ON THE EXISTENCE OF FINITE DIMENSIONAL FILTERS FOR MARKOV MODULATED TRAFFIC

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On the Existence of Finite Dimensional Filters for Markov Modulated Traffic

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Abstract

A Markov-modulated Poisson process (MMPP) is a Poisson process whose rate is a finite Markov chain. The Poisson process is a simple MMPP. An MMPP/M/1 queue is a queue with MMPP arrivals, an infinite capacity, and a single exponential server.

We prove that the output of an MMPP/M/1 queue is not an MMPP process unless the input is Poisson.

We derive this result by analyzing the structure of the nonlinear filter of the state given the departure process of the queue.

The practical relevance of the result is that it rules out the existence of simple finite descriptions of queueing networks with MMPP inputs.

Keywords : Queues, Output Processes, Nonlinear Filtering, Markov Modulated Poisson Process.

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1 Introduction

MMPPs are flexible models of point processes. By suitably fitting parameters, we can approximate many point processes by MMPPs. In addition to its flexibility, this class of processes is closed under Bernoulli sampling and addition. Moreover, the invariant distribution of the MMPP/M/1 queue is easily derived (see [3]). If the output of an MMPP/M/1 queue were an MMPP, then we could analyze MMPP/M/1 queues in tandem, trees of such queues, and possibly more general topologies.

However, we show in this paper that the output of an MMPP/M/1 queue is not an MMPP unless the input is Poisson (in which case, the output is also Poisson in steady-state).

In fact, we prove a more general result. Define a Neuts process (referred to as an N process) to be a process that counts specific transitions in a finite Markov chain. The class of N processes is strictly larger than that of MMPP processes. We prove that the output of an MMPP/M/1 queue is not an N process unless the input is Poisson. We could not prove, although we suspect, that the output of an N/M/1 queue is not an N process unless the input is Poisson.

Our Neuts processes are close relatives of the point processes studied by Marcel Neuts (see e.g., [3]). These processes were also analyzed in [4].

Our negative result continues a series of such results on output processes (see [2]). We use a method of analysis based on the nonlinear filtering equations of the state of the queue given its output. Applications of nonlinear filtering theory in queueing theory were elaborated in [1] and [4]. See also [5] for an introduction to these techniques.

The key idea of our proof is that if the output were an N process, then its stochastic intensity could be expressed in terms of polynomials of bounded degree (by the Cayley-Hamilton Theorem). We show that such a representation is not consistent with the actual structure of the filter for and MMPP/M/1 queue unless it is an M/M/1 queue.

2 Problem statement

2.1 General framework

Consider a continuous time Markov chain on a countable, but not finite, set X of states with transition matrix Q.

Suppose that only the jumps occuring in a subset J of the whole possible transitions are observed. Designate by $\Psi_t(k)$ the probability of being in state k at time t conditioned by the observed jumps up to time t, and Ψ_t the corresponding row-vector. Let Q_J be the transition matrix obtained by removing the observable transitions from matrix Q and Q^J the matrix such that $Q^J + Q_J = Q$. Note that Ψ_t evolves in the hyperplane H_1 such that $\Psi_t.1 = 1$.

The set of equations governing the evolution of Ψ_t is the following [5] :

$$\frac{d\Psi_t}{dt} = \Psi_t Q_J + \lambda_t \Psi_t \text{ between observed jumps;} \tag{1}$$

$$\lambda_t = \Psi_t Q^J.1; \tag{2}$$

$$\Psi_{t+} = (\lambda_{t-})^{-}1\Psi_{t-}Q^J \text{ when a jumps occurs.} \tag{3}$$

This system of equations is equivalent to the following one, where ν_t stands for the unnormalized probability vector :

$$\begin{aligned}
\frac{d\nu_t}{dt} &= \nu_t Q_J \text{ between observed jumps;} \\
\nu_{t+} &= \nu_{t-} Q^J \text{ when a jumps occurs;} \\
\Psi_t &= \frac{\nu_t}{\nu_t.1}.
\end{aligned}$$

Designate by $0 < T_1 < T_2 < \ldots T_n < t$ the times of the observed jumps up to time t. Then the general solution of these sets of equations is the following :

$$\begin{aligned}
\nu_t &= \nu_0 e^{Q_J T_1} Q^J e^{Q_J(T_2-T_1)} \ldots e^{Q_J(T_n-T_{n-1})} Q^J e^{Q_J(t-T_n)}; \\
\Psi_t &= \frac{\nu_t}{\nu_t.1},
\end{aligned}$$

where the exponential is formally defined by

$$e^{Qt} = \sum_{k=0}^{\infty} \frac{(Qt)^k}{k!},$$

this last limit being defined with respect to the chosen topology on X, and assuming that Q is a continuous operator (true for instance if Q entries are uniformly bounded).

Recall that a point process is characterized by its stochastic intensity. That is, the finite dimensional distributions of the point process are completely specified by the stochastic intensity of the process. For the process counting the jumps in J of the Markov chain, the stochastic intensity is equal to

$$\lambda_t := \Psi_t Q^J.1. \tag{4}$$

We define a Neuts process (designated by N) as a process that counts transitions of a finite Markov chain. It is possible for a process counting transitions of an infinite Markov chain to be statistically equivalent to a Neuts process. For instance, the output of a stationary M/M/1 queue is a Poisson process, which is a Neuts process since it counts all the transitions in a 2-state Markov chain.

The point process that counts the transitions in J of the Markov chain with rate matrix Q is a Neuts process if and only if there exists a finite-state Markov chain with rate Q', and corresponding matrices $Q_{J'}, Q^{J'}$ such that

$$\forall t \in R^+, \Psi_t Q^J.1 = \Psi'_t Q^{J'}.1, \tag{5}$$

i.e.,

$$\forall t \in R^+, \lambda_t = \lambda'_t.$$

In (5), 1 stands for the vectors of ones of the proper dimension on both sides of the equation. The existence of such a finite-state Markov chain would allow us to have a finite-dimensional filter at hand for the computation of the expected output rate λ_t, the computation of which *a priori* requires the calculation of the whole infinite conditional probability vector Ψ_t.

2.2 Preliminary analysis

Necessary algebraic conditions for a N process may be first derived from the set of equations (1)-(3) governing the evolution of Ψ_t. We prove the following two results in the Appendix:

Lemma 1 *A necessary condition for 5 to hold is:*

$$\forall x \geq 0, \forall y \geq 0, \qquad \Psi_t (Q_J)^x (Q^J)^y.1 = \Psi'_t (Q_{J'})^x (Q^{J'})^y.1; \tag{6}$$

$$\Psi_t (Q^J)^y (Q_J)^x.1 = \Psi'_t (Q^{J'})^y (Q_{J'})^x.1. \tag{7}$$

Theorem 1 *If (5) holds, then there exist two finite positive integers N_λ, N_μ, two families of complex numbers $\lambda_i, i = 1, \ldots, N_\lambda$ and $\mu_j, j = 1, \ldots, N_\mu$ with respective multiplicity orders $m(i)$ ($m(j)$) and two families of functions of time $t_{ipjq}(t)$ and $t'_{ipjq}(t)$ such that*

$$\begin{aligned} \Psi_t (Q^J)^y Q_J^x.1 &= \sum_{i=1}^{N_\lambda} \sum_{p=0}^{m(i)-1} \sum_{j=1}^{N_\mu} \sum_{q=0}^{m(j)-1} t_{ipjq}(t) \lambda_i^{x(p)} \mu_j^{y(q)}, \\ \Psi_t Q_J^x (Q^J)^y.1 &= \sum_{i=1}^{N_\lambda} \sum_{p=0}^{m(i)-1} \sum_{j=1}^{N_\mu} \sum_{q=0}^{m(j)-1} t'_{ipjq}(t) \lambda_i^{x(p)} \mu_j^{y(q)}, \end{aligned}$$

where the exponent (p) ((q))stands for the p-th (q-th) derivative with respect to x (y).

It is difficult to deal with these matrix products for general partially observed Markov chains (Q_J, Q^J). We now turn to the case of MMPP/M/1 queues.

3 The case of the MMPP/M/1 queue

We suppose now that our Markov chain represents an MMPP/M/1 queue. Following [3] and [5], we consider an augmented Markov chain (x, y), where x stands for the number of customers in the queue at time t and $y \in \{1, ..., Y\}$ is the current state of the Markov chain modulating the arrival rate. Thus (x, y) is a Markov chain with a countable number of states. The augmented rate matrix of this Markov chain is the following:

$$Q = \begin{pmatrix} B_0 & A_0 & 0 & \dots & \dots & 0 \\ A_2 & A_1 & A_0 & 0 & \dots & 0 \\ 0 & A_2 & A_1 & A_0 & 0 & \dots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$A_0 = \begin{pmatrix} \lambda(1) & 0 & \dots & \dots & 0 \\ 0 & \lambda(2) & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \lambda(N) \end{pmatrix};$$

$$A_2 = \begin{pmatrix} \mu & 0 & \dots & \dots & 0 \\ 0 & \mu & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \mu \end{pmatrix};$$

$$B_0 = -A_0 + S;$$

$$A_1 = -A_0 - A_2 + S.$$

We assume that the $\lambda(i)$ are strictly positive. In the above expressions, S is the transition matrix of the input modulating Markov chain, which is assumed to be irreducible. We assume that Q defines an irreducible and regular Markov chain. Necessary conditions for the irreducibility property are that B_0 and A_1 are non singular [3].

3.1 Stationary distribution

From the remark above, we know that this Markov chain has *at most one* invariant distribution Ψ_0, i.e, a probability vector satisfying $\Psi_0 Q = 0$. Following [3], we define the row-vector $\pi_x = [\pi(x,1), \ldots, \pi(x,Y)]$. We look for a stationary distribution Ψ_0 such that:

$$\pi_x = \pi_0 R^x, \tag{8}$$

where R is a $Y \times Y$ non-negative matrix, i.e., for Ψ_0 of the form $\Psi_0 = [\pi_0, \pi_0 R, \ldots, \pi_0 R^x, \ldots]$.

The balance equations then become

$$\begin{aligned} \pi_0(-A_0 + S) + \pi_0 R A_2 &= 0; & (9)\\ \pi_0 R^k A_0 + \pi_0 R^{k+1}(-A_0 - A_2 + S) + \pi_0 R^{k+2} A_2 &= 0 \ \forall k \geq 0. & (10) \end{aligned}$$

The second equation can be replaced by

$$A_0 + R(-A_0 - A_2 + S) + R^2 A_2 = 0. \tag{11}$$

We furthermore know from [3] that if such a solution exists then, since S is irreducible, all the eigenvalues of R will lie stricly inside the unit circle (the spectral radius of R is such that $sp(R) < 1$).

3.2 The main result

We investigate the *departures* from the $MMPP/M/1$ queue. The matrix Q^J then reads

$$Q^J = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ A_2 & 0 & \dots & 0 & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix}.$$

Note that, since $\Psi_0 Q = 0$ and $Q = Q^J + Q_J$, $\Psi_0 Q^J = -\Psi_0 Q_J$.

We now state the main theorem of the paper.

Theorem 2 *The departure process of an MMPP/M/1/ queue is not an N process unless the queue is a stationary M/M/1 queue.*

Proof :

The proof is based on the explicit computation of the terms $\Psi_0(Q^J)^y Q_J^x.1$. A contradiction is shown between their actual expression and the representation theorem stated above.

Let us introduce the two following lemmas. The notation $(M.v)_i$ for a matrix M and a vector v stands for the Y-dimensional subvector of vector $M.v$ corresponding to $x = i$ and 1_Y stands for the Y-dimensional vector of ones.

Lemma 2

$$\forall x \geq 0 \qquad (Q_J^x.1)_0 = (-A_0 + S)^x.1_Y - \sum_{q=1}^{x}(-\mu)^{x-q}(-A_0 + S)^q.1_Y; \tag{12}$$

$$\forall i > 0 (Q_J^x.1)_i = (-\mu)^x 1_Y. \tag{13}$$

The proof of this lemma is given in the appendix.

Lemma 3

$$\forall y \geq 0, \forall i \geq 0, (\Psi_0(Q^J)^y)_i = \pi_0 R^{i+y}\mu^y. \tag{14}$$

The proof of this second lemma is straightforward from the form of Q^J.

These two lemmas lead to the following expression for the terms $\alpha(y,x) = \Psi_0(Q^J)^y Q_J^x.1$:

$$\alpha(y,x) = \pi_0 R^y \mu^y \left((-A_0+S)^x.1_N - \sum_{q=1}^{x}(-\mu)^{x-q}(-A_0+S)^q.1_Y \right) + \sum_{i=1}^{\infty} R^{i+y}\mu^y(-\mu)^x 1_Y. \tag{15}$$

We know from the representation theorem that

$$\alpha(y,x) = \sum_{i=1}^{N_\lambda}\sum_{p=0}^{m(i)-1}\sum_{j=1}^{N_\mu}\sum_{q=0}^{m(j)-1} t_{ipjq}(0)\lambda_i^{x(p)}\mu_j^{y(q)}. \tag{16}$$

Dividing equation (15) by the product $\mu^y(-\mu)^x$ and combining the result with equation (16) we obtain:

$$\begin{aligned}
\frac{\alpha(y,x)}{\mu^y(-\mu)^x} &= \pi_0 R^y \left((\frac{-A_0+S}{-\mu})^x.1_Y - \sum_{q=1}^{x}(\frac{-A_0+S}{-\mu})^q.1_Y \right) + \sum_{i=1}^{\infty} R^{i+y}1_Y \\
&= \sum_{i=1}^{N_\lambda}\sum_{p=0}^{m(i)-1}\sum_{j=1}^{N_\mu}\sum_{q=0}^{m(j)-1} t_{ipjq}(0)(\frac{\lambda_i}{-\mu})^{x^{(p)}}(\frac{\mu_j}{\mu})^{y^{(q)}}.
\end{aligned}$$

Note that, for $x \geq 2$,

$$(\frac{-A_0+S}{-\mu})^x.1_Y - \sum_{q=1}^{x}(\frac{-A_0+S}{-\mu})^q.1_Y = -\sum_{q=1}^{x-1}(\frac{-A_0+S}{-\mu})^q.1_Y.$$

From then on we consider the difference $\frac{\alpha(y,x+1)}{\mu^y(-\mu)^{x+1}} - \frac{\alpha(y,x)}{\mu^y(-\mu)^x}$:

$$\frac{\alpha(y,x+1)}{\mu^y(-\mu)^{x+1}} - \frac{\alpha(y,x)}{\mu^y(-\mu)^x} = \pi_0 R^y(\frac{-A_0+S}{-\mu})^x.1_Y.$$

We then obtain the following representation :

$$\pi_0 R^y(\frac{-A_0+S}{-\mu})^x.1_Y = \sum_{i=1}^{N_\lambda}\sum_{p=0}^{m(i)-1}\sum_{j=1}^{N_\mu}\sum_{q=0}^{m(j)-1} t_{ipjq}(0)\left((\frac{\lambda_i}{-\mu})^{x+1^{(p)}} - (\frac{\lambda_i}{-\mu})^{x^{(p)}}\right)(\frac{\mu_j}{\mu})^{y^{(q)}}. \tag{17}$$

We then sum up all these last equations over y. Since $sp(R) < 1$, we find

$$\begin{aligned}\pi_0(I-R)^{-1}(\frac{-A_0+S}{-\mu})^x.1_Y &= \\ &\sum_{i=1}^{N_\lambda}\sum_{p=0}^{m(i)-1}\left(\sum_{j=1}^{N_\mu}\sum_{q=0}^{m(j)-1} t_{ipjq}(0)\sum_{y=0}^{\infty}(\frac{\mu_j}{\mu})^{y^{(q)}}\right)\left((\frac{\lambda_i}{-\mu})^{x+1^{(p)}} - (\frac{\lambda_i}{-\mu})^{x^{(p)}}\right) \\ &= \\ &\sum_{i=1}^{N_\lambda}\sum_{p=0}^{m(i)-1} S_{ip}(0)\left((\frac{\lambda_i}{-\mu})^{x+1^{(p)}} - (\frac{\lambda_i}{-\mu})^{x^{(p)}}\right),\end{aligned}$$

where $S_{ip}(0) = \sum_{j=1}^{N_\mu}\sum_{q=0}^{m(j)-1} t_{ipjq}(0)\sum_{y=0}^{\infty}(\frac{\mu_j}{\mu})^{y^{(q)}}$.

We now need the following lemma which is proved in the appendix:

Lemma 4

$$\pi_0(I-R)^{-1}S = 0.$$

Using this lemma, and the fact that S admits at most one stationary distribution ν, we conclude that

$$\pi_0(I-R)^{-1} = \nu,$$

where the proportionality constant is 1 because of the normalization on π_0.

Then, we get from the computation above that, for all $x \geq 0$:

$$\nu(\frac{-A_0+S}{-\mu})^x.1_Y = \sum_{i=1}^{N_\lambda}\sum_{p=0}^{m(i)-1} S_{ip}(0)\left((\frac{\lambda_i}{-\mu})^{x+1^{(p)}} - (\frac{\lambda_i}{-\mu})^{x^{(p)}}\right).$$

Since $\nu.S = 0$ and $S.1_Y = 0$ (S being stochastic) :

$$\nu(\frac{-A_0 + S}{-\mu})^x.1_Y = \nu(\frac{A_0}{\mu})^x.1_Y = \sum_{i=1}^{Y} \nu_i(\frac{\lambda(i)}{\mu})^x. \tag{18}$$

This yields :

$$\forall x \geq 0, \sum_{i=1}^{Y} \nu_i(\frac{\lambda(i)}{\mu})^x = \sum_{i=1}^{N_\lambda} \sum_{p=0}^{m(i)-1} S_{ip}(0) \left((\frac{\lambda_i}{-\mu})^{x+1^{(p)}} - (\frac{\lambda_i}{-\mu})^{x^{(p)}} \right).$$

For this to be true for all $x \geq 0$, since $\nu_i > 0$ for all i if S is irreducible, it must be that:

$$\begin{aligned} N_\lambda &\geq Y, \\ \lambda_n &= -\lambda(n), n \leq Y, \end{aligned}$$

i.e., that the Y first λ_i are the opposite of the diagonal elements of matrix A_0.

But it is easy to see that these Y first λ_i must be eigenvalues of matrix $-A_0 + S$ because of the equality (18).

Since the maximum number of distinct eigenvalues is Y (because Y precisely is the dimensionality of this matrix), the $-\lambda(i)$ must be characteristic roots of $M = -A_0 + S$.

Comparing the traces of $-A_0 + S$ and $-A_0$, we get that the trace of S has to be 0. This gives us a contradiction since S is stochastic.

This conclues the proof. $\square$

This proves that the output of an MMPP/M/1 queue is not an N process unless the input is Poisson.

4 Conclusion

This paper presents a negative result regarding the existence of finite-dimensional filters for MMPP/M/1 queues. We conjecture that the result extends to N/PH/1 queues.

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5 Appendix

Proof of Lemma 1 :

The proof of this lemma is three-fold :

The first step is to show that:

$$\forall y \geq 0, \forall x \geq 0, \frac{\Psi_t (Q^J)^y Q_{\bar{J}}^x .1}{\Psi_t (Q^J)^y .1} = \frac{\Psi_t' (Q^{J'})^y Q_{\bar{J}'}^x .1}{\Psi_t' (Q^{J'})^y .1}.$$

The second step is to prove that:

$$\forall x \geq 0, \forall y \geq 0, \Psi_t Q_{\bar{J}}^x (Q^J)^y .1 = \Psi_t' Q_{\bar{J}'}^x (Q^{J'})^y .1.$$

The third step is to combine the previous two to prove the lemma.

For the first step, we consider the derivatives of $\lambda_t = \Psi_t . Q^J .1$, given the equations (1)-(3) governing Ψ_t :

$$\begin{aligned} \frac{d\lambda_t}{dt} &= \Psi_t Q_{\bar{J}} Q^J .1 + \lambda_t^2; \\ \frac{d^2\lambda_t}{dt^2} &= \Psi_t Q_{\bar{J}}^2 Q^J .1 + 3\lambda_t \Psi_t Q_{\bar{J}} Q^J .1 + 2\lambda_t^3; \\ \ldots &= \ldots \end{aligned}$$

It is easy to show by induction that $\frac{d^x \lambda_t}{dt^x} = \Psi_t Q_{\bar{J}}^x Q^J .1 +$ additional terms depending on $\Psi_t Q_{\bar{J}}^q Q^J .1$, $q = 0, \ldots, x-1$. Consequently, equation (5) leads to

$$\forall x \geq 0, \Psi_t Q_{\bar{J}}^x Q^J .1 = \Psi_t' Q_{\bar{J}'}^x Q^{J'} .1.$$

Since for all vector v, $v.Q.1 = 0$, it follows that $v.Q_{\bar{J}}.1 = -v.Q^J.1$. Furthermore $\Psi_t.1 = 1 = \Psi_t'.1$. Therefore,

$$\forall x \geq 0, \Psi_t Q_{\bar{J}}^x .1 = \Psi_t' Q_{\bar{J}'}^x .1.$$

Considering the normalized probabilities vector after an undetermined number y of jumps, we get, in the same vein:

$$\forall y \geq 0, \forall x \geq 0, \frac{\Psi_t(Q^J)^y Q_J^x.1}{\Psi_t(Q^J)^y.1} = \frac{\Psi_t'(Q^{J'})^y Q_{J'}^x.1}{\Psi_t'(Q^{J'})^y.1}.$$

This concludes the first step of the proof.

For the second step, we note that since the expected output rate must be the same after an arbitrary number of jumps, it must be that

$$\forall y \geq 0, \frac{\Psi_t(Q^J)^y.1}{\Psi_t(Q^J)^{y-1}.1} = \frac{\Psi_t'(Q^{J'})^y.1}{\Psi_t'(Q^{J'})^{y-1}.1}.$$

We define $R_t^y = \frac{\Psi_t(Q^J)^y.1}{\Psi_t(Q^J)^{y-1}.1}$, and R'^y_t the equivalent for the finite-state Markov chain. By multiplying the previous equalities we obtain

$$\prod_{i=0}^{y} R_t^i = \prod_{i=0}^{y} R'^i_t,$$

so that, for all y (using $\Psi_t.1 = \Psi_t'.1 = 1$),

$$\forall y \geq 0, \Psi_t(Q^J)^y.1 = \Psi_t'(Q^{J'})^y.1.$$

Since this equality is true for all t, all the derivatives of both sides have to be equal. However,

$$\frac{d\Psi_t(Q^J)^y.1}{dt} = \frac{d\Psi_t}{dt}(Q^J)^y.1,$$

and from (1)-(3) it is easy to show by induction that $\frac{d^x\Psi_t(Q^J)^y.1}{dt^x} = \Psi_t Q_J^x (Q^J)^y.1$ plus additional terms depending on $\Psi_t Q_J^q (Q^J)^y.1$, $q = 0, \ldots, x-1$.

Therefore, the following equalities must hold:

$$\forall x \geq 0, \forall y \geq 0, \Psi_t Q_J^x (Q^J)^y.1 = \Psi_t' Q_{J'}^x (Q^{J'})^y.1.$$

This concludes the second step of the proof.

For the third step, we combine the equalities derived above:

$$\begin{aligned}\forall x \geq 0, \forall y \geq 0, \Psi_t Q_J^x (Q^J)^y.1 &= \Psi'_t Q_{J'}^x (Q^{J'})^y.1;\\ \forall x \geq 0, \forall y \geq 0, \frac{\Psi_t (Q^J)^y Q_J^x.1}{\Psi_t (Q^J)^y.1} &= \frac{\Psi'_t (Q^{J'})^y Q_{J'}^x.1}{\Psi'_t (Q^{J'})^y.1}.\end{aligned}$$

Consider the denominators of the second serie of equalities. They should be equal by taking $x = 0$ in the first serie of equalities. Consequently,

$$\forall x \geq 0, \forall y \geq 0, \Psi_t (Q^J)^y (Q_J)^x.1 = \Psi'_t (Q^{J'})^y (Q_{J'})^x.1,$$

and this ends the proof of the lemma. □

Proof of Theorem 1 :

We shall prove only the first set of equalities, the second being similar. Since $Q_{J'}$ and $Q^{J'}$ are finite dimensional, of dimensionality $N_0 < \infty$, we may use the Cayley-Hamilton theorem twice to get two finite families of coefficients $a^0_{Q_{J'}}, \ldots, a^{N_0}_{Q_{J'}}$ and $a^0_{Q^{J'}}, \ldots, a^{N_0}_{Q^{J'}}$ and such that:

$$\begin{aligned}\sum_{x=0}^{N_0} a^x_{Q_{J'}} Q^x_{J'} &= 0;\\ \sum_{y=0}^{N_0} a^y_{Q^{J'}} (Q^{J'})^y &= 0.\end{aligned}$$

Then, from the last lemma:

$$\begin{aligned}\sum_{x=0}^{N_0} a^x_{Q_{J'}} \Psi_t Q^x_J (Q^J)^y.1 &= 0;\\ \sum_{y=0}^{N_0} a^y_{Q^{J'}} \Psi_t Q^x_J (Q^J)^y.1 &= 0.\end{aligned}$$

Generalizing , $\forall p \geq 0, \forall q \geq 0$:

$$\begin{aligned}
\sum_{x=0}^{N_0} a^x_{Q_{J'}} \Psi_t Q_J^{x+p} (Q^J)^{y+q}.1 &= 0; \\
\sum_{y=0}^{N_0} a^y_{Q^{J'}} \Psi_t Q_J^{x+p} (Q^J)^{y+q}.1 &= 0.
\end{aligned}$$

Designate by λ_i the roots of $Q_{J'}$ characteristic polynomial, their number by N_λ and their multiplicity order by $m(i)$. Similarly, designate by μ_j the roots of $Q^{J'}$ characteristic polynomial, their number by N_μ and their multiplicity order by $m(j)$. Then a general solution of the above equations is

$$\begin{aligned}
\Psi_t Q_J^x (Q^J)^y.1 &= \sum_{i=1}^{N_\lambda} \sum_{p=0}^{m(i)-1} c_{ip}(y) \lambda_i^{x(p)}; \\
\Psi_t Q_J^x (Q^J)^y.1 &= \sum_{j=1}^{N_\mu} \sum_{q=0}^{m(j)-1} d_{jq}(x) \mu_j^{y(q)}.
\end{aligned}$$

For these equalites to be compatible, we must have

$$\begin{aligned}
c_{ip}(y) &= \sum_{j=1}^{N_\mu} \sum_{q=0}^{m(j)-1} t_{ipjq}(t) \mu_j^{y(p)}; \\
d_{jq}(x) &= \sum_{i=1}^{N_\lambda} \sum_{p=0}^{m(i)-1} t_{jqip}(t) \lambda_i^{x(p)}.
\end{aligned}$$

From these relations we get

$$\Psi_t Q_J^x (Q^J)^y.1 = \sum_{i=1}^{N_\lambda} \sum_{p=0}^{m(i)-1} \sum_{j=1}^{N_\mu} \sum_{q=0}^{m(j)-1} t_{ipjq}(t) \lambda_i^{x(p)} \mu_j^{y(q)}.$$

□

<u>Proof of Lemma 3 :</u>

Define $e^x = Q_J^x.1$. After some elementary algebra, we get that $(e^x)^t = [e_0^x, e_1^x, e_1^x, \ldots, e_1^x]$ with the following recursive equations:

$$e_1^x = (-\mu)^x.1_Y;$$

$$e_0^x = (-A_0 + S)e_0^{x-1} + A_0 e_1^{x-1}.$$

It is easy to solve this triangular system for e_0^k, and this yields :

$$e_0^x = (-A_0+S)^x.1 + \sum_{q=1}^{x}(-A_0+S)^{q-1}A.0e_1^{x-q} = (-A_0+S)^x.1_Y - \sum_{q=1}^{k}(-\mu)^{k-q}(-A_0+S)^q.1_Y$$

because $S.1_Y = 0$. □

Proof of Lemma 4 :

Consider equation (11):

$$A_0 + R(-A_0 - A_2 + S) + R^2 A_2 = 0.$$

It can be written

$$(-R(I-R)A_2 + (I-R)A_0 + RS = 0.$$

Equivalently:

$$(I-R)^{-1}R(I-R)A_2 = A_0 + (I-R)^{-1}RS.$$

Since, from [3], $sp(R) < 1$ it follows that

$$\sum_{k=0}^{\infty} R^k R(I-R)A_2 = A_0 + \sum_{k=0}^{\infty} R^{k+1}S.$$

This yields the following intermediate result:

$$A_0 = \mu R - \sum_{k=0}^{\infty} R^{k+1}S. \tag{19}$$

Replacing in equation (9) we find

$$\mu\pi_0 R = \pi_0(\mu R - \sum_{k=0}^{\infty} R^{k+1}S) - \pi_0 S.$$

Consequently,

$$\pi_0 \sum_{k=0}^{\infty} R^k S = 0,$$

which concludes the proof. $\square$