# A SIMPLE ODE WITH MORE THAN 20 STRANGE ATTRACTORS

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## ELECTRONICS RESEARCH LABORATORY

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# A simple ODE with more than 20 strange attractors

*Leon O. Chua*

*University of California, Berkeley*

**Abstract.** This paper presents a glimpse at some complicated nonlinear dynamics and bifurcation phenomena of a remarkably simple 3rd-order ODE whose only nonlinearity is a *scalar* function of a single variable $f(x)$. This ODE is derived from a real electronic circuit imbued with more than 20 strange attractors. This circuit, which contains only 5 *linear* elements (2 resistors, 1 inductor, and 2 capacitors) and a *nonlinear* resistor is the *simplest* electronic circuit that can become chaotic for certain parameter values.

The significance of this 5-parameter ODE is that it is *topologically conjugate* to a 21-parameter family of *piecewise-linear* odd-symmetric vector fields. Moreover, virtually every known bifurcation phenomena from nonlinear dynamics is exhibited by this ODE. Although over a hundred papers on a subclass of this chaotic circuit have already been published, a rigorous and in-depth mathematical study of the nonlinear dynamics of the associated ODE remains a challenging research problem.

## 1. The Canonical ODE

The circuit shown in Table 1 is described by the following system of 3rd-order ordinary differential equations:

$$\begin{array}{rcl} C_1\dot{v}_1 & = & \frac{1}{R}(v_2 - v_1) - \hat{f}(v_1) \\ C_2\dot{v}_2 & = & \frac{1}{R}(v_1 - v_2) + i_3 \\ L\dot{i}_3 & = & -v_2 - R_0 i_3 \end{array} \tag{1}$$

where $C_1$,$C_2$,$L$,$R$,$R_0$ are real numbers, and

$$\hat{f}(v_R) = G_b v_R + \frac{1}{2}(G_a - G_b)\left\{|v_R + B_p| - |v_R - B_p|\right\} \tag{2}$$

denotes the 3-segment odd-symmetric *voltage-current* characteristic of the nonlinear resistor (known as Chua's diode [Ke]) with slopes $G_a$,$G_b$ and breakpoints located at $v_R = -B_p$ and $v_R = B_p$, respectively. By an appropriate change of variables, we can transform Eq (1)-(2) into the following dimensionless form:

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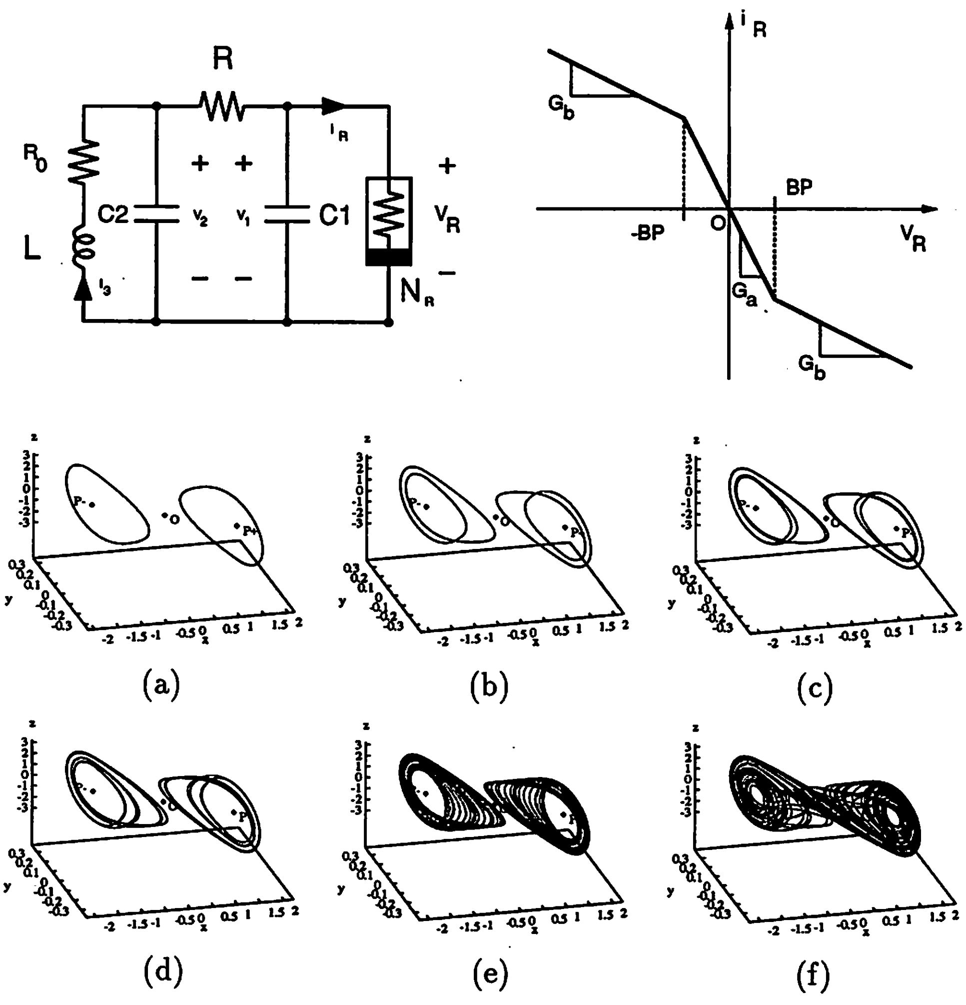
**Table 1** Chua's oscillator circuit and a typical bifurcation sequence with $R_0 = 0$ (Chua's circuit), $\gamma = 0$, $a = -\frac{8}{7}$, $b = -\frac{5}{7}$ and $\beta = 16$. In (a)-(e), the attractor and its twin lying symmetrically with respect to the origin and the three equilibrium points $P^+$, $O$, $P^-$ are also shown. (a) Period 1 limit cycle ($\alpha = 8.8$). (b) Period 2 limit cycle ($\alpha = 9.05$). (c) Period 4 limit cycle ($\alpha = 9.12$). (d) Period 8 limit cycle ($\alpha = 9.162$). (e) Rössler attractor ($\alpha = 9.3$). (f) Double Scroll attractor ($\alpha = 9.8$).



Case I: $RC_2 > 0$

$$\boxed{\begin{array}{rcl} \dot{x} & = & \alpha\,(y - x - f(x)) \\ \dot{y} & = & x - y + z \\ \dot{z} & = & -\beta y - \gamma z \end{array}} \tag{3}$$

Case II: $RC_2 < 0$

$$\boxed{\begin{array}{rcl} \dot{x} & = & \alpha\,(-y + x + f(x)) \\ \dot{y} & = & -x + y - z \\ \dot{z} & = & \beta y + \gamma z \end{array}} \tag{4}$$

$$f(x) = bx + \frac{1}{2}(a - b)\,\{|x + 1| - |x - 1|\} \tag{5}$$

where

$$\begin{array}{lll} x \triangleq \frac{v_1}{B_p}, & y \triangleq \frac{v_2}{B_p}, & z \triangleq i_3\left(\frac{R}{B_p}\right) \\ \alpha \triangleq \frac{C_2}{C_1}, & \beta \triangleq \frac{R^2 C_2}{L}, & \gamma \triangleq \frac{R R_0 C_2}{L} \\ a \triangleq RG_a, & b \triangleq RG_b, & \text{and} \quad \tau \triangleq \frac{t}{|RC_2|} \end{array} \tag{6}$$

and the derivatives in Eq (3)-(4) are with respect to the dimensionless variable $\tau$ which corresponds to time rescaling. Observe that Eq (4) is equivalent to integrating Eq (3) in reverse time. The literature cited in the reference section is concerned almost exclusively with the special case called Chua's circuit [Ma,Ch] where $\gamma = 0$ (corresponding to $R_0 = 0$) and Eq (3) and (5) are used. A typical bifurcation sequence for this case with $a = -\frac{8}{7}$, $b = -\frac{5}{7}$, $\beta = 16$ and $\alpha$ varying from $\alpha = 8.8$ to $\alpha = 9.8$ is shown in Table 1. Observe that the dimensionless system of ODE in (3)-(5) is uniquely specified by the 5 parameters $\{\alpha, \beta, \gamma, a, b\}$. Note that the vector field $\mathbf{F}(\cdot)$ described by (3)-(5) is odd symmetric; namely

$$\mathbf{F}(x, y, z) = -\mathbf{F}(-x, -y, -z) \tag{7}$$

In the case of Chua's circuit ($\gamma = 0$), equations (3)-(5) give the following 3 equilibrium states:

$$\begin{array}{ll} P^+ : & (x, y, z) = (x^*, 0, -x^*) \\ O \;: & (x, y, z) = (0, 0, 0) \\ P^- : & (x, y, z) = (-x^*, 0, x^*) \end{array} \tag{8}$$

where $x^* > 0$ is a solution of the equilibrium equation

$$f(x) = -x \tag{9}$$

**Remark** The nonlinear function $f(x)$ in equations (3)-(4) is represented by the piecewise-linear function in Eq (5) in order to allow the application of linear system analysis. For more analytical studies, it is often desirable to represent $f(x)$ by a $C^\infty$ function. For example,

$$f(x) = a_0 x + a_1 x^3 \tag{10}$$

is used in [Al,Ha] and in several papers in a 2-volume special issue devoted to Chua's circuit [Mda,Mdb]. In fact, Chua's diode characterized by almost any non-linear function, not necessarily piecewise-linear or symmetric, can be fabricated by standard electronic circuit techniques [Chu]. Consequently, from both mathematical and physical point of view, it is meaningful and significant to conduct an in-depth mathematical study of the ODE (3)-(4), where $f(x)$ is replaced by various classes of $C^\infty$ functions.

## 2. A Gallery of Attractors from Chua's Oscillator

In the general case where $R_0 \neq 0$ and hence $\gamma \neq 0$, the circuit in Table 1 is called a *canonical* Chua's circuit, or Chua's oscillator [Md] in the literature. The addition of the *linear* term $\gamma z$ in the original Chua's circuit [Ma] serves as a *global unfolding* of the ODE from Chua's circuit, and allows us to uncover many more non-periodic attractors, 24 of which are exhibited in Table 2, along with their parameter values. It is reasonable to expect that many more attractors will be discovered in the future.

The attractors in Table 2 are arranged according to the relative locations of the eigenvalues $\{\mu_1, \mu_2, \mu_3\}$ in the *inner* linear region. For the first 5 attractors, all eigenvalues are real. For the next 15 attractors, there is a pair of complex-conjugate eigenvalues and a real positive eigenvalue. For the final 4 attractors there is a pair of complex-conjugate eigenvalues and a negative real eigenvalue. In the outer region, there is always 1 real eigenvalue and a pair of complex-conjugate eigenvalues. We will henceforth refer to the location of these eigenvalues as the eigenvalue pattern. Note that the attractors in Table 2 (c), (f), (k)-(n) and (u) are obtained from Chua's circuit ($\gamma = 0$).

The circuit in Table 1 with $R_0 \neq 0$ is said to be *canonical* because given any set of real and/or complex-conjugate eigenvalues

$$\{\mu_1, \mu_2, \mu_3; \nu_1, \nu_2, \nu_3\} \tag{11}$$

there exists a set of parameter values

$$\{\alpha, \beta, \gamma, a, b\} \tag{12}$$

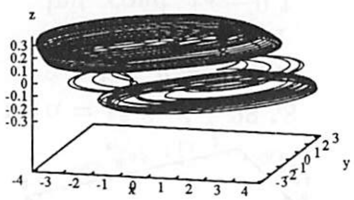
such that Eq (3)-(5) has the above prescribed eigenvalues, except possibly for a set of measure zero, in which case a set of parameter values $\{\alpha, \beta, \gamma, a, b\}$ exists which realizes the nearby eigenvalues

$$\{\mu_1 + \delta\mu_1, \mu_2 + \delta\mu_2, \mu_3 + \delta\mu_3; \nu_1 + \delta\nu_1, \nu_2 + \delta\nu_2, \nu_3 + \delta\nu_3\} \tag{13}$$

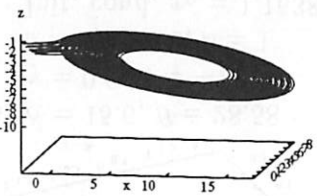
where $\{\delta\mu_1, \delta\mu_2, \delta\mu_3, \delta\nu_1, \delta\nu_2, \delta\nu_3\}$ are arbitrarily small perturbations (all except one may be zero). Indeed, the explicit formulas for calculating $\{\alpha, \beta, \gamma, a, b\}$ are as follows:

$$\begin{aligned}
\alpha &= \tfrac{1}{k_1 k_3^2} \\
\beta &= 1 + \tfrac{1}{k_1^2 k_3^2}\tfrac{p_3 - q_3}{p_1 - q_1} - \tfrac{-p_2 + q_2}{p_1 - q_1}\tfrac{1}{k_1 k_3} \\
\gamma &= -1 + \tfrac{-p_2 + q_2}{p_1 - q_1}\tfrac{1}{k_1 k_3} \\
a &= -1 - \left(p_1 + \tfrac{-p_2 + q_2}{p_1 - q_1}\right) k_3 \\
b &= -1 - \left(q_1 + \tfrac{-p_2 + q_2}{p_1 - q_1}\right) k_3 \\
k &= \operatorname{sgn}(k_1 k_3)
\end{aligned} \tag{14}$$

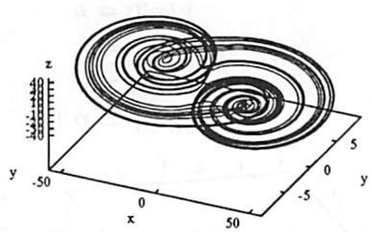
**Table 2** A gallery of 24 attractors from Chua's oscillator circuit.
If $k = 1$, then Eq (3) is used, otherwise Eq (4) is used.



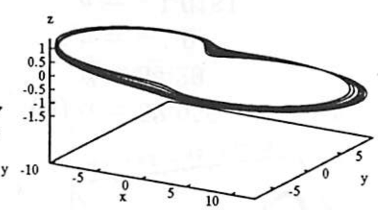
(a) $\alpha = -1.301$
$\beta = -0.0136$
$\gamma = -0.02969$
$a = 0.1690$
$b = -0.4767,\ k = 1$



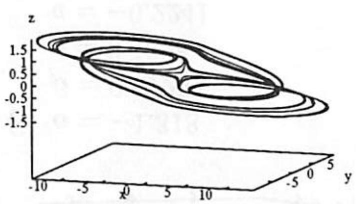
(b) $\alpha = -17.38$
$\beta = 1.413$
$\gamma = 1.045$
$a = -1.525$
$b = -0.4576,\ k = -1$



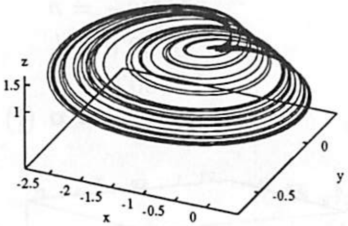
(c) $\alpha = -4.925$
$\beta = -3.649$
$\gamma = 0.0$
$a = -2.497$
$b = -0.9301,\ k = 1$



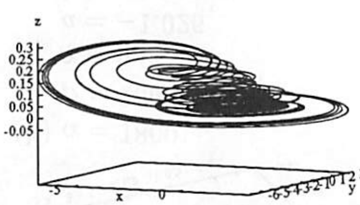
(d) $\alpha = -1.36$
$\beta = -0.0874$
$\gamma = -0.31114$
$a = 1.29215$
$b = -0.49717,\ k = -1$



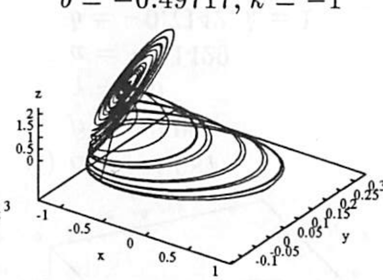
(e) $\alpha = -1.461$
$\beta = -0.09422$
$\gamma = -0.3230$
$a = 1.219$
$b = -0.5141,\ k = -1$



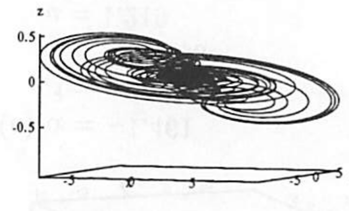
(f) $\alpha = -6.691$
$\beta = -1.520$
$\gamma = 0.0$
$a = -1.1429$
$b = -0.7142,\ k = 1$



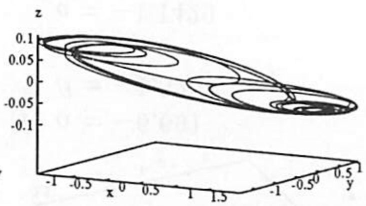
(g) $\alpha = -1.233$
$\beta = 0.007233$
$\gamma = 0.08758$
$a = -0.1767$
$b = -0.01626,\ k = -1$



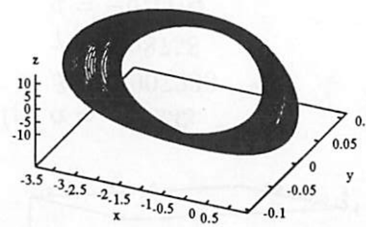
(h) $\alpha = -75.0$
$\beta = 31.25$
$\gamma = -3.125$
$a = -0.98$
$b = -2.4,\ k = 1$



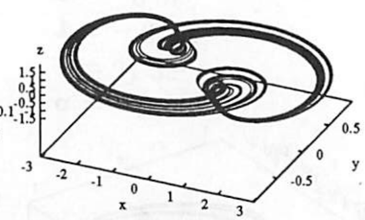
(i) $\alpha = -1.318$
$\beta = 0.01257$
$\gamma = 0.1328$
$a = -0.2241$
$b = -0.02811,\ k = -1$



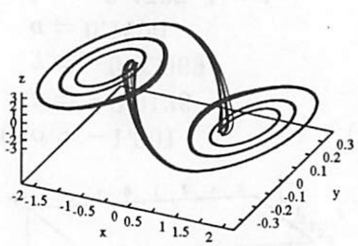
(j) $\alpha = -1.424$
$\beta = 0.02944$
$\gamma = 0.3226$
$a = -0.0715$
$b = -0.1817,\ k = 1$



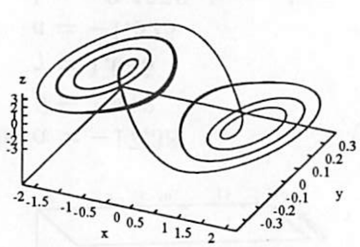
(k) $\alpha = 1800$
$\beta = 10000$
$\gamma = 0.0$
$a = -1.026$
$b = -0.982,\ k = 1$



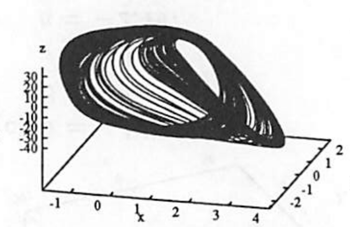
(l) $\alpha = -4.087$
$\beta = -2.00$
$\gamma = 0.0$
$a = -1.1429$
$b = -0.7142,\ k = 1$



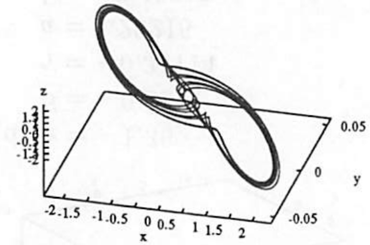
(m) $\alpha = 15.6,\ \beta = 28.58$
$\gamma = 0.0,\ a = -1.143$
$b = -0.7142,\ k = 1$
Init. cond. $x_0 = 0.1$
$y_0 = 0.1$
$z_0 = 0.1$



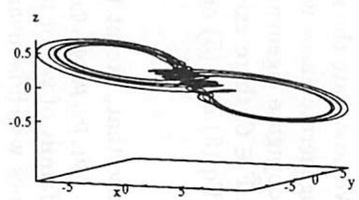
(n) $\alpha = 15.6,\ \beta = 28.58$
$\gamma = 0.0,\ a = -1.143$
$b = -0.7142,\ k = 1$
Init. cond. $x_0 = 1.1638$
$y_0 = -0.0972335$
$z_0 = -0.90565$



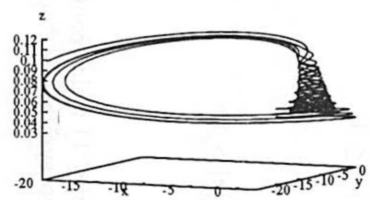
(o) $\alpha = 13.32$
$\beta = 213.1$
$\gamma = -0.9408$
$a = 0.474$
$b = -2.039$
$k = -1$



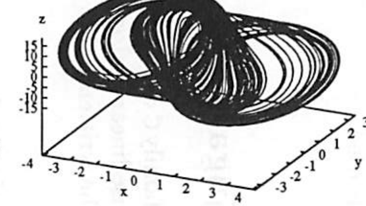
(p) $\alpha = 52.056$
$\beta = 54.29$
$\gamma = -1.0$
$a = -1.0181$
$b = -1.02$
$k = -1$



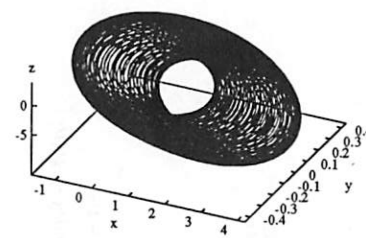
(q) $\alpha = -1.559$
$\beta = 0.01564$
$\gamma = 0.1574$
$a = -0.2438$
$b = -0.04251,\ k = -1$



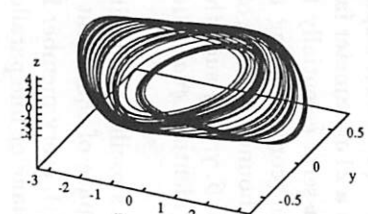
(r) $\alpha = -1.083$
$\beta = 0.0000969$
$\gamma = 0.007327$
$a = -0.09411$
$b = 0.0001899,\ k = -1$



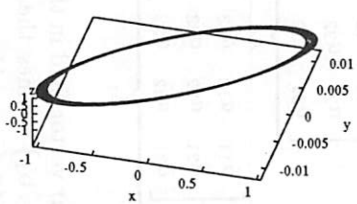
(s) $\alpha = 3.7091$
$\beta = 24.07$
$\gamma = -0.8592$
$a = -2.764$
$b = 0.1805,\ k = 1$



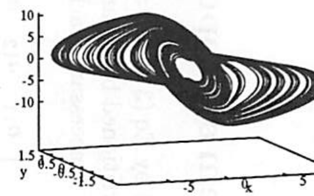
(t) $\alpha = 31.53$
$\beta = 64.25$
$\gamma = -0.6683$
$a = -0.9926$
$b = -1.023,\ k = -1$



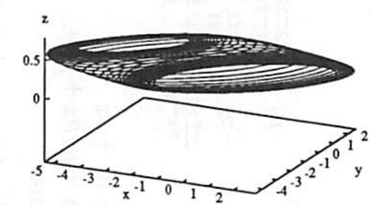
(u) $\alpha = 8.342$
$\beta = 11.925$
$\gamma = 0.0$
$a = -0.7048$
$b = -1.146,\ k = 1$



(v) $\alpha = -100.18$
$\beta = -98.82$
$\gamma = -1.00$
$a = -0.99002$
$b = -0.9893,\ k = -1$



(w) $\alpha = -75.0$
$\beta = 31.25$
$\gamma = -3.125$
$a = -2.4$
$b = -0.98,\ k = 1$



(x) $\alpha = -1.609$
$\beta = 0.0704$
$\gamma = 0.2973$
$a = -0.1392$
$b = -0.2175,\ k = 1$

where

$$\begin{aligned} k_1 &= -p_2 - \left(\frac{p_2-q_2}{p_1-q_1} - p_1\right)\frac{p_2-q_2}{p_1-q_1} + \frac{p_3-q_3}{p_1-q_1} \\ k_2 &= -p_3 + \frac{p_3-q_3}{p_1-q_1}\left(p_1 + \frac{-p_2+q_2}{p_1-q_1}\right) \\ k_3 &= \frac{-p_2+q_2}{p_1-q_1}\frac{1}{k_1} + \frac{k_2}{k_1^2} \end{aligned} \tag{15}$$

and

$$\begin{aligned} p_1 &= \mu_1+\mu_2+\mu_3 & q_1 &= \nu_1+\nu_2+\nu_3 \\ p_2 &= \mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1 & q_2 &= \nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1 \\ p_3 &= \mu_1\mu_2\mu_3 & q_3 &= \nu_1\nu_2\nu_3 \end{aligned} \tag{16}$$

In Eq (14), if $k = 1$, then Eq (3) is used. If $k = -1$, then Eq (4) is used.

## 3. Theorem on Topological Conjugacy

The system of ODE described by Eq (3)-(5) is a special case of the family $\mathcal{C}$ of all continuous odd-symmetric, 3-region (partitioned by 2 parallel planes) piecewise-linear vector fields in $\mathbb{R}^3$. By changing coordinates if necessary, each member of $\mathcal{C}$ can be written in the form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{cases} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathbf{A_0}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, & \text{if } |x| \le 1 \\ \underbrace{\begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} \\ \hat{a}_{21} & \hat{a}_{22} & \hat{a}_{23} \\ \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} \end{bmatrix}}_{\mathbf{A_1}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b_1}}, & \text{if } |x| > 1 \end{cases} \tag{17}$$

where $\mathbf{A_0}$ denotes the *linear* vector field in the "inner" region $M_0$ ($|x| \le 1$) which contains the origin and where $\{\mathbf{A_1}, \mathbf{b_1}\}$ defines the *affine* region in the two symmetric "outer" regions $M^+$ ($x \ge 1$), and $M^-$ ($x \le -1$), respectively.

Equation (17) represents a 21-parameter family of vector fields. We now state the main result of this paper which asserts essentially that, except for the degenerate cases where an eigenvector or eigenplane (corresponding to a pair of complex-conjugate eigenvalues) of $\mathbf{A_0}$ or $\mathbf{A_1}$ is parallel to a boundary plane of the given vector field $\hat{\mathbf{F}} \in \mathcal{C}$, there exists a *unique* set of parameters $\{\alpha, \beta, \gamma, a, b\}$ such that either the ODE in Eq (3) and (5) or the ODE in Eq (4) and (5) has identical *qualitative* behaviors.

**Main Theorem** The Chua's oscillator circuit is *canonical* in the sense that, except for a set of measure zero in the space of equivalent eigenvalue parameters $\{p_1, p_2, p_3; q_1, q_2, q_3\}$ in $\mathbb{R}^6$ (as defined by Eq. (16)), every member $\hat{\mathbf{F}}$ in the 21-parameter family of vector fields in $\mathcal{C}$, such that there is no plane or line parallel to the boundary planes which is invariant under the action of $\hat{\mathbf{F}}$ in the middle region, is *linearly conjugate* to a unique vector field $\mathbf{F}$ defined by the ODE in Eq (3) and (5), or the ODE in Eq (4) and (5).

*Proof.* A vector field in the set $C$ can be rewritten in the following form by a suitable change of basis [CK]:

$$\dot{\mathbf{x}} = \hat{\mathbf{F}}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \frac{1}{2}\{|<\mathbf{w},\mathbf{x}>+1| - |<\mathbf{w},\mathbf{x}>-1|\}\,\mathbf{b} \tag{18}$$

where $\mathbf{A} \in \mathbb{R}^{3\times 3}$ is in a Jordan canonical form and $\mathbf{x}, \mathbf{w}, \mathbf{b} \in \mathbb{R}^3$. Setting $\mathbf{x} = \mathbf{K}^{-1}\mathbf{y}$, equation (18) transforms to

$$\dot{\mathbf{y}} = \mathbf{K}\hat{\mathbf{F}}(\mathbf{K}^{-1}\mathbf{y}) = \mathbf{K}\mathbf{A}\mathbf{K}^{-1}\mathbf{y} + \frac{1}{2}\{|<(\mathbf{K}^{-1})^T\mathbf{w},\mathbf{y}>+1| - |<(\mathbf{K}^{-1})^T\mathbf{w},\mathbf{y}>-1|\}\,\mathbf{K}\mathbf{b} \tag{19}$$

For each of the following Jordan form matrices $\mathbf{A}$ and some conditions on $\mathbf{w} = (w_1, w_2, w_3)^T$, there exists a nonsingular matrix $\mathbf{K}$ such that $\mathbf{K}\mathbf{A}\mathbf{K}^{-1}$ is in *companion* form, namely:

$$\mathbf{K}\mathbf{A}\mathbf{K}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q_3 & -q_2 & q_1 \end{pmatrix} \tag{20}$$

and $(\mathbf{K}^{-1})^T\mathbf{w} = (1,0,0)^T$. The conditions on $\mathbf{w}$ correspond to the assumption that there is no plane or line parallel to the boundary planes which is invariant under the action of $\hat{\mathbf{F}}$ in the middle region. In particular, $\mathbf{K}$ is defined explicitly as follows:

$$\begin{array}{lll} \mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} & \rightarrow & \mathbf{K} = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{pmatrix} \\ w_1 \neq 0, w_2 \neq 0, w_3 \neq 0 & & \\ \mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & \sigma & -\omega \\ 0 & \omega & \sigma \end{pmatrix} & \rightarrow & \mathbf{K} = \begin{pmatrix} 1 & c_1 & c_2 \\ a & c_1\sigma + c_2\omega & c_2\sigma - c_1\omega \\ a^2 & k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & -w_3 \\ 0 & w_3 & w_2 \end{pmatrix} \\ w_1 \neq 0, w_2^2 + w_3^2 \neq 0 & & \end{array} \tag{21}$$

where

$$k_{32} = c_1\sigma^2 + 2c_2\sigma\omega - c_1\omega^2, \quad k_{33} = c_2\sigma^2 - 2c_1\sigma\omega - c_2\omega^2 \tag{22}$$

$$c_1 = \frac{w_2^2 - w_3^2}{w_2^2 + w_3^2}, \quad c_2 = \frac{2w_2w_3}{w_2^2 + w_3^2} \tag{23}$$

and $a$, $b$ and $c$ are distinct and $\omega \neq 0$. The set of eigenvalue parameters which does not correspond to the above Jordan forms has measure zero. Next we show that $\mathbf{K}\mathbf{b} = (e_1, e_2, e_3)^T$ is uniquely determined by the two prescribed sets of eigenvalues $\{\mu_1, \mu_2, \mu_3\}$

and $\{\nu_1, \nu_2, \nu_3\}$, or equivalently $\{p_1, p_2, p_3\}$ and $\{q_1, q_2, q_3\}$. From equation (19) the Jacobian matrix of the vector field in the inner region $M_0$ is given by:

$$\begin{pmatrix} e_1 & 1 & 0 \\ e_2 & 0 & 1 \\ e_3 + q_3 & -q_2 & q_1 \end{pmatrix} \tag{24}$$

Its characteristic polynomial is given by:

$$\lambda^3 - \lambda^2(q_1 + e_1) + \lambda(e_1 q_1 + q_2 - e_2) - (e_3 + q_3 + q_2 e_1 - q_1 e_2) = \lambda^3 - p_1\lambda^2 + p_2\lambda - p_3 \tag{25}$$

Equating the respective coefficients, we obtain

$$\begin{aligned} e_1 &= p_1 - q_1 \\ e_2 &= -p_2 + q_2 + q_1 e_1 \\ e_3 &= p_3 - q_3 - q_2 e_1 + q_1 e_2 \end{aligned} \tag{26}$$

which uniquely defines $\mathbf{Kb} = (e_1, e_2, e_3)^T$. We have thus shown that almost every vector field in the class $\mathcal{C}$ can be transformed into a canonical form with $\mathbf{A}$ in companion form and $\mathbf{w} = (1, 0, 0)^T$. Therefore almost any two vector fields in the class $\mathcal{C}$ with the same eigenvalue parameters can be transformed into the same canonical form and by transitivity these two vector fields are linearly conjugate. The proof is then complete, since by Eq (14)-(16) any set of eigenvalue parameters, except for a set of measure zero, can be realized by Eq (3)-(5). □

# 4. Concluding Remarks

We have presented a simple ODE which represents not only an excellent model of a real physical electronic circuit, but is also imbued with virtually every known dynamical and bifurcation phenomena, including chaos. This ODE is canonical in the sense that it is the simplest ODE imbued with the most complex dynamics. Much future research remains to be done by replacing the piecewise-linear function $f(x)$ by an *arbitrary* $C^\infty$ function.

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