SLIDING MODE CONTROL OF PERTURBED NONLINEAR SYSTEMS

by

A. K. Pradeep

Memorandum No. UCB/ERL M92/33

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Abstract

Professor Shankar Sastry
Chairman

In this dissertation, we present techniques and conditions for the robust control of perturbed nonlinear systems.

First, we develop matching conditions i.e., conditions to be satisfied by perturbations such that the control objective, namely asymptotic regulation, is achieved by the perturbed system, utilizing control laws for the unperturbed system. In the first three chapters of this dissertation we present statements and proofs of matching conditions for :

- Perturbed SISO systems.
- Perturbed, MIMO systems that possess vector relative degree.
- Perturbed MIMO systems that are invertible but do not possess vector relative degrees. We consider control laws for such systems developed in the framework of the zero dynamics algorithm and the dynamic extension method.

In chapter 4, we review in our notation some basic results on existence and uniqueness of systems with discontinuous right hand sides. Finally in chapter 5 we develop techniques that utilize sliding mode control theory to identify unknown parameters for a class of nonlinear systems. We then develop robust control laws using

a Lyapunov control method that ensure stabilization in the presence of mismatched perturbations for a class of nonlinear systems. We utilize sliding mode control theory for the purpose of synchronous regulation utilizing multiple sliding surfaces, and conclude this dissertation with a conjecture on fractional control.

Dedication

This dissertation is dedicated to the following people who have taught me, inspired me, and have given meaning to my life.

- My parents Professor N. Anantha Krishnan, and R. Saraswathy.
- My sister Bhargavi and brother Prasad.
- Professors K. P. Ramakrishnan and S. Narayana Sastry at Madras, and Professor S. Shankara Sastry at Berkeley.
- My sweetheart Caroline L. Winnett.

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Chapter 1

Introduction

In this thesis we develop new and robust control techniques to achieve control objectives for a class of nonlinear systems. Our approach to the problem takes the following paths.

1. We first characterize the set of all perturbations that can be effectively handled by sliding mode control techniques in conjunction with the methodology of exact input-output linearization. We develop the matching conditions that must be satisfied by perturbations of SISO, and invertible MIMO nonlinear systems. In each case, we present proof of achievement of the control objective and the stability of the internal dynamics of the system. Chapter 1 and 4 contain background material on the theory of exact input-output linearization, and sliding mode control. Results, theorems and proofs of the analysis of matching conditions are contained in chapters 2 and 3.

2. While it is useful to know the classes of perturbations that can be effectively handled by existing control theory, the control engineer is often faced with the task of designing control laws for systems that are perturbed by disturbances that do not satisfy the matching conditions. We now design control techniques to handle such cases. We provide systematic robust control design methods using nonlinear identification and Lyapunov control to achieve identification and stabilization for certain classes of nonlinear systems. New sliding mode

nonlinear identifiers ensuring exponential convergence of parameters to true values, and Lyapunov controllers capable of handling mismatched non-lipschitz like perturbations are described in chapter 5.

3. The techniques of the theory of exact input-output linearization assume that full state information is available to the controller. In the absence of full state information, it becomes necessary to design observers to use the measurements to estimate the states. We now extend some results in the theory of planar sliding mode observers to ensure exponential convergence of the estimated state to its true value. An extension to standard, planar sliding mode observer theory is presented in chapter 5.

During the course of our investigations, we came across two novel extensions to classical sliding mode control theory that we present in chapter 5. The extensions are the following.

1. Synchronous control methods that ensure coupled motion through control. Control introduced synchronous motion is useful in many industrial scenarios. Multi-fingered robot hands gripping an object, milling machines working in conjunction with X-Y tables, and a host of other manufacturing processes utilize controlled synchronous motion. We present interesting control techniques that find use in such scenarios.

2. Conjecture on fractional control.

Chapter 2

Basics Of The Theory Of Exact Linearization

In this chapter, we will present a brief description of the theory of exact input-output linearization of nonlinear systems. The material in this chapter is background material liberally adapted from the works of [19].

2.1 Normal Forms for Single Input Single Output Nonlinear Systems

Consider a nonlinear system of the form

$$\dot{x} = f(x) + g(x)u(t) \tag{2.1}$$

$$y = h(x) \tag{2.2}$$

where $x \in \Re^n$, $f(x), g(x) : \Re^n \to \Re^n$ are smooth vector fields, the control input $u(t) : \Re_+ \to \Re$, and the output $h(x) : \Re^n \to \Re$ is a smooth nonlinear function.

We attempt to linearize the input-output behaviour of such a nonlinear system by doing the following.

- By choosing a new set of states, diffeomorphic to the original set of states in which the nonlinear system is described.

- Choosing an input such that in the new state space, the system is linear.

The choice of the new set of states is such that the new system description makes obvious the choice of linearizing input. The methodology is quite similar to the linear state transformations that transform controllable linear systems into the controllable canonical form. Indeed, once the linear system is in the controllable canonical form, the choice of pole-placement control law becomes obvious. We attempt to construct similar canonical forms for nonlinear systems where the choice of control law becomes intuitively obvious. We illustrate the idea with the following simple example.

Example 2.1.1 Exact Input-Output Linearization

Consider a scalar system of the form

$$\dot{x} = f(x) + u(t) \tag{2.3}$$

$$y = x \tag{2.4}$$

where $x \in \Re$, $f(x) : \Re \to \Re$ *and the control input* $u(t) : \Re_+ \to \Re$. *Differentiating the output* y *of system (2.3) once, we obtain*

$$\dot{y} = \frac{dx}{dt} \tag{2.5}$$

$$= f(x) + u(t) \tag{2.6}$$

It is immediately obvious that by choosing

$$u(t) = -f(x) + v(t) \tag{2.7}$$

where $v(t) : \Re_+ \to \Re$ *is unspecified as yet, we recast the system equation (2.3) in the form*

$$\dot{y} = v(t) \tag{2.8}$$

which is a simple linear system. Indeed, in this case there was no necessity to construct a diffeomorphism to a new state space. The particular choice of input (2.7) (cancelling the nonlinearity, $f(x)$ *in equation (2.6)), rendered the system linear and hence the name* input-output linearization. *Also, the input (2.7) cancelled the nonlinearity exactly, and hence the name* exact input-output linearization. *Although the* *methodology for higher dimensional systems is considerably more involved the technique of cancelling nonlinearities through an appropriate choice of inputs and states is retained.*

This simple illustrative example also provides the motivation for the first portion of this thesis. If equation (2.3) was a model for a real physical system, we would expect the real physical system to deviate from the considered model by some small amount. We would then be interested in knowing the classes of such deviations that permit the achievement of the control objective, when the control law designed based on the model is applied to the real physical system. Indeed in scenarios involving inexact knowledge of the controlled system, we wish to quantify in some sense the mismatch between the design model and the physical system so that the methodology of exact cancellation of nonlinearities (which in such situations is bound to be approximate) is still valid. We illustrate this situation with an example.

Example 2.1.2 Matching Conditions

Consider a perturbed design model of the following form.

$$\dot{x} = f(x) + \Delta f(x) + u(t) \tag{2.9}$$

where $x \in \Re$, $f(x) : \Re \rightarrow \Re$, $\Delta f(x) : \Re \rightarrow \Re$ *is a perturbation, and the control input* $u(t) : \Re_+ \rightarrow \Re$. *The choice of input (2.7) applied to the perturbed system (2.9) would yield an equation of the form*

$$\dot{x} = \Delta f(x) + v(t) \tag{2.10}$$

where the control input $v(t) : \Re_+ \rightarrow \Re$ *is to be specified yet. It is clear that* $v(t) : \Re_+ \rightarrow \Re$ *must be chosen to ensure robustness in the presence of the perturbation* $\Delta f(x) : \Re \rightarrow \Re$. *Indeed, if the control objective was stabilization of the system (2.9), and the perturbation* $|\Delta f(x)| \leq K_{\Delta f}$, *then the choice of control*

$$v(t) = -K_s sgn[x] \tag{2.11}$$

where $K_s > K_{\Delta f}$ *would ensure achievement of the control objective despite the perturbation.*

We now present a quick review of the theory of exact input-output linearization for a single-input single-output system, abbrieviated in this dissertation as a SISO system.

$$\begin{aligned} \dot{x} &= f(x) + g(x)u(t) && (2.12) \\ y &= h(x) && (2.13) \end{aligned}$$

with $x \in \Re^n$, $f(x), g(x) : \Re^n \to \Re^n$ are smooth vector fields, the output $h(x) : \Re^n \to \Re$ is a smooth nonlinear function, and $u(t) : \Re_+ \to \Re$ is the control input to the system.

Definition 2.1.1 *The* **Lie derivative** *of a smooth real valued function $h(x) : \Re^n \to \Re$ with respect to a vector field $f(x) : \Re^n \to \Re^n$ is a real valued function denoted by $L_f h(x) : \Re^n \to \Re$ defined as*

$$L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) \tag{2.14}$$

The notations $L_f^2 h(x)$ stand for $L_f(L_f h)(x)$ and $L_g L_f h(x) := L_g(L_f h(x))$ where $g(x) : \Re^n \to \Re^n$ is another smooth vector field.

Definition 2.1.2 *The* **Lie bracket** *of two vector fields $f(x) : \Re^n \to \Re^n$ and $g(x) : \Re^n \to \Re^n$, is a vector field denoted by $[f, g](x) : \Re^n \to \Re^n$ and is given in coordinates by*

$$[f, g] = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \tag{2.15}$$

Definition 2.1.3 *The SISO nonlinear system (2.13) is said to possess* **strict relative degree** γ *at* x_0 *if*

$$\begin{aligned} L_g L_f^i h(x) &\equiv 0 \quad \forall x \in B_r(x_0), \quad i = 0, \ldots, \gamma - 2 \\ L_g L_f^{\gamma-1} h(x_0) &\neq 0 \end{aligned} \tag{2.16}$$

Comment 2.1.1 Such a definition of relative degree is compatible with the usual definition of relative degree for linear systems (as being the excess of poles over zeros).

Comment 2.1.2 The relative degree of some nonlinear systems may not be defined at some points.

Given a SISO system of the form (2.13), with strict relative degree $\gamma \in Z_+$, we will now transform the nonlinear SISO system into a normal form. We commence by defining the components of such a nonlinear state transformation. Define the functions $\xi_i(x) : \Re^n \rightarrow \Re \;\; i = 1, 2, \ldots \gamma$ as follows.

$$\begin{aligned} \xi_1 &= h(x) \\ \xi_2 &= L_f h(x) \\ &\vdots \\ \xi_\gamma &= L_f^{\gamma-1} h(x) \end{aligned} \tag{2.17}$$

It follows from the definition of strict relative degree that

$$\dot{\xi}_1 = \xi_2 \tag{2.18}$$

$$\vdots \quad \vdots \quad \vdots \tag{2.19}$$

$$\dot{\xi}_\gamma = L_f^\gamma h(x) + L_g L_f^{\gamma-1} h(x) \tag{2.20}$$

and $L_g L_f^{\gamma-1} h(x) : \Re^n \rightarrow \Re \neq 0$ everywhere in a ball around x_0.

As the vector field $g(x) : \Re^n \rightarrow \Re^n$ of equation (2.13) is trivially involutive, there exist (by the theorem of Frobenius) functions $\eta_i(x) : \Re^n \rightarrow \Re \;\; i = 1, 2, \ldots . n-1$ such that the matrix

$$\begin{bmatrix} d\eta_1(x) \\ d\eta_2(x) \\ \vdots \\ d\eta_{n-1}(x) \end{bmatrix} \tag{2.21}$$

has rank $n-1$ at x_0 and

$$d\eta_i(x) g(x) = 0 \;\; \forall x \in B_r(x_0) \;\; i = 1, 2, \ldots . n-1 \tag{2.22}$$

Comment 2.1.3 The matrix given by

$$\begin{bmatrix} dh(x) \\ dL_f h(x) \\ \vdots \\ dL_f^{\gamma-1} h(x) \\ d\eta_1(x) \\ \vdots \\ d\eta_{n-1}(x) \end{bmatrix} \tag{2.23}$$

has rank n at $x_0 \in \Re^n$.

We now formally define the coordinate transformation we had been seeking.

$$\Phi : x \in \Re^n \rightarrow \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_\gamma \\ \eta \end{bmatrix} \tag{2.24}$$

Note that $\Phi(x) : \Re^n \rightarrow \Re^n$ is a local diffeomorphism. In the $[\xi, \eta]$ coordinates, the system equations (2.13) are recast in the form

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \vdots \;\; & \vdots \;\; \vdots \\ \dot{\xi}_\gamma &= b(\xi, \eta) + a(\xi, \eta)u(t) \\ \dot{\eta} &= q(\xi, \eta) \\ y &= \xi_1 \end{aligned} \tag{2.25}$$

where

$$b(\xi, \eta) : \Re^n \rightarrow \Re = L_f^\gamma h(x) \tag{2.26}$$

$$a(\xi, \eta) : \Re^n \rightarrow \Re = L_g L_f^{\gamma-1} h(x) \tag{2.27}$$

$$q_i(\xi, \eta) : \Re^n \rightarrow \Re^{n-\gamma} = L_f \eta_i \;\; i = 1, 2, \ldots, n-\gamma \tag{2.28}$$

The choice of linearizing input is now obvious from equation 2.30). Choosing

$$u(t) = \frac{1}{a(\xi,\eta)}[-b(\xi,\eta) + v(t)] \tag{2.29}$$

we exactly cancel the nonlinearity $b(\xi,\eta) : \Re^n \to \Re$ in equation (2.30) to yield,

$$\begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & \xi_3 \\ \vdots & \vdots & \vdots \\ \dot{\xi}_\gamma & = & v(t) \\ \dot{\eta} & = & q(\xi,\eta) \\ y & = & \xi_1 \end{array} \tag{2.30}$$

Thus a portion of the system, the ξ dynamics is now linear. Such a choice of state transformation, and control law is exactly what we attempted to set out to discover.

2.2 Normal Forms for Multi Input Multi Output Nonlinear Systems

We now extend the theory developed for SISO systems to certain classes of square MIMO nonlinear systems. Consider nonlinear systems of the following form

$$\begin{array}{rcl} \dot{x} & = & f(x) + \sum_{i=1}^{m} g_i(x)u_i(t) \\ y_i(x) & = & h_i(x) \;\; i = 1,2,\ldots,m \end{array} \tag{2.31}$$

where the state $x \in \Re^n$, and $f(x) : \Re^n \to \Re^n$, $g_i(x) : \Re^n \to \Re^n$ $i = 1,2,\ldots,m$, are smooth vector fields, $u_i(t) : \Re_+ \to \Re$ $i = 1,2,\ldots,m$ are control inputs the outputs $y_i(x) : \Re^n \to \Re$ $i = 1,2,\ldots,m$ are smooth nonlinear functions.

The development of the theory for MIMO nonlinear systems closely parallels the development of the exact input-output linearizing technique for SISO nonlinear systems outlined in the previous section. We commence by defining the MIMO equivalent of the SISO concept of relative degree.

Definition 2.2.1 *The MIMO system represented by equations (2.31) is said to have vector relative degree* $[\gamma_1, \gamma_2, \ldots, \gamma_m]^T$ *at* x_0 *if*

$$L_{g_i} L_f^k h_i(x) \equiv 0 \;\; for \;\; 1 \le j \le m, 0 \le k \le \gamma_i - 2 \tag{2.32}$$

for $i = 1, \ldots, m$ *and* $x \in B_r(x_0)$, *the matrix* $A(x) : \Re^n \to \Re^{m \times m}$, *referred to as the decoupling matrix, and defined as*

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{\gamma_1 - 1} h_1 & \cdots & L_{g_m} L_f^{\gamma_1 - 1} h_1 \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{\gamma_m - 1} h_m & \cdots & L_{g_m} L_f^{\gamma_m - 1} h_m \end{bmatrix} \tag{2.33}$$

is nonsingular at x_0.

As before, we will now attempt to find a normal form for the square MIMO system (2.31) where the choice of linearizing input will be obvious.

Define the following functions

$$\begin{array}{llll} \xi_1^1 = h_1(x), & \xi_2^1 = L_f h_1(x), & \cdots & \xi_{\gamma_1}^1 = L_f^{\gamma_1 - 1} h_1(x) \\ \xi_1^2 = h_2(x), & \xi_2^2 = L_f h_2(x), & \cdots & \xi_{\gamma_2}^2 = L_f^{\gamma_2 - 1} h_2(x) \\ \cdots & & & \\ \xi_1^m = h_m(x), & \xi_2^m = L_f h_m(x), & \cdots & \xi_{\gamma_m}^m = L_f^{\gamma_m - 1} h_m(x) \end{array} \tag{2.34}$$

where the functions $\xi_j^i : \Re^n \to \Re \;\; i = 1, 2, \ldots, m \;\; j = 1, 2, \ldots, \gamma_i$ qualify as a partial set of coordinates. Complete the basis choosing $n - \gamma$ ($\gamma = \sum_{i=1}^m \gamma_i$) more functions $\eta_i(x) : \Re^n \to \Re \;\; i = 1, 2, \ldots, n - \gamma$. It is no longer possible as in the SISO case to guarantee that

$$L_{g_j} \eta_i(x) \equiv 0 \;\; 1 \le j \le p \;\; 1 \le i \le n - \gamma \tag{2.35}$$

unless the distribution spanned by $g_j(x) : \Re^n \to \Re$ is involutive. Using the notation

$$\xi^i \in \Re^{\gamma_i} = \begin{bmatrix} \xi_1^i \\ \vdots \\ \xi_{\gamma_i}^i \end{bmatrix} \tag{2.36}$$

we recast the system equations (2.31) in the $[\xi^1, \ldots, \xi^m, \eta]$ coordinates as

$$
\begin{aligned}
\dot{\xi}_1^1 &= \xi_2^1 \\
&\cdots \\
\dot{\xi}_{\gamma_1}^1 &= b_1(\xi^1, \ldots, \xi^m, \eta) + \textstyle\sum_{j=1}^m a_j^1(\xi^1, \ldots, \xi^m, \eta) u_j(t) \\
\dot{\xi}_1^2 &= \xi_2^2 \\
&\cdots \\
\dot{\xi}_{\gamma_2}^2 &= b_2(\xi^1, \ldots, \xi^m, \eta) + \textstyle\sum_{j=1}^m a_j^2(\xi^1, \ldots, \xi^m, \eta) u_j(t) \\
&\cdots \\
\dot{\xi}_1^m &= \xi_2^p \\
&\cdots \\
\dot{\xi}_{\gamma_m}^m &= b_m(\xi^1, \ldots, \xi^m, \eta) + \textstyle\sum_{j=1}^m a_j^m(\xi^1, \ldots, \xi^m, \eta) u_j(t) \\
\dot{\eta}_i &= q_i(\xi^1, \ldots, \xi^m, \eta) + \textstyle\sum_{j=1}^m p_{ij}(\xi^1, \ldots, \xi^m, \eta) u_j(t) \quad i = 1, 2, \ldots, n - \gamma \\
y_1 &= \xi_1^1 \\
&\cdots \\
y_m &= \xi_1^m
\end{aligned}
\tag{2.37}
$$

where,

$$b_i(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re = L_f^{\gamma_i} h_i(x) \quad i = 1, 2, \ldots, m \tag{2.38}$$

$$a_j^i(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re = L_{g_j} L_f^{\gamma_i - 1} h_i(x) \quad 1 = 1, 2, \ldots, m \tag{2.39}$$

$$q_i(\xi^1, \ldots, \xi^m, \eta)' : \Re^n \to \Re = L_f \eta_i(x) \quad i = 1, 2, \ldots, n - \gamma \tag{2.40}$$

$$p_{ij}(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re = L_{g_j} \eta_i(x) i = 1, 2, \ldots, n - \gamma \quad j = 1, 2, \ldots, m \tag{2.41}$$

in the $[\xi^1, \ldots, \xi^m, \eta]$ coordinates. Indeed, now choose the control inputs $u_i(t) : \Re_+ \to \Re \;\; i = 1, 2, \ldots, m$ as

$$u_i(t) = -A^{-1}(x) \begin{bmatrix} -b_1(\xi^1, \ldots, \xi^m, \eta) + v_1(t) \\ \vdots \\ -b_m(\xi^1, \ldots, \xi^m, \eta) + v_m(t) \end{bmatrix} \tag{2.42}$$

where $v_i(t) : \Re_+ \to \Re$ are control inputs that are as yet unspecified to yield a *partial* *linear* MIMO system of the following form.

$$
\begin{aligned}
\dot{\xi}_1^1 &= \xi_2^1 \\
\vdots \quad &\vdots \; \vdots \\
\dot{\xi}_{\gamma_1}^1 &= v_1(t) \\
\dot{\xi}_1^2 &= \xi_2^2 \\
\vdots \quad &\vdots \; \vdots \\
\dot{\xi}_{\gamma_2}^2 &= v_2(t) \\
\vdots \quad &\vdots \; \vdots \\
\dot{\xi}_1^m &= \xi_2^p \\
\vdots \quad &\vdots \; \vdots \\
\dot{\xi}_{\gamma_m}^m &= v_m(t) \\
\dot{\eta}_i &= q_i(\xi^1,\dots,\xi^m,\eta) + \textstyle\sum_{j=1}^m p_{ij}(\xi^1,\dots,\xi^m,\eta)u_j(t) \quad i = 1,2,\dots,n-\gamma \\
y_1 &= \xi_1^1 \\
\dots \\
y_m &= \xi_1^m
\end{aligned}
\tag{2.43}
$$

If the $v(t) : \Re_+ \rightarrow \Re \;\; i = 1,2,\dots,m$ are chosen with the objective of stabilizing the nonlinear system, then the dynamics of the η variables with the control law of are given by

$$\dot{\eta} = q(0,\eta) - P(0,\eta)A^{-1}(0,\eta)b(0,\eta) \tag{2.44}$$

If $f(x_0) = 0, h_1(x_0) = \cdots = h_m(x_0) = 0$ then it follows that $\eta = 0$ is an equilibrium point of the zero dynamics of (2.44).

Chapter 3

Generalized Matching Conditions For Perturbed SISO Systems

3.1 Introduction

Matching conditions are conditions that the perturbations of a system must satisfy in order to ensure robustness of the control objective. For instance, in a typical control scenario, the engineer is presented with a model of the plant, and is asked to prescribe a control law *based on the model,* which when applied to the plant would still fulfill the control objective. It is helpful in such situations to view the deviation of the plant from the model as perturbations of the design model. Thus designing a robust control law for the model is equivalent to attaining control objectives in the plant. In this chapter, we pose the dual question. Given a design model, a control objective, and a control law that achieves the control objectives for the model, characterize the set of all perturbations under which the control law is robust. In essence, by specifying the matching conditions, we characterize the set of all plants that can be controlled using the chosen control law and the specified design model.

In this chapter we restrict ourselves to considering robustness of control laws that are based on the theory of exact linearization [17], [20] [3], [19], [26]. Exact cancellations of nonlinearities is seldom achieved in real life and it therefore is quite useful to understand the classes of perturbations that are permissible when an exact linearization methodology is used. The method of exact linearization renders a portion of the system dynamics unobservable. This unobservable dynamics of the system, referred to loosely as the zero dynamics, plays a vital role in detemining the the kinds of perturbations that do not degrade achievement of the control objective.

It is our contention that the matching conditions as they are known in literature [12], [11], [2] today suffer a drawback in that they cannot be naturally extended to MIMO systems. This is especially true for MIMO systems which do not possess a well defined relative degree. (We will refer to such systems as singular MIMO systems, the singularity referred to is the singularity of the decoupling matrix in the neighbourhood of interest.) The reason for the nonextendability of the matching conditions (as known in literature, we will refer to our conditions as the generalized matching conditions), is that they have always arisen out of considering the simple case of SISO systems with unperturbed zero dynamics. Analysis of SISO systems does not reveal the difficulties that arise in the context of MIMO systems as in most cases the zero-dynamics of MIMO systems is perturbed.

It is here that we chose to explore the richer area of SISO systems with perturbed zero dynamics, and come up with a set of generalized matching conditions. Not surprisingly, these conditions have a simple and natural extension to the MIMO nonsingular and singular cases as well.

The organization of the chapter is as follows. Section I presents the matching conditions for SISO systems with perturbed zero dynamics.

3.2 Single Input Single Output Systems

Consider the SISO systems specified by the following equations.

Unperturbed System Equations

$$\dot{x} = f(x) + g(x)u(t) \tag{3.1}$$

$$y = h(x) \tag{3.2}$$

where $x \in \Re^n$ $f(\cdot) : \Re^n \rightarrow \Re^n$ is a smooth vector field, $g(\cdot) : \Re^n \rightarrow \Re^n$ is a smooth vector field, $h(\cdot) : \Re^n \rightarrow \Re$ is a smooth function, $u(\cdot) : \Re_+ \rightarrow \Re$

Perturbed System Equations

$$\dot{x} = f(x) + \Delta f(x) + [g(x) + \Delta g(x)]u(t) \tag{3.3}$$

$$y = h(x) \tag{3.4}$$

where $\Delta f(\cdot) : \Re^n \rightarrow \Re^n$ is a smooth vector field, $\Delta g(\cdot) : \Re^n \rightarrow \Re^n$ is a smooth vector field,

Comment 3.2.1 A wide variety of perturbations ranging from plant noise to parametric uncertainties are included in the structure of the perturbations specified by (3.3) - (3.4). A notable exception however is the class of perturbations known as measurement noise. Throughout this chapter we will assume that the systems under consideration are unaffected by measurement noise, and that full state information is available at all times.

Let x_0 be an equilibrium point of the undriven unperturbed system (3.1) - (3.2), that is $f(x_0) = 0$, such that the output of the nonlinear system is zero at x_0, i.e $h(x_0) = 0$. We will now assume that the system (3.1) - (3.2) has *strict relative degree* γ at x_0 [19] (that is, in an open subset U containing the point x_0, $L_g L_f^i h(x) \equiv 0$ $i = 0, 1, \ldots, \gamma - 2$ and $L_g L_f^{\gamma-1} h(x)$ is bounded away from 0.

Statement Of The Problem

The SISO Matching problem is formally stated as follows:

Given:

- An unperturbed system of the form (3.1) - (3.2) with a relative degree $\gamma \in Z_+ \leq n$.
- The general classes of perturbations of interest specified by (3.3) - (3.4).
- A control objective - asymptotic output regulation. $y = h(x) \rightarrow 0$ as $t \rightarrow \infty$

Determine:

- Conditions that must be satisfied by the perturbations $\Delta f(x)$ and $\Delta g(x)$ of (3.3) - (3.4) such that the control objective of asymptotic output regulation is achieved by the control law developed based on the unperturbed system equations (3.1) - (3.2), when applied to the perturbed system with all the state variables $x \in \Re^n$ remaining bounded.

Matching Condition As Known In Literature

We will now present the matching condition known in literature, [12], [23], [18], [32], [9] and point out the difficulties associated with extending it to MIMO systems. We will then present newer matching assumptions for SISO systems, that can be easily extended to MIMO systems also.

We first develop the standard local normal form for the unperturbed system (3.1) - (3.2) as in [19].

Define the following γ functions.

$$\begin{aligned}
\phi_1(x) &= h(x) && (3.5)\\
\phi_2(x) &= L_f h(x) && (3.6)\\
\vdots &= \vdots && (3.7)\\
\phi_\gamma(x) &= L_f^{\gamma-1} h(x) && (3.8)
\end{aligned}$$

As outlined in the preceding chapter, by the definition of strict relative degree, the functions $\phi_i(\cdot) : \Re^n \to \Re,\ i = 1, \ldots, \gamma < n$ defined in (3.5) - (3.8) possess derivatives that are linearly independent over the ring of smooth functions. Pick $n-\gamma$ other functions $\eta_i(\cdot) : \Re^n \to \Re,\ i = i, \ldots, n-\gamma$ such that $L_g \eta_i(x) \equiv 0,\ i = 1, \ldots, n-\gamma$ and $d\eta_i(x),\ i = 1, \ldots, n-\gamma$ are linearly independent of $d\phi_i(\cdot),\ i = 1, \ldots, \gamma$. The functions $\phi_i(\cdot) : \Re^n \to \Re,\ i = i, \ldots, \gamma < n$, together with $\eta_i(\cdot) : \Re^n \to \Re,\ i = i, \ldots, n-\gamma$ are used to construct a nonlinear change of coordinates so as to exhibit the unperturbed system (3.1) - (3.2) in a local normal form [19].

Denoting $\xi_i = \phi_i(x)$ and $\xi = \begin{bmatrix} \xi_1 & \cdots & \xi_\gamma \end{bmatrix}^T$ and $\eta = \begin{bmatrix} \eta_1(x) & \cdots & \eta_{n-\gamma}(x) \end{bmatrix}^T$ define the map Φ to be

$$\Phi : x \in U \to \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \Phi(U) \tag{3.9}$$

Note that Φ is a diffeomorphism.

Using the diffeomorphism (3.9), construct a local normal form for the system (3.1) - (3.2) as

$$\dot{\xi}_1 = \xi_2 \tag{3.10}$$
$$\vdots = \vdots \tag{3.11}$$
$$\dot{\xi}_\gamma = b(\xi,\eta) + a(\xi,\eta)u \tag{3.12}$$
$$\dot{\eta} = q(\xi,\eta) \tag{3.13}$$

where,

$$\begin{aligned} b(\xi,\eta) : \Re^n \to \Re &= L_f^\gamma h \circ \Phi^{-1}(\xi,\eta) \\ a(\xi,\eta) : \Re^n \to \Re &= L_g L_f^{\gamma-1} h \circ \Phi^{-1}(\xi,\eta) \\ q(\xi,\eta) : \Re^n \to \Re^{n-\gamma} &= L_f \eta \circ \Phi^{-1}(\xi,\eta) \end{aligned}$$

The *zero dynamics* are defined to be the following dynamical system in $\Re^{n-\gamma}$ consistent with the notion of holding the output $y(t)$ to be identically zero, and consequently $\xi_1 = \xi_2 = \cdots = \xi_\gamma = 0$

$$\dot{\eta} = q(0,\eta) \tag{3.14}$$

Note that the specific choice of the η coordinates in (3.9) ensures that the input u does not enter the η dynamics (as the η coordinates were chosen such that $L_g\eta_i(x) \equiv 0 \; i = 1,\ldots,n-\gamma$). Consequently, perturbation vector fields $\Delta f(\cdot), \Delta g(\cdot) : \Re^n \to \Re^n$ that lie in the span of $g(x)$, i.e $\Delta f(x) = \alpha_{\Delta f}(x)g(x)$ and $\Delta g(x) = \alpha_{\Delta g}(x)g(x)$ where $\alpha_{\Delta f}(\cdot) : \Re^n \to \Re$ and $\alpha_{\Delta g}(\cdot) : \Re^n \to \Re$ would also not enter the zero dynamics as $L_{\Delta f(x)}\eta_i(x) = \alpha_{\Delta f}(x)L_g\eta_i(x) \equiv 0 \; i = 1,\ldots,n-\gamma$ and $L_{\Delta g(x)}\eta_i(x) = \alpha_{\Delta g}(x)L_g\eta_i(x) \equiv 0 \; i = 1,\ldots,n-\gamma$ The stability of the η dynamics is therefore unaffected by the presence of perturbations. It is this intution that is captured in the statement of the well known matching conditions presented in [12] and [2].

A formal statement of the classic matching conditions is as follows:

$$\Delta f(x),\ \Delta g(x) \in span\,[g(x)] \tag{3.15}$$

Remarks:

- The matching conditions specify the relative degree of the system under the perturbations. Indeed condition (3.15) ensures that the relative degree of the system with respect to the input is not greater than the relative degree of the system with respect to the perturbation vector fields.

- Matching condition (3.15) guarantees the η dynamics to remain unperturbed. Since the input does not enter the zero dynamics of the system, it is necessary to ensure that the perturbations also do not enter it. Condition (3.15) makes the analysis simpler, but difficulties arise in extending it to MIMO systems except under very special circumstances.

We will now present the theorem that guarantees achievement of the control objective when the classic matching conditions are met.

Theorem 3.1 *Generalized Matching Conditions for SISO systems with unperturbed η dynamics:*

Given *(G1) A perturbed SISO system of the form (3.3) - (3.4).*

(G2) The relative degree of the unperturbed system (3.1) - (3.2) to be γ.

(G3) A control objective of asymptotic output regulation, that is $y = h(x) \rightarrow 0$ as $t \rightarrow \infty$

If *(I1) The zero dynamics of the unperturbed system (3.78) is exponentially stable.*

(I2)

$$\Delta f(x),\ \Delta g(x) \quad \in \quad span[\, g(x)\,] \tag{3.16}$$

(I4)

$$L_{\Delta f} L_f^{\gamma-1} h - L_{\Delta g} L_f^{\gamma-1} h \frac{L_f^{\gamma} h}{L_g L_f^{\gamma-1} h} \circ \Phi^{-1} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \leq K_{\Delta f} \tag{3.17}$$

(I5)

$$\frac{L_{\Delta g} L_f^{\gamma-1} h}{L_g L_f^{\gamma-1} h} \circ \Phi^{-1} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \leq K_{\Delta g} \tag{3.18}$$

everywhere in an open set $\Omega_{\Delta Bound} \subseteq \Phi(U)$

(I7) $L_f \eta$, *satisfy conic continuity in* ξ, *uniformly in* η, *with constants* K_q,*in an open set* $\Omega_{conic} \subseteq \Phi(U)$.

(I8) The control u (3.3) is chosen to be

$$u(t) = \frac{1}{a(\xi,\eta)}[-b(\xi,\eta) + v] \tag{3.19}$$

where,

$$v = a_{\gamma-1}\xi_\gamma + \cdots + a_1\xi_2 - K_s sgn(S) \tag{3.20}$$

$$S = \xi_\gamma + a_{\gamma-1}\xi_{\gamma-1} + \cdots a_1\xi_1 \tag{3.21}$$

$$\text{and is a Hurwitz polynomial} \tag{3.22}$$

$$sgn(S) = \frac{S}{|S|} \, \forall \, |S| > 0 \tag{3.23}$$

Then *(T1) There exist a set* $\Omega \subseteq \Phi(U)$ *and a constant* K^* *and a choice of* K_s *such that for all initial conditions belonging to* Ω, *and* $K_s > K^*$ *the output* $y = h(x)$ *tends to zero asymptotically while all the states* $x \in \Re^n$ *remain bounded.*

Proof: ♠ ▷

We will prove the theorem in two simple steps similar to the proof in [2].

- We will first assume that the system trajectories remain in the set $\Omega_{\Delta Bound} \subseteq \Phi(U)$, satisfying the boundedness of the perturbations, and show that in such a case asymptotic output regulation is achieved.
- We will then show, that there exists a set $\Omega \subseteq \Omega_{\Delta Bound} \subseteq \Phi(U)$ such that for all initial conditions $[\xi(0), \eta(0)]^T \in \Omega$, the system trajectories remain in Ω, and asymptotic output regulation is indeed achieved.

Step 1.

Assumptions *(I2)* - *(I3)* specify a local normal form for the perturbed system of the form,

$$\dot{\xi}_1 = \xi_2 \tag{3.24}$$

$$\vdots = \vdots \tag{3.25}$$

$$\dot{\xi}_\gamma = b(\xi,\eta) + \Delta b(\xi,\eta) + [a(\xi,\eta) + \Delta a(\xi,\eta)]u \tag{3.26}$$

$$\dot{\eta} = q(\xi,\eta) \tag{3.27}$$

Note the presence of an input perturbation in the ξ dynamics in (3.26).

Using the definition of S stated in theorem, we recast the coordinates from (ξ,η) to $[\bar{\xi},S,\eta]^T$ where $\bar{\xi} = [\xi_1,\ldots,\xi_{\gamma-1}]^T$. Now rewrite the system equations (3.24) - (3.27) in the $[\bar{\xi},S,\eta]^T$ coordinates using the control u specified in the theorem.

$$\dot{\bar{\xi}} = A\bar{\xi} + bS \tag{3.28}$$

$$\dot{S} = -K_s sgn(S) + \Delta_1 + \Delta_2 \tag{3.29}$$

$$\dot{\eta} = q(\bar{\xi},S,\eta) \tag{3.30}$$

where

$$A = \begin{bmatrix} 0 & 1 & ,\ldots, & 0 & 0 \\ 0 & 0 & 1 & ,\ldots, & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & -a_2 & ,\ldots, & ,\ldots, & -a_{\gamma-1} \end{bmatrix}_{[\gamma-1]\times[\gamma-1]} \tag{3.31}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{[\gamma-1]\times[1]} \tag{3.32}$$

$$\Delta_1 = \Delta b(\bar{\xi},S,\eta) - \frac{\Delta a(\bar{\xi},S,\eta)}{a(\bar{\xi},S,\eta)} b(\bar{\xi},S,\eta) \tag{3.33}$$

$$\Delta_2 = \frac{\Delta a(\bar{\xi},S,\eta)}{a(\bar{\xi},S,\eta)}[a_{\gamma-1}\xi_\gamma + \cdots + a_1\xi_2 - K_s sgn(S)] \tag{3.34}$$

Consider the Lyapunov function

$$V = \frac{S^2}{2} \tag{3.35}$$

Differentiating V along the flow of (4.37) we obtain,

$$\dot{V} = S[-K_s sgn(S) + \Delta_1 + \Delta_2] \tag{3.36}$$

Assume that the system trajectories remain in the set $\Omega_{\Delta Bound} \subseteq \Phi(U)$. Now note that

$$\Delta_2 \leq K_{\Delta g}|S|[||a||||\bar{\xi}|| + K_s] \tag{3.37}$$

where $a = \begin{bmatrix} a_{\gamma-1} & \cdots & a_1 \end{bmatrix}^T$

Using the bounds on $\Delta b(\bar{\xi}, S, \eta)$ and on $[\frac{\Delta a(\bar{\xi},S,\eta)}{a(\bar{\xi},S,\eta)}$, we obtain

$$\dot{V} \leq -|S|K_s + [K_{\Delta f} + K_{Deltag}[||a||||\bar{\xi}|| + K_s]]|S| \tag{3.38}$$

Now, let $\bar{\xi}_{max} = \sup \bar{\xi} \in \Omega_{\Delta Bound}$, then as we have assumed that the system trajectories remain in the set $\Omega_{\Delta Bound} \subseteq \Phi(U)$ we rewrite $\dot{V}$ as

$$\begin{aligned} \dot{V} &\leq -|S|K_s + [K_{\Delta f} + K_{Deltag}[||a||\bar{\xi}_{max} + K_s]]|S| && (3.39) \\ &\leq -|S|[K_s(1 - K_{\Delta g}) - K_{\Delta f} - K_{\Delta g}||a||\bar{\xi}_{max}] && (3.40) \end{aligned}$$

Let

$$K^* = \frac{K_{\Delta f} + K_{\Delta g}||a||\bar{\xi}_{max}}{K_s[1 - K_{\Delta g}]} \tag{3.41}$$

$\dot{V}$ now is equal to

$$\dot{V} = -|S|[K_s - K^*] \tag{3.42}$$

It is clear that when $K_s > K^*$, $\dot{V}$ is negative definite. Negative definiteness of $\dot{V}$ implies that $S = 0$ is attractive for all trajectories that remain in $\Omega_{\Delta Bound} \subseteq \Phi(U)$. Indeed, for all initial conditions in $\Omega_{\Delta Bound} \subseteq \Phi(U)$, if $\Omega_{\Delta Bound}$ is invariant, the trajectories reach the manifold $S = 0$ in *finite time*. The choice of control renders the manifold $S = 0$ invariant, and the dynamics on the manifold is such that $||\bar{\xi}||$ tends to zero exponentially. (This is evident from setting $S = 0$ in the $\bar{\xi}$ dynamics and noting that A is a Hurwitz matrix.)

However, we need to ensure that the trajectories never leave the set $\Omega_{\Delta Bound}$ thus validating the boundedness of the perturbations. Indeed, we will now attempt to find the largest set $\Omega \subseteq \Omega_{\Delta Bound}$ that would also maintain stability of the internal dynamics. To this end we consider a Lyapunov function that includes both the ξ and the η dynamics.

Step 2:

Since A is a Hurwitz matrix there exists a positive definite symmetric matrix P solving the matrix Lyapunov equation [25]

$$A^T P + PA = -I \tag{3.43}$$

Using a converse Lyapunov theorem [16], the exponential stability of the zero dynamics (3.78) guarantees the existence of a Lyapunov function V_η such that,

$$\begin{aligned} K_1\|\eta\|^2 &\leq V_\eta \leq K_2\|\eta\|^2 && (3.44)\\ \frac{\partial V_\eta}{\partial \eta}[q(0,\eta) &\leq -K_3\|\eta\|^2 && (3.45)\\ \|\frac{\partial V_\eta}{\partial \eta}\| &\leq K_4\|\eta\| && (3.46) \end{aligned}$$

Now consider the composite Lyapunov function given by

$$V = \alpha_1[\bar{\xi}^T P\bar{\xi}] + \frac{\alpha_2}{2}S^2 + \frac{\alpha_2}{4}S^4 + \alpha_3 V_\eta \tag{3.47}$$

where P is the solution of (4.47) and V_η satisfies (4.48) - (4.50).

Differentiating V (4.51) along the flow of (4.36) - (4.41) we obtain

$$\begin{aligned} \dot{V} &= \alpha_1[[A\bar{\xi} + bS]^T P\bar{\xi} + \bar{\xi}^T P[A\bar{\xi} + bS]] && (3.48)\\ &+\alpha_2[S[-K_s sgn(S) + \Delta_1 + \Delta_2]] && (3.49)\\ &+\alpha_2[S^3[-K_s sgn(S) + \Delta_1 + \Delta_2]] && (3.50)\\ &\alpha_3\frac{\partial V_\eta}{\partial \eta}q(\bar{\xi}, S, \eta) && (3.51) \end{aligned}$$

Assuming that $(\xi,\eta) \in \Omega_{\Delta Bound}$, we obtain

$$\begin{aligned} \dot{V} &\leq -\|\bar{\xi}\|^2[\alpha_1 - \frac{1}{4}] - \alpha_2[|S| + |S|^3][K_s - K^*] && (3.52)\\ &+K_6^2|S|^2 && (3.53)\\ &+\alpha_3\frac{\partial V_\eta}{\partial \eta}[q(\bar{\xi}, S, \eta) - q(0,0,\eta)] && (3.54)\\ &+\alpha_3\frac{\partial V_\eta}{\partial \eta}q(0,0,\eta) && (3.55) \end{aligned}$$

where

$$
\begin{aligned}
Sb^T P\bar{\xi} &\leq K_6||S||||\bar{\xi}|| && (3.56)\\
K_6 &= \sigma_{max}(P) && (3.57)\\
K_6||S||||\bar{\xi}|| &\leq \frac{||\bar{\xi}||^2}{4} + K_6^2 S^2 && (3.58)
\end{aligned}
$$

and $\sigma_{max}(P)$ is the maximum singular value of P.

Now define Ω_V to be the largest non-empty subset of $\Phi(U)$ such that

$$\Omega_V = \Omega_{\Delta Bound} \bigcap \Omega_{Conic} \quad (3.59)$$

Choose $c^* \in \Re_+$ such that

$$c^* = sup(c : V \leq c \ \in \Omega_V) \quad (3.60)$$

Define $\Omega = [((\bar{\xi}, S, \eta) : V \leq c^*]$.

For all initial conditions in Ω, we may rewrite $\dot{V}$ as

$$
\begin{aligned}
\dot{V} \leq\ & -||\bar{\xi}||^2[\alpha_1 - \frac{1}{4}] - \alpha_2[K_s - K^*][||S|| - \frac{K_6^2}{\alpha_2[K_s - K^*]}||S||^2 + ||S||^3] && (3.61)\\
& -\alpha_3 K_3||\eta||^2 && (3.62)\\
& +\alpha_3 K_4||\eta||[K_q[||\bar{\xi}|| + ||S||]] && (3.63)
\end{aligned}
$$

where,

$$||q(\bar{\xi}, S, \eta) - q(0, 0, \eta)|| \leq K_q[||\bar{\xi}|| + ||S||] \quad (3.64)$$

Now define

$$K_7 = \alpha_3 K_4 \quad (3.65)$$

and using the fact that

$$K_7||\eta||[||\bar{\xi}|| + ||S||] \leq \frac{||\eta||^2}{2} + K_7^2||\bar{\xi}||^2 + K_7^2||S||^2 \quad (3.66)$$

we rewrite equation (4.58) - (4.163) as

$$
\begin{aligned}
\dot{V} \leq\ & -||\bar{\xi}||^2[\alpha_1 - \frac{1}{4} - K_7^2] && (3.67)\\
& -\alpha_2[K_s - K^*][||S|| - \frac{K_6^2 + K_7^2}{\alpha_2[K_s - K^*]}||S||^2 + ||S||^3] && (3.68)\\
& -||\eta||^2[\alpha_3[K_3 - \frac{1}{2}] && (3.69)
\end{aligned}
$$

Note that the constants α_i have not been chosen yet, and we now choose them in such a manner as to ensure the negative definiteness of the Lyapunov function.

We choose them as

$$\alpha_1 > \frac{1}{4} + K_7^2 \tag{3.70}$$
$$\alpha_2 > K_6^2 + K_7^2 \tag{3.71}$$
$$\alpha_3 > \frac{1}{2K_3} \tag{3.72}$$
$$K_s > K^* \tag{3.73}$$

Such a choice of constants ensure that $\dot{V}$ is negative definite for all initial conditions in Ω, and for all perturbations such that $K_s > K^*$.(Note that the coefficient of $||S||^2$ is always less than unity, and for all $||S|| > 1$, $||S||^2 < ||S||^3$, and for all values of $||S|| < 1$, $||S|| > ||S||^2$.)

We have shown that there exists of an invariant set Ω and a constant K^* and a constant K_s such that Ω is invariant, and for all $K_s > K^*$, the output of the system is asymptotically regulated to the origin. The proof also yields that the states $[\bar{\xi}, S, \eta]^T$ are bounded, and therefore the states $x \in \Re^n$ are also bounded. ⊲♠

3.3 SISO - Generalized Matching Condition

In order to study matching conditions for SISO systems with perturbed η dynamics, we discard the specific choice of η coordinates in (3.9). We now construct another local normal form for the perturbed system by adding to the ξ_i $i = 1, 2, \ldots, \gamma$ a set of $\eta_i(x) : \Re^n \rightarrow \Re$ $i = 1, \ldots, n-\gamma$ whose derivatives $d\eta$ are linearly independent of the $d\xi_i$ $i = 1, 2, \ldots, \gamma$ (over the ring of smooth functions). *We will no longer insist that $d\eta_i(x)g(x) \equiv 0$.* Denote this new diffeomorphism by $\Phi : \Re^n \rightarrow \Re^n$. Under this new coordinate transformation, the equations of the *perturbed system* (3.3) - (3.4) may be recast in the form

$$\dot{\xi}_1 = \xi_2 \tag{3.74}$$
$$\vdots = \vdots \tag{3.75}$$
$$\dot{\xi}_\gamma = b(\xi,\eta) + \Delta b(\xi,\eta) + [a(\xi,\eta) + \Delta a(\xi,\eta)]u \tag{3.76}$$

$$\dot{\eta} = q(\xi,\eta) + \Delta q(\xi,\eta) + [p(\xi,\eta) + \Delta p(\xi,\eta)]u \tag{3.77}$$

where,

$$\begin{aligned}
b(\xi,\eta) : \Re^n \to \Re &= L_f^{\gamma} h \circ \Phi^{-1}(\xi,\eta) \\
\Delta b(\xi,\eta) : \Re^n \to \Re &= L_{\Delta f} L_f^{\gamma-1} h \circ \Phi^{-1}(\xi,\eta) \\
a(\xi,\eta) : \Re^n \to \Re &= L_g L_f^{\gamma-1} h \circ \Phi^{-1}(\xi,\eta) \\
\Delta a(\xi,\eta) : \Re^n \to \Re &= L_{\Delta g} L_f^{\gamma-1} h \circ \Phi^{-1}(\xi,\eta) \\
q(\xi,\eta) : \Re^n \to \Re^{n-\gamma} &= L_f \eta \circ \Phi^{-1}(\xi,\eta) \\
\Delta q(\xi,\eta) : \Re^n \to \Re^{n-\gamma} &= L_{\Delta f} \eta \circ \Phi^{-1}(\xi,\eta) \\
p(\xi,\eta) : \Re^n \to \Re^{n-\gamma} &= L_g \eta \circ \Phi^{-1}(\xi,\eta) \\
\Delta p(\xi,\eta) : \Re^n \to \Re^{n-\gamma} &= L_{\Delta g} \eta \circ \Phi^{-1}(\xi,\eta)
\end{aligned}$$

Comment: As a result of the choice of $\eta_i(x) : \Re^n \to \Re$ $i = 1,2,\ldots,n-\gamma$ coordinates whose derivatives do not annihilate g, both the input and the perturbations enter the η states in (3.77)

The zero-dynamics of the unperturbed system is again consistent with the notion of holding the ouput to zero, and is given by

$$\dot{\eta} = q(0,\eta) + [p(0,\eta)]u(0,\eta) \tag{3.78}$$

Here, $u(0,\eta) = -\frac{b(0,\eta)}{a(0,\eta)}$.

We are now ready to state the conditions on perturbations under which asymptotic regulation is achieved in the perturbed systems using the control law developed based on the unperturbed plant equations.

Theorem 3.2 *Generalized Matching Conditions for SISO systems with perturbed η dynamics:*

Given *(G1) A perturbed SISO system of the form (3.3) - (3.4).*

(G2) The relative degree of the unperturbed system (3.1) - (3.2) to be γ.

(G3) A control objective of asymptotic output regulation, that is $y = h(x) \to 0$ as $t \to \infty$

If *(I1) The zero dynamics of the unperturbed system (3.78) is exponentially stable.*

(I2)

$$\Delta f(x) \in ker[dh(x), dL_f h(x) \cdots dL_f^{\gamma-2} h(x)] \tag{3.79}$$

(I3)

$$\Delta g(x) \in ker[dh(x), dL_f h(x) \cdots dL_f^{\gamma-1} h(x)] \tag{3.80}$$

(I4) $\Delta f(\xi,\eta)$, $\Delta g(\xi,\eta)$ *satisfy conic continuity in* ξ, *uniformly in* η *with constants* $K_{\Delta f-conic}$, $K_{\Delta g-conic}$, *everywhere everywhere in an open set* $\Omega_{\Delta Conic} \subseteq \Phi(U)$. *That is,*

$$||\Delta f(\xi,\eta) - \Delta f(0,\eta)||_2 \leq K_{\Delta f-conic}||\eta|| \tag{3.81}$$

$$||\Delta g(\xi,\eta) - \Delta g(0,\eta)||_2 \leq K_{\Delta g-conic}||\eta|| \tag{3.82}$$

(I5) $L_{\Delta f} L_f^{\gamma-1} h \leq K_b < K_s$ *everywhere in an open set* $\Omega_{\Delta Bound} \subseteq \Phi(U)$

(I6) $f(\xi,\eta)$, $g(\xi,\eta)u(\xi,\eta)$ *satisfy conic continuity in* ξ, *uniformly in* η, *with constants* $K_{f-conic}$, $K_{g-conic}$ *everywhere in an open set* $\Omega_{conic} \subseteq \Phi(U)$.

(I7) $||\Delta f(0,\eta) + \Delta g(0,\eta)u(0,\eta)|| \leq K_{\Delta qp}||\eta||$ *everywhere in an open set*

$$\Omega_{\Delta[f(0,\eta)+g(0,\eta)u(0,\eta)]} \subseteq \Phi(U) \tag{3.83}$$

(I8) The diffeomorphism Φ *has a bounded Jacobian.*

(I9) The control u *(3.3) is chosen to be*

$$u(t) = \frac{1}{a(\xi,\eta)}[-b(\xi,\eta) + v] \tag{3.84}$$

where,

$$v = a_{\gamma-1}\xi_\gamma + \cdots + a_1\xi_2 - K_s sgn(S) \tag{3.85}$$

$$S = \xi_\gamma + a_{\gamma-1}\xi_{\gamma-1} + \cdots a_1\xi_1 \tag{3.86}$$

and is a Hurwitz polynomial (3.87)

$$sgn(S) = \frac{S}{|S|} \; \forall \, |S| > 0 \tag{3.88}$$

Then *(T1) There exist a set $\Omega \subseteq \Phi(U)$ and a constant K^* such that for all initial conditions belonging to Ω, and $K_{\Delta_{qp}} < K^*$ the output $y = h(x)$ tends to zero asymptotically while all the states $x \in \Re^n$ remain bounded.*

Proof:

♠ ▷ Preliminary to proving the theorem, we make the following remarks to clarify the meaning and need for the various matching conditions imposed on the perturbations in the theorem.

Remarks:

Assumption *(I1)* is not a matching assumption to be satisfied by the perturbation, but is needed in order to construct a converse Lyapunov argument [16], as in [29]. Indeed this assumption will be required to prove achievement of the control objective even for an unperturbed system. We will say more about this assumption later.

Asssumption *(I2)* is a matching assumption to be met by the perturbation. The assumption ensures that the relative degree of the perturbation vector field $\Delta f(x)$ is at least as high as the relative degree of the input.

Assumption *(I3)* is again a matching assumption to be satisfied by the perturbation. This assumption on the perturbation of the input vector field is stronger than *(I2)* in that we require the relative degree of the input perturbing vector field $\Delta g(x)$ be *strictly greater* than the relative degree of the input. Thus the input to the ξ dynamics is not corrupted, but the input to the η dynamics may be affected by the presence of $\Delta g(x)$.

Assumption *(I4)* is a matching condition to be satisfied by the perturbation. It is needed to bound certain quantities that show up in the course of proving the theorem. Note that the conic-continuity requirement is not global, but is only needed everywhere in a subset of $\Phi(U)$ containing the point x_0 in which the linearization is performed.

Assumption *(I5)* is again a matching assumption on the perturbation, and is needed to ensure that the size of the sliding mode gain chosen in the control is strictly greater than the size of the perturbation to ensure stability. Such an assumption eliminates the possibility of the perturbation showing up earlier than the input.

(I7) is an assumption on the control input, that in the absence of perturbations would regulate the output to 0. We consider the same control input, and specify the classes of perturbations which would not degrade achievement of the control objective.

We will prove the theorem in two simple steps similar to the proof in [2].

- We will first assume that the system trajectories remain in the set $\Omega_{\Delta Bound} \subseteq \Phi(U)$, satisfying the boundedness of the perturbations, and show that in such a case asymptotic output regulation is achieved.
- We will then show, that there exists a set $\Omega \subseteq \Omega_{\Delta Bound} \subseteq \Phi(U)$ such that for all initial conditions $[\xi(0), \eta(0)]^T \in \Omega$, the system trajectories remain in Ω, and asymptotic output regulation is indeed achieved.

Step 1.

Assumptions *(I2)* - *(I3)* specify a local normal form for the perturbed system of the form,

$$\dot{\xi}_1 = \xi_2 \tag{3.89}$$

$$\vdots = \vdots \tag{3.90}$$

$$\dot{\xi}_\gamma = b(\xi,\eta) + \Delta b(\xi,\eta) + [a(\xi,\eta)]u \tag{3.91}$$

$$\dot{\eta} = q(\xi,\eta) + \Delta q(\xi,\eta) + [p(\xi,\eta) + \Delta p(\xi,\eta)]u \tag{3.92}$$

Note the absence of an input perturbation in the ξ dynamics in (3.91).

Using the definition of S stated in theorem, we recast the coordinates from (ξ,η) to $[\bar{\xi}, S, \eta]^T$ where $\bar{\xi} = [\xi_1, \ldots, \xi_{\gamma-1}]^T$. Now rewrite the system equations (6.274) - (6.281) in the $[\bar{\xi}, S, \eta]^T$ coordinates using the control u specified in the theorem.

$$\dot{\bar{\xi}} = A\bar{\xi} + bS \tag{3.93}$$

$$\dot{S} = -K_s sgn(S) + \Delta b(\bar{\xi}, S, \eta) \tag{3.94}$$

$$\dot{\eta} = q(\bar{\xi}, S, \eta) + \Delta q(\bar{\xi}, S, \eta) + [p(\bar{\xi}, S, \eta) + \Delta p(\bar{\xi}, S, \eta)]u(\bar{\xi}, S, \eta) \tag{3.95}$$

where

$$A = \begin{bmatrix} 0 & 1 & ,\ldots, & 0 & 0 \\ 0 & 0 & 1 & ,\ldots, & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & -a_2 & ,\ldots, & ,\ldots, & -a_{\gamma-1} \end{bmatrix}_{[\gamma-1]\times[\gamma-1]} \tag{3.96}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{[\gamma-1]\times[1]} \tag{3.97}$$

Consider the Lyapunov function

$$V = \frac{S^2}{2} \tag{3.98}$$

Differentiating V along the flow of (3.94) we obtain,

$$\dot{V} = S[-K_s sgn(S) + \Delta b(\bar{\xi}, S, \eta)] \tag{3.99}$$

Assume that the system trajectories remain in the set $\Omega_{\Delta Bound} \subseteq \Phi(U)$. Using the bounds on $\Delta b(\bar{\xi}, S, \eta)$, we obtain

$$\dot{V} \leq -|S|[K_s - K_b sgn(S)] \tag{3.100}$$

Negative definiteness of $\dot{V}$ implies that $S = 0$ is attractive for all trajectories that remain in $\Omega_{\Delta Bound} \subseteq \Phi(U)$. Indeed, for all initial conditions in $\Omega_{\Delta Bound} \subseteq \Phi(U)$, if $\Omega_{\Delta Bound}$ is invariant, the trajectories reach the manifold $S = 0$ in *finite time.* The choice of control renders the manifold $S = 0$ invariant, and the dynamics on the manifold is such that $||\xi||$ tends to zero exponentially. (This is evident from setting $S = 0$ in the $\bar{\xi}$ dynamics and noting that A is a Hurwitz matrix.)

However, we need to ensure that the trajectories never leave the set $\Omega_{\Delta Bound}$ thus validating the boundedness of the perturbations. Indeed, we will now attempt to find the largest set $\Omega \subseteq \Omega_{\Delta Bound}$ that would also maintain stability of the internal dynamics. To this end we consider a Lyapunov function that includes both the ξ and the η dynamics.

Step 2:

Since A is a Hurwitz matrix there exists a positive definite symmetric matrix P solving the matrix Lyapunov equation [25]

$$A^T P + PA = -I \tag{3.101}$$

Using a converse Lyapunov theorem [16], the exponential stability of the zero dynamics (3.78) guarantees the existence of a Lyapunov function V_η such that,

$$\begin{aligned} K_1\|\eta\|^2 &\leq V_\eta \leq K_2\|\eta\|^2 && (3.102)\\ \frac{\partial V_\eta}{\partial \eta}[q(0,\eta)+p(0,\eta)u(0,\eta)] &\leq -K_3\|\eta\|^2 && (3.103)\\ \|\frac{\partial V_\eta}{\partial \eta}\| &\leq K_4\|\eta\| && (3.104) \end{aligned}$$

Now consider the composite Lyapunov function given by

$$V = \alpha_1[\bar{\xi}^T P\bar{\xi}] + \frac{\alpha_2}{2}S^2 + \frac{\alpha_2}{4}S^4 + \alpha_3 V_\eta \tag{3.105}$$

where P is the solution of (3.101) and V_η satisfies (3.102) - (3.104).

Differentiating V (6.42) along the flow of (3.93) - (3.95) we obtain

$$\begin{aligned} \dot{V} = {} & \alpha_1[[A\bar{\xi} + bS]^T P\bar{\xi} + \bar{\xi}^T P[A\bar{\xi} + bS]] \\ & +\alpha_2[S[-K_s sgn(S) + \Delta b(\bar{\xi}.S,\eta)]] \\ & +\alpha_2[S^3[-K_s sgn(S) + \Delta b(\bar{\xi},S,\eta)]] \\ & \alpha_3\frac{\partial V_\eta}{\partial \eta}[q(\bar{\xi},S,\eta) + \Delta q(\bar{\xi},S,\eta) + [p(\bar{\xi},S,\eta) + \Delta p(\bar{\xi},S,\eta)]u(\bar{\xi},S,\eta)] \end{aligned}$$

Assuming that $(\xi,\eta) \in \Omega_{\Delta Bound}$, we obtain

$$\begin{aligned} \dot{V} \leq {} & -\|\bar{\xi}\|^2[\alpha_1 - \frac{1}{4}] - \alpha_2[K_s - K_b][\|S\| - \frac{K_6^2}{\alpha_2[K_s - K_b]}|S|^2 + |S|^3] \\ & +\alpha_3\frac{\partial V_\eta}{\partial \eta}[q(0,0,\eta) + p(0,0,\eta)u(0,0,\eta)] \\ & +\alpha_3\frac{\partial V_\eta}{\partial \eta}[q(\bar{\xi},S,\eta) - q(0,0,\eta)] \end{aligned}$$

$$
\begin{aligned}
&+\alpha_3\frac{\partial V_\eta}{\partial\eta}[p(\bar\xi,S,\eta)u(\bar\xi,S,\eta)-p(0,0,\eta)u(0,0,\eta)]\\
&+\alpha_3\frac{\partial V}{\partial\eta}[\Delta q(\bar\xi,S,\eta)-\Delta q(0,0,\eta)]\\
&+\alpha_3\frac{\partial V}{\partial\eta}[\Delta p(\bar\xi,S,\eta)u(\bar\xi,S,\eta)-\Delta p(0,0,\eta)u(0,0,\eta)]\\
&+\alpha_3\frac{\partial V}{\partial\eta}[\Delta q(0,0,\eta)+\Delta p(0,0,\eta)u(0,0,\eta)]
\end{aligned}
$$

where

$$
\begin{aligned}
Sb^TP\bar\xi &\leq K_6||S||||\bar\xi||\\
K_6 &= \sigma_{max}(P)\\
K_6||S||||\bar\xi|| &\leq \frac{||\bar\xi||^2}{4}+K_6^2S^2
\end{aligned}
$$

and $\sigma_{max}(P)$ is the maximum singular value of P.

Now define Ω_V to be the largest non-empty subset of $\Phi(U)$ such that

$$\Omega_V=\Omega_{\Delta Bound}\bigcap\Omega_{\Delta Conic}\bigcap\Omega_{Conic}\bigcap\Omega_{\Delta[f(0,\eta)+g(0,\eta)u(0,\eta)]} \tag{3.106}$$

Choose $c^*\in\Re_+$ such that

$$c^*=sup(c:V\leq c\ \in\Omega_V) \tag{3.107}$$

Define $\Omega=[((\bar\xi,S,\eta):V\leq c^*]$.

For all initial conditions in Ω, we may rewrite $\dot V$ as

$$
\begin{aligned}
\dot V \leq\ & -||\bar\xi||^2[\alpha_1-\frac{1}{4}]-\alpha_2[K_s-K_b][||S||-\frac{K_6^2}{\alpha_2[K_s-K_b]}||S||^2+||S||^3] && (3.108)\\
&-\alpha_3K_3||\eta||^2 && (3.109)\\
&+\alpha_3K_4||\eta||[||\bar\xi||+||S||][K_q+K_p+K_{\Delta q}+K_{\Delta p}] && (3.110)\\
&+\alpha_3K_4K_{qp}||\eta||^2 && (3.111)
\end{aligned}
$$

where,

$$||q(\bar\xi,S,\eta)-q(0,0,\eta)|| \leq K_q[||\bar\xi||+||S||]$$

$$\begin{aligned}
||p(\bar{\xi}, S, \eta)u(\bar{\xi}, S, \eta) - p(0,0,\eta)u(0,0,\eta)|| &\leq K_p[||\bar{\xi}|| + ||S||] \\
||\Delta q(\bar{\xi}, S, \eta) - \Delta q(0,0,\eta)|| &\leq K_{\Delta q}[||\bar{\xi}|| + ||S||] \\
||\Delta p(\bar{\xi}, S, \eta)u(\bar{\xi}, S, \eta) - \Delta p(0,0,\eta)u(0,0,\eta)|| &\leq K_{\Delta p}[||\bar{\xi}|| + ||S||] \\
||\Delta q(0,0,\eta) + \Delta p(0,0,\eta)u(0,0,\eta)|| &\leq K_{\Delta qp}||\eta||
\end{aligned}$$

Indeed, noting that a bound on the Jacobian of the transformation Φ gives $d\eta \leq K_\eta$, and using assumptions $(I4) - (I7)$, we observe that

$$\begin{aligned}
K_q &\leq K_\eta K_{f-conic} \\
K_p &\leq K_\eta K_{g-conic} \\
K_{\Delta q} &\leq K_\eta K_{\Delta f-conic} \\
K_{\Delta p} &\leq K_\eta K_{\Delta g-conic} \\
K_{qp} &\leq K_\eta K_{\Delta qp}
\end{aligned}$$

Now define

$$K_7 = \alpha_3 K_4[K_q + K_p + K_{\Delta q} + K_{\Delta p}] \tag{3.112}$$

and using the fact that

$$K_7||\eta||[||\bar{\xi}|| + ||S||] \leq \frac{||\eta||^2}{2} + K_7^2||\bar{\xi}||^2 + K_7^2||S||^2 \tag{3.113}$$

we rewrite equation (3.108) - (3.111) as

$$\begin{aligned}
\dot{V} \leq & -||\bar{\xi}||^2[\alpha_1 - \frac{1}{4} - K_7^2] && (3.114) \\
& -\alpha_2[K_s - K_b][||S|| - \frac{K_6^2 + K_7^2}{\alpha_2[K_s - K_b]}||S||^2 + ||S||^3] && (3.115) \\
& -||\eta||^2[\alpha_3[K_3 - K_{qp}] - \frac{1}{2}] && (3.116)
\end{aligned}$$

Note that the constants α_i have not been chosen yet, and we now choose them in such a manner as to ensure the negative definiteness of the Lyapunov function, provided $K_3 > K_{qp}$.

We choose then such that

$$\begin{aligned} \alpha_1 &> \frac{1}{4} + K_7^2 && (3.117) \\ \alpha_2 &> K_6^2 + K_7^2 && (3.118) \\ \alpha_3 &> \frac{1}{2[K_3 - K_{qp}]} && (3.119) \\ K_s - K_b &> 1 && (3.120) \end{aligned}$$

Such a choice of constants ensure that $\dot{V}$ is negative definite for all initial conditions in Ω, and for all perturbations such that $K_{qp} < K_3 \leq K^*$. (Note that the coefficient of $||S||^2$ is always less than unity, and for all $||S|| > 1$, $||S||^2 < ||S||^3$, and for all values of $||S|| < 1$, $||S|| > ||S||^2$.)

We have shown that there exists of an invariant set Ω and a constant K^* such that Ω is invariant, and for all $K_{qp} < K^*$, the output of the system is asymptotically regulated to the origin. The proof also yields that the states $[\bar{\xi}, S, \eta]^T$ are bounded, and therefore the states $x \in \Re^n$ are also bounded. ◁♠

Chapter 4

Generalized Matching Conditions For Perturbed MIMO Systems

We present systematically, matching conditions for the following classes of systems

- Non-singular MIMO systems perturbed zero dynamics.
- Singular MIMO systems, decoupled using the zero-dynamics algorithm.
- Singular MIMO systems, decoupled using the dynamics extension method.

The organization of the chapter is as follows. Section I presents the matching conditions for nonsingular MIMO systems. Section II presents the generalized matching conditions for singular MIMO systems, which are either left or right invertible. The zero-dynamics algorithm is used when the system does not possess a well defined relative degree, but is left invertible. The dynamic extension method is used when the system does not possess a well defined relative degree, but is right invertible. We then conclude this chapter with a brief comparison of the presented methods of system inversion.

4.1 Non-Singular MIMO Systems

Consider square MIMO systems specified by the following equations.

Unperturbed System Equations

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i(t) & (4.1) \\ y_i &= h_i(x) \;\; i = 1, \ldots, m & (4.2) \end{aligned}$$

where $x \in \Re^n$ $f(\cdot) : \Re^n \rightarrow \Re^n$ is a smooth vector field, $g_i(\cdot) : \Re^n \rightarrow \Re^n$, $i = 1, \ldots, m$ are smooth vector fields, $h_i(\cdot) : \Re^n \rightarrow \Re$, $i = 1, \ldots, m$ are smooth functions, $u_i(t) : \Re_+ \rightarrow \Re$, $i = 1, 2, \ldots, m$

Perturbed System Equations

$$\begin{aligned} \dot{x} &= f(x) + \Delta f(x) + \sum_{i=1}^{m} [g_i(x) + \Delta g_i(x)] u_i & (4.3) \\ y_i &= h_i(x) \;\; i = 1, \ldots, m & (4.4) \end{aligned}$$

where $\Delta f(\cdot) : \Re^n \rightarrow \Re^n$ is smooth, $\Delta g_i(\cdot) : \Re^n \rightarrow \Re^n$ $i = 1, \ldots, m$ are smooth,

Comment 4.1.1 As in the case of SISO systems, we assume that the outputs are not corrupted by measurement noise and that full state information is available.

Let x_0 be an equilibrium point of the undriven unperturbed system (4.1) - (4.2), that is $f(x_0) = 0$, such that the output is zero at x_0, i.e $h_i(x_0) = 0$. We will now assume that the system (4.1) - (4.2) has *strict vector relative degree* $\gamma = [\gamma_1, \ldots, \gamma_m]^T \in Z_+^m$ at x_0 [19] (that is, in an open subset U containing the point x_0, $L_{g_i} L_f^k h_j(x) \equiv 0$ $i = 1, \ldots, m$ $j = 1, \ldots, m$ $k = 0, \ldots, \gamma_j - 2$ and the determinant of the decoupling matrix is nonzero, that is,

$$det \begin{bmatrix} L_{g_1} L_f^{\gamma_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{\gamma_1 - 1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{\gamma_m - 1} h_m(x) & \cdots & L_{g_m} L_f^{\gamma_m - 1} h_m(x) \end{bmatrix} \neq 0 \qquad (4.5)$$

Statement Of The Problem

The MIMO Matching problem is formally stated as follows:

Given:

- An unperturbed system of the form (4.1) - (4.2) with vector relative degree $[\gamma_1, \ldots, \gamma_m]^T \in Z_+^m$.
- The general classes of perturbations of interest specifed by (4.3) - (4.4).
- Determine controls $\begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}^T$ such that the outputs $y_i = h_i(x) \to 0$ $i = 1, \ldots, m$ as $t \to \infty$, with the states $x \in \Re^n$ remaining bounded.

Determine:

- Conditions that must be matched by the perturbations $\Delta f : \Re^n \to \Re^n$ and $\Delta g_i : \Re^n \to \Re^n$ $i = 1, \ldots, m$ of (4.3) - (4.4) such that the control objective of asymptotic output regulation is achieved by the control law developed based on the unperturbed system equations (4.1) - (4.2), when applied to the perturbed system and the states $x \in \Re^n$ remain bounded.

We will use the following notation to describe nonlinear coordinate transformations of MIMO systems. As the MIMO system was assumed to have a well defined vector relative degree $\gamma = \begin{bmatrix} \gamma_1 & \cdots & \gamma_m \end{bmatrix}$ there exists a standard transformation Φ of the following form [19]

$$\Phi : x \in U \to \begin{bmatrix} \xi^1 \\ \vdots \\ \xi^m \\ \eta \end{bmatrix} \in \Phi(U) \tag{4.6}$$

where

$$\xi^i : \Re^n \to \Re^{\gamma_i} = \begin{bmatrix} \xi_1^i \\ \vdots \\ \xi_{\gamma_i}^i \end{bmatrix} \tag{4.7}$$

$$\eta : \Re^n \to \Re^{n-\gamma} = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-\gamma} \end{bmatrix} \tag{4.8}$$

$$\gamma = \sum_{i=1}^{m} \gamma_i \tag{4.9}$$

and each $\xi^i_j : \Re^n \rightarrow \Re = L_f^{j-1} h_i(x)$ $i = 1, \ldots, m$ $j = 1, 2, \ldots, \gamma_i$ is a smooth functions of x and each $\eta_i : \Re^n \rightarrow \Re$ $i = 1, 2, \ldots, n - \gamma$ is a smooth function of x such that η is independent of ξ^i $i = 1, 2, \ldots, m$, linear independence being defined over the ring of smooth functions. Note that Φ is a local transformation of coordinates, and is valid for an open region U containing the operating point x_0.

Using the transformation Φ specified by (4.6) a local normal form for the unperturbed system equations (4.1) - (4.2) is written as

$$\dot{\xi}^i_1 = \xi^i_2 \tag{4.10}$$

$$\vdots = \vdots \tag{4.11}$$

$$\dot{\xi}^i_{\gamma_i} = b^i(\xi^1, \ldots, \xi^m, \eta) + \sum_{j=1}^{m} a^i_j(\xi^1, \ldots, \xi^m, \eta) u_j \tag{4.12}$$

$$i = 1, \ldots, m \tag{4.13}$$

$$\dot{\eta} = q(\xi^1, \ldots, \xi^m, \eta) + \sum_{j=1}^{m} p_j(\xi^1, \ldots, \xi^m, \eta) u_j \tag{4.14}$$

where

$$\begin{aligned}
b^i(\xi^1, \ldots, \xi^m, \eta) : \Re^n \rightarrow \Re &= L_f^{\gamma_i} h_i \circ \Phi^{-1}(\xi, \eta) \\
a^i_j(\xi^1, \ldots, \xi^m, \eta) : \Re^n \rightarrow \Re &= L_{g_j} L_f^{\gamma_i - 1} h_i \circ \Phi^{-1}(\xi, \eta) \\
q(\xi^1, \ldots, \xi^m, \eta) : \Re^n \rightarrow \Re^{n-\gamma} &= L_f \eta \circ \Phi^{-1}(\xi, \eta) \\
p_j(\xi^1, \ldots, \xi^m, \eta) : \Re^n \rightarrow \Re^{n-\gamma} &= L_{g_j} \eta \circ \Phi^{-1}(\xi, \eta) \;\; j = 1, \ldots, m
\end{aligned}$$

Comment: The assumption of a well defined relative degree for the unperturbed system (4.1) - (4.2) guarantees the invertibility everywhere in $\Phi(U)$, of the matrix $A : \Re^n \rightarrow \Re^{m \times m}$ defined as

$$A = \begin{bmatrix} a^1_1(\xi^1, \ldots, \xi^m, \eta) & , \ldots, & a^1_m(\xi^1, \ldots, \xi^m, \eta) \\ \vdots & \ddots & \vdots \\ a^m_1(\xi^1, \ldots, \xi^m, \eta) & , \ldots, & a^m_m(\xi^1, \ldots, \xi^m, \eta) \end{bmatrix} \tag{4.15}$$

Comment: Note that the input enters the η dynamics in (4.97). Indeed, we did not assume the involutivity of the vector fields g_j $j = 1, \ldots, m$ in the system equations (4.1). Assuming involutivity of the input vector fields g_j $j = 1, \ldots, m$ and invoking the

Frobenius theorem would give us the choice of a set of η coordinates with derivatives that annihilate the g_j $j = 1, \ldots, m$ vector fields.

Using the same diffeomorphism Φ (4.6), the equations of the perturbed system transform to

$$
\begin{aligned}
\dot{\xi}_1^i &= \xi_2^i && (4.16)\\
\vdots &= \vdots && (4.17)\\
\dot{\xi}_{\gamma_i}^i &= b^i(\xi^1, \ldots, \xi^m, \eta) + \Delta b^i(\xi^1, \ldots, \xi^m, \eta) && (4.18)\\
&\quad + \sum_{j=1}^{m} [a_j^i(\xi^1, \ldots, \xi^m, \eta) + \Delta a_j^i(\xi^1, \ldots, \xi^m, \eta)] u_j && (4.19)\\
&\quad i = 1, \ldots, m && (4.20)\\
\dot{\eta} &= q(\xi^1, \ldots, \xi^m, \eta) + \Delta q(\xi^1, \ldots, \xi^m, \eta) && (4.21)\\
&\quad + \sum_{j=1}^{m} [p_j(\xi^1, \ldots, \xi^m, \eta) + \Delta p_j(\xi^1, \ldots, \xi^m, \eta)] u_j && (4.22)
\end{aligned}
$$

where

$$
\begin{aligned}
b^i(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re &= L_f^{\gamma_i} h_i \circ \Phi^{-1}(\xi, \eta)\\
\Delta b^i(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re &= L_{\Delta f} L_f^{\gamma_i - 1} h_i \circ \Phi^{-1}(\xi, \eta)\\
a_j^i(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re &= L_{g_j} L_f^{\gamma_i - 1} h_i \circ \Phi^{-1}(\xi, \eta)\\
\Delta a_j^i(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re &= L_{\Delta g_j} L_f^{\gamma_i - 1} h_i \circ \Phi^{-1}(\xi, \eta)\\
q(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re^{n - \sum_{j=1}^m \gamma_j} &= L_f \eta \circ \Phi^{-1}(\xi, \eta)\\
\Delta q(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re^{n - \sum_{j=1}^m \gamma_j} &= L_{\Delta f} \eta \circ \Phi^{-1}(\xi, \eta)\\
p_j(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re^{n - \sum_{j=1}^m \gamma_j} &= L_{g_j} \eta \circ \Phi^{-1}(\xi, \eta) \;\; j = 1, \ldots, m\\
\Delta p_j(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re^{n - \sum_{j=1}^m \gamma_j} &= L_{\Delta g_j} \eta \circ \Phi^{-1}(\xi, \eta) \;\; j = 1, \ldots, m
\end{aligned}
$$

The zero dynamics of the unperturbed MIMO system is a dynamical system in $\Re^{n-\gamma}$ given by

$$\dot{\eta} = q(0, \eta) + \sum_{j=1}^{m} p_j(0, \eta) u_j(0, \eta) \tag{4.23}$$

where the control $u(0,\eta)$ is specified to be

$$u(0,\eta) = A^{-1}[\begin{bmatrix} -b^1(0,\eta)+v_1(0,\eta) \\ \vdots \\ -b^m(0,\eta)+v_m(0,\eta) \end{bmatrix} \tag{4.24}$$

The controls $v_i(0,\eta)$ $i=1,2,\ldots,m$ are stabilizing controls for each subsystem.

Observe the similarity between (4.23) and (3.78)

[27] make the observation that the stability of this zero-dynamics is to be maintained in the presence of perturbations. This indeed is a point of departure from the considering SISO systems with unperturbed η dynamics.

We now present the first extension of matching to the MIMO case - the instance when the input vector fields $g_j(x) : \Re^n \rightarrow \Re^n$ $j=1,2,\ldots,m$ are involutive.

Theorem 4.1 *The Generalized matching conditions for MIMO nonsingular systems with perturbed zero dynamics are:*

Given *(G1) A perturbed MIMO system of the form (4.3) - (4.4).*

(G2) The unperturbed MIMO system (4.1) - (4.2) has a vector relative degree $\gamma = [\gamma_1, \ldots, \gamma_m]^T$.

(G3) A control objective of asymptotic output regulation, that is $y_i = h_i(x) \rightarrow 0$ $i=1,\ldots,m$

If *(I1) The zero dynamics of the unperturbed system (4.23) is exponentially stable.*

(I2)

$$\Delta f(x) \in \bigcap_{i=1}^{m} ker[dh_i(x), dL_f h_i(x) \cdots dL_f^{\gamma_i-2} h_i(x)]$$

(I3)

$$\Delta g_j(x) \in \bigcap_{i=1}^{m} ker[dh_i(x), dL_f h_i(x) \cdots dL_f^{\gamma_i-1} h_i(x)] \; j=1,\ldots,m$$

(I4) $\Delta f(\xi^1,\ldots,\xi^m,\eta)$, $\sum_{i=1}^{m} \Delta g_i(\xi^1,\ldots,\xi^m,\eta) u_i(\xi^1,\ldots,\xi^m,\eta)$ *satisfy conic continuity in* $[\xi^1,\ldots,\xi^m]^T$*, uniformly in* η *with constants* $K_{\Delta f-conic}$, $K_{\Delta g-conic}$, *everywhere in an open set* $\Omega_{\Delta Conic} \subseteq \Phi(U)$

(I5) $L_{\Delta f} L_f^{\gamma_i - 1} h_i \leq K_{bi} < K_{s^i}$ *in* $\Omega^i_{\Delta Bound} \subseteq \Phi(U)$

(I6) $f(\xi^1, \ldots, \xi^m, \eta)$, $\sum_{i=1}^m g_i(\xi^1, \ldots, \xi^m, \eta) u_i(\xi^1, \ldots, \xi^m, \eta)$ *satisfy conic continuity in* $[\xi^1, \ldots, \xi^m]^T$, *uniformly in* η, *with constants* $K_{f-conic}$, $K_{g-conic}$ *everywhere in an open set* $\Omega_{conic} \subseteq \Phi(U)$

(I7) $||\Delta f(0,\eta) + \sum_{i=1}^m \Delta g_i(0,\eta) u_i(0,\eta)|| \leq K_{\Delta qp} ||\eta||$ *everywhere in an open set* $\Omega_{\Delta[f(0,\eta)+g(0,\eta)u(0,\eta)]} \subseteq \Phi(U)$

(I8) The diffeomorphism Φ *(4.6) has a bounded Jacobian.*

(I9) The control u *is chosen to be*

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \bar{A}^{-1} \begin{bmatrix} -b^1(\xi^1, \ldots, \xi^m, \eta) + v_1 \\ \vdots \\ -b^m(\xi^1, \ldots, \xi^m, \eta) + v_m \end{bmatrix} \tag{4.25}$$

where, $\bar{A}$ *is specified by (4.15) and* $b^i(\xi^1, \ldots, \xi^m, \eta)$ $i = 1, \ldots, m$ *are specified in (4.12), and*

$$\begin{aligned} v_i &= -a^i_{\gamma_i - 1}\xi^i_{\gamma_i} +, \ldots, +a^i_1 \xi^i_2 - K_{s^i} sgn(S^i) && (4.26) \\ S^i &= \xi^i_{\gamma_i} + a^i_{\gamma_i - 1}\xi^i_{\gamma_i - 1} +, \ldots, +a^i_1 \xi^i_1 && (4.27) \\ & \quad \textit{and is a Hurwitz polynomial} && (4.28) \\ sgn(S^i) &= \frac{S^i}{|S^i|} \; \forall \, |S^i| > 0 && (4.29) \end{aligned}$$

Then *(T1) There exist a set* $\Omega \subseteq \Phi(U)$ *and a constant* K^* *such that for all initial conditions belonging to* Ω, *and* $K_{\Delta qp} < K^*$ *the trajectories of the perturbed system remain in* $\Phi(U)$ *and the output* $y_i = h_i(x)$ *tends to zero asymptotically.*

Proof:♠ ▷

The proof of the theorem, follows along the lines of the proof of theorem (3.2)

Step 1:

We will first assume that system trajectories remain in the set $\bigcap_{i=1}^m \Omega^i_{\Delta Bound} \subseteq \Phi(U)$ where the bounds on the perturbations are satisfied. Under this assumption we will show that asymptotic regulation is indeed achieved. To complete the proof, we will show that for appropriately chosen initial conditions belonging to an invariant set $\Omega \subseteq \bigcap_{i=1}^{m} \Omega^{i}_{\Delta Bound} \subseteq \Phi(U)$ asymptotic regulation is achieved. Indeed, we will then attempt to find the largest invariant set Ω.

Consider the normal form of the perturbed system given by the equations (4.16) - (4.22). Using the assumption *(I2)* and *(I3)* , the normal form equations (4.16) - (4.22) reduce to

$$
\begin{aligned}
\dot{\xi}^i_1 &= \xi^i_2 && (4.30)\\
\vdots &= \vdots && (4.31)\\
\dot{\xi}^i_{\gamma_i} &= b^i(\xi^1,\ldots,\xi^m,\eta) + \Delta b^i(\xi^1,\ldots,\xi^m,\eta) + \sum_{j=1}^{m} a^i_j(\xi^1,\ldots,\xi^m,\eta)u_j && (4.32)\\
& \quad i = 1,\ldots,m && (4.33)\\
\dot{\eta} &= q(\xi^1,\ldots,\xi^m,\eta) + \Delta q(\xi^1,\ldots,\xi^m,\eta) && (4.34)\\
& \quad + \sum_{j=1}^{m} [p_j(\xi^1,\ldots,\xi^m,\eta) + \Delta p_j(\xi^1,\ldots,\xi^m,\eta)]u_j && (4.35)
\end{aligned}
$$

Note the absence of an input perturbation in the ξ^i dynamics in (4.32).
Using the definition of S^i stated in theorem, we split the coordinates $[\xi^1,\ldots,\xi^m,\eta]^T$ as $[(\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta]^T$ where $\bar{\xi}^i = [\xi^i_1,\ldots,\xi^i_{\gamma_i-1}]^T$. Now rewrite the system equations (4.30) - (4.32) in the $[[(\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta]^T$ coordinates using the control u specified in the theorem in chapter 2.

$$
\begin{aligned}
\dot{\bar{\xi}}^i &= A^i\bar{\xi}^i + b^iS^i && (4.36)\\
\dot{S}^i &= -K^i_s sgn(S^i) + \Delta b^i((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) && (4.37)\\
& \quad i = 1,\ldots,m && (4.38)\\
\dot{\eta} &= q((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) + \Delta q((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) && (4.39)\\
& \quad + p((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)u((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) && (4.40)\\
& \quad + \Delta p((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)u((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) && (4.41)
\end{aligned}
$$

where

$$A^i = \begin{bmatrix} 0 & 1 & ,\ldots, & 0 & 0 \\ 0 & 0 & 1 & ,\ldots, & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1^i & -a_2^i & ,\ldots, & ,\ldots, & -a_{\gamma_i-1}^i \end{bmatrix}_{[\gamma_i-1]\times[\gamma_i-1]} \tag{4.42}$$

$$b^i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{[\gamma_i-1]\times[1]} \tag{4.43}$$

Consider the Lyapunov function

$$V = \sum_{i=1}^{m} \frac{[S^i]^2}{2} \tag{4.44}$$

Differentiating V along the flow of (4.37) we obtain,

$$\dot{V} = \sum_{i=1}^{m} S^i[-K_{s^i}sgn(S^i) + \Delta b^i((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta))] \tag{4.45}$$

Assume that the system trajectories remain in the set $\bigcap_{i=1}^m \Omega^i_{\Delta Bound} \subseteq \Phi(U)$. Using the bounds on $b^i((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta))$, we obtain

$$\dot{V} \leq -|S^i|[K_{s^i} - K_{b^i}sgn(S^i)] \tag{4.46}$$

Negative definiteness of $\dot{V}$ implies that each $S^i = 0$ is attractive for all trajectories that remain in $\bigcap_{i=1}^m \Omega^i_{\Delta Bound} \subseteq \Phi(U)$. Indeed, if $\bigcap_{i=1}^m \Omega^i_{\Delta Bound}$ is invariant, the trajectories reach the manifold $\bigcap_{i=1}^m S^i = 0$ in *finite time*. The choice of control renders the manifold $\bigcap_{i=1}^m S^i = 0$ invariant, and the dynamics on the manifold is such that $||\xi^i||$ $i = 1, \ldots, m$ tends to zero exponentially. (This is evident from setting $S^i = 0$ in the $\bar{\xi}^i$ dynamics and noting that A^i is a Hurwitz matrix.)

However, we need to ensure that the trajectories never leave the set defined as $\bigcap_{i=1}^m \Omega^i_{\Delta Bound}$, thus validating the boundedness of the perturbations. Indeed, we will now attempt to find the largest invariant set $\Omega \subseteq \bigcap_{i=1}^m \Omega^i_{\Delta Bound}$ that would also maintain stability of the internal dynamics. To this end we consider a Lyapunov function that includes both the ξ^i and the η dynamics.

Step 2:

Note that as A^i is a Hurwitz matrix there exists a positive definite symmetric matrix P^i solving the matrix Lyapunov equation [25]

$$[A^i]^T P^i + P^i A^i = -I \tag{4.47}$$

Using a converse Lyapunov theorem [16], we assert that the exponential stability of the zero dynamics (4.23) guarantees the existence of a Lyapunov function V_η such that,

$$\begin{aligned} K_1\|\eta\|^2 &\leq V_\eta \leq K_2\|\eta\|^2 && (4.48) \\ \frac{\partial V_\eta}{\partial \eta}[q(0,\eta) + p(0,\eta)u(0,\eta)] &\leq -K_3\|\eta\|^2 && (4.49) \\ \|\frac{\partial V_\eta}{\partial \eta}\| &\leq K_4\|\eta\| && (4.50) \end{aligned}$$

Now consider the composite Lyapunov function given by

$$V = \sum_{i=1}^{m} \alpha_1^i [\bar{\xi}^i]^T P [\bar{\xi}^i] + \sum_{i=1}^{m} \alpha_2^i [\frac{[S^i]^2}{2} + \frac{[S^i]^4}{4}] + \alpha_3 V_\eta \tag{4.51}$$

where P^i is the solution of (4.47) and V_η satisfies (4.48) - (4.50).

Differentiating V (4.51) along the flow of (4.36) - (4.41) we obtain

$$\begin{aligned} \dot{V} = & \sum_{i=1}^{m} [\alpha_1^i [[A^i \bar{\xi}^i + b^i S^i]^T P^i \bar{\xi}^i + [\bar{\xi}^i]^T P^i [A^i \bar{\xi}^i + b^i S^i]]] \\ & + \sum_{i=1}^{m} [\alpha_2^i [S^i [-K_{s^i} sgn(S^i) + \Delta b^i((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta)]]] \\ & + \sum_{i=1}^{m} [\alpha_2^i [[S^i]^3 [-K_{s^i} sgn(S^i) + \Delta b^i((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta)]]] \\ & \alpha_3 \frac{\partial V_\eta}{\partial \eta} [q((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) + \Delta q((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta)] \\ & + \alpha_3 \frac{\partial V_\eta}{\partial \eta} [p((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) u((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta)] \\ & + \alpha_3 \frac{\partial V_\eta}{\partial \eta} [\Delta p((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) u((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta)] \end{aligned}$$

Assuming that $[\xi^1,\ldots,\xi^m,\eta]^T \in \bigcap_{i=1}^m \Omega^i_{\Delta Bound}$, we obtain

$$\begin{aligned}
\dot{V} \;\leq\; & -\sum_{i=1}^{m}[||\bar{\xi}^i||^2[\alpha_1^i - \frac{1}{4}]] \\
& -\sum_{i=1}^{m}[\alpha_2^i[K_{s^i} - K_{b^i}][||S^i|| - \frac{[K_6^i]^2}{\alpha_2^i[K_{s^i} - K_{b^i}]}||S^i||^2 + ||S^i||^3] \\
& +\alpha_3\frac{\partial V_\eta}{\partial\eta}[q(0,\eta) + p(0,\eta)u(0,\eta)] \\
& +\alpha_3\frac{\partial V_\eta}{\partial\eta}[q((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) - q(0,\eta)] \\
& +\alpha_3\frac{\partial V_\eta}{\partial\eta}[p((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)u((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) \\
& -p(0,\eta)u(0,\eta)] \\
& +\alpha_3\frac{\partial V}{\partial\eta}[\Delta q((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) - \Delta q(0,\eta)] \\
& +\alpha_3\frac{\partial V}{\partial\eta}[\Delta p((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)u((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) \\
& -\Delta p(0,\eta)u(0,\eta)] \\
& +\alpha_3\frac{\partial V}{\partial\eta}[\Delta q(0,\eta) + \Delta p(0,\eta)u(0,\eta)]
\end{aligned}$$

where

$$S^i[b^i]^T P^i\bar{\xi}^i \;\leq\; K_6^i||S^i||||\bar{\xi}^i|| \tag{4.52}$$

$$K_6^i \;=\; \sigma_{max}(P^i)||b^i|| \tag{4.53}$$

$$||b^i|| \;=\; 1 \tag{4.54}$$

$$K_6^i||S^i||||\bar{\xi}^i|| \;\leq\; \frac{||\bar{\xi}^i||^2}{4} + [K_6^i]^2[S^i]^2 \tag{4.55}$$

and $\sigma_{max}(P^i)$ is the maximum singular value of P^i.

Now define Ω_V to be the largest non-empty subset of $\Phi(U)$ such that

$$\Omega_V \subseteq \bigcap_{i=1}^{m}[\Omega^i_{\Delta Bound}\bigcap\Omega_{\Delta Conic}\bigcap\Omega_{Conic}\bigcap\Omega_{\Delta[f(0,\eta)+g(0,\eta)u(0,\eta)]}] \tag{4.56}$$

Choose $c^* \in \Re_+$ such that

$$c^* = sup(c : V \leq c \;\in \Omega_V) \tag{4.57}$$

Define $\Omega = [((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) : V \le c^*]$.

For all initial conditions in Ω, we may rewrite $\dot{V}$ as

$$\begin{aligned}
\dot{V} \le\ & -\sum_{i=1}^{m} [||\bar{\xi}^i||^2[\alpha_1^i - \frac{1}{4}]] && (4.58)\\
& -\sum_{i=1}^{m} [\alpha_2^i [K_{s^i} - K_{b^i}][||S^i|| - \frac{[K_6^i]^2}{\alpha_2^i [K_{s^i} - K_{b^i}]}||S^i||^2 + ||S^i||^3] && (4.59)\\
& -\alpha_3 K_3 ||\eta||^2 && (4.60)\\
& +\alpha_3 K_4 ||\eta|| [K_q + K_p + K_{\Delta q} + K_{\Delta p}] \sum_{i=1}^{m} [||\bar{\xi}^i|| + ||S^i||] && (4.61)\\
& +\alpha_3 K_4 K_{qp} ||\eta||^2 && (4.62)
\end{aligned}$$

where,

$$||q((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) - q(0, \eta)|| \le K_q \sum_{i=1}^{m} [||\bar{\xi}^i|| + ||S^i||] \tag{4.63}$$

$$||p((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) u((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) - p(0, \eta) u(0, \eta)|| \le K_p \sum_{i=1}^{m} [||\bar{\xi}^i|| + ||S^i||]$$

$$||\Delta q((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) - \Delta q(0, \eta)|| \le K_{\Delta q} \sum_{i=1}^{m} [||\bar{\xi}^i|| + ||S^i||] \tag{4.64}$$

$$||\Delta p((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) u((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) - \Delta p(0, \eta) u(0, \eta)|| \le K_{\Delta p} \sum_{i=1}^{m} [||\bar{\xi}^i|| + ||S^i||]$$

$$||\Delta q(0, \eta) + \Delta p(0, \eta) u(0, \eta)|| \le K_{\Delta qp} ||\eta|| \tag{4.65}$$

Indeed, noting that a bound on the Jacobian of the transformation Φ gives $\frac{\partial \eta}{\partial x} \le K_\eta$, and using assumptions $(I4) - (I7)$, we observe that

$$\begin{aligned}
K_q &\le K_\eta K_{f-conic} && (4.66)\\
K_p &\le K_\eta K_{g-conic} && (4.67)\\
K_{\Delta q} &\le K_\eta K_{\Delta f-conic} && (4.68)\\
K_{\Delta p} &\le K_\eta K_{\Delta g-conic} && (4.69)\\
K_{qp} &\le K_\eta K_{\Delta qp} && (4.70)
\end{aligned}$$

Now define

$$K_7 = \alpha_3 K_4[K_q + K_p + K_{\Delta q} + K_{\Delta p}] \tag{4.71}$$

and using the fact that

$$K_7||\eta||\sum_{i=1}^{m}[||\bar{\xi}^i|| + ||S^i||] \leq \frac{m||\eta||^2}{4} + K_7^2\sum_{i=1}^{m}||\bar{\xi}^i||^2 + K_7^2\sum_{i=1}^{m}||S^i||^2 \tag{4.72}$$

we rewrite equation (4.58) - (4.163) as

$$\dot{V} \leq -\sum_{i=1}^{m}[||\bar{\xi}||^2[\alpha_1^i - \frac{1}{4} - K_7^2]] \tag{4.73}$$

$$-\sum_{i=1}^{m}[\alpha_2^i[K_{s^i} - K_{b^i}][||S^i|| - \frac{[K_6^i]^2 + K_7^2}{\alpha_2^i[K_{s^i} - K_{b^i}]}||S^i||^2 + ||S^i||^3] \tag{4.74}$$

$$-||\eta||^2[\alpha_3[K_3 - K_{qp}] + \frac{m}{4}] \tag{4.75}$$

Note that the constants α_j^i $i = 1, \ldots, m$ $j = 1, \ldots, 3$ have not been chosen yet, and we now choose them in such a manner as to ensure the negative definiteness of the Lyapunov function, provided $K_3 > K_{qp}$.

We choose them to be

$$\alpha_1^i > \frac{1}{4} + K_7^2 \tag{4.76}$$

$$\alpha_2^i > [K_6^i]^2 + K_7^2 \tag{4.77}$$

$$\alpha_3 > \frac{m}{4[K_3 - K_{qp}]} \tag{4.78}$$

$$K_s - K_b > 1 \tag{4.79}$$

Such a choice of constants ensure that $\dot{V}$ is negative definite for all initial conditions in Ω, and for all perturbations such that $K_{qp} < K_3 \leq K^*$.(Note that the coefficient of $||S^i||^2$ is always less than unity, and for all $||S^i|| > 1$, $||S^i||^2 < ||S^i||^3$, and for all values of $||S^i|| < 1$, $||S^i|| > ||S^i||^2$.)

We have shown that there exists a set Ω and a constant K^* such that for all initial conditions in Ω, and for all $K_{qp} < K^*$, the output of the system is asymptotically regulated to the origin.$\lhd$♠

4.2 Singular MIMO Systems

In this section we will propose matching conditions to ensure asymptotic regulation of the output for systems which do not possess a relative degree. However, assuming either left or right invertibility of such systems provide a means of output regulation using the notion of an extended relative degree. The zero-dynamics algorithm provides a solution under the assumption of left invertibility and the dynamic extension algorithm suggests a decoupling methodology assuming right invertibility. It is useful to prescribe matching conditions to be met by perturbations in either of these two schemes.

Consider square MIMO systems specified by the following equations, as before.

Unperturbed System Equations

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i &(4.80)\\ y_i &= h_1^i(x) \;\; i = 1, \ldots, m &(4.81) \end{aligned}$$

where $x \in \Re^n$ $f(x) : \Re^n \to \Re^n$ is a smooth vector field $g_i(x) : \Re^n \to \Re^n$ $i = 1, 2, \ldots, m$ are smooth vector fields, $h_1^i(x) : \Re^n \to \Re$ $i = 1, 2, \ldots, m$ are smooth functions, $u_i \in \Re$ $i = 1, 2, \ldots, m$

Perturbed System Equations

$$\begin{aligned} \dot{x} &= f(x) + \Delta f(x) + \sum_{i=1}^{m} [g_i(x) + \Delta g_i(x)] u_i &(4.82)\\ y_i &= h_1^i(x) \;\; i = 1, \ldots, m &(4.83) \end{aligned}$$

where $\Delta f(x) : \Re^n \to \Re^n$ is a smooth vector field, $\Delta g_i(x) : \Re^n \to \Re^n$ $i = 1, 2, \ldots, m$ are smooth vector fields.

Comment 4.2.1 A minor change in notation from the previous sections is that the lower indices on the function h_i have been converted to upper indices h_1^i. The lower *index* 1 *indicates that the output has not been been redefined yet. This will be needed later for output redefinition.*

Matching Conditions Assuming Left Invertibility

We present matching conditions for a MIMO system which does not possess a well defined relative degree, but which is left invertible. The statement of the MIMO singular matching problem is as follows.

Statement Of The Problem

Given:

- An unperturbed system of the form (4.80) - (4.81) which does not possess a vector relative degree.
- The unperturbed system (4.80) - (4.81) is left invertible.
- The operating point x_0 is a regular point of the zero dynamics algorithm in the sense of [19]
- A control objective - asymptotic output regulation, that is, $y_i = h_1^i(x) \rightarrow 0$ $i = 1, \ldots, m$ as $t \rightarrow \infty$.
- The zero-dynamics algorithm is utilized to generate a control law to attain the control objective.

Determine:

- Conditions that must be matched by the perturbations $\Delta f(x) : \Re^n \rightarrow \Re^n$ and $\Delta g_i(x) : \Re^n \rightarrow \Re^n$ $i = 1, \ldots, m$ of (4.82) - (4.83) such that the control objective of asymptotic output regulation is achieved by the control law developed based on the unperturbed system equations (4.80) - (4.81), when applied to the perturbed system keeping the states $x \in \Re^n$ bounded.

Assuming that the system is left invertible, and that x_0 is a regular point [9], [6], [8] of the zero dynamics algorithm of [19], for the system (4.80) - (4.81), there exists a transformation Φ exhibiting the system in a local normal form. The transformation Φ is specified by,

$$\Phi : x \rightarrow \begin{bmatrix} \xi^1 \\ \vdots \\ \xi^m \\ \eta \end{bmatrix} \tag{4.84}$$

where

$$\xi^i : \Re^n \rightarrow \Re^{r_i} = \begin{bmatrix} \xi_1^i \\ \vdots \\ \xi_{r_i}^i \end{bmatrix} \tag{4.85}$$

$$\eta : \Re^n \rightarrow \Re^{n-r} = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-r} \end{bmatrix} \tag{4.86}$$

$$r = \sum_{i=1}^{m} r_i \tag{4.87}$$

$$(r_1, \ldots, r_m)^T \in Z_+^m \text{ is a vector of extended relative degrees} \tag{4.88}$$

and each $\xi_j^i : \Re^n \rightarrow \Re$ $i = 1, 2, \ldots, m$ $j = 1, \ldots, r_i$ is a smooth function of x and each $\eta_i : \Re^n \rightarrow \Re$ $i = 1, 2, \ldots, n-r$ is a smooth function of x such that η's and ξ's possess linearly independent differentials, linear independence being defined over the ring of smooth functions. Note that Φ is a local transformation of coordinates, and is valid for an open region U containing the operating point x_0.

Using the transformation Φ specified by (4.84) a local normal form for the unperturbed system equations (4.80) - (4.81), smiliar to the one in [19], is written as:

$$\dot{\xi}_1^1 = \xi_2^1 \tag{4.89}$$

$$\vdots \quad \vdots \quad \vdots \tag{4.90}$$

$$\dot{\xi}_{r_1}^1 = b^1(\xi^1, \ldots, \xi^m, \eta) + \sum_{k=1}^{m} a_k^1(\xi^1, \ldots, \xi^m, \eta) u_k(\xi^1, \ldots, \xi^m, \eta) \tag{4.91}$$

$$\dot{\xi}_1^i = \xi_2^i + \sum_{j=1}^{i-1} \delta_{1j}^i(\xi^1, \ldots, \xi^m, \eta)[b^j(\xi^1, \ldots, \xi^m, \eta) \tag{4.92}$$

$$+ \sum_{k=1}^{m} a_k^j(\xi^1, \ldots, \xi^m, \eta) u_k(\xi^1, \ldots, \xi^m, \eta)] \tag{4.93}$$

$$\vdots = \vdots \tag{4.94}$$

$$\dot{\xi}^i_{r_i} = b^i(\xi^1,\ldots,\xi^m,\eta) + \sum_{k=1}^{m} a^i_k(\xi^1,\ldots,\xi^m,\eta)u_k(\xi^1,\ldots,\xi^m,\eta) \tag{4.95}$$

$$i = 2,\ldots,m \tag{4.96}$$

$$\dot{\eta} = q(\xi^1,\ldots,\xi^m,\eta) + \sum_{k=1}^{m} p_k(\xi^1,\ldots,\xi^m,\eta)u_k(\xi^1,\ldots,\xi^m,\eta) \tag{4.97}$$

where

$$h^i_{j+1} : \Re^n \to \Re = L_f h^i_j - \sum_{k=1}^{i-1}[\delta^i_{jk}(\xi^1,\ldots,\xi^m,\eta)b^k(\xi^1,\ldots,\xi^m,\eta)] \tag{4.98}$$

$$2 \le i \le m \quad 1 \le j \le r_i - 1 \tag{4.99}$$

$$L_{g_p}h^i_j : \Re^n \to \Re = \sum_{k=1}^{i-1} \delta^i_{jk}(\xi^1,\ldots,\xi^m,\eta)a^k_p(\xi^1,\ldots,\xi^m,\eta) \tag{4.100}$$

$$2 \le i \le m \quad 1 \le j \le r_i - 1 \quad 1 \le p \le m \tag{4.101}$$

$$\delta^i_{jk} : \Re^n \to \Re \qquad i = 2,\ldots,m \; j = 1,\ldots,r_i - 1 \; k = 1,\ldots,i-1 \tag{4.102}$$

$$a^1_k : \Re^n \to \Re = L_{g_k}L_f^{r_1-1}h^1_1 \quad k = 1,\ldots,m \tag{4.103}$$

$$a^i_k : \Re^n \to \Re = L_{g_k}h^i_{r_i} \quad 2 \le i \le m \tag{4.104}$$

$$b^1 : \Re^n \to \Re = L_f^{r_1}h^1_1 \tag{4.105}$$

$$b^i : \Re^n \to \Re = L_f h^i_{r_i} \quad 2 \le i \le m \tag{4.106}$$

$$\xi^1_j : \Re^n \to \Re = L_f^{j-1}h^1_1 \; j = 1,\ldots,r_1 \tag{4.107}$$

$$\xi^i_{j+1} : \Re^n \to \Re = h^i_{j+1} \quad 2 \le i \le m \quad 1 \le j \le r_i - 1 \tag{4.108}$$

The zero dynamics of the unperturbed MIMO system is a dynamical system in $\Re^{n-r}$ given by

$$\dot{\eta} = q(0,\eta) + \sum_{j=1}^{m} p_j(0,\eta)u_j(0,\eta) \tag{4.109}$$

where $u_j(0,\eta)$ is the stabilizing control input that renders the output identically zero.

Comment 4.2.2 The first output has a relative degree r_1 in the usual sense of being the number of times it is differentiated before input terms appear. On the other hand, the numbers r_i $i = 2,\ldots,m$ are termed extended relative degrees, as they indicate appearence of input terms after successive redefinition and differentiation.

Comment 4.2.3 The redefined outputs are specified by (4.99). Note that each redefined output is differentiated only once.

Comment 4.2.4 The functions $\delta^i_{qp}(x) : \Re^n \to \Re$ *are chosen based on the linear dependence of rows of the decoupling matrix at each intermediate stage of the algorithm. This is evident from equation (4.101).*

Comment 4.2.5 It is to be noted that the zero-dynamics algorithm provides enough structural information to be able to prescribe a control law to ensure asymptotic output regulation. The actual control law uses steps of the zero-dynamics algorithm, and provability of achievement of control objective imposes some restrictions on the permissible classes of $\delta^i_{jk}(x) : \Re^n \to \Re$ *considered above. Such restrictions limit the scope of the application of the zero-dynamics algorithm.*

Comment 4.2.6 Our assumptions on the functions δ^i_{jk} *are less restrictive that the conditions imposed in [12]. In [12], the functions* δ^i_{jk} *are required to be constants.*

Define $(m-1) \times r_i$ matrices $D_{ik} : \Re^n \to \Re^{i \times m}$ $i = 2, \ldots, m$ $k = 1, \ldots, r_i - 1$ and $(m-1) \times r_i$ vectors $d_{ik} : \Re^n \to \Re^i$ $i = 2, \ldots, m$ $k = 1, \ldots, r_i - 1$ as follows:

$$D_{ik} = \begin{bmatrix} a^1_1(\xi^1, \ldots, \xi^m, \eta) & , \ldots, & a^1_m(\xi^1, \ldots, \xi^m, \eta) \\ \vdots & \vdots & \vdots \\ a^{i-1}_1(\xi^1, \ldots, \xi^m, \eta) & , \ldots, & a^{i-1}_m(\xi^1, \ldots, \xi^m, \eta) \\ L_{g_1} h^i_k & , \ldots, & L_{g_m} h^i_k \end{bmatrix} \tag{4.110}$$

$$d_{ik}[\Delta f] = \begin{bmatrix} L_{\Delta f} L_f^{r_1 - 1} h^1_1 \\ L_{\Delta f} h^2_{r_2} \\ \vdots \\ L_{\Delta f} h^{i-1}_{r_i} \\ L_{\Delta f} h^i_k \end{bmatrix} \tag{4.111}$$

Comment 4.2.7 Note that the matrix D_{ik} *defined in equation (4.110) is such that its last row is always linearly dependent on the other rows. This follows from the construction of the redefined outputs outlined earlier, specifically from equation (4.101).*

Also define the decoupling matrix $\bar{A}_1$ as

$$\bar{A}_1 = \begin{bmatrix} a_1^1(\xi^1,\ldots,\xi^m,\eta) & ,\ldots, & a_m^1(\xi^1,\ldots,\xi^m,\eta) \\ \vdots & \ddots & \vdots \\ a_1^m(\xi^1,\ldots,\xi^m,\eta) & ,\ldots, & a_m^m(\xi^1,\ldots,\xi^m,\eta) \end{bmatrix} \tag{4.112}$$

We now present two classes of matching conditions for asymptotic output regulation for left invertible MIMO systems. The first set of conditions is more restrictive, but ensures the asymptotic regulation of the output. The second set of conditions is less restrictive, but only ensures regulation of the output to an arbitrary ϵ ball about the origin.

Theorem 4.2 *Strict Generalized Matching conditions for asymptotic regulation for left invertible MIMO systems decoupled using the zero dynamics algorithm.*

Given *(G1) A perturbed MIMO system of the form (4.82) - (4.83).*

(G2) The unperturbed MIMO system (4.80) - (4.81) has an extended vector relative degree $[r_1,\ldots,r_m]^T$.

(G3) A control objective of asymptotic output regulation, that is $|\xi_1^i| \to 0$ $i = 1,2,\ldots,m$

If *(I1) The zero dynamics of the unperturbed system, (4.109) is exponentially stable.*

(I2)

$$\Delta f(x) \in ker[dh_1^1(x), dL_f h_1^1(x) \cdots dL_f^{r_1-1} h_1^1(x)] \bigcap [\bigcap_{i=2}^{m} ker[dh_1^i,,\ldots,,dh_{r_i}^i]] \tag{4.113}$$

(I3)

$$\Delta g_j(x) \in ker[dh_1^1(x), dL_f h_1^1(x) \cdots dL_f^{r_1-1} h_1^1(x)] \bigcap [\bigcap_{i=2}^{m} ker[dh_1^i,,\ldots,,dh_{r_i}^i]] \tag{4.114}$$

where $j = 1,\ldots,m$

(I4) $\Delta f(\xi^1,\ldots,\xi^m,\eta)$, $\sum_{i=1}^m \Delta g_i(\xi^1,\ldots,\xi^m,\eta)u_i(\xi^1,\ldots,\xi^m,\eta)$ *satisfy conic continuity in* $[\xi^1,\ldots,\xi^m]^T$*, uniformly in* η *with constants* $K_{\Delta f-conic}$, $K_{\Delta g-conic}$, *everywhere in an open set* $\Omega_{\Delta Conic} \subseteq \Phi(U)$

(I5) $L_{\Delta f} L_f^{\gamma_i - 1} h_i \le K_{bi} < K_{s^i}$ *in* $\Omega^i_{\Delta Bound} \subseteq \Phi(U)$

(I6) $f(\xi^1,\ldots,\xi^m,\eta)$, $\sum_{i=1}^m g_i(\xi^1,\ldots,\xi^m,\eta)u_i(\xi^1,\ldots,\xi^m,\eta)$ *satisfy conic continuity in* $[\xi^1,\ldots,\xi^m]^T$*, uniformly in* η*, with constants* $K_{f-conic}$, $K_{g-conic}$ *everywhere in an open set* $\Omega_{conic} \subseteq \Phi(U)$

(I7) $||\Delta f(0,\eta) + \sum_{i=1}^m \Delta g_i(0,\eta)u_i(0,\eta)|| \le K_{\Delta qp}||\eta||$ *everywhere in an open set* $\Omega_{\Delta[f(0,\eta)+g(0,\eta)u(0,\eta)]} \subseteq \Phi(U)$

(I8) The functions $\delta^i_{pk}(\xi^1,\ldots,\xi^m,\eta) \le K_{\delta^i_{pk}}$ $i = 1,\ldots,m$ $p = 1,\ldots,r_i - 1$ $k = 1,\ldots i-1$ *everywhere in an open set* $\Omega_\delta \subseteq \Phi(U)$

(I9) The diffeomorphism Φ *(4.84) has a bounded Jacobian.*

(I10) The control u *is chosen to be*

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \bar{A}_1^{-1} \begin{bmatrix} -b^1(\xi^1,\ldots,\xi^m,\eta) + v_1 \\ \vdots \\ -b^m(\xi^1,\ldots,\xi^m,\eta) + v_m \end{bmatrix} \qquad (4.115)$$

where, $\bar{A}_1$ *is specified by (4.112) and* $b^i(\xi^1,\ldots,\xi^m,\eta)$ $i = 1,\ldots,m$ *are specified in (4.99) - (4.108), and*

$$v_i = -a^i_{r_i-1}\xi^i_{r_i},\ldots,+a^i_1\xi^i_2 - K_{s_i} sgn(S^i) \qquad (4.116)$$

$$S^i = \xi^i_{r_i} + a^i_{r_i-1}\xi^i_{r_i-1},\ldots,+a^i_1\xi^i_1 \qquad (4.117)$$

$$\text{and is a Hurwitz polynomial} \qquad (4.118)$$

$$sgn(S^i) = \frac{S^i}{|S^i|}\ \forall\, |S^i| > 0 \qquad (4.119)$$

Then *(T1) There exist a set* $\Omega \subseteq \Phi(U)$ *and a constant* K^* *such that for all initial conditions belonging to* Ω*, and* $K_{\Delta qp} < K^*$ *the output* $y_i = h_i(x)$ *tends to zero asymptotically.*

Proof: ♠ ▷

Assuming matching conditions *(I2)* and *(I3)* allows us to write down a local normal form for the perturbed system (4.82) - (4.83) as

$$
\begin{aligned}
\dot{\xi}_1^1 &= \xi_2^1 \\
\vdots \quad &\vdots \quad \vdots \\
\dot{\xi}_{r_1}^1 &= b^1(\xi^1,\ldots,\xi^m,\eta) + \sum_{k=1}^{m} a_k^1(\xi^1,\ldots,\xi^m,\eta)u_k(\xi^1,\ldots,\xi^m,\eta) \\
\dot{\xi}_1^i &= \xi_2^i + \sum_{j=1}^{i-1} \delta_{1j}^i(\xi^1,\ldots,\xi^m,\eta)[b^j(\xi^1,\ldots,\xi^m,\eta) \\
&\quad + \sum_{k=1}^{m} a_k^j(\xi^1,\ldots,\xi^m,\eta)u_k(\xi^1,\ldots,\xi^m,\eta)] \\
\vdots \quad &= \quad \vdots \\
\dot{\xi}_{r_i}^i &= b^i(\xi^1,\ldots,\xi^m,\eta) + \sum_{k=1}^{m} a_k^i(\xi^1,\ldots,\xi^m,\eta)u_k(\xi^1,\ldots,\xi^m,\eta) \\
&\quad i = 2,\ldots,m \\
\dot{\eta} &= q(\xi^1,\ldots,\xi^m,\eta) + \Delta q(\xi^1,\ldots,\xi^m,\eta) \\
&\quad + \sum_{k=1}^{m} [p_k(\xi^1,\ldots,\xi^m,\eta) + \Delta p_k(\xi^1,\ldots,\xi^m,\eta)]u_k(\xi^1,\ldots,\xi^m,\eta)
\end{aligned}
$$

where the definitions follow from (4.99) - (4.108) with

$$
\begin{aligned}
\Delta q(\xi^1,\ldots,\xi^m,\eta) : \Re^n \to \Re^{n-r} &= L_{\Delta f}\eta \\
\Delta p_j(\xi^1,\ldots,\xi^m,\eta) : \Re^n \to \Re^{n-r} &= L_{\Delta g_j}\eta \quad j = 1,2,\ldots,m
\end{aligned}
$$

Using the definition of S^i stated in theorem, we split the coordinates $[\xi^1,\ldots,\xi^m,\eta]^T$ as $[(\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta]^T$ where $\bar{\xi}^i = [\xi_1^i,\ldots,\xi_{\gamma_i-1}^i]^T$. Rewrite the system equations in the $[[(\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta]^T$ coordinates using the control u specified in the theorem.

$$
\begin{aligned}
\dot{\bar{\xi}}^i &= A^i\bar{\xi}^i + \bar{b}^i S^i + c^i && (4.120) \\
\dot{S}^i &= -K_{s^i} sgn(S^i) && (4.121) \\
&\quad i = 1,\ldots,m && (4.122) \\
\dot{\eta} &= q((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) + \Delta q((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) && (4.123)
\end{aligned}
$$

$$+p((\bar{\xi}^1, S^1)\dots(\bar{\xi}^m, S^m), \eta)u((\bar{\xi}^1, S^1)\dots(\bar{\xi}^m, S^m), \eta) \tag{4.124}$$

$$+\Delta p((\bar{\xi}^1, S^1)\dots(\bar{\xi}^m, S^m), \eta)u((\bar{\xi}^1, S^1)\dots(\bar{\xi}^m, S^m), \eta) \tag{4.125}$$

where

$$A^i = \begin{bmatrix} 0 & 1 & ,\dots, & 0 & 0 \\ 0 & 0 & 1 & ,\dots, & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1^i & -a_2^i & ,\dots, & ,\dots, & -a_{r_i-1}^i \end{bmatrix}_{[r_i-1]\times[r_i-1]} \quad i = 1, \dots, m \tag{4.126}$$

$$\bar{b}^i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{[r_i-1]\times[1]} \quad i = 1, \dots, m \tag{4.127}$$

$$c^1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{[r_1-1]\times[1]} \tag{4.128}$$

$$c^i = \begin{bmatrix} \sum_{j=1}^{i-1} \delta_{1j}^i v_j \\ \sum_{j=1}^{i-1} \delta_{2j}^i v_j \\ \vdots \\ \sum_{j=1}^{i-1} \delta_{(r_i-1)j}^i v_j \end{bmatrix}_{[r_i-1]\times[1]} \quad i = 2, \dots, m \tag{4.129}$$

As before, we will prove the theorem in two stages. We will first assume that the system trajectories remain in the open set $\Omega_\delta \subseteq \Phi(U)$ where each of the functions $\delta_{pk}^i(\xi^1, \dots, \xi^m, \eta) \leq K_{\delta_{pk}^i}$ $i = 2, \dots, m$ $p = 1, \dots, r_i - 1$ $k = 1, \dots, i-1$. We will then show that under such an assumption asymptotic output regulation is achieved. We will then consider a composite Lyapunov function and find the largest set of initial conditions such that the system trajectories do not leave the set Ω_δ.

Step 1.

Consider the Lyapunov function

$$V = \sum_{i=1}^{m} \frac{[S^i]^2}{2} \tag{4.130}$$

Differentiating V along the flow of (4.121) we obtain,

$$\dot{V} = \sum_{i=1}^{m} S^i[-K_{s^i} sgn(S^i)] \tag{4.131}$$

$$= -\sum_{i=1}^{m} K_{s^i}|S^i| \tag{4.132}$$

Negative definiteness of $\dot{V}$ implies that each $S^i = 0$ is attractive for all trajectories that remain in $\Phi(U)$. Indeed, the trajectories reach the manifold $\bigcap_{i=1}^{m} S^i = 0$ in *finite time.* The choice of control renders the manifold $\bigcap_{i=1}^{m} S^i = 0$ invariant. Asymptotic regulation to the origin remains to be shown. Performing Filippov averaging [13], we find the average dynamics of the system on the surface $\bigcap_{i=1}^{m} S^i = 0$ to be given by

$$\dot{\bar{\xi}}^i = A^i\bar{\xi}^i + \tilde{c}^i \tag{4.133}$$

$$i = 1, \ldots, m \tag{4.134}$$

where

$$\tilde{c}^1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{[r_1-1]\times[1]} \tag{4.135}$$

$$\tilde{c}^i = \begin{bmatrix} \delta^i_{11} & \cdots & \delta^i_{1(i-1)} \\ \vdots & \vdots & \vdots \\ \delta^i_{(r_i-1)1} & \cdots & \delta^i_{(r_i-1)(i-1)} \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_{i-1} \end{bmatrix}_{[r_i-1]\times[1]} \quad i = 2, \ldots, m \tag{4.136}$$

$$\tilde{v}_j = -a^j_{r_j-1}\xi^j_{r_j}, \ldots, +a^j_1\xi^j_2 \tag{4.137}$$

Note that the difference between $\tilde{v}_j$ and v_j is that the term $-K_{s^j} sgn(S^j)$ is absent in $\tilde{v}_j$. This is due to the fact that when $S^j = 0$, the average value of $sgn(S^j) = 0$.

If the trajectories of the system remain in the open set $\Omega_\delta \subseteq \Phi(U)$ where each of the functions $\delta^i_{pk}(\xi^1, \ldots, \xi^m, \eta) \leq K_{\delta^i_{pk}}$ $i = 2, \ldots, m$ $p = 1, \ldots, r_i - 1$ $k = 1, \ldots, i-1$

then there exist two numbers K_{δ^i} and a^{*i} such that

$$||\bar{c}^i|| \leq K_{\delta^i} \sum_{j=1}^{i-1} a^{*i} ||\bar{\xi}^j|| \; i = 2, \ldots, m \tag{4.138}$$

One choice of the constants is

$$\delta_{sup} = \sup_{k=1,\ldots,r_i-1,\, j=1,\ldots,i-1} \; \sup_{(\xi^1,\ldots,\xi^m,\eta)\in\Omega_\delta} \delta^i_{kj} \; i = 1, \ldots, m \tag{4.139}$$

$$K_{\delta^i} = \delta_{sup}(i-1) \; i = 1, \ldots, m \tag{4.140}$$

$$a^{*i} = [\sum_{k=1}^{r_i-1} [a^i_k]^2]^{\frac{1}{2}} \; i = i, \ldots, m \tag{4.141}$$

Note that as A^i (4.126) is a Hurwitz matrix there exist positive definite symmetric matrices P^i solving the matrix Lyapunov equation [25]

$$[A^i]^T P^i + P^i A^i = -I \tag{4.142}$$

Now consider the Lyapunov function given by

$$V = \sum_{i=1}^{m} \alpha^i [\bar{\xi}^i]^T P^i [\bar{\xi}^i] \tag{4.143}$$

Differentiating V (6.282) along the flow of (4.133) we obtain

$$\dot{V} = \alpha^1[[A^1\bar{\xi}^1]^T P^1 \bar{\xi}^1 + [\bar{\xi}^1]^T P^1 A^1 \bar{\xi}^1] + \sum_{i=2}^{m} \alpha^i [[A^i \bar{\xi}^i + \bar{c}^i]^T P^i \bar{\xi}^i + [\bar{\xi}^i]^T P^i [A^i \bar{\xi}^i + \bar{c}^i]] \tag{4.144}$$

Using (4.142) and (4.138), we get,

$$\dot{V} \leq -\alpha^1 ||\bar{\xi}^1||^2 + -\sum_{i=2}^{m} \alpha^i ||\bar{\xi}^i||^2 + \sum_{i=2}^{m} [2\alpha^i \sigma_{max}(P^i) ||\bar{\xi}^i|| K_{\delta^i} [\sum_{j=1}^{i-1} a^{*j} ||\bar{\xi}^j||]] \tag{4.145}$$

where $\sigma_{max}(P^i)$ is the maximum singular value of P^i.

We rewrite (4.145) as

$$\dot{V} = -[\bar{\xi}^1]^2 [\alpha^1 - \sum_{k=2}^{m} \alpha^k]^2 \tag{4.146}$$

$$-\sum_{i=2}^{m} [\alpha^i - \sigma^2_{max}(P^i) K^2_{\delta^i} - \sum_{k=i+1}^{m} [\alpha^k]^2] ||\bar{\xi}^i||^2 \tag{4.147}$$

where we have made use of the following fact:

$$2\alpha^i\sigma^i_{max}(P^i)||\bar{\xi}^i||[\sum_{j=1}^{i-1}[a^{*j}||\bar{\xi}^j] \leq [\sigma^2_{max}(P^i)[a^{*j}]^2||\bar{\xi}^i||^2 + \sum_{k=1}^{i-1}[\alpha^i]^2||\bar{\xi}^k||^2 \qquad (4.148)$$

Note that we have not chosen the α^i $i = 1, \ldots, m$ yet. We now choose them so as to make the $\dot{V}$ negative definite. Choose

$$\alpha^1 > \sum_{k=2}^{m}[\alpha^k]^2 \qquad (4.149)$$

$$\alpha^i > \sigma^2_{max}(P^i)K^2_{\delta^i} + \sum_{k=i+1}^{m}[\alpha^k]^2]\ i = 2, \ldots, m \qquad (4.150)$$

Note that choice of α^i $i = 1, \ldots, m$ is recursive. We first choose α^m, then α^{m-1}, etc. upto α^1. This implies that on the surface $\bigcap_{i=1}^{m} S^i = 0$, the output $y_i = h^i_1 \rightarrow 0$ as $t \rightarrow \infty$, $i = 1, \ldots, m$.

To ensure that trajectories of the system do not leave the set Ω_δ, we now find a nonempty open subset of initial conditions, $\Omega \subseteq \Omega_\delta \subseteq \Phi(U)$, such that for all initial conditions in the open set Ω, the system trajectories do not leave Ω_δ thus maintaining the validity of the boundedness of δ^i_{pk}. To this end we consider a composite Lyapunov function that includes the η dynamics also:

Using a converse Lyapunov theorem [16], the exponential stability of the zero dynamics (4.109) guarantees the existence of a Lyapunov function V_η such that,

$$K_1||\eta||^2 \leq V_\eta \leq K_2||\eta||^2 \qquad (4.151)$$

$$\frac{\partial V_\eta}{\partial \eta}[q(0,\eta) + p(0,\eta)u(0,\eta)] \leq -K_3||\eta||^2 \qquad (4.152)$$

$$||\frac{\partial V_\eta}{\partial \eta}|| \leq K_4||\eta|| \qquad (4.153)$$

Now consider the composite Lyapunov function given by

$$V = \sum_{i=1}^{m}[\alpha^i_1[\bar{\xi}^i]^T P^i[\bar{\xi}^i]] + \sum_{i=1}^{m}\alpha^i_2[\frac{[S^i]^2}{2} + \frac{[S^i]^4}{4}] + \alpha_3 V_\eta \qquad (4.154)$$

where P^i is the solution of (4.142) and V_η satisfies (4.151) - (4.153).

Differentiating V (4.154) along the flow of (4.120) - (4.125) we obtain

$$
\begin{aligned}
\dot{V} \;=\; & \sum_{i=1}^{m}[\alpha_1^i[[A^i\bar{\xi}^i + b^iS^i]^T P^i\bar{\xi}^i + [\bar{\xi}^i]^T P^i[A^i\bar{\xi}^i + b^iS^i]]] \\
& + \sum_{i=1}^{m} \alpha_2^i S^i[-K_{s^i}sgn(S^i)] \\
& + \sum_{i=1}^{m} \alpha_2^i S^i]^3[-K_{s^i}sgn(S^i)] \\
& \alpha_3\frac{\partial V_\eta}{\partial \eta}[q((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) + \Delta q((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)] \\
& +\alpha_3\frac{\partial V_\eta}{\partial \eta}[p((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)u((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)] \\
& +\alpha_3\frac{\partial V_\eta}{\partial \eta}[\Delta p((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)u((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)]
\end{aligned}
$$

Assuming that $[\xi^1,\ldots,\xi^m,\eta]^T \in \bigcap_{i=1}^m \Omega^i_{\Delta Bound}$, using (4.139) - 4.141), and (4.147) we obtain

$$
\begin{aligned}
\dot{V} \;\leq\; & -[\bar{\xi}^1]^2[\alpha_1^1 - \frac{1}{4} - \sum_{k=2}^{m}\alpha_1^k]^2 \\
& -\sum_{i=2}^{m}[\alpha_1^i - \frac{1}{4} - \sigma_{max}^2(P^i)K_{\delta^i}^2 \\
& -[\sum_{k=i+1}^{m}[\alpha_1^k]^2]]||\bar{\xi}^i||^2 - [2\alpha_1^i\sigma_{max}(P^i)K_{\delta^i}\sum_{j=1}^{i-1}K^*_{\delta^j}]||\bar{\xi}^i|| \\
& -\sum_{i=1}^{m}[\alpha_2^i[K_{s^i}][||S^i|| - \frac{[K_6^i]^2}{\alpha_2^i[K_{s^i}]}||S^i||^2 + ||S^i||^3] \\
& +\alpha_3\frac{\partial V_\eta}{\partial \eta}[q(0,\eta) + p(0,\eta)u(0,\eta)] \\
& +\alpha_3\frac{\partial V_\eta}{\partial \eta}[q((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) - q(0,\eta)] \\
& +\alpha_3\frac{\partial V_\eta}{\partial \eta}[p((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)u((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) \\
& -p(0,\eta)u(0,\eta)] \\
& +\alpha_3\frac{\partial V}{\partial \eta}[\Delta q((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) - \Delta q(0,\eta)] \\
& +\alpha_3\frac{\partial V}{\partial \eta}[\Delta p((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta)u((\bar{\xi}^1,S^1),\ldots,(\bar{\xi}^m,S^m),\eta) \\
& -\Delta p(0,\eta)u(0,\eta)]
\end{aligned}
$$

$$+\alpha_3 \frac{\partial V}{\partial \eta}[\Delta q(0,\eta) + \Delta p(0,\eta)u(0,\eta)]$$

where we have made use of the following fact:

$$2\alpha^i \sigma^i_{max}(P^i)||\bar{\xi}^i||[\sum_{j=1}^{i-1}[a^{*j}||\bar{\xi}^j] \leq [\sigma^2_{max}(P^i)[a^{*j}]^2||\bar{\xi}^i||^2 + \sum_{k=1}^{i-1}[\alpha^i]^2||\bar{\xi}^k||^2 \tag{4.155}$$

and that,

$$\begin{aligned} S^i[b^i]^T P^i \bar{\xi}^i &\leq K_6^i||S^i||||\bar{\xi}^i|| \\ K_6^i &= \sigma_{max}(P^i)||b^i|| \\ ||b^i|| &= 1 \\ K_6^i||S^i||||\bar{\xi}^i|| &\leq \frac{||\bar{\xi}^i||^2}{4} + [K_6^i]^2[S^i]^2 \end{aligned}$$

Now define Ω_V to be the largest non-empty subset of $\Phi(U)$ such that

$$\Omega_V = \bigcap_{i=1}^{m}[\Omega^i_{\Delta Bound} \bigcap \Omega_{\Delta Conic} \bigcap \Omega_{Conic} \bigcap \Omega_{\Delta[f(0,\eta)+g(0,\eta)u(0,\eta)]}] \tag{4.156}$$

Choose $c^* \in \Re_+$ such that

$$c^* = sup(c : V \leq c \ \in \Omega_V) \tag{4.157}$$

Define $\Omega = [((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) : V \leq c^*]$.
For all initial conditions in Ω, we may rewrite $\dot{V}$ as

$$\begin{aligned} \dot{V} \leq\ & -[\bar{\xi}^1]^2[\alpha_1^1 - \frac{1}{4} - \sum_{k=2}^{m}\alpha_1^k]^2 && (4.158) \\ & -\sum_{i=2}^{m}[\alpha_1^i - \frac{1}{4} - \sigma^2_{max}(P^i)K^2_{\delta^i} && (4.159) \\ & -[\sum_{k=i+1}^{m}[\alpha_1^k]^2]]||\bar{\xi}^i||^2 - [2\alpha_1^i\sigma_{max}(P^i)K_{\delta^i}\sum_{j=1}^{i-1}K^*_{\delta^j}]||\bar{\xi}^i|| \\ & -\sum_{i=1}^{m}[\alpha_2^i[K_{s^i}][||S^i|| - \frac{[K_6^i]^2}{\alpha_2^i[K_{s^i}]}||S^i||^2 + ||S^i||^3] && (4.160) \\ & -\alpha_3 K_3||\eta||^2 && (4.161) \\ & +\alpha_3 K_4||\eta||[K_q + K_p + K_{\Delta q} + K_{\Delta p}]\sum_{i=1}^{m}[||\bar{\xi}^i|| + ||S^i||] && (4.162) \\ & +\alpha_3 K_4 K_{qp}||\eta||^2 && (4.163) \end{aligned}$$

where,

$$||q((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) - q(0, \eta)|| \leq K_q \sum_{i=1}^{m} [||\bar{\xi}^i|| + ||S^i||] \tag{4.164}$$

$$||p((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) u((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) - p(0, \eta) u(0, \eta)|| \leq K_p \sum_{i=1}^{m} [||\bar{\xi}^i|| + ||S^i||]$$

$$||\Delta q((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) - \Delta q(0, \eta)|| \leq K_{\Delta q} \sum_{i=1}^{m} [||\bar{\xi}^i|| + ||S^i||] \tag{4.165}$$

$$||\Delta p((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) u((\bar{\xi}^1, S^1), \ldots, (\bar{\xi}^m, S^m), \eta) - \Delta p(0, \eta) u(0, \eta)|| \leq K_{\Delta p} \sum_{i=1}^{m} [||\bar{\xi}^i|| + ||S^i||]$$

$$||\Delta q(0, \eta) + \Delta p(0, \eta) u(0, \eta)|| \leq K_{\Delta qp} ||\eta|| \tag{4.166}$$

Indeed, noting that a bound on the Jacobian of the transformation Φ gives $d\eta \leq K_\eta$, and using assumptions $(I4) - (I7)$, we observe that

$$K_q \leq K_\eta K_{f-conic} \tag{4.167}$$

$$K_p \leq K_\eta K_{g-conic} \tag{4.168}$$

$$K_{\Delta q} \leq K_\eta K_{\Delta f-conic} \tag{4.169}$$

$$K_{\Delta p} \leq K_\eta K_{\Delta g-conic} \tag{4.170}$$

$$K_{qp} \leq K_\eta K_{\Delta qp} \tag{4.171}$$

Now define

$$K_7 = \alpha_3 K_4 [K_q + K_p + K_{\Delta q} + K_{\Delta p}] \tag{4.172}$$

and using the fact that

$$K_7 ||\eta|| \sum_{i=1}^{m} [||\bar{\xi}^i|| + ||S^i||] \leq \frac{m||\eta||^2}{4} + K_7^2 \sum_{i=1}^{m} ||\bar{\xi}^i||^2 + K_7^2 \sum_{i=1}^{m} ||S^i||^2 \tag{4.173}$$

we rewrite equation $\dot{V}$ as

$$\begin{aligned}
\dot{V} \leq\; & -[\bar{\xi}^1]^2[\alpha_1^1 - \frac{1}{4} - -K_7^2 - \sum_{k=2}^{m} \alpha_1^k]^2 \\
& -\sum_{i=2}^{m} ||\bar{\xi}^i||[(\alpha_1^i - \frac{1}{4} - -K_7^2 - \sigma_{max}^2(P^i)K_{\delta^i}^2 - [\sum_{k=i+1}^{m}[\alpha_1^k]^2])||\bar{\xi}^i|| - K_8] \\
& -\sum_{i=1}^{m}[\alpha_2^i[K_{s^i}][||S^i|| - \frac{[K_6^i]^2 + K_7^2}{\alpha_2^i[K_{s^i}]}||S^i||^2 + ||S^i||^3] \\
& -||\eta||^2[\alpha_3[K_3 - K_{qp}] + \frac{m}{4}]
\end{aligned}$$

where

$$K_8 = 2\sigma_{max}(P^i)K_{\delta^i}\sum_{j=1}^{i-1} K_{\delta j}^{*}\; i = 2, \ldots, m \tag{4.174}$$

Note that the constants α_j^i $i = 1, \ldots, m$ $j = 1, \ldots, 3$ have not been chosen yet, and we now choose them in such a manner as to ensure the negative definiteness of the Lyapunov function, provided $K_3 > K_{qp}$.

We choose them to be

$$\alpha_1^1 > \sum_{k=2}^{m}[\alpha^k]^2 + \frac{1}{4} + K_7^2 \tag{4.175}$$

$$\alpha_1^i > \sigma_{max}^2(P^i)K_{\delta^i}^2 + \sum_{k=i+1}^{m}[\alpha^k]^2] > \frac{K_8}{c^*}\; i = 2, \ldots, m \tag{4.176}$$

$$\alpha_2^i > [K_6^i]^2 + K_7^2 \tag{4.177}$$

$$\alpha_3 > \frac{m}{4[K_3 - K_{qp}]} \tag{4.178}$$

Such a choice of constants ensure that $\dot{V}$ is negative definite for all initial conditions in Ω.

We have shown that there exists a set Ω and a constant K^* such that for all initial conditions in Ω, and for all $K_{qp} < K^*$, the output of the system is asymptotically regulated to the origin. ◁♠

Theorem 4.3 *Relaxed Generalized Matching conditions for asymptotic regulation for left invertible MIMO systems decoupled using the zero dynamics algorithm.*

Given *(G1) A perturbed MIMO system of the form (4.82) - (4.83).*

(G2) *The unperturbed MIMO system (4.80) - (4.81) has an extended vector relative degree* $[r_1, \ldots, r_m]^T$.

(G3) *A control objective of asymptotic output regulation, that is* $|\xi_1^i| \to 0 \; i = 1, 2, \ldots, m$

If *(I1)* *The zero dynamics of the unperturbed system, (4.109) is exponentially stable.*

(I2)

$$\Delta f(x) \in ker[dh_1^1(x), dL_f h_1^1(x) \cdots dL_f^{r_1-2} h_1^1(x)] \bigcap [\bigcap_{i=2}^{m} ker[dh_1^i,, \ldots,, dh_{r_i-2}^i]] \tag{4.179}$$

and

$$\begin{aligned}
d_{ik}[\Delta f] &\in row\text{-}span\text{-}of[D_{ik}] \\
d_{ik}[\Delta f] &= \begin{bmatrix} L_{\Delta f} L_f^{r_1-1} h_1^1 \\ L_{\Delta f} h_{r_2}^2 \\ \vdots \\ L_{\Delta f} h_{r_i}^{i-1} \\ L_{\Delta f} h_k^i \end{bmatrix} \\
D_{ik} &= \begin{bmatrix} a_1^1(\xi^1, \ldots, \xi^m, \eta) & , \ldots, & a_m^1(\xi^1, \ldots, \xi^m, \eta) \\ \vdots & \vdots & \vdots \\ a_1^{i-1}(\xi^1, \ldots, \xi^m, \eta) & , \ldots, & a_m^{i-1}(\xi^1, \ldots, \xi^m, \eta) \\ L_{g_1} h_k^i & , \ldots, & L_{g_m} h_k^i \end{bmatrix} \\
& \quad i = 1, 2, \ldots, m \;\; k = 1, 2, \ldots, r_i - 1 \\
L_{\Delta f} L_f^{r_1-1} h_1^1 &\leq K_1 \in \Re_+ \\
L_{\Delta f} h_{r_i}^i &\leq K_i \;\; i = 2, 3, \ldots, m
\end{aligned}$$

(I3)

$$\Delta g_j(x) \in ker[dh_1^1(x), dL_f h_1^1(x) \cdots dL_f^{r_1-1} h_1^1(x)] \bigcap [\bigcap_{i=2}^{m} ker[dh_1^i,, \ldots,, dh_{r_i}^i]] \tag{4.180}$$

where $j = 1, \ldots, m$

(I4) $\Delta f(\xi^1,\ldots,\xi^m,\eta)$, $\sum_{i=1}^m \Delta g_i(\xi^1,\ldots,\xi^m,\eta)u_i(\xi^1,\ldots,\xi^m,\eta)$ *satisfy conic continuity in* $[\xi^1,\ldots,\xi^m]^T$*, uniformly in* η *with constants* $K_{\Delta f-conic}$, $K_{\Delta g-conic}$, *everywhere in an open set* $\Omega_{\Delta Conic} \subseteq \Phi(U)$

(I5) $L_{\Delta f} L_f^{\gamma_i - 1} h_i \leq K_{bi} < K_{s^i}$ *in* $\Omega^i_{\Delta Bound} \subseteq \Phi(U)$

(I6) $f(\xi^1,\ldots,\xi^m,\eta)$, $\sum_{i=1}^m g_i(\xi^1,\ldots,\xi^m,\eta)u_i(\xi^1,\ldots,\xi^m,\eta)$ *satisfy conic continuity in* $[\xi^1,\ldots,\xi^m]^T$*, uniformly in* η*, with constants* $K_{f-conic}$, $K_{g-conic}$ *everywhere in an open set* $\Omega_{conic} \subseteq \Phi(U)$

(I7) $||\Delta f(0,\eta) + \sum_{i=1}^m \Delta g_i(0,\eta)u_i(0,\eta)|| \leq K_{\Delta qp}||\eta||$ *everywhere in an open set* $\Omega_{\Delta[f(0,\eta)+g(0,\eta)u(0,\eta)]} \subseteq \Phi(U)$

(I8) The functions $\delta^i_{pk}(\xi^1,\ldots,\xi^m,\eta) \leq K_{\delta^i_{pk}}$ $i = 1,\ldots,m$ $p = 1,\ldots,r_i - 1$ $k = 1,\ldots i-1$ *everywhere in an open set* $\Omega_\delta \subseteq \Phi(U)$

(I9) The diffeomorphism Φ *(4.84) has a bounded Jacobian.*

(I10) The control u *is chosen to be*

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \bar{A}_1^{-1} \begin{bmatrix} -b^1(\xi^1,\ldots,\xi^m,\eta) + v_1 \\ \vdots \\ -b^m(\xi^1,\ldots,\xi^m,\eta) + v_m \end{bmatrix} \tag{4.181}$$

where, $\bar{A}_1$ *is specified by (4.112) and* $b^i(\xi^1,\ldots,\xi^m,\eta)$ $i = 1,\ldots,m$ *are specified in (4.99) - (4.108), and*

$$\begin{aligned} v_i &= -a^i_{r_i-1}\xi^i_{r_i},\ldots,+a^i_1\xi^i_2 - K_{s_i} sgn(S^i) && (4.182)\\ S^i &= \xi^i_{r_i} + a^i_{r_i-1}\xi^i_{r_i-1},\ldots,+a^i_1\xi^i_1 && (4.183)\\ & \quad \text{and is a Hurwitz polynomial} && (4.184)\\ sgn(S^i) &= \frac{S^i}{|S^i|}\ \forall\ |S^i| > 0 && (4.185) \end{aligned}$$

Then *(T1) There exist a set* $\Omega \subseteq \Phi(U)$ *and a constant* K^* *such that for all initial conditions belonging to* Ω*, and* $K_{\Delta qp} < K^*$ *the output* $y_i = h_i(x)$ *tends to zero asymptotically.*

Proof: ♠ ▷

Choosing the control law prescribed by the theorem, subject to the prescribed matching conditions, the equations (??) - (??) may be rewritten as

$$\begin{aligned}
\dot{\xi}_1^1 &= \xi_2^1 \\
\vdots & \vdots \vdots \\
\dot{\xi}_{r_1}^1 &= v_1 + \Delta b^1(\xi^1, \ldots, \xi^m, \eta) \\
\dot{\xi}_1^i &= \xi_2^i + \sum_{j=1}^{i-1} \delta_{1j}^i [v_j + \Delta b^j(\xi^1, \ldots, \xi^m, \eta)] + L_{\Delta f} h^i \\
\vdots &= \vdots \\
\dot{\xi}_{r_i}^i &= v_i + \Delta b^i(\xi^1, \ldots, \xi^m, \eta) \\
& \quad i = 2, \ldots, m \\
\dot{\eta} &= q(\xi^1, \ldots, \xi^m, \eta) + \Delta q(\xi^1, \ldots, \xi^m, \eta) \\
& \quad + \sum_{k=1}^{m} [p_k(\xi^1, \ldots, \xi^m, \eta) + \Delta p_k(\xi^1, \ldots, \xi^m, \eta)] u_k
\end{aligned}$$

Comment 4.2.8 Assumption $I(2)$ guarantees that the term

$$[\sum_{j=1}^{i-1} \delta_{1j}^i \Delta b^1(\xi^1, \ldots, \xi^m, \eta)] + L_{\Delta f} h^i = 0 \tag{4.186}$$

The equations (??) - (??) may be rewritten as

$$\begin{aligned}
\dot{\xi}_1^1 &= \xi_2^1 \\
\vdots & \vdots \vdots \\
\dot{\xi}_{r_1}^1 &= v_1 + \Delta b^1(\xi^1, \ldots, \xi^m, \eta) \\
\dot{\xi}_1^i &= \xi_2^i + \sum_{j=1}^{i-1} \delta_{1j}^i v_j \\
& \quad + \\
\vdots &= \vdots \\
\dot{\xi}_{r_i}^i &= v_i + \Delta b^i(\xi^1, \ldots, \xi^m, \eta) \\
& \quad i = 2, \ldots, m \\
\dot{\eta} &= q(\xi^1, \ldots, \xi^m, \eta) + \Delta q(\xi^1, \ldots, \xi^m, \eta) \\
& \quad + \sum_{k=1}^{m} [p_k(\xi^1, \ldots, \xi^m, \eta) + \Delta p_k(\xi^1, \ldots, \xi^m, \eta)] u_k
\end{aligned}$$

Furthermore, from the definition of each v_i $i = 1, 2, \ldots, m$ it is clear that the problem has been reduced to the hypothesis of the theorem on strict matching. Invoking the results of the theorem on strict matching, the conclusions of the theorem on relaxed matching are proved. $\lhd\spadesuit$

Matching Conditions Assuming Right Invertibility

We now present matching conditions for a MIMO system which does not possess a well defined relative degree, but which is right invertible [10].

Statement Of The Problem

Given:

- An unperturbed system of the form (4.1) - (4.2) which does not possess a well defined vector relative degree.
- The unperturbed system (4.1) - (4.2) is right invertible.
- A control objective - Asymptotic output regulation $y_i = h_1^i(x) \rightarrow 0$ $i = 1, \ldots, m$ as $t \rightarrow \infty$.
- The dynamic extension algorithm [5], [4], is utilized to generate a control law to attain the control objective.

Determine:

- Conditions that must be matched by the perturbations $\Delta f(x) : \Re^n \rightarrow \Re^n$ and $\Delta g_i(x) : \Re^n \rightarrow \Re^n$ $i = 1, \ldots, m$ of (4.3) - (4.4) such that the control objective of asymptotic output regulation is achieved by the control law developed based on the unperturbed system equations (4.1) - (4.2), when applied to the perturbed system.

Assuming that x_0 is a regular point [19] of (4.1) - (4.2) and that the system is right invertible [10] there exists a transformation Φ and a local normal form for the system generated as follows:

The dynamic extension algorithm [8] systematically extends the dimension of the state vector, at each step, and therefore we will start the algorithm with the state $x \in \Re^n = x^e$. The algorithm proceeds in the following manner. [4]

Step 1 Let r_i be the relative degree of the ith output of (4.1) - (4.2), i.e the largest integer such that

$$L_{g_j}L_f^l h_i(x^e) = 0 \forall\, l < r_i - 1 \;\forall 1 \leq j \leq m \tag{4.187}$$

and for all x^e near x_0^e. Define the decoupling matrix $A_k(x^e)$ to hae as its ijth entry,

$$a_{ij}(x) = L_{g_j}L_f^{r_i-1}h_i(x^e) \tag{4.188}$$

and denote its normal rank by s_k. If $s_k = m$, stop.

Step 2 If $s_k < m$ assume that the first s_k columns of $A_k(x^e)$ are linearly independent at each point of an open, dense set of X^e (this can always be achieved by a permutation of the components of the output.) Apply the regular static state feedback law

$$u = \alpha_k(x^e) + \beta_k(x^e)v \tag{4.189}$$

with α_k, β_k analytic functions of x^e such that the decoupling matrix with the control law (6.134) is of the form

$$A_{k1}(x^e) = \begin{bmatrix} I_{s_k \times s_k} & 0 \\ M(x^e) & 0 \end{bmatrix} \tag{4.190}$$

This may be achieved by choosing α_k, β_k to be the solutions of the equations

$$\begin{aligned} dL_f^{r_i-1}h(x^e)(f(x^e) + g(x^e)\alpha)k(x^e)) &= 0 \,\forall 1 \leq i \leq s_k & (4.191) \\ dL_f^{r_i-1}h(x^e)(g(x^e)\beta_k(x^e))_j &= \delta_{ij} \,\forall 1 \leq i \leq s_k \;\; 1 \leq j \leq m & (4.192) \end{aligned}$$

where $(g(x^e)\beta_k(x^e))_j$ denotes the jth column of the matrix $g(x^e)\beta_k(x^e)$.

Step 3 There exist q_k columns of the matrix $A_1(x^e)$ (without loss of generality, the first q_k) with two or more non zero elements. Put an integrator in series with q_k corresponding input channels. Define the dynamic extension as

$$\dot{\zeta}_i = v_i \; i = 1, \ldots, q_k \tag{4.193}$$

Extend the original system with new inputs $v_1, \ldots, v_k, u_{q_k+1}, \ldots, u_m$, and return to step 1 to resume the procedure with $k \rightarrow k+1$ are new state variables $x^e \bigcup \zeta_i$.

If the original system is right invertible, the procedure converges in a finite number of steps to a system having vector relative degree $[r_1^e, \ldots, r_m^e]^T$. Let the triple (f^e, g_j^e, h_j^e) characterize the new extended system thus obtained, with $x^e = (x, \zeta)$ as its state, u_e as its input, and y^e as its output. Constructing a local change of coordinates $\Phi(x) = (\xi, \eta)$ with $\xi = col(\xi_i)$ by setting

$$\xi_i = col(h_i^e(x^e).L_{f^e}h_i^e(x^e), , \ldots, , L_{f^e}^{r_i^e-1}h_i^e(x^e)) \tag{4.194}$$

$$= col(\xi_1^i, \ldots, \xi_{r_i^e}^i) \tag{4.195}$$

and using complementary coordinates η. Then the transformed system equations are given by

$$\dot{\xi}_1^i = \xi_2^i \tag{4.196}$$

$$\vdots = \vdots \tag{4.197}$$

$$\dot{\xi}_{r_i^e}^i = b_i^e(\xi^1, \ldots, \xi^m, \eta) + \sum_{j=1}^{m} a_{ij}^e(\xi^1, \ldots, \xi^m, \eta)u_j^e \tag{4.198}$$

$$i = 1, \ldots, m \tag{4.199}$$

$$\dot{\eta} = q^e(\xi^1, \ldots, \xi^m, \eta) + \sum_{j=1}^{m} p_j^e(\xi^1, \ldots, \xi^m, \eta)u_j \tag{4.200}$$

where

$$\begin{aligned}
b_i^e(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re &= L_{f^e}^{r_i^e} h_i^e \\
a_{ij}^e(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re &= L_{g_j^e} L_{f^e}^{r_i^e-1} h_i^e \\
q^e(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re^{n-r} &= L_{f^e}\eta \\
p_j^e(\xi^1, \ldots, \xi^m, \eta) : \Re^n \to \Re^{n-r} &= L_{g_j^e}\eta \;\; j = 1, \ldots, m \\
r &= \sum_{i=1}^{m} r_i^e
\end{aligned}$$

The zero-dynamics is a dynamical system in $\Re^{n-\sum_{i=1}^{m} r_i^e}$ described by the following equations.

$$\dot{\eta} = q^e(0, \eta) + \sum_{j=1}^{m} p_j^e(0, \eta)u_j \tag{4.201}$$

It is clear that for the unperturbed system, the output can be regulated to zero asymptotically by choosing first a decoupling and linearizing control law, and then applying any standard linear control law that guarantees asymptotic regulation of the output.

We now prescribe matching conditions similar to [7] for the perturbed system (4.3)- (4.4) such that the control law developed using the dynamic extension algorithm will still ensure asymptotic output regulation in the presence of perturbations.

Theorem 4.4 *Generalized Matching conditions for asymptotic regulation for right invertible systems decoupled using the dynamic extension algorithm.*

Given *(G1) A perturbed MIMO system of the form (4.3) - (4.4).*

(G2) The unperturbed MIMO system (4.1) - (4.2) has an extended vector relative degree $[r_1^e, \dots, r_m^e]^T$.

(G3) A control objective of asymptotic output regulation, that is $y_i^e = h_i^e(x) \to 0 \;\; i = 1, \dots, m$

If *(I1) The zero dynamics of the unperturbed system (4.201) is exponentially stable.*

(I2)

$$\Delta f(x) \;\; \in \;\; \bigcap_{i=1}^{m} ker[dh_i^e(x^e), dL_{f^e}h_i^e(x^e) \cdots dL_{f^e}^{r_i^e-2}h_i^e(x^e)] \quad (4.202)$$

(I3)

$$\Delta g_j^e(x^e) \;\; \in \;\; \bigcap_{i=1}^{m} ker[dh_i^e(x^e), dL_{f^e}h_i^e(x^e) \cdots dL_{f^e}^{r_i^e-1}h_i^e(x^e)] \; j = 1, \dots, m$$

(I4) $\Delta f^e(\xi^1, \dots, \xi^m, \eta)$, $\sum_{i=1}^m \Delta g_i^e(\xi^1, \dots, \xi^m, \eta) u_i^e(\xi^1, \dots, \xi^m, \eta)$ *satisfy conic continuity in* $[\xi^1, \dots, \xi^m]^T$*, uniformly in* η *with constants* $K_{\Delta f-conic}$, $K_{\Delta g-conic}$, *everywhere in an open set* $\Omega_{\Delta Conic} \subseteq \Phi(U)$

(I5) $L_{\Delta f^e} L_{f^e}^{r_i-1} h_i^e \leq K_{bi} < K_{s^i}$ *in* $\Omega^i_{\Delta Bound} \subseteq \Phi(U)$

(I6) $f^e(\xi^1, \dots, \xi^m, \eta)$, $\sum_{i=1}^m g_i^e(\xi^1, \dots, \xi^m, \eta) u_i^e(\xi^1, \dots, \xi^m, \eta)$ *satisfy conic continuity in* $[\xi^1, \dots, \xi^m]^T$*, uniformly in* η*, with constants* $K_{f-conic}$, $K_{g-conic}$ *everywhere in an open set* $\Omega_{conic} \subseteq \Phi(U)$

(I7) $||\Delta f(0,\eta) + \sum_{i=1}^{m} \Delta g_i(0,\eta)u_i(0,\eta)|| \leq K_{\Delta qp}||\eta||$ *everywhere in an open set* $\Omega_{\Delta[f(0,\eta)+g(0,\eta)u(0,\eta)]} \subseteq \Phi(U)$

(I8) The diffeomorphism Φ *(4.6) has a bounded Jacobian.*

(I9) The control u *is chosen to be*

$$\begin{bmatrix} u_1^e \\ \vdots \\ u_m^e \end{bmatrix} = \bar{A}^{-1} \begin{bmatrix} -b^1(\xi^1,\ldots,\xi^m,\eta) + v_1 \\ \vdots \\ -b^m(\xi^1,\ldots,\xi^m,\eta) + v_m \end{bmatrix} \tag{4.203}$$

and

$$\begin{aligned} v_i &= -a^i_{\gamma_i-1}\xi^i_{\gamma_i},\ldots,+a^i_1\xi^i_2 - K_{s^i}sgn(S^i) && (4.204)\\ S^i &= \xi^i_{\gamma_i} + a^i_{\gamma_i-1}\xi^i_{\gamma_i-1},\ldots,+a^i_1\xi^i_1 && (4.205)\\ & \quad \textit{and is a Hurwitz polynomial} && (4.206)\\ sgn(S^i) &= \frac{S^i}{|S^i|}\ \forall\,|S^i| > 0 && (4.207) \end{aligned}$$

Then *(T1) There exist a set* $\Omega \subseteq \Phi(U)$ *and a constant* K^* *such that for all initial conditions belonging to* Ω*, and* $K_{\Delta qp} < K^*$ *the trajectories of the perturbed system remain in* $\Phi(U)$ *and the output* $y_i = h_i(x)$ *tends to zero asymptotically.*

Proof:♠ ▷

The proof the theorem is obvious once we realize that the matching conditions cast the extended system into the exact form of a perturbed non-singular MIMO system, satisfying all the hypotheses of the theorem that guarantees output regulation for perturbed nonsingular MIMO systems. Indeed the system equations now look like

$$\begin{aligned} \dot{\xi}^i_1 &= \xi^i_2 \\ \vdots &= \vdots \\ \dot{\xi}^i_{r^e_i} &= b^e_i(\xi^1,\ldots,\xi^m,\eta) + \Delta b^e_i(\xi^1,\ldots,\xi^m,\eta) + \sum_{j=1}^{m} a^e_{ij}(\xi^1,\ldots,\xi^m,\eta)u^e_j \\ & \quad i = 1,\ldots,m \end{aligned}$$

$$\begin{aligned}\dot{\eta} &= q^e(\xi^1,\ldots,\xi^m,\eta) + \Delta q^e(\xi^1,\ldots,\xi^m,\eta) \\ &\quad + \sum_{j=1}^{m}[p_j^e(\xi^1,\ldots,\xi^m,\eta) + \Delta p_j^e(\xi^1,\ldots,\xi^m,\eta)]u_j^e\end{aligned}$$

The system equations and the hypotheses of the theorem (4.4) satisfy the hypotheses of the earlier proven theorem (4.1). Therefore invoking the conclusions of theorem (4.1), the proof is complete. ◁♠

Example 4.2.1 *Comparing inversion techniques*

Consider the system represented by the following equations,

Unperturbed System Equations:

$$\dot{x}_1 = x_3 \sin x_1 + u_1 \tag{4.208}$$

$$\dot{x}_2 = x_3 \cos^2 x_1 - u_1 \sin x_1 \tag{4.209}$$

$$\dot{x}_3 = u_2 \tag{4.210}$$

$$y_1 = x_1 \tag{4.211}$$

$$y_2 = x_2 \tag{4.212}$$

where $x \in \Re^3$*, the controls* $u_i(t) : \Re_+ \to \Re \;\; i = 1, 2, 3.$

Perturbed System Equations:

$$\dot{x}_1 = x_3 \sin x_1 + u_1 + \Delta_1(x) \tag{4.213}$$

$$\dot{x}_2 = x_3 \cos^2 x_1 - u_1 \sin x_1 + \Delta_2(x) \tag{4.214}$$

$$\dot{x}_3 = u_2 + \Delta_3(x) \tag{4.215}$$

$$y_1 = x_1 \tag{4.216}$$

$$y_2 = x_2 \tag{4.217}$$

where the perturbations $\Delta_i(x) : \Re^3 \to \Re \;\; i = 1, 2, 3$ *are unknown.*

The control objective is to ensure that the states are regulated to the origin commencing from arbitrary initial conditions. We will choose two control laws based on the zero dynamics algorithm, and the dynamic extension method, and the prescribe *matching conditions to be met by the perturbations* $\Delta_i(x) : \Re^3 \to \Re \;\; i = 1,2,3$ *such that the control objective is still attained.*

Control Using The Zero-Dynamics Algorithm

$$\begin{aligned}
h_1^1 &= x_1 \\
h_1^2 &= x_2 \\
\dot{y}_1 &= L_f h_1^1 \\
&= x_3 \sin x_1 + u_1 \\
\dot{y}_2 &= L_f h_1^2 \\
&= x_3 \cos^2 x_1 - u_1 \sin x_1
\end{aligned}$$

Now defining

$$h_2^2 = x_3 \tag{4.218}$$

$$\delta_{11}^2(x) = \sin x_1 \tag{4.219}$$

we note that

$$\dot{y}_2 = x_3 \cos^2 x_1 - \delta_{11}^2(x)[\dot{y}_1 - x_3 \sin x_1] \tag{4.220}$$

$$= x_3[\cos^2 x_1 + \sin^2 x_1] - \delta_{11}^2(x)\dot{y}_1 \tag{4.221}$$

$$= x_3 - \dot{y}_1 \sin x_1 \tag{4.222}$$

$$= h_2^2 - \dot{y}_1 \sin x_1 \tag{4.223}$$

Now note that

$$L_f h_2^2 = u_2 \tag{4.224}$$

Consider the following change of coordinates

$$\Phi : x \in \Re^3 \to \begin{bmatrix} \xi_1^1 \\ \xi_1^2 \\ \xi_2^2 \end{bmatrix} \in \Re^3 \tag{4.225}$$

where

$$\xi_1^1 = h_1^1 = x_1 \tag{4.226}$$

$$\xi_1^2 = h_1^2 = x_2 \tag{4.227}$$

$$\xi_2^2 = h_2^2 = x_3 \tag{4.228}$$

In these new coordinates, the system equations look like

$$\dot{\xi}_1^1 = x_3 \sin \xi_1^1 + u_1 \tag{4.229}$$

$$\dot{\xi}_2^1 = \xi_2^2 - \dot{\xi}_1^1 \sin \xi_1^1 \tag{4.230}$$

$$\dot{\xi}_2^2 = u_2 \tag{4.231}$$

Now choose the control inputs u_1, *to be*

$$u_1 = -x_3 \sin \xi_1^1 + v_1 \tag{4.232}$$

where $v_1(t) : \Re_+ \rightarrow \Re$ *is to be chosen later. Such a choice of control yields,*

$$\dot{\xi}_1^1 = v_1 \tag{4.233}$$

$$\dot{\xi}_2^1 = \xi_2^2 - \dot{\xi}_1^1 \sin \xi_1^1 \tag{4.234}$$

$$\dot{\xi}_2^2 = u_2 \tag{4.235}$$

The same proceedure for the perturbed system equations would have yielded a modified set of equations.

$$\begin{aligned} \dot{\xi}_1^1 &= x_3 \sin \xi_1^1 + u_1 + \Delta_1(x) \\ \dot{\xi}_2^1 &= \xi_2^2 - \dot{\xi}_1^1 \sin \xi_1^1 + [\sin \xi_1^1 \Delta_1(x) + \Delta_2(x)] \\ \dot{\xi}_2^2 &= u_2 + \Delta_3(x) \end{aligned}$$

Now choose the control inputs as before to be

$$u_1 = -x_3 \sin \xi_1^1 + v_1 \tag{4.236}$$

where $v_1(t) : \Re_+ \rightarrow \Re$ *is to be chosen later. Such a choice of control yields,*

$$\begin{aligned} \dot{\xi}_1^1 &= v_1 + \Delta_1(x) \\ \dot{\xi}_2^1 &= \xi_2^2 - \dot{\xi}_1^1 \sin \xi_1^1 + [\sin \xi_1^1 \Delta_1(x) + \Delta_2(x)] \\ \dot{\xi}_2^2 &= u_2 + \Delta_3(x) \end{aligned}$$

The matching conditions that perturbations must satisfy are now obvious by inspecting equations (4.237) - (4.237).

The matching conditions that could be imposed on the perturbations are of the following three kinds.

Strict Matching Conditions

$$\begin{aligned}\Delta_1(x) &= 0\\ \Delta_2(x) &= 0\\ \Delta_3(x) &= 0\end{aligned}$$

Relaxed Matching Conditions

$$\begin{aligned}\Delta_1(x) &\le K_{\Delta_1} \in \Re_+\\ \sin\xi_1^1\Delta_1(x) + \Delta_2(x) &= 0\\ \Delta_3(x) &\le K_{\Delta_3} \in \Re_+\end{aligned}$$

Choosing the controls in (4.237) and (4.237) to be

$$\begin{aligned}v_1 &= -K_1 sgn[\xi_1^1]\\ u_2 &= -a_1\xi_2^2 - K_2 sgn[a_1\xi_1^2 + \xi_2^2]\\ K_1 &> K_{\Delta_1}\\ K_2 &> K_{\Delta_2}\\ a_1\xi_1^2 + \xi_2^2 & \text{ is a Hurwitz polynomial}\end{aligned}$$

Note that the matching conditions specified in I satisfy the strict matching assumption, while the conditions specified in II satisfy the relaxed matching assumptions. The stabilization objective is therefore realized for cases I and II.

Dynamic Extension Algorithm

We will now prescribe matching conditions for the system if instead of the zero dynamics algorithm, we had used the method of dynamic extension. As before, *we first develop a change of coordinates for the unperturbed system, and transform the perturbed system equations into the new coordinates.*

Differentiating the outputs y_1 and y_2 of (4.211) - (4.212), we get

$$h_1^1 = x_1 \tag{4.237}$$

$$h_1^2 = x_2 \tag{4.238}$$

$$\dot{y}_1 = L_f h_1^1 \tag{4.239}$$

$$= x_3 \sin x_1 + u_1 \tag{4.240}$$

$$\dot{y}_2 = L_f h_1^2 \tag{4.241}$$

$$= x_3 \cos^2 x_1 - u_1 \sin x_1 \tag{4.242}$$

Now using the methodology of dynamic-extension outlined in the previous section, we set the following dynamic extension

$$\dot{u}_1 = w_1 \tag{4.243}$$

and thus make u_1 an element of the extended state vector. The extended state vector is given by $\begin{bmatrix} x_1 & x_2 & x_3 & u_1 \end{bmatrix}^T$. Using this extended definition of the state vector, it is obvious that the inputs w and u_2 have not yet entered the equations (4.240) - (4.242). So we differentiate them once again to obtain,

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} \cos x_1[x_3 \sin x_1 + u_1] \\ -2\cos x_1 \sin x_1[x_3 \sin x_1 + u_1] - u_1 \cos x_1[x_3 \sin x_1 + u_1] \end{bmatrix} \tag{4.244}$$

$$+ \begin{bmatrix} 1 & \sin x_1 \\ -\sin x_1 & \cos^2 x_1 \end{bmatrix} \begin{bmatrix} w_1 \\ u_2 \end{bmatrix} \tag{4.245}$$

Note that

$$det \begin{bmatrix} 1 & \sin x_1 \\ -\sin x_1 & \cos^2 x_1 \end{bmatrix} = 1 \tag{4.246}$$

Therefore the decoupling matrix is invertible. Now consider the following change of coordinates,

$$\Phi : \begin{bmatrix} x \\ u_1 \end{bmatrix} \in \Re^4 \rightarrow \begin{bmatrix} \xi_1^1 \\ \xi_2^1 \\ \xi_1^2 \\ \xi_2^2 \end{bmatrix} \tag{4.247}$$

where the coordinates are given by

$$\xi_1^1 = x_1 \quad (4.248)$$
$$\xi_2^1 = x_3 \sin x_1 + u_1 \quad (4.249)$$
$$\xi_1^2 = x_2 \quad (4.250)$$
$$\xi_2^2 = x_3 \cos^2 x_1 - u_1 \sin x_1 \quad (4.251)$$

Note that

$$x_1 = \xi_1^1 \quad (4.252)$$
$$x_2 = \xi_1^2 \quad (4.253)$$
$$x_3 = \xi_2^1 \sin \xi_1^1 + \xi_2^2 \quad (4.254)$$
$$u_1 = \xi_2^1 \cos^2 \xi_1^1 - \xi_2^2 \sin \xi_1^1 \quad (4.255)$$

In the new coordinates, the system equations are

$$\dot{\xi}_1^1 = \xi_2^1 \quad (4.256)$$
$$\dot{\xi}_1^2 = \xi_2^2 \quad (4.257)$$
$$\begin{bmatrix} \dot{\xi}_2^1 \\ \dot{\xi}_2^1 \end{bmatrix} = \begin{bmatrix} -b_1(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) \\ -b_2(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) \end{bmatrix} + \begin{bmatrix} 1 & \sin x_1 \\ -\sin x_1 & \cos^2 x_1 \end{bmatrix} \begin{bmatrix} w_1 \\ u_2 \end{bmatrix} \quad (4.258)$$

Indeed choosing the controls to be

$$\begin{bmatrix} w_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & \sin x_1 \\ -\sin x_1 & \cos^2 x_1 \end{bmatrix}^{-1} \begin{bmatrix} -b_1(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) + v_1 \\ -b_2(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) + v_2 \end{bmatrix}$$

where

$$\begin{aligned} b_1(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) &= \cos \xi_1^1 [\sin \xi_1^1 + \xi_2^1 \cos^2 \xi_1^1 - \xi_2^2 \sin \xi_1^1] \\ b_2(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) &= -2 \cos \xi_1^1 \sin \xi_1^1 [[\xi_2^1 \sin \xi_1^1 + \xi_2^2] \sin \xi_1^1 + \xi_2^1 \cos^2 \xi_1^1 - \xi_2^2 \sin \xi_1^1] \\ &\quad - u_1 \cos \xi_1^1 [[\xi_2^1 \sin \xi_1^1 + \xi_2^2] \sin \xi_1^1 + \xi_2^1 \cos^2 \xi_1^1 - \xi_2^2 \sin \xi_1^1] \end{aligned}$$

and the control inputs are

$$v_1 = -a_1 \xi_2^1 - K_1 sgn[a_1 \xi_1^1 + \xi_2^1]$$

$$\begin{aligned} v_2 &= -a_1\xi_2^2 - K_2 sgn[a_2\xi_1^2 + \xi_2^2] \\ a_1\xi_1^1 + \xi_2^1 & \textit{ is a Hurwitz polynomial} \\ a_1\xi_1^2 + \xi_2^2 & \textit{ is a Hurwitz polynomial} \end{aligned}$$

Such a choice of controls yields a system of the form

$$\dot{\xi}_1^1 = \xi_2^1 \tag{4.259}$$

$$\dot{\xi}_2^1 = -a_1\xi_2^1 - K_1 sgn[a_1\xi_1^1 + \xi_2^1] \tag{4.260}$$

$$\dot{\xi}_1^2 = \xi_2^2 \tag{4.261}$$

$$\dot{\xi}_2^2 = -a_1\xi_2^2 - K_2 sgn[a_1\xi_1^2 + \xi_2^2] \tag{4.262}$$

$$\tag{4.263}$$

The system is thus stabilized.

We will now consider the effects of dynamic extension on the perturbed system equations.

In the new coordinates, the system equations are

$$\dot{\xi}_1^1 = \xi_2^1 + \Delta_1(x) \tag{4.264}$$

$$\dot{\xi}_1^2 = \xi_2^2 + \Delta_2(x) \tag{4.265}$$

$$\begin{bmatrix} \dot{\xi}_2^1 \\ \dot{\xi}_2^1 \end{bmatrix} = \begin{bmatrix} -b_1(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) \\ -b_2(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) \end{bmatrix} + \begin{bmatrix} 1 & \sin x_1 \\ -\sin x_1 & \cos^2 x_1 \end{bmatrix} \begin{bmatrix} w_1 \\ u_2 \end{bmatrix} \tag{4.266}$$

$$+ \begin{bmatrix} \frac{d\Delta_1(x)}{dt} + \Delta_3(x)\sin\xi_1^1 \\ \frac{d\Delta_2(x)}{dt} + \Delta_3(x)\cos^2\xi_1^1 \end{bmatrix} \tag{4.267}$$

Indeed choosing the controls to be

$$\begin{aligned} \begin{bmatrix} w_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} 1 & \sin x_1 \\ -\sin x_1 & \cos^2 x_1 \end{bmatrix}^{-1} \begin{bmatrix} -b_1(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) + v_1 \\ -b_2(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) + v_2 \end{bmatrix} \\ b_1(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) &= \cos\xi_1^1[\sin\xi_1^1 + \xi_2^1\cos^2\xi_1^1 - \xi_2^2\sin\xi_1^1] \\ b_2(\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2) &= -2\cos\xi_1^1\sin\xi_1^1[[\xi_2^1\sin\xi_1^1 + \xi_2^2]\sin\xi_1^1 + \xi_2^1\cos^2\xi_1^1 - \xi_2^2\sin\xi_1^1] \\ & \quad -u_1\cos\xi_1^1[[\xi_2^1\sin\xi_1^1 + \xi_2^2]\sin\xi_1^1 + \xi_2^1\cos^2\xi_1^1 - \xi_2^2\sin\xi_1^1] \\ v_1 &= -a_1\xi_2^1 - K_1 sgn[a_1\xi_1^1 + \xi_2^1] \end{aligned}$$

$$\begin{aligned} v_2 &= -a_1\xi_2^2 - K_2 sgn[a_2\xi_1^2 + \xi_2^2] \\ a_1\xi_1^1 + \xi_2^1 & \textit{ is a Hurwitz polynomial} \\ a_1\xi_1^2 + \xi_2^2 & \textit{ is a Hurwitz polynomial} \end{aligned}$$

Such a choice of controls yields a system of the form

$$\dot{\xi}_1^1 = \xi_2^1 + \Delta_1(x) \tag{4.268}$$

$$\dot{\xi}_2^1 = -a_1\xi_2^1 - K_1 sgn[a_1\xi_1^1 + \xi_2^1] + \frac{d\Delta_1(x)}{dt} + \Delta_3(x)\sin\xi_1^1 \tag{4.269}$$

$$\dot{\xi}_1^2 = \xi_2^2 + \Delta_2(x) \tag{4.270}$$

$$\dot{\xi}_2^2 = -a_1\xi_2^2 - K_2 sgn[a_1\xi_1^2 + \xi_2^2] + \frac{d\Delta_2(x)}{dt} + \Delta_3(x)\cos^2\xi_1^1 \tag{4.271}$$

Matching Conditions

Inspecting equations (4.268) - (4.271) reveals that the matching conditions sufficient to ensure stabilization are

- $\Delta_1(x) \equiv 0 \Rightarrow \frac{d\Delta_1(x)}{dt} \equiv 0$
- $\Delta_2(x) \equiv 0 \Rightarrow \frac{d\Delta_2(x)}{dt} \equiv 0$
- $\Delta_3(x) \leq K_{\Delta_3} \in \Re_+$

4.3 Comparing Inversion Strategies

From a designer's perspective, it becomes necessary to decide upon a strategy for controlling an invertible MIMO system with no vector relative degree.

- Stabilization involving the dynamic extension method allows for uncertainties only in the dynamic compensator that is being built. It is less tolerant of plant uncertainties in those equations that involve an extension of the state vector.
- The relaxed generalized matching conditions of the zero-dynamics algorithm permit disturbances that satsify an algebraic constraint that is a consequence

of the algorithm. The price exacted for such a relaxation, is that in addition to satisfying conditions for invertibility, stabilization objective requires the functions δjk^i to satisfy additional constraints. This algorithm however has the flavour of a MIMO extension of the usual SISO matching condition that requires the disturbances to lie in the span of the input vector field.

- The simplicity of the dynamic extension method is an attractive feature in the design of control laws. Instead of searching for elements of the left null-space of the decoupling matrix, as with the zero-dynamics algorithm, the method extends the dimension of the state-space, and attempts to eliminate the singularity of the system by embedding it in a higher dimensional space. This is a conceptually elegant technique.

- There are however, systems that violate the conditions of the theorems, which can still be stabilized. This is by virtue of the fact that the conditions of the theorems are merely sufficient conditions, and provide ample scope for improvement.

4.4 Closure

We presented the generalized matching conditions for SISO systems with perturbed zerodynamics and MIMO nonsingular and MIMO singular systems. It is to be noted that these are only sufficient conditions, and therefore are bound to be conservative. Aside from helping to understand the classes of tolerable perturbations, the matching conditions are useful when a choice is to be made between two competing algorithms for control. It is prudent to choose a control methodology whose matching assumptions are less restrictive, or more suited for a class of applications.

Chapter 5

Foundations Of Sliding Mode Control Theory

In this chapter we present a brief collection of facts and results that form the basis of sliding mode control theory. We will present theorems concerning the existence of solutions to differential equations with discontinuous righthand sides. We present the proofs of existence, first for continuous righthand sides, and then relax the continuity requirements to illustrate how the proof techniques change. We introduce differential inclusions, and illustrate their use in proving existence for differential equations with discontinuous righthand sides.

Finally we introduce the notion of a sliding mode and show that it is just a special case of the presented theory. We will also demonstrate the construction of simple sliding mode control laws for SISO and MIMO linear, and linearizable systems. Finally we will show how the theory of input-output linearization [19] can be seen as a way of constructing nonlinear sliding surfaces.

5.1 Mathematical Preliminaries

Qualitatively, sliding mode control theory involves dynamical systems controlled by control inputs that are defined almost everywhere, excepting possibly on sets of zero measure. Such discontinuous control inputs are designed to render a subspace of the state-space of the dynamical system, attractive and invariant. The subspace rendered invariant by control, is such that, trajectories evolving on this invariant subspace achieve the required control objective. We will say more about this later. We will now present some examples intended to stimulate interest in this topic, and to illustrate the qualitative principles behind sliding mode control. We will say more about these examples later.

Example 5.1.1 *Linear Invariant Sliding Surfaces*

Consider the simple double-integrator given by

$$\dot{x}_1 = x_2 \tag{5.1}$$

$$\dot{x}_2 = u(t) \tag{5.2}$$

where $x \in \Re^2$, *and the control* $u(t) : \Re_+ \rightarrow \Re$. *Choosing*

$$u = -ax_2 - Ksgn[ax_1 + x_2] \tag{5.3}$$

where the function $sgn[x] : \Re^2 - \{0\} \rightarrow [-1, 1]$ *is undefined at the origin, renders the* 1 *dimensional subspace*

$$ax_1 + x_2 = 0 \tag{5.4}$$

attractive and invariant. That is to say, trajectories commencing from arbitrary initial conditions reach this subspace (actually in finite time), and that once they reach this subspace, they continue to remain on this subspace. This is an example of a subspace rendered attractive and invariant through control.

Example 5.1.2 *Finite Time Control With Saturation*

Consider the double integrator system again, given by the equations

$$\dot{x}_1 = x_2 \tag{5.5}$$

$$\dot{x}_2 = u(t) \tag{5.6}$$

where $x \in \Re^2$, *and the control* $u(t) : \Re_+ \rightarrow \Re$. *Consider the following choice of control*

$$u(t) = -sgn[x_1 + \frac{x_2|x_2|}{2}] \;\; if\, |x_1 + \tfrac{x_2|x_2|}{2}| > 0 \tag{5.7}$$

$$= -sgn[x_2] \;\; if\, |x_1 + \tfrac{x_2|x_2|}{2}| = 0 \tag{5.8}$$

It is easy to show that such a choice of control renders the 1 *dimensional subspace*

$$x_1 + \frac{x_2|x_2|}{2} = 0 \tag{5.9}$$

attractive and invariant. Indeed, trajectories commencing from aribitrary initial conditions reach this subspace in finite time, *and slide on this surface to reach the origin in finite time.*

Example 5.1.3 *Winding Algorithm*

Consider the double integrator system given by the equations

$$\dot{x}_1 = x_2 \tag{5.10}$$

$$\dot{x}_2 = u(t) \tag{5.11}$$

where $x \in \Re^2$*, and the control* $u(t) : \Re_+ \rightarrow \Re$*. Consider the following choice of control*

$$u(t) = -k_1 sgn[x_1] - k_2 sgn[x_2] \tag{5.12}$$

$$k_1 > k_2 > 0 \tag{5.13}$$

$$\tag{5.14}$$

Such a choice of discontinuous control guarantees that the 1 *dimensional surface*

$$x_2 = 0 \tag{5.15}$$

is invariant and attractive. Indeed the surface $x_2 = 0$ *is attained in finite time, from arbitrary initial conditions, and the states of the system slide to the origin in* finite time, *along this surface.*

The examples are interesting in that they admit a control input that involves the function $sgn[x] : \Re \rightarrow [-1, 1]$, which has the following attributes.

$$sgn[\cdot] = \frac{[\cdot]}{||[\cdot]||} \text{ if } ||[\cdot]|| > 0 \tag{5.16}$$

$$\text{undefined if } ||[\cdot]|| = 0 \tag{5.17}$$

The aforementioned examples are instances of using discontinuous control for the purpose of stabilization of systems in the controllable canonical form.

Comment 5.1.1 It is quite possible to define upper and lower semi-continuous versions of the function $sgn[x] : \Re^n \rightarrow [-1, 1]$. *We will use such versions later in constructing differential inclusions that model our system. We will say more about this later.*

We wish to formalize, the intutive notion that for dynamical systems involving discontinuous control inputs, if the control inputs are undefined on sets of measure zero, they are not too pathological and we might expect the dynamical system to exhibit the desired behaviour in a general sense. There are a number of technical issues that have to be resolved before we could quantify this intutive notion. The major issues that have to be dealt with are really the definition of what we might call an acceptable solution to differential equations whose righthand sides may not be defined. As the righthand sides of some of the differential equations we consider may be undefined at some points, we no longer define the trajectories of the dynamical system using the Riemann integral, (as Riemann integration depends on the considered function being defined everywhere on its domain), but resort to measure theory and Lesbegue integration. (which is more based on properties of functions based on the properties of the projection of its graph onto its range) To do so, we need to invoke simple notions about measurable functions. The usual technique in the theory of differential equations is to construct approximate solutions, and show properties of the solution using the properties of the constructed approximations. However, when the righthandside of a differential equation is undefined on sets of zero measure, it becomes critical to understand what the notion of an approximate solution might very well be on such sets of zero measure where the vector field is not defined. Finally, given certain approximate solutions to these differential equations with discontinuous righthand sides, we wish to formalize the notion of convergence of these functions, in measure, to some limit function. We then pass to the limit and consider properties of the limit functions.

5.2 Existence Of Solutions To Differential Equations

In this section we present basic results for the local existence of solutions of differential equations with discontinuous righthand sides. We define a sliding mode, and present conditions for the existence of a sliding mode. We then present briefly the development of the sliding mode control law, and the various regularizations of it.

We will now state without proof the following two important results from analysis that we will need.

Arzela-Ascoli Theorem:

Let K be a compact subset of R^p and let F be a collection of functions which are continuous on K and have values in R^q. The following properties are equivalent.

1. *The family F is uniformly bounded and equicontinuous on K.*

2. *Every sequence from F has a subsequence which is uniformly convergent on K.*

The theorem allows us to define a sequence of approximate solutions of a differential equation, and guarantees convergence of the approximate solutions to a limit function of the sequence is equicontinuous and uniformly bounded.

Filippov Convergence Lemma:

Given a differential inclusion of the form $\dot{x} = \mathcal{F}(x,t)$. If the inclusion $\mathcal{F}(x,t)$ is closed, bounded, convex, and uppersemicontinuous, the limit of any uniformly convergent sequence of approximate solutions of the differential inclusion, is also a solution of this inclusion, in the domain of convergence.

That the limit function satisfies the differential inclusion, is the main reason for invoking the lemma.

As a prelude we compare and classify ordinary differential equations based on the nature of the right hand sides. Consider a differential equation of the following form.

$$
\begin{aligned}
\dot{x} &= f(x,t) && (5.18)\\
x(0) &= x_0 && (5.19)\\
& x \in \Re^n \; t \in \Re_+ && (5.20)\\
& f(x,t) : \Re^n \times \Re_+ \to \Re^n && (5.21)\\
& && (5.22)
\end{aligned}
$$

The smoothness assumptions on $f(x,t)$ determine the kind of differential system referred to by (5.18).

The three major kinds of differential systems are

1. **Cauchy Differential Systems:** In the domain D of the (x,t) space,
   - $f(x,t) : \Re^n \times \Re_+ \to \Re^n$ is continuous in x.
   - $f(x,t) : \Re^n \times \Re_+ \to \Re^n$ is continuous in t.
2. **Caratheodory Differential Systems:** In the domain D of the (x,t) space,
   - $f(x,t) : \Re^n \times \Re_+ \to \Re^n$ is continuous in x.
   - $f(x,t) : \Re^n \times \Re_+ \to \Re^n$ is discontinuous in t on sets of zero measure.
3. **Filippov Differential Systems:** In the domain D of the (x,t) space,
   - $f(x,t) : \Re^n \times \Re_+ \to \Re^n$ is discontinuous in x and t on sets of zero measure.

We will henceforth refer to the assumptions made on the vector field $f(x,t) : \Re^n \times \Re_+ \to \Re^n$ by the assigned system names. For each kind of differential system, we define the solution concept, and present further assumptions necessary to ensure the local existence of the defined solution concept in a domain D of the (x,t) space.

5.2.1 Cauchy Differential Systems

In this subsection we will first state the Cauchy problem and proceed to define the relevant solution concept. We will then state the equivalent integral equation to the Cauchy problem. We will then derive conditions for the existence of a solution to the Cauchy problem.

$$
\begin{aligned}
\dot{x} &= f(x,t) && (5.23)\\
x(0) &= x_0 && (5.24)\\
& x \in \Re^n \textit{ and } t \in \Re_+ && (5.25)\\
& f(x,t) : \Re^n \times \Re_+ \to \Re^n && (5.26)\\
& && (5.27)
\end{aligned}
$$

where, in a domain D of the (x,t) space

- $f(x,t)) : \Re^n \times \Re_+ \to \Re^n$ is continuous in x.
- $f(x,t)) : \Re^n \times \Re_+ \to \Re^n$ is continuous in t.

Cauchy Solution Concept: *A continuous vector function $s(t) : \Re_+ \to \Re^n$ is a solution of the Cauchy problem (5.23) if for any $(x,t) \in D$*

$$\frac{ds(t)}{dt}|_{t=t^*} = f(s(t^*), t^*) \qquad (5.28)$$

and $s(0) = x_0$

We will now show the equivalence between solutions of the differential equation (5.23) and the solutions of an integral equation, through the following proposition.

Proposition 5.1 *We may state the conditions for equivalence of the Integral and Differential forms of the Cauchy problem as follows.*

Given *(G1) A Cauchy differential system of the form (5.23).*

If *(I1) A continuous function $s(t) : \Re_+ \to \Re^n$ is a solution of the vector integral equation*

$$s(t) = x_0 + \int_0^t f(s(\tau), \tau) d\tau \qquad (5.29)$$

where integration is in the sense of Riemann.

Then *(T1)* *The continuous function $s(t) : \Re_+ \rightarrow \Re^n$ given by (5.29) is also a solution of the Cauchy differential system given by (5.23).*

Proof:

♠ ⊳ The proof is trivially obvious using the Fundamental Theorem Of Calculus. Indeed as $s(0) = x_0$, and $\frac{ds(t)}{dt} = f(s(t),t)$, it is clear that $s(t)$ is a solution of the Cauchy differential system. Conversely, for all continuously differentiable functions $s(t) : \Re_+ \rightarrow \Re^n$, the Fundamental Theorem Of Calculus shows that $s(t) = x(0) + \int_0^t \frac{ds(\tau)}{d\tau} d\tau$. If $s(t)$ satisfies (5.23) then indeed $s(t) = x_0 + \int_0^t f(s(\tau),\tau)d\tau$. We obtain the integral equation (5.29). ⊲♠

We will now prove the Peano-Existence Theorem for the *local* existence of solutions to Cauchy differential systems.

Theorem 5.1 *Local Existence Of Solutions To Cauchy Differential Systems*

Given *(G1)* *A Cauchy differential system of the form (5.23).*

If *(I1)* *The domain D of continuity of $f(x,t)$ is specified for $\{(x,t) \in \Re^n \times \Re_+ : ||x - x_0|| \leq K_x$ and $t \leq K_t\}$ where $K_x \in \Re_+$ and $K_t \in \Re_+$.*

(I2) $||f(x,t)|| \leq K_f \; \forall \, (x,t) \in D$ *where* $K_f \in \Re_+$.

Then *(T1)* *The Cauchy differential system (5.23) has atleast one solution $s(t) : \Re_+ \rightarrow \Re^n$ for $t \leq min(K_t, \frac{K_x}{K_f})$ satisfying the initial condition $s(0) = x_0$.*

Proof:

♠ ⊳ The method of proof will be used repeatedly in the solutions of the Caratheodory and Filippov differential systems also.

The proof consists of the following steps.

Step 1. We will define a sequence of functions that form approximate solutions of the integral equation equivalent to the Cauchy differential system.

Step 2. We will show that this sequence is uniformly bounded and equicontinuous.

Step 3. Invoking the Arzela-Ascoli theorem, we will show that there exists a uniformly convergent subsequence.

Step 4. We will now pass to the limit of this subsequence, and show that the limit function satisfies the integral equation, and hence the Cauchy differential system.

Step 1.

Set $T_{min} = min(K_t, \frac{K_x}{K_f})$. For $t \in [0, T_{min}]$, we construct a sequence of functions $s^i(t), i = 1, 2, \ldots$ in the following manner.

$$\begin{aligned} s^i(t) &= x_0 \;\forall\, 0 \leq t \leq \frac{T_{min}}{i} && (5.30) \\ &= x_0 + \int_0^{t-\frac{T_{min}}{i}} f(s^i(\tau), \tau) d\tau \;\; \forall \frac{T_{min}}{i} \leq t \leq T_{min} && (5.31) \\ & \quad i = 1, 2, \ldots && (5.32) \end{aligned}$$

The geometric interpretation of this formula is the method of constructing *Euler broken lines*. Let us clarify by evaluating $s^i(t)$ for $i = 1, 2$. For $i = 1$, we get

$$\begin{aligned} s^1(t) &= x_0 \;\forall\, 0 \leq t \leq \frac{T_{min}}{1} && (5.33) \\ &= x_0 + \int_0^{t-\frac{T_{min}}{1}} f(s^1(\tau), \tau) d\tau \;\; \forall \frac{T_{min}}{1} \leq t \leq T_{min} && (5.34) \end{aligned}$$

Which is the initial condition itself over the entire interval. This indeed is the crudest approximate solution, satisfying the initial condition. When $i = 2$, we get,

$$\begin{aligned} s^2(t) &= x_0 \;\forall\, 0 \leq t \leq \frac{T_{min}}{2} && (5.35) \\ &= x_0 + \int_0^{t-\frac{T_{min}}{2}} f(s^2(\tau), \tau) d\tau \;\; \forall \frac{T_{min}}{2} \leq t \leq T_{min} && (5.36) \end{aligned}$$

It follows that the functions $s^i(t)$ are defined for $0 \leq t \leq T_{min}$.

Step 2:

We show uniform boundedness as follows. For any integer $i \in Z_+$, we have,

$$\begin{aligned} ||s^i(t)|| &\leq ||x_0|| + ||\int_0^{T_{min}} f(s^i(\tau), \tau) d\tau|| && (5.37) \\ &\leq ||x_0|| + \int_0^{T_{min}} ||f(s^i(\tau), \tau)|| d\tau && (5.38) \\ &\leq ||x_0|| + \int_0^{T_{min}} K_f d\tau && (5.39) \\ &\leq ||x_0|| + T_{min} K_f && (5.40) \end{aligned}$$

Hence the sequence of functions $s^i(t)$ $i = 1, 2, \ldots$ is uniformly bounded.

We show equicontinuity as follows. For any $i \in Z_+$, for all $t_1 \in \Re_+$ and $t_2 \in \Re_+$ such that $\frac{T_{min}}{i} < t_1 < t_2 \leq T_{min}$ we have

$$\begin{aligned}
s^i(t_1) &= x_0 + \int_0^{t_1 - \frac{T_{min}}{i}} f(s^i(\tau), \tau) d\tau && (5.41) \\
s^i(t_2) &= x_0 + \int_0^{t_2 - \frac{T_{min}}{i}} f(s^i(\tau), \tau) d\tau && (5.42) \\
\|s^i(t_2) - s^i(t_1)\| &\leq \int_{t_1 - \frac{T_{min}}{i}}^{t_2 - \frac{T_{min}}{i}} \|f(s^i(\tau), \tau)\| d\tau && (5.43) \\
&\leq K_f |t_2 - t_1| && (5.44)
\end{aligned}$$

Therefore, given any $\epsilon \in \Re_+$, it is possible to choose a $\delta \leq \frac{\epsilon}{K_f}$ so that for all t_2, t_1 such that $\|t_2 - t_1\| \leq \delta$, we can ensure, by (5.44) that $\|s^i(t_2) - s^i(t_1)\| \leq \epsilon$. Equicontinuity is therefore shown for the family of functions $s^i(t)$ $i = 1, 2, \ldots$

Step 3:

Having shown the uniform boundedness, and equicontinuity of the sequence of functions $s^i(t)$ $i = 1, 2, \ldots$ on the closed interval $[0, T_{min}]$, we invoke the Arzela-Ascoli Theorem to guarantee the existence of a subsequence $s^{i_k}(t)$ $k = 1, 2, \ldots$ that is uniformly convergent in the interval $[0, T_{min}]$.

We therefore claim that the sequence $s^{i_k}(t)$ $k = 1, 2, \ldots$ converges to a continuous function $s(t)$ as $i_k \to \infty$.

Step 4:

Having passed to the limit, we now verify that the limit function $s(t)$ satisfies the integral equation (5.29)

We rewrite equation (5.31) in the form

$$\begin{aligned}
s^{i_k}(t) &= x_0 \;\forall\; 0 \leq t \leq \frac{T_{min}}{i_k} && (5.45) \\
s^{i_k}(t) &= x_0 + \int_0^{t - \frac{T_{min}}{i_k}} f(s^{i_k}(\tau), \tau) d\tau \;\forall\; \frac{T_{min}}{i_k} \leq t \leq T_{min} && (5.46) \\
&= x_0 + \int_0^t f(s^{i_k}(\tau), \tau) d\tau - \int_0^{\frac{T_{min}}{i_k}} f(s^{i_k}(\tau), \tau) d\tau && (5.47)
\end{aligned}$$

Now as $i_k \to \infty$, $\int_0^t f(s^{i_k}(\tau),\tau)d\tau \to \int_0^t f(s(\tau),\tau)d\tau$ because $f(x,t)$ is uniformly continuous. Furthermore the last term of (5.47) tends to zero because,

$$\begin{aligned} \|\int_0^{\frac{T_{min}}{i_k}} f(s^{i_k}(\tau),\tau)d\tau\| &\leq \int_0^{\frac{T_{min}}{i_k}} K_f d\tau && (5.48) \\ &\leq K_f \frac{T_{min}}{i_k} && (5.49) \\ &\to 0 \; as \; i_k \to \infty && (5.50) \end{aligned}$$

We have thus shown the local existense of a solution to the Cauchy differential system satisfying the specified initial condition. ◁♠

5.2.2 Caratheodory Differential Systems

We will now develop the theory in a manner quite analogous to the development for Cauchy differential systems. We will formulate the problem, and prescribe the solution concept. We will then state the equivalent integral equation, and proceed to derive conditions for existence of solutions.

We will now state and prove the existense theorem for Caratheodory Differential Systems of the following form.

$$\begin{aligned} \dot{x} &= f(x,t) && (5.51) \\ x(t=0) &= x_0 && (5.52) \\ & x \in \Re^n \; t \in \Re_+ && (5.53) \\ & f(x,t) : \Re^n \times \Re_+ \to \Re^n && (5.54) \\ & && (5.55) \end{aligned}$$

where In the domain D of the (x,t) space,

- $f(x,t) : \Re^n \times \Re_+ \to \Re^n$ is continuous in $x \in D$.
- $f(x,t) : \Re^n \times \Re_+ \to \Re^n$ is discontinuous in $t \in D$ on sets of zero measure.
- $f(x,t) : \Re^n \times \Re_+ \to \Re^n$ is measurable in $t \in D$ for each $x \in D$

- $||f(x,t)|| \leq K_f(t) \; \forall \, (x,t) \in D$ where $K_f(t) : \Re_+ \rightarrow \Re$ is summable.

The aforementioned conditions on the function $f(x,t)$ are also called Caratheodory conditions.

We now formally define the solution of a Caratheodory differential system.

Caratheodory Solution Concept: *An absolutely continuous vector function $s(t) : \Re_+ \rightarrow \Re^n$ is defined to be a Caratheodory solution of the Caratheodory differential system (5.51) if for almost all $t \in D$,*

$$\frac{ds}{dt}|_{t=t^*} = f(s(t^*), t^*) \tag{5.56}$$

Indeed, we require the Caratheodory solution to satisfy equations (5.51) only in the domain of continuity the function $f(x,t)$. Furthermore, the solution concept requires that the Caratheodory solution $s(t) : \Re_+ \rightarrow \Re^n$ be an absolutely continuous function, instead of merely being continuous. Absolute continuity is needed to ensure the equivalence between the differential and integral formulations. Also absolute continuity eliminates pathologies as ones proposed by Vitali [1], [21].

We now bring out the equivalence between Caratheodory solutions of the differential system and equivalent integral equation formulations.

Proposition 5.2 *We may state the conditions for equivalence of the Integral and Differential forms of the Caratheodory problem as follows.*

Given *(G1) A Caratheodory differential system of the form (5.51).*

If *(I1) An* absolutely continuous function $s(t) : \Re_+ \rightarrow \Re^n$ *is a solution of the vector integral equation*

$$s(t) = x_0 + \int_0^t f(s(\tau), \tau) d\tau \tag{5.57}$$

where integration is in the sense of Lesbegue.

Then *(T1) The absolutely continuous function $s(t) : \Re_+ \rightarrow \Re^n$ given by (5.57) is also a Caratheodory solution of the Caratheodory differential system given by (5.51).*

Proof:

♠ ▷ The proof is a virtual repetition of the proof provided earlier in this chapter for Cauchy differential systems, and will not be repeated here. ◁♠

We now present the conditions for existense (local) of Caratheodory solutions for differential systems of the form (5.51).

Theorem 5.2 *Local Existence Of Solutions To Caratheodory Differential Systems*

Given *(G1) A Caratheodory differential system of the form (5.51).*

If *(I1) The domain D where $f(x,t)$ is specified for almost all t*
$(x,t) \in \Re^n \times \Re_+ : ||x - x_0|| \leq K_x$ *and* $t \leq K_t$

(I2) $f(x,t) : \Re^n \times \Re_+ \rightarrow \Re^n$ *is measurable in* $t \in \Re_+$ *for all* $x \in \Re^n$

(I3) $||f(x,t)|| \leq K_f(t) \; \forall \, (x,t) \in D$ *where* $K_f(t) : \Re_+ \rightarrow \Re$ *is summable.*

(I4) There exists $0 < K_{tmin} \leq K_t$ *such that*

$$\int_0^{K_{tmin}} K_f(\tau) d\tau \leq K_x \tag{5.58}$$

Then *(T1) The differential system (5.51) has atleast one Caratheodory solution* $s(t) : \Re_+ \rightarrow \Re^n$ *for* $t \leq min(K_t, K_{tmin})$ *satisfying the initial condition* $s(0) = x_0$.

Proof:

The proof consists of the following steps.

Step 1. We will define a sequence of functions that form approximate solutions of the integral equation equivalent to the Caratheodory differential system. Note that the solutions are Caratheodory solutions.

Step 2. We will show that this sequence of proposed Caratheodory solutions is both uniformly bounded and equicontinuous.

Step 3. Invoking the Arzela-Ascoli theorem, we will show that there exists a uniformly convergent subsequence of Caratheodory solutions whose limit is the Caratheodory solution to the Caratheodory differential system.

Step 4. We will now pass to the limit of this subsequence, and show that the limit function satisfies the integral equation, and hence the Caratheodory differential system.

First we note two properties that follow from the assumption $(I2)$ and $(I3)$ made on the vector function $f(x,t)$. We will use these properties later in the proof. The properties follow from elementary results in real analysis.

Property 1.

Given that $f(x,t) : \Re^n \times \Re_+ \to \Re^n$ satisfies assumptions $(I1)$, $(I2)$ and $(I3)$, and that the function $s(t) : \Re_+ \to \Re^n$ is measurable for all $0 \leq t \leq K_t$, then the composite function $f(s(t),t) : \Re^n \times \Re_+ \to \Re^n$ is summable. (Proof is by invoking the implicitness result in measure theory.)

Property 2.

The function $\int_0^t K_f(\tau)d\tau : \Re_+ \to \Re$ is continuous on the closed interval $[0, K_t]$ and is therefore uniformly continuous. (Proof is by invoking the result that continuous functions on compact metric spaces are uniformly continuous.)

Step 1.

Set $T_{min} = min(K_t, K_{tmin})$. For $t \in [0, T_{min}]$, we construct a sequence of functions $s^i(t), i = 1, 2, \ldots$ in the following manner.

$$\begin{aligned} s^i(t) &= x_0 \;\forall\, 0 \leq t \leq \frac{T_{min}}{i} && (5.59)\\ &= x_0 + \int_0^{t-\frac{T_{min}}{i}} f(s^i(\tau),\tau)d\tau \;\; \forall\, \frac{T_{min}}{i} \leq t \leq T_{min} && (5.60)\\ & \quad i = 1, 2, \ldots && (5.61)\end{aligned}$$

Using assumption $(I2)$, we note that the Lesbegue integral in (5.60) has meaning.

It follows that the functions $s^i(t)$ $i = 1, 2, \ldots$ are defined for $0 \leq t \leq T_{min}$.

Step 2:

We show uniform boundedness as follows. For any integer $i \in Z_+$, we have,

$$\begin{aligned} ||s^i(t)|| &\leq ||x_0|| + ||\int_0^{T_{min}} f(s^i(\tau),\tau)d\tau|| && (5.62)\\ &\leq ||x_0|| + ||\int_0^{T_{min}} K_f(\tau)d\tau|| && (5.63)\end{aligned}$$

$$\leq \quad ||x_0|| + K_x \tag{5.64}$$

Here we have made use of assumption ($I3$) on the summability of the function $\int_0^{T_{min}} ||K_f(\tau)||d\tau$ with the assumption $\int_0^{T_{min}} ||K_f(\tau)||d\tau \leq K_{K_f} \in \Re_+$

Hence the sequence of functions $s^i(t)$ $i = 1, 2, \ldots$ is uniformly bounded.

We show equicontinuity as follows. For any $i \in Z_+$, for all $t_1 \in \Re_+$ and $t_2 \in \Re_+$ such that $\frac{T_{min}}{i} < t_1 < t_2 \leq T_{min}$ we have

$$s^i(t_1) \quad = \quad x_0 + \int_0^{t_1 - \frac{T_{min}}{i}} f(s^i(\tau), \tau)d\tau \tag{5.65}$$

$$s^i(t_2) \quad = \quad x_0 + \int_0^{t_2 - \frac{T_{min}}{i}} f(s^i(\tau), \tau)d\tau \tag{5.66}$$

$$||s^i(t_2) - s^i(t_1)|| \quad \leq \quad ||\int_{t_1 - \frac{T_{min}}{i}}^{t_2 - \frac{T_{min}}{i}} K_f(\tau)d\tau|| \tag{5.67}$$

Therefore, given any $\epsilon \in \Re_+$, it must be possible to choose a δ so that for all t_2, t_1 such that $||t_2 - t_1|| \leq \delta$, we can ensure, by (5.67) that $||s^i(t_2) - s^i(t_1)|| \leq \epsilon$. By invoking Property 2, we see from (5.67) that this is indeed the case. Equicontinuity is therefore shown for the family of functions $s^i(t)$ $i = 1, 2, \ldots$

Step 3:

Having shown the uniform boundedness, and equicontinuity of the sequence of functions $s^i(t)$ $i = 1, 2, \ldots$ on the closed interval $[0, T_{min}]$, we invoke the Arzela-Ascoli Theorem to guarantee the existence of a subsequence $s^{i_k}(t)$ $k = 1, 2, \ldots$ that is uniformly convergent in the interval $[0, T_{min}]$.

We therefore claim that the sequence $s^{i_k}(t)$ $k = 1, 2, \ldots$ converges to an absolutely continuous function $s(t)$ as $i_k \to \infty$. Here we use an elementary result from analysis that the limit of a convergent sequence of absolutely continuous functions is also absolutely continuous.

Step 4:

Having passed to the limit, we now verify that the limit function $s(t)$ satisfies the integral equation (5.57)

We rewrite equation (5.60) in the form

$$s^{i_k}(t) \quad = \quad x_0 \; \forall \; 0 \leq t \leq \frac{T_{min}}{i_k} \tag{5.68}$$

$$
\begin{aligned}
s^{i_k}(t) &= x_0 + \int_0^{t-\frac{T_{min}}{i_k}} f(s^{i_k}(\tau),\tau)d\tau \;\; \forall \frac{T_{min}}{i_k} \leq t \leq T_{min} && (5.69)\\
&= x_0 + \int_0^t f(s^{i_k}(\tau),\tau)d\tau - \int_0^{\frac{T_{min}}{i_k}} f(s^{i_k}(\tau),\tau)d\tau && (5.70)
\end{aligned}
$$

Now as $i_k \to \infty$, $\int_0^t f(s^{i_k}(\tau),\tau)d\tau \to \int_0^t f(s(\tau),\tau)d\tau$ because $f(x,t)$ is continuous in x. Furthermore the last term of (5.70) tends to zero because,

$$
\begin{aligned}
\left\|\int_0^{\frac{T_{min}}{i_k}} f(s^{i_k}(\tau),\tau)d\tau\right\| &\leq \int_0^{\frac{T_{min}}{i_k}} K_f(\tau)d\tau && (5.71)\\
&\to 0 \; as \; i_k \to \infty && (5.72)
\end{aligned}
$$

We have thus shown the local existense of a Caratheodory solution to the differential system satisfying the specified initial condition. ◁♠

5.2.3 Filippov Differential Systems

In this section we will develop solution concepts and conditions for existence of solutions to differential equations with discontinuous right hand sides. Such equations represent physical systems governed by switching behaviours.

Our method of analysis would be the following kind. Instead of describing solutions for differential equations with discontinuous right hand sides, we will consider differential inclusion which include the said discontinuity as a special case. We will then describe generalized solution concepts for these differential inclusions, and will present conditions for existense of generalized solutions to differential inclusions.

We consider Filippov Differential Systems of the following form.

$$
\begin{aligned}
\dot{x} &= f(x,t) && (5.73)\\
x(t=0) &= x_0 && (5.74)\\
& x \in \Re^n \; t \in \Re_+ && (5.75)\\
& f(x,t) : \Re^n \times \Re_+ \to \Re^n && (5.76)\\
& && (5.77)
\end{aligned}
$$

where In the domain D of the (x,t) space,

- $f(x,t) : \Re^n \times \Re_+ \rightarrow \Re^n$ is discontinuous in $x \in D$ on sets of zero measure.
- $f(x,t) : \Re^n \times \Re_+ \rightarrow \Re^n$ is discontinuous in $t \in D$ on sets of zero measure.
- $f(x,t) : \Re^n \times \Re_+ \rightarrow \Re^n$ is measurable in $t \in D$ for each $x \in D$
- $\|f(x,t)\| \leq K_f(t) \; \forall \, (x,t) \; \in \; D$ where $K_f(t) : \Re_+ \rightarrow \Re$ is summable.

The aforementioned conditions on the function $f(x,t)\Re^n \times \Re_+ \rightarrow \Re^n$ are also called Filippov conditions.

We will now consider a differential inclusion that adequately describes the discontinuous system. Though the function $f(x,t) : \Re^n \times \Re_+ \rightarrow \Re^n$ of equation (5.73) is undefined on sets of zero measure, we choose instead to represent the function $f(x,t) : \Re^n \times \Re_+ \rightarrow \Re^n$ by a set valued map on such sets of zero measure. That is to say, if for instance the function is undefined at a point $(x^*, t^*) \in \Re^n \times \Re_+$, we formally define the function to be set valued at the point (x^*, t^*). Indeed, depending on the set-value attributed to the function at the point (x^*, t^*), we may show the existense of certain generalized solutions to the system (5.73). To construct the inclusion intelligently, we need some knowledge about the behaviour of the function $f(x,t) : \Re^n \times \Re_+ \rightarrow \Re^n$, in a neighbourhood of the point of discontinuity. To justify the use of the inclusion, we must show that given any arbitrary $\epsilon \in \Re_+$, there exists a small enough $\delta \in \Re_+$ neighbourhood of the point of discontinuity, such that, the trajectories of the differential equation in this δ neighbourhood are ϵ close to the solutions of the differential inclusion. Furthermore, as the size of the set containing the point of discontinuity shrinks to zero, that is $\delta \rightarrow 0$, the solutions of the differential equation tend to the solution of the differential inclusion. That is to say, that the trajectories of the differential equation weakly converge to the solution of the differential inclusion. We will say more about this later.

Indeed, given a discontinuous differential system of the form (5.73), henceonward we will replace it (whenever possible) with a *differential inclusion* of the following form.

$$\dot{x} \;\in\; \mathcal{F}(x,t) \tag{5.78}$$

$$
\begin{aligned}
x(t=0) &= x_0 && (5.79)\\
&x \in \Re^n \; t \in \Re_+ && (5.80)\\
&\mathcal{F}(x,t) : \Re^n \times \Re_+ \rightarrow S \in \Re^n && (5.81)\\
& && (5.82)
\end{aligned}
$$

where S is a set in $\Re^n$ and in the domain D of the (x, t) space,

- the set valued map $\mathcal{F}(x,t) : \Re^n \times \Re_+ \rightarrow S \in \Re^n$ is *upper semi-continuous.*
- The *Range*$[\mathcal{F}(x,t)] \in \Re^n$ is *compact and convex.*

Comment 5.2.1 The definition of the inclusion $\mathcal{F}(x,t)$ is such that it is single-valued in the domain of continuity of the function $f(x,t)$, indeed it is equal to $f(x,t)$ in the domains of continuity, but is set valued in the domains of discontinuity of $f(x,t)$.

Comment 5.2.2 It is important to note the properties of the set $S \in \Re^n$ which will be used for the existence of solutions.

We now formally define the solution of a Filippov differential system.

Filippov Solution Concept: *An absolutely continuous vector function $s(t) : \Re_+ \rightarrow \Re^n$ is defined to be a Filippov solution of the Filippov differential system (5.78) if for almost all $t \in D$,*

$$\frac{ds}{dt}|_{t=t^*} \in \mathcal{F}(s(t^*), t^*) \tag{5.83}$$

where

$$
\begin{aligned}
\mathcal{F}(s(t^*), t^*) &= f(s(t^*), t^*) \;\; \textit{in the domains of continuity} && (5.84)\\
\mathcal{F}(s(t^*), t^*) &= \bigcap_{\delta>0} \bigcap_{\mu N=0} \textit{convex-hull}(B(x,\delta) - N, t) && (5.85)
\end{aligned}
$$

and $\bigcap_{\mu N=0}$ denotes the intersection over all sets N of Lebesgue measure zero where the functionf(x,t) is either undefined or discontinuous.

Comment 5.2.3 In the domains of continuity of $f(x,t) : \Re^n_+ \rightarrow \Re^n$, the inclusion $\mathcal{F}(x,t)$ is the same as the function and therefore the set operation $\in$ in equation (6.165) must be replaced with the strict equality $=$

The utility of the Filippov solution concept is that it is indeed the limit of solutions to (5.73) averaged over neighbourhoods of diminishing size. The key point to be understood is that the Filippov trajectories of the discontinuous system remain close to the true trajectories.

As was evidenced earlier in the proofs of the Cauchy and Caratheodory systems, the method of constructing solutions to differential equations begins by constructing sequences of approximating solutions, and then ensuring that the approximations converge in some sense. Indeed, it now becomes important to formalize the notion of what an acceptable definition of an *approximate Filippov solution* might be. We now introduce some additional notation to help facilitate the definition.

We first denote the closed δ-neighborhood of a set, by M^{δ}.

The δ-neighborhood of a function is a set valued map, associating to each point in the projection of the graph to the range, a closed set of size δ containing the point. Formally stated, given a function $f(x) : \Re^m \rightarrow \Re^n$, the $\delta - neighborhood$ of the function is the set-valued map denoted by $[f(x)]^{\delta} : xin\Re^m \rightarrow B(f(x), \delta) \in \Re^n$

Example 5.2.1 *Neighborhoods of Functions*

Let $p(x) : \Re^n \rightarrow \Re$ be a real-valued function, then $[p(x)]^{\delta} : \Re^n \rightarrow [p(x) - \delta, p(x) + \delta]$ is a real, set-valued function, that maps every point $\Re^n$ to an interval in $\Re$ of length 2δ.

Qualitatively, we wish to describe an approximate solution of a differential inclusion in the following manner. Given an instant of time $t^* \in \Re_+$, and a candidate approximate solution $s(t) : \Re_+ \rightarrow \Re^n$, defined almost everywhere, we consider two closed sets.

$$s^{\delta}(t^*) \in \Re^n \tag{5.86}$$

$$(t^*)^{\delta} \in \Re_+ \tag{5.87}$$

Note that $[s(t^*)]^{\delta}$

We formalize the notion of an approximate solution in the following manner. *An absolutely continuous vector function $s(t) : \Re_+ \rightarrow \Re^n$, $\delta \in \Re_+$ is defined to be an* *approximate Filippov solution of the Filippov differential system (5.78) if for almost all $t \in D$,*

$$\frac{ds}{dt}|_{t=t^*} \in \mathcal{F}^\delta(B(s(t^*),\delta), B(t^*,\delta)) \tag{5.88}$$

where

$$\mathcal{F}^\delta(B(s(t^*),\delta), B(t^*,\delta)) = [f(s(t^*),t^*)]^\delta \text{ in the domains of continuity} \tag{5.89}$$
$$\mathcal{F}^\delta(B(s(t^*),\delta), B(t^*,\delta)) = [\text{convex-hull } f(B(s(t^*),\delta) - N, B(t^*,\delta))]^\delta \tag{5.90}$$

where the intersection over all sets N of Lebesgue measure zero where the functionf(x,t) is either undefined or discontinuous.

We now state the theorem that guarantees the local existence of Filippov solutions.

Theorem 5.3 *Local Existence Of Filippov Solutions To Filippov Differential Systems*

Given *(G1) A Filippov differential system of the form (5.78).*

If *(I1) The domain D where $f(x,t)$ is specified for almost all t*
$(x,t) \in \Re^n \times \Re_+ : ||x - x_0|| \leq K_x$ *and* $t \leq K_t$

(I2) $f(x,t) : \Re^n \times \Re_+ \rightarrow \Re^n$ *is measurable in* $t \in \Re_+$ *for all* $x \in \Re^n$

(I3) $||f(x,t)|| \leq K_f(t)\ \forall\ (x,t) \in D$ *where* $K_f(t) : \Re_+ \rightarrow \Re$ *is summable. Furthermore there exists* $K_f \in \Re_+$ *such that* $K_f > |K_f(t)|\ \forall\, t \in D$

(I4) The differential inclusion in (5.78), $\mathcal{F}(x,t) : \Re^n \times \Re_+ \rightarrow S \in \Re^n$, *where S is a set in $\Re^n$ and in the domain D of the (x,t) space satisfies the following two assumptions.*

* *the set valued map* $\mathcal{F}(x,t) : \Re^n \times \Re_+ \rightarrow S \in \Re^n$ *is* upper semi-continuous.
* *the set* $S \in \Re^n$ *is* compact and convex.

Then *(T1) The differential system (5.78) has atleast one Filippov solution* $s(t) : \Re_+ \rightarrow \Re^n$ *for* $t \leq min(K_t, \frac{K_x}{K_f})$ *satisfying the initial condition* $s(0) = x_0$.

Proof:

The proof consists of the following steps.

Step 1. We will define a sequence of functions that form approximate Filippov solutions to the Filippov differential system. We will use the method of Euler broken lines to generate such approximate solutions.

Step 2. We will show that this sequence of approximate Filippov solutions is both uniformly bounded and equicontinuous.

Step 3. Invoking the Arzela-Ascoli theorem, we will show that there exists a uniformly convergent subsequence of approximate Filippov solutions whose limit is a Filippov solution to the Filipov differential system.

Step 4. We will now pass to the limit of this subsequence, and show that the limit function is a solution of the inclusion and hence of the the Filippov differential system.

Step 1:

Set $T_{min} = min(K_t, \frac{K_x}{K_f})$. For $t \in [0, T_{min}]$, we construct a sequence of functions $s^i(t), i = 1, 2, \ldots$ in the following manner. We consider a partitioning of the interval $[0, T_{min}]$ given by,

$$\begin{aligned}
\Delta^i t &= \frac{T_{min}}{i} \; i = 1, 2, \ldots & (5.91)\\
t^i_j &= j\Delta^i t \; j = 0, 2, \ldots, i & (5.92)
\end{aligned}$$

Note here that the size of the ith partition is given by $\Delta^i t$ and represents the the fineness of the discretization of the interval. The step, or instant of time j, given a step size or discretization $\Delta^i t$ is referred to by t^i_j. We now construct an Euler broken line in the following manner.

$$\begin{aligned}
s^i(0) &= x_0 & (5.93)\\
s^i(t) &= s^i(t^i_j) + [t - t^i_j]v^i_j & (5.94)\\
v^i_j &\in \mathcal{F}(s^i(t^i_j), t^i_j) \; t^i_j < t < t^i_{j+1} & (5.95)\\
& i = 1, 2, \ldots \;\; j = 0, 2, \ldots, i & (5.96)
\end{aligned}$$

To understand the constructed approximations, let us explicitly write out the expressions for $i = 1$ and $i = 2$

The first($i = 1$) function in the sequence $s^1(t)$ is constructed with 1 partition of the interval $[0, T_{min}]$, the length of the partition being $\Delta^1 t = T_{min}$. Indeed the interval over which the function $s^1(t)$ would be defined is $[t_0^1, t_1^1]$ where

$$
\begin{aligned}
t_0^1 &= 0 && (5.97)\\
t_1^1 &= T_{min} && (5.98)
\end{aligned}
$$

The first function in the sequence, $s^1(t)$, is explicitly written as

$$
\begin{aligned}
s^1(0) &= x_0 && (5.99)\\
s^1(t) &= s^1(0) + [t - 0]v_0^1 \;\; 0 < t < T_{min} && (5.100)\\
v_0^1 &\in \mathcal{F}(s^1(0), 0) && (5.101)
\end{aligned}
$$

Comment 5.2.4 The the main difference between differential equations and inclusions is that in equation (5.101) we would have strict equality = in the case of differential equations.

The second ($i = 2$) function in the sequence, $s^2(t)$, is constructed with 2 partitions of the the interval $[0, T_{min}]$, the length of each partitioned interval being $\Delta^2 t = \frac{T_{min}}{2}$. Indeed the two intervals over which the function $s^2(t)$ would be defined are $[t_0^2, t_1^2]$ and $[t_1^2, t_2^2]$ where

$$
\begin{aligned}
t_0^2 &= 0 && (5.102)\\
t_1^2 &= \frac{T_{min}}{2} && (5.103)\\
t_2^2 &= T_{min} && (5.104)
\end{aligned}
$$

The second function in the sequence, $s^2(t)$, is explicitly written as

$$
\begin{aligned}
s^2(0) &= x_0 && (5.105)\\
s^2(t) &= s^2(0) + [t - 0]v_0^2 \;\; 0 < t < \frac{T_{min}}{2} && (5.106)\\
&= s^2(\frac{T_{min}}{2}) + [t - \frac{T_{min}}{2}]v^2 1 \;\; \frac{T_{min}}{2} < t < T_{min} && (5.107)\\
v_0^2 &\in \mathcal{F}(s^2(0), 0) && (5.108)\\
v_1^2 &\in \mathcal{F}(s^2(\frac{T_{min}}{2}), \frac{T_{min}}{2}) && (5.109)
\end{aligned}
$$

Though we are yet to show that any of these functions so constructed are approximate *solutions* of the differential inclusion, the flavour of solution construction using Euler broken lines is fairly obvious. However we will show later in the proof that the functions $s^i(t)$ are indeed approximate solutions, (approximate in the sense of our earlier definition of approximate Filippov solutions)

Step 2:

Now we note some properties of the functions $s^i(t) : \Re_+ \rightarrow \Re^n$ defined in equations (5.93) - (5.96).

We show first that the graph of the functions $s^i(t) : \Re_+ \rightarrow \Re^n$ is contained in $D(x,t)$. Indeed,

$$\begin{aligned}
||s^i(t) - s^i(0)||_2 &= 0 \text{ for } t = 0 && (5.110)\\
||s^i(t) - s^i(0)||_2 &= ||s^i(t^i_j) + [t - t^i_j]v^i_j - s^i(0)||_2 && (5.111)\\
&= ||s^i(t^i_j) - s^i(0)||_2 + ||[t - t^i_j]v^i_j||_2 && (5.112)\\
&\leq K_f||t^i_j - 0||_2 + K_f||t - t^i_j||_2 && (5.113)\\
&\leq K_f||t - 0||_2 && (5.114)\\
&\leq K_f||t||_2 && (5.115)\\
& \quad t > 0 \;\; i = 1, 2, \ldots \;\; j = 0, 2, \ldots, i && (5.116)
\end{aligned}$$

It is therefore obvious that $s^i(t) : \Re_+ \rightarrow \Re^n \in D(x,t)$.

We will now show uniform boundedness of the functions $s^i(t) : \Re_+ \rightarrow \Re^n$ as follows. For any $i \in Z_+$, and for all t, we have

$$\begin{aligned}
||s^i(t)||_2 &= ||x_0||_2 \text{ for } t = 0 && (5.117)\\
||s^i(t)||_2 &= ||s^i(t^i_j) + [t - t^i_j]v^i_j||_2 && (5.118)\\
&\leq ||s^i(t^i_j||_2 + ||[t - t^i_j]v^i_j||_2 && (5.119)\\
&\leq ||x_0||_2 + K_f T_{min} && (5.120)\\
& \quad t > 0 \;\; i = 1, 2, \ldots \;\; 0 = 1, 2, \ldots, i && (5.121)
\end{aligned}$$

Consequently the functions $s^i(t) : \Re_+ \rightarrow \Re^n$ are uniformly bounded.

We will show equicontinuity as follows. For any $i \in Z_+$, given an $\epsilon \in \Re_+$, we will show that there exists a positive constant $\delta \in \Re_+$ such that for any two instants of time $t_1 \leq t_2 \in \Re_+$, satisfying the condition $|t_2 - t_1| < \delta \rightarrow ||s^i(t_2) - s^i(t_1)||_2 < \epsilon$. We first note that the two instants of time t_1, t_2 satisfy the following property. By virtue of the construction of intervals of time Δ^i outlined earlier there exists $j_1 < i \in Z_+$ such that

$$j_1 \frac{T_{min}}{i} \leq \; t_1 \; \leq [j_1 + 1] \frac{T_{min}}{i} \tag{5.122}$$
$$(j_1 + m) \frac{T_{min}}{i} \leq \; t_2 \; \leq (j_1 + m + 1) \frac{T_{min}}{i} \tag{5.123}$$

where $m \in Z_+$ may take integer values satisfying $0 \leq m \leq i - j_1$. Indeed, $m = 0$ corresponds to the two instants of time t_1, t_2 lying in the same interval, and the case when $m = 1$ corresponds to the instants t_1, t_2 lying in adjacent intervals of time. We will say more about these two cases later. Now write

$$t_2 - t_1 = [t_2 - t_{j_1+m}] + [t_{j_1+m} - t_{j_1+m-1}] + \cdots + [t_{j_1+1} - t_1] \tag{5.124}$$
$$s^i(t_2) - s^i(t_1) = [s^i(t_2) - s^i(t_{j_1+m})] + [s^i(t_{j_1+m}) - s^i(t_{j_1+m-1})] + \cdots \tag{5.125}$$
$$+ \cdots + [s^i(t_{j_1+1}) - s^i(t_1)] \tag{5.126}$$
$$||s^i(t_2) - s^i(t_1)||_2 \leq ||s^i(t_2) - s^i(t_{j_1+m})||_2 + ||s^i(t_{j_1+m}) - s^i(t_{j_1+m-1})||_2 + \tag{5.127}$$
$$+ \cdots + ||s^i(t_{j_1+1}) - s^i(t_1)||_2 \tag{5.128}$$
$$\leq K_f[[t_2 - t_{j_1+m}] + K_f[[t_{j_1+m} - t_{j_1+m-1}] + \cdots \tag{5.129}$$
$$+ \ldots + K_f[t_{j_1+1} - t_1] \tag{5.130}$$
$$||s^i(t_2) - s^i(t_1)||_2 \leq K_f[t_2 - t_1] \tag{5.131}$$

Therefore, given any $\epsilon \in \Re_+$, we can choose a $\delta = \frac{\epsilon}{K_f}$ such that the following is true

$$[t_2 - t_1] < \delta \tag{5.132}$$
$$< \frac{\epsilon}{K_f} \tag{5.133}$$
$$||s^i(t_2) - s^i(t_1)||_2 \leq K_f[t_2 - t_1] \tag{5.134}$$
$$< K_f \frac{\epsilon}{K_f} \tag{5.135}$$
$$||s^i(t_2) - s^i(t_1)||_2 < \epsilon \tag{5.136}$$

Equicontinuity is therefore shown by equations (5.132) and (5.136).

To clarify the proof, we will illustrate the cases when t_1, t_2 lie in the same interval and the case when the instants t_1, t_2 lie in adjacent intervals.

Indeed, when we have $m = 0$ and the two instants of time lie in the same interval, we have.

$$j_1 \frac{T_{min}}{i} \leq \; t_1 \; \leq [j_1 + 1] \frac{T_{min}}{i} \tag{5.137}$$

$$(j_1 + 0) \frac{T_{min}}{i} \leq \; t_2 \; \leq (j_1 + 0 + 1) \frac{T_{min}}{i} \tag{5.138}$$

$$s^i(t_1) \;=\; s^i(t^i_{j_1}) + [t_1 - t^i_{j_1}] v^i_{j_1} \tag{5.139}$$

$$s^i(t_2) \;=\; s^i(t^i_{j_1}) + [t_2 - t^i_{j_1}] v^i_{j_1} \tag{5.140}$$

$$||s^i(t_2) - s^i(t_1)||_2 \;\leq\; ||[t_2 - t_1] v^i_{j_1} \tag{5.141}$$

$$\leq K_f [t_2 - t_1] \tag{5.142}$$

Equicontinuity is trivially obvious. Similarly, for the case when $m = 1$, we have

$$j_1 \frac{T_{min}}{i} \leq \; t_1 \; \leq (j_1 + 1) \frac{T_{min}}{i} \tag{5.143}$$

$$(j_1 + 1) \frac{T_{min}}{i} \leq \; t_2 \; \leq (j_1 + 1 + 1) \frac{T_{min}}{i} \tag{5.144}$$

$$0 \leq m \leq i - j_1 \tag{5.145}$$

$$[t_2 - t_1] \;=\; [t_2 - t_{j_1+1}] + [t_{j_1+m} - t_1] \tag{5.146}$$

$$s^i(t_1) - s^i(t_2) \;=\; s^i(t_1) - s^i(t^i_{j_1+1}) + s^i(t^i_{j_1+1}) - s^i(t_2) \tag{5.147}$$

$$||s^i(t_2) - s^i(t_1)||_2 \;\leq\; ||s^i(t_2) - s^i(t^i_{j_1+1})||_2 + ||s^i(t^i_{j_1+1}) - s^i(t_1)||_2 \tag{5.148}$$

$$\leq\; K_t [t_2 - t^i_{j_1+1}] + K_t [t^i_{j_1+1} - t_1] \tag{5.149}$$

$$\leq\; K_t [t_2 - t_1] \tag{5.150}$$

Equicontinuity is hence shown.

We will now show that functions $s^i(t) : \Re_+ \rightarrow \Re^n$ are indeed Δ^i approximate solutions of the differential inclusion (5.78). Indeed, it would suffice to show that

$$\frac{ds^i}{dt}|_{t=t^*} \in \mathcal{F}^{\Delta^i}(B(s^i(t^*), \Delta^i), B(t^*, \Delta^i)) \tag{5.151}$$

From the definition of the functions $s^i(t) : \Re_+ \to \Re^n$ in (5.95) it is clear that

$$\begin{aligned}
\frac{ds^i}{dt}|_{t=t^*} &\in \mathcal{F}(s^i(t^i_j), t^i_j) \;\; |t^* - t^i_j| < \Delta^i \\
&\quad \text{we only strengthen the inclusion by averaging over a neighborhood} \\
&\in \mathcal{F}(B(s^i(t^*), \Delta^i), B(t^*, \Delta^i)) \\
&\quad \text{we strengthen the inclusion further by} \\
&\in \mathcal{F}^{\Delta^i}(B(s^i(t^*), \Delta^i), B(t^*, \Delta^i)) \\
&\quad i = 1, 2, \ldots \;\; j = 0, 2, \ldots, i
\end{aligned}$$

Hence the $s^i(t) : \Re_+ \to \Re^n$ in (5.95) are indeed Δ^i approximate solutions of the differential inclusion (5.78).

Step 3:

Having shown the uniform boundedness, and equicontinuity of the sequence of functions $s^i(t)$ $i = 1, 2, \ldots$ on the closed interval $[0, T_{min}]$, we invoke the Arzela-Ascoli Theorem to guarantee the existence of a subsequence $s^{i_k}(t)$ $k = 1, 2, \ldots$ that is uniformly convergent in the interval $[0, T_{min}]$.

We therefore claim that the sequence $s^{i_k}(t)$ $k = 1, 2, \ldots$ converges to an absolutely continuous function $s(t)$ as $i_k \to \infty$. Here we use an elementary result from analysis that the limit of a convergent sequence of absolutely continuous functions is also absolutely continuous.

Step 4:

As the inclusion satisfies the assumption ($I4$) of the theorem, we now invoke the Filippov lemma to conclude that the limit function $s^i(t) \to s(t)$ as $\Delta^i \to 0$ also satisfies the differential inclusion. This concludes the proof of the theorem. $\triangleleft\spadesuit$

5.3 Design Of Sliding Mode Controls

In this section, we specialize the preceding theory to a special class of systems of the following form.

$$\dot{x} = f_+(x) \text{ for } [x : s(x) > 0] \tag{5.152}$$

$$= \quad f_{-}(x) \text{ for } [x : \; s(x) < 0] \tag{5.153}$$

where $x \in \Re^n$, and $f(x) : \Re^n \rightarrow \Re^n$ and $s(x) : \Re^n \rightarrow \Re$. Note that $S = \{x : \; s(x) = 0\}$ is a manifold of dimension $n-1$. This manifold S is called the *sliding manifold* or *sliding surface.* The dynamics of the system on this manifold S is called the *sliding dynamics* or *sliding modes* of the system. The design of the manifold S is such that it is globally attractive, and trajectories commencing from arbitrary initial conditions reach S in finite time. Furthermore, the dynamics on S achieves the control objective.

Local existence of solutions is verified by modelling the system represented by equations (5.152) - (5.153) by the appropriate differential inclusions and verifying whether the inclusion satisfies the hypotheses of the theorem concerning local existence of Filippov solutions.

Uniqueness, in the sense of the Filippov solution is shown if either $\frac{\partial s(x)}{\partial x} f_{+}(x) < 0$ or $\frac{\partial s(x)}{\partial x} f_{-}(x) > 0$. This is shown in [30], [15], [14]. The physical interpretation of these conditions is simply that the trajectories of the system are always directed towards S, thus rendering it attractive.

Example 5.3.1

$$\dot{x} \quad = \quad -ksgn[x] \tag{5.154}$$

$$sgn[x] \quad = \quad 1 \; if x > 0 \tag{5.155}$$

$$sgn[x] \quad = \quad -1 \; if x < 0 \tag{5.156}$$

Modelling the system (5.154) by a simple differential inclusion, we rewrite (5.154) as

$$\dot{x} \in \mathcal{F}(x) \tag{5.157}$$

where

$$\mathcal{F}(x) \quad = \quad sgn[x] \; if \, x \neq 0 \tag{5.158}$$

$$\mathcal{F}(x) \quad \in \quad [-1, 1] \; if \, x = 0 \tag{5.159}$$

The inclusion in (5.157) is closed, bounded, convex and uppersemicontinuous and therefore by the theorem on existence of Filippov solutions, Filippov solutions exist for this system.

The sliding modes of a system, defined to be the Filippov solutions to the system on the manifold S, are calculated by performing Filippov averaging, which is a convex combination of dynamics on either side of the manifold S. Indeed, by dynamics on either side of the manifold S, we merely refer to $f_+(x)$ and $f_-(x)$. The simple extension of the notion of sliding manifolds to non-autonomous systems is shown in [30].

While the theory of existense of solutions has been developed for general nonlinear systems with discontinuous controls, the methodology to design sliding mode controls to achieve stabilization or tracking is well understood only for a restricted class of systems. In the following sections, we will present the theory for Linear Time Invariant Systems - SISO and MIMO.

Comment 5.3.1 The design of sliding surfaces for linearizable nonlinear systems, follows exactly the design of sliding controls for linear time invariant systems. The only difference is that the controls are designed using the transformed variables.

5.3.1 Sliding Mode Design For LTI Systems

Consider linear time invariant systems represented by the following equations

$$\dot{\bar{x}} = A\bar{x} + Bu \tag{5.160}$$

where $x \in \Re^n$, $A \in \Re^{nxn}$, $B \in \Re^{mxn}$ and the controls $u \in \Re^m$. We will now prescribe the sliding mode controller design procedure in a sequence of steps.

Step 1.

Check to see if the system is completely controllable. If the system is not completely controllable, a sliding mode controller cannot be designed.

Step 2.

If the system is completely controllable, find a linear transformation of the state that recasts the system in the controllable canonical form. That is find a transformation

$$x = T\bar{x} \quad T \in \Re^{nxn} \tag{5.161}$$

such that the state equations are of the form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \cdots & 0 & \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \\ b_1 & b_2 & \cdots & b_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{5.162}$$

Step 3.

We define $S(x) : \Re^n \rightarrow \Re$ as

$$S(x) = a_1 x_1 + a_2 x_2 + \cdots + a_{n-1} x_{n-1} + x_n \tag{5.163}$$

where the coefficients a_i $i = 1, 2, \ldots, n-1$ of (5.163) are such that the polynomial $S(x)$ is a Hurwitz polynomial. Furthermore, note that $S = 0$ is an $n-1$ dimensional manifold, called the sliding surface.

Indeed now choose the control input u of (??) to be

$$\begin{aligned} u(t) &= -b_1 x_1 - b_2 x_2 \cdots - b_n x_n - v_1(t) && (5.164) \\ v(t) &= -a_1 x_2 - a_2 x_3 - \cdots - a_{n-1} x_n - ksgn[S(x)] && (5.165) \end{aligned}$$

Choice of control u enables us to rewrite system (??) in the form

$$\begin{aligned} \dot{x}_1 &= x_2 && (5.166) \\ \dot{x}_2 &= x_3 && (5.167) \\ \dot{x}_{n-1} &= -a_1 x_1 - a_2 x_2 - \cdots - a_{n-1} x_{n-1} + S(x) && (5.168) \\ \dot{S}(x) &= -ksgn[s(x)] && (5.169) \end{aligned}$$

Indeed utilizing the theorems developed in the preceding section it is easy to show that Filippov solutions exist, and that $S(x) = 0$ is reached in finite time from arbitrary initial conditions. Furthermore on the $n-1$ dimensional manifold $S = 0$, the reduced order dynamics is exponentially stable. Consequently global exponential stability of the system is shown.

The choice of discontinuous input induces chatter in the system. To reduce the chatter, we utilize various regularizations and smoothings of the discontinuous *sgn* function. The common smoothing technique is the use of the saturation function, which is presented in [30].

We now present a choice of *continuous control input,* that enables us to reach the sliding surface $S = 0$ in finite time. Indeed, consider the control given by

$$\begin{aligned} u(t) &= -b_1x_1 - b_2x_2 \cdots - b_nx_n - v_1(t) && (5.170)\\ v(t) &= -a_1x_2 - a_2x_3 - \cdots - a_{n-1}x_n - k|S(x)|^{\frac{1}{m}} sgn[S(x)] && (5.171)\\ m &> 1 && (5.172) \end{aligned}$$

Such a choice of control u enables us to recast the system equations in the form

$$\begin{aligned} \dot{x}_1 &= x_2 && (5.173)\\ \dot{x}_2 &= x_3 && (5.174)\\ \dot{x}_{n-1} &= -a_1x_1 - a_2x_2 - \cdots - a_{n-1}x_{n-1} + S(x) && (5.175)\\ \dot{S}(x) &= -k|S(x)|^{\frac{1}{m}} sgn[s(x)] && (5.176) \end{aligned}$$

Indeed utilizing the theorems developed in the preceding section it is easy to show that Filippov solutions exist, and that the $n-1$ dimensional manifold $S(x) = 0$ is reached in finite time. Furthermore on the $n-1$ dimensional manifold given by $S = 0$, we see that the reduced order dynamics is exponentially stable. Consequently global exponential stability of the system is shown. This control law u is interesting in that it is continuous, but not differentiable.

Comment 5.3.2 The disturbance rejection properties of the discontinuous control law are significantly better than that of the continuous control law. This indeed is the design tradeoff involved in designing continuous control laws.

Comment 5.3.3 The extension of the sliding mode control techniqes to controllable MIMO systems that are decouplable is trivial. Once the system equations are transformed into decoupled systems, each of which is in the controllable canonical form, we apply the design method outlined earlier to design sliding surfaces for the decoupled system. Note however that sliding occurs not at the individual surfaces, but at the intersection of all these surfaces.

In the following section, we will present a class of non-conventional discontinuous control laws for a class of mechanical systems.

5.3.2 Novel Discontinuous Control Laws for Mechanical Systems

Many mechanical systems are described by differential equations which are essentially of the second order. We will now describe a set of control laws for such systems, the control objective being regulation to the origin. While the extension of these control laws to systems of higher dimensions is non-trivial, these laws by themselves are quite important from an applications point of view. They provide the engineer with an additional set of nonlinear tools to control mechanical systems. In this section, we will concern ourselves with planar dynamical systems of the form.

$$\dot{x}_1 = x_2 \tag{5.177}$$

$$\dot{x}_2 = u \tag{5.178}$$

The various control laws that ensure finite time stabilization for the system (6.316) - (6.317) are as follows.

$$u_{optimal} = \begin{cases} -sgn[x_1 + \frac{x_2|x_2|}{2}] & \text{if } |x_1 + \frac{x_2|x_2|}{2}| > 0 \\ -sgn[x_2] & \text{if } |x_1 + \frac{x_2|x_2|}{2}| = 0 \end{cases} \tag{5.179}$$

$$u_{winding} = -k_1 sgn[x_1] - k_2 sgn[x_2] \quad k_1 > k_2 > 0 \tag{5.180}$$

$$u_{nested} = -k_2 sgn[x_2 - k_1 sgn[x_1]] \tag{5.181}$$

$$u_{switching} = -k_2 sgn[x_2 + k_1 |x_1|^{\frac{1}{m}} sgn[x_1]] \quad k_2 \text{ is large} \tag{5.182}$$

We will now show that for all the aforementioned control laws the states of the system are regulated to the origin in finite time.

We will examine each control law briefly, show the relevant properties and present a phase portrait of the system subjected to the control law for a variety of initial conditions.

Optimal Control Viewpoint

Consider the minimum time optimal control problem with the functional to be minimized, given by

$$J = \int_0^{t_f} dt \tag{5.183}$$

Using standard method of optimal control, we write down the Hamiltonian function $H(x, u, \lambda, t)$ as

$$H(x, u, \lambda, t) = 1 + \lambda_1 x_2 + \lambda_2 u(t) \tag{5.184}$$

Indeed mere inspection of equation reveals that the control $u(t)$ that minimizes the Hamiltonian is given by

$$u(t) = -sgn[\lambda_2]u_{max} \tag{5.185}$$

where u_{max} is the maximum permissible value of control. Without loss of generality, we will assume that $u_{max} = 1$.

where λ_1 and λ_2 are the co-state variables. The co-state equations are given by

$$\dot{\lambda}_1 = 0 \tag{5.186}$$

$$\dot{\lambda}_2 = -\lambda_1 \tag{5.187}$$

Integrating the co-state equations yields

$$\lambda_2(t) = -\lambda_1(0)t - \lambda_2(0) \tag{5.188}$$

Therefore the optimal control is given as

$$u = sgn[-\lambda_1(0)t - \lambda_2(0)] \tag{5.189}$$

The control can assume only two values +1 or −1. When $u = +1$, we integrate the state equations to obtain

$$x_2(t) = t + x_2(0) \tag{5.190}$$

$$x_1(t) = \frac{t^2}{2} + x_2(0)t + x_1(0) \tag{5.191}$$

Eliminating t we obtain (5.192)

$$x_1 = \frac{x_2^2}{2} + x_1(0) - \frac{x_2^2(0)}{2} \tag{5.193}$$

Similarly, when $u = -1$, integrating the state equations we obtain

$$\begin{aligned} x_2(t) &= -t + x_2(0) && (5.194) \\ x_1(t) &= -\frac{t^2}{2} + x_2(0)t + x_1(0) && (5.195) \\ &\text{Eliminating } t \text{ we obtain} && (5.196) \\ x_1 &= -\frac{x_2^2}{2} + x_1(0) - \frac{x_2^2(0)}{2} && (5.197) \end{aligned}$$

These curves describe a family of parabolas, whose switching curve may be written as

$$S(x,t) = x_1 + \frac{x_2|x_2|}{2} \tag{5.198}$$

In terms of the switching curve, the control $u_{optimal}$ may be written as

$$u_{optimal} = \begin{cases} -sgn[x_1 + \frac{x_2|x_2|}{2}] & \text{if } |x_1 + \frac{x_2|x_2|}{2}| > 0 \\ -sgn[x_2] & \text{if } |x_1 + \frac{x_2|x_2|}{2}| = 0 \end{cases} \tag{5.199}$$

The phase portrait of trajectories subject to the optimal control $u_{optimal}$ is given below. Note the trajectories converging to the switching curve, which is non-linear. (while the switching curve in conventional sliding mode systems is linear)The chosen control gains are

$$\begin{aligned} k_1 &= 1 && (5.200) \\ k_2 &= 2 && (5.201) \end{aligned}$$

Winding Algorithm

The winding algorithm was introduced by [22] and makes use of continuous switching between the surfaces $x_1 = 0$ and $x_2 = 0$ to reach the origin. The interesting feature of this control technique is that the control has two switches. One switch is used to change the direction, and the other is used to change the magnitude. By repeatedly switching between the surfaces $x_1 = 0$ and $x_2 = 0$, we wind closer to the origin.

Let us first prove the stability and finite time stabilization of the algorithm. To show stability, we use the extended Lyapunov theorem, proofs for which may be

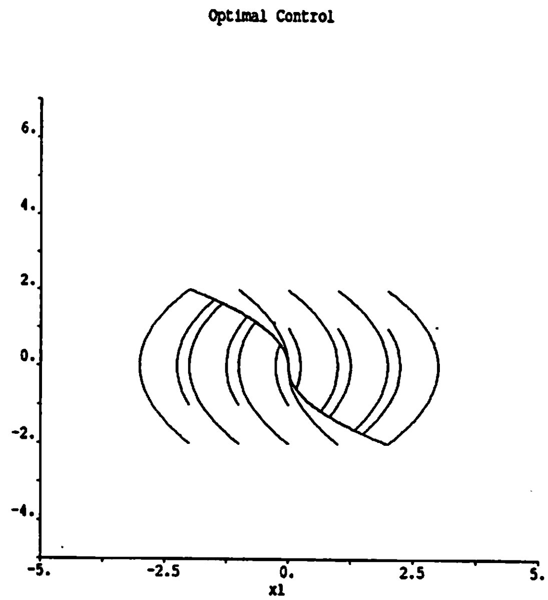

Figure 5.1: Finite Time Stabilization With Optimal Control

found in [1]. The theorem is primarily used to conclude *weak-stability* of differential inclusions by investigating generalized gradients of non-differentiable Lyapunov functions. A brief statement of the theorem would be as follows.

Given a differential inclusion $\dot{x} \in F(x,t)$ and a nondifferentiable Lyapunov function $V(x)$. If for every element v in the generalized gradient of V, there exists atleast one element $f \in F(x,t)$, such that $L_F V \leq 0$, then the zero-solution is weakly asymptotically stable. Indeed, weak asymptotic stability is the best we could hope for when dealing with set-valued differential inclusions.

Now consider the system (6.316) - (6.317) subject to the controls $u_{winding}$. The system equations are

$$\dot{x}_1 = x_2 \tag{5.202}$$

$$\dot{x}_2 = -k_1 sgn[x_1] - k_2 sgn[x_2] \tag{5.203}$$

Consider a candidate Lyapunov function

$$V = |x_1| + \frac{x_2}{2k_1} \tag{5.204}$$

The derivative for $x_1, x_2 \neq 0$ is given by

$$\begin{aligned} \dot{V} &= -\frac{k_2 x_2^2}{k_1} && (5.205) \\ &\leq 0 && (5.206) \end{aligned}$$

Therefore $x_2 \to 0$, and the reduced dynamics is such that $x_1 \to 0$. However, when $x_1 = 0$, it is clear that we have to investigate the properties of the generalized gradient of V. However, it is obvious that when $x_1 = 0$, for every element v of the generalized gradient of V, (which in this case happens to be any real number in (-1,1)) there exists an element of the inclusion $F(x,t)$ (indeed, choose $f = v$) such that the generalized gradient of V along the flow of the inclusion $F(x,t)$ is negative definite. The conditions of the generalized Lyapunov theorem are satisfied, and hence the result.

Finite time is shown by considering the state equations of the planar dynamical system in the various quadrants. Indeed, if the portrait of the system were to be drawn with x_1 along the x axis and x_2 along the y axis, we would note the following.

$$\begin{aligned} x_1 &= \pm\frac{x_2^2}{k_1 + k_2} \text{ in the first and third quadrants} && (5.207) \\ x_2 &= \mp\frac{x_2^2}{k_1 - k_2} \text{ in the second and fourth quadrants} && (5.208) \end{aligned}$$

Every instance the trajectory moves from the first quadrant through the fourth quadrant to hit the y axis, we see a contraction occurring in the magnitude of x_2 in the following manner.

$$x_2^2(t_1) = \frac{k_1 - k_2}{k_1 + k_2} x_2^2(0) \tag{5.209}$$

From the third quadrant through the second to strike the y axis again, we see the following contraction.

$$x_2^2(t_2) = \frac{k_1 - k_2}{k_1 + k_2} x_2^2(t_1) \tag{5.210}$$

The state trajectory therefore winds to the origin.

The phase portrait of the planar dynamical system subject to the winding algorithm is illustrated in below. Note the very interesting way in which the state trajectories wind to the origin.

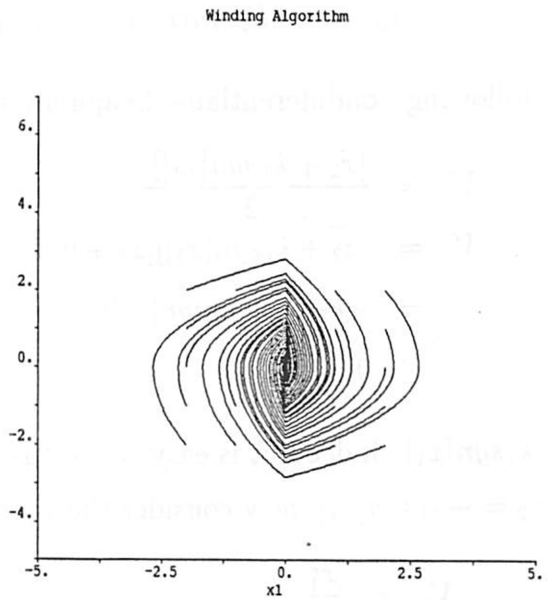

Figure 5.2: Finite Time Stabilization With The Winding Algorithm

The values of chosen control gains are

$$k_1 = 2 \quad (5.211)$$

$$k_2 = 1 \quad (5.212)$$

Nested Switching Control

Nested switching controls work well for planar dynamical systems. The basic approach is to permit chatter about the dual sliding surfaces $x_1 = 0$ and $x_2 = 0$. It is to be noted that chatter for multiple sliding surfaces is the equivalent of limit-cycle like behaviour. Consequently, by utilizing multiple sliding surfaces, and nondifferentiable controls, we are willing to tolerate limit-cycle like behavious at the origin. Indeed, the problems associated with eliminating chatter in one-dimensional systems naturally extend to the higher order systems also. The use of saturation functions to perform nested switching is an extension of the idea of using saturation functions in one-dimensional systems, to many dimensions. The basic control technique is well understood in considering the following non-differentiable Lyapunov function. Consider the system (6.316)-(6.317) subject to the nested switching control law given by

$$\dot{x}_1 = x_2 \quad (5.213)$$

$$\dot{x}_2 = -k_2 sgn[x_2 + k_1 sgn[x_1] \tag{5.214}$$

Now consider the following nondifferentiable Lyapunov function

$$V = \frac{[x_2 + k_1 sgn[x_1]]^2}{2} \tag{5.215}$$
$$\dot{V} = [x_2 + k_1 sgn[x_1]][\dot{x}_2 + 0] \text{if } |x_1| > 0 \tag{5.216}$$
$$= -k_2 |x_2 + k_1 sgn[x_1]| \tag{5.217}$$
$$\leq 0 \tag{5.218}$$

Therefore $x_2 \rightarrow -k_1 sgn[x_1]$. Indeed, it is easy to see that this happens in finite time. As in finite time $x_2 = -k_1 sgn[x_1]$, now consider the Lyapunov function

$$V_1 = \frac{x_1^2}{2} \tag{5.219}$$
$$\dot{V}_1 = x_1 x_2 \tag{5.220}$$
$$= x_1[-k_1 sgn[x_1]] \text{ in finite time} \tag{5.221}$$
$$\leq k_1 |x_1| \tag{5.222}$$
$$\leq 0 \tag{5.223}$$

It is clear that $x_1 \rightarrow 0$ in finite time. However, when $x_1 = 0$, $x_2 \in [-k_1, k_1]$, and is not equal to 0. This is where chatter commences, and the system limit cycles between the surfaces $x_1 = 0$ and $x_2 = k_1 sgn[x_1]$. Such limit cycling behaviour is present as the gain k_1 is not slowly reduced as $x_1 \rightarrow 0$. Indeed if the multiplicand of $sgn[x_1]$ was to decrease in magnitude and finally equal 0 when $x_1 = 0$, we can expect x_2 to also be equal to 0 without chatter. This indeed is the principle behind using saturation functions as opposed to *sgn* functions in nested control. We will now show an extension of this method, without using saturation functions.

The phase plot shown below clearly shows the behaviour of the system subject to nested control. The values of chosen gains are

$$k_1 = 0.5 \tag{5.224}$$
$$k_2 = 5 \tag{5.225}$$

Switching Control

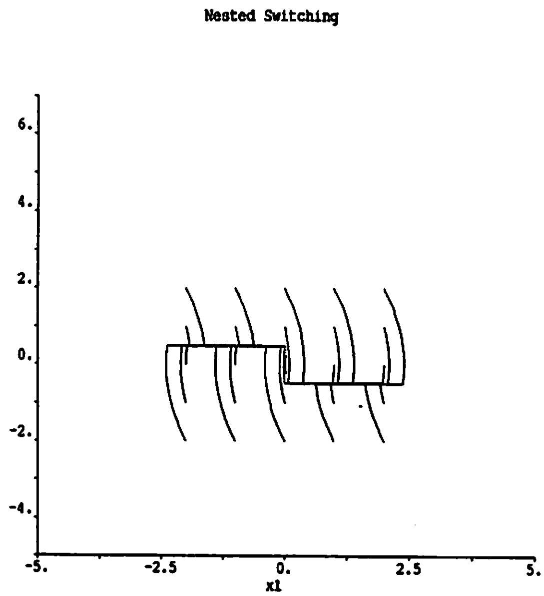

Figure 5.3: Finite Time Stabilization Using Nested Control

We now try to eliminate the problem of limit cycling between switching surfaces that was mentioned earlier. We do this using the switching control law mentioned earlier which is of the form.

$$u_{switching} = -k_2 sgn[x_2 + k_1 |x_1|^{\frac{1}{m}} sgn[x_1]] \tag{5.226}$$

Denote $S = x_2 + k_1 |x_1|^{\frac{1}{m}} sgn[x_1]$. Note that S is not differentiable at $x_1 = 0$. However, almost everywhere, the derivative of S may be written as

$$\dot{S} = -k_2 sgn[S] + k_1 \frac{x_2}{|x_1|^{1-\frac{1}{m}}} \tag{5.227}$$

By choosing a large value of k_2, we hope to swamp the term $k_1 \frac{x_2}{|x_1|^{1-\frac{1}{m}}}$. Indeed, only in cases when this is possible, it is possible to conclude that

$$x_2 = -k_1 |x_1|^{\frac{1}{m}} sgn[x_1] \tag{5.228}$$

And the conclusions of the previous section follow, without the limit cycle behaviour.

The phase portrait shown below illustrates the properties of the control law. The values of chosen gains are

$$k_1 \;=\; 0.5 \tag{5.229}$$

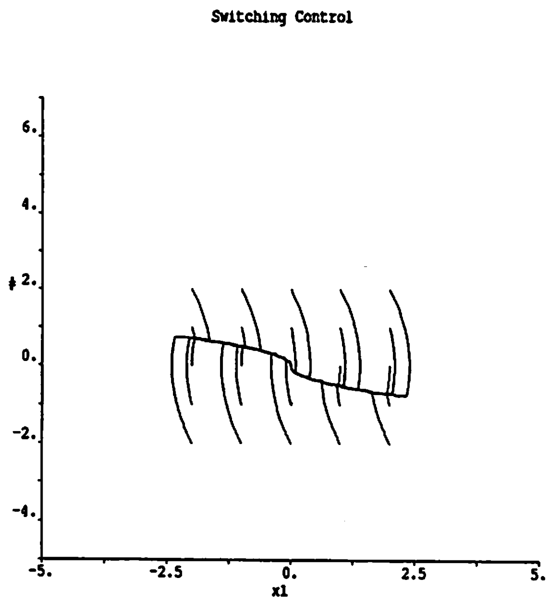

Figure 5.4: Finite Time Stabilization With Switching Control

$$k_2 = 5 \tag{5.230}$$

$$m = 2 \tag{5.231}$$

For the same values of control gains, it is possible to choose a higher order fractional index, and the resulting phase portrait is shown below.

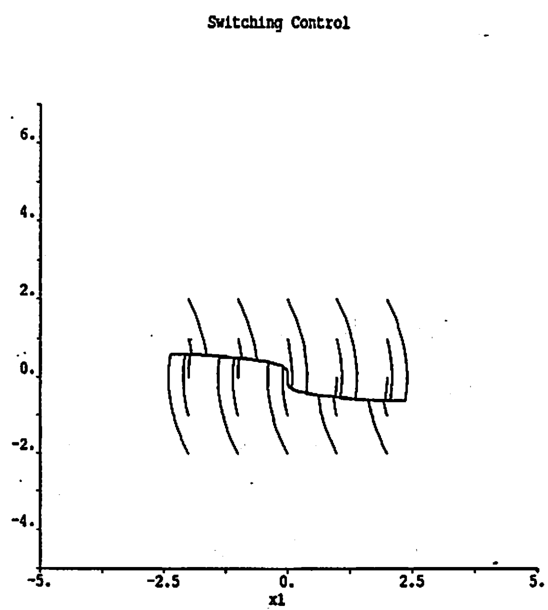

Figure 5.5: Finite Time Stabilization With Switching Control

Chapter 6

New Applications Of Sliding Mode Theory

6.1 Introduction

In this chapter we will present new and varied applications of sliding mode control theory. By applications, we refer to the solution of certain theoretical problems using the technique of sliding modes, as opposed to the control of a physical system using sliding mode control theory. We will use sliding mode theory in the solution of certain classes of problems in the areas of nonlinear identification, synchronous control, lyapunov control, and in the construction of observers. We will conclude this chapter with a conjecture that opens an interesting avenue for research in control using sliding modes.

The organization of this chapter is as follows. Section 1 presents the new theory of sliding mode identifiers. We attempt to identify bounded but unknown parameters using a sliding mode identifier. Section 2 presents results in synchronous sliding modes. Section 3 presents the lyapunov control of certain benchmark problems, and section 4 presents the extension of sliding mode theory to solving some problems in sliding mode observers. Finally we present an interesting conjecture, and its verification using simulation in section 5. Each section is self contained, and has simple examples and simulation plots to elucidate the theory. Each section concludes with a critical appraisal of the presented methodology, possible advantages to the technique and the implementation difficulties associated with it. We also present the scope for future extensions of the technique.

6.2 Sliding Mode Identifiers

In previous chapters we identified *matching conditions* to be satisfied by perturbations to ensure attainment of the control objective. In this chapter we will present methods to identify unknown parameters in a nonlinear system. We view the process of controlling a system by identifying unknown parameters as a way of aysmptotically enforcing the matching conditions.

In this chapter we present a novel identifier that guarantees exponentially convergent identification for a class of nonlinear systems affine in the unknown parameters.The identifier uses variable structure control methodology to ensure exponential reduction of the parameter identification error. We show existence of generalized Filippov solutions for the identifier equations and show parameter convergence using standard Fillipov averaging techniques and the method of equivalent control.

Throughout this section, we assume that unknown parameters of the considered nonlinear systems are bounded, and the bounds are known. Given *full state* information of a nonlinear plant, the identifier then uses this state information, and the bounds on unknown parameters to ensure *exponential convergence* of the identified parameters. We also note here that by exponential convergence of the identified parameter, we refer to the average value of the parameter as prescribed by the generalized Filippov solution to the discontinuous differential system.

We present theoretical results for general nonlinear systems and present simulation results for a simple scalar example to illustrate the theory.

6.2.1 Scalar Nonlinear Systems

To develop the intution embedded in the construction of the identifier, it is useful to consider a simple scalar case. Consider the following scalar system.

$$\dot{x} = \theta f(x) + g(x)u \tag{6.1}$$

where $x \in \Re$, $\theta \in \Re$ and is unknown, $|\theta| < k_\theta$, $u \in \Re$ and $f(x), g(x)\ \Re \to \Re$.

Comment 6.2.1 The unknown parameter enters the system equation affinely.

This simple system (6.1) could represent the velocity dynamics of a mass-spring-damper system with an unknown nonlinear damping coefficient. We will say more about this example later.

The statement of the problem is as follows. *Given the values of the state variable $x \in \Re$, construct a nonlinear identifier that adaptively identifies the parameter $\theta \in \Re$ so that the estimation error goes to zero exponentially.*

To achieve this objective, we construct an identifier of the following form

$$\begin{aligned}
\dot{\hat{x}} &= \hat{\theta} f(x) + g(x)u + v_1 && (6.2)\\
\dot{\hat{\theta}} &= w_1 && (6.3)\\
v_1 &= 2k_\theta |f(x)| sgn[\tilde{x}] && (6.4)\\
w_1 &= k_1 sgn[\tilde{x}] sgn[f(x)] && (6.5)\\
\tilde{x} &= x - \hat{x} && (6.6)\\
k_1 &> 0 && (6.7)
\end{aligned}$$

Comment 6.2.2 The value of theta used in equation (6.2) is given as $[\hat{\theta} \bmod k_\theta]$

In this section and throughout this chapter, we will define the function $sgn[(\cdot)] : \Re \to [-1, 1]$ as follows

$$\begin{aligned}
sgn[(\cdot)] &= \frac{(\cdot)}{|(\cdot)|}\ if\ |(\cdot)| > 0 && (6.8)\\
sgn[(\cdot)] &\in [-1, 1]\ if\ |(\cdot)| = 0 && (6.9)
\end{aligned}$$

Comment 6.2.3 Such a definition of the function $sgn[(\cdot)] : \Re \rightarrow [-1, 1]$ *merely asserts that the function is single valued when the argument is nonzero, but is set-valued when the argument is equal to zero.*

Subtracting the plant equation (6.321) and the identifier equation (6.2), and using the identifier control inputs specified in (6.4) - (6.5) the error equations are written as

$$\dot{\tilde{x}} = \tilde{\theta} f(x) - 2k_\theta |f(x)| sgn[\tilde{x}] \tag{6.10}$$

$$\dot{\tilde{\theta}} = -k_1 sgn[\tilde{x}] sgn[f(x)] \tag{6.11}$$

$$\tilde{x} = x - \hat{x} \tag{6.12}$$

$$\tilde{\theta} = \theta - \hat{\theta} \tag{6.13}$$

We will concern ourselves with the state estimation error, and parameter estimation error dynamics henceforth. We will show existence of generalized Filippov solutions, and parameter convergence to true values using this state estimation error, and parameter estimation error dynamics, We now state the main result of this section.

Theorem 6.1 *Existence of trajectories for identifier and state estimation error dynamics, stability of state estimation error dynamics, and exponential convergence of identified parameters to their true values in the sense of Filippov.*

Given *(G1) A nonlinear system of the form (6.321)*

(G2) A nonlinear identifier of the form (6.2) resulting in state estimation and parameter identification error dynamics of the form (6.10) - (6.11)

If *(I1)* $|\theta| < k_\theta$

(I2) $|f(x)| \neq 0$ *along the system trajectories.*

(I3) In any compact region D, $f(x)$ *is bounded. That is there exists* $k_f = sup[f(x)] \; \forall \; x \in D$.

Then *(T1) Filippov solutions exist for the system (6.10) - (6.11)*

(T2) The surface $\tilde{x} = 0$ is attractive.

(T3) The sliding dynamics on the surfaces $\tilde{x} = 0$ are such that the parameter estimate $\hat{\theta}$ converges to the true value θ exponentially in the sense of Filippov.

Proof: ♠ ▷ We will prove the theorem in three steps. First we show existence of solutions, then we show the existence of attractive sliding surfaces, and finally we show parameter convergence to true values.

Step 1: Existence Of Filippov-Solutions

In a compact domain D of the $(x, \tilde{x}, \tilde{\theta})$ space, the righthandside of equations (6.10) - (6.11) can be modelled by a differential inclusions of the following form

$$\dot{\tilde{x}} \in \mathcal{F}_x(x, \tilde{x}, \tilde{\theta}) \tag{6.14}$$

$$\dot{\tilde{\theta}} \in \mathcal{F}_\theta(x, \tilde{x}, \tilde{\theta}) \tag{6.15}$$

where the inclusions $\mathcal{F}_x(x, \tilde{x}, \tilde{\theta})$ and $\mathcal{F}_\theta(x, \tilde{x}, \tilde{\theta})$ are defined to be

$$\mathcal{F}_x(x, \tilde{x}, \tilde{\theta}) = \tilde{\theta} f(x) - 2k_\theta |f(x)| sgn[\tilde{x}] \;\; if \; \tilde{x} \neq 0 \tag{6.16}$$

$$\in [-3k_\theta k_f, 3k_\theta k_f] \;\; if \; \tilde{x} = 0 \tag{6.17}$$

$$\mathcal{F}_\theta(x, \tilde{x}, \tilde{\theta}) = -k_1 sgn[\tilde{x}] sgn[f(x)] \;\; if \; \tilde{x} \neq 0 \tag{6.18}$$

$$\in [-k_1, k_1] \;\; if \; \tilde{x} = 0 \tag{6.19}$$

where k_f is supremum of the function $f(x) : \Re \to \Re$ over all values of $x \in D$. The inclusions $\mathcal{F}_x(x, \tilde{x}, \tilde{\theta})$ and $\mathcal{F}_\theta(x, \tilde{x}, \tilde{\theta})$ are

- closed, convex, bounded and upper-semicontinuous.

Therefore, invoking the theorem on existense of Filippov-solutions we show that solutions exist for the system represented by equations (6.10) and (6.11).

Step 2: Attractivity Of Sliding Surface

We will now show that the surface $\tilde{x} = 0$ is attractive. Consider a candidate Lyapunov function $V(\tilde{x}) : \Re \to \Re_+$ given by

$$V = \frac{\tilde{x}^2}{2} \tag{6.20}$$

Differentiating (6.20) along the flow of (6.10), we get

$$\begin{aligned} \dot{V} &= \tilde{x}\tilde{\theta}f(x) - 2k_\theta|f(x)||\tilde{x}| & (6.21)\\ &\leq -|\tilde{x}|[2k_\theta|f(x)| - \tilde{\theta}f(x)] & (6.22)\\ &\leq 0 & (6.23) \end{aligned}$$

We have used the fact that $2k_\theta|f(x)| > \tilde{\theta}f(x)$, and that $f(x) \neq 0$ along the system trajectories to conclude that $\dot{V}$ is negative definite and therefore the surface $\tilde{x} = 0$ is globally attractive.

Furthermore, as the term $2k_\theta|f(x)| > \tilde{\theta}f(x)$, the surface $\tilde{x} = 0$ is attained in *finite time* and sliding occurs on the surface $\tilde{x} = 0$.

Step 3: Parameter Convergence

We will prove convergence of the identified parameter to its true value using two methods of proof. The first method will be the method of Filippov averaging and the second method will be using the Equivalent control method.

Proof By Filippov Averaging

The sliding mode on the surface $\tilde{x} = 0$ may be estimated to be a convex combination of the the dynamics on either side of the surface. This indeed is the principle behind the Filippov averaging technique. The averaged dynamics of the estimation error on the surface $\tilde{x} = 0$ takes the following form.

$$\begin{aligned} \dot{\tilde{x}}_{average} &= \gamma[\tilde{\theta}f(x) - 2k_\theta|f(x)|] + [1-\gamma][\tilde{\theta}f(x) + 2k_\theta|f(x)|] & (6.24)\\ \dot{\tilde{\theta}}_{average} &= \gamma[-k_1 sgn[f(x)]] + [1-\gamma][k_1 sgn[f(x)]] & (6.25) \end{aligned}$$

where $0 \leq \gamma \leq 1$. But as $\tilde{x}_{average} = 0$ in finite time, set the left hand side of equation (6.24) to 0 and solve for γ.

$$\begin{aligned} \dot{\tilde{x}}_{average} &= \gamma[\tilde{\theta}f(x) - 2k_\theta|f(x)|] + [1-\gamma][\tilde{\theta}f(x) + 2k_\theta|f(x)|] & (6.26)\\ 0 &= \gamma[\tilde{\theta}f(x) - 2k_\theta|f(x)|] + [1-\gamma][\tilde{\theta}f(x) + 2k_\theta|f(x)|] & (6.27)\\ \gamma &= \frac{\tilde{\theta}sgn[f(x)]}{4k_\theta} + \frac{1}{2} & (6.28) \end{aligned}$$

Comment 6.2.4 It is obvious from (6.28) that $0 \leq \gamma \leq 1$.

To find the sliding dynamics of $\tilde{\theta}_{average}$ along the surface $\tilde{x} = 0$, substitute the value of γ obtained in equation (6.28) in equation (6.25) to obtain,

$$\dot{\tilde{\theta}}_{average} = \gamma[-k_1 sgn[f(x)]] + [1-\gamma][k_1 sgn[f(x)]] \quad (6.29)$$

$$= -2\gamma[k_1 sgn[f(x)]] + k_1 sgn[f(x)] \quad (6.30)$$

$$= -2k_1 sgn[f(x)][\frac{\tilde{\theta} sgn[f(x)]}{4k_\theta} + \frac{1}{2}] + k_1 sgn[f(x)] \quad (6.31)$$

$$\dot{\tilde{\theta}}_{average} = -\frac{k_1}{2k_\theta}\tilde{\theta} \quad (6.32)$$

Comment 6.2.5 Here we have made use of the fact that $sgn[f(x)]sgn[f(x)] = 1$. *This is true only as long as* $f(x) \neq 0$. *Hence we needed the assumption that* $f(x) \neq 0$ *along the trajectories of the system.*

It is clear from equation (6.32) that the averaged dynamics of the estimation error $\tilde{\theta}$ is such that the parameter estimation error is *exponentially diminishing* in the sense of Lyapunov.

Proof By Equivalent Control

The equivalent control method finds the average symbolic value of the function $sgn[\tilde{x}] : \Re \rightarrow [-1,1]$ necessary to ensure the invariance of the sliding surface. This value of $sgn[\tilde{x}] : \Re \rightarrow [-1,1]$ is used to find the reduced order dynamics on the invariant sliding surface.

Using this method, we set the lefthandside of equation (6.10) to 0, and find the value of $sgn[\tilde{x}] : \Re \rightarrow [-1,1]$ necessary to ensure that $\tilde{x} = 0$ is invariant. That is

$$\dot{\tilde{x}} = \tilde{\theta} f(x) - 2k_\theta |f(x)| sgn[\tilde{x}] \quad (6.33)$$

$$0 = \tilde{\theta} f(x) - 2k_\theta |f(x)| sgn[\tilde{x}] \quad (6.34)$$

$$sgn[\tilde{x}] = \frac{1}{2k_\theta}\tilde{\theta} sgn[f(x)] \quad (6.35)$$

Substituting the value of $sgn[\tilde{x}] : \Re \rightarrow [-1,1]$ from equation (6.35) into (6.11), we get,

$$\dot{\tilde{\theta}} = -k_1 sgn[\tilde{x}] sgn[f(x)] \tag{6.36}$$

$$= -\frac{k_1}{2k_\theta}\tilde{\theta} \tag{6.37}$$

Indeed the average dynamics of the estimation error is exponentially convergent to 0 in the sense of Lyapunov.

This concludes the proof of the theorem. ◁♠

Comment 6.2.6 It is interesting to note that we require that the term $|f(x)| \neq 0$ *along the solution trajectories. It is clear that such a requirement is not unreasonable as there is no necessity for identification in the regions of the state space when* $|f(x)| = 0$ *!*

The simplicity of the identifier is obvious from the equations (6.2) - (6.7). It is very simple to implement, but at the same time gives exponential parameter convergence.

Example 6.2.1 *Identifying Friction Coefficient*

Consider a simple mechanical system represented by the following dynamical equations.

$$\dot{x}_1 = x_2 \tag{6.38}$$

$$\dot{x}_2 = \mu sgn[x_2] + f \tag{6.39}$$

where $x \in \Re^2$, $\mu \in \Re$, *is unknown but bounded with a known bound* $|\mu| < k_\mu \in \Re_+$, $f \in \Re$ *is the mechanical force which is the control input to the system. The goal of identification is to find the nonlinear damping coefficient* μ *of the system.*

We construct a sliding mode identifier for this system of the following form.

$$\dot{\hat{x}}_2 = \hat{\mu} sgn[x_2] + f + v_1 \tag{6.40}$$

$$\dot{\hat{\mu}} = w_1 \tag{6.41}$$

$$v_1 = 2k_\mu sgn[\tilde{x}_2] \tag{6.42}$$

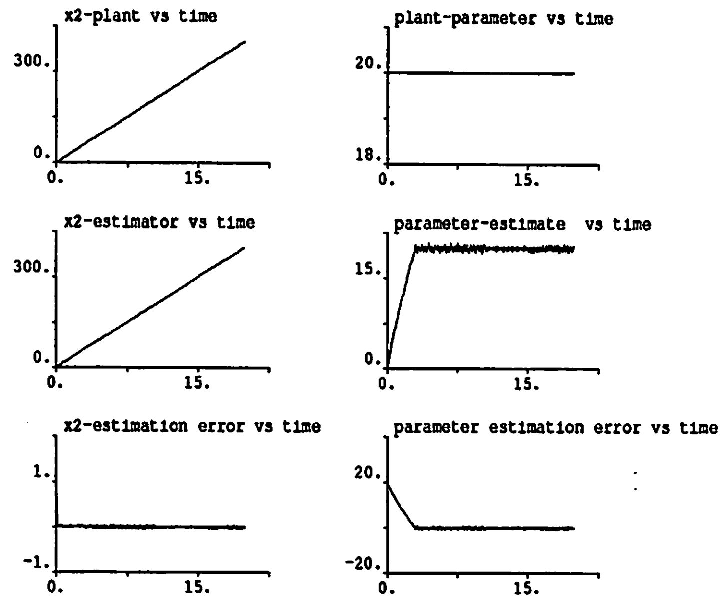

Figure 6.1: Convergence Of Parameter To True Values

$$w_1 = k_1 sgn[\tilde{x}_2] sgn[x_2] \tag{6.43}$$

$$\tilde{x}_2 = x_2 - \hat{x}_2 \tag{6.44}$$

$$k_1 > 0 \tag{6.45}$$

The convergence of the estimated parameter $\hat{\mu}$ to the true value μ is easily shown invoking Theorem 5.1.

Simulation of the system show the interesting features of the identifier. First we note that the state estimation error goes to zero in finite time, as predicted. Then the identified parameter converges to its true value, and the parameter identification error gradually goes to zero. But as Filippov solutions are only solutions averaged over neighbourhoods of diminishing size, the average value of the parameter error is zero, though the actual value chatters about its true value.

6.2.2 Vector Nonlinear Systems - Special Structure

It is now possible to extend the identifier equations to more general systems. As a first extension, consider the system of $n \in \Re_+$ equations containing unknown but bounded, affine parameters θ_i $i = 1, 2, \ldots, n$ The special structure endowed to these systems is *each state equation contains utmost one unknown parameter.* That is, consider systems of the following form

$$\dot{x}_1 = \theta_1 f_1(x) + \sum_{j=1}^{m} g_{1j}(x)u_i \quad (6.46)$$

$$\dot{x}_2 = \theta_2 f_2(x) + \sum_{j=1}^{m} g_{2j}(x)u_i \quad (6.47)$$

$$\vdots \quad \vdots \quad \vdots \quad (6.48)$$

$$\dot{x}_n = \theta_n f_n(x) + \sum_{j=1}^{m} g_{nj}(x)u_i \quad (6.49)$$

where, $x \in \Re^n$, and $\theta_i \in \Re$ are unknown, but constant, and *bounded.* That is to say, $|\theta_i| < k_{\theta_i}$ and the bounds, k_{θ_i} are *known.* The functions $f_i(x), g_{ki} : \Re^n \rightarrow \Re \;\; k = 1, 2, \ldots, m.$ are smooth

Construct n identifiers, of the following structure. The ith identifier identifying the parameter θ_i has the following form.

$$\dot{\hat{x}}_i = \hat{\theta}_i f_i(x) + [\sum_{j=1}^{m} g_{ij}(x)u_j] + v_i \quad (6.50)$$

$$\dot{\hat{\theta}}_i = w_i \quad (6.51)$$

$$v_i = 2k_{\theta_i}|f_i(x)|sgn[\tilde{x}_i] \quad (6.52)$$

$$w_i = k_i sgn[\tilde{x}_i]sgn[f_i(x)] \quad (6.53)$$

$$\tilde{x}_i = x_i - \hat{x}_i \quad (6.54)$$

$$k_i > 0 \quad (6.55)$$

$$i = 1, 2, \ldots, n \quad (6.56)$$

Given such an identifier structure, the state and parameter estimation error dynamics may be written in the following form,

$$\dot{\tilde{x}}_i = \tilde{\theta}_i f_i(x) - 2k_{\theta_i}|f_i(x)|sgn[\tilde{x}_i] \quad (6.57)$$

$$\dot{\tilde{\theta}}_i = -k_i sgn[\tilde{x}_i]sgn[f_i(x)] \quad (6.58)$$

$$i = 1, 2, \ldots, n \quad (6.59)$$

We now show existense of solutions for the identifier, stability of state estimation error dynamics, and the convergence of the identified parameter to its true value.

Theorem 6.2 *Existence of trajectories for identifier and state estimation error dynamics, stability of state estimation error dynamics, and exponential convergence of identified parameters to their true values in the sense of Filippov.*

Given *(G1) A nonlinear system of the form (6.46) - (6.49).*

(G2) A nonlinear identifier of the form (6.50) - (6.56) resulting in state estimation and parameter identification error dynamics of the form (6.57) - (6.58)

If *(I1)* $|\theta_i| < k_{\theta_i} \forall\, i = 1, 2, \ldots, n$

(I2) $|f_i(x)| \neq 0$ *along the system trajectories.*

(I3) In any compact region $D \in \Re^n$*, each* $f_i(x) : \Re^n \to \Re$ $i = 1, 2, \ldots, n$ *is bounded from above and from below.*

Then *(T1) Filippov solutions exist for the system (6.57) - (6.58).*

(T2) The surfaces $\tilde{x}_i = 0$ $i = 1, 2, \ldots n$ *are attractive.*

(T3) The sliding dynamics on the surfaces $\tilde{x}_i = 0$ $i = 1, 2, \ldots n$ *are such that the parameter estimate* $\hat{\theta}_i \to \theta_i$ $i = 1, 2, \ldots, n$ *exponentially in the sense of Filippov.*

Proof: ♠ ▷ As before we will prove the theorem in three steps. First we show existence of solutions, then we show attractivity of the sliding surfaces, and finally we show parameter convergence to true values using two methods of proof.

Step 1: Existence of Filippov Solutions

In compact domains D_i $i = 1, 2, \ldots, n$ of the $(x, \tilde{x}_i, \tilde{\theta}_i)$ space, that is $D_i \in \Re^n \times \Re \times [\Re \bmod k_{\theta_i}]$, the righthandside of equations (6.57) - (6.58) can be modelled by a differential inclusions of the following form

$$\dot{\tilde{x}}_i \in \mathcal{F}_{x_i}(x, \tilde{x}_i, \tilde{\theta}_i) \tag{6.60}$$

$$\dot{\tilde{\theta}}_i \in \mathcal{F}_{\theta_i}(x, \tilde{x}_i, \tilde{\theta}_i) \tag{6.61}$$

where the inclusions $\mathcal{F}_{x_i}(x,\tilde{x}_i,\tilde{\theta}_i)$ $i = 1,2,\ldots,n$ and $\mathcal{F}_{\theta_i}(x,\tilde{x}_i,\tilde{\theta}_i)$ $i = 1,2,\ldots,n$ are defined to be

$$\mathcal{F}_{x_i}(x,\tilde{x}_i,\tilde{\theta}_i) = \tilde{\theta}f_i(x) - 2k_{\theta_i}|f_i(x)|sgn[\tilde{x}_i] \ \ if \ \tilde{x}_i \neq 0 \tag{6.62}$$

$$\in [-3k_{\theta_i}k_{f_i}, 3k_{\theta_i}k_{f_i}] \ \ if \ \tilde{x}_i = 0 \tag{6.63}$$

$$\mathcal{F}_{\theta_i}(x,\tilde{x}_i,\tilde{\theta}_i) = -k_1 sgn[\tilde{x}_i]sgn[f_i(x)] \ if \ \tilde{x}_i \neq 0 \tag{6.64}$$

$$\in [-k_i, k_i] \ \ if \ \tilde{x}_i = 0 \tag{6.65}$$

$$i = 1,2,\ldots,n \tag{6.66}$$

where k_{f_i} is supremum of the function $f_i(x) : \Re \rightarrow \Re$ $i = 1,2,\ldots,n$ over all values of $x \in D_i$. The inclusions $\mathcal{F}_{x_i}(x,\tilde{x}_i,\tilde{\theta}_i)$ $i = 1,2,\ldots,n$ and $\mathcal{F}_{\theta_i}(x,\tilde{x}_i,\tilde{\theta}_i)$ $i = 1,2,\ldots,n$ so defined are

- closed, convex, bounded and upper-semicontinuous.

Therefore, invoking the theorem on existense of Filipov-solutions we show that solutions exist for the system represented by equations (6.57) and (6.58).

Step 2: Attractivity of Sliding Surfaces

We will now show that the surfaces $\tilde{x}_i = 0$ $i = 1,2,\ldots,n$ are attractive. Consider a candidate Lyapunov function $V(\tilde{x}_1,\ldots,\tilde{x}_n) : \Re^n \rightarrow \Re_+$ given by

$$V = \sum_{i=1}^{n} \frac{\tilde{x}_i^2}{2} \tag{6.67}$$

Differentiating (6.67) along the flow of (6.57), we get

$$\dot{V} = \sum_{i=1}^{n} \tilde{x}_i\tilde{\theta}_i f_i(x) - 2k_{\theta_i}|f_i(x)||\tilde{x}_i| \tag{6.68}$$

$$\leq -\sum_{i=1}^{n} |\tilde{x}_i|[2k_{\theta_i}|f_i(x)| - \tilde{\theta}_i f_i(x)] \tag{6.69}$$

$$\leq 0 \tag{6.70}$$

Here we have used the fact that $2k_{\theta_i}|f_i(x)| > \tilde{\theta}f_i(x)$, and that $f_i(x) \neq 0$ along the system trajectories. Thus $\dot{V}$ is shown to be negative definite thus showing the global attractivity of the surface $\tilde{x}_i = 0$ $i = 1,2,\ldots,n$.

As the term $2k_{\theta_i}|f_i(x)| > \tilde{\theta}_i f_i(x)$, it is clear that the surface $\tilde{x}_i = 0$ is attained in *finite time* and that sliding occurs at the surface $\tilde{x}_i = 0$.

Step 3: Parameter Convergence

As before, we will show convergence of estimated parameter values to true values using two methods of proof. The first method will be the method of Filippov averaging, and the second one is by the equivalent control method.

Proof By Filippov Averaging

The sliding mode on the surface $\tilde{x}_i = 0$ may be estimated to be a convex combination of the the dynamics on either side of the surface. This indeed is the principle behind the Filippov averaging technique. The averaged dynamics of the estimation error on the surface $\tilde{x}_i = 0$ takes the following form.

$$\begin{aligned}
\dot{\tilde{x}}_{i_{average}} &= \gamma_i[\tilde{\theta}_i f_i(x) - 2k_{\theta_i}|f_i(x)|] + [1-\gamma_i][\tilde{\theta}_i f_i(x) + 2k_{\theta_i}|f_i(x)|] && (6.71)\\
\dot{\tilde{\theta}}_{i_{average}} &= \gamma_i[-ksgn[f_i(x)]] + [1-\gamma_i][k_i sgn[f_i(x)]] && (6.72)\\
& i = 1, 2, \ldots, n && (6.73)
\end{aligned}$$

where $0 \leq \gamma_i \leq 1$ $i = 1, 2, \ldots, n$ But as $\tilde{x}_{i_{average}} = 0$ in finite time, set the left hand side of equation (6.71) to 0 and solve for γ_i.

$$\begin{aligned}
\dot{\tilde{x}}_{i_{average}} &= \gamma_i[\tilde{\theta}_i f_i(x) - 2k_{\theta_i}|f_i(x)|] + [1-\gamma_i][\tilde{\theta}_i f_i(x) + 2k_{\theta_i}|f_i(x)|] && (6.74)\\
0 &= \gamma_i[\tilde{\theta}_i f_i(x) - 2k_{\theta_i}|f_i(x)|] + [1-\gamma_i][\tilde{\theta}_i f_i(x) + 2k_{\theta_i}|f_i(x)|] && (6.75)\\
\gamma_i &= \frac{\tilde{\theta}_i sgn[f_i(x)]}{4k_{\theta_i}} + \frac{1}{2} && (6.76)\\
& i = 1, 2, \ldots, n && (6.77)
\end{aligned}$$

To find the sliding dynamics of $\tilde{\theta}_{i_{average}}$ along the surface $\tilde{x}_i = 0$, substitute the value of γ_i obtained in equation (6.76) into equation (6.72) to obtain,

$$\begin{aligned}
\dot{\tilde{\theta}}_{i_{average}} &= \gamma_i[-k_i sgn[f_i(x)]] + [1-\gamma_i][k_i sgn[f_i(x)]] && (6.78)\\
&= -2\gamma_i[k_i sgn[f_i(x)]] + k_i sgn[f_i(x)] && (6.79)\\
&= -2k_i sgn[f_i(x)][\frac{\tilde{\theta}_i sgn[f_i(x)]}{4k_{\theta_i}} + \frac{1}{2}] + k_i sgn[f_i(x)] && (6.80)
\end{aligned}$$

$$\dot{\tilde{\theta}}_{i\,average} = -\frac{k_i}{2k_{\theta_i}}\tilde{\theta}_i \tag{6.81}$$

$$i = 1, 2, \ldots, n \tag{6.82}$$

It is clear from equation (6.81), from the averaged dynamics of the estimation error that the parameter estimation error $\tilde{\theta}_i$ $i = 1, 2, \ldots, n$ is *exponentially diminishing* in the sense of Lyapunov. This in particular ensures that the average value of the estimated parameter $\hat{\theta}_i \rightarrow \theta_i$ $i = 1, 2, \ldots, n$ exponentially.

Proof By Equivalent Control

As before, we will find the value of the n functions $sgn[\tilde{x}_i] : \Re \rightarrow [-1, 1]$ $i = 1, 2, \ldots, n$ necessary to ensure the invariance of the sliding surfaces $\tilde{x}_i = 0$ $i = 1, 2, \ldots, n$. The values are obtained by setting the lefthandsides of equation (6.57) to 0. That is,

$$\dot{\tilde{x}}_i = \tilde{\theta}_i f_i(x) - 2k_{\theta_i}|f_i(x)|sgn[\tilde{x}_i] \tag{6.83}$$

$$0 = \tilde{\theta}_i f_i(x) - 2k_{\theta_i}|f_i(x)|sgn[\tilde{x}_i] \tag{6.84}$$

$$sgn[\tilde{x}_i] = \frac{1}{2k_{\theta_i}}\tilde{\theta}_i sgn[f_i(x)] \tag{6.85}$$

$$i = 1, 2, \ldots, n \tag{6.86}$$

Using equation (6.85) in equation (6.58), we rewrite the parameter estimation error dynamics as

$$\dot{\tilde{\theta}}_i = -k_i sgn[\tilde{x}_i] sgn[f_i(x)] \tag{6.87}$$

$$\dot{\tilde{\theta}}_i = -\frac{k_i}{2k_{\theta_i}}\tilde{\theta}_i \tag{6.88}$$

$$i = 1, 2, \ldots, n \tag{6.89}$$

The exponential convergence of the parameter identification error is obvious. This concludes the proof of the theorem. $\triangleleft$♠

6.2.3 General Nonlinear Systems

As a final extension, we consider the general case when there are $n \in \Re_+$ state equations with $m < n \in \Re_+$ affine, unknown but bounded parameters whose bounds are known.. We now make a crucial assumption related to observability of the system, which states that *there are as many dynamical equations containing the unknown parameters, as there are unknown parameters.* That is to say, the system equations are of the following form.

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{bmatrix} = \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_m \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{r} g_{1j}(x)u_j \\ \vdots \\ \sum_{j=1}^{r} g_{mj}(x)u_j \end{bmatrix} \tag{6.90}$$

$$\begin{bmatrix} \dot{x}_{m+1} \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_{(m+1)}(x) + \sum_{j=1}^{r} g_{m+1j}(x)u_j \\ \vdots \\ f_n(x) + \sum_{j=1}^{r} g_{nj}(x)u_j \end{bmatrix} \tag{6.91}$$

where $x \in \Re^n$ $\theta_i \in \Re$ $i = 1, \ldots, m$ are unknown but bounded,$|\theta_i| < k_{\theta_i}$ $i = 1, \ldots, m$, and $f_i(x), g_{ij}(x) : \Re^n \to \Re$ $i = 1, 2, \ldots, n$ $j = 1, 2, \ldots, r$.

Now consider the following identifier structure.

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \vdots \\ \dot{\hat{x}}_m \end{bmatrix} = \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \vdots \\ \hat{\theta}_m \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{r} g_{1j}(x)u_j \\ \vdots \\ \sum_{j=1}^{r} g_{mj}(x)u_j \end{bmatrix} \tag{6.92}$$

$$+ \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \tag{6.93}$$

$$\begin{bmatrix} \dot{\hat{\theta}}_1 \\ \vdots \\ \dot{\hat{\theta}}_m \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \tag{6.94}$$

$$v_i = k_{sup} sgn[\tilde{x}_i] \quad i = 1, \ldots, m \tag{6.95}$$

$$k_{sup} = 2m sup[k_{\theta_i}][\sum_{i=1}^{m}\sum_{j=1}^{m} |f_{ij}|] \tag{6.96}$$

$$\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} l_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_m \end{bmatrix} \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix}^{-1} \tag{6.97}$$

$$\begin{bmatrix} k_{sup}sgn[\tilde{x}_1] \\ \vdots \\ k_{sup}sgn[\tilde{x}_m] \end{bmatrix}$$

$$\tilde{x}_i = x_i - \hat{x}_i \tag{6.98}$$

$$l_i > 0 \quad i = 1, \ldots, m \tag{6.99}$$

The resulting state-estimation and parameter identification error dynamics has the following forms

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \vdots \\ \dot{\tilde{x}}_m \end{bmatrix} = \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix} \begin{bmatrix} \tilde{\theta}_1 \\ \vdots \\ \tilde{\theta}_m \end{bmatrix} - \begin{bmatrix} k_{sup}sgn[\tilde{x}_1] \\ \vdots \\ k_{sup}sgn[\tilde{x}_m] \end{bmatrix} \tag{6.100}$$

$$\begin{bmatrix} \dot{\tilde{\theta}}_1 \\ \vdots \\ \dot{\tilde{\theta}}_m \end{bmatrix} = -\begin{bmatrix} l_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_m \end{bmatrix} \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix}^{-1} \tag{6.101}$$

$$\begin{bmatrix} k_{sup}sgn[\tilde{x}_1] \\ \vdots \\ k_{sup}sgn[\tilde{x}_m] \end{bmatrix} \tag{6.102}$$

We now show existense of solutions for the identifier, stability of state estimation error dynamics, and the convergence of the identified parameter to its true value.

Theorem 6.3 *Existence of trajectories for identifier and state estimation error dynamics, stability of state estimation error dynamics, and exponential convergence of identified parameters to their true values in the sense of Filippov.*

Given *(G1) A nonlinear system of the form (6.90) - (6.91).*

(G2) A nonlinear identifier of the form (6.93) - (6.99) resulting in state estimation and parameter identification error dynamics of the form (6.100) - (6.102)

If *(I1)* $|\theta_i| < k_{\theta_i} \forall\, i = 1, \ldots, m$

(I2) The matrix $\begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix}$ *has rank* m *along the trajectories of the system.*

(I3) In any compact region $D \in \Re^n$, *each* $f_{ij}(x) : \Re^n \to \Re$ $i = 1,2,\ldots,m$ $j = 1,2,\ldots,m$ *is bounded from above and from below.*

Then *(T1) Filippov solutions exist for the system (6.100) - (6.102).*

(T2) The surfaces $\tilde{x}_i = 0$ $i = 1,2,\ldots n$ *are attractive.*

(T3) The sliding dynamics on the surfaces $\tilde{x}_i = 0$ $i = 1,2,\ldots n$ *are such that the parameter estimate* $\hat{\theta}_i$ *converges to the true value* θ_i *exponentially in the sense of Filippov.*

Proof: ♠ ▷ As before we will prove the theorem in three steps. First we show existence of solutions, then we show the existence of attractive sliding surfaces, and finally we show parameter convergence to true values.

Step 1: Existence of Solutions

In compact domains D_i $i = 1,2,\ldots,n$ of the $(x, \tilde{x}_i, \tilde{\theta}_i)$ space, the righthand-side of equations (6.100) - (6.102) can be modelled by a differential inclusions of the following form

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \vdots \\ \dot{\tilde{x}}_m \end{bmatrix} \in \begin{bmatrix} \mathcal{F}_{x_1}(x, \tilde{x}_1, \tilde{\theta}_1, \ldots, \tilde{\theta}_m) \\ \vdots \\ \mathcal{F}_{x_m}(x, \tilde{x}_m, \tilde{\theta}_1, \ldots, \tilde{\theta}_m) \end{bmatrix} \tag{6.103}$$

$$\begin{bmatrix} \dot{\tilde{\theta}}_1 \\ \vdots \\ \dot{\tilde{\theta}}_m \end{bmatrix} \in \begin{bmatrix} \mathcal{F}_{\theta_1}(x, \tilde{x}_1, \tilde{\theta}_1, \ldots, \tilde{\theta}_m) \\ \vdots \\ \mathcal{F}_{\theta_m}(x, \tilde{x}_m, \tilde{\theta}_1, \ldots, \tilde{\theta}_m) \end{bmatrix} \tag{6.104}$$

where the inclusions $\mathcal{F}_{x_i}(x, \tilde{x}_i, \tilde{\theta}_1, \ldots, \tilde{\theta}_m)$ $i = 1,2,\ldots,m$ and $\mathcal{F}_{\theta_i}(x, \tilde{x}_i, \tilde{\theta}_1, \ldots, \tilde{\theta}_m)$ $i = 1,2,\ldots,m$ are defined as follows

$$\mathcal{F}_{x_i}(x, \tilde{x}_i, \tilde{\theta}_1, \ldots, \tilde{\theta}_m) = \sum_{j=1}^{m} [\tilde{\theta}_j f_{ij}(x) - k_{sup} sgn[\tilde{x}_i]] \quad if\ \tilde{x}_i \neq 0$$

$$
\begin{aligned}
&\in [-2k_{sup}, 2k_{sup}] \ if\ \tilde{x}_i = 0 \\
\mathcal{F}_{\theta_i}(x, \tilde{x}_i, \tilde{\theta}_i) &= -\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & l_i & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix}^{-1} \begin{bmatrix} sgn[\tilde{x}_1] \\ \vdots \\ sgn[\tilde{x}_m] \end{bmatrix} \\
&\quad if\ \tilde{x}_i \neq 0 \\
&\in [-k_1 k_{f_{min}}, k_1 k_{f_{min}}] if\ \tilde{x}_i = 0
\end{aligned}
$$

The inclusions are

- closed, convex, bounded and upper-semicontinuous.

Therefore, invoking the theorem on existense of Filipov-solutions we show that solutions exist for the system represented by equations (6.100) and (6.102).

Step 2: Attractivity of Sliding Surfaces

We will now show that the surfaces $\tilde{x}_i = 0$ are attractive. Consider a candidate Lyapunov function

$$
V = \sum_{i=1}^{n} \frac{\tilde{x}_i^2}{2} \tag{6.105}
$$

Differentiating (6.105) along the flow of (6.100), we get

$$
\begin{aligned}
\dot{V} &= \begin{bmatrix} \tilde{x}_1 & \cdots & \tilde{x}_n \end{bmatrix} [\begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix} \begin{bmatrix} \tilde{\theta}_1 \\ \vdots \\ \tilde{\theta}_m \end{bmatrix} - \begin{bmatrix} k_{sup} sgn[\tilde{x}_1] \\ \vdots \\ k_{sup} sgn[\tilde{x}_m] \end{bmatrix}] \\
&\leq -\begin{bmatrix} |\tilde{x}_1| & \cdots & |\tilde{x}_n| \end{bmatrix} [\begin{bmatrix} k_{sup} \\ \vdots \\ k_{sup} \end{bmatrix} - \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix} \begin{bmatrix} \tilde{\theta}_1 sgn[\tilde{x}_1] \\ \vdots \\ \tilde{\theta}_m sgn[\tilde{x}_m] \end{bmatrix}] \\
&\leq -\begin{bmatrix} |\tilde{x}_1| & \cdots & |\tilde{x}_n| \end{bmatrix} [\begin{bmatrix} k_{sup} - \sum_{j=1}^{m} f_{1j} \tilde{\theta}_j sgn[\tilde{x}_1] \\ \vdots \\ k_{sup} - \sum_{j=1}^{m} f_{mj} \tilde{\theta}_j sgn[\tilde{x}_m] \end{bmatrix} \\
&\leq 0
\end{aligned}
$$

Here we have used the fact that $k_{sup} > \sum_{j=1}^{m} f_{ij} \tilde{\theta}_j sgn[\tilde{x}_i]$ $i = 1, 2, \ldots, m$, and that

$det \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix} \neq 0$ along the system trajectories. Thus $\dot{V}$ is shown to be negative definite.

As the term $k_{sup} > \sum_{j=1}^{m} f_{ij}\tilde{\theta}_j \; i = 1, 2, \ldots, m$, it is clear that the surface $\tilde{x}_i = 0 \; i = 1, 2, \ldots, m$ is attained in *finite time* and that sliding occurs on the surfaces $\tilde{x}_i = 0 \; i = 1, 2, \ldots, m$.

Step 3: Parameter Convergence

The sliding mode on the surfaces $\tilde{x}_i = 0 i = 1, 2, \ldots, m$ may be estimated to be a convex combination of the the dynamics on either side of the surface. This indeed in the principle behind the Filippov averaging technique. We will show the stability first using Filippov averaging and then by the equvalent control method.

Proof By Filippov Averaging

Here again we perform Filippov averaging using convex combinations of dynamics on either sides of the sliding surface. The averaging is a little more involved, and sharply contrasts to the simplicity of the equivalent control method of proof. Let

$$\begin{bmatrix} q_{11}(x) & \cdots & q_{1m}(x) \\ \vdots & \ddots & \vdots \\ q_{m1}(x) & \cdots & q_{mm}(x) \end{bmatrix} = \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix}^{-1} \tag{6.106}$$

where $q_{ij}(x) : \Re^n \to \Re \; i = 1, 2, \ldots, m$.

The averaged dynamics of the estimation error on the surface $\tilde{x}_i = 0 \; i = 1, 2, \ldots, m$ takes the following form.

$$\dot{\tilde{x}}_{1average} = \gamma_1[\sum_{j=1}^{m} f_{1j}\tilde{\theta}_j - k_{sup}] + [1 - \gamma_1][\sum_{j=1}^{m} f_{1j}\tilde{\theta}_j + k_{sup}] \tag{6.107}$$

$$\vdots \quad \vdots \quad \vdots \tag{6.108}$$

$$\dot{\tilde{x}}_{maverage} = \gamma_m[\sum_{j=1}^{m} f_{mj}\tilde{\theta}_j - k_{sup}] + [1 - \gamma_m][\sum_{j=1}^{m} f_{mj}\tilde{\theta}_j + k_{sup}] \tag{6.109}$$

$$\begin{bmatrix} \dot{\tilde{\theta}}_{1average} \\ \vdots \\ \dot{\tilde{\theta}}_{maverage} \end{bmatrix} = - \begin{bmatrix} l_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_m \end{bmatrix} \begin{bmatrix} q_{11}(x) & \cdots & q_{1m}(x) \\ \vdots & \ddots & \vdots \\ q_{m1}(x) & \cdots & q_{mm}(x) \end{bmatrix}^{-1} \begin{bmatrix} k_{sup}[1 - 2\gamma_1] \\ \vdots \\ k_{sup}[1 - 2\gamma_m] \end{bmatrix}$$

Setting the lefthandside of (6.107) - (6.109) to 0, we solve for γ_i $i = 1, 2, \ldots, m$ and use it to find the average dynamics of the parameter estimation error. That is,

$$\begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{bmatrix} = \frac{1}{2k_{sup}}[\begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix}\begin{bmatrix} \tilde{\theta}_1 \\ \vdots \\ \tilde{\theta}_m \end{bmatrix} + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}] \tag{6.110}$$

$$\begin{bmatrix} \dot{\tilde{\theta}}_{1average} \\ \vdots \\ \dot{\tilde{\theta}}_{maverage} \end{bmatrix} = -\begin{bmatrix} \frac{l_1}{k_{sup}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{l_m}{k_{sup}} \end{bmatrix}\begin{bmatrix} q_{11}(x) & \cdots & q_{1m}(x) \\ \vdots & \ddots & \vdots \\ q_{m1}(x) & \cdots & q_{mm}(x) \end{bmatrix}$$
$$\begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix}\begin{bmatrix} \tilde{\theta}_1 \\ \vdots \\ \tilde{\theta}_m \end{bmatrix}$$

Now using equation (6.106), we rewrite the above equation as

$$\begin{bmatrix} \dot{\tilde{\theta}}_{1average} \\ \vdots \\ \dot{\tilde{\theta}}_{maverage} \end{bmatrix} = \begin{bmatrix} -\frac{l_1}{k_{sup}}\tilde{\theta}_1 \\ \vdots \\ -\frac{l_m}{k_{sup}}\tilde{\theta}_m \end{bmatrix} \tag{6.111}$$

Exponential convergence of the average value of the identifier to 0 is evident. We will now show the same result from an equivalent control view point.

Proof By Equivalent Control

As before we estimate the values taken by the functions $sgn[\tilde{x}_i] : \Re \to [-1, 1]$ $i = 1, 2, \ldots, m$. Indeed, we see from equation (6.100) that

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \vdots \\ \dot{\tilde{x}}_m \end{bmatrix} = \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix}\begin{bmatrix} \tilde{\theta}_1 \\ \vdots \\ \tilde{\theta}_m \end{bmatrix} - \begin{bmatrix} k_{sup}sgn[\tilde{x}_1] \\ \vdots \\ k_{sup}sgn[\tilde{x}_m] \end{bmatrix} \tag{6.112}$$

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix}\begin{bmatrix} \tilde{\theta}_1 \\ \vdots \\ \tilde{\theta}_m \end{bmatrix} - \begin{bmatrix} k_{sup}sgn[\tilde{x}_1] \\ \vdots \\ k_{sup}sgn[\tilde{x}_m] \end{bmatrix} \tag{6.113}$$

$$\begin{bmatrix} sgn[\tilde{x}_1] \\ \vdots \\ sgn[\tilde{x}_m] \end{bmatrix} = \frac{1}{k_{sup}} \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix} \begin{bmatrix} \tilde{\theta}_1 \\ \vdots \\ \tilde{\theta}_m \end{bmatrix} \tag{6.114}$$

Using (6.114) in the parameter error equation (6.102), we find

$$\begin{bmatrix} \dot{\tilde{\theta}}_1 \\ \vdots \\ \dot{\tilde{\theta}}_m \end{bmatrix} = -\begin{bmatrix} l_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_m \end{bmatrix} \begin{bmatrix} f_{11}(x) & \cdots & f_{1m}(x) \\ \vdots & \ddots & \vdots \\ f_{m1}(x) & \cdots & f_{mm}(x) \end{bmatrix}^{-1} \begin{bmatrix} sgn[\tilde{x}_1] \\ \vdots \\ sgn[\tilde{x}_m] \end{bmatrix} \tag{6.115}$$

$$= \begin{bmatrix} -\frac{l_1}{k_{sup}}\tilde{\theta}_1 \\ \vdots \\ -\frac{l_m}{k_{sup}}\tilde{\theta}_m \end{bmatrix} \tag{6.116}$$

It is obvious that the parameter estimation error is exponentially diminishing. This concludes the proof of the theorem. ⊲♠

6.2.4 Illustrative Example

Consider a system represented by the following dynamical equations.

$$\dot{x}_1 = \theta_1 \cos x_1 - \theta_2 \sin x_1 + u_1 \tag{6.117}$$

$$\dot{x}_2 = \theta_1 \sin x_1 + \theta_2 \cos x_1 + u_2 \tag{6.118}$$

where $x \in \Re^2$, $\theta_i \in \Re$ $i = 1, 2$, is unknown but bounded with a known bound $|\theta_1 < k_{\theta_i} \in \Re_+$ $i = 1, 2$, $u_i \in \Re$ $i = 1, 2$ are control inputs to the system. The goal of identification is to estimate θ_1 and θ_2 of the system.

We construct a sliding mode identifier for this system of the following form.

$$\dot{\hat{x}}_1 = \hat{\theta}_1 \cos x_1 - \hat{\theta}_2 \sin x_1 + u_1 + v_1 \tag{6.119}$$

$$\dot{\hat{x}}_2 = \hat{\theta}_1 \sin x_1 + \hat{\theta}_2 \cos x_1 + u_2 + v_2 \tag{6.120}$$

$$\dot{\hat{\theta}}_1 = w_1 \tag{6.121}$$

$$\dot{\hat{\theta}}_2 = w_2 \tag{6.122}$$

$$v_1 = 2k_{sup}[|(\cos x_1)| + |\sin x_1|]sgn[\tilde{x}_1] \tag{6.123}$$

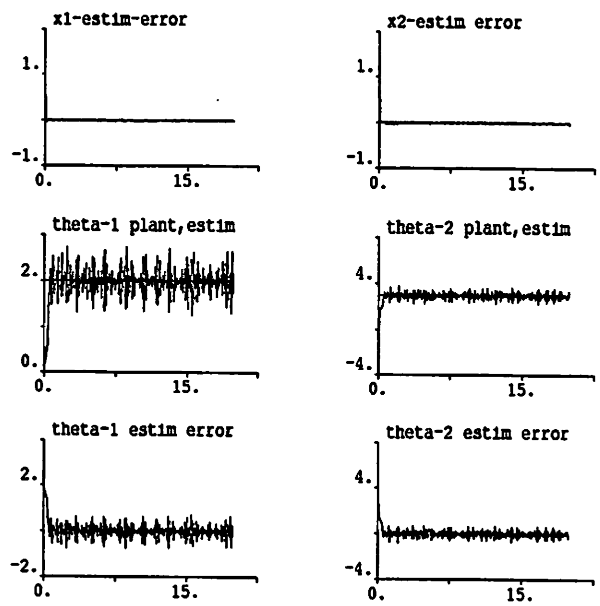

Figure 6.2: Convergence Of Parameter To True Values

$$v_2 = 2k_{sup}[|(\cos x_1)| + |\sin x_1|]sgn[\tilde{x}_2] \tag{6.124}$$

$$k_{sup} = k_{\theta_1} + k_{\theta_2} \tag{6.125}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} \cos x_1 & -\sin x_1 \\ \sin x_1 & \cos x_1 \end{bmatrix}^{-1} \begin{bmatrix} sgn[\tilde{x}_1] \\ sgn[\tilde{x}_2] \end{bmatrix} \tag{6.126}$$

$$\tilde{x}_1 = x_1 - \hat{x}_1 \tag{6.127}$$

$$\tilde{x}_2 = x_2 - \hat{x}_2 \tag{6.128}$$

$$l_1 > 0 \tag{6.129}$$

$$l_2 > 0 \tag{6.130}$$

The convergence of the estimated parameters $\hat{\theta}_1$, $\hat{\theta}_2$ to their true values is easily shown invoking Theorem 5.1.

Simulation of the system show the interesting features of the identifier. First we note that the state estimation errors goes to zero in finite time, as predicted. Then the identified parameters converges to their true value, and the parameter identification error gradually goes to zero. But as Filippov solutions are only solutions averaged over neighbourhoods of diminishing size, the average value of the parameter error is zero, though the actual values chatter about their true values.

6.2.5 Application And Commercial Importance

Identification of the operating parameters of a machine help us design better and more robust control laws. Each operation of a machine is an opportunity to identify the parameters of the machine, either for purposes of control, or even as a system check to ensure that all the parameters are within operating ranges. For the typical operation of an automobile, it could very well be possible to adaptively identify the mass of the vehicle, and the tyre friction coefficient. Such identification performed in *real time* allows the designer the opportunity to choose control laws that depend in *real time* on the identification process. The class of identifiers which formed the subject of this section are exciting in that they are nonlinear identifiers that are capable of providing such exponential parameter convergence.

6.2.6 Criticism And Future Prospects

The following are the implementational difficulties associated with the identifier.

- There is no a-priori guarantee that the multiplicand of the unknown parameters will be non-zero along the system trajectories. This is a major assumption, and its elimination virtually impossible.
- Parameter convergence is guaranteed only in the sense of Filippov. That is the average value of the parameter converges to its true value, but there is no bound on the chatter in the value of the parameter. However, it is possible to design sliding mode identifiers such that the chatter in parameter values is minimized.
- Extending the theory to systems where the number of parameters exceeds the number of dynamical equations is non-trivial.
- The use of the sliding mode technique provides a certain degree of robustness to the identifier, though more careful analysis is necessary to identify those perturbations which do not adversely affect robust identification.

- While the parameter identifier specifies exponentially accurate values of the parameter, incorporating the parameter identifier into a closed loop feedback scheme, is considerably more involved.

6.3 Synchronous Sliding Modes

6.3.1 Introduction

In this section, we present an interesting property of a modified vector sliding mode control law, and its possible application. The property of this modified vector sliding mode control law is such that it achieves simultaneous regulation for a group of n scalar systems with n inputs. The control law has the interesting property that it can be prescribed without *explicit* reference to the initial conditions of the system. The law is interesting in that it introduces coupling between decoupled systems to achieve the synchronization objective.

We present a simple example to verify the synchronization result. We then apply this control law to the problem of tracking trajectories by a system of robots or multifingered hands.

6.3.2 Synchronous Sliding

Consider a group of n scalar decoupled systems of the form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \tag{6.131}$$

$$\begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ \vdots \\ x_{n0} \end{bmatrix} \tag{6.132}$$

where the states $x_i \in \Re$, the controls $u_i \in \Re$ $i = 1, 2, \ldots, n$ the initial conditions $x_{i0} \in \Re$ $i = 1, 2, \ldots, n$. With minor abuse of notation, we create a new state vector $x \in \Re^n$, where $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$.

The control objective is to regulate the states from non-zero initial conditions to the origin, in finite time. That is, that there exist instants of time $t_i^* < \infty \in \Re_+ \ i = 1, 2, \ldots, n$ such that the following is true.

$$x_i(t) = 0 \; \forall \, t \geq t_i^* \; i = 1, 2, \ldots, n \tag{6.133}$$

We choose n sliding mode control laws of the following form to ensure achievement of the control objective.

$$u_i = -k_i \frac{x_i}{|x_i|} \text{ if } |x_i| \neq 0 \quad i = 1, 2, \ldots, n \tag{6.134}$$

where $k_i \in \Re_+$

Comment 6.3.1 We note here that the controls u_i $i = 1, 2, \ldots$ are decoupled, in that u_i is a function only of x_i. Also note that the time taken by each state x_i $i = 1, 2, \ldots, n$ to reach the origin is a function of its initial value $x_i(0)$ $i = 1, 2, \ldots, n$ and the control gains k_i $i = 1, 2, \ldots, n$.

Comment 6.3.2 Also note in equation (6.134) we did not specify the control law at $|x_i| = 0$ $i = 1, 2, \ldots, n$. Indeed, $u_i = -k_i sgn[x_i]$ $|x_i| \neq 0$. We do not specify the control at $|x_i| = 0$. As the control is not specified only on sets of zero measure, it does not affect the existence of Filippov solutions shown by modelling the system by a differential inclusion.

We now present some interesting properties of a modified sliding mode control law that deliberately introduces coupling between the decoupled systems. We present proof of existence of solutions, proof of stability, and proof of synchronous finite time convergence for the modified sliding mode control law. In order to do so, we formalize the notion of *synchronous finite time convergence.*

Definition 6.3.1 *A set of $n \in Z_+$ variables $x_i(t) : \Re_+ \rightarrow \Re$ $i = 1, 2, \ldots, n$ are said to reach the origin* synchronously *commencing from nonzero initial conditions $x_i(0) \neq 0$ $i = 1, 2, \ldots, n$ if there exists an instant of time $t^* < \infty \in \Re_+$ such that the* *following is true.*

$$x_i(t) \neq 0 \,\forall\, t < t^* \tag{6.135}$$

$$x_i(t) = 0 \,\forall\, t \geq t^* \tag{6.136}$$

$$i = 1, 2, \ldots, n \tag{6.137}$$

That is to say, that the states with nonzero initial conditions (an assumption we make without loss of generality) are regulated to 0 at the *same* instant of time t^*. There are many practical applications where such *synchronous* regulation is important. A typical application is a multifingered robot hand that grips an object. It is important to ensure that the fingers touch the object *synchronously* and thus cause force closure without imparting motion to the object. We will say more about this later.

It is possible to ensure synchronous motion using a simple sliding mode feedback where the control gains are chosen with explicit dependence on initial conditions. Indeed, given the initial conditions exactly, we choose a decoupled control law that uses the values of initial conditions to derive control gains that guarantee synchronous reaching of the origin. For the sake of completeness we state the control law as follows.

Theorem 6.4 *Synchronous regulation with explicit dependence on initial conditions.*

Given *(G1) A nonlinear system of the form (6.131) - (6.132).*

(G2) A control law of the form (6.134)

If *(I1)* k_i $i = 1, 2, \ldots, n$ *are chosen such that*

$$\frac{|x_i(0)|}{k_i} = \frac{|x_j(0)|}{k_j} \; i = 1, 2, \ldots, n \;\; j = 1, 2, \ldots, n \tag{6.138}$$

Then *(T1) Filippov solutions exist for the system (6.131) - (6.132) subject to the control law (6.134).*

(T2) The surfaces $x_i = 0$ $i = 1, 2, \ldots n$ *are reached synchronously at a time* $t^* = \frac{x_1(0)}{k_1}$.

Proof: ♠ ▷ The proof is quite straightforward and utilizes standard facts from sliding mode control theory.

The existence of Filippov solutions is shown using the fact that the modelling differential inclusions $\mathcal{F}_i(x) : \Re \rightarrow [-1, 1]$ are closed, bounded, convex and uppersemicontinuous. Note that $\mathcal{F}_i(x) : \Re \rightarrow [-1, 1]$ are defined as follows

$$\mathcal{F}_i(x) = -k_i \frac{x_i}{|x_i|} \text{ if } |x_i| \neq 0 \quad (6.139)$$

$$\in [-1, 1] \text{ if } |x_i| = 0 \quad (6.140)$$

Stability is shown using the candidate Lyapunov function $V(x) : \Re^n \rightarrow \Re_+$ given by $V(x) = \sum_{i=1}^{n} \frac{x_i^2}{2}$ whose derivative along the flow of (6.131) - (6.132) is given by $\dot{V} = -\sum_{i=1}^{n} |x_i|$. Indeed $\dot{V}$ is negative definite proving global exponential stability of the origin.

Finally, the time taken to reach the origin is given by $t_i^\star = \frac{|x_i(0)|}{k_i}$ $i = 1, 2, \ldots, n$. Now using the assumption that $\frac{|x_i(0)|}{k_i} = \frac{|x_j(0)|}{k_j}$ $i = 1, 2, \ldots, n$ $j = 1, 2, \ldots, n$, we see that $t_1^\star = t_2^\star \ldots = t_n^\star = t^\star$.

This completes the proof of the theorem. ◁♠

Comment 6.3.3 The control law is inelegant to implement as it explicitly depends on the initial conditions. It would be desirable to develop a state feedback control law that would achieve the same objective, but one whose control gains do not explicitly depend on initial conditions.

We now propose a state feedback control law that would ensure synchronous regulation.

Theorem 6.5 *Synchronous regulation with state feedback.*

Given *(G1) A nonlinear system of the form (6.131) - (6.132).*

If *(I1) The controls u_i $i = 1, 2, \ldots, n$ in equations (6.131) - (6.132) are chosen to be*

$$u_i = -k^\star \frac{x_i}{||x||_2} \textit{ if } ||x||_2 > 0 \;\; i = 1, 2, \ldots, n \quad (6.141)$$

$$||x||_2 = [\sum_{i=1}^{n} x_i^2]^{\frac{1}{2}} \tag{6.142}$$

$$\tag{6.143}$$

where $k^* \in \Re_+$

Then *(T1) Filippov solutions exist for the system (6.131) - (6.132) subject to the control law (6.141).*

(T2) The surfaces $x_i = 0$ $i = 1, 2, \ldots n$ are reached synchronously at a time $t^* = \frac{||x(0)||_2}{k^*}$ where $||x(0)||_2$ is the 2-norm of the vector of initial conditions, given by $||x(0)||_2 = [\sum_{i=1}^{n} x_i^2(0)]^{\frac{1}{2}}$

Proof: ♠ ▷ We prove the theorem in three steps. First we show existence of generalized Filippov solutions to the system (6.131) - (6.132) subject to the control law (6.141). We then show attractivity of the origin when subject to the control law using a simple Lyapunov argument. Finally we show the achievement of synchronous regulation, by explicitly computing the times taken to reach the origin. We first make the following comments.

Comment 6.3.4 It is interesting to compare the control laws given by equations (6.134) and (6.141). While the control specified by (6.134) decouples the system entirely, the control specified by (6.141) introduces a coupling between the through the 2-norm of the state vector $||x||_2$. Furthermore, note that the control gains k^* remain the same for all u_i $i = 1, 2, \ldots, n$.

Comment 6.3.5 The discontinuous control law (6.141) is not defined at the origin, the same way the function $sgn[(\cdot)] : \Re \to [-1, 1]$ is not defined when $(\cdot) = 0$. But also note that the control law specified by (6.141) is bounded by k^*. Indeed, as $\frac{x_i}{||x||_2} \leq 1$ $i = 1, 2, \ldots, n$, $u_i \leq k^*$ $i = 1, 2, \ldots, n$.

Step 1: Existence Of Filippov Solutions

To show the existence of generalized Filippov solutions we model the system (6.131) - (6.132) subject to the control law (6.141) by the following differential inclusion.

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} \in \begin{bmatrix} \mathcal{F}_1(x) \\ \vdots \\ \mathcal{F}_n(x) \end{bmatrix} \tag{6.144}$$

where the inclusions $\mathcal{F}_i(x) : \Re \to [-k^*, k^*]$ are specified as

$$\begin{aligned} \mathcal{F}_i(x) &= -k^* \frac{x_i}{||x||_2} \text{ if } ||x||_2 > 0 && (6.145) \\ &\in [-k^*, k^*] \text{ if } ||x||_2 = 0 && (6.146) \\ & \quad i = 1, 2, \ldots, n && (6.147) \end{aligned}$$

The inclusions $\mathcal{F}_i(x)$ $i = 1, 2, \ldots, n$ are

- closed, bounded, convex and uppersemicontinuous.

Invoking the theorem on the existence of generalized Filippov solutions, we conclude that Filippov solutions exist for the system (6.131) - (6.132) subject to the control law (6.141).

Step 2: Attractivity Of The Origin

Consider a candidate Lyapunov function $V(x) : \Re^n \to \Re_+$ given by

$$V = \frac{x^T x}{2} \tag{6.148}$$

Differentiating V along the flow of (6.131) - (6.132) subject to the control law (6.141), we find

$$\begin{aligned} \dot{V} &= \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} \begin{bmatrix} -k^* \frac{x_1}{||x||_2} \\ \vdots \\ -k^* \frac{x_n}{||x||_2} \end{bmatrix} && (6.149) \\ &= -k^* \frac{||x||_2^2}{||x||_2} && (6.150) \\ &= -k^* ||x||_2 \text{ if } ||x||_2 \neq 0 && (6.151) \\ &\leq 0 && (6.152) \end{aligned}$$

Negative definiteness of $\dot{V}$ confirms the global exponential stability of the origin.

Step 3: Synchronous Reaching

From system (6.131) - (6.132) subject to the control law (6.141) the following is true for any i, j

$$\begin{aligned} \dot{x}_i &= -k^* \frac{x_i}{\|x\|_2} && (6.153) \\ \dot{x}_j &= -k^* \frac{x_j}{\|x\|_2} && (6.154) \\ \frac{dx_i}{dx_j} &= \frac{x_i}{x_j} && (6.155) \\ & \forall\, i, j \leq n \quad i \neq j \ \|x\|_2 \neq 0 && (6.156) \end{aligned}$$

Solving (6.155), we obtain explicit expressions for constraints on state trajectories as

$$x_i(t) = \frac{x_i(0)}{x_j(0)} x_j(t) \forall\, i, j \leq n \quad i \neq j \ \|x\|_2 \neq 0 \tag{6.157}$$

Using (6.157) in (6.153), we recast (6.153) in the form

$$\begin{aligned} \dot{x}_i &= -k^* \frac{x_i}{\|x\|_2} && (6.158) \\ &= -k^* \frac{x_i}{[\sum_{k=1}^{n} x_k^2]^{\frac{1}{2}}} && (6.159) \\ &= -k^* \frac{x_i}{[x_i^2 + \sum_{k=1,\, k\neq i}^{n} x_k^2]^{\frac{1}{2}}} && (6.160) \\ &= -k^* \frac{x_i}{[x_i^2 + \sum_{k=1,\, k\neq i}^{n} \frac{x_k^2(0)}{x_i^2(0)} x_i^2]^{\frac{1}{2}}} && (6.161) \\ &= -k^* \frac{x_i}{x_i[1 + \sum_{k=1,\, k\neq i}^{n} \frac{x_k^2(0)}{x_i^2(0)}]^{\frac{1}{2}}} && (6.162) \\ &= -k^* \frac{x_i(0)}{[x_i^2(0) + \sum_{k=1,\, k\neq i}^{n} x_k^2(0)]^{\frac{1}{2}}} && (6.163) \\ \dot{x}_i &= -k^* \frac{x_i(0)}{\|x(0)\|_2} \quad i = 1, 2, \ldots, n && (6.164) \end{aligned}$$

The righthandside of (6.164) is a real constant, and therefore the solution of (6.164) is given by

$$x_i(t) = -k^* \frac{x_i(0)}{\|x(0)\|_2} t + x_i(0) \quad i = 1, 2, \ldots, n \tag{6.165}$$

From (6.165), we obtain the time t^* taken by $x_i(t)$ $i = 1, 2, \ldots, n$ to reach the origin, starting from arbitrary nonzero initial conditions by setting the righthandside of

(6.165) to 0.

$$0 = -k^* \frac{x_i(0)}{||x(0)||_2} t^* + x_i(0) \tag{6.166}$$

$$t^* = \frac{||x(0)||_2}{k^*} \quad i = 1, 2, \ldots, n \tag{6.167}$$

Synchronous convergence of state trajectories commencing from nonzero initial conditions is thus shown. This concludes the proof of the theorem. ◁♠

6.3.3 Design Of Tracking Control Laws

The control laws that we have developed, are discontinuous. As a prelude to presenting tracking control laws that involve discontinuities, let us analyze a simple linear pole-placement control law from another perspective. Consider a system represented as a chain of integrators of the form,

$$\dot{x}_1 = x_2 \tag{6.168}$$

$$\vdots \quad \vdots \quad \vdots \tag{6.169}$$

$$\dot{x}_{n-1} = x_n \tag{6.170}$$

$$\dot{x}_n = u \tag{6.171}$$

where the state vector $x \in \Re^n$ and the control input $u \in \Re$. Given a desired smooth trajectory $x_{1d}(t) : \Re_+ \rightarrow \Re$ to be tracked by the state x_1 we present a tracking control law that uses succesive derivative of desired trajectories. We define recursively, a set of desired trajectories for the states as

$$x_{id}(t) = \frac{dx_{i-1,d}(t)}{dt} - k_{i-1}[x_i(t) - x_{i,d}(t)] \quad i = 2, 3, \ldots, n \tag{6.172}$$

While we are *given* a desired trajectory to be tracked by the state $x_1(t)$, we define desired trajectories for the remaining states the tracking of which automatically ensures the original tracking objective for $x_1(t)$. Indeed, the intution behind such a definition of desired trajectories becomes clear when we look at $x_{2d}(t)$.

$$x_{2d}(t) = \frac{dx_{1d}(t)}{dt} - k_1[x_1(t) - x_{1d}(t)] \tag{6.173}$$

From (6.173) it is clear that when the surface $x_2 = x_{2d}$ the resulting dynamics for $x_1(t)$ is given as

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) && (6.174) \\ &= x_{2d}(t) && (6.175) \\ &= \frac{dx_{1d}(t)}{dt} - k_1[x_1(t) - x_{1d}(t)] && (6.176) \end{aligned}$$

The dynamics of the system is such as to ensure that $x_1(t) \to x_{1d}(t)$ exponentially. However, if the surface $x_2 - x_{2d} = 0$ can only be reached exponentially, then the dynamics of $\dot{x}_1$ is perturbed by an exponentially decaying signal, and therefore invoking the result on the exponentially stable systems perturbed by exponentially decaying perturbations, we conclude exponential convergence of $x_1(t)$ to $x_{1,d}(t)$. We now show the relationship between control laws developed using the recursively defined desired trajectories and the standard pole-placement control law.

Theorem 6.6 *Connection between pole-placement and recursive trajectory definition.*

Given *(G1) A nonlinear system of the form (6.168) - (6.171).*

(G2) Given a set of desired trajectories of the form (6.172)

If *(I1) The controls u in equation (6.171) are chosen to be*

$$u = \frac{dx_{nd}(t)}{dt} - k_n[x_n(t) - x_{nd}(t)] \qquad (6.177)$$

where $x_{i,d}(t) : \Re^n \times \Re_+ \to \Re$ $i = 2, 3, \ldots, n$ is specified by (6.172) and $k_n \in \Re_+$.

Then *(T1) The control law specified by (6.177) is a stable pole-placement control with the n eigenvalues each being equal to $-k_i$ $i = 1, 2, \ldots, n$*

Proof: ♠ ▷ The proof is obvious by writing the dynamics for $\dot{x}_1$ and $\dot{x}_2$. Indeed,

$$\begin{aligned} \dot{x}_1 &= x_2 && (6.178) \\ \dot{x}_2 &= \frac{dx_{2d}(t)}{dt} - k_2[x_2(t) - x_{2d}(t)] && (6.179) \end{aligned}$$

Using the definition of $x_{2d}(t)$ provided by (6.172), we rewrite (6.179) as

$$\dot{x}_1 = x_2 \tag{6.180}$$

$$\dot{x}_2 = \frac{d[\frac{dx_{1d}(t)}{dt} - k_1[x_1(t) - x_{1d}(t)]]}{dt} \tag{6.181}$$

$$-k_2[x_2(t) - [\frac{dx_{1d}(t)}{dt} - k_1[x_1(t) - x_{1d}(t)]]] \tag{6.182}$$

Which may be rewritten as

$$\dot{x}_1 = x_2 \tag{6.183}$$

$$\dot{x}_2 = \frac{d^2x_{1d}(t)}{dt^2} + [k_1 + k_2]\frac{dx_{1d}(t)}{dt} + [k_1k_2]x_{1d}(t) \tag{6.184}$$

$$-[k_1 + k_2]x_2(t) - [k_1k_2]x_1(t) \tag{6.185}$$

That is to say

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{d^2x_{1d}(t)}{dt^2} + [k_1 + k_2]\frac{dx_{1d}(t)}{dt} + [k_1k_2]x_{1d}(t) \end{bmatrix} \tag{6.186}$$

$$-\begin{bmatrix} 0 & 1 \\ -[k_1k_2] & -[k_1 + k_2] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6.187}$$

The placement of poles through recursive trajectory definition is trivially obvious by inspection of equation (6.187). This concludes the proof of the theorem. ◁♠

Comment 6.3.6 It is to be noted that this tracking control law is valid for any specification of desired trajectories that are smooth, the tracking of which guarantees achievement of the control objective. That is, we are free to specify any smooth set of trajectories $x_{id}(t)$ $i = 2, 3, \ldots, n$, the only constraint being $x_i(t) = x_{id}(t)$ $i = 2, 3, \ldots, n \Rightarrow x_{i-1}(t) \rightarrow x_{i-1,d}(t)$. Indeed, the linear pole-placement control law is just a special case *of control laws that achieve this tracking objective.*

Comment 6.3.7 We now ask if it is possible to relax the smoothness assumption on the desired trajectories $x_{id}(t)$. Indeed, the first relaxation would be to consider desired trajectories that are differentiable almost everywhere, except possibly on sets of zero measure. The Nested and Switching control laws presented in the previous chapter *are examples of such discontinuous control laws, the discontinuities existing on sets of zero measure. The proofs of such control laws are much harder in general, though the regularization of such control laws that involve saturation functions have been used in the recent literature. We have been inspired by the attempts of [31] in developing control laws that use Filippov averaging instead of regularization. That is to say, that we are prepared to tolerate chatter and limit cycling by using discontinuous control laws. The drawback however is that we can show finite time synchronous stabilization only on the average, whereas a regularized control law, by eliminating the discontinuity would permit smooth stabilization, though exponentially, without the chatter.*

Comment 6.3.8 Our interest in relaxing the smoothness assumption on the desired trajectories is merely enable us to utilize the discontinuous, synchronous control law for a practical mechanical system.

We will first present the control law for a group of $n \in Z_+$ and then mechanical systems, apply it to a well known example of a two fingered robotic hand. Many mechanical systems are represented by Newtons force and torque balance equations that assume the form

$$\dot{x}_1^i = x_2^i \tag{6.188}$$

$$\dot{x}_2^i = u^i \tag{6.189}$$

where $x^i \in \Re^2$ is the state of the ith mechanical system where $i \leq n \in Z_+$, and $u^i(x,t) : \Re^2 \times \Re_+ \rightarrow \Re$ is the input force. Typically, x_1^i represents the generalized position coordinate of the mechanical system, and x_2^i represents the generalized velocity coordinate. These equations, though simple in form, serve to illustrate the application of the theory, and also represent a large class of useful physical systems. Given desired trajectories $x_{1d}^i(t) : \Re_+ \rightarrow \Re$ to be tracked by the states $x_1^i(t)$, we attempt to find control laws u^i that ensure synchronous tracking for the states $x_1^i(t)$.

We now state the theorem that ensures synchronous tracking for the systems of the form (6.188)-(6.189).

Theorem 6.7 *Synchronous tracking for a class of mechanical systems.*

Given *(G1) n mechanical systems, each of the form (6.188) - (6.189).*

(G2) Given a set of desired trajectories of the form $x^i_{1d}(t) : \Re_+ \rightarrow \Re$ $i = 1, 2, \ldots, n$

If *(I1) The controls $u^i(x,t)$ $i = 1, 2, \ldots, n$ in equation (6.189) are chosen to be*

$$u^i = \frac{dx^i_{2d}}{dt} - k^\star_2 \frac{x^i_2 - x^i_{2d}}{[\sum_{j=1}^n [x^j_2 - x^j_{2d}]^2]^{\frac{1}{2}}} \tag{6.190}$$

$$x^i_{2d} = \frac{dx^i_{1d}}{dt} - k^\star_1 \frac{x^i_1 - x^i_{1d}}{[\sum_{j=1}^n [x^j_1 - x^j_{1d}]^2]^{\frac{1}{2}}} \tag{6.191}$$

where$k_1, k_2 \in \Re_+$.

Then *(T1) Filippov solutions exist for system (6.188) - (6.189) subject to control (6.190).*

(T2) States $x^i_1(t)$ track their respective trajectories $x^i_{1d}(t)$ synchronously.

Proof: ♠ ▷ The proof is simple once we realize the validity of the system equations (6.188) - (6.189) subject to the control law (6.190) for arbitrarily small neighborhoods of the origin. Indeed, the control law is undefined only on a set of zero measure. As this set of zero measure is indeed the set we desire to make invariant, and the control law directs system trajectories to this set, and hence maintain invariance, the conclusions of the theorem naturally follow. The theorem can also be proved invoking the results of the nested, and switching control laws mentioned in the previous chapter. ◁♠

6.3.4 Application To Robotics

In this subsection we apply the proposed tracking control law to two robotic manipulators, in order to have the joint angles track desired trajectories at the same instant of time. Each manipulator is assumed to be a two link planar arm. The practical visualization of such a system would a two fingered robotic hand when they are contacting the same object.

First we present a very brief description of a typical robotic manipulator. The equations of motion of an n degree of freedom rigid robotic manipulator in the joint space may be written down as

$$M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + B\dot{q}(t) + g(q(t)) = T(t) \tag{6.192}$$

where the vectors $q(t) : \Re_+ \to \Re^n$ are the joint angles, $\dot{q}(t) : \Re_+ \to \Re^n$ are joint angular velocities, and $\ddot{q}(t) : \Re_+ \to \Re^n$ are joint angular accelerations. $M(q(t)) : \Re_+ \to \Re^{n\times n}$ denotes the symmetric inertia matrix, which is positive definite for all $q \in \Re^n$; the vector $C(q(t),\dot{q}(t))\dot{q}(t)$ denotes Coriolis and centripetal torques, while the vector $g(q(t))$ denotes gravitational effects and the matrix $B \in \Re^{n\times n}$ is a constant positive definite (diagonal) matrix representing damping in the system. $T(t)$ represents the vector of generalized forces applied to the manipulator joints.

Different tasks are accomplished by the robotic manipulators by designing different control forces $T(t)$. A widely used, and perhaps simplest control scheme is the *computed torque* technique. This technique is based on the exact knowledge of the manipulator dynamics, and results in a controller that achieves tracking of desired trajectories. One possible computed torque control is

$$T(t) = C(q(t),\dot{q}(t))\dot{q}(t) + B\dot{q}(t) + g(q(t)) + M(q(t))u(t) \tag{6.193}$$

where $M(q(t))u(t) \in \Re^n$ is an input applied at the manipulator joints. The computed torque T is realized by measuring $q(t)$, and $\dot{q}$ for all $t > 0$ and constructing the Coriolis, damping, gravitational terms and the inertia matrix. The torque T applied to the system (6.192) results in

$$\ddot{q}(t) = u(t) \tag{6.194}$$

Now consider the two fingered robotic arm. The joint angles are denoted by $q_1^1, q_1^2, q_1^3, q_1^4$. The equations of motion are of course similar to (6.192). The dynamics of these manipulators therefore are of a form similar to (6.194), given by,

$$\dot{q}_1^i = q_2^i \tag{6.195}$$

$$\dot{q}_2^i = u^i \tag{6.196}$$

$$i = 1, 2, \ldots, 4 \tag{6.197}$$

Given desired trajectories $q^i_{1d}(t)$ $i = 1, 2, \ldots, 4$ to be tracked by the respective state variables $q^i_1(t)$ $i = 1, 2, \ldots, 4$, We now define the following set of vectors.

$$e^i_j(t) = q^i_j(t) - q^i_{jd}(t) \; i = 1, 2, \ldots, 4 \;\; j = 1, 2 \tag{6.198}$$

$$e_1(t) = \begin{bmatrix} e^1_1(t) & \cdots & e^4_1(t) \end{bmatrix}^T \tag{6.199}$$

$$e_2(t) = \begin{bmatrix} e^1_2(t) & \cdots & e^4_2(t) \end{bmatrix}^T \tag{6.200}$$

where

$$q^i_{2d}(t) = \frac{dq^i_{1d}(t)}{dt} - k^\star_1 \frac{q^i_1(t) - q^i_{1d}(t)}{||e_1(t)||} \text{ if } ||e_1(t)||_2 \neq 0 \tag{6.201}$$

$$= \frac{dq^i_{1d}(t)}{dt} \text{ if } ||e_1(t)||_2 = 0 \tag{6.202}$$

Now note that $q^i_{2d}(t)$, is not strictly differentiable at the origin, but has a derivative that exists almost everywhere. Indeed define the generalized derivative as

$$\dot{q}^i_{2d}(t) = \frac{dq^i_{2d}(t)}{dt} - \frac{k^\star_1 N^i_1(t)}{||e_1(t)||^{\frac{3}{3}}} \text{ if } ||e_1(t)||_2 \neq 0 \tag{6.203}$$

$$= \frac{dq^i_{2d}(t)}{dt} \text{ if } ||e_1(t)||_2 = 0 \tag{6.204}$$

where

$$N^i_1(t) = \sum e^j_1(t)[e^j_1(t)\dot{e}^i_1(t) - \dot{e}^j_1(t)e^i_1(t)] \; j = 1, 2, 3, 4 \;\; j \neq i \tag{6.205}$$

$$\dot{e}^i_1(t) = q^i_2(t) - \dot{q}^i_{id}(t) \; i = 1, 2, \ldots, 4 \tag{6.206}$$

We now choose u^i $i = 1, 2, \ldots, 4$ in the following manner.

$$u^i(t) = \dot{q}^i_{2d}(t) - k^\star_2 \frac{q^i_2(t) - q^i_{2d}(t)}{||e_2(t)||} \text{ if } ||e_2(t)||_2 \neq 0 \tag{6.207}$$

$$= \frac{dq^i_{1d}(t)}{dt} \text{ if } ||e_2(t)||_2 = 0 \tag{6.208}$$

Claim 6.1 *Synchronous tracking for a system of robotic hands.*

Given *(G1) n mechanical systems, each of the form (6.197).*

(G2) Given a set of desired trajectories of the form $x^i_1 d(t) : \Re_+ \rightarrow \Re$ $i = 1, 2, \ldots, n$

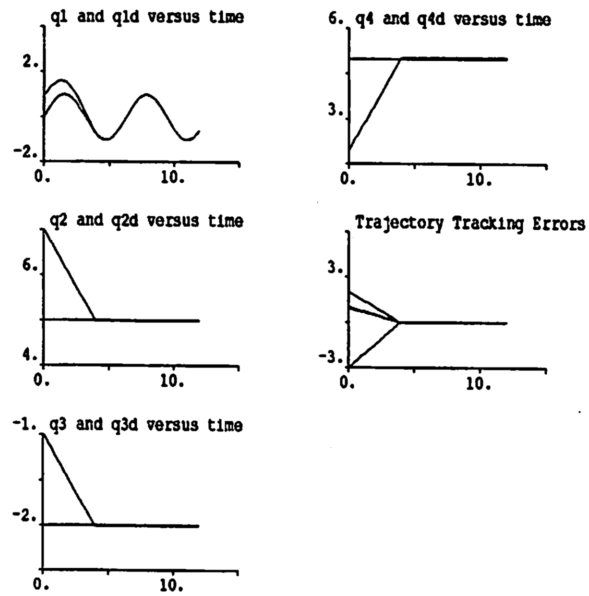

Figure 6.3: Synchronous Convergence Of Trajectories To Desired Values

If *(I1) The controls $u^i(x,t)$ $i = 1,2,\ldots,n$ in equation (6.197) are specified by (6.207). where$k_1, k_2 \in \Re_+$.*

Then *(T1) Filippov solutions exist for system (6.197) - subject to control (6.207).*

(T2) States $x_1^i(t)$ track their respective trajectories $x_{1d}^i(t)$ synchronously.

Proof: ♠ ▷ The proof of the claim is by invoking the theorem proved earlier for the more general case of a group of mechanical systems.

Indeed, it is easily seen that the application of control (6.207) would cause the states $x_2^i(t)$ $i = 1,2,\ldots,4$ to reach their desired values in finite time, and the desired trajectories are so chosen that the reduced dynamics ensures finite time tracking for $x_{1d}^i(t)$. ◁♠

Results of simulation are shown for the following conditions. The chosen desired trajectories were as follows. $q_{1d}^1(t) = \sin t$, $q_{1d}^2(t) = 5$, $q_{1d}^3(t) = -2$, $q_{1d}^4(t) = 5$. The initial conditions were as follows $q_1^1(0) = 1$, $q_1^2(0) = 7$, $q_1^3(0) = -1$, $q_1^4(0) = 2$, $q_2^1(0) = 0$, $q_2^2(0) = 0$, $q_2^3(0) = 0$, $q_2^4(0) = 0$

Simulation results are in excellent agreement with the predicted behaviour.

Indeed, note that the trajectory errors vanish identically at the same instant of time. This indeed was the motivation for considering the synchronous tracking control law.

6.3.5 Criticism And Future Prospects

- Though proof of robustness of synchronous control is hard, the simulation results indicate a high degree of tolerance of perturbations.
- The control law though bounded, tends to be ill-conditioned when implemented on a computer. As a consequence some regularization has to be done.
- The control law works very well with a fractional control approach presented later in this chapter. The rate of convergence is fast, as would be expected.
- The extensions of such an approach to chains of integrators of arbitrary length is natural and can be accomplished, though it is very difficult to avoid chattering about the intersection of the sliding surfaces.

6.4 Variable Structure Lyapunov Control Of Certain Benchmark Problems

6.4.1 Introduction

In this section, we will present techniques to control a class of benchmark problems. This class of control techniques are called Lyapunov control methods as they result in a choice of control law that ensures the negative definiteness of a chosen candidate Lyapunov function. In choosing the control law, we choose variable structure control laws to ensure negative definiteness of the derivative of the candidate Lyapunov function.

The class of benchmark problems we will consider are essentially linear systems in the controllable canonical form, which are perturbed by additive non-lipschitz, mismatched perturbations with a triangular structure. We will impose further structure on them by specifying that the uncertainties enter as affine and unknown but bounded parameters with known bounds multiplying known non-lipschitz functions. Furthermore, we will assume that the class of non-lipschitz functions is such that, if the parameters were known exactly, then the system is linearized by the methodology of input-output linearization.

We will consider systems of the following form

$$\begin{aligned}
\dot{x}_1 &= x_2 + \theta_1 f_1(x_1) && (6.209)\\
\dot{x}_2 &= x_3 + \theta_2 f_2(x_1, x_2) && (6.210)\\
\vdots &\ \ \vdots\ \ \vdots && (6.211)\\
\dot{x}_n &= u + \theta_n f_n(x_1, x_2, \ldots, x_n) && (6.212)
\end{aligned}$$

where $x \in \Re^n$, $f_i(x_1, \ldots, x_i) : \Re^i \to \Re$ $i = 1, 2, \ldots, n$ are smooth, and $u \in \Re$. The following comments are in order.

Comment 6.4.1 The system comprises of an underlying controllable canonical form (a chain of integrators) perturbed by a vector of non-lipschitz, mismatched perturbations. If each θ_i $1 = 1, 2, \ldots, n$ were known, we can linearize the system through a change of coordinates choosing the output to be x_1.

While the methodology we describe will be true in general for $n \in Z_+$ dimensional systems, we restrict our attention to 3 dimensinal systems for the sake of clarity, i.e.,

$$\begin{aligned}
\dot{x}_1 &= x_2 + \theta_1 f_1(x_1) && (6.213)\\
\dot{x}_2 &= x_3 + \theta_2 f_2(x_1, x_2) && (6.214)\\
\dot{x}_3 &= u + \theta_3 f_3(x_1, x_2, x_3) && (6.215)
\end{aligned}$$

where $x \in \Re^3$, $\theta_i \in \Re$ $i = 1, 2, 3$ are unknown but bounded, that is $|\theta_i| < k_{\theta_i}$ $i = 1, 2, 3$ and the control $u \in \Re$.

Our method of stabilization is a constructive proceedure:

1. Find a nonlinear transformation of the state that helps to swamp out the unknown perturbations through a Lyapunov argument. Such a transformation may be difficult to prescribe ahead of time, so we incorporate enough freedom in our prescription of the transformation to help us achieve the stabilization objective.

2. Consider a candidate Lyapunov function that is essentially quadratic, but has some other positive definite terms to assist in the proof.

Step 1:

Consider following change of state variables $\Phi : x \in \Re^3 \rightarrow y \in \Re^3$, given by,

$$y_1 = x_1 \tag{6.216}$$

$$y_2 = x_2 + h_1(x_1) \tag{6.217}$$

$$y_3 = x_3 + h_2(x_1, x_2) \tag{6.218}$$

where $h_1(x_1) : \Re \rightarrow \Re$ is smooth, and $h_2(x_1, x_2) : \Re^2 \rightarrow \Re$ is smooth. We reserve the freedom to specify the functions $h_1(x_1) : \Re \rightarrow \Re$, and $h_2(x_1, x_2) : \Re^2 \rightarrow \Re$ later, but only make the assumption that they are smooth.

Proposition 6.1 *The coordinate transformation given by (6.216) - (6.218) is a diffeomorphism.*

Proof: ♠ ▷ Utilizing the fact that $\Phi : x \in \Re^3 \rightarrow y \in \Re^3$ is invertible everywhere, we explicitly compute $\Phi^{-1} : y \in \Re^3 \rightarrow x \in \Re^3$ to yield,

$$x_1 = y_1 \tag{6.219}$$

$$x_2 = y_2 - h_1(y_1) \tag{6.220}$$

$$x_3 = y_3 - h_3(y_1, y_2) \tag{6.221}$$

where $h_3(y_1, y_2) = h_2 \circ \Phi^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Therefore the transformation $\Phi : x \in \Re^3 \rightarrow y \in \Re^3$ is invertible everywhere, and is therefore a globally valid diffeomorphism. ◁♠

Differentiating (6.216) and using equations (6.219) - (6.221), we obtain

$$\dot{y}_1 = y_2 - h_1(y_1) + \theta_1 f_1(y_1) \tag{6.222}$$

It is clear from (6.222) that a candidate quadratic Lyapunov function would result in $y_1[-h_1(y_1) + \theta_1 f_1(y_1)]$ that has to be made negative definite. This fact gives us enough information to make some further assumptions on $f_1(y_1)$ and to be able to specify the function $h_1(y_1)$.

Assumption (A1):

We now assume that $f_1(y_1)$ is such that there exists a function $p_1(y_1)$ such that the following is true

$$\begin{aligned} y_1 p_1(y_1) \quad &\text{is} \quad \text{positive definite} && (6.223) \\ y_1 f_1(y_1) \quad &< \quad y_1 p_1(y_1) && (6.224) \end{aligned}$$

The assumption merely indicates that $y_1 f_1(y_1)$ is swamped by a passive function. We now define the function $h_1(y_1)$ in the following manner.

Choice (H1):

$$h_1(y_1) = y_1 + k_{\theta_1} p_1(y_1) \tag{6.225}$$

Utilizing (6.225) in (), we rewrite (6.4.1) as

$$\dot{y}_1 = -y_1 + y_2 - [k_{\theta_1} p_1(y_1) - \theta_1 f_1(y_1)] \tag{6.226}$$

Set $[k_{\theta_1} p_1(y_1) - \theta_1 f_1(y_1)] = F_1(\theta_1, y_1)$ and note that $y_1 F_1(\theta_1, y_1) > 0$.

Define the set Ω to be the set of all functions $q(\star) : \Re \rightarrow \Re_+$ such that the following is true.

$$\begin{aligned} q(y_1) \quad &\text{is} \quad \text{positive definite} && (6.227) \\ \frac{\partial q(y_1)}{\partial y_1} F_1(\theta_1, y_1) \quad &> \quad 0 && (6.228) \\ y_1 \frac{\partial q(y_1)}{\partial y_1} \quad &> \quad 0 && (6.229) \end{aligned}$$

Assumption (A2):

There exists $q_1(y_1) \in \Omega$ such that the following is true

$$[f_1(y_1)]^2 < y_1 \frac{\partial q_1}{\partial y_1} \tag{6.230}$$

We will use a simple example to clarify the nature of the assumptions made.

Example 6.4.1 *Illustrate Assumptions Made*

For instance, let the function $f_1(y_1) = y_1^2$. That is, the differential equation is of the form,

$$\dot{y}_1 = y_1^2 \tag{6.231}$$

Now choose

$$p_1(y_1) = \alpha_1 y_1 + \alpha_2 y_1^3 \quad \alpha_1, \alpha_2 > 1 \tag{6.232}$$

Note the following

$$\begin{aligned}
y_1 p_1(y_1) &= \alpha_1 y_1^2 + \alpha_2 y_1^4 && (6.233)\\
&\geq 0 && (6.234)\\
y_1[f_1(y_1)] &= y_1^3 && (6.235)\\
&\leq \alpha_1 y_1^2 + \alpha_2 y_1^4 && (6.236)\\
&\leq y_1 p_1(y_1) && (6.237)
\end{aligned}$$

Thus Assumption (A1) is satisfied.

Note that $[f_1(y_1)]^2 = y_1^4$. Choose

$$q_1(y_1) = \alpha_3 \frac{y_1^4}{4} \quad \alpha_3 > 1 \tag{6.238}$$

The following are then true.

$$\begin{aligned}
y_1 \frac{\partial q_1}{\partial y_1} &= \alpha_3 y_1^4 && (6.239)\\
&> y_1^4 && (6.240)\\
&> [f_1(y_1)]^2 && (6.241)\\
& && (6.242)
\end{aligned}$$

Thus Assumption (A2) is also satisfied.

Continuing the example, let $\theta_1 f(y_1) = \theta_1 y_1^2$. Choose $p_1(y_1) = k_{\theta_1}[\alpha_1 y_1 + \alpha_2 y_1^3]$ where $k_{\theta_1} > |\theta_1|$, and $\alpha_1, \alpha_2 > 1$. The following are true

$$\begin{aligned}
F_1(\theta_1, y_1) &= k_{\theta_1}[\alpha_1 y_1 + \alpha_2 y_1^3] - \theta_1 y_1^2 && (6.243)\\
q_1(y_1) &= \alpha_3 \frac{y_1^4}{4} \; \alpha_3 > 1 && (6.244)\\
\frac{\partial q_1}{\partial y_1} F_1(\theta_1, y_1) &= \alpha_3 y_1^3 [k_{\theta_1}[\alpha_1 y_1 + \alpha_2 y_1^3] - \theta_1 y_1^2] && (6.245)\\
&> 0 && (6.246)
\end{aligned}$$

Thus it is possible to swamp the unknown nonlinearity.

Now returning to the problem at hand, we differentiate y_2 of equation (6.217) to get

$$\begin{aligned}
\dot{y}_2 &= y_3 - h_3(y_1, y_2) + \theta_2 f_2'(y_1, y_2) && (6.247)\\
&+ \frac{\partial h_1(y_1)}{\partial y_1}[-y_1 + y_2 - k_{\theta_1} p_1(y_1) - \theta_1 f_1(y_1)] && (6.248)
\end{aligned}$$

where $f'(y_1, y_2) = f_2 \circ \Phi^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ We now make the following assumption on $f'(y_1, y_2)$.

Assumption (A3):

There exist smooth functions $q_2(y_2) \in \Omega : \Re \to \Re$ and $p_2(y_2) : \Re \to \Re$, such that,

$$\begin{aligned}
[f_2'(y_1, y_2)]^2 &< y_1 \frac{\partial q_2(y_1)}{\partial y_1} + y_2 p_2(y_2) && (6.249)\\
y_1 \frac{\partial q_2(y_1)}{\partial y_1} &> 0 && (6.250)\\
y_2 p_2(y_2) &> 0 && (6.251)
\end{aligned}$$

The reason for these assumptions become clearer during the course of the proof. We now choose the function $h_3(y_1, y_2)$ as

Choice (H3):

$$\begin{aligned}
h_3(y_1, y_2) &= \frac{\partial h_1(y_1)}{\partial y_1}[-y_1 + y_2 - k_{\theta_1} p_1(y_1)] + \alpha_2 y_2 && (6.252)\\
&+\alpha_3 p_2(y_2) + \alpha_1 [\frac{\partial h_1(y_1)}{\partial y_1}]^2 y_2 + \alpha_4 \frac{\partial q_2(y_1)}{\partial y_1} + \frac{\partial q_1(y_1)}{\partial y_1} + y_1 && (6.253)
\end{aligned}$$

where the constants α_i $i = 1, 2, \ldots, 4$ < 1 and will be specified later. Such a choice of the function $h_3(y_1, y_2)$ results in equation (6.248) being recast in the form

$$\dot{y}_2 = -\alpha_2 y_2 - \alpha_3 p_2(y_2) + y_3 \tag{6.254}$$

$$+\theta_2 f_2'(y_1, y_2) + \theta_1 \frac{\partial h_1(y_1)}{\partial y_1} - \alpha_1 [\frac{\partial h_1(y_1)}{\partial y_1}]^2 y_2 \tag{6.255}$$

$$-\alpha_4 \frac{\partial q_2(y_1)}{\partial y_1} - \frac{\partial q_1(y_1)}{\partial y_1} - y_1 \tag{6.256}$$

We now differentiate (6.218) to get

$$\dot{y}_3 = u + \theta_3 f_3'(y_1, y_2, y_3) \tag{6.257}$$

$$+\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1}[-y_1 + y_2 - k_{\theta_1} p_1(y_1) + \theta_1 f_1(y_1)] \tag{6.258}$$

$$+\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1}[-\alpha_2 y_2 - \alpha_3 p_2(y_2) + y_3] \tag{6.259}$$

$$+\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1}[\theta_2 f_2'(y_1, y_2) + \theta_1 \frac{\partial h_1(y_1)}{\partial y_1} + \alpha_1 [\frac{\partial h_1(y_1)}{\partial y_1}]^2 y_2] \tag{6.260}$$

$$+\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1}[\alpha_4 \frac{\partial q_2(y_1)}{\partial y_1} + \frac{\partial q_1(y_1)}{\partial y_1} + y_1] \tag{6.261}$$

We now choose the control input u to be

$$u = -\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1}[-y_1 + y_2 \tag{6.262}$$

$$-k_{\theta_1} p_1(y_1)] - k_{\theta_1} |\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1}||f_1(y_1)| sgn[y_3] \tag{6.263}$$

$$-\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1}[-\alpha_2 y_2 - \alpha_3 p_2(y_2) + y_3] \tag{6.264}$$

$$-\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1}[\theta_2 f_2'(y_1, y_2) + \theta_1 \frac{\partial h_1(y_1)}{\partial y_1} + \alpha_1 [\frac{\partial h_1(y_1)}{\partial y_1}]^2 y_2] \tag{6.265}$$

$$-\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1}[\alpha_4 \frac{\partial q_2(y_1)}{\partial y_1} + \frac{\partial q_1(y_1)}{\partial y_1} + y_1] \tag{6.266}$$

$$-k_{\theta_1} |\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_2}||\frac{\partial h_1(y_1)}{\partial y_1} f_1(y_1)| sgn[y_3] \tag{6.267}$$

$$-k_{\theta_2} |\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_2}||f_2'(y_1, y_2)| sgn[y_3] \tag{6.268}$$

$$-y_3 - y_2 - k_{\theta_3} |f_3'(y_1, y_2, y_3 sgn[y_3] \tag{6.269}$$

Comment 6.4.2 The control law (6.269) is implementable as it does not involve the unknown parameters $\theta_1, \theta_2, \theta_3$.

Using the control law specified by (6.269) in the equation (6.261), we rewrite (6.261) as

$$\begin{aligned}
\dot{y}_3 &= -y_3 - y_2 + \theta_3 f_3'(y_1,y_2,y_3) - k_{\theta_3}|f_3'(y_1,y_2,y_3 sgn[y_3] && (6.270)\\
&+\theta_1\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_1}f_1(y_1) - k_{\theta_1}|\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_1}||f_1(y_1)|sgn[y_3] && (6.271)\\
&\theta_2\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_1}f_2'(y_1,y_2) - k_{\theta_2}|\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_2}||f_2(y_1,y_2)|sgn[y_3] && (6.272)\\
&\theta_1\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_2}\frac{\partial h_1(y_1)}{\partial y_1} - k_{\theta_1}|\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_2}||\frac{\partial h_1(y_1)}{\partial y_1}|sgn[y_3] && (6.273)
\end{aligned}$$

To summarize, let us collect the transformed state equations as

$$\begin{aligned}
\dot{y}_1 &= -y_1 + y_2 - [k_{\theta_1}p_1(y_1) - \theta_1 f_1(y_1)] && (6.274)\\
\dot{y}_2 &= -\alpha_2 y_2 - \alpha_3 p_2(y_2) + y_3 && (6.275)\\
&+\theta_2 f_2'(y_1,y_2) + \theta_1\frac{\partial h_1(y_1)}{\partial y_1} - \alpha_1[\frac{\partial h_1(y_1)}{\partial y_1}]^2 y_2 && (6.276)\\
&-\alpha_4\frac{\partial q_2(y_1)}{\partial y_1} - \frac{\partial q_1(y_1)}{\partial y_1} - y_1 && (6.277)\\
\dot{y}_3 &= -y_3 - y_2 + \theta_3 f_3'(y_1,y_2,y_3) - k_{\theta_3}|f_3'(y_1,y_2,y_3)|sgn[y_3] && (6.278)\\
&+\theta_1\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_1}f_1(y_1) - k_{\theta_1}|\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_1}||f_1(y_1)|sgn[y_3] && (6.279)\\
&\theta_2\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_1}f_2'(y_1,y_2) - k_{\theta_2}|\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_2}||f_2(y_1,y_2)|sgn[y_3] && (6.280)\\
&\theta_1\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_2}\frac{\partial h_1(y_1)}{\partial y_1} - k_{\theta_1}|\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_2}||\frac{\partial h_1(y_1)}{\partial y_1}|sgn[y_3] && (6.281)
\end{aligned}$$

We will now show global exponential stability of the system represented by (6.274) - (6.281).

Theorem 6.8 *Proof of global exponential stability of system with mismatched non-lipschitz perturbations.*

Given *(G1) A system of equations represented by (6.274) - (6.281)*

If *(I1) The unknown parameters* $\theta_1, \theta_2, \theta_3$ *are bounded by constants* $k_{\theta_1}, k_{\theta_2}, k_{\theta_3}$.

Then *(T1)* $||x||_2 \rightarrow 0$ *as* $t \rightarrow \infty$

Proof: ♠ ▷ Consider a candidate Lyapunov function $V(y_1, y_2, y_3) : \Re^3 \to \Re_+$ of the form,

$$V = \frac{y_1^2}{2} + \frac{y_2^2}{2} + \frac{y_3^2}{2} + \alpha_4 q_2(y_1) + \alpha_5 q_1(y_1) \tag{6.282}$$

where $\alpha_4, \alpha_5 \in \Re$, but will be chosen later. Note that V is postive definite. Differentiating (6.282) along the flow of (6.274)-(6.281), we obtain,

$$\begin{aligned}
\dot{V} \;=\; & -y_1^2 - y_1[k_{\theta_1} p_1(y_1) - \theta_1 f_1(y_1)] \\
& -\alpha_2 y_2^2 - \alpha_3 y_2 p_2(y_2) + \theta_2 f_2'(y_1, y_2) + \theta_1 \frac{\partial h_1(y_1)}{\partial y_1} - \alpha_1 [\frac{\partial h_1(y_1)}{\partial y_1}]^2 y_2^2 \\
& -y_3^2 - |y_3||f_3'(y_1, y_2, y_3)|[k_{\theta_3} - \theta_3 sgn[f_3'(y_1, y_2, y_3)] sgn[y_3]] \\
& -|y_3||\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1}||f_1(y_1)|[k_{\theta_1} - \theta_1 sgn[y_3 \frac{\partial h_3(y_1, y_2, y_3)}{\partial y_1} f_1(y_1)]] \\
& -|y_3||\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_2}||f_2'(y_1, y_2)| \\
& [k_{\theta_2} - \theta_2 sgn[y_3 \frac{\partial h_3(y_1, y_2, y_3)}{\partial y_2} f_2(y_1, y_2)]] \\
& -|y_3||\frac{\partial h_3(y_1, y_2, y_3)}{\partial y_2}||\frac{\partial h_1(y_1)}{\partial y_1} f_1(y_1)| \\
& [k_{\theta_2} - \theta_2 sgn[y_3 \frac{\partial h_3(y_1, y_2, y_3)}{\partial y_2} \frac{\partial h_1(y_1)}{\partial y_1} f_1(y_1)]] \\
& -\alpha_4 \frac{\partial q_2(y_1)}{\partial y_1} y_1 - \alpha_4 \frac{\partial q_2(y_1)}{\partial y_1}[k_{\theta_1} p_1(y_1) - \theta_1 f_1(y_1)] \\
& -\alpha_5 \frac{\partial q_1(y_1)}{\partial y_1} y_1 - \alpha_5 \frac{\partial q_1(y_1)}{\partial y_1}[k_{\theta_1} p_1(y_1) - \theta_1 f_1(y_1)]
\end{aligned}$$

Now we use the fact that

$$\theta_2 f_2'(y_1, y_2) \;\le\; k_{\theta_2} \frac{y_2^2}{4} + [f_2'(y_1, y_2)]^2 \tag{6.283}$$

$$\;\le\; k_{\theta_2} \frac{y_2^2}{4} + y_2 p_2(y_2) + y_1 \frac{\partial q_2}{\partial y_1} \tag{6.284}$$

$$\theta_1 \frac{\partial h_1(y_1)}{\partial y_1} \;\le\; \frac{k_{\theta_1}^2}{4}[\frac{\partial h_1(y_1)}{\partial y_1}]^2 y_2^2 + [f_1(y_1)]^2 \tag{6.285}$$

$$\;\le\; \frac{k_{\theta_1}^2}{4}[\frac{\partial h_1(y_1)}{\partial y_1}]^2 y_2^2 + y_1 \frac{\partial q_1(y_1)}{\partial y_1} \tag{6.286}$$

Choosing the constants α_i $i = 1, 2, \ldots, 5$ to be

$$\alpha_1 \;>\; \frac{k_{\theta_2}^2}{4} \tag{6.287}$$

$$\begin{aligned}\alpha_2 &> \frac{k_{\theta_2}^2}{4} && (6.288)\\ \alpha_3 &> 1 && (6.289)\\ \alpha_4 &> 1 && (6.290)\\ \alpha_5 &> 1 && (6.291)\end{aligned}$$

We rewrite $\dot{V}$ as

$$\begin{aligned}\dot{V} \;=\; & -y_1^2 - y_1[k_{\theta_1}p_1(y_1) - \theta_1 f_1(y_1)]\\ & -[\alpha_2 - \frac{k_{\theta_2}^2}{4}]y_2^2 - [\alpha_3 - 1]y_2p_2(y_2) - [\alpha_1 - \frac{k_{\theta_2}^2}{4}][\frac{\partial h_1(y_1)}{\partial y_1}]^2 y_2^2\\ & -y_3^2 - |y_3||f_3'(y_1,y_2,y_3)|[k_{\theta_3} - \theta_3 sgn[f_3'(y_1,y_2,y_3)]sgn[y_3]]\\ & -|y_3||\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_1}||f_1(y_1)|[k_{\theta_1} - \theta_1 sgn[y_3\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_1}f_1(y_1)]]\\ & -|y_3||\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_2}||f_2'(y_1,y_2)|\\ & [k_{\theta_2} - \theta_2 sgn[y_3\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_2}f_2(y_1,y_2)]]\\ & -|y_3||\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_2}||\frac{\partial h_1(y_1)}{\partial y_1}f_1(y_1)|\\ & [k_{\theta_2} - \theta_2 sgn[y_3\frac{\partial h_3(y_1,y_2,y_3)}{\partial y_2}\frac{\partial h_1(y_1)}{\partial y_1}f_1(y_1)]]\\ & -[\alpha_4 - 1]\frac{\partial q_2(y_1)}{\partial y_1}y_1 - \alpha_4\frac{\partial q_2(y_1)}{\partial y_1}[k_{\theta_1}p_1(y_1) - \theta_1 f_1(y_1)]\\ & -[\alpha_5 - 1]\frac{\partial q_1(y_1)}{\partial y_1}y_1 - \alpha_5\frac{\partial q_1(y_1)}{\partial y_1}[k_{\theta_1}p_1(y_1) - \theta_1 f_1(y_1)]\end{aligned}$$

Note that by the assumptions on the functions $p_1(y_1), p_2(y_2), q_1(y_1), q_2(y_2)$, the following are true.

$$\begin{aligned}y_1[k_{\theta_1}p_1(y_1) - \theta_1 f_1(y_1)] &\geq 0 && (6.292)\\ y_2p_2(y_2) &\geq 0 && (6.293)\\ \frac{\partial q_2(y_1)}{\partial y_1}y_1 &\geq 0 && (6.294)\\ \frac{\partial q_1(y_1)}{\partial y_1}y_1 &\geq 0 && (6.295)\\ \frac{\partial q_1(y_1)}{\partial y_1}[k_{\theta_1}p_1(y_1) - \theta_1 f_1(y_1)] &\geq 0 && (6.296)\end{aligned}$$

It is therefore obvious that $\dot{V}$ is indeed negative definite. This confirms the global exponential stability of the origin. ◁♠

Example 6.4.2 *Variable Structure Lyapunov Control Technique.*

Consider a simple example to clarify the methodology outlined earlier. Indeed, consider the following two state example given by

$$\dot{x}_1 = x_2 + \theta_1 x_1^2 \quad (6.297)$$

$$\dot{x}_2 = u \quad (6.298)$$

where $x \in \Re^2$ *and* $|\theta_1| \leq k_{\theta_1}$ *Now consider a coordinate change given by*

$$y_1 = x_1 \quad (6.299)$$

$$y_2 = x_2 + h_1(y_1) \quad (6.300)$$

Note that the function $f_1(y_1)$ *satisfies assumptions (A1) and (A2) prescribed in the previous section. Indeed, choosing* $p_1(y_1) = ky_1^3$ $k > 1$, *we see that*

$$y_1 f_1(y_1) < y_1 p_1(y_1) \quad (6.301)$$

Thus assumption (A1) is satisfied. Furthermore, choosing

$$q_1(y_1) = k[y_1^6] \; k > 1 \quad (6.302)$$

$$f_1^2(y_1) = y_1^6 \quad (6.303)$$

$$\leq k[y_1^6] \; k > 1 \quad (6.304)$$

$$\leq y_1 \frac{\partial q_1(y_1)}{\partial y_1} \quad (6.305)$$

Thus assumption (A2) is also satisfied. Now in accordance with the theory outlined in the earlier section choose $h_1(y_1)$ *to be*

$$h_1(y_1) = k_{\theta_1} y_1^3 \quad (6.306)$$

Indeed, in the new coordinate system, the system equations are given by

$$\dot{y}_1 = y_2 - y_1^3[k_{\theta_1} - \theta_1] \quad (6.307)$$

$$\dot{y}_2 = u + 3k_{\theta_1} y_1^2[y_2 - y_1^3[k_{\theta_1} - \theta_1]] \quad (6.308)$$

Now choose the control u to be

$$u = -y_2 - y_1 - 3k_{\theta_1} y_1^2 y_2 - 6k^* |y_1^5| sgn[y_2] \tag{6.309}$$

where $k^* = max(k_{\theta_1}, k_{\theta_1}^2)$. *Such a choice of control yields system equations of the form*

$$\dot{y}_1 = y_2 - y_1^3[k_{\theta_1} - \theta_1] \tag{6.310}$$

$$\dot{y}_2 = -y_2 - y_1 - |y_1^5|[6k^* sgn[y_2] + 3k_{\theta_1}\theta_1 sgn[y_1^5] \tag{6.311}$$

Choosing a candidate Lyapunov function of the form

$$V = \frac{y_1^2}{2} + \frac{y_2^2}{2} \tag{6.312}$$

and differentiating V along the flow of (6.310) - (6.311) we find

$$\dot{V} = -y_1^4[k_{\theta_1} - \theta_1] \tag{6.313}$$

$$-y_2^2 - |y_1^5||y_2[6k^* + 3k_{\theta_1}\theta_1 sgn[y_1^5 y_2] \tag{6.314}$$

$$\leq 0 \tag{6.315}$$

Negative definiteness of $\dot{V}$ *guarantees global exponential stability.*

Simulation results indicate that the states are regulated to the origin. Note however the large control effort required.

6.4.2 Criticism And Future Prospects

The methodology presented in this section is a systematic means of stabilizing this class of benchmark problems. The technique however suffers from the following shortcomings.

- The class of systems that can be handled are limited, to those that can be exactly linearized should the unknown parameters be known.
- The feedback strategy, being the result of a Lyapunov analysis is extraordinarily conservative, and asks for unrealistic control efforts.

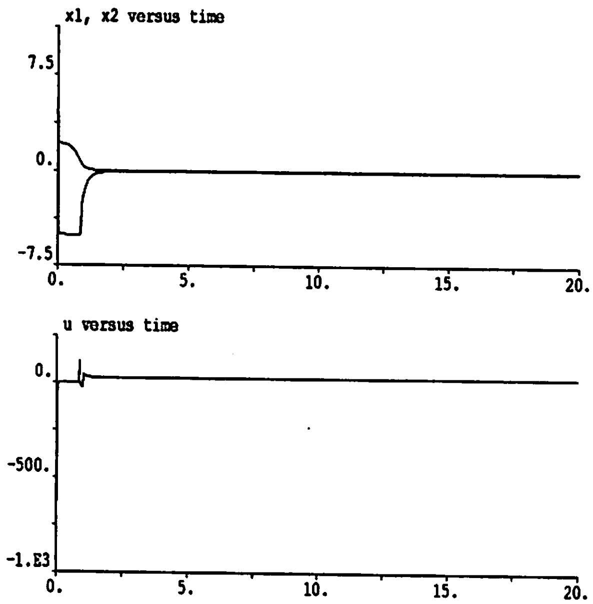

Figure 6.4: Stabilization Of Mismatched Systems

- The strategy assumes full state feedback which is a serious setback in an industrial scenario.
- More work needs to be done in identifying the class of perturbations that can be handled by a Lyapunov design.
- The problem begins to be increasingly difficult if instead of the stabilization objective, the goal were to be tracking.

6.5 Sliding Mode Observers For Mechanical Systems

6.5.1 Introduction

In this section, we present a correction to an existing result in sliding mode observer theory, [24] and remark on using the result and its modifications as helpful design rules towards designing observers for mechanical systems. The sliding mode observer problem for systems with more that 2 states is yet unsolved, and the available results require a number of assumptions to be made on the system. Even in the important class of planar dynamical systems, the theory is incomplete, in that the observers are very sensitive to measurement noise.

We first present the basic theory of sliding mode observers for mechanical systems, and prove the existence of generalized Filippov solutions and stability. We then show the convergence of the observer state errors to zero. We then present the problem with existing theory, and present bounds on variables that would prevent observer failure. Finally we remark on the utilization of the computed bounds as a design rule to help design such sliding mode observers.

The problem of designing observers using sliding mode theory was first introduced and studied by [24]. Here the observation problem is treated as a special case of a state regulation problem. Sliding surfaces are designed based on the error dynamics, and reaching a sliding surface is equivalent to the error in the estimate of the measured state decaying to zero. In sliding mode control, the surface $S = 0$ is reached in finite time, and on that surface the states decay exponentially. Similarly, in sliding mode observer theory, the error in the estimate of the measured state decays in finite time. All other state errors decay exponentially.

Consider a simple mechanical system of the form

$$\dot{x}_1 = x_2 \tag{6.316}$$

$$\dot{x}_2 = u \tag{6.317}$$

where $x \in \Re^2$ and $u \in \Re$. Now consider an observer of the following form.

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1 sgn[\tilde{x}_1] \tag{6.318}$$

$$\dot{\hat{x}}_2 = k_2 sgn[\tilde{x}_2] \tag{6.319}$$

$$\tilde{x} = x - \hat{x} \tag{6.320}$$

Such an observer structure equation leads to error dynamics of the form

$$\dot{\tilde{x}}_1 = \tilde{x}_2 - k_1 sgn(\tilde{x}_1) \tag{6.321}$$

$$\dot{\tilde{x}}_2 = -k_2 sgn[\tilde{x}_1] \tag{6.322}$$

Theorem 6.9 *Convergence of the state estimation errors:*

Given

(G1) Error dynamics of the form (6.321)- (6.322)

If

(I1) $|\tilde{x}_2| < k_1$

Then

(T1) Generalized Filippov Solutions exist for the system (6.321)- (6.322)

(T2) The one-dimensional manifold $\tilde{x}_1 = 0$ *is attractive*

(T3) The averaged dynamics of $\tilde{x}_2$ *about the surface* $\tilde{x}_1 = 0$ *decays exponentially.*

Proof: ♠ ▷ Existence of Filippov solutions is due to the fact that the governing differential inclusions are closed, bounded, convex and uppersemicontinuous.

We will prove the theorem using simple Lyapunov analysis. Consider the candidate Lyapunov function,

$$V = \frac{\tilde{x}_1^2}{2} \tag{6.323}$$

Differentiating V along the flow of the system(6.321), we get,

$$\begin{aligned} \dot{V} &= \tilde{x}_1[\hat{x}_2 + k_1 sgn[\tilde{x}_1]] & (6.324) \\ &< -||\tilde{x}_1||[k_1 - \hat{x}_2 sgn[\tilde{x}_1]] & (6.325) \end{aligned}$$

Thus as long $\tilde{x}_2 < k_1$, $\dot{V} < 0$, indeed the surface $\tilde{x} = 0$ is attractive.

Comment 6.5.1 The Theorem asserts the existence of a tubular neighbourhood around the $\tilde{x}_2 = 0$ *axis where, the trajectories converge to the manifold given by* $\tilde{x}_1 = 0$. *It is to be noted that* $\tilde{x}_2$ *must not be greater than* k_1 *until the trajectories converge to* $\tilde{x}_1 = 0$. *Some additional conditions are necessary to prevent such an occurence.*

The dynamics of the system when constrained to evolve on the surface $\tilde{x}_1 = 0$, can be derived using the Fillipov solution concept. Thus, taking a convex combination of the dynamics on either side of the sliding surface, we get,

$$\dot{\tilde{x}}_1 = \gamma[\tilde{x}_2 + k_1] + (1-\gamma)[\tilde{x}_2 - k_1] \quad (6.326)$$

$$\dot{\tilde{x}}_2 = \gamma k_2 + (1-\gamma)(-k_2] \quad (6.327)$$

From the above equations, we eliminate γ, and from the invariance of the sliding surface, we get,

$$\dot{\tilde{x}}_1 = 0 \quad (6.328)$$

$$\dot{\tilde{x}}_2 = -\frac{k_2}{k_1}\tilde{x}_2 \quad (6.329)$$

Exponential decay of $\tilde{x}_2$ is clear from the above equation. The proof of the theorem is complete. $\triangleleft$♠

Comment 6.5.2 It is interesting to note the roles played by the constants k_1 and k_2. Increasing k_1, increases the region of of attractivity of the surface $\tilde{x}_1 = 0$. But the same time, it decreases the rate of decay of the state $\tilde{x}_2$. Thus there is an obvious design tradeoff to be considered here.

Comment 6.5.3 The hypothesis of this theorem is weak. Current literature makes an important omission in this regard. The hypothesis asserts that the surface $\tilde{x}_1 = 0$ is attractive, only as long as $\tilde{x}_2 < k_1$. Now it is not clear, that the condition $\tilde{x}_2 < k_1$ will not be violated before $\tilde{x}_1 = 0$ If this condition is violated, then the surface $\tilde{x}_1 = 0$ is no longer attractive. It is therefore necessary to clearly understand the conditions under which such a pathology may not occur. By choosing gains k_1 and k_2 carefully, we may prevent the occurrence of such circumstances. Furthermore, assumptions on maximum bounds on the initial conditions become necessary to the analysis.

We now state the following theorem that ensures the stability of the perturbed error dynamics.

$$\dot{\tilde{x}}_1 = \tilde{x}_2 - k_1 sgn[\tilde{x}_1] \quad (6.330)$$

$$\dot{\tilde{x}}_2 = -k_2 sgn[\tilde{x}_1] + w(t) \quad (6.331)$$

$$|w(t)| \leq w_{max} \quad (6.332)$$

Theorem 6.10 *Sufficient Conditions For Attractivity of Sliding Surface:*

Given

(G1) Error dynamics of the form (6.330)- (6.331)

If

(I1)

$$\epsilon_2 \le k_1 - \epsilon_1^{\frac{1}{2}} [\frac{\sqrt{2}[k_2 + w_{max}]^{\frac{3}{2}}}{[k_2 - w_{max}]}] \tag{6.333}$$

where,

$$|\tilde{x}_1(0)| \le \epsilon_1 \tag{6.334}$$

$$|\tilde{x}_2(0)| \le \epsilon_2 \tag{6.335}$$

$$|w(t)| \le w_{max} \tag{6.336}$$

Then

(T1) The surface $\tilde{x}_1 = 0$ is locally attractive.

Proof: ♠ ▷ We will prove the theorem in the following manner. Firstly, we will find the *minimum* time t_{min} it takes for $\tilde{x}_2$ to get outside the tubular neighbourhood defined by $||\tilde{x}_2(t)|| \le k_1$. Then we will find the maximum time it takes for $\tilde{x}_1$ to become zero, given the time evolution of the $\tilde{x}_2$. Then we will derive the condition that $t_{min} \ge t_{max}$. This ensures local attractivity of the manifold $\tilde{x}_1 = 0$.

As the first step, let us find the time it takes for $||\tilde{x}_2(t)|| = k_1$.

Integrating the $\dot{\tilde{x}}_2$, equation, it is clear that

$$\tilde{x}_2(t) = [-k_2 sign[\tilde{x}_1] + w]t + \tilde{x}_2(0) \tag{6.337}$$

Therefore, the minimum time for $||\tilde{x}_2(t)|| = k_1$ is given by

$$t_{min} = \frac{k_1 - \epsilon_2}{||k_2 + w_{max}||} \tag{6.338}$$

Substituting for $\tilde{x}_2(t)$ in the equation (6.330) and solving for the maximum time $\tilde{x}_1 = 0$, we get,

$$t_{max} = \epsilon_1^{\frac{1}{2}}[\frac{\sqrt{2}[k_2 + w_{max}]^{\frac{1}{2}}}{[k_2 - w_{max}]}] \tag{6.339}$$

For the surface $\tilde{x}_1 = 0$ to be attractive, it is sufficient that $t_{min} > t_{max}$. Comparing (6.338) and (6.339), it is clear that

$$\frac{k_1 - \epsilon_2}{||k_2 + w_{max}||} > \epsilon_1^{\frac{1}{2}}[\frac{\sqrt{2}[k_2 + w_{max}]^{\frac{1}{2}}}{[k_2 - w_{max}]}] \tag{6.340}$$

$$\epsilon_2 \leq k_1 - \epsilon_1^{\frac{1}{2}}[\frac{\sqrt{2}[k_2 + w_{max}]^{\frac{3}{2}}}{[k_2 - w_{max}]}] \tag{6.341}$$

The theorem is therefore proved. ◁♠

Comment 6.5.4 For the case of no uncertainity, meaning $w(t) = 0$, we get

$$\epsilon_2 \leq k_1 - \sqrt{2\epsilon_1 k_2} \tag{6.342}$$

This equations makes sense. The intution is that an increase initial conditions on $\tilde{x}_2$ lead to a higher value of k_1. Futhermore, if the initial value, or a bound on the initial value of $\tilde{x}_2$ is known, k_1 can be selected based on the a knowledge of $x_1(0)$.

Comment 6.5.5 The extension of the results of the theorem to higher dimensions is nontrivial.

6.5.2 Increasing Regions of Attractivity

It is clear from the results of the previous section that the domain of attraction for the sliding surface $\tilde{x}_1 = 0$ is the chosen sliding gain k_1. Now with an increase in the perturbation of the initial condition of $\tilde{x}_2$, it becomes necessary to increase k_1. Such an increase helps to give sufficient time for $\tilde{x}_1$ to decay to zero, while at the same time enforcing a bound on the norm of $\tilde{x}_2(t)$. The price to be paid, as seen by comments in previous sections is a sharp decrease in the decay of $\tilde{x}_2(t)$. In this section, we present a simple way to alleviate this problem. We now consider a slight modification of the observer structure with the introduction of the linear term,

$$\dot{\hat{x}}_1 = \hat{x}_2 + h_1\tilde{x}_1 + k_1 sgn(\tilde{x}_1) \quad (6.343)$$

$$\dot{\hat{x}}_2 = k_2 sgn(\tilde{x}_2) \quad (6.344)$$

Such an observer structure equation leads to error dynamics of the form

$$\dot{\tilde{x}}_1 = \tilde{x}_2 - h_1\tilde{x}_1 - k_1 sgn[\tilde{x}_1] \quad (6.345)$$

$$\dot{\tilde{x}}_2 = -k_2 sgn[\tilde{x}_2] \quad (6.346)$$

We now state the result concerning the domain of attraction and stability of the modified observer.

Theorem 6.11 *Domain of Attraction and Stability of the Modified Observer:*

Given

(G1) Error dynamics of the form (6.345)- (6.346)

If

(I1) $||\tilde{x}_2(t)|| \leq k_1 + h_1||\tilde{x}_1(0)||$

Then

(T1) The one dimensional manifold $\tilde{x}_1 = 0$ *is attractive.*

(T2) $\tilde{x}_2$ *decays to 0 exponentially.*

The proof of the proposition follows verbatim the proof of the previous theorem, and therefore will not be repeated.

Comment 6.5.6 Note that for large values of $\tilde{x}_1$*, the system behaves as a linear system, meaning a Luenberger observer. Then as the* $\tilde{x}_1$ *decreases in magnitude, the observer operates as a nonlinear observer owing to the presence of the switching term. This is interesting as there is a clear demarcation of linear and nonlinear regimes, and we deliberately introduce a nonlinear regime in order to ensure the reduction of the error of the observed variable to zero in finite time. Operation in the nonlinear regime has the additional effect of ensuring the stability of the averaged dynamics.*

Comment 6.5.7 Also note that the introduction the linear term $h_1\tilde{x}_1$ has no effect on the Fillipov averaged dynamics of the observer, as this term vanishes when $\tilde{x}_1 = 0$.

Comment 6.5.8 The bounds derived on initial conditions derived using the theorem are unaffected. Only that now a significantly larger value of perturbation in initial conditions can be tolerated. The introduction of the h_1 term permits a reduction in the value of k_1 and k_2.

6.5.3 Smoothing And Reduction of Chattering

The use of the signum function in the observer equations leads to a lot of chattering in the estimates of the observer. There is a simple technique to overcome this. We replace the signum function with the saturation function, with a specified boundary layer. Analogous to sliding mode control, the observer dynamics also has a boundary layer, and Fillipov averaging is performed about the boundary layer. Furthermore, the use of the boundary layer indicates that in the presence of very small errors; the filter again operates in the linear regime, and functions essentially as a steady state Kalman estimator. The sliding gains can be chosen such that for very small innovations the filter uses steady state Kalman gains.

With a very large estimation error, the h_1 term predominates, and the filter starts out as a linear observer. Then, as the estimation error decreases, the filter starts to perform as nonlinear observer with the signum term predominating. This ensures the confinement of the observed variable to within the boundary layer. Furthermore, the averaged dynamics of the the system about the boundary layer is exponentially stable, leading to a decay of estimation errors in the other state variables.

The sliding mode observer tries to capture the essential features of the Luenberger observer, the sliding mode observer, and the steady state Kalman filter. For all this, and more, the increase in computational complexity is minimal. Comparison of the Kalman filter equations and the equations of the sliding mode observer reveal the salient feature of the sliding mode observer - operational and computational simplicity with no loss of robustness.

6.5.4 Criticism And Future Prospects

The sliding mode observers presented, while being applicable to most mechanical systems, suffer from serious technical problems when applied to systems of higher dimensions. The major problems with sliding mode observers are the following.

- Sensitivity to measurement noise.
- The chattering nature of the observer prevents achievement of the desired levels of accuracy. The average value of the observer is zero, but to be useful, the instantaneous values of the observer estimate must be used. However this instantaneous value is corrupted with the noise that results from chattering, and consequently the purpose is not served.

6.6 Fractional Control - Conjecture, Open Problem

6.6.1 Introduction

In this section, we will present an interesting variable structure control law for a vector dynamical system, that is a bounded control law, but whose convergence rate is faster than a comparable linear control law, and whose robustness properties are much better than comparable linear control laws. We will clarify what we mean by *comparable linear control laws* in the following subsections. We use the term fractional control law to indicate that this is a particular form of variable structure control law where the powers of indices are positive fractions.

We will present qualitative arguments for the conjecture, and will provide simulation results that are in agreement with the conjecture. However the proof of this conjecture has been quite elusive, and we have been unable to present anything more tangible than this conjecture. We leave the proof of this control method as an open problem to the reader.

6.6.2 Finite Time With Continuous Control - Scalar Systems

Consider a scalar dynamical system of the form

$$\dot{x} = u \tag{6.347}$$

where $x \in \Re$ and the control $u \in \Re$. Given the control objective of regulating the state of the system (6.347) to the origin commencing from arbitrary initial conditions in finite time, we choose u in the following manner.

$$u = -k|x|^{\frac{1}{r}} sgn[x] \tag{6.348}$$

where $k \in \Re_+$ and $r > 1$.

Comment 6.6.1 The choice of u is novel since the control is obviously continuous, but not differentiable at the origin. Also note that the control law involves raising the power of $|x|$ to a fraction, and hence the term fractional control.

We now make the following claim regarding existence of trajectories, stability and convergence for the system (6.347)

Claim 6.2 *Existence of solutions, stability and convergence for fractional control of scalar systems.*

Given

(G1) System dynamics of the form (6.347)

If

(I1) The control u is specified as in (6.348)

Then

(T1) Cauchy solutions exist for (6.347) subject to (6.348).

(T2) $x = 0$ is stable.

(T3) Indeed $x \to 0$ in finite time t^*, given by $t^* = \frac{|x(0)|^{1-\frac{1}{k}}}{k[1-\frac{1}{k}]}$

Proof: ♠ ▷ Existence of Cauchy solutions is easily seen by the fact that the righthandside of the differential system is continuous.

Considering the candidate Lyapunov function $V(x) : \Re \to \Re_+$ given by

$$V = \frac{x^2}{2} \tag{6.349}$$

Indeed $\dot{V} = -k|x|^{1+\frac{1}{r}} \leq 0$. Attractivity of the origin is therefore confirmed.

To show finite time convergence we solve the equation

$$\dot{x} = -k|x|^{\frac{1}{r}} sgn[x] \tag{6.350}$$

to obtain that $t^* = \frac{|x(0)|^{1-\frac{1}{k}}}{k[1-\frac{1}{k}]}$ The proof of the claim is complete. ◁♠

We now make a comparison between three kinds of control laws that regulate the state of the system (6.347) to the origin.

$$\begin{aligned} u_{linear} &= -kx && (6.351)\\ u_{sliding} &= -\frac{k}{|x|}x \text{ if } |x| > 0 && (6.352)\\ u_{fractional} &= -\frac{k}{|x|^{1-\frac{1}{r}}}x \text{ if } |x| > 0 && (6.353) \end{aligned}$$

Comparison of control efforts reveals something interesting. For all $|x| > 1$, the linear controller has the maximum gain, closely followed by the fractional controller, and the sliding mode controller has the smallest gain. However the situation is reversed when $|x| < 1$. .

Similarly, the times taken to reach the origin from initial conditions $x(0) \neq 0$ are

$$\begin{aligned} t_{linear} &= \infty && (6.354)\\ t_{sliding} &= \frac{|x(0)|}{k} && (6.355)\\ t_{fractional} &= \frac{|x(0)|^{1-\frac{1}{k}}}{k[1-\frac{1}{k}]} && (6.356) \end{aligned}$$

We now formulate an alternative control law that combines the best of both the linear and the fractional control law to give

$$
\begin{aligned}
u^* &= -kx \text{ if } |x| > 1 && (6.357)\\
&= -\frac{k}{|x|^p}x \text{ if } 0 < |x| \leq 1 && (6.358)\\
p &> 1 && (6.359)
\end{aligned}
$$

Note that we do not bother to define the control law at the origin.

There is yet another viewpoint as to why this control law does better than a linear control law when $|x| < 1$. The linear control law has an eigenvalue $-k$, and but u^* has an eigenvalue $\frac{-k}{|x|^{\frac{1}{p}}}$ (we use the term eigenvalue very loosely here, since strictly speaking even the term eigenvalues does not make sense in a nonlinear context) that is increasing to ∞ as $|x| \to 0$. Though both control laws are bounded, qualitatively, the fractional control law converges much faster to the origin as seen in the following scalar example.

Example 6.6.1 *Fractional Control - Scalar Case*

Consider the simple scalar example given by the equations

$$\dot{x} = u \qquad (6.360)$$

Choose

$$
\begin{aligned}
u_{linear} &= -kx && (6.361)\\
u^* &= -kx \text{ if } |x| > 1 && (6.362)\\
&= -\frac{k}{|x|^p}x \text{ if } 0 < |x| \leq 1 && (6.363)\\
k &= 2 && (6.364)\\
p &= 2 && (6.365)
\end{aligned}
$$

It is clear from the simulation plots that the modified fractional control law outperforms the linear control law.

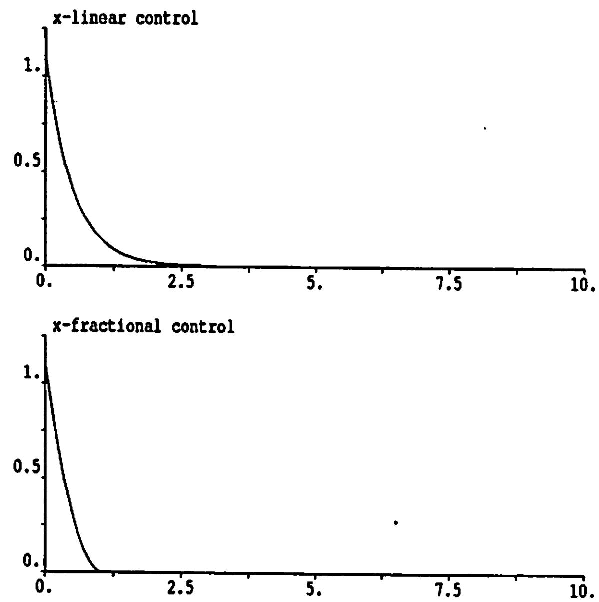

Figure 6.5: Comparison of Linear and Fractional Control Laws

Now consider a linear system in the controllable canonical form, given by the following equations.

$$\dot{x}_1 = x_2 \tag{6.366}$$

$$\vdots \quad \vdots \quad \vdots \tag{6.367}$$

$$\dot{x}_n = u \tag{6.368}$$

where $x \in \Re^n$, $u \in \Re$.

Now choose the control u to be of the following form

$$u = -k_1 x_1 - k_2 x_2 - \cdots - k_n x_n \text{ if } ||x||_2 > 1 \tag{6.369}$$

$$= -\frac{k_1}{||x||_2^{\frac{n}{r}}} x_1 - \frac{k_2}{||x||_2^{\frac{n-1}{r}}} x_2 - \cdots - \frac{k_n}{||x||_2^{\frac{1}{r}}} x_2 \text{ if } 0 < ||x||_2 < 1 \tag{6.370}$$

where

$$||x||_2 = \sum_{i=1}^{n} x_i^2 \tag{6.371}$$

$$r > n \tag{6.372}$$

$$s^n + k_n s^{n-1} + \cdots + k_1 \text{ is a stable Hurwitz polynomial} \tag{6.373}$$

We now formulate the following conjecture.

Conjecture 6.1 *Existence of solutions, stability and convergence for fractional control of controllable linear systems.*

Given

(G1) System dynamics of the form (6.366) - (6.368)

If

(I1) The control u is specified as in (6.369) - (6.373)

Then

(T1) Filippov solutions exist for systems (6.366) - (6.368) subject to control (6.369) - (6.373)

(T2) $x = 0$ is globally stable

(T3) Indeed $x \to 0$ faster than a comparable linear control law of the form $u_{linear} = -k_1x_1 - k_2x_2 - \cdots - k_nx_n$

Qualitative Proof:

First we note that within the unit ball ($||x||_2 < 1$), the control effort is bounded by

$$|u| \leq \sum_{i=1}^{n} k_i \tag{6.374}$$

So the control does not blow up at any instant of time. We have used the notion that in the nonlinear setting, within the unit ball, we have each eigenvalue $\lambda_i \;\; i = 1, 2, \ldots, n$ of this system being replaced by $\frac{\lambda_i}{||x||_2^{\frac{1}{r}}}$ where $r > n$. Consequently, from the way the $\lambda_i \;\; i = 1, 2, \ldots, n$ combine to form the k_i of the control law, the form of the control law is intutively obvious.

We find by simulation that the robustness, and rate of convergence of the proposed nonlinear law are much superior to a linear control law. The proof of this conjecture, however, has eluded us.

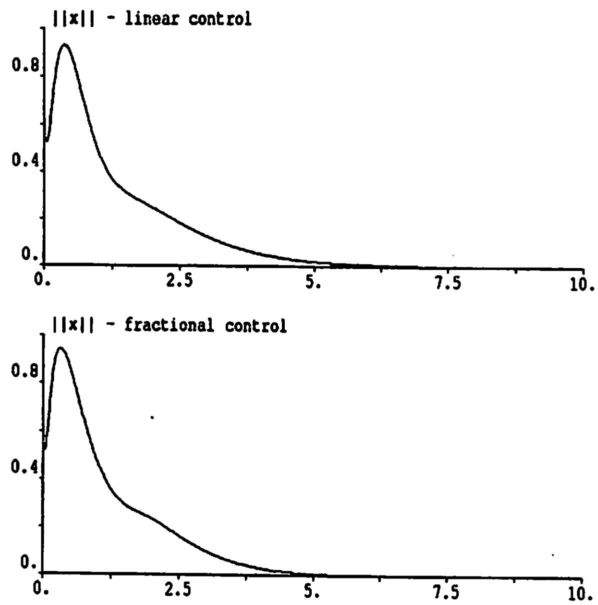

Figure 6.6: Fractional Control for Systems in Canonical Form

Example 6.6.2 *Fractional Control for Linear System in Controllable Canonical Form.*

We present simulation results for a system of the form

$$\dot{x}_1 = x_2 \tag{6.375}$$

$$\dot{x}_2 = x_3 \tag{6.376}$$

$$\dot{x}_3 = u \tag{6.377}$$

where

$$u = -k_1 x_1 - k_2 x_2 - k_3 x_3 \quad \text{if } ||x||_2 > 1 \tag{6.378}$$

$$= -\frac{k_1}{||x||_2^{\frac{3}{4}}} x_1 - \frac{k_2}{||x||_2^{\frac{1}{2}}} x_2 - \frac{k_3}{||x||_2^{\frac{1}{4}}} x_3 \quad \text{if } 0 < ||x||_2 < 1 \tag{6.379}$$

$$k_1 = 6 \tag{6.380}$$

$$k_2 = 11 \tag{6.381}$$

$$k_3 = 6 \tag{6.382}$$

The results show the faster convergence of the state subject to fractional control.

6.6.3 Criticism And Future Prospects

The modified fractional control law is interesting in that it seems to provide some desired features of both linear and nonlinear control laws. The major problems with this control law however are implementational difficulties.

- Computating the fractional powers of $||x||_2$ requires significant real time computing power.
- Accuracy of fractional control starts diminishing rapidly with lower fractional exponents.
- While the robustness of fractional control is quite high, it is not significantly better than a well designed H_∞ controller designed to minimize the effect of the disturbance on the regulation error. In simulation runs, the performance of a comparable H_∞ controller was just as good.
- While we were unable to come up with an acceptable proof of the control law, this control law seems to be a good alternative to a standard pole-placement control law.

Chapter 7

Conclusions And Future Work

7.1 Conclusions of This Thesis

We conclude this thesis with a brief summary of the contributions of this thesis. We presented the following extensions to nonlinear control theory.

- Generalized matching conditions for perturbed SISO systems with perturbed zero dynamics.
- Generalized matching conditions for perturbed nonsingular MIMO systems.
- Generalized matching conditions for perturbed singular MIMO systems which are left invertible.
- Generalized matching conditions for perturbed singular MIMO systems which are right invertible.
- Existence theorems for differential equations and inclusions with discontinuous righthand sides.
- Novel discontinuous control laws for finite time stabilization of nonlinear systems.
- Novel sliding mode identifiers for affine nonlinear systems.

- Synchronous sliding mode control theory.
- Variable structure lyapunov control of controllable systems perturbed by mismatched, non-lipschitz perturbations.
- Extensions to planar sliding mode observer theory.
- Conjecture regarding the use of fractional control for controllable linear systems.

7.2 Scope For Future Work

There is a lot of scope for future work in the areas mentioned above. We will outline some interesting problems that are worth looking into.

- Matching conditions relaxing the requirement of exponential stability of the zero dynamics.
- Combining sliding mode identification with control.
- Utilizing chattering control as a form of persistent excitation.
- Developing the generalization of the winding algorithm.
- Develop generalized state space sliding using $n-1$ switches to regulate to the origin.
- Extending Lyapunov control techniques to handle systems that cannot be input-output linearized.
- Extending sliding mode observer theory to handle non-planar systems.
- Extensions of the work on synchronous sliding and fractional control using other forms of nonlinearities.
- Formulating the Lyapunov and invariant set theorems for non-differentiable Lyapunov functions and differential inclusions.

- Adaptive control techniques for differential inclusions.
- Combining sliding mode identification with adaptive control.
- Applying synchronous control to physical examples.

We conclude this dissertation with a deep sense of fulfillment of having explored to some depth some aspects of nonlinear control theory.

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