STEERING A THREE-INPUT NONHOLONOMIC SYSTEM USING MULTI-RATE CONTROLS

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Steering a Three-Input Nonholonomic System using Multi-rate Controls

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Abstract

In this paper we examine a multi-rate control scheme for nonholonomic path planning using constant control inputs over different time periods. For chained systems, an exact point-to-point trajectory is generated. Simulation results are presented for a three-input system, and comparisons are made with a sinusoidal method for path planning.

1 Introduction

Our interests in this paper are in finding paths for nonholonomic systems. These systems are characterized by having nonintegrable constraints on the state velocities which do not restrict the reachable configuration space. There is a classical result from nonlinear control theory [8] which can be used to prove that such systems are completely controllable, implying that a feasible path exists between any two points in free space. This result, however, does not have a constructive proof. The problem we consider in this paper is the construction of a path between the start and goal states.

We transform the path-planning problem with constraints on the velocities into the dual control problem with fewer control inputs than degrees of freedom. Many researchers have attacked this control problem by looking at specific classes of input functions. Some early work by Murray and Sastry [6, 7] used sinusoidal inputs at varying frequencies to plan paths for systems in a special chained canonical form. More recently, it has been shown by Monaco and Normand-Cyrot [5] that piecewise constant controls may be used for nonholonomic path planning. In fact, a digital control procedure, referred to as multi-rate control, was initially introduced within the nonlinear control context when digital control of a continuous time system was investigated [4]. This method is based on a faster sampling of the control variables than the state ones, and gives the necessary degrees of freedom on the control in order to solve some specific problems. For instance, a multi-rate control strategy is efficient for preserving under sampling properties such as feedback linearization or input-output decoupling of a continuous time system.

The system that we use as our example was originally examined in [1]. It has three control inputs and six states, and can be shown to be completely nonholonomic or completely controllable.

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In Section 2, we define this system and show how it can be "nilpotentized" through a change of coordinates. In fact, the system can be transformed into a specific class of nilpotent systems called chained form systems. In Section 3, we apply the multi-rate control scheme to the example system, using the chained form coordinates as defined in the previous section. We show how to choose the inputs to steer the system from any initial point to any other final position. Finally, we present some simulation results in Section 4, comparing the multi-rate and sinusoidal algorithms on three different trajectories. Our findings are that the paths generated by the multi-rate control scheme are shorter and more direct than the paths generated by the sinusoidal path-planning algorithm.

2 The Firetruck System

One example of a non-holonomic system with three inputs is a firetruck. The truck is driven exactly like an ordinary car. The ladder is carried on a long trailer, the rear wheels of which can be steered, allowing extra maneuverability around tight corners. A sketch of the system is shown in Figure 1. There are six states in the system: the (x, y) position of the truck, the angles of the truck and trailer with respect to the inertial frame (θ_0 and θ_1 respectively), the angle of the front wheels with respect to the truck (ϕ_0), and the angle of the rear wheels with respect to the trailer (ϕ_1). The position of the rear wheels of the trailer, (x_1, y_1) can be uniquely expressed in terms of the other six states. There are two length constants: l_0 is the distance between the front and rear wheels of the truck, and l_1 is the distance between the rear wheels of the truck and the rear wheels of the trailer. The kinematic equations are derived in [1], and are stated here for convenience:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi}_0 \\ \dot{\theta}_0 \\ \dot{\phi}_1 \\ \dot{\theta}_1 \end{pmatrix} = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ 0 \\ \frac{\tan\phi_0}{l_0} \\ 0 \\ \frac{-\sin(\phi_1-\theta_0+\theta_1)}{l_1\cos\phi_1} \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_3 \tag{1}$$

The three inputs correspond to driving and steering: u_1 is the linear velocity of the truck, u_2 is the angular velocity of the front wheels, and u_3 is the angular velocity of the rear wheels. As shown in [1], this system can be put into a special three-input chained form by a change of coordinates and state feedback. We will find it convenient to use this chained form for our path-planning algorithm, since the Lie algebra generated by the transformed input vector fields is nilpotent. We state here the coordinate transformation and the resulting chained form equations. More details on the derivation can be found in [1].

$$\begin{aligned} z_1 &= x \\ z_2 &= \frac{\tan\phi_0}{l_0\cos^3\theta_0} \\ z_3 &= \frac{-\sin(\phi_1-\theta_0+\theta_1)}{l_1\cos\phi_1\cos\theta_0} \\ z_4 &= \tan\theta_0 \\ z_5 &= \theta_1 \end{aligned}$$

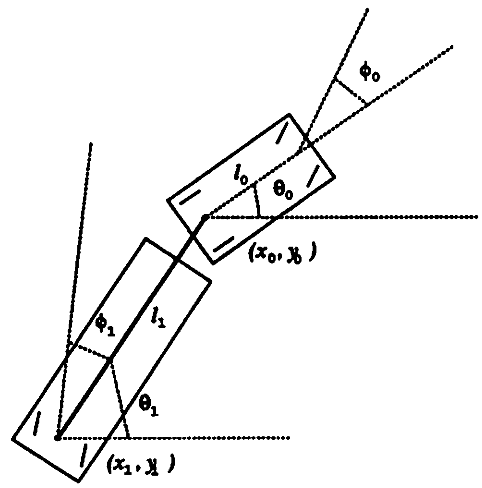


Figure 1: A sketch of the Firetruck System showing the state variables

$$z_6 = y$$

The corresponding input transformation is:

$$\begin{aligned}
v_1 &= \cos\theta_0 u_1 \\
v_2 &= \frac{3\tan^2\phi_0 \sin\theta_0}{l_0^2\cos^4\theta_0}u_1 + \frac{1}{l_0\cos^2\phi_0\cos^3\theta_0}u_2 \\
v_3 &= \left(\frac{\cos(\phi_1+\theta_1)\sin\phi_0}{l_0 l_1\cos\phi_0\cos\phi_1\cos^2\theta_0} + \frac{\cos(\phi_1-\theta_0+\theta_1)\sin(\phi_1-\theta_0+\theta_1)}{l_1^2\cos^2\phi_1\cos^2\theta_0}\right)u_1 - \frac{\cos(\theta_1-\theta_0)}{l_1\cos^2\phi_1\cos\theta_0}u_3
\end{aligned}$$

It can be shown that this coordinate transformation is valid on the entire state space, except where θ_0, ϕ_0, or ϕ_1 is equal to $\frac{\pi}{2}$. With a little algebra and some trigonometric identities, one can check that the system equations in these coordinates reduce to:

$$\begin{aligned}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= v_3 \\
\dot{z}_4 &= z_2 v_1 \\
\dot{z}_5 &= z_3 v_1 \\
\dot{z}_6 &= z_4 v_1
\end{aligned} \tag{2}$$

From this point on we will consider the system in these coordinates.

3 Multi-Rate Controls for the Firetruck Example

The idea of using piecewise constant inputs with different holding times for nonholonomic motion planning was first introduced in [5]. Although this multi-rate control can be applied to almost any system, it is especially powerful for systems admitting an exact discretization, since in this case, an exact point-to-point trajectory can be generated. The derivation for a system in chained canonical form, introduced in [7], has a particularly simple expression which will be detailed here for the firetruck system.

First, we recall some theoretical tools used within the framework of the discrete time control system. Let X be any formal operator. Then the exponential Lie operator is defined as follows:

$$e^X = \sum_{k\geq 0} \frac{1}{k!} X^k = 1 + X + \frac{1}{2!} X^2 + \cdots + \frac{1}{k!} X^k + \cdots \tag{3}$$

where "1" represents the identity operator, and X^k the k^{th} iterated composition of the operator X.

Proposition 1 ([3]) *Let us consider the driftless continuous time system:*

$$\Sigma: \quad \dot{x} = v_1 g_1(x) + v_2 g_2(x) + \cdots + v_m g_m(x) \tag{4}$$

driven by piecewise constant controls of the form:

$$v_i(t) = v_i^{\mathrm{D}}(k) \quad \textit{for } t \in [k\delta, (k+1)\delta[, \;\; k \geq 0, \;\; i = 1, \cdots, m \tag{5}$$

then the exact discretized system is given by

$$\Sigma_{\mathrm{D}}: \quad x^{\mathrm{D}}(k+1) = F^{\delta}(\delta, x^{\mathrm{D}}(k), v_i^{\mathrm{D}}(k)) = e^{\delta(\sum_{i=1}^m v_i^{\mathrm{D}} L_{g_i})}(I_d)_{|x^{\mathrm{D}}(k)} \tag{6}$$

where "I_d" is the identity function, and $\delta \in]0, \delta_0[$ is the sampling period. For the same initialization ($x^{\mathrm{D}}(0) = x(0)$) and $x^{\mathrm{D}}(k) = x_{|t=k\delta}$, Σ_{D} reproduces at the sampling instants the input-state behavior of the continuous time system Σ. □

Under the controllability condition of the continuous time system, the idea which was developed in [5] is used to provide the extra degrees of freedom needed to span all of $\mathbb{R}^n$.

If we denote by n_i the multi-rate order on the controls v_i ($i = 1, \cdots, m$), then $\sum_{i=1}^m n_i$ must be at least equal to the dimension of the state space, namely 6 for the case of the firetruck system.

Choosing the following configuration for the controls:

$$v_1(t) = v_1^{\mathrm{D}} \quad \text{for } t \in [k\delta, (k+1)\delta[$$

$$v_2(t) = \begin{cases} v_{21}^{\mathrm{D}} & \text{for } t \in \left[k\delta, (k+\frac{1}{3})\delta\right[\\ v_{22}^{\mathrm{D}} & \text{for } t \in \left[(k+\frac{1}{3})\delta, (k+\frac{2}{3})\delta\right[\\ v_{23}^{\mathrm{D}} & \text{for } t \in \left[(k+\frac{2}{3})\delta, (k+1)\delta\right[\end{cases} \tag{7}$$

$$v_3(t) = \begin{cases} v_{31}^{\mathrm{D}} & \text{for } t \in \left[k\delta, (k+\frac{1}{3})\delta\right[\\ v_{32}^{\mathrm{D}} & \text{for } t \in \left[(k+\frac{1}{3})\delta, (k+1)\delta\right[\end{cases}$$

one can write the associated sampled dynamics as the composition of exponential forms. Indeed, using the discretization formula (6), we have

$$z(k+\frac{i}{3}) = e^{\frac{\delta}{3} X_i}(I_d)_{|z(k+\frac{i-1}{3})} \tag{8}$$

where $\bar{\delta} = \frac{\delta}{3}$, $X_i = v_1^{\mathrm{D}} L_{g_1} + v_{2i}^{\mathrm{D}} L_{g_2} + v_{3i}^{\mathrm{D}} L_{g_3}$, and $v_{33}^{\mathrm{D}} = v_{32}^{\mathrm{D}}$. Thus,

$$z(k+\frac{2}{3}) = e^{\bar{\delta} X_2}(I_d)_{|z(k+\frac{1}{3})} \tag{9}$$

which may be rewritten, using a property of the exponential operator

$$z(k+\frac{2}{3}) = e^{\bar{\delta} X_1} \circ e^{\bar{\delta} X_2}(I_d)_{|z(k)} \tag{10}$$

where "$\circ$" is the composition operator. By iteration, one finally obtains

$$z(k+1) = e^{\bar{\delta} X_1} \circ e^{\bar{\delta} X_2} \circ e^{\bar{\delta} X_3}(I_d)_{|z(k)} \tag{11}$$

The sampled dynamics (11) defined on $\mathbb{R}^6$ is a function of the six multi-rate controls:

$$(v_{11}^{\mathrm{D}}, v_{12}^{\mathrm{D}}, v_{21}^{\mathrm{D}}, v_{22}^{\mathrm{D}}, v_{23}^{\mathrm{D}}, v_{31}^{\mathrm{D}}, v_{32}^{\mathrm{D}})$$

and is referred to as the multi-rate sampled dynamics.

Using the formula (11), the discretization of the firetruck system in chained form coordinates leads to an exact form given by:

$$\left\{\begin{array}{lcl} z_1(k+1) & = & z_1(k) + 3\bar{\delta} v_1^{\mathrm{D}} \\ z_2(k+1) & = & z_2(k) + \bar{\delta}(v_{21}^{\mathrm{D}} + v_{22}^{\mathrm{D}} + v_{23}^{\mathrm{D}}) \\ z_3(k+1) & = & z_3(k) + \bar{\delta}(v_{31}^{\mathrm{D}} + 2v_{32}^{\mathrm{D}}) \\ z_4(k+1) & = & z_4(k) + 3\bar{\delta} v_1^{\mathrm{D}} z_2(k) + \frac{\bar{\delta}^2}{2!}(5v_{21}^{\mathrm{D}} + 3v_{22}^{\mathrm{D}} + v_{23}^{\mathrm{D}}) v_1^{\mathrm{D}} \\ z_5(k+1) & = & z_5(k) + 3\bar{\delta} v_1^{\mathrm{D}} z_3(k) + \frac{\bar{\delta}^2}{2!}(5v_{31}^{\mathrm{D}} + 4v_{32}^{\mathrm{D}}) v_1^{\mathrm{D}} \\ z_6(k+1) & = & z_6(k) + 3\bar{\delta} v_1^{\mathrm{D}} z_4(k) + \frac{9\bar{\delta}^2}{2!}(v_1^{\mathrm{D}})^2 z_2(k) + \frac{\bar{\delta}^3}{3!}(19v_{21}^{\mathrm{D}} + 7v_{22}^{\mathrm{D}} + v_{23}^{\mathrm{D}})(v_1^{\mathrm{D}})^2 \end{array}\right. \tag{12}$$

The motion planning problem can be stated as: given an initial state $z^0 = (z_1^0, z_2^0, z_3^0, z_4^0, z_5^0, z_6^0)^{\mathrm{T}}$ and a desired final state $z^f = \left(z_1^f, z_2^f, z_3^f, z_4^f, z_5^f, z_6^f\right)^{\mathrm{T}}$, find control inputs $(v_1^{\mathrm{D}}, v_{21}^{\mathrm{D}}, v_{22}^{\mathrm{D}}, v_{23}^{\mathrm{D}}, v_{31}^{\mathrm{D}}, v_{32}^{\mathrm{D}})$ such that the system described above with initial condition $z(k) = z^0$ satisfies the endpoint condition $z(k+1) = z^f$.

Proposition 2 *Given a pair of states $\{z^0, z^f\}$ with $z_1^f - z_1^0 \neq 0$, then there exists a multi-rate control steering exactly z^0 to z^f in one step of amplitude δ.*

□

Proof : The proof is straightforward. Indeed, solving the set of equations (12), one obtains

$$v_1^{\mathrm{D}} = (\Delta_1^{\bar{\delta}})^{-1}\left(z_1^f - z_1^0\right) \tag{13}$$

$$\begin{pmatrix} v_{21}^{\mathrm{D}} \\ v_{22}^{\mathrm{D}} \\ v_{23}^{\mathrm{D}} \end{pmatrix} = (\Delta_2^{\bar{\delta}})^{-1} \begin{pmatrix} z_2^f - z_2^0 \\ z_4^f - z_4^0 - 3\bar{\delta} v_1^{\mathrm{D}} z_2^0 \\ z_6^f - z_6^0 - 3\bar{\delta} v_1^{\mathrm{D}} z_4^0 - \frac{9\bar{\delta}^2}{2!}(v_1^{\mathrm{D}})^2 z_2^0 \end{pmatrix} \tag{14}$$

$$\begin{pmatrix} v_{31}^{\mathrm{D}} \\ v_{32}^{\mathrm{D}} \end{pmatrix} = (\Delta_3^{\bar{\delta}})^{-1} \begin{pmatrix} z_3^f - z_3^0 \\ z_5^f - z_5^0 - 3\bar{\delta} v_1^{\mathrm{D}} z_3^0 \end{pmatrix} \tag{15}$$

with

$$\Delta_1^{\bar{\delta}} = 3\bar{\delta} \tag{16}$$

$$\Delta_2^{\bar{\delta}} = \begin{pmatrix} \bar{\delta} & \bar{\delta} & \bar{\delta} \\ \frac{5}{2!}\bar{\delta}^2 v_1^{\mathrm{D}} & \frac{3}{2!}\bar{\delta}^2 v_1^{\mathrm{D}} & \frac{1}{2!}\bar{\delta}^2 v_1^{\mathrm{D}} \\ \frac{19}{3!}\bar{\delta}^3 (v_1^{\mathrm{D}})^2 & \frac{7}{3!}\bar{\delta}^3 (v_1^{\mathrm{D}})^2 & \frac{1}{3!}\bar{\delta}^3 (v_1^{\mathrm{D}})^2 \end{pmatrix} \tag{17}$$

$$\Delta_3^{\bar{\delta}} = \begin{pmatrix} \bar{\delta} & \bar{\delta} \\ \frac{5}{2!}\bar{\delta}^2 v_1^{\mathrm{D}} & 2\bar{\delta}^2 v_1^{\mathrm{D}} \end{pmatrix} \tag{18}$$

The nonsingularity of the matrices $\Delta_2^{\bar{\delta}}$ and $\Delta_3^{\bar{\delta}}$ is ensured by the condition v_1^{D} different from zero, in other words $z_1^f \neq z_1^0$. □□

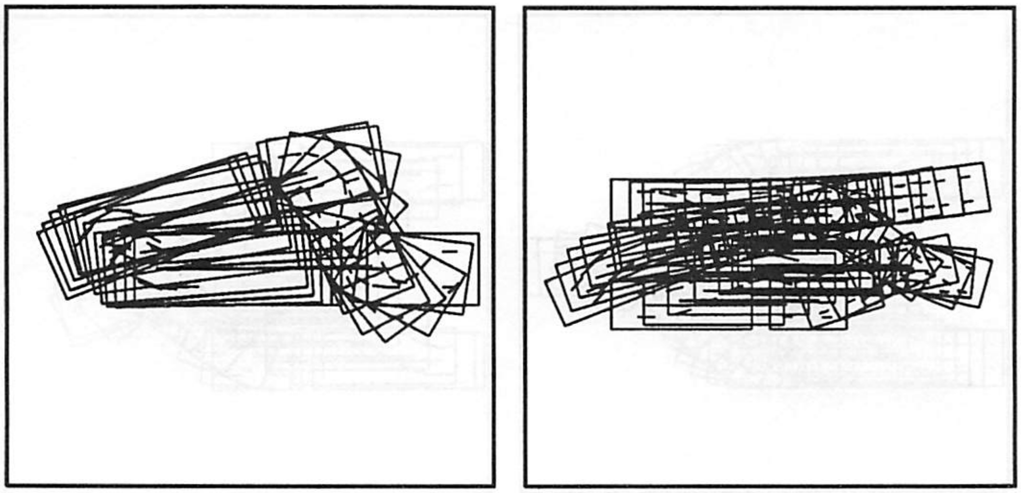
The case $z_1^f = z_1^0$, when $z_6^f \neq z_6^0$, is generally referred in the literature as the parallel parking problem. In the real system, this corresponds to moving sideways, without changing the x-coordinate. There are several approaches that can be taken to solve the problem of parallel parking. The first one consists of a higher multi-rate order on the controls, giving more degrees of freedom. The number of inputs is then greater than the number of equations to be solved, and an optimization method may be used to find the "best solution" according to some criterion, for example, least norm.

The second approach, chosen here, is somewhat simpler. When a parallel parking trajectory is required, we plan the path in two stages. An intermediate point is chosen that is halfway between the initial and final values in all the coordinates except x, and the intermediate x value is chosen to be the initial x increased by the difference between the initial and final y values.

4 Simulation Results and Comparisons

We include three trajectories here. In all cases, the simulations were done in the chained form coordinates, and the results have been transformed back into the standard coordinates for ease of presentation and interpretation. We note here that we have chosen the lengths $l_0 = 1$ and $l_1 = 3$.

For each pair of start and goal points, we have planned a trajectory using two different methods. One method is the multi-rate scheme which we propose in this paper. As a means of comparison, we have also followed the step-by-step sinusoids algorithm presented in [1]. This algorithm which plans paths for systems in chained form requires one step for each level of a chain. For example, the firetruck system has a three-level chain and so three steps are needed. At the first step, constant inputs v_1, v_2, v_3 are chosen to steer states z_1, z_2, and z_3 to their desired values. The next step calls for out-of-phase sinusoids to steer the first level states in the two chains. These inputs are $v_1 = \alpha \sin(\omega t)$, $v_2 = \beta \cos(\omega t)$, and $v_3 = \gamma \cos(\omega t)$. After this step, the states $z_1, \ldots, z_5$ are in their final positions. The last step uses a double-frequency sinusoid to steer the state z_6, the second level of the chain. The inputs $v_1 = \alpha \sin(\omega t)$, $v_2 = \beta \cos(2\omega t)$, and $v_3 = 0$ will result in the system arriving at the goal position.



The multi-rate path The sinusoidal path

Figure 2: The trajectories planned by the two algorithms for the arbitrary path. Note the directness of the multi-rate path. The start and goal points are the same in the two cases, and indeed, both algorithms plan paths that successfully reach the goal configuration.

This sinusoidal scheme is interesting because it relies on frequency cancellations to achieve the desired motions. It is particularly powerful on "parallel parking" trajectories, since the parallel parking direction corresponds exactly to z_6 and so only one step is needed. However, as we will see, it does not take into account any of the properties of the system, and can therefore plan somewhat impractical trajectories.

The first trajectory was rather arbitrarily chosen to go from an initial position of

$$(x, y, \phi_0, \theta_0, \phi_1, \theta_1) = (-2, 2, 0.1, 0.2, 0.5, 0.4)$$

to a final position of $(0, 0, 0, 0, 0, 0)$. The traces of this trajectory as planned by both the multi-rate and sinusoidal algorithms can be seen in Figure 2.

The second trajectory shown here is a parallel parking trajectory, illustrating our technique of choosing a point distant from the given points, and then using two multi-rate controls. The corresponding trajectory has therefore twice as many input variables, but is planned as two separate connecting paths. The initial position here is

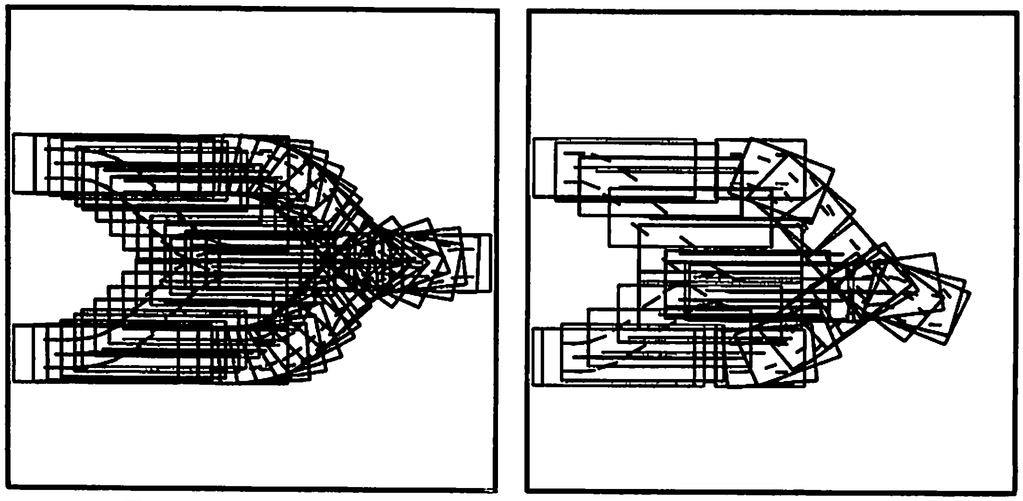
$$(x, y, \phi_0, \theta_0, \phi_1, \theta_1) = (0, 5, 0, 0, 0, 0)$$

and the goal is once again the origin, $(0, 0, 0, 0, 0, 0)$. For the multi-rate path, the intermediate point is chosen to be $(5, 2.5, 0, 0, 0, 0)$. See Figure 3 for the traces of these two trajectories.

The third path we have chosen to plan is a corner trajectory, where the firetruck starts out more or less aligned with the y axis, and ends up aligned with the x axis (due to the singularity in the coordinate transformation, we have not planned a full 90^o turn). The initial position in both cases is

$$(x, y, \phi_0, \theta_0, \phi_1, \theta_1) = (-5, -5, 0, 1.27, 0, 1.27)$$

and the goal point is $(0, 0, 0, 0, 0, 0)$. Both algorithms successfully steer the system from the start to the goal, but this example illustrates very clearly the advantages of the multi-rate algorithm.



The multi-rate path The sinusoidal path

Figure 3: The trajectories planned by the two algorithms for parallel parking. Note the similarities between the two methods for this case. This is partly due to the way the multi-rate algorithm as proposed deals with the special case of parallel parking. Two paths are planned, the first from the start to an intermediate point halfway between the start and the goal in all variables except x, with the x-coordinate increased to a value of the total desired change in y. In this case, the intermediate point is $(5, 2.5, 0, 0, 0, 0)$. The second path leads from the intermediate point to the goal.

The sinusoidal algorithm in the first step drives the system forwards until the x-coordinate is at its desired final position, and in doing so, diverges radically from the area of interest.

5 Conclusion

In this paper we have examined the multi-rate control algorithm as applied to the problem of path planning for nonholonomic systems. We have presented the theory for this control algorithm, and applied it to the specific example of a three-input nonholonomic system. The simulation results presented here attest to the practicality of this method for planning short, direct paths from the start to the goal. This is especially evident when contrasted with the method using sinusoids, also discussed here.

It should be noted that the multi-rate control is most useful for planning short paths which satisfy the nonholonomic constraints. If the robot were required to traverse a long distance or avoid obstacles, a holonomic path could first be planned using well-known standard techniques (see for example [2]). The main controller could choose a series of points along this path, which could then be connected using feasible paths generated by the multi-rate algorithm.

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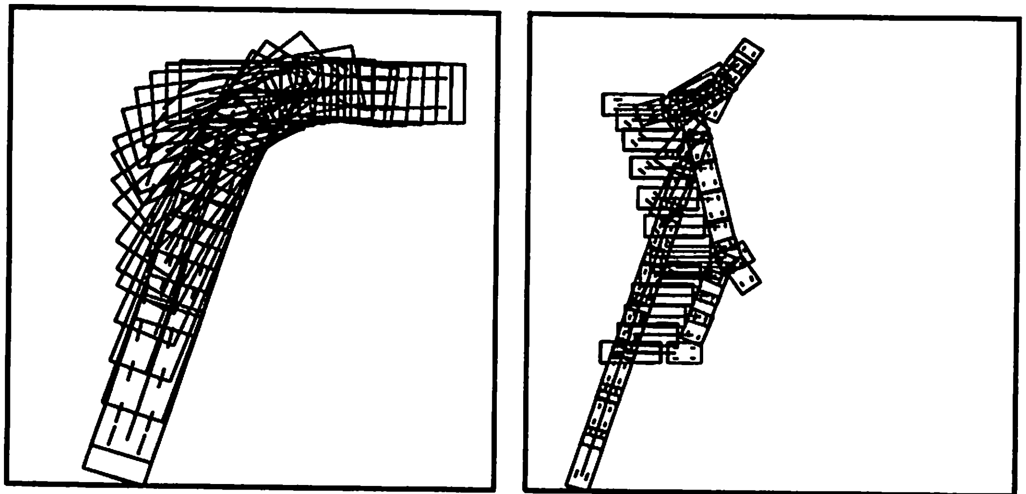

The multi-rate path The sinusoidal path

Figure 4: The trajectories planned by the two algorithms for going around a corner. Note how smooth and direct the multi-rate path is. The start and goal points are the same in the two cases, although the scale has been greatly reduced in the sinusoidal trace in order to show the entire trajectory. This example perhaps best exemplifies the problems with the sinusoidal algorithm and the reasons for choosing the driving input constant in the multi-rate scheme.

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