase $\phi_1(p_1)$. By (50) this will decrease β_1. Next, by induction assumption, suppose an increase in H decreases $\beta_1, \cdots, \beta_{i-1}$; $i \geq 2$. Then by the same argument and (51) β_i will also be decreased. This completes the proof. □

An immediate implication of Proposition 3 is that there exists a *unique* $H^* > 0$ such that $\sum_{i=1}^{n} \beta_i(H^*) = 1$. We use the algorithm described earlier to obtain λ_i^* and p_i^* that correspond to this H^*. Then the n contracts $\{(p_i^*, R^i) \ ; \ i = 1, 2, \cdots, n\}$ are optimal for the consumers since by (41) and (43) each consumer has net surplus H^*. By (49) supply is equal to rationed demand in all contingencies, so the complementary slackness condition (16) is satisfied. We summarize these conclusions in the following theorem.

Theorem 2 *There exist H^* and λ^* such that the contracts $\{(p_i^*, R^i) \ ; \ i = 1, 2, \cdots, n\}$, where $p_i^* = \sum_{j \geq i} \pi_j \lambda_j^* / \sum_{j \geq i} \pi_j$, are optimal. The set of consumers who have picked contract (p_i^*, R^i) is of Lebesgue measure $\beta_i(H^*)$, and the β_i's satisfy $\sum_{i=1}^{n} \beta_i(H^*) = 1$. Furthermore, these n contracts sustain the optimum of the welfare problem (8)-(11) as an equilibrium provided that the amount $C_i(\lambda^*)$ is charged (if $C_i(\lambda^*)$ is positive) or reimbursed (if $C_i(\lambda^*)$ is negative) to consumers who have chosen contract (p_i^*, R^i).* □

5 Concluding Remarks

In this paper we have considered a two-period pricing model for an electric power system. The power company offers a set of contracts in period 1. Each customer picks a contract in period 1 and then decides her demand after the random element is observed in period 2. This is a decentralized decision problem. The supplier, on the other hand, needs to design a rationing scheme so that the demand can be met by the supply available in period 2, and each contract can be fulfilled. We have shown that it is possible to design a set of contracts that induce customers and the supplier to act optimally. Each contract consists of a price p_i^*, a cost/reimbursement term $C_i(\lambda^*)$, and a vector $R^i \in \{0,1\}^n$ that specifies the contingencies under which the service will be interrupted. It is shown in §4.1 that the specification of service interruption depends on an ordering of the demand contingencies.

The type of contracts considered in this paper is of the form (p, R), where $R \in \{0,1\}^n$. This permits us to assume that consumer preferences are characterized only by utility functions. Moreover, the welfare function is simplified and does not contain a loss term as discussed below. In general, demand preferences are characterized by both utility and loss functions. This is because when there is *independent* random supply, a consumer will suffer a loss when the service is cut.

Suppose there are independent supply and demand contingencies. The supply takes random values $s_1 < \cdots < s_m$ with probabilities $\eta_1, \cdots, \eta_m$. The market works as follows. In period 1 the power company announces a set of contracts $\{(p_{ij}, \rho_{ij})\}$, where $\rho_{ij} = \sum_{l \geq i} \sum_{k \geq j} \eta_l \pi_k$ is the probability that a consumer who picks contract (p_{ij}, ρ_{ij}) will receive her service. Suppose random demand preference k is revealed to consumer t at the beginning of period 2, and the supply available in period 2 is s_l. Then consumer t's demand is:

$$\arg\max_{d \geq 0} \rho_{ij} U_k(d) - (1 - \rho_{ij}) L_k(d) \ - \ p_{ij} d \tag{53}$$

where U_k and L_k are the utility and loss functions in contingency k, respectively. As in the model considered in §2 the company selects a rationing function $R_{ij}(t)$ so that the supply constraint can be met.

The problem is complicated by the presence of the loss term in (53). In this formulation, the contract prices are no longer given by the conditional expectation of scarcity costs. However, the analysis becomes easier when the demand functions corresponding to each pair (U_k, L_k) are "horizontal" shifts of each other. That is, the preference functions are

$$U_k(d) := U(d - \gamma_k)\ ,\ L_k(d) := L(d - \gamma_k)\ ,\ k = 1, \cdots, n \tag{54}$$

where the random numbers $\gamma_1 > \cdots > \gamma_n > 0$ occur with probabilities $\pi_1, \cdots, \pi_n$. The functions U and L satisfy $U(0) = L(0) = 0$. Also U is strictly concave and L is convex. Then the problem is equivalent to the case of deterministic demand preferences (U, L), and random supply with values $s_i - t_{ij}\gamma_{ij}$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, occurring with probabilities $\eta_i \pi_j$ (see [5] for details). The number t_{ij} is the number of consumers who are assigned to contract (p_{ij}, ρ_{ij}).

References

[1] P. T. Ahlstrand. Demand-side real-time pricing. In *1986 Annual Report*, Pacific Gas & Electric Co., 1987.

[2] R. E. Bohn, M. C. Caramanis, and F. C. Schweppe. Optimal pricing in electrical networks over space and time. *Rand Journal of Economics*, 15(3):360–376, Autumn 1984.

[3] M. C. Caramanis, R. E. Bohn, and F. C. Schweppe. Optimal spot pricing: practices and theory. *IEEE Transactions on Power Apparatus and Systems*, PAS-101(9):3234–3245, 1982.

[4] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*. Interscience Division of John Wiley & Sons, Inc., 1962.

[5] C.-W. Tan. *Prices for Interruptible Electric Power Service*. PhD thesis, Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, December 1990.

[6] C.-W. Tan and P. Varaiya. *Interruptible Electric Power Service Contracts*. UCB/ERL/IGCT Memorandum No. M90/108, Electronics Research Laboratory, University of California, Berkeley, 1990.

[7] W. Vickrey. Responsive pricing of public utility service. *Bell Journal of Economics and Management Science*, 2(1):337–346, 1971.