ON CHAOS IN DIGITAL FILTERS:
CASE b = −1

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ON CHAOS IN DIGITAL FILTERS: CASE b = -1

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Abstract

In this note we extend the symbolic analysis of digital filters with overflow nonlinearity to all values of the parameter a. While behavior of digital filters for $| a | \leq 2$ is not chaotic, it is completely chaotic for $| a | > 2$.

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1. Introduction

The chaotic phenomena in digital filters [1] has attracted much recent interest (see the references in [2]). In [1] symbolic dynamics was used to explain the complex behavior of second-order digital filters with overflow nonlinearity for a = 0.5. It is conjectured that the trajectories starting from the set I_γ (see [1] for a definition) is chaotic. *In this* paper we extend the usage of symbolic dynamics to all values of the parameter a. We show that the behavior of the digital filter with overflow nonlinearity is in fact not chaotic if $|a| \leq 2$, and that it is completely chaotic when $|a| > 2$.

2. Symbolic dynamics

Consider a second-order digital filter with overflow nonlinearity. Its dynamics is governed by the following nonlinear map:

$$F: I^2 \rightarrow I^2, \; F(x, y) = (y, f(-x + ay)) \tag{1}$$

where

$$f(v) = v - 2m, \quad \text{for } -1 + 2m \leq v < 1 + 2m, \text{ m an integer}$$

The phase space is:

$$I^2 = \{ \mathbf{x} = (x, y): -1 \leq x < 1, -1 \leq y < 1 \}$$

The associated linear model is:

$$G: R^2 \rightarrow R^2, \; G(x, y) = (y, -x + ay) \tag{2}$$

where

$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let q_1 and q_2 be the eigenvalues of the matrix A:

$$q_1, q_2 = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

The linear system (2) maps the unit square onto a parallelogram in such a way that $G(1,1) = (1, -1 + a)$, $G(1,-1) = (-1,-1 - a)$, $G(-1,1) = (1,a + 1)$ and $G(-1,-1) = (-1,-a + 1)$. Suppose that $2l - 1 \leq \max \{|a - 1|, |a + 1|\} < 2l + 1$. Then the map (1) can be rewritten as

$$F(x, y) = (y, -x + ay + 2s)$$

or

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} s$$

where

$$s = \mathrm{Int}(-x + ay)$$

and Int(•) is defined as:

$$\mathrm{Int}(v) = -m, \qquad \text{if } 2m - 1 \leq v < 2m + 1, \quad m = -l, \ldots, -1, 0, 1, \ldots, l$$

The integer s is the vertical translation required to return a point to the unit square. It tells us in which of the $2l + 1$ regions

$$I_m = \{ (x, y) \in I^2 : 2m - 1 \leq -x + ay < 2m + 1\}, \quad m = l, \ldots, -1, 0, 1, \ldots, l$$

lies the point (x,y). Since F is one-to-one and onto [1], $F^{-1}(\cdot)$ exists and is given explicitly by

$$F^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} s$$

where $s = \mathrm{Int}(ax - y)$.

The trajectory of (1) starting from $x(0) = x$ is defined as

$$\Gamma(x) = \{ x_n = F^n(x), \text{ n an integer}\}$$

where

$$F^0(x) = x,\ F^k(x) = F(F^{k-1}(x)) \text{ and } F^{-k}(x) = F^{-1}(F^{-k+1}(x))$$

Define Σ as the set of bi-infinite sequences consisting of $2l + 1$ symbols: $-l, \ldots, -1, 0, 1, \ldots, l$. Given an initial condition $x \in I^2$ we can generate a symbolic sequence $s \in \Sigma$ corresponding to the trajectory $\Gamma(x)$ by

$$s_i = \begin{cases} \mathrm{Int}(-x_i + ay_i) & i \geq 0 \\ \mathrm{Int}(ax_{i+1} - y_{i+1}) & i \leq -1 \end{cases}$$

Therefore we obtain a well-defined map

$$S: I^2 \rightarrow \Sigma = \{ s = (\ldots, s_{-1}, s_0, s_1, \ldots): s_i = -l, \ldots, -1, 0, 1, \ldots, l \}.$$

Let $\Sigma_F = S(I^2)$. We say that the sequence $\bar{s}$ is admissible if $\bar{s} \in \Sigma_F$.

The dynamics of second-order digital filters can be described as a second-order difference equation [3]; indeed the values of coordinate y satisfy the following equation:

$$y_{n+1} - ay_n + y_{n-1} = 2s_n \tag{3}$$

It is clear that the trajectory $\{y_n\}$ and the symbol sequence $\{s_n\}$ are uniquely determined for a given initial condition: y_i and y_{i-1}. This immediately follows from:

$$s_n = \mathrm{Int}(ay_n - y_{n-1}),\quad n \geq i$$

$$y_{n+1} = ay_n - y_{n-1} + 2s_n,\quad n \geq i$$

For $n \leq i$, we have:

$$s_{n-1} = \mathrm{Int}(-y_n + ay_{n-1}),\quad n \leq i$$

$$y_{n-2} = -(y_n - ay_{n-1} - 2s_{n-1}),\quad n \leq i$$

The following theorem gives the conditions when an admissible sequence determines a unique trajectory of (1).

Theorem 1. If $|q_1| \neq 1$ and $|q_2| \neq 1$, then any admissible sequence determines a unique trajectory of (1). Moreover, the explicit solution of the trajectory is given by:

$$y_n = \sum_{k=-\infty}^{+\infty} Y_{n-k}\, s_k, \tag{4}$$

$$Y_n = \frac{2}{\sqrt{a^2 - 4}}\, \rho^{|n|} \tag{5}$$

where

$$\rho = \left| \frac{a - \sqrt{a^2 - 4}}{2} \right| .$$

Remark 1. Since $|q_1| \neq 1$ and $|q_2| \neq 1$, it follows that $|\, a \,| > 2$. Thus, q_1 and q_2 are real numbers, $q_1 \neq q_2$ and $q_1 q_2 = 1$.

Proof of Theorem 1.

The general solution of the inhomogeneous equation (3) has the form

$$y_n = \bar{y}_n + y_n^*$$

where y_n^* is some particular solution of (3), and $\bar{y}_n$ is the general solution of the corresponding homogeneous equation

$$y_{n+1} - ay_n + y_{n-1} = 0 \tag{6}$$

The general solution of (6) is given by:

$$\bar{y}_n = C_1 q_1^n + C_2 q_2^n$$

where C_1 and C_2 are arbitrary constants, and q_1 and q_2 are the roots of the characteristic equation:

$$q^2 - aq + 1 = 0$$

i.e., q_1 and q_2 are the eigenvalues of the matrix A.

Now, let us consider equation (3) with s_n of the special form:

$$s_n = \delta_0^n = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{7}$$

The solution of (3) with s_n given by (7) will be denoted by Y_n. We will look for the bounded solution of Y_n of the following group of equations:

$$Y_{n+1} - aY_n + Y_{n-1} = 0 \quad , n \le -1 \tag{8}$$

$$Y_1 - aY_0 + Y_{-1} = 2 \tag{9}$$

$$Y_{n+1} - aY_n + Y_{n-1} = 0 \quad , n \ge 1 \tag{10}$$

Let us assume that $|q_1| < 1$ and $|q_2| > 1$. The particular solution of equations (8) and (10) is:

$$Y_n = \begin{cases} C_1^{(1)} q_1^n + C_2^{(1)} q_2^n & n \le 0 \\ \\ C_1^{(2)} q_1^n + C_2^{(2)} q_2^n & n \ge 0 \end{cases} \tag{11}$$

Since Y_n is a bounded solution, it follows that $C_1^{(1)} = C_2^{(2)} = 0$. Therefore:

$$Y_n = \begin{cases} C_2^{(1)} q_2^n & n \le 0 \\ \\ C_1^{(2)} q_1^n & n \ge 0 \end{cases} \tag{12}$$

For n = 0, (12) gives one and the same value Y_0; so $C_1^{(2)} = C_2^{(1)}$. From (9) we have:

$$C_1^{(2)} = \frac{2}{q_1 - a + q_2^{-1}}$$

Since

$$q_1 - a + q_2^{-1} = (q_2 - a + q_2^{-1}) + q_1 - q_2 = q_1 - q_2 = \sqrt{a^2 - 4}$$

and using the constraint $q_1 q_2 = 1$, we obtain (5). In a similar fashion we can obtain the same result when $|q_1| > 1$ and $|q_2| < 1$.

For an arbitrary right-hand side $\{ s_n \}$, the particular solution y_n^* of (3) can be written in the following way:

$$y_n^* = \sum_{k=-\infty}^{+\infty} Y_{n-k} \, s_k \tag{13}$$

where Y_{n-k} is the solution of the following equation:

$$Y_{n+1-k} - a\, Y_{n-k} + Y_{n-1-k} = 2\, \delta_k^n$$

and δ_k^n is defined by:

$$\delta_k^n = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases}$$

Indeed, substituting the series (13) into the left-hand side of (3) we get

$$y_{n+1} - ay_n - by_{n-1} = \sum_{k=-\infty}^{+\infty} Y_{n+1-k}\, s_k - a \sum_{k=-\infty}^{+\infty} Y_{n-k}\, s_k + \sum_{k=-\infty}^{+\infty} Y_{n-1-k}\, s_k =$$

$$= \sum_{k=-\infty}^{+\infty} (Y_{n+1-k} - a\, Y_{n-k} + Y_{n-1-k})\, s_k = \sum_{k=-\infty}^{+\infty} 2\, \delta_k^n\, s_k = 2s_n$$

From (5) we can see that Y_n decreases exponentially for $n \to \pm \infty$

$$| Y_n | < Y \rho^{|n|} \tag{14}$$

where $Y = \dfrac{2}{\sqrt{a^2 - 4}} > 0$ and $0 < \rho < 1$ are constants. In the case of second-order digital filters we have:

$$| s_n | \leq l \tag{15}$$

Using (14) and (15) we obtain:

$$| y_n^* | = \left| \sum_{k=-\infty}^{+\infty} Y_{n-k}\, s_k \right| \leq \sum_{k=-\infty}^{n} | Y_{n-k} s_k | + \sum_{k=n+1}^{+\infty} | Y_{n-k} s_k |$$

$$< Yl \left[\sum_{k=-\infty}^{n} \rho^{n-k} + \sum_{k=n+1}^{+\infty} \rho^{k-n} \right] < \frac{2Y}{1 - \rho}\, l$$

Hence, the series (13) certainly converges.

Let $\tilde{y}_n$ be another bounded solution of (3). Then:

$$\tilde{y}_{n+1} - a\tilde{y}_n + \tilde{y}_{n-1} = 2s_n$$

$$y_{n+1}^* - ay_n^* + y_{n-1}^* = 2s_n$$

and the difference $\bar{y}_n = \tilde{y}_n - y_n^*$ is the solution of the homogeneous equation (6). But, we can see that the unique bounded solution for (6) is $\bar{y}_n = 0$. Thus, the solution (4) is the only bounded solution for a given right-hand side. ▮

Remarks:

2. Theorem 1 shows that if $|a| > 2$, the map S is one-to-one. In the case $|a| \leq 2$, the map S is neither one-to-one nor onto [1].

3. The condition that $\{y_n\}$ should lie in the interval $[-1, 1)$ puts an infinite number of constraints on the sequence $\{s_n\}$:

$$-1 \leq \sum_{k=-\infty}^{+\infty} \frac{2}{\sqrt{a^2 - 4}} \rho^{|n-k|} s_k < 1$$

3. Chaotic behavior

Because values of y differing by integers are identified, whereas the corresponding values of x are not, the phase space of second-order digital filters can be considered as a cylinder. Then (1) is an area-preserving map known as a sawtooth map. The differential of F in the points where it is defined is given by the constant matrix A. The product of the eigenvalues q_1 and q_2 of A is equal to 1, and either q_1 and q_2 are both real, or they are complex conjugate. In the first case we have

$$|q_1| < 1 < |q_2| \text{ (or } |q_2| < |q_1|) \tag{16}$$

In the second case, we have

$$1 = q_1 q_2 = |q_1|^2 = |q_2|^2,\ q_1 \neq q_2 \tag{17}$$

The third case is

$$q_1 = q_2 = \pm 1 \tag{18}$$

Since A is a constant matrix, all periodic points of F for a given a belong to the same class:

(i) If $|a| > 2$, then all periodic points are hyperbolic (the first case (16));

(ii) If $a = \pm 2$, then all periodic points are parabolic (the third case (18)); and

(iii) If $|a| < 2$, then all periodic points are elliptic (the second case (17)).

Let $a = 2$. Then the system (1) is integrable; i.e. the line

$$y = x + y_0 - x_0$$

is invariant. Indeed, (1) can be written in the form:

$$z_{n+1} = z_n$$

$$x_{n+1} = x_n + z_n + 2s_n$$

where $z = y - x$. The trajectories of (1) can have different qualitative behaviors depending upon the rationality, or irrationality, of $z_0 = x_0 - y_0$. If z_0 is a rational number, then each trajectory consists of only a finite number of parabolic periodic points. If z_0 is an irrational number, then each trajectory consists of an infinite number of points which are dense on a circle.

Let $a = 2 \pm \varepsilon$, for ε small enough. What happens to the parabolic periodic points?

If $\varepsilon = 0$, then every point on the circle $z = m_1/m_2$ is a fixed point with period m_2. In *typical* area preserving maps [4], after the perturbation, only a finite number of points remain (generally, this number is $2m_2$). They form a chain of alternating elliptic and hyperbolic points, with regular trajectories (or KAM curves) encircling the elliptic fixed points and chaotic trajectories in the neighborhood of hyperbolic points.

But, the map (1) is not a typical area-preserving map. For $\varepsilon > 0$, all periodic points are hyperbolic, while for $\varepsilon < 0$, they are elliptic.

It is easy to evaluate the Lyapunov exponents for the dynamical system (1); indeed, they are given by:

$$\lambda_1 = \ln |q_1|, \lambda_2 = \ln |q_2| \tag{19}$$

We recall that the Lyapunov exponents are defined by [4]:

$$\lambda_i = \lim_{n\to\infty} \ln \mid q_i(n) \mid$$

where $q_i(n)$ are the eigenvalues of the matrix

$$[\ B(x_n)B(x_{n-1}) \ \ldots\ B(x_1)\ ]^{1/n}$$

and $B(x_n)$ is the Jacobian matrix of F:

$$B(x_n) = \frac{\partial}{\partial x} F(x_n)$$

Since the Jacobian matrix of F is a constant matrix **A**, (19) follows immediately.

If $\mid a \mid \leq 2$, the behavior of digital filters is not chaotic; indeed, in this case the Lyapunov exponents are $\lambda_1 = \lambda_2 = 0$ and all periodic orbits are elliptic.

Now we shall deal with the case $|a| > 2$.

The dynamical system φ defined on torus T^2 by

$$\varphi(x, y) = (\ a_{11}x + a_{12}y,\ a_{21}x + a_{22}y) \pmod 1$$

is an ergodic automorphism if a_{11}, a_{12}, a_{21}, a_{22} are integers, $a_{11}a_{22} - a_{12}a_{21} = 1$ and the eigenvalues of the matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

are real numbers. Ergodic automorphisms on the torus T^2 are Bernoulli shifts [5].

If the parameter a is an integer, $|a| > 2$, the map (1) is an ergodic automorphism on the torus [6]; $a = 3$ is the well known Arnold - Sinai cat map [4]. When a is not an integer, $|a| > 2$, the map (1) becomes discontinuous and its phase space is filled by a dense countable set of discontinuity lines for the powers of the map and its inverses. Very recently Vaienti [7] proved that the discontinuous sawtooth map is a Bernoulli system. Thus, even though (1) evolves in a completely deterministic way, if one makes a measurement with only a finite number of possible outcomes (e.g. the phase space of the digital filter is covered by a partition with a finite number of disjoint sets), then the resulting process is random, and is essentially indistinguishable from a finite coding of a roulette wheel.

3. Conclusions

We have shown that the symbolic dynamics can be used to analyzed the behavior of digital filters for all values of the parameter a. It was proven that when $|a| > 2$ a symbolic sequence determines a unique trajectory of map (1). The dynamics of the digital filter is not chaotic when $|a| \leq 2$, but it is completely chaotic when $|a| > 2$. In the case when a is an integer, $|a| > 2$, the digital filter is described by a torus automorphism isomorphic to the Bernoulli shifts. For $a = 3$, the Arnold - Sinai cat map governs the behavior of the digital filters.

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