A STRUCTURE THEOREM FOR PARTIALLY ASYNCHRONOUS RELAXATIONS

Summary of Results

by

Reza Gharavi and Venkat Anantharam

Memorandum No. UCB/ERL M92/143

15 December 1992

A STRUCTURE THEOREM FOR PARTIALLY ASYNCHRONOUS RELAXATIONS

Summary of Results

by

Reza Gharavi and Venkat Anantharam

Memorandum No. UCB/ERL M92/143

15 December 1992

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

A STRUCTURE THEOREM FOR PARTIALLY ASYNCHRONOUS RELAXATIONS

Summary of Results

by

Reza Gharavi and Venkat Anantharam

Memorandum No. UCB/ERL M92/143

15 December 1992

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

A STRUCTURE THEOREM FOR PARTIALLY ASYNCHRONOUS RELAXATIONS *
Summary of Results

Reza Gharavi and Venkat Anantharam

School of Electrical Engineering
Cornell University
Ithaca, NY 14853, U.S.A.

Abstract

We consider partially asynchronous computations with stationary ergodic interprocessor communication delays. We focus on parallel iteration of a fixed matrix. Under these conditions, by Oseledec's multiplicative ergodic theorem, there will exist only a finite number of nonrandom rates, each with its own corresponding random subspace, such that the computation will converge at the rate corresponding to the subspace which the initial condition lies in. However, neither the convergence rates nor their corresponding subspaces can be easily specified. Here we construct a computation graph by following the interprocessor dependencies. By studying this graph, we discover a number of invariant properties, and give a direct demonstration that there is almost surely a constant number of subspaces of initial conditions corresponding to different dynamics of the computation. We then relate our subspaces with Oseledec's subspaces. Our results are particularly strong when both the matrix and the initial conditions are non-negative. Under such assumptions, we can relate each coordinate of the initial condition to a unique convergence rate based on the invariant sets of initial conditions it belongs to.

1. Introduction

We consider the asymptotic behavior of iterative linear systems in partially asynchronous parallel computation environments. Let A be a $p \times p$ matrix. $x(n) \in \mathbf{C}^p$, $n \in \mathbf{Z}^+ = \{1, 2, \ldots\}$, is to be iteratively computed via the equation $x(n+1) = Ax(n)$, with some given initial condition.

Such systems arise in a wide range of applications, the most common of them being in solving systems of linear equations $My = c$, where M is an $p \times p$ matrix, c is a vector in $\mathbf{C}^p$, and y is the desired solution to be determined. Even though direct methods, such

* Research supported by NSF under PYI award NCR 8857731 and IRI 9005849, by IBM under a Faculty Development award, by an AT&T Foundation Award, and by BellCore Inc.

as Gaussian elimination, yield exact solutions after a finite number of steps, it may be advantageous to use iterative methods when M is large and sparse. The reason is twofold– iterative methods can yield very accurate results after a relatively small number of steps and they have smaller memory requirements. Such large and sparse M arise naturally in many important applications, one of the more important ones being in discretization of partial linear differential equations. See [2] and [4].

In a parallel processing environment, there are p processors, each responsible for computing a component of the vector $x(n)$. Due to communication delays among the processors, or between each processor and a shared memory, at each time n, processor i may compute $x_i(n)$ using possibly outdated values, that is,

$$x_i(n) = \sum_{j=1}^{p} a_{ij} x_j(n - d_{ij}(n))$$

where $d_{ij}(n) > 0$ is the communication delay from processor j to processor i at processor i at time n.

The advantages of such asynchronous algorithms are well known. Firstly, since fast processors don't have to wait for messages from every processor, they can execute more iterations, and so a potential speed advantage is gained over synchronous algorithms. See subsection 6.3.5 in [2] for such cases. Secondly, synchronization overhead is reduced. Therefore, such algorithms are usually preferable to synchronous ones, specially in distributed systems. See [2] for a thorough discussion of such algorithms.

In this study, we impose two conditions on the delays. Firstly, we will assume all the delays to be bounded by some integer $B > 0$, that is, our system will be only partially asynchronous, as defined in [2].

Secondly, we will assume that the statistics of the delays can be described by a stationary process. By ergodic decomposition theorem (see [6]), we lose little generality by assuming ergodicity, so we do so. We impose no other condition on the delays; in particular, messages between two processors may overtake each other and d_{ii} may be bigger than one, which may occur, for example, when processors share a common memory. On the other hand, the statistical model is sufficiently versatile that such assumptions can be imposed if they are appropriate for the situation being modeled. For example if $d_{ii} \equiv 1$, we would use a statistical model where this event has probability 1.

Since the assumed bound on the delays makes $x(n)$ depend linearly on the components of $x(n-1), \ldots, x(n-B)$, it is possible to construct a Bp-dimensional system with $\tilde{x}(n) = [x^T(n)\ x^T(n-1)\ \ldots\ x^T(n-B+1)]^T \in \mathbf{C}^{Bp}$, and a sequence of $Bp \times Bp$ matrices $\tilde{A}(n)$ satisfying $\tilde{A}(n+1)\tilde{x}(n) = \tilde{x}(n+1)$ for $n \geq 0$. The statistical assumptions on the delays make the sequence of matrices $\{\tilde{A}(n), n \geq 1\}$ stationary and ergodic. In addition, since $\tilde{A}(1)$ attains at most B^{p^2} different values and its elements are either zero, one, or those of

A, $E(\max(0, \log \|\tilde{A}(1)\|)) < \infty$. It is therefore natural to apply Oseledec's multiplicative ergodic theorem (OMET) to our system (see [1],[3]).

OMET. *If $\{\tilde{A}(n)\}$ is a stationary ergodic sequence such that $E(\max(0, \log \|\tilde{A}(1)\|)) < \infty$, then there exists an integer $s \leq Bp$ with* **nonrandom** *constants $-\infty \leq \lambda_s < \ldots < \lambda_1 < \infty$ called Lyapunov exponents, and nonrandom nonzero integers $\delta_1, \ldots, \delta_s$ satisfying $\sum \delta_k = Bp$, such that the following hold almost surely:*

1. *The random sets*

$$V(k) \stackrel{\text{def}}{=} \{y \in \mathbf{C}^{Bp}\colon \lim_{n \to \infty} n^{-1} \log \|\tilde{A}(n) \cdots \tilde{A}(1) y\| \leq \lambda_k\}$$

are subspaces and if θ is the shift on the probability space for which $\tilde{A}(n, \theta\omega) = \tilde{A}(n+1, \omega)$, then $\tilde{A}(1, \omega)\; V(k, \omega) \subset V(k, \theta\omega)$,

2. *$\dim V(k) = \sum_{i \geq k} \delta_i$,*
3. *If $y \in V(k) \setminus V(k+1)$, with $V(s+1) \stackrel{\text{def}}{=} 0$, then*

$$\lim_{n \to \infty} n^{-1} \log \|\tilde{A}(n) \cdots \tilde{A}(1) y\| = \lambda_k$$

for $k = 1, \ldots, s$.

Therefore, depending on the initial condition, our computation may converge at one of only s growth rates $\lambda_1, \ldots, \lambda_s$. In particular, $x(n)$ converges to zero for all initial conditions if and only if $\lambda_1 < 0$. (For a sufficient condition on A for this to occur, see [5]). It is thus seen that if a reasonable model of statistics of the interprocessor delay is available, the Lyapunov exponents and associated subspaces give substantial information about the convergence properties of the asynchronous iteration.

In general, besides their existence, little else is known about the Lyapunov exponents and their corresponding subspaces. Our aim in this paper is to examine characteristics of the iteration by constructing a computation graph which describes the interprocessor influences in time. In section 2 we describe our model and construct our graphical representation. In section 3 we state a number of invariant structural properties of our representation. In section 4 we relate this structure to the random subspaces of the OMET. Section 5 is devoted to the case where A is a positive matrix and the iteration is started from positive initial conditions. Compared to the general results of section 4, these results are particularly strong.

2. Model

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\theta\colon \Omega \to \Omega$ be an invertible transformation on Ω. We assume (P, θ) to be stationary and ergodic. We characterize the delays by the measurable mapping $d: \mathbf{Z} \times \Omega \to \{1, \ldots, B\}^{p\times p}$, which is compatible with θ, that is, $d(n, \theta\omega) = d(n+1, \omega)$ for all ω and $n \in \mathbf{Z}$ (and thus by induction, $d(n, \theta^m\omega) = d(n+m, \omega)$ for all ω and $n, m \in \mathbf{Z}$.) Given a *scenario* $\omega \in \Omega$ and some initial condition $\{x_i(n); n = -B+1, \ldots, 0; 1 \leq i \leq p\}$, the value of processor i at time n, $x_i(n, \omega) \in \mathbf{C}$, evolves according to the relation

$$x_i(n, \omega) = \sum_{j=1}^{p} a_{ij} x_j(n - d_{ij}(n, \omega)), \tag{2.1}$$

with $i = 1, \ldots, p$, $n \in \mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, and $x_i(n, \omega) = x_i(n)$ for $-B+1 \leq n \leq 0$. Recalling the definition of $\tilde{A}(n)$ from the first section, we can verify that $\{\tilde{A}(n); n \in \mathbf{Z}^+\}$ are compatible with θ and that the sequence is indeed stationary and ergodic.

We now describe a graphical representation of our system. For each ω, we define a directed graph $G(\omega) = (U, E(\omega))$ by

$$\begin{aligned} U &= \{1, \ldots, p\} \times \{-B+1, -B+2, -B+3, \ldots\} \\ E(\omega) &= \{<(q_1, n_1), (q_2, n_2)>: a_{q_2, q_1} \neq 0,\, n_1 = n_2 - d_{q_2, q_1}(n_2, \omega),\, n_2 > 0\} \end{aligned}$$

In words, $<(q_1, n_1), (q_2, n_2)> \in E(\omega)$ if processor q_2 at time n_2 directly uses the value of processor q_1 at time n_1 in computation (2.1). In the definition of $E(\omega)$, it is required that $n_2 > 0$ because the value of the initial condition nodes are fixed and thus cannot influence each other. To simplify notation, we use u, v as typical members of U, and define $T((q, n)) = n$, $u + i = (q, n+i)$ for $u = (q, n)$ whenever defined and say that node u is *shifted* by i time units. We also define $I(u, \omega)$ to be the set $\{v : <v, u> \in E(\omega)\}$, and the *base* W to be the set $\{u \in U: T(u) \leq 0\}$. With no loss of generality we assume A to have no zero rows, making $I(u, \omega) \neq \emptyset$ for all $u \in U \setminus W$. We will later find it useful to order the elements of U by the following relation: We say $(q_1, n_1) > (q_2, n_2)$ if and only if a) $n_1 > n_2$ or b) $n_1 = n_2$ but $q_1 > q_2$. Using the compatibility of d with θ, it is easy to show that:

Lemma L1.

$$E(\theta\omega) = \{<u, v>: <u+1, v+1> \in E(\omega) \ \textit{and} \ T(v) > 0\}.$$

In words, to obtain $G(\theta\omega)$ from $G(\omega)$, shift every edge $<u, v>$ in $E(\omega)$ with $T(v) > 1$ to $<u-1, v-1>$. If $T(v) = 1$, then the shifted version of $<u, v>$ does not appear in $G(\theta\omega)$.

We next define $H: U \times \Omega \to 2^U$ by

$$H(u, \omega) = \begin{cases} \emptyset, & \text{if } T(u) \leq 0; \\ \bigcup_{u' \in I(u, \omega)} (H(u', \omega) \cup \{u'\}), & \text{otherwise}; \end{cases}$$

$C: (U \setminus W) \times \Omega \to 2^W$ by

$$C(u,\omega) = \{v \in H(u,\omega): T(v) \leq 0\},$$

and $U_C(\omega)$ by $\{u \in U \setminus W: C(u,\omega) = C\}$.

We call $H(u,\omega)$ and $C(u,\omega)$ the *history* and *color* of node u in $G(\omega)$, respectively and call $U_C(\omega)$ the *universe* of color C. Note that since $I(u,\omega) \neq \emptyset$ for all $u \in U \setminus W$, $H(u,\omega) \neq \emptyset$ and $C(u,\omega) \neq \emptyset$ for all $u \in U \setminus W$. It is also straightforward to see that $v \in H(u,\omega)$ if and only if there exists a nontrivial directed path $v = w_n, w_{n-1}, \ldots, w_0 = u$ from v to u in $G(\omega)$. By a nontrivial path we mean $w_{i+1} \in I(w_i,\omega)$ for $i = 0, \ldots, n-1$ and $n \geq 1$. Therefore, if $v \notin H(u,\omega)$, then the value of v in no way affects the value of u, and in particular, if $(q,n) \notin C((q',n'),\omega)$, then the initial condition $x_q(n)$ in no way affects the value of $x_{q'}(n',\omega)$.

Using L1, the next lemma describes the behavior of H under θ:

Lemma L2. *For* $u \in U \setminus W$,

$$H(u,\theta\omega) = \big\{v : \exists v' \text{ such that } T(v') > 1,\ < v+1, v' > \in E(\omega), \\ \text{and } v' \in H(u+1,\omega) \cup \{u+1\}\big\}.$$

It is possible for $v \in H(u,\omega)$ while $v-1 \notin H(u-1,\theta\omega)$. This happens if all influences from v to u in ω travel through nodes with time index 1. In particular this always happens for v with $T(v) = -B+1$. The lemma asserts that if there is some influence from v to u that does not travel through nodes with time index 1, then $v-1 \in H(u-1,\theta\omega)$.

Next, define $S: \Omega \to 2^{2^W}$ by

$$S(\omega) = \{C \in 2^W: \forall n_0 \geq 1,\ \exists u \in U \text{ such that} \\ T(u) \geq n_0 \text{ and } C(u,\omega) = C\}$$

and for each $C \in S(\omega)$, we define $F_C(\omega)$ by

$$F_C(\omega) = \{u: T(u) \geq 1 \text{ and } \forall n_0 \geq 1,\ \exists v \in U \text{ such that} \\ T(v) \geq n_0, C(v,\omega) = C \text{ and } u \in H(v,\omega)\}.$$

$S(\omega)$ is called the *spectrum* and $F_C(\omega)$ is called the *filament* of the color C.

In words, $C \in S(\omega)$ if and only if there are infinitely many C–colored nodes in $G(\omega)$, and $v \in F_C(\omega)$ if and only if v is in the history of infinitely many C–colored nodes in $G(\omega)$. Note that there are only a finite ($\leq 2^{Bp}$) number of possible colors for infinitely many nodes, implying that $S(\omega) \neq \emptyset$, and, furthermore, by definition of $S(\omega)$,

$$|\{u: C(u,\omega) \notin S(\omega)\}| < \infty.$$

Example E1: We provide here a simple example to better illustrate these concepts. Let A be a 2×2 matrix with no zero elements. Let $B = 2$ and $\Omega = \{\omega_0, \omega_1\}$ with $P(\omega_i) = .5$ and $\theta\omega_i = \omega_{(i+1)\bmod 2}$. Clearly, (P, θ) is stationary and ergodic. Let $d(n, \omega_i) = d(n \bmod 2, \omega_i)$ (i.e., the delays are periodic with period 2), with

$$d(1, \omega_0) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad \text{and} \qquad d(2, \omega_0) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

To make d compatible with θ, we let $d(n, \omega_1) = d\big((n+1) \bmod 2, \omega_0\big)$.

After constructing $G(\omega_0)$ (see Fig. 1a), it is easy to show that, for example,

$$\begin{aligned} H\big((2,1), \omega_0\big) &= \{(1,0), (2,-1)\} \\ H\big((1,2), \omega_0\big) &= \{(2,0), (1,0)\} \\ H\big((2,3), \omega_0\big) &= \big(H((2,1), \omega_0) \cup \{(2,1)\}\big) \cup \big(H((1,2), \omega_0) \cup \{(1,2)\}\big) \\ &= \big\{(1,2), (2,1), (2,0), (1,0), (2,-1)\big\} \end{aligned}$$

and so on, which implies

$$\begin{aligned} C\big((2,1), \omega_0\big) &= \{(1,0), (2,-1)\} \\ C\big((1,2), \omega_0\big) &= C_1 \\ C\big((2,3), \omega_0\big) &= C_2 \end{aligned}$$

with $C_1 = \big\{(2,0), (1,0)\big\}$ and $C_2 = \big\{(2,0), (1,0), (2,-1)\big\}$.

In a similar way, we can show that the colors of $(1,3), (2,2)$, and $(1,2)$ in $G(\omega_0)$ are all C_1. Then using $C(u, \omega_0) = \cup_{v \in I(u,\omega_0)} C(v, \omega_0)$ (easy to show) and the periodicity of the delays we can show that

$$\begin{aligned} U_{C_1} &= \big\{(1, 1+2n), (2, 2+2n), (1, 2+2n) \colon n \geq 0\big\} \\ U_{C_2} &= \big\{(2, 1+2n) \colon n \geq 1\big\}. \end{aligned}$$

Since $|U \setminus (U_{C_1} \cup U_{C_2})| < \infty$, we get $S(\omega_0) = \big\{C_1, C_2\big\}$.

With slightly more work, we can show that

$$\begin{aligned} F_{C_1}(\omega_0) &= \big\{(2, 2+2n), (1, 2+2n) \colon n \geq 0\big\} \\ F_{C_2}(\omega_0) &= \big\{(2, 2+2n), (1, 2+2n), (2, 1+2n) \colon n \geq 0\big\}. \end{aligned}$$

Now consider $G(\omega_1)$ (see Fig 1b). L1 can be visually verified. Now note that

$$H\big((2,2), \omega_1\big) = \big\{(1,1), (2,0), (2,-1), (1,-1))\big\},$$

as predicted by $H\big((2,3),\omega_0\big)$ and L2.

Repeating what was done for $G(\omega_0)$, (see Fig 2b), we get $S(\omega_1) = \{C_1', C_2'\}$ with $C_1' = \{(2,-1), (1,-1)\}$ and $C_2' = \{(1,-1),(2,-1),(2,0)\}$. Similarly, we get

$$U_{C_1'} = \{(1,1+2n),(2,1+2n),(1,2+2n): n \geq 0\}$$
$$U_{C_2'} = \{(2,2+2n): n \geq 0\},$$

and

$$F_{C_1'}(\omega_1) = \{(2,1+2n),(1,1+2n): n \geq 0\}$$
$$F_{C_2'}(\omega_1) = \{(2,2+2n),(2,1+2n),(1,1+2n): n \geq 0\}.$$

Returning to our general model, we next observe the following :

Lemma L3. *For any $C \in S(\omega)$, $F_C(\omega)$ contains a node in every* B–Block *of the type $W + n$, $n \geq B$, implying $|F_C(\omega)| = \infty$.*

We remark at this point that $S(\omega)$ is defined for each ω individually and that there is no obvious correspondence among colors of $S(\omega)$ for different ω's. In particular, $C \in S(\omega)$ is not *a priori* a random function, (unless $|S| = 1$), as there is no canonical way of assigning C to ω. However, there are certain important properties of $S(\omega)$ that belong to an almost sure set. We study these properties in the next section.

3. Preliminary Results

In this section we will examine some of the invariant properties of colors in $S(\omega)$. Using ergodicity, we can show that these properties must be uniform in an almost sure set. This invariance will allow us to obtain some of the above introduced concepts at scenario $\theta\omega$, namely the spectrum and the filaments of the colors in the spectrum, directly in terms of those at scenario ω. We will finally state some of the invariant structures related to the colors of the spectrum in P4.

Given $C \in S(\omega)$, let us define

$$S_C(\omega) = \{C' \in 2^W : |\{u \in U_C(\omega) : T(u) > 1 \textit{ and } C(u-1, \theta\omega) = C'\}| = \infty\}.$$

It is easy to see that $S_C(\omega) \neq \emptyset$ for all $C \in S(\omega)$ and that

$$\cup_{C \in S(\omega)} S_C(\omega) \subset S(\theta\omega).$$

In words, $S_C(\omega)$ is the collection of all colors that show up infinitely often as the colors under scenario $\theta\omega$ of the nodes that have colors C under ω. With some work, we can show that for any two distinct colors $C_1, C_2 \in S(\omega)$, $S_{C_1} \cap S_{C_2} = \emptyset$. We can thus make the following key observation:

Proposition P1. *For all* ω, $|S(\omega)| \leq |S(\theta\omega)|$.

An immediate consequence of P1 and ergodicity is the following:

Proposition P2. *For some nonrandom constant* $\sigma \leq 2^{Bp}$, $|S| = \sigma$ *almost surely.*

P2 indicates that the spectrum $S(\omega)$ is central to understanding invariant properties of the computation. In order to examine the relation between the spectrums of scenarios ω and $\theta\omega$, we need the following definition. Given $C \in S(\omega)$, define $\theta C = C^1 + C^0$, where

$$C^1 = \{u - 1: u \in C, \exists v \in F_C(\omega) \text{ such that } T(v) > 1 \text{ and } < u, v > \in E(\omega)\},$$
$$C^0 = \{(q, 0): (q, 1) \in F_C(\omega)\}.$$

Note the implicit dependence of θC on $G(\omega)$ and $F_C(\omega)$. Also note that if u is in C^1 (C^0), then $T(u) < 0$ ($T(u) = 0$, respectively). The next result describes the behavior of $S(\omega)$ under θ.

Proposition P3. *For almost all (a.a.)* ω, *if* $C \in S(\omega)$, *then* $S_C(\omega)$ *consists of the single element* θC. *Hence for a.a.* ω, *if* $S(\omega) = \{C_1, \ldots, C_\sigma\}$, *then* $S(\theta\omega) = \{\theta C_1, \ldots, \theta C_\sigma\}$ *with* $\theta C_i \neq \theta C_j$ *for all* $i \neq j$.

According to P3, in an a.s. set we can characterize $S(\theta\omega)$ directly in terms of notions defined for scenario ω. In a similar vein, in an a.s. set we can characterize $F_{\theta C}(\theta\omega)$ directly in terms of $F_C(\omega)$. The next Lemma describes the behavior of $F_C(\omega)$ under θ.

Lemma L4. *For a.a.*ω, *if* $C \in S(\omega)$, *then*

$$F_{\theta C}(\theta\omega) = \{u \in F_C(\omega): T(u) > 1\} - 1.$$

Let Ω_{P3} be the a.s. set satisfying P3. We now make two remarks. Firstly, by applying P3 to $\theta\Omega_{P3}$, we see that for an a.s. subset of Ω_{P3} (recall that $P\theta^{-1} = P$), if $C \in S(\omega)$, then $S_{\theta C}(\theta\omega)$ is well-defined (since $\theta C \in S(\theta\omega)$) and is equal to $\{\theta^2 C\}$, where $\theta^2 C = \theta\theta C$ is defined as a function of θC and $F_{\theta C}(\theta\omega)$ in the same way as θC was defined as a function of C and $F_C(\omega)$.

Secondly, since θ is invertible and (P, θ) stationary, for an a.s. subset of Ω_{P3}, if $C \in S(\omega)$, then there must exist a unique map $C \mapsto \theta^{-1}C \in S(\theta^{-1}\omega)$ such that $S_{\theta^{-1}C}(\theta^{-1}\omega) = \{C\}$. Using this fact in conjunction with L4, we get

$$F_C(\omega) = \{u \in F_{\theta^{-1}C}(\theta^{-1}\omega): T(u) > 1\} - 1.$$

We can by induction extend the two remarks to obtain a generalization of P3 and L4 to all shifted scenarios $\theta^n\omega$.

Corollary C1. *For a.a.ω, for all $n \in \mathbf{Z}$, if $S(\omega) = \{C_1, \ldots, C_\sigma\}$, then:*

1. $S(\theta^n\omega) = \{\theta^n C_1, \ldots, \theta^n C_\sigma\}$ *and* $\theta^n C_i \neq \theta^n C_j$ *for all* $i \neq j$, *and*
2. $F_{\theta^n C}(\theta^n\omega) = \{u \in F_C(\omega)\colon T(u) > n\} - n$ *for* $n > 0$ *and* $F_C(\omega) = \{u \in F_{\theta^n C}(\theta^n\omega)\colon T(u) > -n\} + n$ *for* $n < 0$.

In particular, it is easy to show that for $n > B$,

$$\theta^n C = \{u \in F_C(\omega) \cap (W+n)\colon <u,v> \in E(\omega) \text{ for some } v \in F_C(\omega) \text{ with } T(v) > n\} - n.$$

Our next result reveals more invariant properties relating to the structure of colors in the spectrum.

Proposition P4. *Let C_1, $C_2 \in S(\omega)$ and $C_1 \neq C_2$. Then the following hold:*

1. *If $C_1 \cap C_2 = \emptyset$, then $F_{C_1} \cap F_{C_2} = \emptyset$ and $\theta C_1 \cap \theta C_2 = \emptyset$ for all ω.*
2. *If $C_1 \cap C_2 \neq \emptyset$, then $\theta C_1 \cap \theta C_2 \neq \emptyset$ for a.a. ω.*
3. *If $C_1 \cap C_2 \neq \emptyset$, then there exists $C_3 \in S(\omega)$ such that $C_3 \subset C_1 \cap C_2$ for a.a. ω.*
4. *If $C_1 \not\subset C_2$, then $\theta C_1 \not\subset \theta C_2$ for a.a. ω.*
5. *If $C_1 \subset C_2$, then $\theta C_1 \subset \theta C_2$ for a.a. ω.*

It is also easy to generalize P4 to $\theta^n C_1$ and $\theta^n C_2$ for arbitrary n. To illustrate P4, we give another example.

Example E2. Let A be a 2×2 matrix with no zero elements. Let $B = 2$ and $\Omega = \{\omega_0, \omega_1, \omega_2, \omega_3\}$ with $P(\omega_i) = 1/4$ and $\theta\omega_i = \omega_{(i+1)\bmod 4}$ for $0 \leq i \leq 3$. Clearly, (P, θ) is stationary and ergodic. Let $d(n, \omega_i) = d(n \bmod 4, \omega_i)$ (i.e., the delays are periodic with period 4) with

$$d(1,\omega_0) = d(2,\omega_0) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad d(3,\omega_0) = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}, \qquad d(4,\omega_0) = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix},$$

and $d(n, \omega_i) = d((n+1) \bmod 4, \omega_{(i-1)\bmod 4})$ to make the delays compatible with θ. (See Fig 2a). Then by induction (on sections of the graph with temporal length of 4) we can show that $S(\omega_0) = \{C_1, C_2, C_3\}$, with

$$C_1 = \{(1,0),(2,0)\},$$
$$C_2 = \{(1,0),(2,0),(1,-1)\},$$
$$C_3 = \{(1,0),(2,0),(2,-1)\},$$

and

$$F_{C_1}(\omega_0) = \{(1,4+4n),(2,4+4n): n \geq 0\},$$
$$F_{C_2}(\omega_0) = \{(1,1+4n),(2,1+4n),(2,2+4n),(1,3+4n),(1,4+4n),(2,4+4n): n \geq 0\},$$
$$F_{C_3}(\omega_0) = \{(1,1+4n),(2,1+4n),(1,2+4n),(2,3+4n),(1,4+4n),(2,4+4n): n \geq 0\}.$$

Therefore, we get

$$\theta C_1 = \{(1,-1),(2,-1)\},$$
$$\theta C_2 = \{(1,0),(2,0),(1,-1),(2,-1),(1,-2)\},$$
$$\theta C_3 = \{(1,0),(2,0),(1,-1),(2,-1),(2,-2)\}.$$

Note that since $\omega_1 = \theta\omega_0$, using $G(\omega_1)$ one can verify that $S(\omega_1) = \{\theta C_1, \theta C_2, \theta C_3\}$, as predicted by P3. See Fig 2b.

For certain cases, P4 will allow us to canonically distinguish the colors of $S(\omega)$ a.s. For example, as in E1, say $|S| = 2$ and for a set of positive probability, one color of S is the subset of the other color. By part (5), (and ergodicity), this property must apply to an a.s. set. Therefore, given an ω in this set, we can define random functions

$$C_a(\omega) = \{C \in S(\omega): C \subset C' \;\; for \;\; C' \in S(\omega) \setminus \{C\}\}$$

and

$$C_b(\omega) = \{C \in S(\omega): C \not\subset C_a(\omega)\},$$

which imply $C_a(\omega) \subset C_b(\omega)$ for a.a. ω.

As a second example, say $|S| = 3$ with the property that for a.a. ω, $C_{\pi(1)} \not\subset C_{\pi(2)}$, $C_{\pi(2)} \not\subset C_{\pi(1)}$, and $C_{\pi(3)} \subset C_{\pi(1)} \cap C_{\pi(2)}$, where $C_i \in S(\omega)$ and π is a (random) permutation. This was the situation in E2. By parts (2), (3), and (4) of the theorem, this happens with either probability one or zero. Again, we can canonically differentiate $C_{\pi(3)}$ from the other two colors by its unique feature, while the other two colors cannot be differentiated from each other. In this sense, in many situations, we may be able to canonically differentiate between colors that have some unique properties in an a.s. set.

4. Main Results

In this section we make the connection between the structure of the computation graph described above and the random subspaces of OMET. The structure is particularly clear when $p = 1$. This is because each node is influenced by exactly one other node, i.e., $|I(u,\omega)| = 1$ for all u, ω. Thus, $C(u,\omega) \in \{-B+1,\ldots,0\}$. (Note that when $p = 1$, we can simplify the notation and identify U with $\{-B+1,-B+2,-B+3,\ldots\}$.) A consequence of this observations is the following Theorem:

Theorem T1. *For a.a. ω, there exists a partition $\{S^k(\omega),\ k=1,\ldots,s\}$ of W such that:*

1. $|S^k(\omega)| = \delta_k$,
2. $S^s(\omega) = W \setminus S(\omega)$ *if* $\lambda_s = -\infty$ *and* $S(\omega) = W$ *otherwise, and*
3. $V(k,\omega) = \bigoplus_{i\geq k}(\bigoplus_{j\in S^i(\omega)} \operatorname{span} e_j)$.

Therefore, for the case $p=1$, T1 completely characterizes Oseledec's subspaces in terms of colors of the spectrum.

The case $p>1$ is much more complicated. Here we cannot expect the simple picture of each color having a single Lyapunov exponent. The main difficulty is due to the fact that each node is affected by more than one node. It thus possible to have the values to two nodes canceling each other at a future node. Indeed, even for deterministic delays the situation is significantly deeper. For example, let

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix},$$

$B=1$, and $d_{ij}=1$ for all $1 \leq i,j \leq 2$, which corresponds to a synchronous iteration. There is clearly only one color $\{(1,0),(2,0)\}$ in $S(\omega)$, but there are two growth rates, namely the eigenvalues 2 and 3.

Before stating our main theorem for $p>1$, we make the following definitions. Let Ω_1 be the a.s. set satisfying both C1 and P4. Recalling OMET, let $\overline{V}(k,\omega), k=1,\ldots,s$, be the orthogonal complement of $V(k+1,\omega)$ in $V(k,\omega)$. Note that $\overline{V}(s,\omega)=V(s,\omega)$. Thus $\dim \overline{V}(k,\omega)=\delta_k$ and $\bigoplus_{k=1}^{s}\overline{V}(k,\omega)=\mathbf{C}^{Bp}$ for a.a. ω. Define $P(k,\omega)$, $k=1,\ldots,s$ to be the projection mapping onto the subspace $\overline{V}(k,\omega)$.

Theorem T2. *For any subset $C \subset S(\omega)$,*

1. $\tilde{A}(1,\omega)$ *maps* $\mathbf{C}^{W\setminus\cup_{c\in C} c}$ *into* $\mathbf{C}^{W\setminus\cup_{c\in C}\theta c}$, *and*
2. *For all $k=1,\ldots,s$, for which $\lambda_k > -\infty$,*

$$\dim P(k,\omega)\mathbf{C}^{W\setminus\cup_{c\in C} c} = \dim P(k,\theta\omega)\mathbf{C}^{W\setminus\cup_{c\in C}\theta c}$$

for a.a. ω.

One interprets the theorem as follows: Initial conditions whose support is disjoint from any color can never influence nodes of that color. The inclusion partial order on the 2^{σ} subsets of $S(\omega)$ induces a containment partial order on the corresponding subspace $\mathbf{C}^{W\setminus\cup_{c\in C} c}$. Any initial vector in the minimal subspace $\mathbf{C}^{W\setminus\cup_{c\in S(\omega)} c}$ influences only finitely many nodes and so lies in the subspace corresponding to the Lyapunov exponent $-\infty$. For any $C \subset S(\omega)$ the subspace $\mathbf{C}^{W\setminus\cup_{c\in C} c}$ may differ in dimension from $\mathbf{C}^{W\setminus\cup_{c\in C}\theta c}$, but for each

Lyapunov exponent $\lambda_k > -\infty$, $\dim\big(V(k,\omega)\cap \mathbf{C}^{W\setminus\cup_{C\in c}C}\big) - \dim\big(V(k+1,\omega)\cap \mathbf{C}^{W\setminus\cup_{C\in c}C}\big)$ is the same as $\dim\big(V(k,\theta\omega)\cap \mathbf{C}^{W\setminus\cup_{C\in c}\theta C}\big) - \dim\big(V(k+1,\theta\omega)\cap \mathbf{C}^{W\setminus\cup_{C\in c}\theta C}\big)$ for a.a. ω.

Recall the example of the previous section that had $|S| = 2$. In cases such as this, we can define $\mathcal{C}$ canonically by a random function. In our example, we can set $\mathcal{C}_a(\omega) = \{C_a(\omega)\}$ and associate scalars $\dim P(k,\omega)\mathbf{C}^{W\setminus\cup_{C\in c}C}$ to the nodes of $C_b(\omega) \setminus C_a(\omega)$ for k with $\lambda_k > -\infty$ in an a.s. set. We used OMET here only to associate such nonrandom scalars to a subspace which is characterized independently of OMET.

5. Positive Case

When A is positive (when all elements of A are nonnegative), the influence of two initial condition nodes with non-negative values cannot cancel each other at a future node. For this reason, one expects that, in some sense, subspaces corresponding to each color give rise only to a single Lyapunov exponent, and a more detailed picture along the lines of T1 should be possible. For the remainder of this section we will assume both A and the initial conditions to be positive.

In the following discussion, we will limit ourselves to the a.s. subset Ω_1 that satisfies OMET. Next, we will order the elements of $S(\omega)$ using the lexicographic convention that was introduced in Section 2. For $|S| = \sigma$ we then write

$$\hat{S}(\omega) = \big(C_1(\omega), \ldots, C_\sigma(\omega)\big),$$

an ordered set, where $C_1(\omega), \ldots, C_\sigma(\omega)$ are the lexicographically ordered colors in $S(\omega)$. Thus, $C_i, i = 1, \ldots, \sigma$ are now a.s. well defined random functions. (Previously, for each ω, C_i was just a picked member of $S(\omega)$). We also define $\theta C_i(\omega)$ using $C_i(\omega) \in S(\omega)$ as θC was defined using $C \in S(\omega)$. Let $\mathcal{S}_\sigma$ be the set of permutations of a set of size σ. It is important to note that in general, $C_i(\theta\omega) \neq \theta C_i(\omega)$. However, since $C \mapsto \theta C$ is a bijection, there must exist a unique permutation $\hat{\pi}(\omega) \in \mathcal{S}_\sigma$ such that

$$\theta C_i(\omega) = C_{\hat{\pi}(i,\omega)}(\theta\omega) \tag{5.1}$$

for $i = 1, \ldots, \sigma$.

We next define the random function $\mathcal{C}: \{0,1\}^\sigma \times \Omega \to 2^{2^W}$ by

$$\mathcal{C}(b,\omega) = \big\{C_i(\omega) \colon b_i = 1\big\},$$

where b_i is the i-th component of b. For $\pi \in \mathcal{S}_\sigma$, we define $\pi(b) = \pi(b_1 \cdots b_\sigma) = b_{\pi^{-1}(1)} \cdots b_{\pi^{-1}(\sigma)}$. The action of π on b is chosen so that $(\pi_1\pi_2)(b) = \pi_1(\pi_2(b))$. We may write $\mathcal{C}(b)$ and C_i whenever ω is fixed. As an example, let $\sigma = 4$; then $\mathcal{C}(0101) = \{C_2, C_4\}$.

For notational simplicity, we write $\theta\mathcal{C}(b,\omega)$ for $\{\theta C_i(\omega): b_i = 1\}$, write $\mathcal{I}(n,\mathcal{C},\omega)$ for $\cap_{C\in\mathcal{C}}\theta^n C \setminus \cup_{C\in S(\omega)\setminus\mathcal{C}}\theta^n C$, and write $L(y,\omega)$ for $\lim_{n\to\infty} n^{-1}\log\|\tilde{A}(n,\omega)\cdots\tilde{A}(1,\omega)y\|$.

We will next state that for a.a.ω, the following hold for all b (i.e., $b\in\{0,1\}^\sigma$) and for all $\pi\in\mathcal{S}_\sigma$.

(A1) $\theta\mathcal{C}(b,\omega) = \mathcal{C}(\hat{\pi}(b,\omega),\theta\omega)$,

(A2) $\mathcal{I}(n,\mathcal{C}(b,\omega),\omega) = \mathcal{I}(n-1,\mathcal{C}(\hat{\pi}(b,\omega),\theta\omega),\theta\omega)$ for $n>0$, where $\hat{\pi}$ was defined in (5.1).

With the help of (A1) and (A2), we can show the following:

Theorem T3. *Let A be positive. Then for a.a.ω, the following hold: If $\mathcal{I}(n,\mathcal{C}(b,\omega),\omega)\neq\emptyset$ for some n, then $\mathcal{I}(n,\mathcal{C}(b,\omega),\omega)\neq\emptyset$ for infinitely many n. Moreover, if $n_1<n_2$ are such that $\mathcal{I}(n_i,\mathcal{C}(b,\omega),\omega)\neq\emptyset$, for $i=1,2$, then:*

1. *For all $y\in\mathbf{C}^{\mathcal{I}(n_1,\mathcal{C}(b,\omega),\omega)}$ such that $y>0$, $L(y,\theta^{n_1}\omega)=\lambda\in\{\lambda_1,\ldots,\lambda_s\}$ with $\lambda>-\infty$, and*

2. *For all $y\in\mathbf{C}^{\mathcal{I}(n_2,\mathcal{C}(b,\omega),\omega)}$ such that $y>0$, $L(y,\theta^{n_2}\omega)=\lambda$, which is the same λ as the one in part (1).*

To demonstrate the need for the first part of the theorem, recall E2. Let $\mathcal{C}=\{C_2,C_3\}$. Then $\mathcal{I}(0,\mathcal{C},\omega_0) = (C_2\cap C_3)\setminus C_1 = \emptyset$ whereas $\mathcal{I}(1,\mathcal{C},\omega_0) = (\theta C_2\cap\theta C_3)\setminus\theta C_1 = \{(1,0),(2,0)\}$. It is therefore possible to have $\mathcal{I}(n,\mathcal{C}(b,\omega),\omega)\neq\emptyset$ for some n while $\mathcal{I}(m,\mathcal{C}(b,\omega),\omega)=\emptyset$ for some m. Let $\Omega_{T3}\subset\Omega_1$ be the a.s. set that satisfies T3 and OMET.

We now make a number of remarks. Firstly, note that every node $u\in S(\omega)$ belongs to $\mathcal{I}(0,\mathcal{C}(u),\omega)$ where $\mathcal{C}(u)=\{C\in S(\omega): u\in C\}$. Therefore, every node in $S(\omega)$ has an associated Lyapunov exponent which gives the growth rate of the norm of B-blocks with initial condition e_u. Furthermore, if two initial condition nodes lie in precisely the same set of colors, then they must have the *same* Lyapunov exponent. In particular, if there is a single color, all nodes in that color have the same Lyapunov exponent (with the rest of the nodes having the exponent $-\infty$). This contrasts with the situation in T2 for matrices A that have negative entries. In our example with $|S|=2$, we can associate an exponent λ_b to the nodes of $C_b\setminus C_a$ and λ_a to the nodes of C_a (recall that $C_a\subset C_b$). In this example, by P4, since $\cap_{C\in\mathcal{C}}C\setminus\cup_{C\in\overline{\mathcal{C}}}C\neq\emptyset$, necessarily $\cap_{C\in\mathcal{C}}\theta C\setminus\cup_{C\in\overline{\mathcal{C}}}\theta C\neq\emptyset$ for $\mathcal{C}=\{C_b\}$ and $\{C_a,C_b\}$.

Secondly, it is clear that for any $u_1,u_2\in S(\omega)$, if $\mathcal{C}(u_1)\subset\mathcal{C}(u_2)$, then $L(e_{u_1},\omega)\leq L(e_{u_2},\omega)$. That is, the set-theoretic relationship among colors can provide some information concerning the rates of convergence for different subspaces. Going back to our example again, we get $\lambda_a\geq\lambda_b$.

Fix $\omega \in \Omega_{T3}$, and for $\mathcal{C} \in 2^{S(\omega)}$, define $\Lambda(\omega)$: $2^{S(\omega)} \to \{\lambda_1, \ldots, \lambda_s, \phi\}$ by

$$\Lambda(\mathcal{C}, \omega) = \begin{cases} L(y, \theta^n \omega) & \text{for all } y \in \mathbf{C}^{\mathcal{I}(n,\mathcal{C},\omega)}, \text{ if } \mathcal{I}(n, \mathcal{C}, \omega) \neq \emptyset \text{ for some } n \geq 0; \\ \phi, & \text{if } \mathcal{I}(n, \mathcal{C}, \omega) = \emptyset \text{ for all } n \geq 0. \end{cases}$$

According to T3, $\Lambda(\mathcal{C}(b, \omega), \omega)$ is well defined for all $\omega \in \Omega_{T3}$ and if $\mathcal{I}(n_0, \mathcal{C}, \omega) \neq \emptyset$ for some n_0, then $\mathcal{I}(n, \mathcal{C}, \omega) \neq \emptyset$ for infinitely many n. Let $\mathcal{C}(u) = \{C \in S(\omega): u \in C\}$. Then T3 implies that

$$L(\tilde{x}, \omega) = \max\{\Lambda(\mathcal{C}(q, n), \omega): x_q(n) > 0\}.$$

The next result describes the behavior of Λ under θ.

(A3) For all $\omega \in \Omega_{T3}$, the following holds for all b, and for all $\pi \in \mathcal{S}_\sigma$:

$$\Lambda(\mathcal{C}(b, \omega), \omega) = \Lambda(\mathcal{C}(\hat{\pi}(b, \omega), \theta\omega), \theta\omega).$$

Now let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_s, \phi\}^{2^\sigma}$. Then $\big(\Lambda(\mathcal{C}(b, \omega), \omega)\big)_{b=\{0,1\}^\sigma} \in \mathcal{L}$ for all $\omega \in \Omega_{T3}$. We next lexicographically order elements of $\mathcal{L}$ using the ordering $\lambda_1 > \cdots > \lambda_s > \phi$. Given any $\Lambda \in \mathcal{L}$ define Λ_L to be the largest element of $\{\pi(\Lambda): \pi \in \mathcal{S}_\sigma\}$. Here $\pi(\Lambda) \in \mathcal{L}$ is defined as $\big(\pi(\Lambda)\big)_b = \Lambda_{\pi^{-1}(b)}$ for all $b \in \{0,1\}^\sigma$. So if for some $\Lambda, \Lambda' \in \mathcal{L}$ we have $\Lambda_L \neq \Lambda'_L$, there exists no permutation π such that $\pi(\Lambda) = \Lambda'$. Therefore, if we let $\mathcal{L}_L = \{\Lambda_L: \Lambda \in \mathcal{L}\}$, we can partition $\mathcal{L}$ into equivalence classes $\big(\{\pi(\Lambda): \pi \in \mathcal{S}_\sigma\}\big)_{\Lambda \in \mathcal{L}_L}$. This is turn induces a partition on Ω_{T3} with elements

$$\Omega_\Lambda = \{\omega \in \Omega_{T3}: \exists \pi \in \mathcal{S}_\sigma \textit{ such that } \Lambda(\mathcal{C}(\pi(b), \omega), \omega) = \Lambda_b \text{ for all } b\}$$

with $\Lambda \in \mathcal{L}_L$.

Let $\omega \in \Omega_{T3}$ and $\hat{\pi} = \hat{\pi}(\omega)$. Now by (A3), if $\Lambda(\mathcal{C}(\pi(b), \omega), \omega) = \Lambda$ for some $\pi \in \mathcal{S}\sigma$ and some $\Lambda \in \mathcal{L}_L$, then $\Lambda(\mathcal{C}((\hat{\pi} \cdot \pi)(b), \theta\omega), \theta\omega) = \Lambda$, that is, $\Omega_\Lambda \subset \theta\Omega_\Lambda$. We therefore have by ergodicity that for every $\Lambda \in \mathcal{L}_L$, $P(\Omega_\Lambda)$ is either zero or one. Since $\mathcal{L}_L$ has finite number of elements, and since $\{\Omega_\Lambda: \Lambda \in \mathcal{L}_L\}$ partitions the almost sure set Ω_{T3}, we get:

Theorem T4. *There exists a constant Λ^*, largest in its equivalence class, such that for a.a.ω, there exists a $\pi(\omega) \in \mathcal{S}_\sigma$ with*

$$\Lambda\big(\mathcal{C}(\pi(b, \omega), \omega), \omega\big) = \Lambda^*_b$$

for all $b \in \{0,1\}$.

Therefore, each initial condition node will activate a certain convergence rate. The convergence rate is determined by the set of colors to which the node belongs to. Furthermore, such set of colors, modulo some permutation, are invariant under θ. So given any arbitrary positive initial condition, let $\mathcal{U}$ be all the nodes (q,n) of the initial condition that have values that are strictly positive. To each of these node we can attribute an exponent. Then our system's convergence rate, i.e., the convergence rate of the norm of the B-blocks, will equal to the maximum of these exponents.

We next define

$$\Pi(\omega) = \{\pi \in \mathcal{S}_\sigma \colon \Lambda(\mathcal{C}(\pi(b),\omega),\omega) = \Lambda_b^* \text{ for all } b\},$$

for all ω satisfying T4 and let $\pi \in \Pi(\omega)$. Then by definition of $\Pi(\omega)$ and (A3) we can show that $|\Pi(\omega)| \le |\Pi(\theta\omega)|$. We again use ergodicity to obtain the following result.

Corollary C2. $|\Pi| = const$ *almost surely.*

Note that if $|\Pi| = 1$ we can in a canonical way differentiate among all the colors using OMET.

Until now, all of our results dealt with the convergence rate of the norm of the B-blocks, which provides very little information concerning the asymptotic values of individual processors themselves. All we can infer is the existence of a sequence of nodes which have values that converge at that rate. Our last result, which has an involved proof, describes in detail the asymptotic behavior of individual processors. In particular, it asserts that every increasing sequence of nodes belonging to certain color has values that do have a rate of convergence. This fact further emphasizes the importance of colors in our construction.

Theorem T5. *Let A be positive. Then for a.a.ω and a given nonnegative $\tilde{x}(0)$, the following holds: For any sequence of nodes $(q_k, n_k) \in U_C$, $C \in S$, with $\lim_{k\to\infty} n_k = \infty$,*

$$\lim_{k\to\infty} n_k^{-1} \log x_{q_k}(n_k) = \zeta,$$

where ζ is either in $\mathbf{R} \cup \{-\infty\}$ or is $-\infty$, depending respectively on whether the initial condition $\tilde{x}(0)$ makes the value of any node $u \in C \in S(\omega)$ positive or not.

At this point, we do not know whether these ζ's are Lyapunov exponents or not. However, when $\sigma = 1$, T3 asserts that the rate of convergence of the norm of B-blocks is either a Lyapunov exponent larger than $-\infty$ or $-\infty$, depending on whether the initial condition $\tilde{x}(0)$ makes the value of any node in the unique color of the spectrum positive or not. Then T5 implies that any increasing sequence of nodes will have that same rate of convergence.

Acknowledgment. The authors would like to thank Professors Pravin Varaiya and Jean Walrand for their warm hospitality at University of California at Berkeley in Fall 1992, where the final portions of this work were done.

6. References

[1] L. Arnold and V. Wihstutz (editors), Lyapunov Exponents, Springer–Verlag, Berlin, New York, 1986.

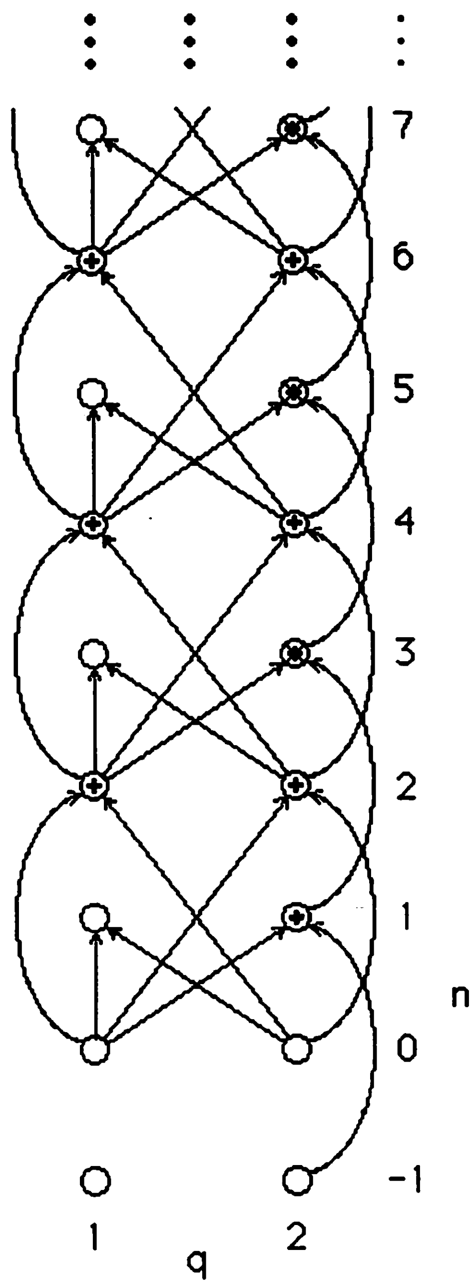
[2] D. P. Bertsekas and J. N. Tsitsiklis, Parallel and Distributed Computation, Prentice Hall, Englewood Cliffs, 1989.

[3] J. E. Cohen, H. Kesten, and C. M. Newman, "Oseledec's Multiplicative Ergodic Theorem: A proof," in Random Matrices and their Application, J. E. Cohen, H.Kesten, and C. M. Newman (editors), AMS, Providence, 1986.

[4] D. Chazan and W. Miranker, "Chaotic relaxations," Linear Algebra and its Applications, vol. 2, pp. 199-222, 1969.

[5] G. H. Golub and C. F. Van Loan, Matrix Computations, John Hopkins University Press, Baltimore and London, 1989.

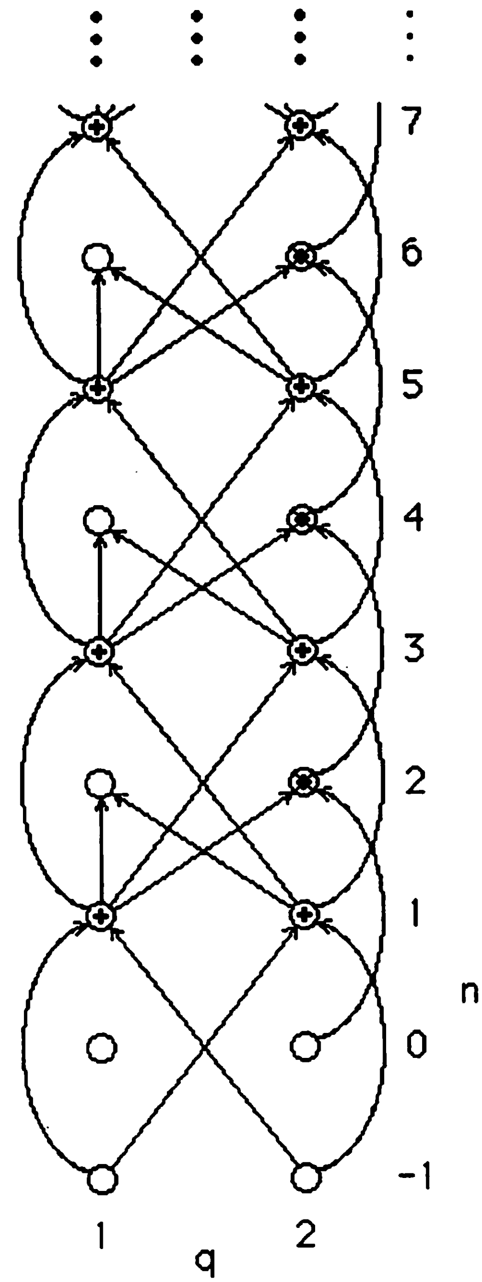
[6] Y. Kifer, Ergodic Theory of Random Transformations, Birkhauser, Boston, Basel, Stuttgart, 1986.



$\otimes\, U_{C_2}(\omega_0)$

$\oplus\, F_{C_2}(\omega_0)$

$G(\omega_0)$ **Fig. 1a**

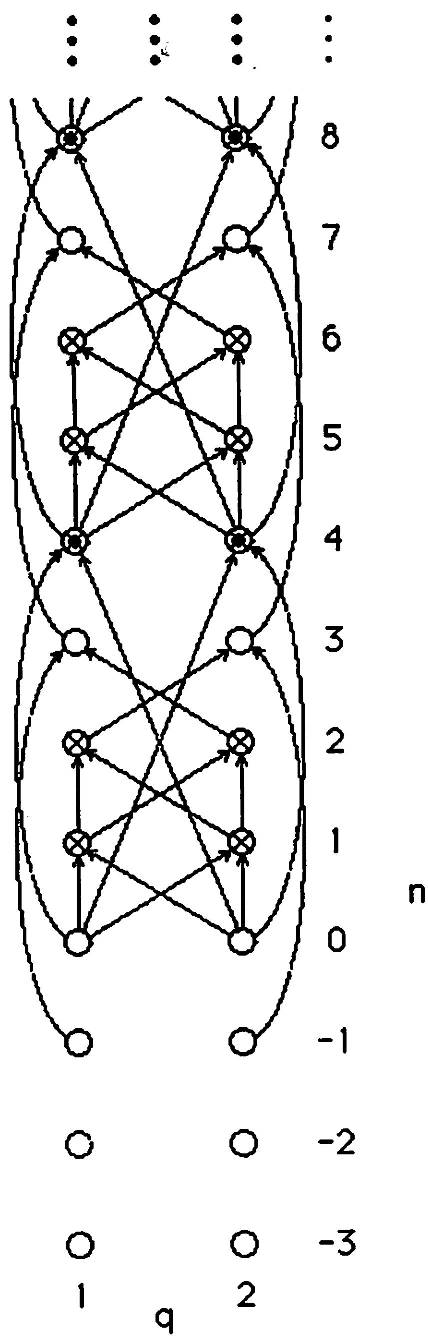


$\otimes\, U_{C_2'}(\omega_1)$

$\oplus\, F_{C_2'}(\omega_1)$

$G(\omega_1)$ **Fig. 1b**

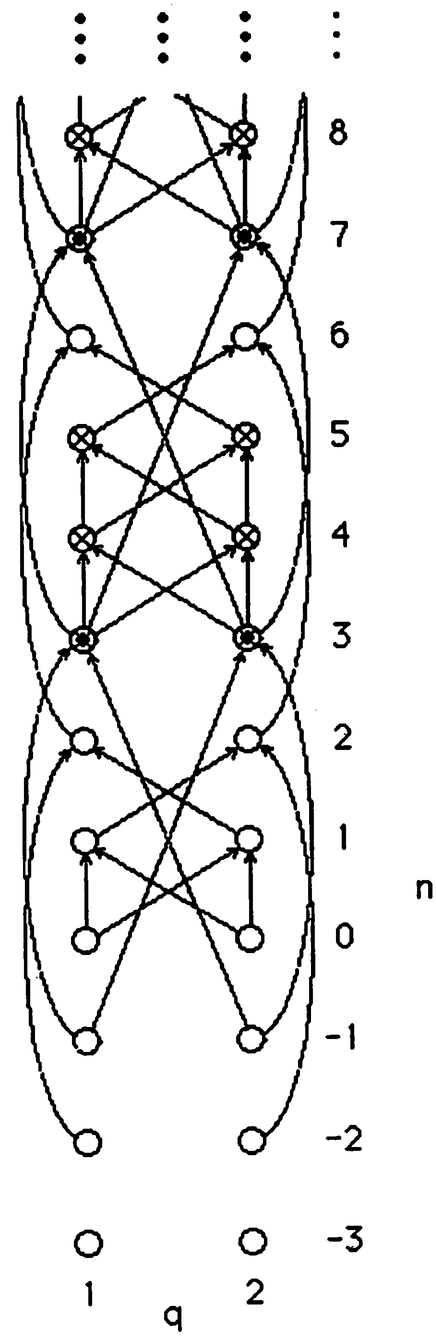
Figure 1: The graphs $G(\omega_0)$ and $G(\omega_1)$ of E1. In Fig. 1a, the nodes belonging to the universe and filament of color $C_2 = \{(2,0),(1,0),(2,-1)\}$, and in Fig. 1b, the nodes belonging to the universe and filament of color $C_2' = \{(2,0),(2,-1),(1,-1)\}$ are marked.



$\otimes\, U_{C_1}(\omega_0)$

$\oplus\, F_{C_1}(\omega_0)$

$G(\omega_0)$ **Fig. 2a**



$\otimes\, U_{\theta C_1}(\omega_1)$

$\oplus\, F_{\theta C_1}(\omega_1)$

$G(\omega_1)$ **Fig. 2b**

Figure 2: The graphs $G(\omega_0)$ and $G(\omega_1)$ of E2. In Fig. 2a, the nodes belonging to the universe and filament of color $C_1 = \{(1,0),(2,0)\}$, and in Fig. 2b, the nodes belonging to the universe and filament of color $\theta C_1 = \{(1,-1),(2,-1)\}$ are marked.