MISCELLANEOUS ON RULE-BASED INCREMENTAL CONTROL: FROM STABILITY TO LEARNABILITY

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Dominique Luzeaux

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Contents

Abstract

This report will address some issues on *rule-based incremental control* like: is it possible for a given class of systems to find a rule-based incremental controller that stabilizes these systems? Is it possible to find a rule-based incremental controller that tracks any reference trajectory? These two questions are particularly interesting in robotics: for instance if some method can generate a trajectory [Luz92b], it is necessary to be able to control the given robot in order to follow that trajectory.

We did not focus only on these two questions, but they build the core of the first chapter, and the search for an affirmative answer has given us new insights on the mechanisms of rule-based incremental control. This has led us to reconsider the current definition of rule-based incremental control; in [Luz92b], we have discussed the use of artificial intelligence techniques in order to learn rule-based incremental controllers for a given process, but the learnability has not been proved yet, although experimental results seem to confirm it. A theoretical frame has been developed in order to try to confirm or infirm the learnability [MLZ92, Mar92] (Eric Martin is currently finishing his PhD on the subject [Mar92]) and one of our aims has been to find an adequate definition of rule-based incremental control, which encompasses the current definition (we do not want to lose the benefit of the results we have already proved!), and which can be used in that theoretical frame. The second and the third chapters are the result of that search for a new definition of rule-based incremental control; they only give a temptative answer and much work has still to be done.

The general form of rule-based incremental control has been discussed in [Luz91]; other references and results can be found in [Luz92b]. Let us only recall that the control laws considered have the following form: $u_{k+1} = u_k + \epsilon_k \Delta$ where ϵ_k (called the *sign*) is an integer between $-m$ and $+m$, and Δ (called the *increment*) is any non null positive real. A rule-based incremental controller is a finite set of *rules*: *if* some test on the state at time k *then* add $\epsilon_k \Delta$ to the current input. Usually, the test in the if-part consists in computing some function of the state and comparing it with given values, in other words the value of ϵ_k is computed by a step-function applied on an eventually nonlinear function of the state. Actually this definition yields *state rule-based incremental controllers*, and by replacing in the rules the word *state* by *output*, we

could define *output rule-based incremental controllers*, which are used for instance in the learning program CANDIDE [Luz92b].

We would like to make a few remarks at this point.

The word *rule-based* should not confuse the reader: we are not dealing here with expert systems for instance. Of course we are dealing with rules, but that is not the key-point of our approach; much more important is the fact that the input at time k will not be chosen arbitrarily, but will be deduced from the previous input by adding (or subtracting, or more generally, by applying) a given amount of a fixed increment. The rules are only a convenient way in the final stage of the computation of that amount. This had to be made precise, as the term *rule-based* is often used without discernment: without being too sarcastic, we could imagine a rule like "if the system is controllable, then find a control law", this would be rule-based control too! The crucial point when using rules is to define precisely the language used when one writes rules, and although it seems trivial, it is not always observed...

Another remark is: although we are using rules and basically, the action on the input between two successive time steps could be interpretated as "add some large increment" or "subtract a small increment", we are not using fuzzy control either.

The definition of a rule-based incremental controller makes the next remark void, but we would rather appear as insisting too much than be misunderstood (and misjudged...): the coefficient ϵ_k is the result of the comparison of a nonlinear function of the state or the output with given values; it is not obtained by mere discretization of the output space or the state space.

We have always been guided by one principle: only claim as true that which can be theoretically proved, and use experimental results only as a guide for further proofs, not as a show-off. We hope this report keeps to this principle.

Chapter 1

Rule-based incremental control: chaos, stability, tracking

In the first section, we will study a particular rule-based incremental controller when applied in a feedback loop to a linear system; we will then observe the distribution of the different components of the state vector, which seems to point at chaotic behavior. When the state vector space is 1-dimensional, we will study more precisely the relationship between the evolution of the state vector, the initial state and the state matrix in order to understand the underlying mechanisms [Luz92a]. The next sections will be dedicated to other rule-based incremental controllers, used to stabilize first time-invariant linear systems, and then time-varying linear systems. This will lead ultimately to a rule-based incremental controller used for trajectory tracking for a class of nonlinear systems.

1.1 Rule-based incremental control leading to chaotic behavior

1.1.1 A state-feedback rule-based incremental controller

Proposition 1 *For a controllable linear single input system ($X_{k+1} = AX_k + Bu_k$), a given desired trajectory $(X_k^{(d)})_{k\in\mathbf{N}}$ and a sequence of inputs $(u_k^{(d)})_{k\in\mathbf{N}}$ yielding this trajectory ($X_{k+1}^{(d)} = AX_k^{(d)} + Bu_k^{(d)}$) such that $(u_k^{(d)} - u_{k-1}^{(d)})_{k\in\mathbf{N}}$ is bounded, then, for any positive real ϵ, there exists a rule-based incremental controller tracking the trajectory with an error smaller than ϵ.*

Proof: First let us introduce some notations: any real number x can be written $x = [x] + \{x\}$ where $[x] \in \mathbf{Z}$ and $\{x\} \in]-1, 1[$[1]. For positive x, let $x = \sum_{i=-\infty}^{p} d_i 10^i$ be a decimal expansion of x,

[1]Our definition differs from the usual integer and fractional part for negative reals

where the digits d_i are integers in $[0,9]$; we shall take for $\{x\}$ the real number $\sum_{i=-\infty}^{-1} d_i 10^i$ and $[x]$ will be the integer $\sum_{i=0}^{p} d_i 10^i$; for negative x, we take respectively: $\{x\} = -\{-x\}$ and $[x] = -[-x]$ (for instance $[1.2] = 1$, $\{1.2\} = 0.2$ and $[-2.5] = -2$, $\{-2.5\} = -0.5$).

As the system is controllable, it has a canonical controllable form (see for instance [Oga87] or appendix):

$$\widehat{X}_{k+1} = \widehat{A}\widehat{X}_k + \widehat{B}u_k$$

Let us define v_k and u_k by (Δ is a non null positive real and obviously $\widehat{X}_k^{(d)}$ corresponds to $X_k^{(d)}$ expressed in the new coordinates):

$$\begin{cases} v_k &= [\frac{-\widehat{B}^\top \widehat{A}(\widehat{X}_k - \widehat{X}_k^{(d)}) + u_k^{(d)}}{\Delta}]\Delta + \widehat{B}^\top \widehat{A}(\widehat{X}_k - \widehat{X}_k^{(d)}) - u_k^{(d)} \\ u_k &= -\widehat{B}^\top \widehat{A}(\widehat{X}_k - \widehat{X}_k^{(d)}) + u_k^{(d)} + v_k \end{cases}$$

We notice that $|v_k| \leq \Delta$. A straightforward computation yields:

$$\widehat{X}_{k+1} - \widehat{X}_{k+1}^{(d)} = J(\widehat{X}_k - \widehat{X}_k^{(d)}) + \widehat{B}v_k$$

where J is a $n \times n$ Jordan matrix. We conclude that for any k:

$$\begin{aligned} \| \widehat{X}_{k+n} - \widehat{X}_{k+n}^{(d)} \| &= \| \textstyle\sum_{i=1}^{n} J^{n-i} \widehat{B} v_{k+i} \| \\ &\leq (\textstyle\sum_{i=1}^{n} \| J^{n-i} \|) \| \widehat{B} \| \Delta \end{aligned}$$

Take Δ such that $(\sum_{i=1}^{n} \| J^{n-i} \|) \| \widehat{B} \| \Delta < \epsilon$ and the error on the trajectory is smaller than ϵ. Let us now verify that (u_k) is an incremental control law:

$$\begin{aligned} u_k = u_{k-1} + & \\ & \left(-\widehat{B}^\top \widehat{A}(\widehat{X}_k - \widehat{X}_k^{(d)}) + u_k^{(d)} + v_k\right) + \left(\widehat{B}^\top \widehat{A}(\widehat{X}_{k-1} - \widehat{X}_{k-1}^{(d)}) - u_{k-1}^{(d)} - v_{k-1}\right) \end{aligned}$$

The two last terms of the right part of the previous equality are, by construction, multiples of Δ. Furthermore:

$$|u_k - u_{k-1}| \leq |u_k^{(d)} - u_{k-1}^{(d)}| + |v_k| + |v_{k-1}| + \| \widehat{B}^\top \widehat{A} \| (\| \widehat{X}_k - \widehat{X}_k^{(d)} \| + \| \widehat{X}_{k-1} - \widehat{X}_{k-1}^{(d)} \|)$$

which is uniformly bounded in k. There exists thus an integer m such that for all k, there exists an integer ϵ_k with $|\epsilon_k| \leq m$ verifying $u_k = u_{k-1} + \epsilon_k \Delta$. The corresponding rule-based controller is obvious to write: compute $[\frac{-\widehat{B}^\top \widehat{A}(\widehat{X}_k - \widehat{X}_k^{(d)}) + u_k^{(d)}}{\Delta}] - [\frac{-\widehat{B}^\top \widehat{A}(\widehat{X}_{k-1} - \widehat{X}_{k-1}^{(d)}) + u_{k-1}^{(d)}}{\Delta}]$ which will be equal to some integer p (bounded by a constant as we saw previously) and add then $p\Delta$ to u_{k-1}. □

<u>Remark</u>: the assumption of boundedness for the difference of two successive reference inputs is not too restrictive, as in practice reference signals are usually bounded and thus their variation between two time steps is bounded too.

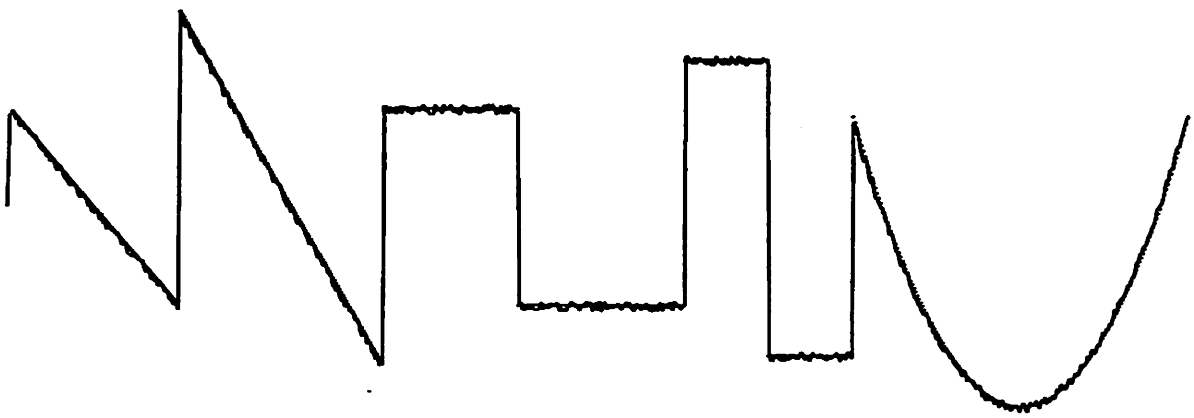

Figure 1.1: **Tracking a 1-dimensional reference signal**

An illustration of this proposition can be seen on figure 1.1, where a reference signal consisting in steps, ramps and second-degree polynomials is tracked.

Let us look more closely at the previous proposition when $u_k^{(d)} = 0$ and $X_k^{(d)} = 0$. We can then state that for all ϵ, there exists a state feedback rule-based incremental controller such that for the controlled system we have: $\exists M, \forall k, \| X_k \| \leq M$.

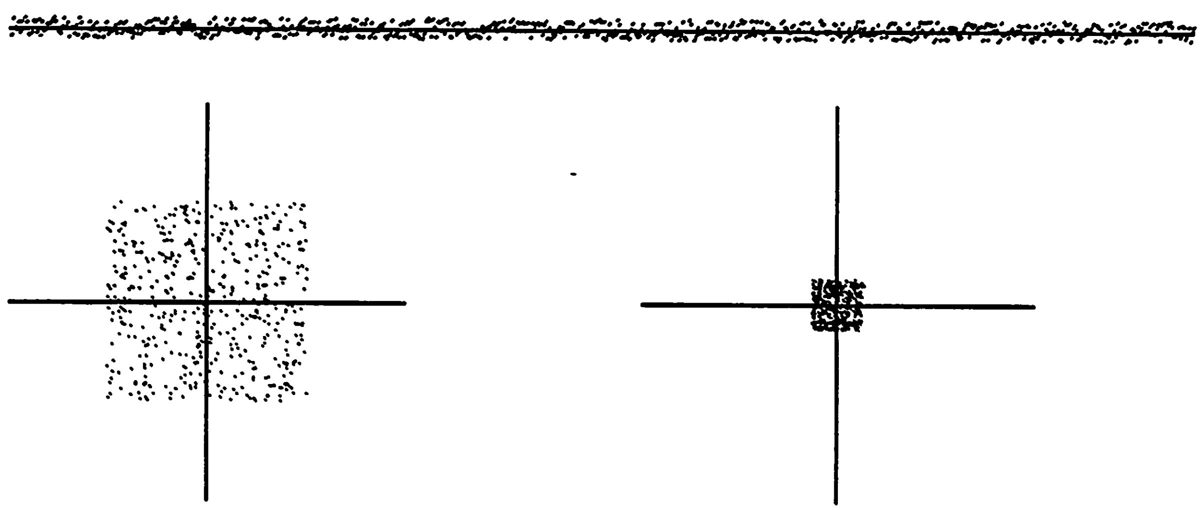

Figure 1.2: **1 and 2-dimensional systems in closed loop**

Figure 1.2 represents the state vector for 1-dimensional and 2-dimensional systems as a function of time. We see the values of the state vector are spread respectively inside a strip and inside a square. For the 2-dimensional case, two different values of Δ have been chosen, and the figure shows clearly that the error is proportional to Δ.

Looking at the form of the state feedback incremental law, it is obvious that as soon as some X_k is null, the state vector is null for any $k' \geq k$. We will now try to find a condition such that there exists some k with $X_k = 0$. Rewriting the incremental control law, we have:

$$\begin{cases} u_k &= [-\dfrac{\widehat{B}^\top \widehat{A} X_k}{\Delta}]\Delta \\ v_k &= \{\dfrac{\widehat{B}^\top \widehat{A} X_k}{\Delta}\}\Delta \end{cases}$$

The expression $-\widehat{B}^\top \widehat{A} X_k$ equals $\sum_{i=1}^{n} a_{n+1-i} X_k^{(i)}$. The state vector can be rewritten as:

$$X_k = \begin{pmatrix} v_{k-n} \\ \vdots \\ v_{k-1} \end{pmatrix}$$

This shows that a necessary condition for $X_k = 0$ is the array of equalities $v_{k-i} = 0$ for $1 \leq i \leq n$. But then $X_{k-n+i} = J^i X_{k-n}$ and all the equalities $v_{k-i} = 0$ can be rewritten as $\sum_{j=n-i+1}^{n} a_{2n-i-j+1} X_{k-n}^{(j)} = q_i \Delta$, where q_i is some integer in $\mathbb{Z}$. Every such equality is in fact in $\mathbb{R}^n$ (seen as an affine space) the equation of a family of parallel affine hyperplanes (the corresponding direction in the vector space is given by the equation $\sum_{j=n-i+1}^{n} a_{2n-i-j+1} X_{k-n}^{(j)} = 0$), and the condition $X_k = 0$ reduces to: the point with coordinates $X_{k-n}^{(i)}$ must be on one of the vertices of the previously defined lattice (the intersection of these n infinite families of hyperplanes)[2].

The previous paragraph only tells us how to choose a state vector such that it goes to 0 after at most n steps; it would be interesting to have a general condition on X_0 such that $X_k = 0$ for $k > n$, but for multi-dimensional vectors, this cannot be written in an easy way. However for 1-dimensional systems, there are very simple conditions, as we will see in the next section (the 1-dimensional case is much easier to write down, as X_k and v_k can then be identified).

1.1.2 Distribution of the state vector

Figure 1.3 is another illustration of the 2-dimensional case; values of (a_1, a_2) are from left to right: $(-1.045, -1.0032)$, $(-1.03, -1.01)$, $(-1.2, -1.2)$, $(-2.48, -1.2)$, $(-10, -12)$ and the initial state has been chosen small. A quick look at this figure shows that the values taken by the state vector inside the square do not seem to be distributed in the same way.

The aim of this section is to see how this distribution behaves when $\widehat{A}$ and X_0 vary. In order to simplify the equations, we will consider the 1-dimensional case and the system is - trivially, as B is then a scalar and has to be non null - under its canonical controllable form. Furthermore we assume that the scalar state matrix (written now a) and the initial state (x_0) are positive.

[2]Actually, as soon as $k > n$, X_{k-n} cannot be chosen arbitrarily as its norm is bounded, and only a finite part of the lattice is admissible

Figure 1.3: **Other 2-dimensional systems in closed loop**

We have:

$$\begin{cases} x_{k+1} &= ax_k + u_k \\ u_k &= [\frac{-ax_k}{\Delta}]\Delta \end{cases}$$

The recurrent equation can be rewritten as:

$$\frac{x_{k+1}}{\Delta} = \{a\frac{x_k}{\Delta}\}$$

Without any loss of generality, we can thus assume that $\Delta = 1$ (the sequence $(\frac{x_k}{\Delta})$ has the same distribution as the sequence (x_k)). For $k \geq 1$ we have obviously $x_k < 1$; we can then assume $x_0 < 1$ (as the distribution of the sequence (x_k) does not change if we skip the first element of the sequence). Let us write q_k for $[-ax_k]$; it follows from our definition of the integral part, that $|q_k| \leq [a] - 1$. The recurrent expression of the state vector is:

$$x_{k+1} = q_k + ax_k = a^{k+1}x_0 + q_k + aq_{k-1} + \cdots + a^k q_0$$

Rewritten as:

$$x_0 = \frac{(-q_0)}{a} + \cdots + \frac{(-q_{k-1})}{a^k} + \frac{(-q_k)}{a^{k+1}} + \frac{1}{a^{k+1}}x_{k+1}$$

this expression is similar to an expansion of x_0 in base a that begins with $(-q_0)\ldots(-q_k)$ and where x_{k+1} is the number whose fractional part is the remainder of the expansion; in other words, x_{k+1} is obtained by shifting the expansion of x_0 to the left and losing the $k+1$ leading digits.

It is then obvious that x_{k+1} is null for some k if and only if this expansion in base a is finite. This cannot occur for all (x_0, a) as can be seen if one takes $x_0 \in \mathbb{Q}$ and a transcendental.

Remark: if a is not a natural number, the existence and unicity of the expansion in base a of any real number in $[0, 1[$ do not hold any longer: if the digits used in the expansion are in $\{0, \cdots, [a] - 1\}$, the whole interval $[0, 1[$ is not covered by such expansions, and if the digits are in $\{0, \cdots, [a]\}$, some expansions are outside $[0, 1[$. However we may consider the "expansion" we encountered as canonical, as it is similar to the usual expansion in a natural base. Such expansions have been studied intensively; among the main results:

- the map: $x \mapsto \{ax\}$ is ergodic (any Lebesgue measurable subset of $[0,1]$ invariant by this map with non null Lebesgue measure has necessarily measure 1) [Ren57, Par60]
- the set of all fractional expansions $F = \{\sum_{i=-1}^{-\infty} \alpha_i a^i\}$ where all α_i are integers in $\{d_1, \ldots, d_k\}$ has similarity dimension $\log k / \log a$ [Edg90]
- concerning the existence of periodicity in the expansion, only sufficient conditions are known [Bla89] (other known facts and open questions can be found in this reference).

As x_{k+i} can be deduced from x_k by shifting the expansion in base a of x_k by i steps to the left (and losing of course the leading digits), the existence of a cycle in the distribution of the state vector is equivalent to the existence of periodicity in the expansion in base a of x_k; the last remark shows that even for a 1-dimensional system, the existence of cycles in the distribution of the state vector is an open question: in fact, almost nothing is known in the general case when $a \in \mathbf{R}$ (some fractional and some complex values of a have been studied in [Knu81, Edg90] because of their relations to fractal sets: for instance the set of all fractional expansions F associated to $a = -1 + i$ and $d_1 = 0, d_2 = 1$ yields a fractal made of two adjacent Heighway dragons; furthermore as every complex number can be represented in this number system, the whole plane is covered with countably many such dragons which can be shown to overlap only in their boundaries; other fractal sets exist for instance for $a = -2$ and $d_1 = 0, d_1 = 1, d_3 = \omega, d_4 = \omega^2$ where $\omega = e^{2i\pi/3}$). Even for a priori very simple cases, such as $a = 2$, there are lots of open questions: in absence of any theoretical result, some try to find some pattern in the distribution of the orbit of π (i.e. the sequence $x_{n+1} = \{2x_n\}$ with $x_0 = \pi$) or e with help of computers; even equidistribution has not been proved or refuted yet. Although nothing is known in the general case if one picks some real and tries to understand the distribution of its orbit, there are some intermediate results:

- [Fra63] let $p(X)$ be a polynomial with real coefficients. Suppose that for some x_0, the sequence $(x_{n+1} = \{p(x_n)\})$ for $n \geq 0$ is equidistributed[3] in $[0, 1[$. Then either $p(X) = X + \alpha$ with $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ or $p(X) = NX + \theta$ with $N \in \mathbf{Z} \setminus \{0, 1, -1\}$ and $\theta \in \mathbf{R}$

- [Fra63] the previous result is only a necessary condition of equidistribution: a sequence $(x_{n+1} = \{Nx_n\})$ may fail to be equidistributed even if x_0 is transcendental. In fact, take the Liouville number $x_0 = \sum_{\nu=1}^{\infty} N^{-\nu!}$ which is known to be transcendental; then $x_n < N^{-1} + N^{-2} + N^{-3} < 1$ and the sequence (also known as *Multiply sequence*) cannot be thus equidistributed in $[0, 1[$

In the next proposition, we will study the case $a = 2$ (but all results can be extended to any integer value of a) and instead of considering the orbit of a particular real and trying to see

[3] (x_n) is *equidistributed* or *uniformly distributed modulo 1* if for all a, b in $[0, 1[$, where $a < b$, we have $\lim_{N \to \infty} \sum_{n=1}^{N} \mathbf{I}_{[a,b[}(x_n) = b - a$, where $\mathbf{I}_{[a,b[}$ is the characteristic function of the interval $[a, b[$

whether it is dense or whatever, we will consider the set of all reals which have a dense orbit, and its complement and look for their respective properties.

Let $D : [0,1] \to [0,1]$ with $D(x) = 2x \bmod 1$ and consider the following set:

$$B = \{x \in [0,1] \mid \text{the orbit of } x \text{ under } D \text{ is not dense}\}$$

Let us denote B^c its complement. Of course, B is the set of all x_0 such that the sequence $(\{2^n x_0\})$ is not dense in $[0,1]$.

Proposition 2 *For integer values of a, we have the following results:*

1. *B is uncountable*
2. *B^c is uncountable*
3. *B and B^c are dense in* $[0,1]$
4. *B has Lebesgue measure* 0 *(and of course B^c has Lebesgue measure* 1*)*
5. *B and B^c are second category*

Proof:

<u>B is uncountable.</u>

Since every rational number is eventually periodic, its orbit visits but a finite number of points in $]0,1[$ and hence cannot be dense. But there are other points which do not have a dense orbit: let S be the set of all nonperiodic strings of 0's and 1's not having 11 as a substring; then no point on the orbit of $x \in S$ can be close to the number 0.11 and hence S is included in B; a similar argument holds for $S[b_1 b_2 \cdots b_k]$, the set of all nonperiodic strings of 0's and 1's not having $b_1 b_2 \cdots b_k$ as a substring (S is actually $S[11]$). B is hence obviously uncountable.

<u>B^c is uncountable.</u>

Let $x = 0.0\ 1\ \ 00\ 01\ 10\ 11\ \ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111\ \ldots$, in other words x contains every finite sequence of 0's and 1's; then the orbit of x visits every neighborhood in $]0,1[$ infinitely often and is therefore dense; the same holds of course for any x which has all finite binary sequences as substrings in its binary representation arranged in any order; conversely, if the orbit of x is dense in $]0,1[$, then some point in its orbit must be arbitrarily close to any $0.b_1 b_2 \cdots b_k$ (any finite binary expansion in $]0,1[$) and hence this point must begin its binary expansion with the same string of 0's and 1's; hence x contains any finite string of 0's and 1's. B^c is characterized by this property and is hence uncountable.

<u>B and B^c are dense in $[0,1]$.</u>

Let us introduce H_1 and H_2 defined from $[0,1]$ into $[0,1]$ with $H_1(x) = x/2 \bmod 1$ and $H_2(x) = (1+x)/2 \bmod 1$. It is obvious that B and B^c are preserved by D, H_1 and H_2. If we look at the effect of any of these three maps on the binary expansion of some x, we see that D shifts the expansion to the left and loses the leading digit, H_1 shifts the expansion to the right (therefore inserts a 0 between the comma and the binary expansion of x) and H_2 inserts a 1 between the comma and the binary expansion of x. By applying a finite number of times alternatively these three maps, it is obvious that starting from x we can obtain any number with a binary expansion which starts with some $b_1 b_2 \cdots b_k$ and goes on with the binary expansion of x. As we have noticed that B and B^c are preserved by a finite application of these maps, we conclude easily that B and B^c are dense in $[0,1]$.

B has Lebesgue measure 0.

Let s be a finite string of 0's and 1's. We will show that the set of x which have a finite binary expansion, where s does not appear, has measure 0. Suppose s has length n. The number of possible binary strings of length nN in which s does not appear is at most $(2^n-1)^N$, since there are in each block of length n only $2^n - 1$ strings different from s (the number of such possible strings is smaller because we do not consider overlapping possibilities of s between the different blocks). Thus the measure of the set of x such that s does not appear within the first nN bits is smaller than $(2^n-1)^N/2^{nN} = (1-2^{-n})^N$. As N goes to infinity, this number goes to 0, as $1-2^{-n} < 1$. The set of all x which have a binary expansion in which s does not appear is contained in all these sets, whose measure becomes arbitrarily small, so it itself has measure zero. As there are countably many strings s and the union of a countable collection of sets of measure zero has measure zero, and as B^c is the set of all x which have a binary expansion that contains all finite strings, we conclude that B has measure 0. Of course, B^c has Lebesgue measure 1.

B and B^c are second category.

This follows from the fact that they are dense in $[0,1]$, hence the interior of their closure is not empty. B is another example of a set which has measure zero and is not first category, thus non important in a measure sense, but still important in a topological sense [HY61].

These results can be proved in the same way for other integer values of a, mutatis mutandis. As key points of some of the previous proofs include unicity of the expansion in base a of any real number in $[0,1]$ and periodicity of the expansion for a well-known class of real numbers, and these questions are either false or unsettled for any real a, the previous results on B and B^c are not necessarily true for non integer a. □

These different results show that if we consider a 1-dimensional system controlled by the incremental control law we defined at the beginning of this chapter, the distribution of the values taken by the state vector (inside the interval depending on the choice of Δ) depends on the initial state and for arbitrarily close initial states, the behavior will be radically different (loss of density, loss of equidistribution, ...). Some generalizations to higher dimensional cases can be found in [KN74] where references are given concerning the study of the map: $x \mapsto \{Ax\}$ which is ergodic with respect to Lebesgue measure if and only if A is a non singular matrix and none of its eigenvalues is a root of unity.

As we have seen, the study of the distribution of the state vector of our feedback system leads to many number theoretic open questions, in spite of the apparent simplicity of the problem. If we refer to the study we could make in the easiest cases and to the experiments performed, we see that this distribution varies drastically for different initial states, and although the state vector is kept inside well-known boundaries, the behavior of the controlled system is chaotic.

Remark: in [FC90, FC91b, FC91a], the structure of a single-loop $\Sigma - . \Delta$ modulator, consisting of a quantizer and a discrete-time integrator in a feedback loop with constant input is analyzed; this system is described by:

$$x_{n+1} = px_n + g(u - sign(x_n))$$

where the input u is constant and the output is $sign(x_n)$. This equation is very similar to the equation we just studied in the 1-dimensional case; limit cycles for the output have been searched for and the characteristic of input versus average output has been described. Instead of considering only $sign(x_n)$, these papers consider also $Q(x_n)$, which corresponds physically to the case when the number of quantization levels is changed (Q is a step function between -1 and $+1$). The interesting part is that this study points at fractal behaviors, which has kind of familiarity with our study.

1.2 Another way to stabilize a linear time-invariant system

The preceding discussion discussed a rule-based incremental controller which stabilized a discrete linear time-invariant system; the only drawback is that this controller involves the canonical controllable form. In this section, we will give another controller which does not make use of that canonical form for the system. The rule-based incremental controller is inspired from [Kle74, KP75] (the first paper deals with non singular state matrices, while the second extends the result to any state matrix), and the proof will follow very closely these two papers.

Let us consider the system: $X_{k+1} = AX_k + Bu_k$, with a single input, a n-dimensional state space (it can be $\mathbf{R}^n$ or $\mathbf{C}^n$; we will write in all cases $A^\top$ for the adjoint of A), and let us assume this system is completely controllable. With the same previous definitions of $[x]$ and $\{x\}$ for any real scalar x, we define the control law:

$$u_k = \left[-\frac{B^\top (A^\top)^n (\sum_{i=0}^{n} A^i B B^\top (A^\top)^i)^{-1} A^{n+1} X_k}{\Delta}\right]\Delta \tag{1.1}$$

This expression is well defined, as the system is completely controllable (if we define $W_m = \sum_{i=0}^{m} A^i B B^\top (A^\top)^i$ for any m, it is obvious that W_m is positive semi-definite for any m; it can be shown that complete controllability implies W_{n-1} is invertible, hence positive definite and as a consequence the same holds for W_m for all $m \geq n-1$). As in the previous section, we define v_k by:

$$u_k = v_k - B^\top (A^\top)^n (\sum_{i=0}^{n} A^i B B^\top (A^\top)^i)^{-1} A^{n+1} X_k$$

It is obvious that u_k takes only values which are multiples of Δ and v_k is smaller in norm than Δ. Actually u_k is the sum of v_k and of the control law introduced in [Kle74, KP75], where it has been shown that it stabilizes any discrete linear time-invariant system. We will first recall this proof and then, making use of this result, we will proof the following proposition.

Proposition 3 *Let $X_{k+1} = AX_k + Bu_k$ be a discrete linear time-invariant single input system and let us assume this system is completely controllable. Then the control 1.1 can be rewritten as a rule-based incremental controller and stabilizes the system.*

Proof: The proof consists in two parts; the first part follows [Kle74, KP75], while the second part deals with the fact that we are using incremental control laws.

- Part 1

Using the control law 1.1, we have: $X_{k+1} = \overline{A} X_k + B v_k$, where:

$$\overline{A} = A - BB^\top (A^\top)^n (\sum_{i=0}^{n} A^i B B^\top (A^\top)^i)^{-1} A^{n+1}$$

Let p be the algebraic multiplicity of the zero eigenvalue of the matrix A. There exists a similarity transformation S which transforms A into $\hat{A}$ with:

$$\hat{A} = S^{-1} A S = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}$$

where A_0 is a nonsingular $(n-p) \times (n-p)$ matrix and A_1 is a $p \times p$ matrix with only zero eigenvalues (hence $A_1^i = 0$ for all $i \geq p$). Taking $\hat{B} = S^{-1} B$ and $\hat{\overline{A}} = S^{-1} \overline{A} S$, we have obviously:

$$\begin{aligned} \hat{B} &= \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} \\ \hat{\overline{A}} &= \hat{A} - \hat{B}\hat{B}^\top (\hat{A}^\top)^n (\sum_{i=0}^{n} \hat{A}^i \hat{B} \hat{B}^\top (\hat{A}^\top)^i)^{-1} \hat{A}^{n+1} \end{aligned}$$

The column vectors B_0 and B_1 are respectively $(n-p) \times 1$ and $p \times 1$. As we are dealing with a controllable single-input system, the subsystems (A_1, B_1) and (A_0, B_0) are controllable single-input systems too, with respectively p and $n-p$ dimensional state spaces (this comes from the fact that $(B, AB, \cdots, A^{n-1}B)$ is a basis for the n-dimensional state space, and this space can be decomposed into $N_0 \oplus N_1$, where N_1 is spanned by the eigenvectors of A corresponding to the zero eigenvalue, and N_0 is spanned by the eigenvectors of A corresponding to non zero eigenvalues; it is then obvious that $(B_0, A_0, B_0, \cdots, A_0^{n-p-1}B_0)$ is a basis for N_0 and $(B_1, A_1, B_1, \cdots, A_1^{p-1}B_1)$ is a basis for N_1). There exists thus a similarity transformation S_1 (see appendix) such that:

$$S_1^{-1}A_1S_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad S_1^{-1}B_1 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

The similarity matrix S' defined as:

$$S' = \begin{pmatrix} I_{n-p} & 0 \\ 0 & S_1 \end{pmatrix} S$$

transforms the state matrix A into a diagonal matrix with blocks A_0' and A_1' where A_1' is a Jordan matrix, and B is transformed into B_0' and B_1' where B_1' has null entries except for the last entry which is 1. For sake of simplicity, let us write A_0, A_1, B_0 and B_1 instead of A_0', A_1', B_0' and B_1'; in other words let us assume the similarity transformation S had already done the whole job and S_1 was thus reduced to the identity I_p.

If $p = n$, we have $\widehat{\widehat{A}} = A_1$, since $\widehat{A} = A_1$ and $\widehat{A}^{n+1} = 0$; thus $\widehat{\widehat{A}}$ is a stable matrix. In this case, proceed directly to part 2 for the end of the proof.

Assume $p < n$. By direct computation we have:

$$\sum_{i=0}^{n} \widehat{A}^i \widehat{B}\widehat{B}^{\mathsf{T}}(\widehat{A}^{\mathsf{T}})^i = \begin{pmatrix} \sum_{i=0}^{n} {A_0}^i B_0 {B_0}^{\mathsf{T}}({A_0}^{\mathsf{T}})^i & \sum_{i=0}^{p-1} {A_0}^i B_0 {B_1}^{\mathsf{T}}({A_1}^{\mathsf{T}})^i \\ \sum_{i=0}^{p-1} {A_1}^i B_1 {B_0}^{\mathsf{T}}({A_0}^{\mathsf{T}})^i & \sum_{i=0}^{p-1} {A_1}^i B_1 {B_1}^{\mathsf{T}}({A_1}^{\mathsf{T}})^i \end{pmatrix}$$

Recognizing that $A_1^i B_1$ is a column vector with all null entries but the $p - i$ entry which equals 1, the following identities can be established:

$$\sum_{i=0}^{p-1} A_1^i B_1 B_1^{\mathsf{T}} (A_1^{\mathsf{T}})^i = I_p$$

$$\sum_{i=0}^{n} A_0^i B_0 B_0^{\mathsf{T}} (A_0^{\mathsf{T}})^i - (\sum_{i=0}^{p-1} A_0^i B_0 B_1^{\mathsf{T}} (A_1^{\mathsf{T}})^i)(\sum_{i=0}^{p-1} A_1^i B_1 B_0^{\mathsf{T}} (A_0^{\mathsf{T}})^i) = \sum_{i=p}^{n} A_0^i B_0 B_0^{\mathsf{T}} (A_0^{\mathsf{T}})^i$$

Using these identities, we have:

$$(\sum_{i=0}^{n}\widehat{A}^i\widehat{B}\widehat{B}^\top(\widehat{A}^\top)^i)^{-1} = \begin{pmatrix} (\sum_{i=p}^{n} A_0^i B_0 B_0^\top (A_0^\top)^i)^{-1} & \cdots \\ \cdots & \cdots \end{pmatrix}$$

From the fact that $A_1^i = 0$ for all $i \geq p$, we have finally:

$$\widehat{\overline{A}} = \begin{pmatrix} A_0 - B_0 B_0^\top (A_0^\top)^n (\sum_{i=p}^{n} A_0^i B_0 B_0^\top (A_0^\top)^i)^{-1} A_0^{n+1} & 0 \\ 0 & A_1 \end{pmatrix}$$

The first block matrix can be rewritten as: $A_0 - B_0 B_0^\top (A_0^\top)^{n-p} (\sum_{i=0}^{n-p} A_0^i B_0 B_0^\top (A_0^\top)^i)^{-1} A_0^{n-p+1}$. In order to show that this $(n-p)\times(n-p)$ matrix is stable, it is sufficient to show that for any nonsingular $(n \times n)$ matrices A, the corresponding matrix $\overline{A}$ is stable, independently of n (in fact, this first block matrix can be interpreted as $\overline{A_0}$, where A_0 is a $(n-p)\times(n-p)$ matrix, hence the sufficiency).

Let us now assume A is a $(n \times n)$ invertible matrix. We have:

$$\overline{A} = A - BB^\top (A^\top)^n W_n^{-1} A^{n+1} = A^{-n-1}(A - A^{n+1} BB^\top (A^\top)^n W_n^{-1}) A^{n+1}$$

This can be rewritten as $A^{-n-1}\widetilde{A}A^{n+1}$; as A is invertible, the matrix $\widetilde{A}$ is thus similar to $\overline{A}$. Let us study this new matrix more closely:

$$\begin{aligned}&\widetilde{A} W_n \widetilde{A} - W_n = \\ &\quad A W_n A^\top - 2A^{n+1} BB^\top (A^{n+1})^\top + A^{n+1} BB^\top (A^n)^\top W_N^{-1} A^n BB^\top (A^{n+1})^\top - W_n\end{aligned}$$

But we have: $AW_nA^\top = A^{n+1}BB^\top(A^{n+1})^\top + W_n - BB^\top$, hence:

$$\widetilde{A} W_n \widetilde{A} - W_n = -A^{n+1}[BB^\top - BB^\top (A^n)^\top W_N^{-1} A^n BB^\top](A^{n+1})^\top - BB^\top = -Q_1 - BB^\top$$

The first term Q_1 thus defined can be rewritten as: [4]

$$Q_1 = A^{n+1} B (I - B^\top (A^n)^\top (A^n BB^\top (A^n)^\top + W_{n-1})^{-1} A^n B) B^\top (A^{n+1})^t op$$

Using the following matrix identity:

$$I - Y^\top (ZY^\top + X)^{-1} Z = (I + Y^\top X^{-1} Z)^{-1}$$

we have:

$$Q_1 = A^{n+1} B (I + B^\top (A^n)^\top W_{n-1}^{-1} A^n B)^{-1} B^\top (A^{n+1})^\top$$

[4] In fact this part of the proof with A invertible holds for higher dimensional control vectors, i.e. B can be a $(n \times r)$ matrix, if the system has r inputs; in the next expression, the identity matrix is a $(r \times r)$ matrix, and in the case of our single-input system, it is obviously a scalar, but we kept the matrix notation on purpose

As W_{n-1}^{-1} is positive definite, $B^{\top}(A^n)^{\top}W_{n-1}^{-1}A^nB$ is positive semi-definite; the same holds then for $(I+B^{\top}(A^n)^{\top}W_{n-1}^{-1}A^nB)^{-1}$, hence Q_1 is positive semi-definite too. We conclude that $\widetilde{A}W_n\widetilde{A}-W_n$ is negative semi-definite. Let X be an eigenvector of $\widetilde{A}^{\top}$ corresponding to the eigenvalue λ; we have then:

$$X^{\top}\widetilde{A}W_n\widetilde{A}^{\top}X - X^{\top}W_nX = (|\lambda|^2-1)X^{\top}W_nX$$

All eigenvalues of $\widetilde{A}^{\top}$ have thus modulus smaller or equal to 1, because of the negative semi-definite character of $\widetilde{A}W_n\widetilde{A}^{\top} - W_n$ and since W_n is positive definite. Let us assume one of these eigenvalues has modulus equal to 1; then we have for some X, the corresponding eigenvector:

$$X^{\top}Q_1X + X^{\top}BB^{\top}X = 0$$

This implies $X^{\top}Q_1X = 0$, because both Q_1 and $BB^{\top}$ are positive semi-definite. Instead of X, we could have considered $(\widetilde{A}^{\top})^iX$ for any i; this implies that $X^{\top}\widetilde{A}^iQ_1(\widetilde{A}^{\top})^iX = 0$ for all i. As we have seen that Q_1 can be rewritten as $A^{n+1}BVB^{\top}(A^{n+1})^{\top}$, this implies that $X^{\top}\widetilde{A}^iA^{n+1}B$ is zero for all i. We have then:

$$X^{\top}(A^{n+1}B|\widetilde{A}A^{n+1}B|\cdots|\widetilde{A}^{n-1}A^{n+1}B) = 0$$

But we have: $\widetilde{A}^iA^{n+1}B = A^{n+1}(A - BB^{\top}(A^n)^{\top}W_n^{-1}A^{n+1})^iB$, so the previous expression transforms into:

$$X^{\top}A^{n+1}(B|\cdots|(A - BB^{\top}(A^n)^{\top}W_n^{-1}A^{n+1})^{n-1}B) = 0$$

But, as for any C, the controllability of the system $X_{k+1} = AX_k + Bu_k$ is equivalent to the controllability of the system $X_{k+1} = (A+BC)X_k + Bu_k$ (see appendix), this expression contradicts the controllability assumption of our initial system. Hence no eigenvalues of $\widetilde{A}^{\top}$ have modulus equal to 1, and this matrix is stable; the same holds then for $\widetilde{A}$, and as $\overline{A}$ is similar to $\widetilde{A}$, we conclude that $\overline{A}$ is stable.

- **Part 2**

The system controlled by the control law 1.1 can be described as: $X_{k+1} = \overline{A}X_k + Bv_k$, where $\overline{A}$ has been shown to be stable, whether A is invertible or not. We have obviously:

$$X_{k+1} = \overline{A}^{k+1}X_0 + Bv_k + \overline{A}Bv_{k-1} + \cdots + \overline{A}^kBv_0$$

As B is bounded and $\forall i, |v_i| < \Delta$, we have:

$$\| X_{k+1} \| \leq \| \overline{A} \|^{k+1} \| X_0 \| + \| B \| \Delta \sum_{i=0}^{k} \| \overline{A} \|^i$$

As $\| \overline{A} \| < 1$, we conclude that for all k, X_k is bounded, and more precisely:

$$\exists M > 0, \exists k_0 \in \mathbb{N}, \forall k > k_0, \| X_k \| < M\Delta$$

As in 1.1 all terms are uniformly bounded in k (for all k, $\| X_k \| \leq \max(X_0, \cdots, X_{k_0}, M\Delta)$), and as this control law takes only values which are multiples of Δ, there exists some m such that we have: $u_k = \epsilon_k \Delta$, with $|\epsilon_k| < m$. It is obvious then to write a rule-based controller yielding this control law (do as in a former section where another law was studied). □

<u>Remark</u>: as is discussed in [Kle74], the previous law could be modified slightly in order to tackle some optimization problems; we could have considered for instance $u_k = -[\frac{R^{-1}B^T(A^T)^n W_n^{-1} A^{n+1} X_k}{\Delta}]\Delta$, where R is some positive definite matrix, related to the cost functional of the optimization problem. For systems with multiple inputs, the proposition holds for A invertible; if A is not invertible, in various special cases, the proposition holds still, or a slightly modified version of the control law can be found [Kle74, KP75].

1.3 From linear time-invariant to linear time-varying systems

We will now consider discrete linear time-varying systems: $X_{k+1} = A_k X_k + B_k U_k$, where X_k is the n-dimensional state, and A_k and B_k are time-varying matrices. Notice that there may be several inputs: U_k is not restricted to scalar values. Let us take the following convention: U_k stands for the whole input vector, while u_k stands for any component of that vector (no subscripts to indicate which component, as we only need to distinguish the whole vector from any of its components); this holds for all other vectors trivially related to U_k (e.g. $U_k^{(g)}, U_k^{(i)}, V_k$). We will proceed as in the preceding sections: in order to find an incremental law $U_k^{(i)}$, we will look for some general[5] control law $U_k^{(g)}$ and take $u_k^{(i)} = [\frac{u_k^{(g)}}{\Delta}]\Delta$, where Δ is a non zero positive real; like previously we can define $V_k = U_k^{(i)} - U_k^{(g)}$. This allows us to rewrite $X_{k+1} = AX_k + BU_k^{(i)}$ as $X_{k+1} = AX_k + BU_k^{(g)} + BV_k$; if we have some stability result for the system controlled in closed loop by the general control law $U_k^{(g)}$, we can deduce stability results for the incremental control law $U_k^{(i)}$ by applying adequate theorems on perturbed difference equations (which are given in appendix). In a first step we will give some results with general control laws, and in a second step, we will give the incremental versions of these results.

1.3.1 General control laws

Many general control laws stabilizing linear time-varying systems have been given in literature, and we will only recall a few. The first one extends the method used in [Kle74, KP75] for stabilizing discrete linear time-invariant systems, described in the preceding section. In [KP78], the following proposition is stated.

[5] This is really an awkward notation, but will help us to avoid confusion; we want to state clearly when we deal with *incremental* control laws and when we do not

Let us consider a completely uniformly controllable and observable linear time-varying discrete system $X_{k+1} = A_k X_k + B_k U_k$, $y_k = C_k X_k$, where y_k is the output vector. Complete uniform controllability of the pair (A_k, B_k) implies the existence of a fixed integer l_c such that, for all k and for some positive scalars (α_1, α_2):

$$\alpha_1 I \leq \sum_{i=k}^{k+l_c-1} A(k,i+1)B_i B_i^\top A^\top(k,i+1) \leq \alpha_2 I$$

where $A(k,j) = \prod_{i=j}^{k-1} A_i$. The integer l_o is similarly defined using the complete uniform observability of the pair (A_k, C_k).

Proposition 4 *Let* $U_k^{(g)} = -R_k^{-1} B_k^\top P_{k+1,k+1+N}^{-1} A_k X_k$, *where N is an integer, R_k is a positive definite matrix, and the double indexed matrix P is defined implicitly by $P_{j,j} = 0$ and:*

$$\begin{aligned} P_{i,j} = {} & A_i^{-1} P_{i+1,j} (A_i^{-1})^\top - \\ & A_i^{-1} P_{i+1,j} (A_i^{-1})^\top C_i^\top D_i^\top (I + D_i C_i (A_i^{-1})^\top P_{i+1,j} (A_i^{-1})^\top C_i^\top D_i^\top)^{-1} D_i C_i A_i^{-1} P_{i+1,j} (A_i^{-1})^\top + \\ & B_{i-1} R_{i-1}^{-1} B_{i-1}^\top \end{aligned}$$

where D_i is some matrix, and $Q_i = D_i^\top D_i$. Then:

1) Assume that R_k and Q_k satisfy $\alpha_3 I \leq R_k \leq \alpha_4 I$ and $0 \leq Q_k \leq \alpha_6 I$ for positive scalars $(\alpha_3, \alpha_4, \alpha_6)$ and for all k. Under the conditions that the pair (A_k, B_k) is uniformly completely controllable and C_k is bounded such that $\| C \| \leq \alpha_7$ for all k, the linear system with this feedback control is uniformly asymptotically stable when the horizon length N is chosen to satisfy $l_c + 1 \leq N < \infty$

2) Assume that R_k and Q_k satisfy $\alpha_3 I \leq R_k \leq \alpha_4 I$ and $\alpha_5 I \leq Q_k \leq \alpha_6 I$ for positive scalars $(\alpha_3, \alpha_4, \alpha_5, \alpha_6)$ and for all k. Under the conditions that the pair (A_k, B_k) is uniformly completely controllable and the pair (A_k, C_k) is uniformly completely observable, the linear system with this feedback control is uniformly asymptotically stable when the horizon length N is chosen to satisfy $1 + \max(l_c, l_o) \leq N < \infty$

The proof of this proposition uses the adjoint system; a further assumption is made: namely, the state matrix of the closed loop system (in the time-invariant case, it was $\overline{A}$) is assumed invertible, in order to define the adjoint system. The proof involves finding lower and upper bounds for the double indexed matrix P. The matrices Q_k and R_k can be interpreted as cost matrices, associated to the optimization problem: $J = \sum_{i=k}^{k+N-1} y_i^\top Q_i y_i + u_i^\top R_i u_i$. For further discussion, refer to [KP78].

The next general control law, presented in [Che78, Che79], makes the zero-solution of the closed-loop system uniformly exponentially stable at a rate at least μ (i.e. $\exists \mu > 1, \exists M > 0, \forall k_0 \in \mathbb{N}, \forall k \geq k_0, \forall x_{k_0}, \| x_k \| \leq \| x_{k_0} \| M \mu^{k_0 - k}$).

Let us consider a discrete linear time-varying system, and assume that A_k is invertible for all k. For $k \geq k_0 \geq 0$ and $\nu > 1$, define:

$$\begin{aligned} C(k_0,k) &= (A_{k_0}^{-1}B_{k_0}|A_{k_0}^{-1}A_{k_0+1}^{-1}B_{k_0+1}|\cdots|A_{k_0}^{-1}A_{k_0+1}^{-1}\cdots A_k^{-1}B_k) \\ S(k_0,k) &= C(k_0,k)C(k_0,k)^\top = \sum_{j=k_0}^{k}(A_{k_0}^{-1}\cdots A_j^{-1})B_jB_j^\top(A_{k_0}^{-1}\cdots A_j^{-1})^\top \\ S_\nu(k_0,k) &= \sum_{j=k_0}^{k}\nu^{4(k_0-k)}(A_{k_0}^{-1}\cdots A_j^{-1})B_jB_j^\top(A_{k_0}^{-1}\cdots A_j^{-1})^\top \end{aligned}$$

We have then the following proposition.

Proposition 5 *Assume that there exists $l \in \mathbb{N}\setminus\{0\}$ such that:*

1) the rank of $C(k,k+l-1)$ is n for all k

2) $\exists s_M, \exists s_m, s_M \geq s_m > 0$ such that: $0 < s_m I \leq S(k,k+l) \leq s_M I$ for all k

Then, for any $\nu > 1$, the linear time-varying state-feedback general control law:

$$U_k^{(g)} = -B_k^\top(A_k^{-1})^\top S_\nu(k,k+l)^{-1}X_k$$

is such that the zero solution of the closed-loop system:

$$X_{k+1} = A_k[I - A_k^{-1}B_kB_k^\top(A_k^{-1})^\top S_\nu(k,k+l)^{-1}]X_k$$

is uniformly exponentially stable at a rate at least ν.

Proof: Let us rewrite the closed-loop system as: $X_{k+1} = \overline{A_k}X_k$, where:

$$\overline{A_{k+1}} = A_k[I - A_k^{-1}B_kB_k^\top(A_k^{-1})^\top S_\nu(k,k+l)^{-1}]X_k$$

It is easy to see that if the zero solution of the ν-scaled system $\xi_{k+1} = \nu\overline{A_k}\xi_k$, with $\nu > 1$, is uniformly exponentially stable at a rate $\mu > 1$, then the zero solution of the system $X_{k+1} = \overline{A_k}X_k$ is uniformly exponentially stable at a rate at least $\nu\mu$. Let us define a real-valued function[6] V by: $V(k,\xi) = \xi^\top S_\nu(k,k+l)^{-1}\xi$. From the second assumption in the proposition, we have:

$$0 < \nu^{-4l}s_m I \leq S_\nu(k,k+l) \leq s_M I$$

and of course:

$$0 < s_M^{-1}I \leq S_\nu(k,k+l)^{-1} \leq \nu^{4l}s_m^{-1}I$$

[6]Actually this function is a Lyapunov function and will be shown to be negative definite, hence uniform stability of the zero solution [LT88].

By straightforward computation, we compute the difference of the function V along the solutions of the scaled system:

$$\begin{aligned}\Delta V(k,\xi_k) &= V(k+1,\xi_{k+1}) - V(k,\xi_k) \\ &= \xi_k^{\mathsf{T}}[(\nu\overline{A_k})^{\mathsf{T}} S_\nu(k+1,k+1+l)^{-1}(\nu\overline{A_k}) - S_\nu(k,k+l)^{-1}]\xi_k\end{aligned}$$

Using the definition of $\overline{A_k}$ and the identity:

$$A_k S_\nu(k,k+l)A_k^{\mathsf{T}} = B_k B_k^{\mathsf{T}} + \nu^{-4} S_\nu(k+1,k+l)$$

we have:

$$\begin{aligned}&(\nu\overline{A_k})^{\mathsf{T}} S_\nu(k+1,k+1+l)^{-1}(\nu\overline{A_k}) \\ &\quad= \nu^{-6} S_\nu(k,k+l)^{-1} A_k^{-1} S_\nu(k+1,k+l) S_\nu(k+1,k+1+l)^{-1} \\ &\qquad S_\nu(k+1,k+l)(A_k^{-1})^{\mathsf{T}} S_\nu(k,k+l)^{-1}\end{aligned}$$

As we have: $S_\nu(k+1,k+1+l) \geq S_\nu(k+1,k+l) > 0$, we have obviously: $S_\nu(k+1,k+1+l)^{-1} \leq S_\nu(k+1,k+l)^{-1}$, and the previous equality becomes:

$$\begin{aligned}&(\nu\overline{A_k})^{\mathsf{T}} S_\nu(k+1,k+1+l)^{-1}(\nu\overline{A_k}) \\ &\leq \nu^{-6} S_\nu(k,k+l)^{-1} A_k^{-1} S_\nu(k+1,k+l)(A_k^{-1})^{\mathsf{T}} S_\nu(k,k+l)^{-1} \\ &\leq \nu^{-2} S_\nu(k,k+l)^{-1} A_k^{-1}[B_k B_k^{\mathsf{T}} + \nu^{-4} S_\nu(k+1,k+l)](A_k^{-1})^{\mathsf{T}} S_\nu(k,k+l)^{-1} \\ &= \nu^{-2} S_\nu(k,k+l)^{-1}\end{aligned}$$

The last inequality holds because $B_k B_k^{\mathsf{T}}$ is positive semi-definite and the last equality is a consequence of the identity given previously. Hence we have:

$$\Delta V(k,\xi_k) \leq (\nu^{-2}-1)\xi_k^{\mathsf{T}} S_\nu(k,k+l)^{-1}\xi_k \leq -(1-\nu^{-2})s_M^{-1} \parallel \xi_k \parallel^2$$

which is negative definite[7]. We have then:

$$\frac{\Delta V(k,\xi_k)}{V(k,\xi_k)} \leq \frac{-(1-\nu^{-2})s_M^{-1}}{\nu^{4l} s_m^{-1}}$$

This yields for all $k \geq k_0 \geq 0$ and all ξ_{k_0}:

$$V(k+1,\xi_{k+1}) \leq [1-(1-\nu^{-2})s_m s_M^{-1}\nu^{-4l}]V(k,\xi_k)$$

and finally:

$$\parallel \xi_k \parallel \leq \parallel \xi_{k_0} \parallel \nu^{2l}\sqrt{\frac{s_M}{s_m}}[1-(1-\nu^{-2})s_m s_M^{-1}\nu^{-4l}]^{(k-k_0)/2}$$

The convergence rate of the closed-loop system is then at least $\nu[1-(1-\nu^2)s_m s_M^{-1}\nu^{-4l}]^{-1/2}$ which is obviously greater than ν. □

[7]If the ν-scaled system had not been used, we would only have obtained 0 as an upper bound, and the function would only have been negative semi-definite

Remark: further discussion can be found in [Che78, Che79], for instance on robustness of closed-loop stability (an issue we did not study at all in this report).

1.3.2 Incremental control laws

Starting from the general control laws $U_k^{(g)}$ studied in the previous section, we define the corresponding incremental control laws $U_k^{(i)}$ by: $u_k^{(i)} = [\frac{u_k^{(g)}}{\Delta}]\Delta$. We define $v_k = u_k^{(i)} - u_k^{(g)} = -\{\frac{u_k^{(g)}}{\Delta}\}\Delta$; the linear time-varying system controlled by such an incremental control law verifies then: $X_{k+1} = A_k X_k + B_k U_k^{(g)} + B_k V_k$ and can be seen as a perturbation of $X_{k+1} = A_k X_k + B_k U_k^{(g)}$.

As a consequence of the previous section, we have two possible definitions for V_k:

$$v_k = \{\frac{R_k^{-1} B_k^\top P_{k+1,k+1+N}^{-1} A_k X_k}{\Delta}\}\Delta$$

$$v_k = \{\frac{B_k^\top (A_k^{-1})^\top S_\nu(k,k+l)^{-1} X_k}{\Delta}\}\Delta$$

It is obvious that in both cases, for $X_k = 0$, we have $V_k = 0$; furthermore, V_k takes its values in the ball[8] of radius Δ. As for all real x and y, we have: $|\{xy\}| \leq |x||y|$, we have in both cases: $\| B_k V_k \| \leq M_k \| X_k \|$, where M_k is a positive real that does not depend on X_k but only on the different matrices which define the system (A_i, B_i, C_i, D_i, R_i for all i). Under the hypotheses used in the previous section to prove the stability of the corresponding general control laws, we have the following proposition.

Proposition 6 *Assuming the hypotheses of propositions 4 or 5, depending on which incremental control law is used, we have:*

1) *If $\sum_{i=0}^{\infty} M_i < \infty$, then the controlled system is uniformly asymptotically stable.*

2) *If there exists some $L > 0$ sufficiently small such that: $\forall k, M_k \leq L$, then the controlled system is exponentially asymptotically stable.*

3) *If there exists some $L > 0$ such that: $\forall k, M_k \leq L$, then the controlled system is totally stable.*

4) *If there exists some $0 < L < 1$ such that: $\forall k, M_k \leq L$, then the controlled system is practically stable.*

[8] We consider the following norm: the norm of a vector is the sum of the absolute values of its components divided by the dimension of the vector space.

Proof: The proof uses results on difference equations given in the appendix. We notice first that both general control laws when applied to the system yield uniform asymptotic stability (uniform exponential stability implies uniform asymptotic stability).

1) Use proposition 12.

2) Use proposition 13. As was stated in the proof of proposition 4, the matrix P can be shown to be upper bounded, and in proposition 5, the matrix S_ν is upper bounded too; hence the new assumption on the existence of L needs only further constraints on A_k and B_k, which can then be stated easily.

3) As in both cases V_k depends additively on X_k, the existence of L implies that proposition 14 can be applied.

4) For $X_k = 0$, we have obviously V_k bounded; furthermore the existence of L implies as in the previous case that proposition 15 can be applied. □

> Remark: the additional assumptions in the previous proposition imply that the u_k are uniformly bounded in k and a rule-based incremental controller can then be written, in the way described in a previous section, that yields the previous incremental control laws. The previous proposition uses the rather brute-force inequality: $|\{xy\}| \leq |xy|$, and any finer inequality would of course yield propositions with a wider scope.

1.4 Nonlinear systems and trajectory tracking

This section is the discrete-time counterpart to [WTS+92] with the additional constraint that our control laws cannot be chosen freely, we have to keep to rule-based incremental controllers.

We will consider nonlinear systems with the state equation: $X_{k+1} = f(X_k) + g(X_k)U_k$, where X_k are the n-dimensional states, U_k the p-dimensional inputs, and f and g are two differentiable functions modeling respectively the *drift* and the *nonholonomy* of the system. This class of system covers many models used in robotics [RM91, TLM+92, WTS+92, Luz92b]. Given a desired trajectory (i.e. a sequence $X_k^{(d)}$) and nominal inputs (a sequence $U_k^{(d)}$ yielding this trajectory), we will compute first the linearization of the system around this desired trajectory, obtaining a linear time-varying system; using the results of the previous section, we will then be able to control the nonlinear system with rule-based incremental controllers.

Proposition 7 *Given a nonlinear system $X_{k+1} = f(X_k)+g(X_k)U_k$, a desirable trajectory $(X_k^{(d)})$ and nominal inputs $(U_k^{(d)})$, under some assumptions on the linearization of this system given in proposition 6, there exist rule-based incremental controllers which exponentially asymptotically stabilize the nonlinear system to the desired trajectory.*

Proof: Let us define the error signal E_k and the error input I_k as:

$$\begin{aligned} E_k &= X_k - X_k^{(d)} \\ I_k &= U_k - U_k^{(d)} \end{aligned}$$

We solve for the dynamics of these error signals using the Taylor expansions:

$$\begin{aligned} E_{k+1} &= f(X_k) - f(X_k^{(d)}) + [g(X_k) - g(X_k^{(d)})]U_k + g(X_k^{(d)})[U_k - U_k^{(d)}] \\ &= \frac{Df}{Dx}(X_k^{(d)})(X_k - X_k^{(d)}) + \frac{Dg}{Dx}(X_k^{(d)})(X_k - X_k^{(d)})(U_k^{(d)} + I_k) + g(X_k^{(d)})I_k + R_k \end{aligned}$$

We define then A_k and B_k as:

$$\begin{aligned} A_k &= \frac{Df}{Dx}(X_k^{(d)}) + \frac{Dg.U_k^{(d)}}{Dx}(X_k^{(d)}) \\ B_k &= g(X_k^{(d)}) \end{aligned}$$

The error signal satisfies then:

$$E_{k+1} = A_k E_k + B_k I_k + \frac{Dg}{Dx}(X_k^{(d)}) E_k I_k + R_k$$

As we have been using the Taylor expansion, $R_k = o(\| E_k \|)$. Let us take for I_k one of the incremental control laws described previously; we have then: $E_k I_k = O(\| E_k \|^2)$ and the equation for the error signal becomes:

$$E_{k+1} = A_k E_k + B_k I_k + R'_k \tag{1.2}$$

where $R'_k = o(\| E_k \|)$. We know that under the assumptions 1) or 2) on A_k and B_k given in proposition 6, any of the two incremental control laws taken for I_k stabilizes uniformly the system: $E_{k+1} = A_k E_k + B_k I_k$. As $R'_k = o(\| E_k \|)$, we can use proposition 13, hence the system 1.2 can be exponentially asymptotically stabilized.

Using the remark at the end of the previous section, we conclude that there exist rule-based incremental controllers (yielding any of the two previous incremental control laws), which exponentially asymptotically stabilize the nonlinear system around the desired trajectory. □

<u>Remark</u>: in [WTS+92], applications are given in the continuous case, which show that the corresponding control laws are not too difficult to compute; three systems are studied, namely a system whose control Lie algebra is the Heisenberg algebra with two generators, a simple nonholonomic mobile robot HILARE with two parallel wheels, and a front-wheel drive car (control of this last system with rule-based incremental controllers has been studied in [Luz92b]).

1.5 Concluding remarks

In this chapter, we have been looking for stability and tracking results, when using rule-based incremental controllers on different classes of systems. Starting with time-invariant linear systems, we finished with a large class of nonlinear systems; it turns out that theory confirms what experiments had hinted at [FKB+85, FL89, Fou90], namely rule-based incremental control, in spite of its apparent simplicity, has a broad scope of applications.

Looking back at previous research shows how much the theoretical results have gained in scope and depth: in [Luz91] only linear time-invariant systems were considered and the propositions stated that for any given time interval, there existed an incremental control law that could bring the state vector at the end of that time interval arbitrarily close to the origin; but during the time interval, the only thing that was guaranteed was that the state vector was uniformly bounded by an affine function (i.e. the state trajectory remained in a cone depending only on the system). Although this incremental control law could be constructed, the various ϵ_i were found backwards in time: given the time interval $[0, k]$, first ϵ_k was constructed, then ϵ_{k-1} and so on. Such a construction could not allow to find a rule-based incremental controller, as the rules depended explicitly on the length of the time interval, and to compute the first coefficient ϵ_1, you already needed all the other coefficients, so you had to compute the whole control law just to start!

The propositions of the current chapter give an explicit construction of rule-based incremental controllers, where each coefficient ϵ_k depends on the state X_k. We have actually been studying *state rule-based incremental controllers* and further work could focus on *output rule-based incremental controllers*, where the coefficient ϵ_k would this time depend on the output y_k (and the previous outputs). We do not think such a study would change radically the propositions, but would only give additional constraints related to the observability of the system. However this will have to be done, in order to characterize only in terms of inputs and outputs the class of systems controllable by rule-based incremental controllers.

The study of the chaotic behavior of the 1-dimensional linear time-invariant system controlled by a state feedback rule-based incremental controller shows that it will not be easy to have a smooth behavior of the system: although stability or tracking with arbitrarily small error are guaranteed, the system chatters with various and almost unpredictable frequency. This is of course one of the inherent drawbacks of discrete control, but it has to be noticed (actually, due to relaxation delays, this chattering is very often damped in real-world applications, but it still exists theoretically).

As we have already said, no robustness issues have been discussed here and this will of course have to be done, in order to confirm experimental results.

Chapter 2

Other possible formulations of rule-based incremental control

In this chapter, we present different formulations of rule-based incremental control that are an alternative to the algebraic definition of incremental control used for instance in the previous chapter. There will be no propositions in this chapter related to the behavior of a system controlled by such controllers as, till now, we have focussed our attention only on the algebraic definition which seemed so easier to exploit. This chapter should be seen as a transition between the analysis of controlled systems, as developed in the previous chapter and former papers or reports, and the search for a unified theory of rule-based incremental control and machine learning, as presented in the next chapter.

2.1 Operator based formulation

Instead of considering the input at time k as the sum of the input at time $k-1$ and the product of the increment by a uniformly bounded integer, we can consider the input at time k as the result of some operator which acts on the input at time $k-1$ and on the increment. This formulation has been presented in [FL89, Fou90, LZ90, Luz91].

We define an operator-based incremental law as:

$$\begin{cases} u_k^0 & = u_{k-1}^0 \oplus_k^0 u_k^1 \\ u_k^1 & = u_{k-1}^1 \oplus_k^1 u_k^2 \\ \vdots & \\ u_k^{n-1} & = u_{k-1}^{n-1} \oplus_k^{n-1} u_k^n \\ u_k^n & = \Delta \end{cases}$$

u_k^0 is the command vector at time k, u_k^1 its increment, and u_k^n the n^{th}-order increment. All these increments are taken in the same space as the input vector. The operators $\oplus_k^j$ are taken in a finite set of operators, and Δ is a positive real constant.

These operators satisfy the following rewriting rules (x, y, z are taken in the set where the u_k^p are defined):

1. $(x \oplus^i y) \oplus^j z = x \oplus^i y \oplus^j z$
2. There exists an internal law on the finite set of operators $\oplus^i$ called $\bullet$ which is associative and commutative
3. $x \oplus^i (y \oplus^j z) = x \oplus^i y \oplus^i \bullet \oplus^j z$
4. $x \oplus^i y \oplus^j z = x \oplus^j z \oplus^i y$

The first and the third rule are only rewriting rules, they do not authorize reasonings like : as $\oplus^k = \oplus^i \bullet \oplus^j$, let us apply the third rule to the expression $x \oplus^i y \oplus^k z$.

By induction, it is obvious to show:

$$\begin{cases} u_k^i &= u_{k-1}^i \quad \oplus_k^i \quad u_{k-1}^{i+1} \quad \cdots \quad \oplus_k^i \bullet \oplus_k^{i+1} \ldots \bullet \oplus_k^{n-2} \quad u_{k-1}^{n-1} \quad \oplus_k^i \bullet \oplus_k^{i+1} \ldots \bullet \oplus_k^{n-1} \quad \Delta \\ u_k^i &= u_0^i \oplus_1^0 u_1^{i+1} \cdots \oplus_k^0 u_k^{i+1} \end{cases}$$

The rewriting rules have been chosen in order to allow some recombining of terms in complex expressions with parentheses; of course one could forget these additional rules, and only consider the operators $\oplus_k^i$ as some elements of an eventually infinite family, but it would not be very practical. One could certainly take advantage of some more structure on the input space and on the controlled system, as in [Sai81]; some work could perhaps be done in this direction.

This operator-based formulation is a direct generalization of the algebraic definition: just take $x + \epsilon_k^i y$ as definition for $x \oplus_k^i y$ (if $i \geq 1$) and $x + \epsilon_k^0 y + \epsilon_k' \Delta'$ as definition of $x \oplus_k^0 y$, and the rewriting rules are satisfied, as the ϵ_k^i are in $\{-1, 0, 1\}$, and ϵ_k' is in $\{-m, \cdots, +m\}$. This algebraic definition, used in [Luz91] to define the (n, m) incremental control laws – n refers to the maximum order of the increment and m to the maximum absolute value of ϵ_k' – is more general than the algebraic definition used for instance in chapter one.

Actually in the first applications like the laser cutting robot [FKB+85, ZFG84], only $(1, 0)$ incremental control laws were used. Such a crude definition of rule-based incremental control has been shown to be insufficient [Luz91] (linear time-invariant systems could not be stabilized if their state matrix had a norm greater than 3) and experimentally a need for a better definition had been felt too [FL89]. This led to $(n, 0)$ incremental control laws. Although it has been shown in [Luz91] that a stable linear time-invariant system controlled by a $(n, 0)$ incremental control law could follow all reference signals $k \mapsto k^i$ for $i \leq n$, no result on stabilization could easily be proved.

This can be better understood when one goes from the recurrent definition of u_k to the explicit form of u_k depending on the initial conditions:

$$\begin{aligned} u_k &= u_0 + \sum_{i_0=1}^{k} \epsilon^0_{i_0} u^1_0 + \cdots + \sum_{i_0=1}^{k} \sum_{i_1=1}^{i_0} \cdots \sum_{i_{n-2}=1}^{i_{n-3}} \epsilon^0_{i_0} \epsilon^1_{i_1} \ldots \epsilon^{n-2}_{i_{n-2}} u^{n-1}_0 + \\ & \sum_{i_0=1}^{k} \sum_{i_1=1}^{i_0} \cdots \sum_{i_{n-1}=1}^{i_{n-2}} \epsilon^0_{i_0} \epsilon^1_{i_1} \ldots \epsilon^{n-1}_{i_{n-1}} \Delta \end{aligned}$$

which can be rewritten when $u^1_0 = \cdots = u^{n-1}_0 = 0$ as:

$$u_k = u_0 + \sum_{i_0=1}^{k} \sum_{i_1=1}^{i_0} \cdots \sum_{i_{n-1}=1}^{i_{n-2}} \epsilon^0_{i_0} \epsilon^1_{i_1} \ldots \epsilon^{n-1}_{i_{n-1}} \Delta$$

The coefficient of Δ is a complex function of k; when all ϵ^i_j are equal to 1, its value is C^n_{n+k-1}, but it can be observed that all values between $-C^n_{n+k-1}$ and C^n_{n+k-1} are not taken, when the ϵ^i_j vary inside $\{-1, 0, 1\}$. Although it is possible to infer different recurrent descriptions of the set of all values taken by this coefficient when k varies, we could not find an explicit description depending only on k. The idea was then to introduce the (n, m) incremental control laws, where an additional term $\epsilon'_k \Delta'$ was added to the recurrent definition of u_k, which overcame the previous difficulties. As all theoretical results on stabilization or reference signal tracking have only used $(0, m)$ incremental control laws (i.e. used only this new additional term), and as such a definition models the intuitive idea of incremental control, we think that $(0, m)$ incremental control laws are the algebraic definition we have been really looking for. The only objection could be that in such a definition, the coefficient of the increment is uniformly bounded in k, whereas in the (n, m) definition, there was no such limitation; but neither in theoretical proofs, nor in experimental investigations, there has ever been made use of that feature.

2.2 Distance based formulation

Instead of considering operators on the input space, we can consider metrics on this space, and choose the input at time k such that the distance between this input and the input at time $k-1$ is the product of an increment by a uniformly positive integer. Actually, such a formulation tells us that the input at time k will be on a sphere with center the input at time $k-1$; it would be nice to have some more information on the location on this sphere: typically, we would like to have an order on successive inputs, so as to be able to say whether the input increases or decreases. This will lead us to consider functions instead of distances, but as the underlying idea is to keep two successive inputs inside some ball, we call that formulation distance-based.

According to the previous remarks, u_k and u_{k-1} are related by:

$$f(u_k, u_{k-1}) = \epsilon_k \Delta$$

where, as usual, ϵ_k is a uniformly bounded integer and Δ a positive real. In [Fou90], some particular values of f have been proposed, although no further study has been undertaken, like: $f(a,b) = (a+b)/(1-ab)$. Of course, if we take $f(x,y) = x - y$, we have the usual algebraic definition of incremental control laws. Without adding further constraints on f, it is almost impossible to achieve anything with such a definition. The question is then, of course, how to choose these constraints? The remainder of this section will be dedicated to the consequences of one constraint which seemed "natural" to us.

Let us consider a linear scalar controllable system $x_{k+1} = ax_k + u_k$; we will look for all continuous f such that: $f(x_{k+1}, x_k) = af(x_k, x_{k-1}) + f(u_k, u_{k-1})$; in other words, in the same way as f reduces the study of the inputs to the study of the increments ϵ_k, we would like f to reduce the study of the state vector to the study of a "state increment" that satisfies further a linear equation. We present all these developments in the scalar case, as it is much easier and we do not need to bother about the dimension problems.

Such a function f is defined by the next equation:

$$\forall x_1, x_2, x_3,\ \ f(x_1 - ax_2, x_2 - ax_3) = f(x_1, x_2) - af(x_2, x_3) \tag{2.1}$$

If we make particular choices for (x_1, x_2, x_3), we can derive the following identities (when no further restriction is given, x and y are any real):

- $f(0,0) = 0$
- $f(x - ay, 0) = f(x,y) - af(y, y/a)$
- $f(x - a, 1) = f(x, 1) - af(1, 0)$
- $f(ax, x) = f(0, x - ay) + af(x, y)$
- $f(0, x) = f(ax, x) - af(x, 0) = f(0, x - a) + af(x, 1) - af(x, 0)$
- $f(a^2x, ax) = af(ax, x)$

Let us consider $g(x,y) = f(x,y) - \lambda x - \mu y$, where $\lambda = f(1,0)$ and $\mu = af(1,0) - f(a,1)$. Then g satisfies obviously the same identities as f. Furthermore, $g(1,0) = 0$, hence $g(x-a,1) = g(x,1)$. We have:

$$\begin{aligned} g(x-a, y) &= g(x - a - ay, 0) + ag(y, y/a) \\ &= g(x - ay, 1) - ag(1, 0) + ag(y, y/a) \\ &= g(x - ay + a, 1) + ag(y, y/a) \\ &= g(x, y) \end{aligned}$$

Thus $g(x-a,0) = g(x,0)$, but we had $g(x-a,0) = g(x,1) - ag(1,1/a)$ and $0 = g(0,1) = g(a,1) = ag(1,1/a)$; all this implies $g(x,0) = g(x,1)$.

$$\begin{aligned} g(x,y+1) &= \frac{1}{a}g(ax,x) - \frac{1}{a}g(0,x-ay-a) \\ &= \frac{1}{a}g(ax,x) - \frac{1}{a}g(0,x-ay) + g(x-ay,1) - g(x-ay,0) \\ &= g(x,y) \end{aligned}$$

$$\begin{aligned} g(x+1,y) &= \frac{1}{a}g(ax+a,x+1) - \frac{1}{a}g(0,x+1-ay) \\ &= \frac{1}{a}g(ax+a,x) - \frac{1}{a}g(0,x-ay) \\ &= g(x,y) \end{aligned}$$

Let us assume a is not an integer, then we have just shown that $g_y : x \mapsto g(x,y)$ has 1 and a as periods; as g_y is continuous, we conclude that for each y, g_y is constant. In particular, we have then $g(x,0) = g(y,0) = g(0,0)$.

$$\begin{aligned} g(x,y+a) &= g(x-ay-a^2,0) + ag(y+a,1+y/a) \\ &= g(x-ay,0) + ag(y,y/a) \\ &= g(x,y) \end{aligned}$$

We have just shown that, if a is not an integer, $g_x : y \mapsto g(x,y)$ has 1 and a as periods; as g_x is continuous, we conclude that for each x, g_x is constant. In particular, $g(0,x) = g(0,y) = g(0,0)$. As $g(0,0) = 0$, all this proves that g is null everywhere.

If a in an integer, then a is a period of g_x and g_y, as 1 is a period. As, obviously, $-a$ is a period too, we can further assume a is a positive integer. Let us consider any rational number in $[0,1[$ and its expansion in base a: $\sum_{i=1}^{N} p_i a^{-i}$, where p_i are positive integers in $\{0,\cdots,a-1\}$. Then:

$$g(0,\sum_{i=1}^{N} p_i a^{-i}) = a^{-N} g(0,\sum_{i=1}^{N} p_i) = g(0,0) = 0$$

We have:

$$g(a\sum_{i=1}^{N} p_i a^{-i},\sum_{i=1}^{N} p_i a^{-i}) = a^{-N} g(a\sum_{i=1}^{N} p_i,\sum_{i=1}^{N} p_i) = g(0,0) = 0$$

By continuity of g, we have then $g(0,x) = 0$ and $g(ax,x) = 0$ for any real x in $[0,1[$, and more generally, as 1 is a period for both g_x and g_y, for any real x; as furthermore $g(ax,x) = g(0,x-ay) + ag(x,y)$, we conclude that g is null on $\mathbf{R}^2$.

We have proved that the continuous functions f that verify 2.1 have the following form: $f(x,y) = \lambda x + \mu y$, where λ and μ are arbitrary.

This form generalizes the usual form: $f(x,y) = x - y$. Using the explicit form of f in the equation $x_{k+1} = ax_k + u_k$, we have:

$$\left\{\begin{array}{lcl} u_k & = & \frac{\Delta}{\lambda}\sum_{i=1}^{k}\epsilon_i(\frac{-\mu}{\lambda})^{k-i} + (\frac{-\mu}{\lambda})^k u_0 \\ x_{k+1} & = & a^{k+1}x_0 + \frac{u_0}{a+\frac{\mu}{\lambda}}(a^{k+1} - (\frac{-\mu}{\lambda})^{k+1}) + \frac{\Delta}{\lambda}\sum_{i=1}^{k}\epsilon_i\frac{a^{k+1-i} - (\frac{-\mu}{\lambda})^{k+1-i}}{a+\frac{\mu}{\lambda}} \end{array}\right.$$

A study similar to the one in [Luz91] can then be performed, and the results are qualitatively the same.

All this shows that the constraint we chose does not yield a very satisfactory generalization. We have tried other functions f, but the recurrent equations look very quickly like the garden of Sleeping Beauty's castle, and we did not find yet the charming Prince ...

Chapter 3

A look into abstract systems theory

We have been looking at the different approaches in abstract systems theory, in order to find a general frame in which we could express rule-based control with enough generality, so as to be able to infer then connections with other formalisms like in [MLZ92, Mar92], where learning paradigms in process control are discussed.

We will first present some approaches we found in literature and will then concentrate on a particular approach which is much more developed and has already given non trivial results. With help of this formalism, we will then try to describe rule-based incremental control and give hints for further developments.

3.1 A tour through literature

Our aim is not to justify abstract systems theory or even to give a precise definition of it. Let us just say that we looked for a theory which could be used to describe very large classes of systems and behaviors - we do not even try at this point to define these concepts - without using specific equations, but rather by giving some array of axioms which formalize rather "naturally" usual concepts. It appears in literature that the word *systems theory* is used very widely and covers many topics which were of no use to us; focusing on *abstract systems theory* and on *formal theories of general systems*, the topics covered were more or less what we wanted. We will describe in this section some approaches we did not use directly, although the concepts introduced are of course almost the same as the concepts we will use, but their formalization did not suit well to our task.

In [Kli69, Orc72], the theory relies on the existence of five fundamental traits which are shared by all systems; the last reference introduces a sixth fundamental definition for time-varying systems and proposes an application to a cellular automaton. Let us summarize these fundamental characteristics: a given phenomenon under investigation is known via measure values of some *quantities* associated to the phenomenon. These measures are made at a given *space-time resolution level.* Once the quantities have been chosen and a resolution level assigned to each, values are measured starting at a reference time; the variation in time of these values in the *activity* of the system. If the system is observed during a sufficient time, three types of behavior may be observed: *permanent behavior*, which is the real behavior of the system, the absolute relation satisfied over the entire time interval, *relatively permanent behavior*, which is the relative relation consistent with the data, and *temporary local behavior*, which corresponds to a relation satisfied only during some time interval. After study of the different behaviors, it should be possible to have an idea of the structure of the system: a *universe of discourse* is given, the collection of all elements of the system, and the *coupling* of two elements is the set of all common external quantities; corresponding to the different behaviors, there are *real* and *hypothetic* couplings. The *structure of universe of discourse and couplings* is the set of all elements or behaviors and their couplings or compositions. Finally, the *state* of a system is the set of instantaneous values of all quantities, and the *state-transition structure* is the set of all states and transitions between these states.

The main five traits are thus: *quantity and resolution level*, *activity*, *permanent behavior*, *universe of discourse and coupling structure* and *state transition structure.* In [Orc72], these traits may be considered as changing during time (indexed by a new time space), in order to describe time-varying systems, and a system is then defined by a trajectory in fundamental nonfundamental time-space!

In [Wym72], a set of *input functions* is first defined; this set is invariant by *translation*, as an input function should not depend on the arbitrary origin of the time scale, and it must be invariant by *segmentation*: if f and g are two input functions, the function defined at t as being $f(t)$ if t is negative and as $g(t)$ if t is positive, must be an input function. This last condition allows us to experiment on a system: if an experiment is to begin at time 0 and an input function is then to be generated, the total input history composed of the history up to time 0 and the history from time 0, must be a legitimate input function. Once we have this set of *admissible input functions*, a system is defined as an *assemblage* (S, P, F, M, T, σ) where S is a set of states, P a set of inputs, F a set of admissible input functions defined from $\mathbf{R}$ into P, M a set of functions (*behaviors*) defined from S into S, which contains the identity, T a set of time steps ($T \subset \mathbf{R}$, $0 \in T$) and σ a function from $F \times T$ onto M that satisfies additional constraints ($\sigma(f,0) = \mathrm{Id}_S$; $\sigma(f,t_1+t_2) = \sigma(f_{\to t_1},t_2)\circ\sigma(f,t_1)$ where $f_{\to\tau}(t) = f(t+\tau)$; $f_{|[0,\tau[} = g_{|[0,\tau[} \Longrightarrow \sigma(f,\tau) = \sigma(g,\tau)$), namely *initial state consistency*, *composition property* and *causality*, in order to rule out anticipatory systems. Starting from this definition, "recipes" (sic) are given to account for concepts as coupling, subsystems and components, and systems classification considerations, with definition of system homomorphisms. The main objective in [Wym72] is not to give a rigorous mathematical formulation but rather to present a frame which can "represent any engineering phenomenon of interest"(sic).

It is obvious that one of the most immediate fields of application of topology in a theory of general systems relies on an adequate definition of proximity (of systems, trajectories, or behaviors), in order to address problems of stability, or optimality. Some work has been done in this direction, as we will see in the next sections. In [Cor72], a "natural topology" is introduced within the framework of [Wym72]; actually, the introduction of the topology concerns the admissibility of input functions: the method of generating admissible input functions suggests the introduction of the *admissible set operator*, which satisfies generalized closure conditions (not the well-known Kuratowsky closure conditions), as developed in [Ham72], and the introduction of a generalized topological structure, namely an *Appert space*. Although a natural topological extension is thus presented (with new definitions of continuity, connectedness, convergence [Ham72]), no other properties, more closely related to general systems, seem to have been explored since.

In order to conclude this brief review of general systems theory in literature, let us give some additional references: in [Win71], time processes and time processors, contracting and expanding processes are defined, and the book ends with the definition of a state space; finally, [MMT70] is dedicated to hierarchical and multilevel systems.

3.2 A formal theory of general systems

A constructive specification of an input-output system can be done in two ways: either by considering the systems objects as functions of time and defining a system by restricting the domains of these functions and adding some analytical properties, or by considering these objects as sets and introducing additional algebraic structure. The first approach is developed in [Mes72, MT75, MT85] and the second approach in [Sai81]. Actually it is rather nonsense to erect a solid wall between these two approaches: all the algebraic developments in [Sai81], although very appealing, are actually guided by a translation into algebraic terms of the state equation of a linear time-invariant system, and the only non trivial result is the *internal model principle*, which states that under some – rather strong – assumptions, an internal model of the exosystem is present in the controller of a regulator (in other words, the dynamical action of the controller includes a copy in some sense of the dynamical action of the exosystem). This explains why a mixed approach is used in [MT85] too, which starts from the same concepts as in [MT75] but uses intensively category theory and other algebraic concepts. As our aim is to use this formalism in order to define rule-based control and eventually lay a bridge between systems theory and learning theory, and as this report is only a first step, some definitions given may seem useless, and some may be lacking. The given references will fill the gaps, and we hope we can achieve our goal, if not in this report then at least in a rather short time.

A *general system* S is a relation on nonempty sets: $S \subset \times_{i \in I} V_i$, where $\times$ denotes the cartesian product and I is the index set. A component set V_i is referred to as a system object. Let $I_x \subset I$ and $I_y \subset I$ be a partition of I; the set $X = \times_{i \in I_x} V_i$ is termed the *input* object, while $Y = \times_{i \in I_y} V_i$ is termed the *output* object. The system $S \subset X \times Y$ is referred to as an *input-output system.* $X = \mathcal{D}(S)$ is the domain of S and $Y = \mathcal{R}(S)$ is the range.

Given a general system S, let C be an arbitrary set and R a partial function defined from $C \times X$ in Y, such that: $(x, y) \in S \Rightarrow (\exists c)[R(c, x) = y]$. Then C is a *global state object*, its elements are global states, and R is a *global response function* for S. This function provides a way to express the output as a function of the input even when the system is not initially functional. It can be shown that *every system has a total global response function.* However if further requirements are imposed either on C or on R, it is not guaranteed any longer that R is total.

Let A and B be arbitrary sets, T a time set (i.e. a linearly ordered set), A^T and B^T the set of all maps on T into A and B respectively, $X \subset A^T$ and $Y \subset B^T$, then a *general time system* S is a relation on X and Y. The elements of X and Y are *abstract time functions*; for $x \in X$, its value at t is denoted by $x(t)$ (the same notation holds for y of course). Let us introduce now the following notational convention for *time segments*:

$$\begin{array}{lll} T_t = \{t^* \mid t^* \geq t\} & T^t = \{t^* \mid t^* < t\} & T_{tt'} = \{t^* \mid t \leq t^* < t'\} \\ \overline{T}_{tt'} = T_{tt'} \cup \{t'\} & \overline{T}^t = T^t \cup \{t\} & \end{array}$$

Corresponding to various time segments, the restriction of $x \in A^T$ will be defined as follows: $x_t = x_{|T_t}$, $x^t = x_{|T^t}$, $x_{tt'} = x_{|T_{tt'}}$, $\overline{x}_{tt'} = x_{|\overline{T}_{tt'}}$, $\overline{x}^t = x_{|\overline{T}^t}$. The following restrictions of X are defined: $X_t = \{x_t \mid x_t = x_{|T_t}, x \in X\}$; X^t and $X_{tt'}$ are defined in a similar way; $X(t) = \{x(t) \mid x \in X\}$. The following convention is used: $x_{tt} = \emptyset$ and $X_{tt} = \{\emptyset\}$.

Based on the restriction operation, another operation is defined: the *concatenation.* Let $x \in A^T$ and $x^* \in A^T$; for any t we define $\hat{x}$ by: $\hat{x}(\tau) = x(\tau)$ if $\tau < t$ and $\hat{x}(\tau) = x^*(\tau)$ if $\tau \geq t$. $\hat{x}$ is represented by $\hat{x} = x^t.x^*_t$ and is called the concatenation of x_t and x^*_t.

A time system is *input complete* if and only if:

$$\left\{ \begin{array}{l} (\forall x)(\forall x^*)(\forall t \in T)(x, x^* \in \mathcal{D}(S) \Rightarrow x^t.x^*_t \in \mathcal{D}(S)) \\ (\forall t)(\{x(t) \mid x \in X\} = A) \end{array} \right.$$

Every time system is now assumed input-complete unless explicitly stated otherwise.

The restrictions of a time system S are defined in reference to the restrictions of inputs and outputs: for instance S_t is the set of all (x_t, y_t) for which $(x, y) \in S$. The other restrictions S^t and $S_{tt'}$ are defined in a similar way. For a time system, the global state object and the global response function are called *initial state object* and *initial response function*, and denoted by C_0 and ρ_0. The *state object* at t, denoted by C_t, is an initial state object for the restriction S_t; the function ρ_t defined from $C_t \times X_t$ into Y_t such that: $(x_t, y_t) \in S_t \Leftrightarrow (\exists c)[\rho_t(c, x_t) = y_t]$ is referred to as the *response function* at t. A family $\overline{\rho}$ of response functions for a given system is a *response family* for S, while $\overline{C}$, the set of all C_t for $t \in T$, is a family of state objects.

Let S be a time system, $S \subset X \times Y$, and ρ_t an arbitrary function defined from $C_t \times X_t$ into Y_t. The function ρ_t is termed *consistent with* S if and only if it is a response function at t for S. A family of arbitrary functions $\overline{\rho}$ is consistent with a time system S if and only if $\overline{\rho}$ is a response family for S.

A time system $S \subset X \times Y$ has a *predynamical representation* if and only if there exist two families of mappings: $\overline{\rho} = \{\rho_t : C_t \times X_t \rightarrow Y_t\}$ and $\overline{\phi} = \{\phi_{tt'} : C_t \times X_{tt'} \rightarrow C_{t'}\}$ for $t' \geq t$, such that:

(i) $\overline{\rho}$ is a response family consistent with S

(ii) the functions $\phi_{tt'}$ in the family $\overline{\phi}$ satisfy: (for all $t, t', t"$ in T such that $t \leq t' \leq t"$)

(α) $\rho_t(c_t, x_t)_{|T_{t'}} = \rho_{t'}(\phi_{tt'}(c_t, x_{tt'}), x_{t'})$ where $x_t = x_{tt'}.x_{t'}$

(β) $\phi_{tt'}(c_t, x_{tt'}) = \phi_{t"t'}(\phi_{tt"}(c_t, x_{tt"}), x_{t"t'})$ where $x_{tt'} = x_{tt"}.x_{t"t'}$

(γ) $\phi_{tt}(c_t, x_{tt}) = c_t$

$\phi_{tt'}$ is the *state-transition function* on $T_{tt'}$, while $\overline{\phi}$ is referred to as the *state-transition family.*

Condition (α) represents the consistency property of the state-transition family with the given response family, while (β) represents the state-transition composition property; if the response family is reduced (i.e. $\rho_t(c_t, x_t) = \rho_t(\hat{c}_t, x_t)$ implies $c_t = \hat{c}_t$), then condition (β) is implied by (α). If there exists a set C such that $C_t = C$ for every $t \in T$, the pair $(\overline{\rho}, \overline{\phi})$ is termed a *dynamical representation*, and C is called a *state space* for S.

It can be shown that *every system S has a predynamical representation.* Given a predynamical representation $(\overline{\rho}, \overline{\phi})$, if one takes for C the union of all C_t, and selects for each t a fixed element c_t^* from C_t, and defines ρ'_t and $\phi'_{tt'}$ by: $\rho'_t(c, x_t) = \rho_t(c, x_t)$ if $c \in C_t$, $\rho'_t(c, x_t) = \rho_t(c_t^*, x_t)$ if not, and $\phi'_{tt'}(c, x_{tt'}) = \phi_{tt'}(c, x_{tt'})$ if $c \in C_t$, $\phi'_{tt'}(c, x_{tt'}) = \phi_{tt'}(c_t^*, x_{tt'})$ if not, then it can be shown that $(\overline{\rho'}, \overline{\phi'})$ is a dynamical representation with C as state space.

Other functions can be defined, like the output-generating function (defined from $C_t \times \overline{X}_{tt'}$ into $Y(t')$), the output function (defined from $C_t \times X(t)$ into $Y(t)$) and the state-generating function (defined from $X^t \times Y^t$ into C_t); the set of all functions for $t \in T$ of any type is a family of the corresponding functions. The existence of such families is related to certain conditions which can can be interpreted as causality concepts (nonanticipation and past-determinacy). We will not go into more details, but this could be useful in further research.

Other concepts, like minimal representation, stability, controllability, can be developed in this frame. The problem of uniqueness of representation up to an isomorphism can be addressed too.

3.3 Intelligent control and abstract systems theory

In [MLZ92, Mar92] the following problem is discussed: let us consider some generic family of controllers and let a system S be, which will be controlled by one of the controllers of that family. The only two assumptions of the family of controllers are that it is recursively enumerable (i.e. it is the range of a partial recursive function, or informally: there is a "standard" algorithm that can enumerate all these controllers) and any controller has a finite number of ways to act on the inputs of a system (that does not mean that the input takes only a finite number of values, it only means that the action which a controller can perform at any time is chosen in a finite predefined set of actions). Of course the family of rule-based incremental controllers satisfies these assumptions (this is also true for the generalizations discussed in chapter two), because any such controller has only a finite number of rules, and therefore two different controllers can be ordered by lexicographical order; but fuzzy controllers (see [Luz92b] for some references on the topic) or other so-called *intelligent* controllers can be considered too.

At any time k, one action e_k – chosen among the allowed finite actions – will be performed; a sequence $(e_1, \cdots, e_k)$ represents then the whole input history of a system, starting from some initial conditions and evolving during a finite time; such sequences will be called *evolutions*. The set of all evolutions is denoted by $\mathcal{E}$. Actually for sake of simplicity, e_k is assumed to take only 2 values, but that does not change anything on a formal point of view.

The interesting issue is that some evolutions are acceptable while some are not, because there are additional constraints on the behavior of the system, like stability, or seeking to fulfill a particular goal.

This has led to the two following definitions: *set of rules* and *set of constraints*. The first definition characterizes the set of evolutions a controller can produce when applied to a particular system (with this definition there is no need for a further formalization of the notion of a controller), while the second definition characterizes the unsatisfactory evolutions. More formally, we write $x \sqsubseteq y$ (respectively $x \sqsubset y$) if x is an initial (respectively a proper initial) segment of y, and we consider a recursive coding of the set of all evolutions upon $\mathbb{N}$, i.e. to any evolution will be associated an integer in a constructive way, and sets of integers will be considered instead of sets of evolutions[1].

A *set of rules* is a recursively enumerable set R of elements of $\mathcal{E}$ such that: $(\forall x \in R)(\forall y \in \mathcal{E})(x \sqsubseteq y \Rightarrow y \in R)$. A set of rules is *total* if every evolution can be extended: $(\forall x \in R)(\exists y \in R)(x \sqsubset y)$.

A *set of constraints* is a recursively enumerable set R of elements of $\mathcal{E}$ such that:

i) $(\forall x \in C)(\forall y \in \mathcal{E})(x \sqsubseteq y \Rightarrow y \in C)$

ii) $(\forall x \in \mathcal{E})[(\forall y \in \mathcal{E})(x \sqsubset y \Rightarrow y \in C) \Rightarrow x \in C]$

[1] Recursivity theory appears thus as the "natural" tool to describe and classify sets of evolutions via this coding

All these definitions formalize intuitive features: a set of rules is a set of evolutions generated by a controller, hence every sub-evolution of an evolution should possibly be generated by that controller too; concerning the set of constraints, condition i) rules out forbidden evolutions (once you are "out", you will not be allowed back "in"!) and condition ii) rules out catastrophic evolutions (if an evolution cannot be extended so as to avoid C, that evolution is already doomed...). Furthermore both sets have a tree-like structure (with possibly infinite depth); hence one of the evolutions in such a set will sometimes be referred to as a path. It should be noticed that R and C are not always disjoint; when they are, R is called *coherent.*

The learning problem is then to find some way to be able to generate only the acceptable evolutions, or at least to find a controller that maximizes the number of its evolutions in $\bar{C}$ and minimizes the number of its evolutions in C; this is formalized in [Mar92] and we do not need to know more of it at this point.

What interests us here is for instance to characterize the class of systems for which a given set of evolutions is acceptable; this is the inverse problem of chapter one, where we were looking for evolutions that were acceptable for a given class of systems. Another interesting issue would be to characterize the set of evolutions such that a rule-based incremental controller can generate these and only these evolutions.

It has been noted that R and C are not always disjoint; this means that a controller can generate "bad" evolutions. The question is then: are there always "good" evolutions and can they be generated by some controller?

Let C be a set of constraints distinct of $\mathbb{N}$. Then $\bar{C}$ is infinite and contains an infinite path; but if this path is not recursive, it cannot be produced by a controller. If any infinite path in $\bar{C}$ is not recursive, then no coherent set of rules is total, and the system can only be controlled on a finite horizon: every mechanical controller must fail at a given time. More precisely, we have to distinguish between two cases: either every coherent set of rules is finite, or there exists an infinite one. In the first case, any controller can control the system only on a bounded horizon (the maximal length of the evolution that controller allows), but however large the horizon may be, a controller exists; in the second case, there exists a controller which can control the system only on a finite horizon, but this horizon can be of any length. If we want to push away the horizon, we have to change the controller in the first case, not in the second (this does not mean that we can control up to some time t and then go on with another controller, it means that there is another controller that could have controlled the system up to a time $t' > t$). In [Mar92] different set of constraints are built that illustrate this informal talk, in other words:

- there exists a set of constraints C, $C \neq \mathbb{N}$, such that every set of rules which is coherent relative to C is finite;
- there exists a set of constraints C, $C \neq \mathbb{N}$, such that there exists a set of rules which

is infinite and coherent relative to C, and such that every set of rules which is coherent relative to C is not total.

These two results may seem weird in regard to the results of chapter one; the key point is that we considered stability or reference tracking in that chapter, but there exist much more complex control goals for which only control on finite horizons may be possible. One interesting question is then to find a set of constraints that verifies either of the two previous results and corresponds actually to the forbidden evolutions of a system: it is the problem of the physical realizability of such sets of constraints, the search of a model for the formal theory of [Mar92]. Trying to find a model within the systems controlled by rule-based incremental controllers is our aim, as that would imply this type of control works and can be learned, at least within a well-known class of systems.

No definitive answer will be given to all these questions in this report, although we think we have the means to provide an answer with the help of abstract formal theory. We have for now only a partial answer concerning the problem of characterizing "good" and "bad" evolutions; the next result can be related to the notion of coherent sets of rules. We will use the notations of the previous section introducing a formal theory of general systems.

Let us consider a time system $S \subset X_S \times Y_S \subset X \times Y$ and let C be its initial state object and ρ be its initial response function (ρ is defined from $C \times X_S$ into Y_S such that: $(x, y) \in S \Leftrightarrow (\exists c)[y = \rho(c, x)]$). Let us define now *consistency* and *completeness* in reference to Y and two subsets of Y, $R \subset Y$ and $W \subset Y$. The subset R consists of all desirable outputs while W is the subset of "undesirable" outputs (forbidden or unacceptable outputs for instance).

The system is W-consistent if and only if: $(R \cap Y_S) \cap W = \emptyset$

The system is W-complete if and only if: $(R \cap Y_S) \cup W = Y$

We would like to characterize these concepts differently; this will be done by using a function g called in [MT85] a generalized Goedel function, because of the self-referencing character of its use in the next proof; g is an injective map from C into X. Two other definitions will be useful. To every state $c \in C$, there will correspond a set of inputs X_c for which ρ produces desirable outputs: $X_c = \{x \mid \rho(c, x) \in R\}$. Let now X' be an arbitrary subset of inputs, $X' \subset X$; then X' is *acceptable* if and only if there exists $c \in C$ such that $X' = X_c$. Finally, let us define $C^{d,W}$ and $X^{d,W}$ by:

$$\begin{cases} c \in C^{d,W} \Leftrightarrow \rho(c, g(c)) \in W \\ X^{d,W} = g(C^{d,W}) \end{cases}$$

The funny notation comes from the fact that $C^{d,W}$ can be seen as the set of all states whose diagonalization[2] is in W.

We can now state the following proposition.

[2]This terminology is taken from [MT85] because of the similarity with the well-known Goedel's proof...

Proposition 8 *Given $\rho : C \times X_S \to Y_S$ and $R, W \subset Y$. The system is either W-inconsistent or W-incomplete whenever $X^{d,W}$ is an acceptable set.*

Proof: As $X^{d,W}$ is acceptable, there exists $c^* \in C$ such that:

$$x \in X^{d,W} \Leftrightarrow \rho(c^*, x) \in R \cap Y_S$$

In particular, for $g(c^*)$, we have:

$$g(c^*) \in X^{d,W} \Leftrightarrow \rho(c^*, g(c^*)) \in R \cap Y_S$$

By definition of $X^{d,W}$ (i.e. $g(c^*) \in X^{d,W} \Leftrightarrow \rho(c^*, g(c^*)) \in W$), it follows that:

$$\rho(c^*, g(c^*)) \in W \Leftrightarrow \rho(c^*, g(c^*)) \in R \cap Y_S$$

Denote by y^* the output of the system such that $y^* = \rho(c^*, g(c^*))$; it follows then that either $y^* \in (R \cap Y_S) \cap W$, i.e. the system is W-inconsistent, or $y^* \in (R \cap Y_S) \cup W$, i.e. the system is W-incomplete. □

<u>**Remark**</u>: the roles played by the inputs and the states in the previous definitions can be exchanged, and one can define C_x, C', $X^{d,W}$ and $C^{d,W}$ with $g : X \to C$. The proposition becomes then: *if $C^{d,W}$ is an acceptable set, the system is either W-inconsistent or W-incomplete.*

Appendix A

Some useful results on linear control

A.1 Canonical form for a discrete linear time-invariant controllable system

Let us consider a controllable discrete linear time-invariant system, given by its state equation: $X_{k+1} = AX_k + Bu_k$, where the state vector is n-dimensional. We introduce the matrices M, W and T:

$$M = (B|AB|\cdots|A^{n-1}B)\ ;\ W = \begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}\ ;\ T = MW$$

The elements of matrix W are the coefficients of the characteristic polynomial of A ($\chi(A)(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$). The matrix M is full rank because of the controllability assumption: hence T is obviously invertible and we define:

$$\widehat{A} = T^{-1}AT, \widehat{B} = T^{-1}B, \widehat{X}_k = T^{-1}X_k$$

The controllable form is then:

$$\widehat{X}_{k+1} = \widehat{A}\widehat{X}_k + \widehat{B}u_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix} \widehat{X}_k + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u_k$$

A.2 Controllability under linear feedback

Let us consider a discrete linear time-invariant system $X_{k+1} = AX_k + BU_k$, which we will call system (A, B), where the state space is n-dimensional and the control vector is r-dimensional (B is now a $(n \times r)$ matrix). Let C be any matrix $(r \times n)$ matrix, and let us consider the system $X_{k+1} = (A + BC)X_k + BU_k$, i.e. system $(A + BC, B)$. We will prove in this section the following proposition:

Proposition 9 *Controllability of* $(A + BC, B) \Longrightarrow$ *controllability of* (A, B).

Proof: Let us define, for sake of simplicity, $Y(i)$ recursively by:

$$\begin{cases} Y(0) &= B \\ Y(i) &= (Y(i-1)|A^iB) \end{cases}$$

The matrix $Y(n-1)$ is the matrix M encountered in the previous section, which must be full rank for a controllable system. Actually, controllability for a (A, B) system can be alternatively expressed as: $\forall i \geq 0,\ v^{\mathsf{T}}Y(i) = 0 \Longrightarrow v = 0$, which can be restated as: $\forall i \geq 0,\ v^{\mathsf{T}}A^iB = 0 \Longrightarrow v = 0$.

Let us now assume $(A + BC, B)$ is controllable. We want to show that (A, B) is then controllable. Assume then: $\forall i, v^{\mathsf{T}}A^iB = 0$. If we compute $v^{\mathsf{T}}(A + BC)^iB$, we notice that we obtain a sum of matrix products which all start with some A^jB, where $0 \leq j \leq i$. This sum is then zero with the assumption, and as $(A + BC, B)$ is assumed controllable, this implies that $v = 0$; hence the controllability for (A, B) follows.

As the previous proof is valid for any C, it shows that $((A + BC) + B(-C), B)$ controllable implies $((A + BC), B)$ controllable. □

Appendix B

Difference equations

The following results are all taken from [LT88] and so are all notations. We will denote by $B(y,\delta)$ the open ball having center at y and radius δ; if $y = 0$, we shall use the notation B_δ.

B.1 Several stability concepts

Let $y_0 \in B_a$ and $f : \mathbb{R}^s \to \mathbb{R}^s$ be a bounded function in B_a. The solution $y(n, n_0, y_0)$ of the difference equation:

$$\begin{cases} y_{n+1} = f(n, y_n) \\ y_{n_0} = y_0 \end{cases} \tag{B.1}$$

will remain in B_a for all $n_0 \geq n$ such that $f(n, y_n) \in B_a$. The points $\overline{y} \in \mathbb{R}^s$ which satisfy $f(n, \overline{y}) = \overline{y}$ for all n are called *fixed points* of B.1. For simplicity, we will suppose that there is now a fixed point at the origin (this can always be achieved by an appropriate coordinate change).

The solution $y = 0$ of B.1 is said to be:

1. *stable* if given $\epsilon > 0$, there is a $\delta(\epsilon, n_0)$ such that for any $y_0 \in B_\delta$, the solution y_n remains in B_δ
2. *uniformly stable* if it is stable and δ can be chosen independently of n_0
3. *attractive* if there is $\delta(n_0) > 0$ such that for $y_0 \in B_\delta$, one has $\lim_{n\to\infty} y_n = 0$
4. *uniformly attractive* if it is attractive and δ can be chosen independently of n_0
5. *asymptotically stable* if it is stable and attractive
6. *uniformly asymptotically stable* if it is uniformly stable and uniformly attractive

7. *exponentially stable* if there exists $\delta > 0, a > 0, \eta \in]0,1[$ such that if $y_0 \in B_\delta$, then $\| y_n \| \leq a \| y_0 \| \eta^{n-n_0}$

8. *lp-stable* if it is stable and moreover for some $p > 0$, $\sum_{j=n_0}^{\infty} \| y(j, n_0, y_0) \| < \infty$

9. *uniformly lp-stable* if the previous summation converges uniformly with respect to n_0

These different stability notions are not equivalent: obviously any uniform property implies the property; asymptotic stability implies stability; lp-stability implies asymptotic stability, but not the converse; exponential stability implies lp-stability.

B.2 Stability of perturbed equations

Let us now consider the linear case:

$$y_{n+1} = A(n)y_n \qquad y_{n_0} = y_0 \tag{B.2}$$

where $A(n)$ is a $(s \times s)$ matrix. The *fundamental matrix* $\Phi(n, n_0)$ is by definition $\prod_{i=n_0}^{n-1} A(i)$. We have then the following propositions.

Proposition 10 *The solution $y = 0$ of B.2 is uniformly stable if there exists $M > 0$ such that for $n \geq n_0$:*

$$\| \Phi(n, n_0) \| < M$$

Proof: The sufficiency follows from the fact that $y_n = \Phi(n, n_0)y_0$, for we have:

$$\| Y_n \| \leq \| \Phi(n, n_0) \| \| y_n \| \leq M \| y_0 \|$$

and hence: $\| y_n \| < \epsilon$, if $\| y_0 \| < \epsilon M^{-1}$.

To prove necessity, if there is uniform stability, then $\| \Phi(n, n_0)y_0 \| < 1$ for $\| y_0 \| < \delta$. Taking $x_0 = y_0 / \| y_0 \|$, we have then: $\sup_{\|x_0\| \leq 1} \| \Phi(n, n_0)x_0 \|$ is bounded, which means that the norm of the fundamental matrix is bounded. □

Proposition 11 *The solution $y = 0$ of B.2 is uniformly asymptotically stable if there exist two positive numbers a and η, with $\eta < 1$ such that:*

$$\| \Phi(n, n_0) \| \leq a\eta^{n-n_0}$$

Proof: The proof of sufficiency is direct like in the previous proposition. The necessity follows by considering that if there is uniform asymptotic stability, then fixing $\epsilon > 0$, there exists $\delta > 0$ and N (depending only on ϵ) such that for $y_0 \in B_\delta$ and for $n \geq n_0 + N$, we have: $\| \Phi(n, n_0)y_0 \| < \epsilon$.

As before, it is then easy to see that $\| \Phi(n, n_0) \| < \eta'$, where this time η' can be chosen arbitrarily small, and this holds for any n_0 because of the uniform property. Moreover, as uniform asymptotic stability implies uniform stability, the previous proposition tells that $\| \Phi(n, n_0) \|$ is bounded by a positive number a' for all $n \geq n_0$. We then have for $n \in [n_0 + mN, n_0 + (m+1)N]$:

$$\begin{aligned} \| \Phi(n, n_0) \| &\leq \| \Phi(n, n_0 + mN) \| \| \Phi(n_0 + mN, n_0 + (m-1)N) \| \cdots \| \Phi(n_0 + N, n_0) \| \\ &< a'\eta'^m = a'\eta'^{-1}(\eta'^{\frac{1}{N}})^{(m+1)N} \leq a\eta^{n-n_0} \end{aligned}$$

with $\eta = \eta'^{\frac{1}{N}}$ and $a = a'\eta'^{-1}$. □

As a result of this proposition, **for linear systems, uniform asymptotic stability is equivalent to exponential asymptotic stability.**

The next propositions deal with perturbed linear equations. We consider:

$$y_{n+1} = A(n)y_n + f(n, y_n) \tag{B.3}$$

where $A(n)$ is a $(s \times s)$ matrix and f is defined for $n \geq n_0$ and $y_n \in B_a$ and takes values in B_a and $f(n, 0) = 0$. This equation can be seen as a perturbation of:

$$x_{n+1} = A(n)x_n \tag{B.4}$$

and the question arises whether the properties of B.4 are preserved for B.3, when f is small in the sense to be specified.

Proposition 12 *Assume that:*

$$\| f(n, y_n) \| \leq g_n \| y_n \|$$

where g_n are positive and $\sum_{n=n_0}^{\infty} g_n < \infty$. Then if the zero solution of B.4 is uniformly stable (or uniformly asymptotically stable), then the zero solution of B.3 is uniformly stable (or uniformly asymptotically stable).

Proof: As we have: $y_n = \Phi(n, n+0)y_0 + \sum_{j=0}^{n-1} \Phi(n, j+1)f(j, y_j)$, using proposition 10, we have:

$$\| y_n \| \leq M \| y_0 \| + M \sum_{j=n_0}^{n-1} g_j \| y_j \|$$

which implies, by Gronwall inequality:

$$\| y_n \| \leq M \| y_0 \| \exp\left(M \sum_{j=n_0}^{n-1} g_j\right)$$

from which follows the proof for the case of uniform stability.

In the case of uniform asymptotic stability, it follows that for $n > N$, $\| \Phi(n, n_0) \| < \epsilon$, for every $\epsilon > 0$, and the previous inequality can be written:

$$\| y_n \| \leq \epsilon \exp\left(M \sum_{j=n_0}^{\infty} g_j\right)$$

from which follows $\lim_n y_n = 0$. □

Proposition 13 *Assume that:*

$$\| f(n, y_n) \| \leq L \| y_n \|$$

where $L > 0$ is sufficiently small, and the zero solution of B.4 is uniformly asymptotically stable. Then the zero solution of B.3 is exponentially asymptotically stable.

Proof: By using the proposition 11, we have:

$$\exists H > 0, \exists \eta \in]0,1[\quad \| \Phi(n; n_0) \| < H\eta^{n-n_0}$$

We have then, using the assumption of the proposition:

$$\| y_n \| \leq H\eta^{n-n_0} \| y_0 \| + LH\eta^{n-1} \sum_{j=n_0}^{n-1} \eta^{-j} \| y_j \|$$

Introducing the new variable: $p_n = \eta^{-n} \| y_n \|$, we have:

$$p_n \leq H\eta^{-n_0} \| y_0 \| + LH\eta^{-1} \sum_{j=n_0}^{n-1} p_j$$

which implies, by Gronwall inequality:

$$p_n \leq H\eta^{-n_0} \| y_0 \| \prod_{j=n_0}^{n-1} (1 + LH\eta^{-1}) = H\eta^{-n_0} \| y_0 \| (1 + LH\eta^{-1})^{n-n_0}$$

Hence, $\| y_n \| \leq H \| y_0 \| (\eta + LH)^{n-n_0}$, and if $\eta + LH < 1$ (hence the assumption on L in the proposition), the conclusion follows. □

We will conclude this appendix on difference equations with two other notions of stability. Let us consider the equations:

$$y_{n+1} = f(n, y_n) + R(n, y_n) \tag{B.5}$$

$$y_{n+1} = f(n, y_n) \tag{B.6}$$

where R is a bounded Lipschitz function in B_a and $R(n, 0) = 0$. We shall consider B.5 as a perturbation of equation B.6.

The zero solution of B.6 is said to be *totally stable* (or stable with respect to permanent perturbations) if for every $\epsilon > 0$, there exist two positive numbers δ_1 and δ_2 such that every solution of B.5 lies in B_ϵ for $n \geq n_0$ provided that:

$$\begin{cases} \| y_0 \| < \delta_1 \\ \| R(n, y_n) \| < \delta_2 \quad \text{for } y_n \in B_\epsilon, n \geq n_0 \end{cases}$$

The other concept of stability is of *practical stability.* In this case, we no longer require that $R(n, 0) = 0$, so that B.5 does not have the fixed point at the origin, but we assume $\| R(n, 0) \|$ is bounded for all n. The solution $y = 0$ of B.6 is said to be practically stable if there exists a neighborhood A of the origin and $N \geq n_0$ such that for $n \geq N$, the solution of B.5 remains in A.

We have the following proposition, which we give without proof (see [LT88] for more details).

Proposition 14 *Suppose that the zero solution of B.6 is uniformly asymptotically stable and moreover, for $y', y'' \in B_a$:*

$$\| f(n, y') - f(n, y'') \| \leq L \| y' - y'' \|$$

where $L > 0$. Then it is totally stable.

Actually proposition 14, as given here, is a weaker form of the proposition to be found in [LT88], where the assumption is: $\| f(n, y') - f(n, y'') \| \leq L_r \| y' - y'' \|$ for $y', y'' \in B_r \subset B_a$, i.e. the Lipschitz coefficient depends on the ball where it is defined. But the proof given in the reference uses the fact that for any $\epsilon > 0$, ϵL_ϵ can be taken arbitrarily small, which does not seem correct. However the proof is correct for our weaker proposition.

The next proposition deals with practical stability.

Proposition 15 *Suppose that the zero solution of B.6 is uniformly asymptotically stable and moreover that in a set $D \subset \mathbf{R}$ the following conditions are satisfied:*

(1) There exists $0 < L < 1$ such that: $\| f(n, y) - f(n, y') \| \leq L \| y - y' \|$

(2) $\| R(n, y) \| < \delta$

Then the origin is practically stable for B.6.

Proof: Let y_n and $\tilde{y}_n$ be the solution of B.5 and B.6 respectively. Set $m_n = \| y_n - \tilde{y}_n \|$; then by hypothesis: $m_{n+1} \leq L m_n + \delta$ from which it follows:

$$\| y_n - \tilde{y}_n \| \leq L^n \| y_0 - \tilde{y}_0 \| + \delta \sum_{j=0}^{n-1} L^j = L^n \| y_0 - \tilde{y}_0 \| + \frac{\delta}{1 - L}$$

As $L < 1$, this last expression is obviously bounded and as $\tilde{y}_n$ is uniformly asymptotically stable, there exists N such that the solution of B.5 remains in the ball $B(0, \frac{\delta}{(1-L)} + 1)$ for $n > N$. □

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