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Beyond Linear Feedback for the Nonlinear Regulator *

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Abstract

We examine the benefits of using a nonlinear feedback in the nonlinear regulator problem to induce larger domains of attraction to the zero-error manifold. We illustrate such benefits on the "ball and beam" example, using a recently developed semi-globally stabilizing control law for this system that employs saturation functions [8].

1 Introduction

As in [7], we seek to expand the region of attraction of the zero-error manifold of nonlinear regulator theory developed in [1]. In [7], we approached this problem by deforming the manifold so that the initial state of the system started close to the deformed manifold and then allowed the deformed manifold to decay slowly to the zero-error manifold. We then used standard linear feedback to regulate to this deformed manifold. For some systems this approach yielded dramatic improvements. Nevertheless, the result was still inherently local. A further drawback to this approach was that (approximate) knowledge of the initial state of the system was needed. Also, dynamic states equal to the number of states of the system were added to the compensator.

In this paper, we seek to expand the region of attraction without deforming the manifold. We propose replacing the standard linear feedback used to regulate to the manifold with nonlinear feedback based on global or semi-globally stabilizing control laws for unperturbed systems. We show that this approach yields theoretical reasons for an increased domain of attraction. We demonstrate an application of this approach using the frequently studied "ball and beam" example presented in [3]. For this example, we choose a recently developed semi-globally stabilizing control law recently developed in [8].

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2 Problem Statement

The task at hand is to achieve (perhaps approximate) tracking for the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)w \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and $w \in W \subset \mathbf{R}^s$ is a disturbance. As usual, f and the columns of g and p are assumed to be smooth vector fields and $h(x)$ is a smooth mapping on $\mathbf{R}^n$. We assume that the desired trajectory and the disturbance are generated by an autonomous, Poisson stable exosystem

$$\begin{aligned} \dot{w} &= s(w) \\ y_d &= -q(w) \end{aligned} \tag{2}$$

where s is a smooth vector field and $q(w)$ is a smooth mapping defined on W. The Poisson stability of the exosystem implies that the eigenvalues of the linear approximation of the exosystem lie on the imaginary axis. For simplicity we assume that $f(0) = 0$, $s(0) = 0$, $h(0) = 0$ and $q(0) = 0$ so that, for $u = 0$ the composite system (1), (2) has an equilibrium state $(x, w) = (0, 0)$ which yields zero tracking error.

We will focus on finding a state feedback $u = \alpha(x, w)$ that yields (perhaps approximate) tracking.

3 The solution

As is standard in nonlinear regulator theory (see [1], [5]) our starting point will be to assume that we can solve the following partial differential and algebraic equations (at least approximately) for $\pi(w)$ and $c(w)$ which characterize the zero-error manifold and the feedforward that renders the manfold invariant, respectively:

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w \\ h(\pi(w)) + q(w) &= 0 \end{aligned} \tag{3}$$

The standard nonlinear regulator solution is then to choose the feedback

$$u = c(w) + K[x - \pi(w)] \tag{4}$$

where K is a linear gain matrix that stabilizes the Jacobian linear approximation of (1). This, of course, implies that the linear approximation of (1) is controllable. For the nonlinear regulator problem, the feedback (4) solves the tracking problem for sufficiently small $(x(0), w(0))$. We will retain the requirement that $w(0)$ is sufficiently small, but we will allow $x(0)$ to be large.

Consider the system (1) disconnected from the exosystem:

$$\dot{x} = f(x) + g(x)u \tag{5}$$

Let $u = \varphi(x)$, with $\varphi(0) = 0$, be a smooth control that renders the equilibrium $x = 0$ of (5) globally asymptotically stable and locally *exponentially* stable. We then have the following result.

Theorem 3.1 $\exists \epsilon_o$ *such that for any* $\epsilon < \epsilon_o$, *if* $|w(t)| < \epsilon$ *for all* $t \geq 0$, *then the control* $u = c(w) + \varphi(x - \pi(w))$ *solves the nonlinear regulator problem with basin of attraction containing the ball* $|x(0)| \leq \kappa(\frac{1}{\epsilon})$ *for some class-K function* $\kappa(\cdot)$.

Proof. The proof uses the total stability result of Sontag [6]. Define

$$F(x,w) := f(x) + g(x)[c(w) + \varphi(x - \pi(w))] + p(x)w \tag{6}$$

Since $c(0) = 0$ and $\pi(0) = 0$, we have that $\dot{x} = F(x,0)$ is globally asymptotically stable. Therefore, there exists a smooth positive definite and proper Lyapunov function

$$V : \mathbf{R}^n \to \mathbf{R}$$

such that

$$dV(x) \cdot F(x,0) < 0$$

for all nonzero x. It then follows that

$$dV(x) \cdot F(x,w) < 0 \tag{7}$$

for all $|w| < \theta(|x|)$ for some continuous function $\theta : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\theta(0) = 0$ and that is decreasing on $[1, \infty)$. (See [6, Lemmas 3.1,3.2].) Then, for some ϵ_o sufficiently small and any $\epsilon < \epsilon_o$, we can deduce from the function $\theta(\cdot)$ two class-K functions κ_1 and κ_2 such that

$$dV(x) \cdot F(x,w) < 0 \tag{8}$$

for all $x \in \mathbf{R}^n$ satisfying

$$\kappa_1(\epsilon) \leq |x| \leq \kappa_2(\frac{1}{\epsilon}) \tag{9}$$

If ϵ_o sufficiently small then for all $\epsilon < \epsilon_o$ we have returned to the local nonlinear regulator problem. Since, $u = \varphi(x)$ locally *exponentially* stabilizes the origin of (5) the linear approximation of the composite closed loop is in the form for which center manifold theory applies. Since $\varphi(0) = 0$, $u = c(w) + \varphi(x - \pi(w))$, and $c(w)$ and $\pi(w)$ satisfy (3) and since φ is a locally exponential stabilizer, $x = \pi(w)$ is an attractive, invariant manifold for the closed loop. Finally, also from (3), the tracking error approaches zero asymptotically. □

Remark. Although we will not show it here, the results of the theorem extend readily to the approximate regulator problem (where the manifold equation (3) is solved up to some arbitrary order), and to the use of semi-globally stabilizing controls ($u = \varphi(x,p)$ where the basin of attraction of the system (5) can be made arbitrarily large by choice of p.)

4 Example

We demonstrate the capabilities of this approach on the "ball and beam" example which has been studied with regard to approximate tracking in [3], [2], [4] and [7]. The dynamics of this system can be modeled as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 x_4^2 - G\sin(x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \\ y &= x_1 \end{aligned} \tag{10}$$

where x_1 is ball position, x_2 is ball velocity, x_3 is the angle of the beam, and x_4 is the beam's angular velocity. (For a derivation of these equations, see [3].) For simplicity, we have normalized the acceleration due to gravity to be $G = 1$ in our simulations. In [8] it was shown that the control law

$$u = \varphi(x) = -4x_3 - 4x_4 - \sigma(y_1 + y_2) \tag{11}$$

where

$$\begin{aligned} y_1 &= -\tfrac{4}{G}x_1 - \tfrac{8}{G}x_2 + 5x_3 + x_4 \\ y_2 &= -\tfrac{4}{G}x_2 + 4x_3 + x_4 \end{aligned} \tag{12}$$

and $\sigma(\cdot)$ satisfies

1. $\sigma(s) = s$ for all $|s| \leq \delta$
2. $|\sigma(s)| = \delta$ for all $|s| > \delta$

for some $\delta > 0$ is an example of a semi-globally stabilizing control law for (10). The basin of attraction for $x = 0$ can be made arbitrarily large by making δ arbitrarily small.

The task at hand is to cause the ball position x_1 to (at least almost) track a sinusoid generated by the exosystem

$$\begin{aligned} \dot{w}_1 &= -\lambda w_2 \\ \dot{w}_2 &= \lambda w_1 \\ q(w) &= -w_1 \end{aligned} \tag{13}$$

As seen in [2] and [4] approximating the manifold to either first or third order yields nice approximate tracking results. A first order approximation to the mappings $x = \pi(w)$ and $u = c(w)$ are given by

$$\begin{aligned} \pi_1(w) &= w_1 \\ \pi_2(w) &= -\lambda w_2 \\ \pi_3(w) &= \tfrac{1}{G}\lambda^2 w_1 \\ \pi_4(w) &= -\tfrac{1}{G}\lambda^3 w_2 \\ c(w) &= -\tfrac{1}{G}\lambda^4 w_1 \end{aligned} \tag{14}$$

For simulation purposes, for the exosystem (13), we chose $\lambda = \frac{\pi}{30}$, $w_1(0) = 15$ and $w_2(0) = 0$. Consequently, the task is for the ball position, x_1, to track $15 \cdot \cos(\frac{\pi}{30}t)$. We choose the control

$$u = c(w) + \varphi(x - \pi(w)) \tag{15}$$

with $c(w)$ and $\pi(w)$ specified in (14) and φ specified in (11). To demonstrate regulation from a difficult initial condition, we choose the initial angle of the beam to be 90° and the ball to be a position slightly below the pivot of the beam at $x_1 = -1$. We give the ball zero initial velocity and the beam zero initial angular velocity. The results of the simulation are demonstrated in figures 1, 2 and 3.

5 Conclusion

We have demonstrated that the use of nonlinear feedback in place of linear feedback in the nonlinear regulator problem expands the domains of attraction when the nonlinear feedback is known to be a global or semi-global stabilizer.

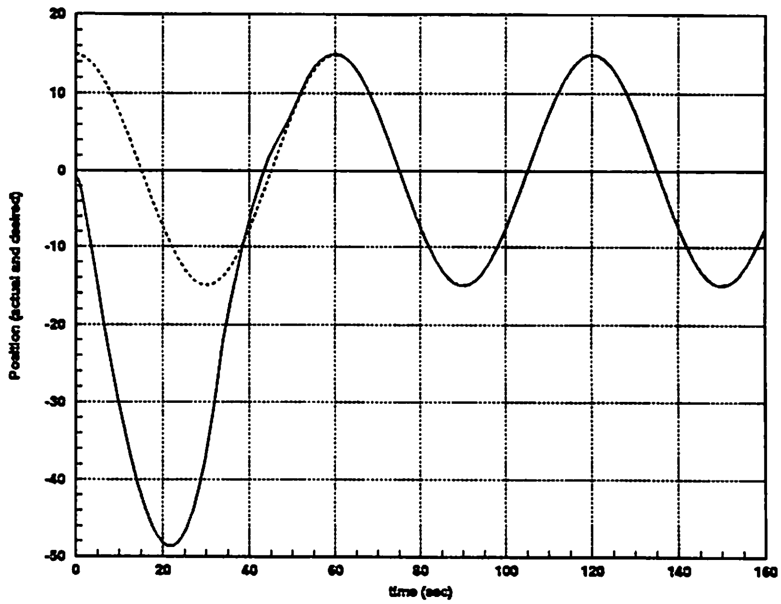

Figure 1: Tracking Results for the "ball and beam"

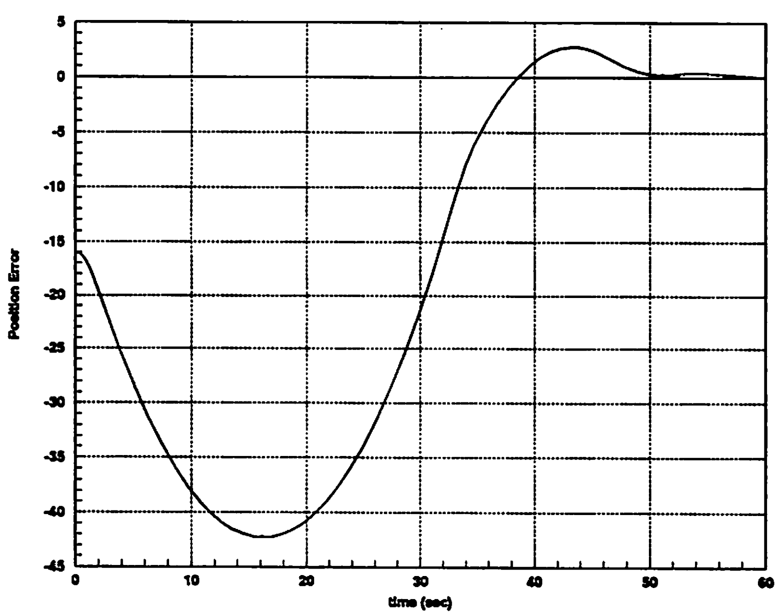

Figure 2: Transient performance

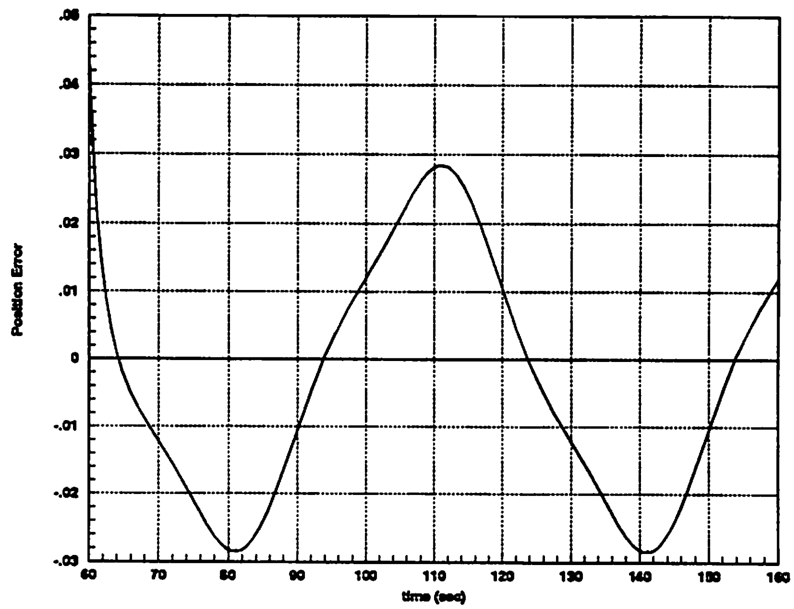

Figure 3: Steady-state performance

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