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Cycles of Chaotic Intervals in a 1-D Piecewise-Linear Map

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Abstract

We study the bifurcations of attractors of a one-dimensional 2-segment piecewise linear map. We prove that the parameter regions of existence of stable *point* cycles γ are separated by regions of existence of stable *interval* cycles Γ containing *chaotic* trajectories. Moreover, we show that the period-doubling phenomenon for stable *interval* cycles is characterized by two universal constants α and δ, whose values are calculated from explicit formulas.

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Introduction.

In this work we consider the endomorphisms of the *interval* $I = [0, 1]$:

$$f_{l,p} : x \mapsto f_{l,p}(x), \quad x \in I,$$

where $f_{l,p}$ denotes a 2-segment piecewise-linear function with one extremum and having slopes l, p as parameters. These maps arise in the consideration of the time-delayed Chua's circuits modeled by a difference equations with a continuous argument

$$x(t+1) = f_{l,p}(x(t)), \quad t \in R^+.$$

Since the dynamics of this *difference* equation is governed by the dynamics of the trajectories of the 1-D map $f_{l,p}$, we will consider only the 1-D map $f_{l,p} : I \mapsto I$ in this paper.

There are many publications dealing with one-dimensional piecewise-linear maps. In particular the kneading theory is developed in [Misiurewicz & Visinescu, 1988] and [Marcuard & Visinescu, 1989]. The paper [Sharkovsky et al., 1993] considered an ideal model of Chua's circuits containing a time delay and proved the existence of stable *point* cycles γ_n of all periods n. Moreover, the conditions for the existence of stable *interval* cycles Γ and some results for a two-dimensional generalization of this one-dimensional model are given in [Maistrenko et al., 1992].

The order of the bifurcation sequence in piecewise-linear maps $f_{l,p}$ is different from that of smooth maps. In the case of our piecewise-linear maps, when a period-n *point* cycle γ_n loses its stability, a "rigid" period-doubling bifurcation occurs which leads to the emergence of not *point* cycles but *interval* cycles $\Gamma_{n,2n}$ of double period having chaotic trajectories. This is followed by an inverse period-doubling bifurcation; i.e., *interval* cycles $\Gamma_{n,2n}$ of period $2n$ are merged pairwise, giving birth to a period-n interval cycle $\Gamma_{n,n}$. Finally, in the next bifurcation all intervals of *interval* cycles $\Gamma_{n,n}$ will merge into an *interval* cycle $\Gamma_{n,1} = I$. In this case, there are no subintervals of I which recur periodically under the map f.

The bifurcation of a period-2 *point* cycle ($n = 2$) is different from the above scenario and is therefore somewhat special. When a period-2 *point* cycle γ_2 loses its stability, an *interval* cycle $\Gamma_{2,2^k}$ of period-2^k occurs, where k is *any* integer, depending on the values of the parameters l, p. In this case, the next bifurcation consists of a pairwise merging of period-2^k *interval* cycles, giving birth to an *interval* cycle $\Gamma_{2,2^{k-1}}$ of period 2^{k-1}.

At the point $(l, p) = (1, -1)$ two ***universal constants*** associated with period-doubling *interval* cycles ($\delta = 2$ and $\alpha = \infty$) are obtained which are analogous to the "point cycle" period-doubling Feigenbaum's universal constants.

Therefore, for general one-dimensional piecewise-linear maps with one extremum, the following ordering of attractor bifurcations must occur:

$$\gamma_1 \Rightarrow \gamma_2 \Rightarrow (\Gamma_{2,2^k} \Rightarrow \Gamma_{2,2^{k-1}} \Rightarrow \cdots \Rightarrow \Gamma_{2,2} \Rightarrow I) \Rightarrow \gamma_3 \Rightarrow (\Gamma_{3,6} \Rightarrow \Gamma_{3,3} \Rightarrow I) \Rightarrow \gamma_4 \Rightarrow$$
$$\Rightarrow (\Gamma_{4,8} \Rightarrow \Gamma_{4,4} \Rightarrow I) \Rightarrow \cdots \Rightarrow \gamma_n \Rightarrow (\Gamma_{n,2n} \Rightarrow \Gamma_{n,n} \Rightarrow I) \Rightarrow \gamma_{n+1} \Rightarrow \cdots.$$

This result is similar to the well-known "period-adding" phenomenon [Pei et al., 1986], [Kennedy & Chua, 1986], [Chua, 1986], observed in non-autonomous circuits where the period increases consecutively: i.e., by "addition" of the unit integer, i.e.,

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow \cdots \Rightarrow n \Rightarrow n+1 \Rightarrow \cdots,$$

and not by multiplication, as in the period-doubling route to chaos. Here, every two consecutive stable periodic orbits are separated by a chaotic region.

1 Stable point cycles γ_n in natural ordering

In this paper, we will consider a continuous, 2-parameter piecewise-linear map $f : [0,1] \mapsto [0,1]$ with one extremum (maximum) point defined by:

$$f = f_{l,p} : x \mapsto f_{l,p}(x) = \begin{cases} f_1(x) \overset{def}{=} lx + a, & x \in [0,b], \\ f_2(x) \overset{def}{=} px - p, & x \in (b,1]. \end{cases} \tag{1}$$

We assume that the parameters l, p belong to the region

$$\Pi = \{(l,p) : \quad 0 \leq l \leq \frac{p}{p+1}, \quad p \in (-\infty, -1)\}. \tag{2}$$

Since $f_{l,p}$ in (1) is assumed to be continuous, the constants a and b are defined by the formulas

$$a = 1 - l(1 + \frac{1}{p}), \quad b = 1 + \frac{1}{p}. \tag{3}$$

It should be noted, that any continuous piecewise-linear 1-D map with one breakpoint, having a nontrivial invariant interval, can be reduced to the map (1) by a linear transformation of the real line (see Appendix).

The graphs of the map $f_{l,p}$ and its next two iterations are shown at Fig. 1(a)-1(c).

Let $\gamma_n = \{x_1, \ldots, x_n\}$, $n = 2, 3, \ldots$ denote a period-n cycle, i.e,

$$x_i < x_{i+1}, \quad f(x_i) = x_{i+1}, \quad i = 1, \ldots, n-1, \quad f(x_n) = x_1. \tag{4}$$

Let us denote by

$$L_n \overset{def}{=} 1 + l + l^2 + \cdots + l^n = \frac{1 - l^{n+1}}{1 - l}.$$

We need later on the following basic theorem which was proved in [Sharkovsky et al., 1993].

Theorem 1 *A point cycle γ_n of the 1-D map $f_{l,p}$ in (1) exists if, and only if,*

$$p \leq -\frac{L_{n-2}}{l^{n-2}}; \tag{5}$$

and is attracting if, and only if,

$$p > -\frac{1}{l^{n-1}}. \tag{6}$$

It follows from Theorem 1 that for each "n", the existence and stability region of the *point* cycles γ_n in the (l,p)-parameter space is defined by

$$\Pi_n = \{(l,p) : -\frac{1}{l^{n-1}} \leq p \leq -\frac{L_{n-2}}{l^{n-2}}\}, \quad n = 2, 3, \ldots \tag{7}$$

To avoid clutter, the regions Π_n are plotted in the (l,p^*)-parameter plane in Fig. 2, where $p^* = log_2(-p)$.

Each region Π_n is bounded from below by an "existence curve", denoted by $[E,n]$, and from above by a "stability curve", denoted by $[S,n]$, as shown in Fig. 2. These two curves intersect at a point $O_n = (l_n, p_n)$, $n = 2, 3, \ldots$, which defines the end point (*apex*) of the stability region Π_n, where the first coordinate $l = l_n$ is the root of the algebraic equation

$$lL_{n-2} = 1, \quad (l^n - 2l + 1 = 0) \tag{8}$$

in the interval $(1/2, 1)$, The second coordinate of the point O_n is located at $p_n = -l_n^{-(n-1)}$. The apex points O_n, $n = 2, 3, \ldots$, are situated on a branch of the hyperbola

$$p = -\frac{1}{2} - \frac{1}{4(l - 1/2)} = \frac{l}{1 - 2l}. \tag{9}$$

The coordinate p_n has the asymptotic property

$$p_n \sim -2^{n-1} + 1, \qquad n \to \infty. \tag{10}$$

The formula (10) is derived from the properties that the curve $[E, n]$ passes through the point $(1/2, 1 - 2^{n-1})$ and the curve $[S, n]$ passes through the point $(1/2, -2^{n-1})$). In particular, it follows from (9) and (10) that

$$\lim_{n \to \infty} l_n = \frac{1}{2}, \qquad \lim_{n \to \infty} p_n = -\infty$$

Therefore, if we fix some parameter value $l \in (0, 1/2)$ and vary the parameter p from -1 to $-\infty$ then the stable *point* cycles (separated by chaotic regions) of all integer periods will be observed for the map $f_{l,p}$:

$$2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow \ldots \Rightarrow n \Rightarrow n + 1 \Rightarrow \ldots \tag{11}$$

These cycles arise as the parameter (l, p) passes through the regions $\Pi_2, \Pi_3, \ldots, \Pi_n, \Pi_{n+1}, \ldots$. This phenomenon is known in electronic circuits as the "period adding" phenomenon, which consists of the appearance of periodic oscillations whose period increases consecutively through all integers as a system parameter is tuned continuously. Observe that the period increases according to a natural ordering. In particular, as $p \to -\infty$ and $l \in (0, 1/2)$, the period of the cycle must tend to infinity.

On the other hand if we fix some parameter value $p \in (-\infty, -1]$ and increase the parameter l from 0 to 1 , then the period-adding phenomenon will also be observed, however, in this case, the period will increase only up to the some finite integer, depending on the value of p .

It should be noted, that the Schwarzian derivative $Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ is equal to zero everywhere except at extremum point in which case it is not defined. This is one reason which leads to the period-adding bifurcation (11). It is known, that if the Schwarzian derivative of a one-dimensional map is not equal to zero, then a period-doubling *point* cycle bifurcation must occur as a parameter changes.

2 Stable interval cycles $\Gamma_{n,2n}$, $\Gamma_{n,n}$ for $n \geq 3$

The map $f_{l,p}$ does *not* have attracting *point* cycle for $(l, p) \in \Pi \setminus \bigcup_{n=2}^{\infty} \Pi_n$. However, in this case, it has attracting cycles of intervals with chaotic dynamics, i.e. an invariant measure exists; it is concentrated on intervals and is absolutely continuous with respect to the Lebesgue measure.

We will show that the stability regions of *interval* cycles of periods $2n$ and n, respectively, exist in the parameter space Π for all $n \geq 2$ (see fig. 3). These regions are denoted by $\Pi_{n,2}$ and $\Pi_{n,1}$ respectively. The bifurcation curve which separates the regions $\Pi_{n,2}$ and $\Pi_{n,1}$ is denoted by $[D,n]$. The curve which bounds the region $\Pi_{n,1}$ from above is denoted by $[C,n]$. The equations of the curves $[D,n]$ and $[C,n]$ will be obtained in the proof of the following theorem.

Theorem 2. *Let* $n \geq 3$.

1) If $(l,p) \in \Pi_{n,1}$, *then the map* $f_{l,p}$ *in the form of (1) will have a stable interval cycle* $\Gamma_{n,n}$ *of period* n.

2) If $(l,p) \in \Pi_{n,2}$, *then the map* $f_{l,p}$ *in the form of (1) will have a stable interval cycle* $\Gamma_{n,2n}$ *of period* $2n$.

3) If $(l,p) \in \Pi \setminus (\bigcup_{n=2}^{\infty}(\Pi_n \cup \Pi_{n,1} \cup \Pi_{n,2}))$, *then the map* $f_{l,p}$ *will have a stable interval cycle* $\Gamma_{n,1} = [0,1]$ *of period* 1.

Proof. Consider a parameter point $(l,p) \in \Pi$. Let this point cross the curve $[E,n]$ and enter the region Π_n. It is easy to see that at the moment (critical bifurcation parameter) where one crosses the curve $[E,n]$, two period-n cycles $\gamma_n = \{x_1, \ldots, x_n\}$ and $\bar{\gamma}_n = \{\bar{x}_1, \ldots, \bar{x}_n\}$ emerged.

These cycles satisfy the following condition:

$$x_i \leq \bar{x}_i, \quad i = 1, \ldots, n-1, \quad \bar{x}_n \leq x_n. \tag{12}$$

At the above critical bifurcation point, these two cycles coincide with each other, and then split off into two distinct cycles (see fig.4). The cycle $\bar{\gamma}_n$ is always unstable, but the cycle γ_n is stable for $(l,p) \in \Pi_n$. Consider next the case where the parameter point (l,p) leave the region Π_n and cross the stability curve $[S,n]$. It is easy to see that an *interval* cycle of double period, i.e., $2n$, is born at this bifurcation point.

Indeed, let us consider the rightmost upper angle of the graph of the function f^n shown in Fig.4 and expanded in Fig.5 over the subinterval $[\bar{x}_n, 1]$ at the moment when the point (l,p) crossed the curve $[S,n]$. The slope of the right segment of the function f^n, denoted by l', is slightly less than -1, and the slope of the left segment is equal to $l'\frac{p}{l}$. Obviously, $f^{2n}(1) > x_n$ in some neighborhood of the curve $[S,n]$. Therefore the map f^n has an *interval* cycle of period 2:

$$\Gamma_{n,2n} \overset{def}{=} \{ \quad [f^n(1), f^{3n}(1)], \quad [f^{2n}(1), 1] \quad \}.$$

This *interval* cycle is attracting as soon as it is born, but at the precise bifurcation point $(l,p) \in [S,n]$ it coincides with the *point* cycle $\gamma = \{f^n(1), 1\}$ of period 2. The *interval* cycle $\Gamma_{n,2n}$ of period $2n$ is obtained by iterating the interval $[f^{2n}(1), 1]$ under the action of the map f.

If we continue to vary the parameter values so that the slopes of f^n increases then at some bifurcation parameter, the intervals $[f^n(1), f^{3n}(1)]$ and $[f^{2n}(1), 1]$ of cycle $\Gamma_{n,2n}$ touched each other and merged into one, as shown in Fig. 6. This bifurcation parameter defines the bifurcation curve $[D,n]$ and the onset of an inverse period-doubling bifurcation of *interval* cycles: $\Gamma_{n,2n} \Longrightarrow \Gamma_{n,n}$. The period-$n$ *interval* cycle $\Gamma_{n,n}$ is obtained by iterating the interval $[f^n(1), 1]$ under the action of f.

It is easy to see that the bifurcation phenomenon $\Gamma_{n,2n} \Longrightarrow \Gamma_{n,n}$ occurs when the $2n$-th iteration of the point $x = 1$ maps into the point x_n of the cycle γ_n. Figures 6 and 7 illustrate this situation for f in the case of $n = 4$. The analytical expression defining this condition, shown in Fig. 7, is

$$f_2 f_1^{n-2} f_2 f_1^{n-1}(0) = x_n, \tag{13}$$

where f_1 and f_2 denote the linear parts of the map $f_{l,p}$ (see Fig. 1). Here we used the property $f_2(1) = 0$.

We will derive formula (13) in term of the parameters l and p later, but for now let us continue to vary the values of the parameters l, p further. As the magnitude of the slope l and the magnitude of the slope p increases (in general, this involves a decrease of the value of a and an increase of the value of b as shown in Fig. 1(a)), we come to a situation when $f^n(1) = \bar{x}_n$ (see Fig.8).

At this moment the bifurcation phenomenon $\Gamma_{n,n} \Longrightarrow \Gamma_{n,1} = [0,1]$ occurs (curve [C,n]). The stable *interval* cycle of period n bifurcates into a stable *interval* cycle of period 1. It is easy to see that the condition for this bifurcation is

$$f^{n-1}(0) = \bar{x}_n. \tag{14}$$

Figure 9 shows this situation for a cycle of period 4.

To derive conditions (13) and (14) in terms of the parameters (l, p), we must first derive the formulas for the points x_n and $\bar{x}_n$ belonging to the cycles γ_n and $\bar{\gamma}_n$. The point x_n is defined by the equation

$$f_1^{n-1} f_2(x) = x; \tag{15}$$

The point $\bar{x}_n$ is defined by the equation

$$f_2 f_1^{n-2} f_2(x) = x. \tag{16}$$

Since these equations are linear, we can solve them for x_n and $\bar{x}_n$ as follow:

$$x_n = 1 + \frac{1}{p} + \frac{L_{n-1}}{(l^{n-1}p - 1)p}, \tag{17}$$

$$\bar{x}_n = 1 + \frac{1}{p} + \frac{1 + pL_{n-2}}{(l^{n-2}p^2 - 1)p}. \tag{18}$$

Substituting (17) into (13) and using the expression for f_1 and f_2 (see (1)), we obtain the following relation between l and p, which defines the bifurcation $\Gamma_{n,2n} \Longrightarrow \Gamma_{n,n}$:

$$l^{3n-4}p^4 + l^{2(n-1)}L_{n-2}p^3 - l^{n-2}p^2 + (l^{n-1} - L_{n-2})p + lL_{n-2} = 0. \tag{19}$$

It should be noted that the bifurcation curve $[E, n]$ satisfies the relation (19) (the relation (13) is satisfied upon the birth of the cycles γ_n and $\bar{\gamma}_n$). Therefore, if we eliminate the factor $l^{n-2} + L_{n-2}$, we will obtain

$$l^{2(n-1)}p^3 - p + l = 0. \tag{20}$$

Equation (20) defines the bifurcation curve $[D, n]$. As an example, for $n = 2$ we obtain the curve $[D, 2]$

$$l^2p^3 - p + l = 0, \tag{21}$$

which can be solved explicitly for l :

$$l = \frac{-1 - \sqrt{1 + 4p^4}}{2p^3}.$$

The equation for the curve $[D, 3]$ is given by

$$l^4 p^3 - p + l = 0. \tag{22}$$

Substituting (18) into (14), we obtain a relation between l and p, which defines the bifurcation phenomenon $\Gamma_{n,n} \Longrightarrow \Gamma_{n,1}$:

$$l^{2n-3} p^3 + l^{n-2} L_{n-1} p^2 + (L_{n-2} - l^{n-1}) p - l L_{n-2} = 0. \tag{23}$$

The curve $[E, n]$ satisfies the relations (23) and (19). Therefore, if we divide the left side of the relation by $l^{n-2} p + L_{n-2}$ we will obtain

$$l^{n-1} p^2 + p - l = 0. \tag{24}$$

Equation (24) defines the bifurcation curve $[C, n]$. In particular, we have the curve $[C, 2]$

$$l p^2 + p - l = 0, \quad \left(l = \frac{p}{1 - p^2}\right) \tag{25}$$

for $n = 2$, and the curve $[C, 3]$

$$l^2 p^2 + p - l = 0 \tag{26}$$

for $n = 3$.

Therefore, the map $f_{l,p}$ has a stable *interval* cycle of period $2n$ in the regions $\Pi_{n,2}$, bounded by the curves $[S, n]$, $[E, n]$, $[D, n]$, and a period-n stable *interval* cycle for the region $\Pi_{n,1}$, bounded by the curves $[D, n]$, $[E, n]$, $[C, n]$, for all $n = 2, 3, \ldots$.

This completes our proof of theorem 2.

Remark: Although theorem 2 was formulated for $n > 2$, it is also true for $n = 2$ except for some neighborhood of the point

$$(l, p) = (1, -1). \tag{27}$$

The curves $[E, n]$, $[S, n]$ $[D, n]$, $[C, n]$ for the first 3 values of n $(n = 2, 3, 4)$ and the regions Π_n, $\Pi_{n,2}$, $\Pi_{n,1}$ are shown in the Fig. 10.

3 Period-doubling bifurcation of interval cycles (n = 2)

In this section we consider in detail the case $n = 2$. We will study the bifurcations phenomena which are observed when a period-2 *point* cycle γ_2 loses its stability. We will show that this case is different from the cases $n > 2$, which were described by theorem 2. The difference is in the appearance of an attracting *interval* cycle of period 2^m for all integers m. This bifurcation sequence occurs when the point (l, p) passes through the curve $[S, 2]$. Moreover, if the curve $[S, 2]$ is crossed by varying the parameter (l, p) though the point $(l, p) = (1, -1)$, then an *interval* cycle of period 2^∞ appears. Subsequent parameter variations lead to an inverse period-doubling bifurcation of *interval* cycles.

In section 2 we have given the formulas for the bifurcation curves $[D,n]$ and $[C,n]$, $n = 2,3,\ldots$. As Fig. 10 shows, the curve $[D,n]$ separates regions of stable *interval* cycles $\Gamma_{n,2n}$ and $\Gamma_{n,n}$ of periods $2n$ and n, respectively. Analogously, the curve $[C,n]$ separates regions of stable *interval* cycles $\Gamma_{n,n}$ and $\Gamma_{n,1}$ of periods n and 1, respectively.

Let us consider in detail a parameter point on the curve $[S,n]$ where a period-n cycle γ_n loses its stability. Figure 5 shows a part of the graph of the map f^n at this parameter point; namely, the "tent-like" map from the extreme right position in Fig. 4. Let us examine the f^{2n} graph (see Fig. 11) at once after crossing this parameter bifurcation point. Here $\{x_{n,1}, x_{n,2}\}$ is a *point* cycle of period 2 for the map f^n. Does there exist a stable *interval* cycle of period 2 for the map f^{2n}? It follows from the arguments in the preceding section that this *interval* cycle exists if, and only if, the value of the second iteration of the point $x = 1$ under the action of f^{2n} is greater than $x_{n,2}$; i.e., $f^{4n}(1) > x_{n,2}$. This inequality must be fulfilled at the bifurcation point $(l,p) \in [S,n]$, when the slope of the extreme right segment of the graph, shown in Fig. 12, is equal to p/l, but the slope of the second rightmost segment is equal to 1.

Let us consider an auxiliary map g in the form of (1) with slopes 1 and $p' = p/l$ (Fig. 12), respectively. If the map g has an *interval* cycle of period 2, then the original map f will have an *interval* cycle of period $4n$ as (l,p) crosses the curve $[S,n]$. It follows from (25) that the condition for the existence of an *interval* cycle of period 2 is $(p')^2 + p' - 1 < 0$, i.e.

$$\left(\frac{p}{l}\right)^2 + \frac{p}{l} - 1 < 0, \tag{28}$$

or:

$$p > -\frac{1+\sqrt{5}}{2}l. \tag{29}$$

Therefore, if at the parameter point where the cycle γ_n loses its stability (i.e. for $(l,p) \in [S,n]$) the condition (29) is violated, then, the *interval* cycle $\Gamma_{n,4n}$ of period $4n$ will not occur. Instead, we will have an *interval* cycle $\Gamma_{n,2n}$ of period $2n$. It is easy to see (Fig. 10) that in region Π the straight line

$$p = -\frac{1+\sqrt{5}}{2}l \tag{30}$$

is situated above the regions $\Pi_3, \Pi_4, \ldots$. Therefore the loss of stability of the cycle γ_n leads to the birth of a stable *interval* cycle $\Gamma_{n,2n}$ of period $2n$, for any $n = 3,4,\ldots$.

Let us consider the cycle of period 2. The straight line (30) passes through the curve $(S,2)$ at the point

$$(l,p) = \left(\sqrt{\frac{2}{1+\sqrt{5}}} \;\; ; \;\; -\sqrt{\frac{1+\sqrt{5}}{2}} \right). \tag{31}$$

Consequently, if $l < \sqrt{2/(1+\sqrt{5})}$, when this cycle loses its stability at $(l,p) \in [S,2]$, then a period-doubling bifurcation of the *interval* cycle $\Gamma_{2,4}$ will occur. Otherwise, a stable *interval* cycle of some periods $2^3, 2^4, \ldots$ will occur.

Let us consider the map f and its iterated maps $f^{2^m}, m = 1,2,\ldots.$ The slope of the rightmost segment of the graph f^{2^m} is equal to

$$p^{(2^m)} = l^{\alpha_m} p^{\alpha_m + (-1)^m}, \quad m = 0,1,\ldots, \tag{32}$$

where α_m is a solution of the difference equation

$$\alpha_{m+1} = \alpha_m + 2\alpha_{m-1}, \quad m = 1,2,\ldots, \tag{33}$$

with initial conditions $\alpha_0 = 0$, and $\alpha_1 = 1$. This solution is equal to

$$\alpha_m = \frac{1}{3}\left(2^m + (-1)^{m+1}\right), \quad m = 0,1,\ldots. \tag{34}$$

The slope of the second rightmost segment of the graph f^{2^m} is equal to

$$l^{(2^m)} = l^{2\alpha_{m-1}} p^{2(\alpha_{m-1} + (-1)^m)}, \quad m = 0,1,\ldots. \tag{35}$$

As an example, the slopes for $m = 1,2,\ldots,6$ are given below :

m	$l^{(2^m)}$	$p^{(2^m)}$
0	l	p
1	l^2	lp
2	l^2p^2	lp^3
3	l^2p^6	l^3p^5
4	l^6p^{10}	l^5p^{11}
5	$l^{10}p^{22}$	$l^{11}p^{21}$
6	$l^{22}p^{42}$	$l^{21}p^{43}$

In order that the original map $f_{l,p}$ has an *interval* cycle of period 2^{m+1}, it is necessary and sufficient that f^{2^m} has an *interval* cycle of period 2. Granting this and using formulas (32), (35) and (25) we obtain the following equation of the curve for the bifurcation phenomenon $\Gamma_{2,2^m} \Longrightarrow \Gamma_{2,2^{m+1}}$:

$$p^{\delta_{m+1}} l^{\delta_m} + (-1)^m (p - l) = 0 \quad m = 0,1,\ldots, \tag{36}$$

where $\delta_m, \quad m = 0,1,\ldots,$ is the solution of the inhomogeneous difference equation

$$\delta_{m+1} = 2\delta_m + \frac{1}{2}(1 + (-1)^m), \quad m = 1,2,\ldots, \tag{37}$$

with initial condition $\delta_0 = 1$.

The bifurcation curves defined by equations (36) are denoted by $[D,2,2^m]$, for any $m = 0,1,\ldots.$ It should be noted that $[D,2,2^0] = [C,2]$, $[D,2,2^1] = [D,2]$. The regions bounded by the curves $[D,2,2^{m-1}]$, $[D,2,2^m]$, $[S,2]$ and $[E,2]$, are denoted by $\Pi_{2,2^m}$.

It follows that the following theorem is true for any $m = 1,2,\ldots.$

Theorem 3 *Let $(l,p) \in \Pi_{2,2^m}$. Then the map $f_{l,p}$ has a stable interval cycle of period* 2^m.

Figure 13 shows the bifurcation curves $[D, 2, 2^m]$ converge to the point $(l, p) = (1, -1)$. As an example, the equations of these curves for $m = 0, 1, 2, \ldots, 6$ are as follow:

$$\begin{aligned} p^2 l + p - l = 0, &\quad [D,2,1]; \\ p^3 l^2 - p + l = 0, &\quad [D,2,2]; \\ p^6 l^3 + p - l = 0, &\quad [D,2,2^2]; \\ p^{11} l^6 - p + l = 0, &\quad [D,2,2^3]; \\ p^{22} l^{11} + p - l = 0, &\quad [D,2,2^4]; \\ p^{43} l^{22} - p + l = 0, &\quad [D,2,2^5]; \\ p^{86} l^{43} + p - l = 0, &\quad [D,2,2^6]. \end{aligned} \tag{38}$$

Theorems 1-3 allow us to conclude that in the general case of a one-dimensional piecewise-linear map with one extremum, the following ordering of attractor bifurcations must occur:

$$\begin{aligned} &\gamma_1 \Rightarrow \gamma_2 \Rightarrow (\Gamma_{2,2^k} \Rightarrow \Gamma_{2,2^{k-1}} \Rightarrow \cdots \Rightarrow \Gamma_{2,2} \Rightarrow I) \Rightarrow \gamma_3 \Rightarrow \\ &\Rightarrow (\Gamma_{3,6} \Rightarrow \Gamma_{3,3} \Rightarrow I) \Rightarrow \gamma_4 \Rightarrow (\Gamma_{4,8} \Rightarrow \Gamma_{4,4} \Rightarrow I) \Rightarrow \\ &\Rightarrow \cdots \Rightarrow \gamma_n \Rightarrow (\Gamma_{n,2n} \Rightarrow \Gamma_{n,n} \Rightarrow I) \Rightarrow \gamma_{n+1} \Rightarrow \ldots. \end{aligned}$$

4 Universal constants of period-doubling bifurcation of interval cycles

Since the period-doubling bifurcation curves have been found in explicit forms (see (36), (37)), we can derive two *universal constants* δ and α for period-doubling bifurcations of *interval* cycles, just like the Feigenbaum's constants, for period-doubling *point* cycles. To define the constants δ and α we consider in the (l, p) parameter space any straight line $p = k(l-1)+1$, which passes through the point $(l, p) = (1, -1)$. Let $(l^{(m)}, p^{(m)})$, $m = 0, 1, \ldots$, be the intersection point of this straight line with the bifurcation curve in the form of (37) for some given fixed m. The distance between the points $(l^{(m)}, p^{(m)})$ and $(l^{(m+1)}, p^{(m+1)})$ is denoted by d_m for any $m = 0, 1, \ldots$. Then the constant δ is defined as

$$\delta = \lim_{m \to \infty} \frac{d_m}{d_{m+1}}. \tag{39}$$

Analogously the constant α is defined as

$$\alpha = \lim_{m \to \infty} \frac{1 - x_m}{1 - x_{m+1}}, \tag{40}$$

where $x_m = x_m(l^{(m)}, p^{(m)})$ and $x_{m+1} = x_{m+1}(l^{(m+1)}, p^{(m+1)})$ are *point* cycles of periods 2^m and 2^{m+1}, defined by formulas (32) and (35), respectively. These points are calculated with the following bifurcation conditions: x_m at $(l, p) = (l^{(m)}, p^{(m)})$ and x_{m+1} at $(l, p) = (l^{(m+1)}, p^{(m+1)})$.

We will say that *the family of maps $f_{l,p}$ at the point $(l, p) = (1, -1)$ is characterized by an* **universal behavior** *with constant δ and α, if the limits in (39) and (40) exist and do not depend on choice of the straight line through the point $(l, p) = (1, -1)$.*

Theorem 4 *The family of maps $f_{l,p}$ is characterized by an universal behavior at the point $(l,p) = (1,-1)$ with universal constants $\delta = 2$ and $\alpha = \infty$.*

Proof. Let us first prove the existence of the universal constant $\delta = 2$. The proof will be carried out for the case $l = 1$, i.e. when the slope of the straight line is equal to ∞ (see fig. 13).

The intersection point of the straight line $l = 1$ and the bifurcation curve $[D, 2\cdot 2^{m-1}]$ is denoted by p_m for all $m = 1, 2, \ldots$. This bifurcation curve is the curve of the *interval* cycle of period 2^m. Then

$$\delta = \lim_{m\to\infty} \frac{|p_{m-1} - p_m|}{|p_m - p_{m+1}|}.$$

We will prove, that this limit exists and is equal to 2.

Let us consider the family of functions $y_n(x) = x^n - 1, x > 1, n = 1, 2, \ldots$. Let x_n be the root of the equation $x_n = x_n^n - 1$, which is nearest to $x = 1$ with $x_n \geq 1$. Graphically, x_n is the abscissa of the intersection point of the graph $y = y_n(x)$ and the bisectrix $y = x$ (Fig. 14).

Lemma 1. *The sequence x_n, $n = 1, 2, \ldots$, has the property*

$$\lim_{n\to\infty} \frac{|x_{2n} - x_n|}{|x_{2n} - x_{4n}|} = 2.$$

Proof. Let us estimate the distance between the points x_n and $\sqrt[n]{2}$. Using the boundary condition $y_n(\sqrt[n]{2}) = 1$, we find the derivative

$$y_n' = nx^{n-1}\,|_{x=\sqrt[n]{2}} \;= n2^{\frac{n-1}{n}} = \frac{2n}{\sqrt[n]{2}}.$$

Then the equation of the tangent at the point $(\sqrt[n]{2}, 1)$ has the form $y = (2n/(\sqrt[n]{2}))x - 2n+1$. The tangent crosses the bisectrix at the point $x = x_n^*$, where

$$x_n^* = (2n-1)/(\frac{2n}{\sqrt[n]{2}} - 1).$$

Assuming $x_n^* > x_n$, then

$$x_n^* - \sqrt[n]{2} = \sqrt[n]{2}\frac{\sqrt[n]{2} - 1}{2n - \sqrt[n]{2}} \stackrel{def}{=} \varepsilon_n.$$

Let us prove that ε_n is a higher-degree infinitesimal than $\sqrt[n]{2} - \sqrt[2n]{2}$. Indeed we have

$$\lim_{n\to\infty} \frac{\sqrt[n]{2}(\sqrt[n]{2} - 1)}{(2n - \sqrt[n]{2})(\sqrt[n]{2} - \sqrt[2n]{2})} = \lim_{n\to\infty} \frac{(\sqrt[2n]{2} - 1)(\sqrt[2n]{2} + 1)}{(2n - \sqrt[n]{2})(\sqrt[n]{2} - 1)} = 0.$$

Moreover, it is easy to see that $\varepsilon_n \sim 1/n$, as $n \to \infty$. Then

$$\lim_{n\to\infty} \frac{|x_n - x_{2n}|}{|x_{2n} - x_{4n}|} \leq \lim_{n\to\infty} \frac{\sqrt[n]{2} - \sqrt[2n]{2} + (\varepsilon_n + \varepsilon_{2n})}{\sqrt[2n]{2} - \sqrt[4n]{2} - (\varepsilon_{2n} + \varepsilon_{4n})} = 2.$$

This completes our proof of lemma 1.

Our calculations give the following results for p_i, $i = 1, 2, \ldots, 10$:

$$\begin{aligned} p_1 &= -1.618022, & p_2 &= -1.324698, \\ p_3 &= -1.134732, & p_4 &= -1.068296, \\ p_5 &= -1.032771, & p_6 &= -1.016444, \\ p_7 &= -1.008140, & p_8 &= -1.004074, \\ p_9 &= -1.002032, & p_{10} &= -1.001017. \end{aligned}$$

Using these numbers, we obtain the following approximations for δ:

$$\begin{aligned} \delta_1 &= 1.544, & \delta_2 &= 2.860, \\ \delta_3 &= 1.87, & \delta_4 &= 2.18, \\ \delta_5 &= 1.97, & \delta_6 &= 2.04, \\ \delta_7 &= 1.99, & \delta_8 &= 2.02. \end{aligned}$$

To obtain the universal constant α we consider *point* cycles x_{2^m} of period 2^m on the bifurcation curves $[D, 2, 2^{m-1}]$. Then

$$\alpha = \lim_{m \to \infty} \frac{x_{2^{m-1}} - x_{2^m}}{x_{2^m} - x_{2^{m+1}}}.$$

The constant α was obtained by using x_{2^m} in the following algorithm. Let x_{2^m} be a root of the linear equation $a_m x + b_m = x$, where (a_m, b_m) is the result obtained after m iterations of the map

$$G_n : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} p^{(-1)^{m+1}} a^2 \\ p^{(-1)^{n+1}}(b - ab - 1) + 1 \end{pmatrix},$$

where $n = 1, 2, \ldots, m$. The map G_n is employed at the point $(a, b) = (p, -p - 1/p)$ for the value $p = p_n$, on the bifurcation curve $[D, 2 \cdot 2^{m-1}]$. That is

$$\begin{pmatrix} a_m \\ b_m \end{pmatrix} = G_m \cdots G_2 G_1 \begin{pmatrix} p_m \\ -p_m - 1/p_m \end{pmatrix},$$

Then we find $x_{2^m} = -b_m/(a_m - 1)$ for all $m = 1, 2, \ldots$. It should be noted that the initial condition $(p_m, -p_m - 1/p_m)$ varies with m.

Using this algorithm the following results are obtained

$$\begin{aligned} \alpha_1 &= 2.820, & \alpha_2 &= 14.058, & \alpha_3 &= 17.777, \\ \alpha_4 &= 50.462, & \alpha_5 &= 84.501, & \alpha_6 &= 190.896, \\ \alpha_7 &= 358.839, & \alpha_8 &= 672.111. \end{aligned}$$

It follows from the above result that

$$\alpha = \infty,$$

where

$$\alpha_n \sim \alpha_0 \cdot 2^n, \quad n \to \infty, \qquad \alpha_0 \simeq \sqrt{2}.$$

This completes our proof of theorem 4.

Four one-dimensional bifurcation diagrams for $l = 1$ in successively enlarged scale are shown in the Fig. 15(a-d).

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6 Appendix

There are two cases where a continuous piecewise linear 1D-map g with one breakpoint has a nontrivial invariant interval. Both are for the slopes l and p such as:

$$(l,p) \in \Pi = \{0 \le l \le \frac{p}{p+1}, \quad p < -1\}.$$

In the first case

$$g : x \mapsto g_{l,p}(x) = \begin{cases} lx + A, & x \le \frac{B-A}{l-p}, \\ px + B, & x > \frac{B-A}{l-p}, \end{cases}$$

where A and B satisfy

$$A > \frac{1-l}{1-p}B.$$

In the second case

$$g : x \mapsto g_{l,p}(x) = \begin{cases} px + B, & x \le \frac{B-A}{l-p}, \\ lx + A, & x > \frac{B-A}{l-p}, \end{cases}$$

where A and B satisfy

$$A < \frac{1-l}{1-p}B.$$

It is easy to see that in both cases the map g can be reduced by the linear transformation

$$\sigma : x \mapsto \sigma(x) = 1 + \frac{(1-2p)(l-p)}{[A(1-p)+B(l-1)]p}\left[x - \frac{lB-pA}{l-p}\right]$$

to obtain a map f in the form (1) with an invariant interval $[0,1]$:

$$f = \sigma \circ g \circ \sigma^{-1}.$$

7 Figure captions

Fig. 1(a). Graph of piecewise-linear function $f_{l,p}(x)$, with two slopes l, and p.

Fig. 1(b),(c). Graphs of iterations $f_{l,p}^2 = f(f(x))$ and $f_{l,p}^3 = f(f(f(x)))$ of the piecewise-linear map $f : x \mapsto f_{l,p}$.

Fig. 2. The existence and stability regions Π_n of the *point* cycles γ_n in the parameter space (p^*, l), where $p^* = log_2(-p)$. Each region Π_n is bounded from below by an existence curve $[E, n]$ and from above by a stability curve $[S, n]$.

Fig. 3. The stability regions Π_n of the *point* cycle γ_n and $\Pi_{n,2}$, $\Pi_{n,1}$ of the *interval* cycles $\Gamma_{n,2n}$ and $\Gamma_{n,n}$ of periods $2n$ and n respectively, in the parameter space (p^*, l) for all $n > 2$. The regions $\Pi_{n,2}$ and $\Pi_{n,1}$ are separated by the bifurcation curve $[D, n]$. The curve $[C, n]$ bounds the region $\Pi_{n,1}$ from above.

Fig. 4. The graph of the function $f_{l,p}^n$ when the *point* (l, p) crosses the curve $[E, n]$ and enters the region Π_n. The points of the period-n stable cycle are given by $\gamma_n = \{x_1, \ldots, x_n\}$. Those for a period-$n$ unstable cycle are given by $\bar{\gamma}_n = \{\bar{x}_1, \ldots, \bar{x}_n\}$.

Fig. 5 The rightmost upper angle of the graph of the function $f_{l,p}^n$, from Fig. 4 at the moment when the point (l, p) crosses the curve $[S, n]$ and enters the region $\Pi_{n,2}$. At this moment each point x_n of a stable period-n cycle $\gamma_n = \{x_1, \ldots, x_n\}$ creates the *interval* cycle $\Gamma_{n,2n}$ of period $2n$.

Fig. 6. The rightmost upper angle of the graph of the function $f_{l,p}^n$ from Fig. 4 at the moment when *interval* cycle $\Gamma_{n,n}$ of period-n is born.

Fig. 7. The graph of the function $f_{l,p}$ $(n = 4)$ when the point (l, p) crosses the curve $[D, n]$ and an *interval* cycle $\Gamma_{n,n}$ of period-n was born. At this moment each pair of intervals $[f^n(1), f^{3n}(1)]$ and $[f^{2n}(1), 1]$ of the cycle $\Gamma_{n,2n}$ touched each other and merged into one interval.

Fig. 8. The rightmost upper angle of the graph of the function $f_{l,p}^n$ from Fig. 4 at the moment when *interval* cycle $\Gamma_{n,1} = [0, 1]$ is born.

Fig. 9. The graph of the function $f_{l,p}$ $(n = 4)$ when the point (l, p) crosses the curve $[C, n]$ and all intervals of the *interval* cycle $\Gamma_{n,n}$ merged into one interval $[0, 1]$.

Fig. 10. The stability region Π_n of the *point* cycles γ_n and the stability regions $\Pi_{n,2}$ and $\Pi_{n,1}$ of the *interval* cycles $\Gamma_{n,2n}, \Gamma_{n,n}$ of periods $2n$ and n, respectively in the parameter space (p^*, l) for $n = 2, 3, 4$. The regions $\Pi_{n,2}$ and $\Pi_{n,1}$ are separated by the bifurcation curve $[D, n]$. The curve $[C, n]$ bounds the region $\Pi_{n,1}$ from above.

Fig. 11. The graph of the function $f_{l,p}^{2n}$ at the moment when the point (l, p) crossed the curve $[S, n]$ and the cycle γ_n lost its stability. Here $\{x_{n,1}, x_{n,2}\}$ is a period-2 cycle of f^n.

Fig. 12. The rightmost upper angle of the graph of the function $f_{l,p}^{2n}$ from Fig. 11.

Fig. 13. The stability regions Π_2 of the *point* cycles γ_2 and $\Pi_{n,2^m}$ of the *interval* cycles $\Gamma_{2,2^m}$ in the parameter space (p^*, l) for $m = 0, 1, \ldots$ The regions $\Pi_{n,2^m}$ and $\Pi_{n,2^{m+1}}$ are bounded by the bifurcation curves $[D, 2, 2^m]$.

Fig. 14. The graph of the functions $y_n(x) = x^n - 1$. The point x_n is the abscissa of the intersection point of the graph $y = y_n(x)$ and the bisectrix $y = x$.

Fig. 15(a-d). Four parameter bifurcation diagrams in successively enlarged scale, which illustrate the cascade of period-doubling bifurcations of *interval* cycles for $l = 1$. The bifurcation points $p_m^* = log_2(-p_m), m = 1, 2, 3\ldots$ belong to the curves $[D, 2, 2^m]$.

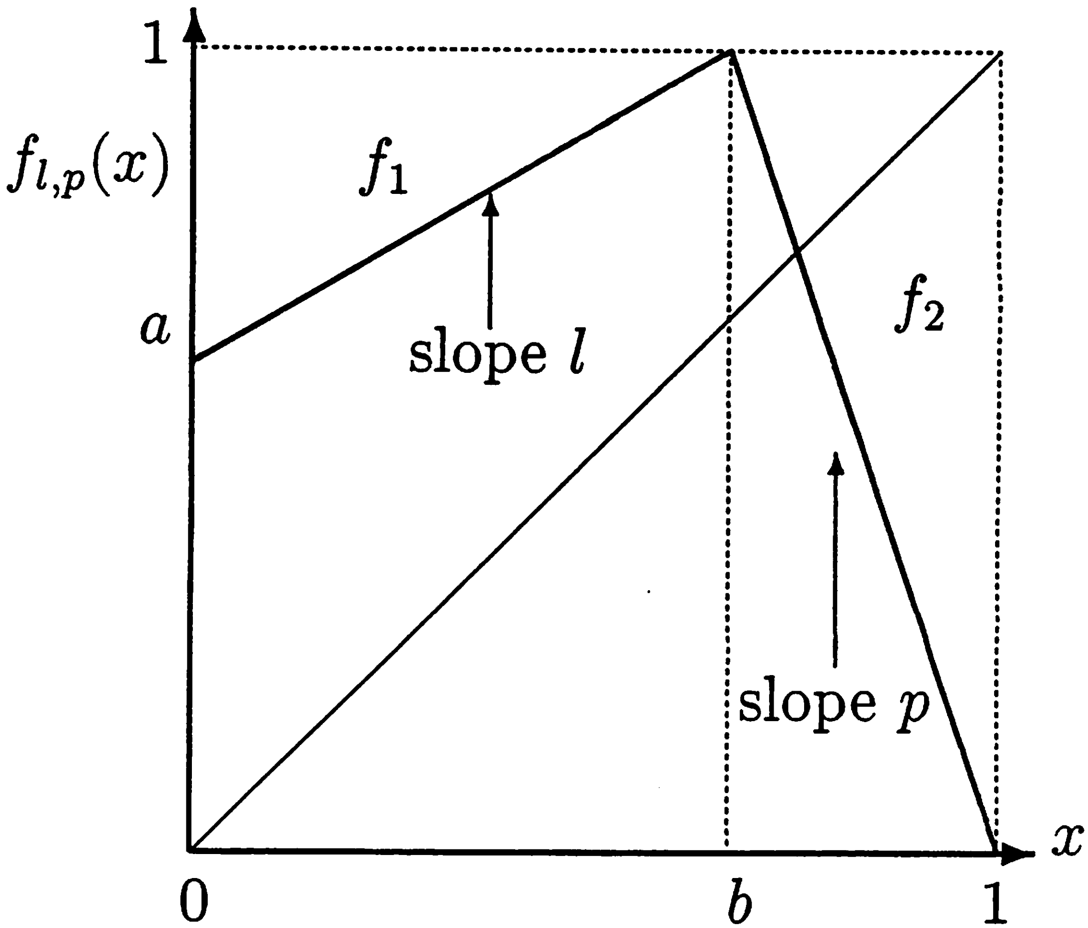


Fig. 1(a)

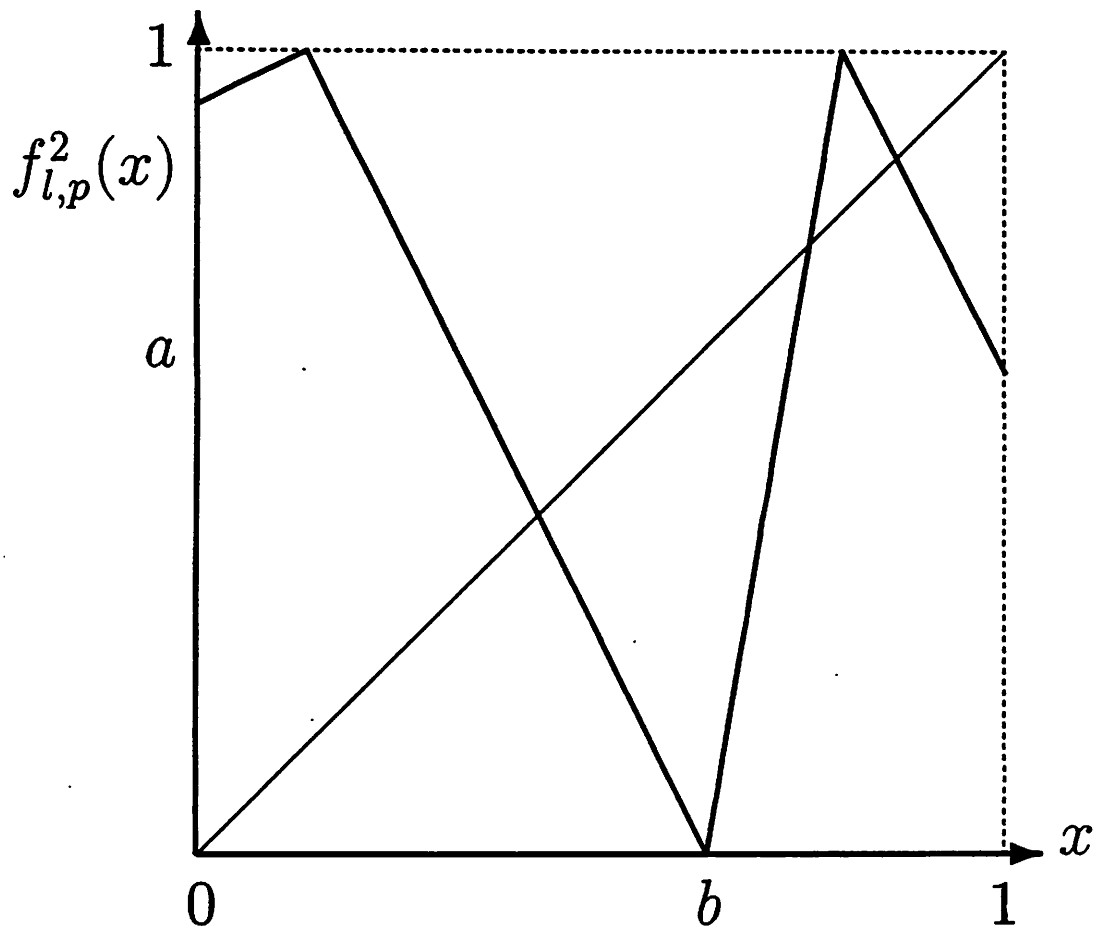


Fig. 1(b)

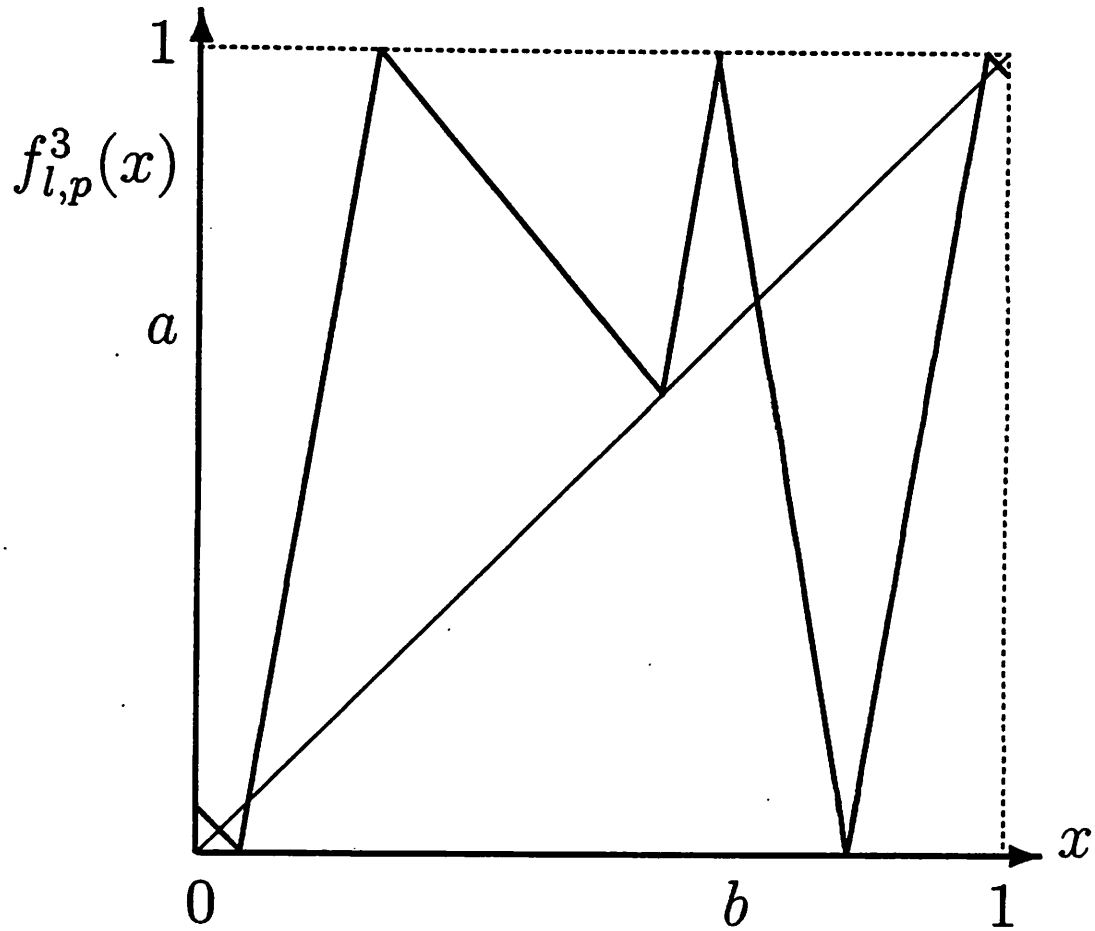


Fig. 1(c)

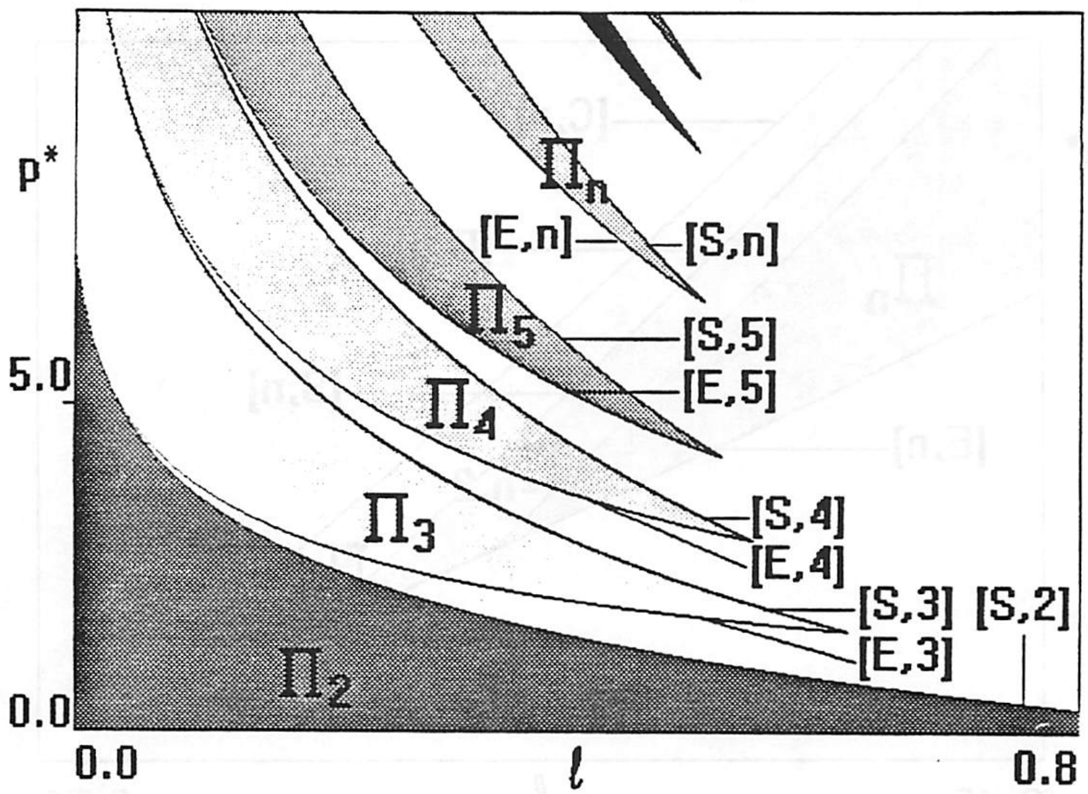


Fig. 2

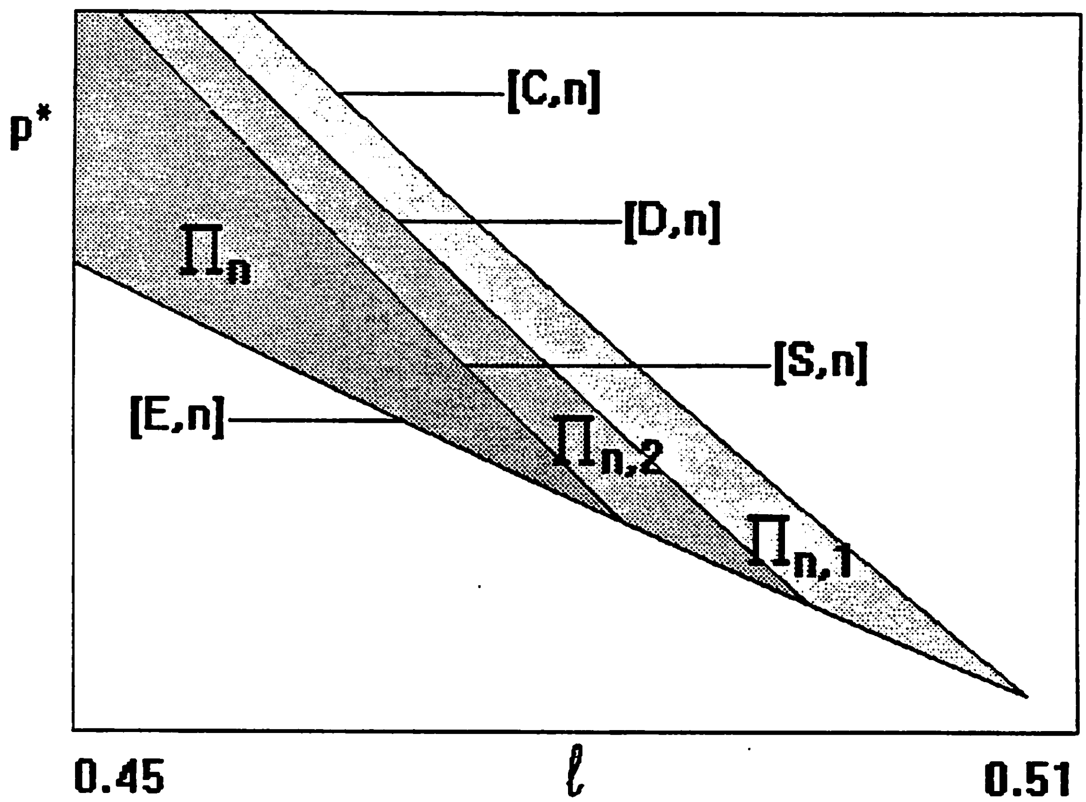


Fig. 3

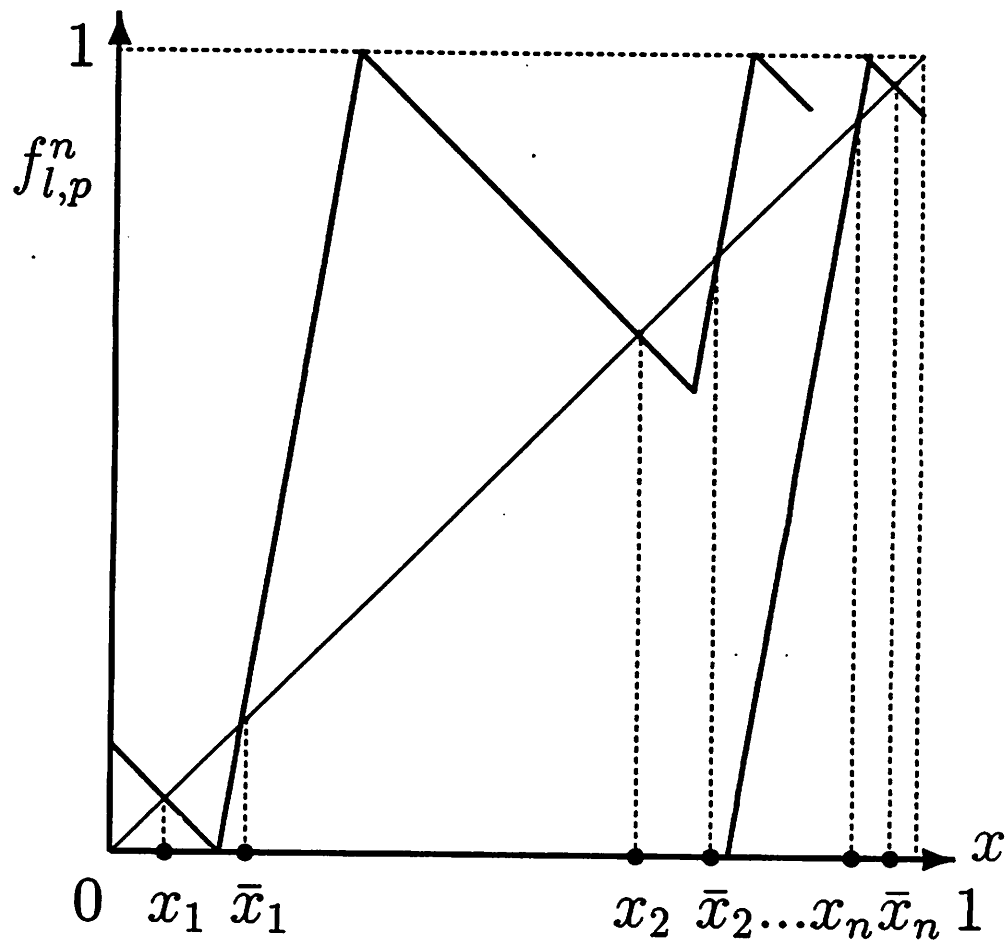


Fig. 4

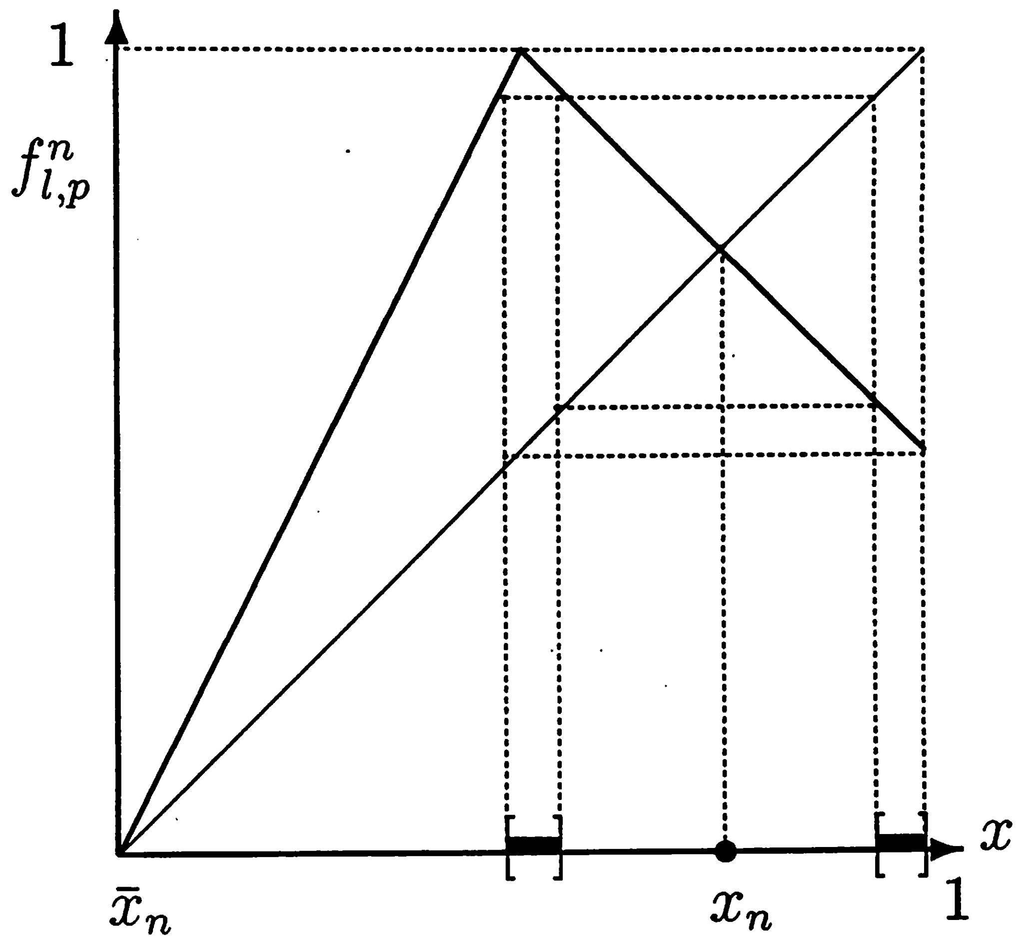


Fig. 5

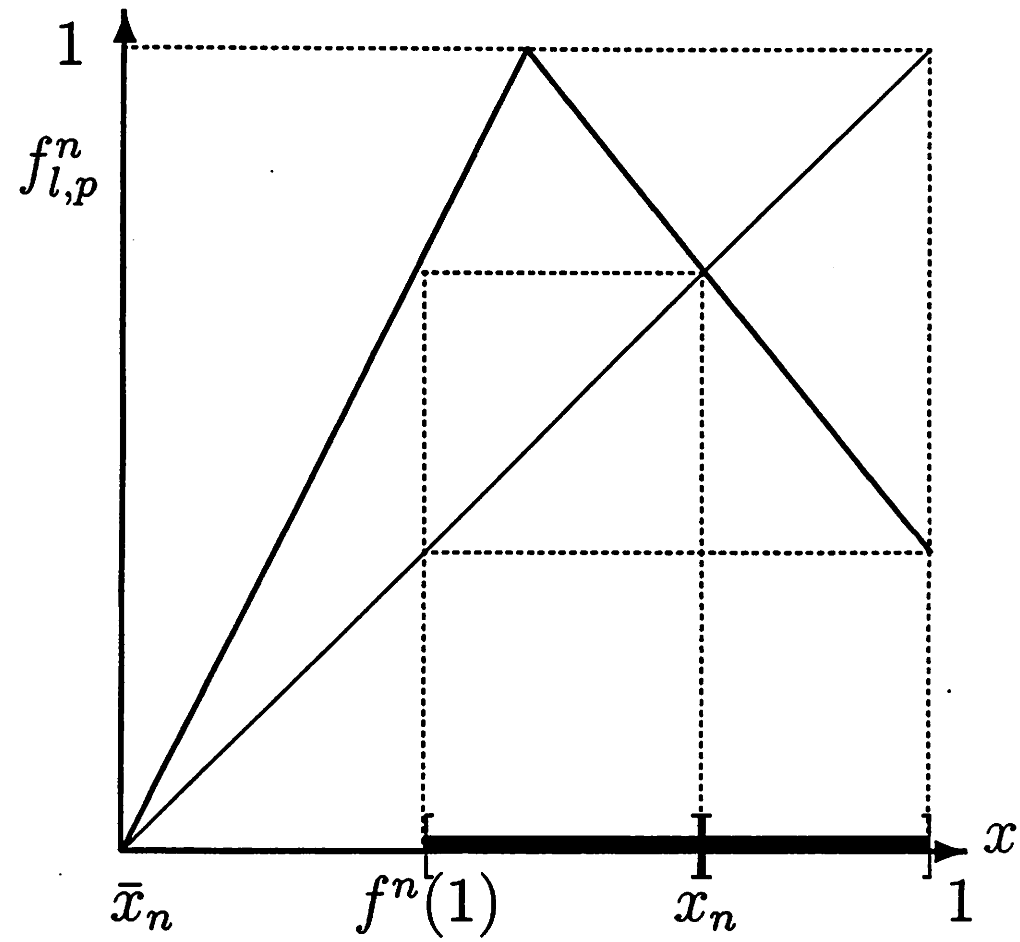


Fig. 6

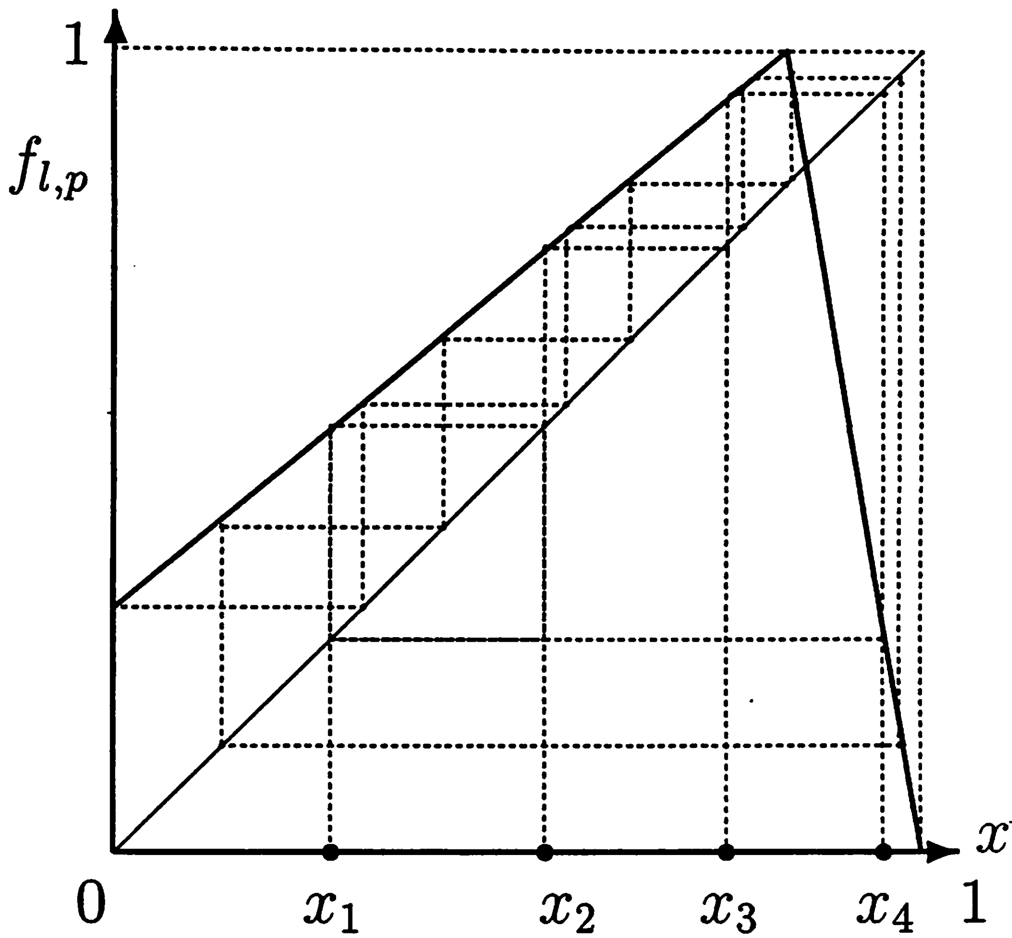


Fig. 7

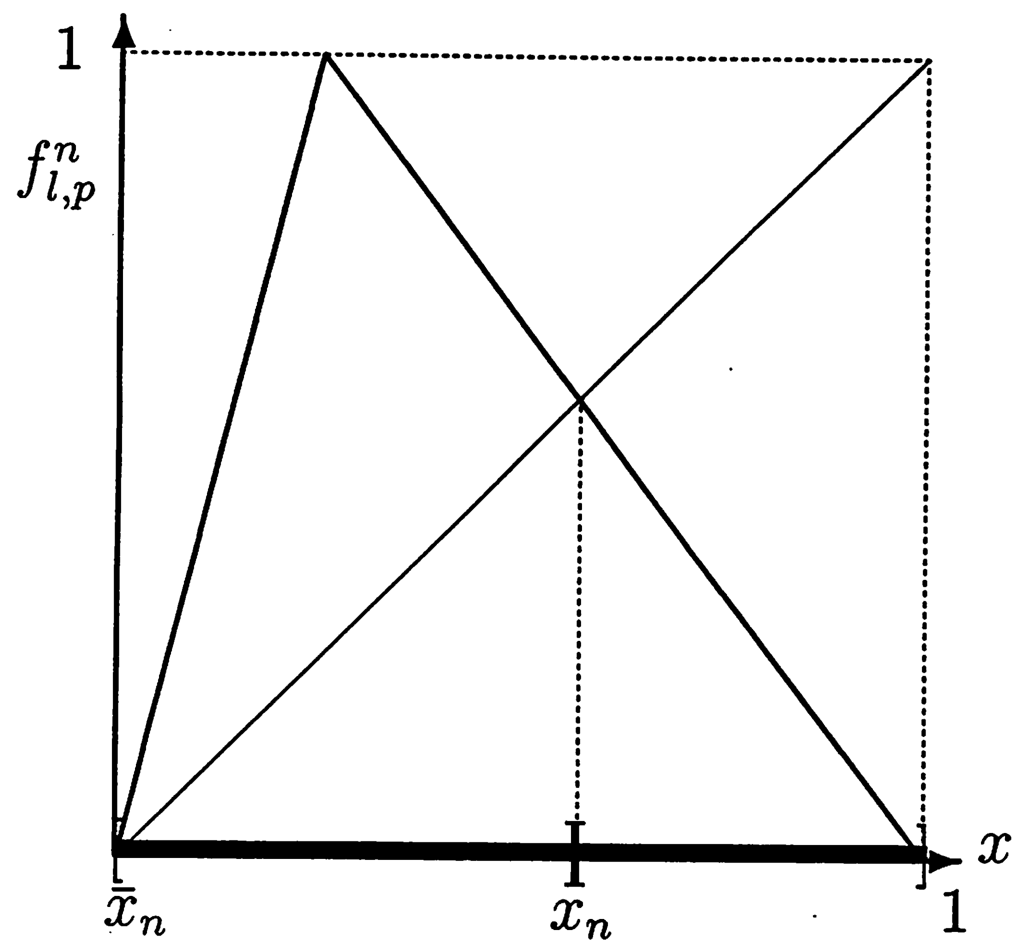


Fig. 8

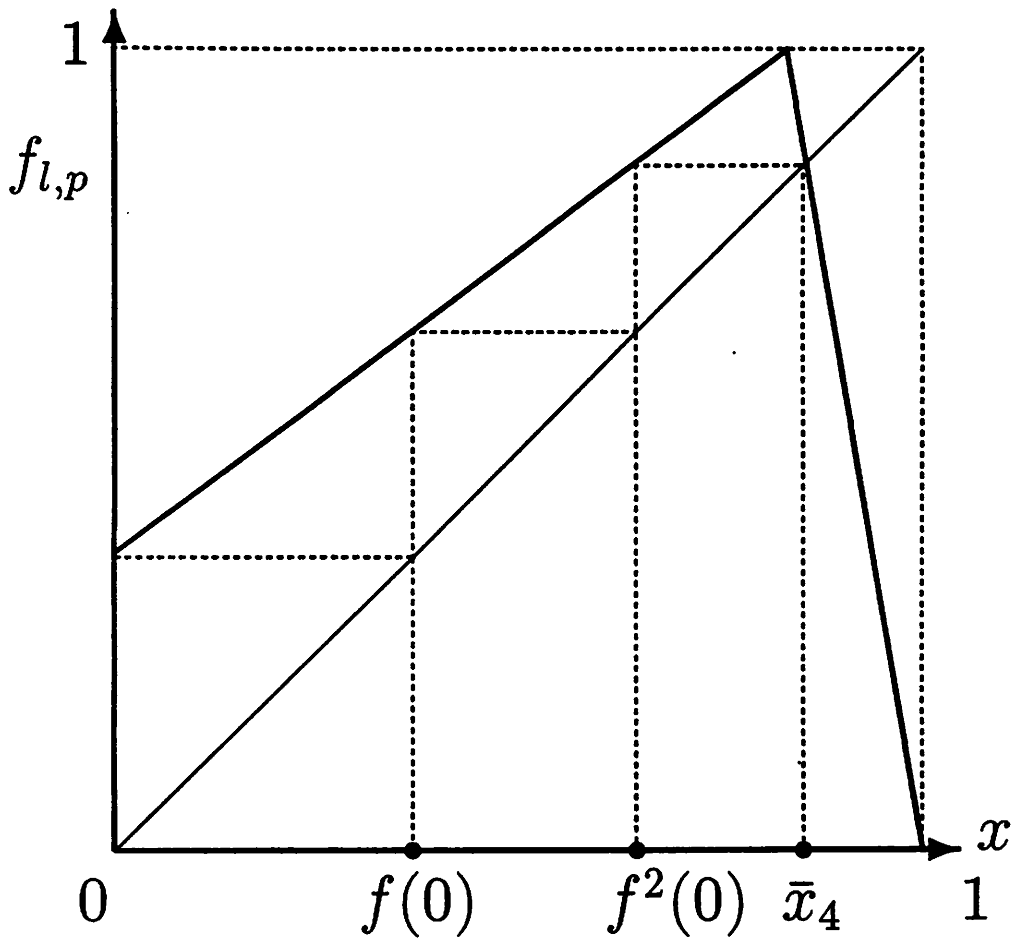
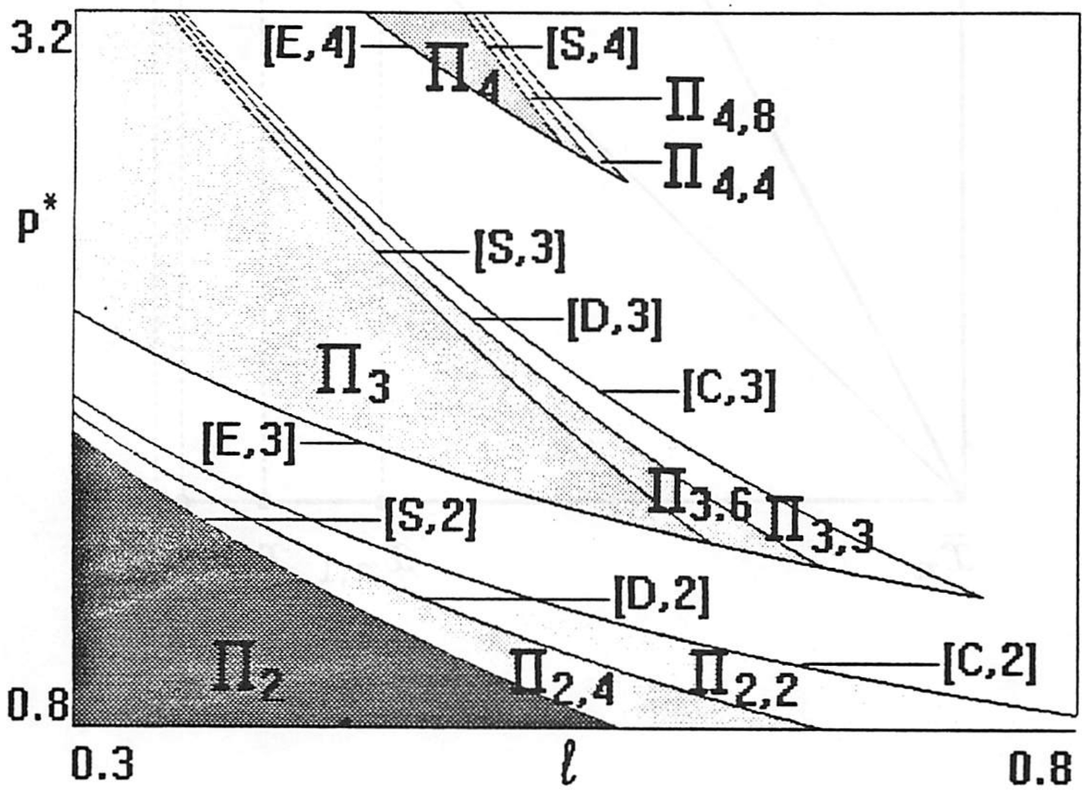


Fig. 9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p*
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ℓ
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[S,4]
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Fig. 10

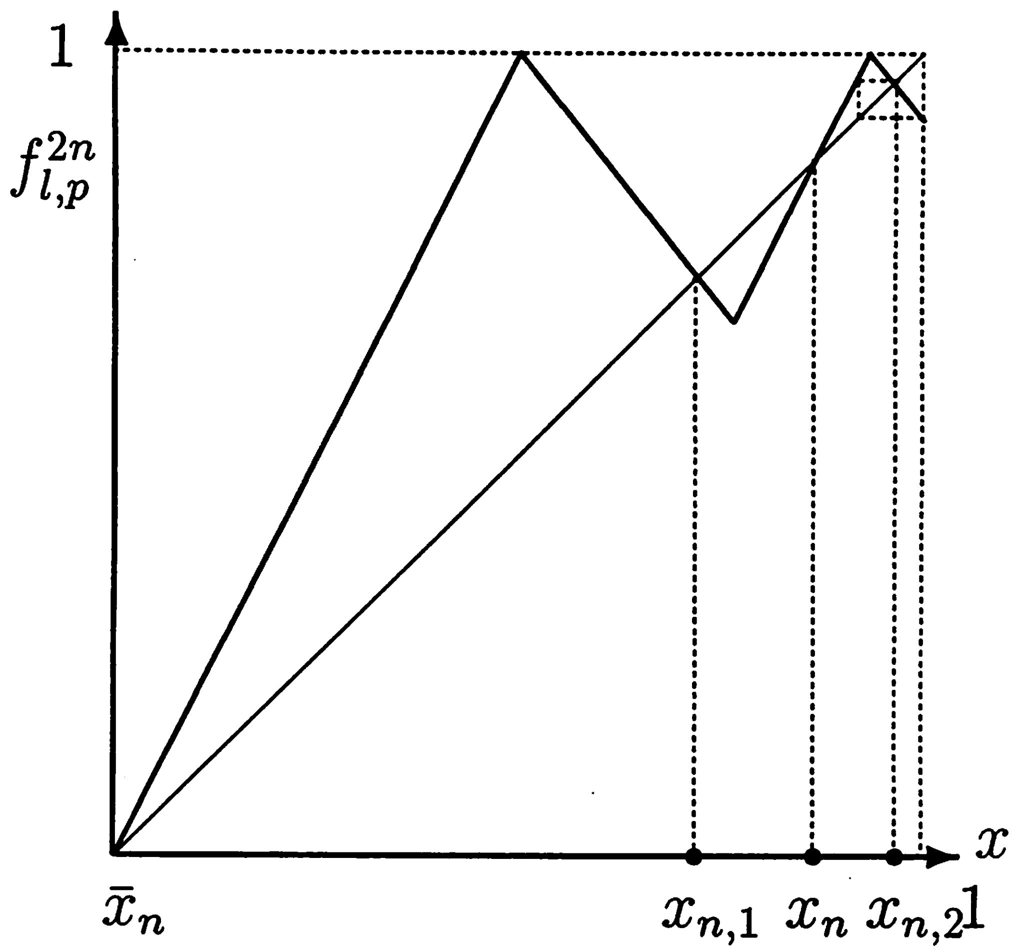

Fig. 11

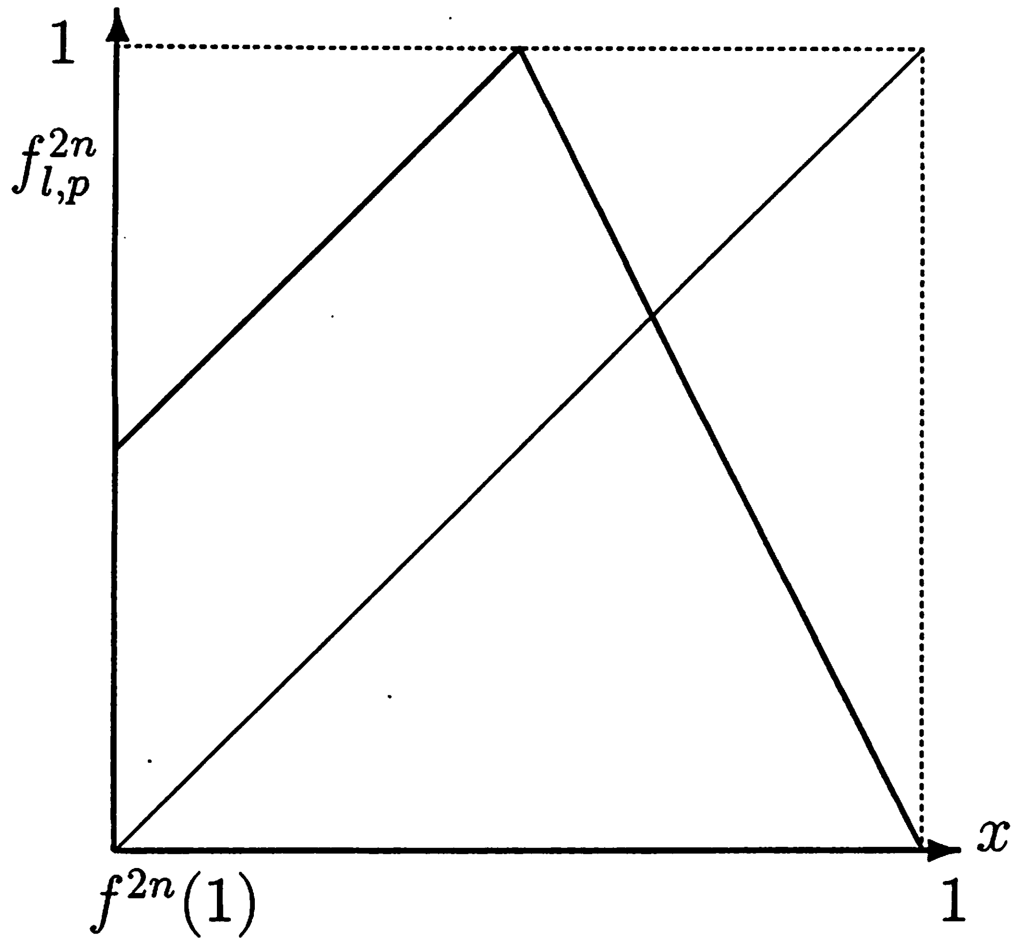

Fig. 12

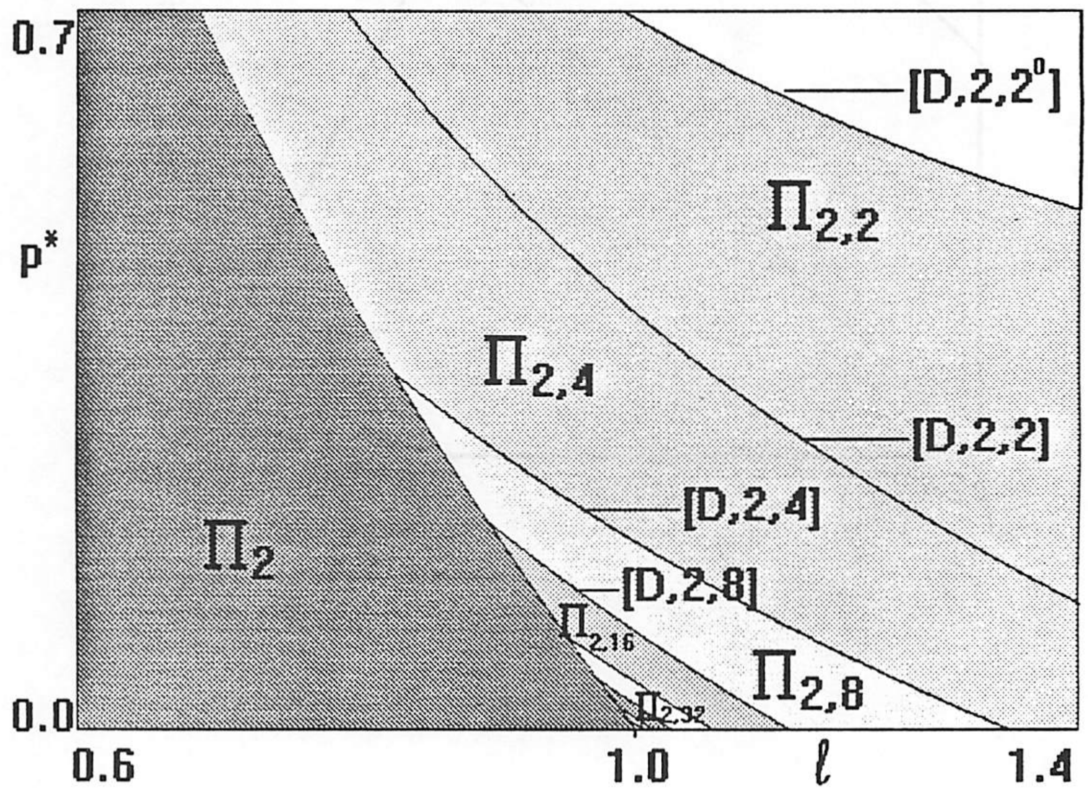

Fig. 13

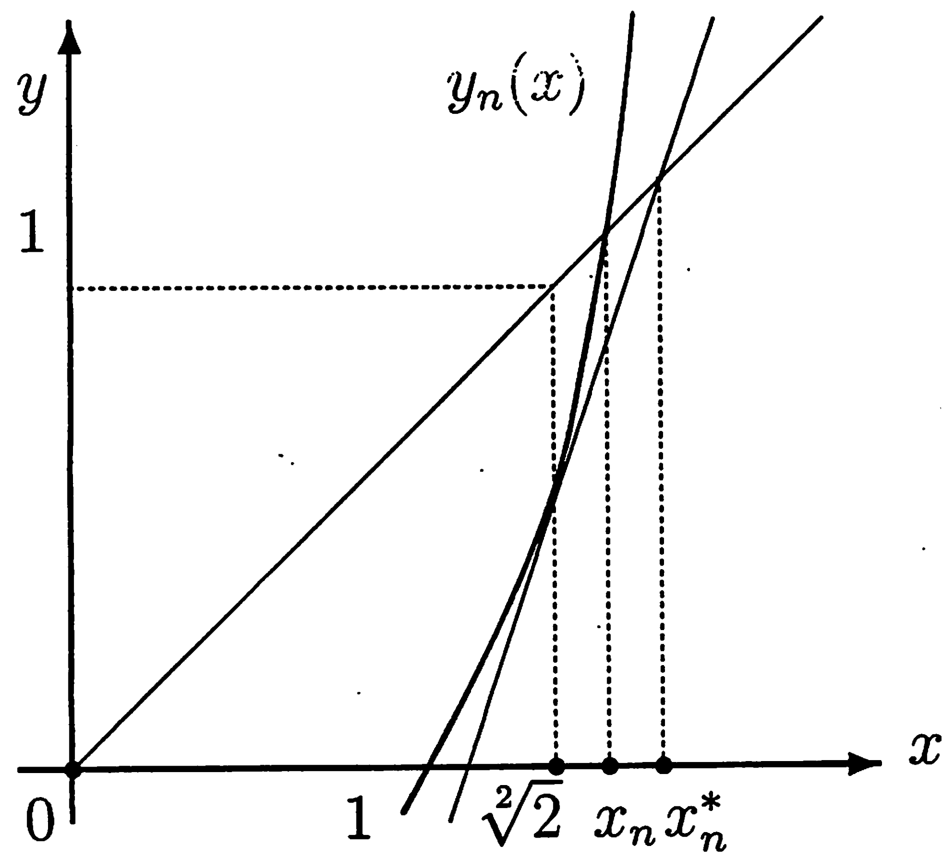

Fig. 14

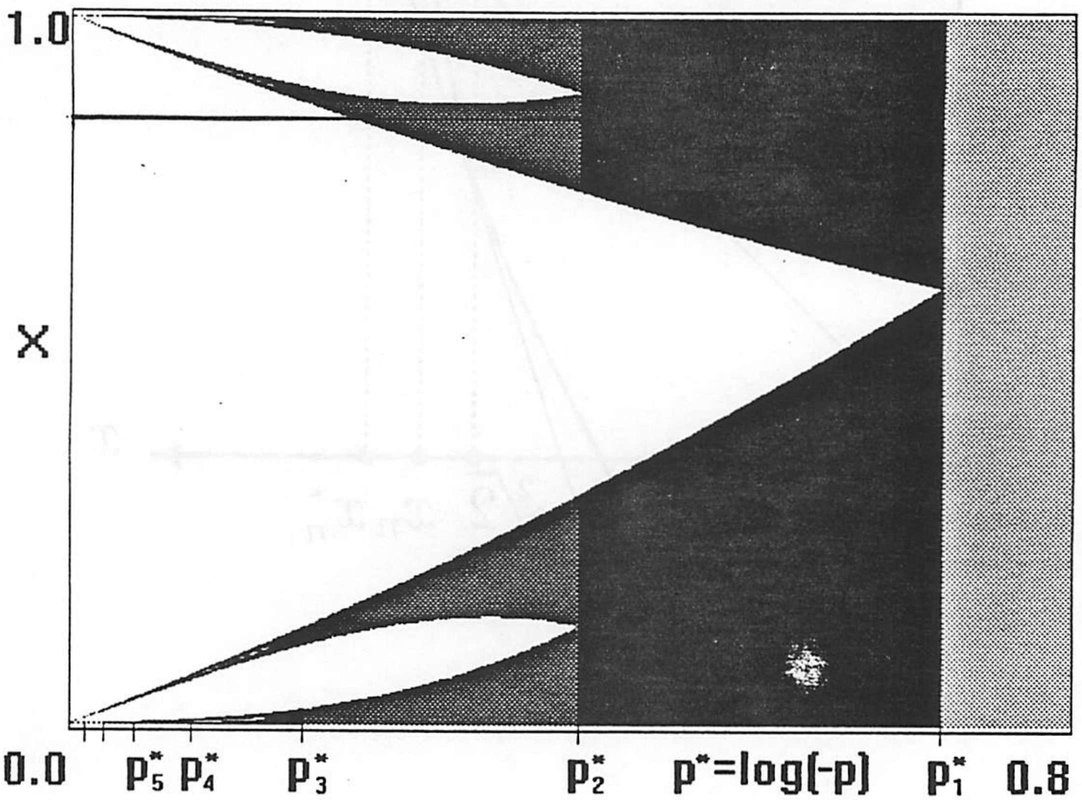

Fig. 15(a)

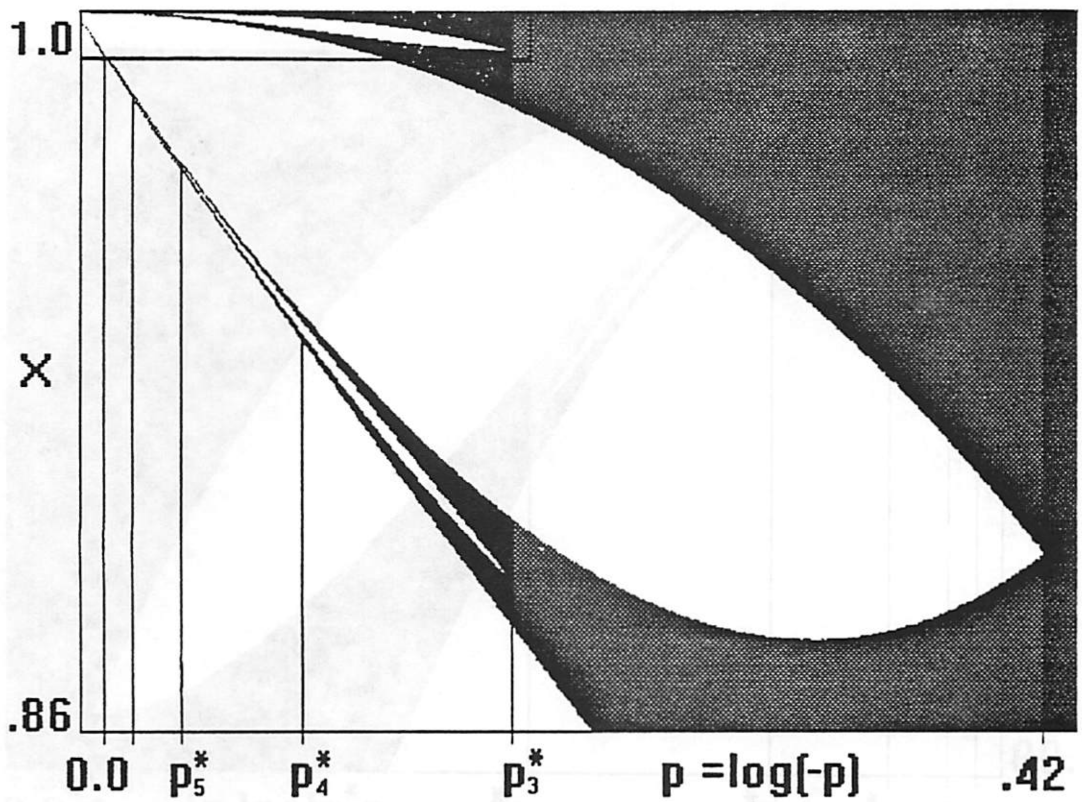

Fig. 15(b)

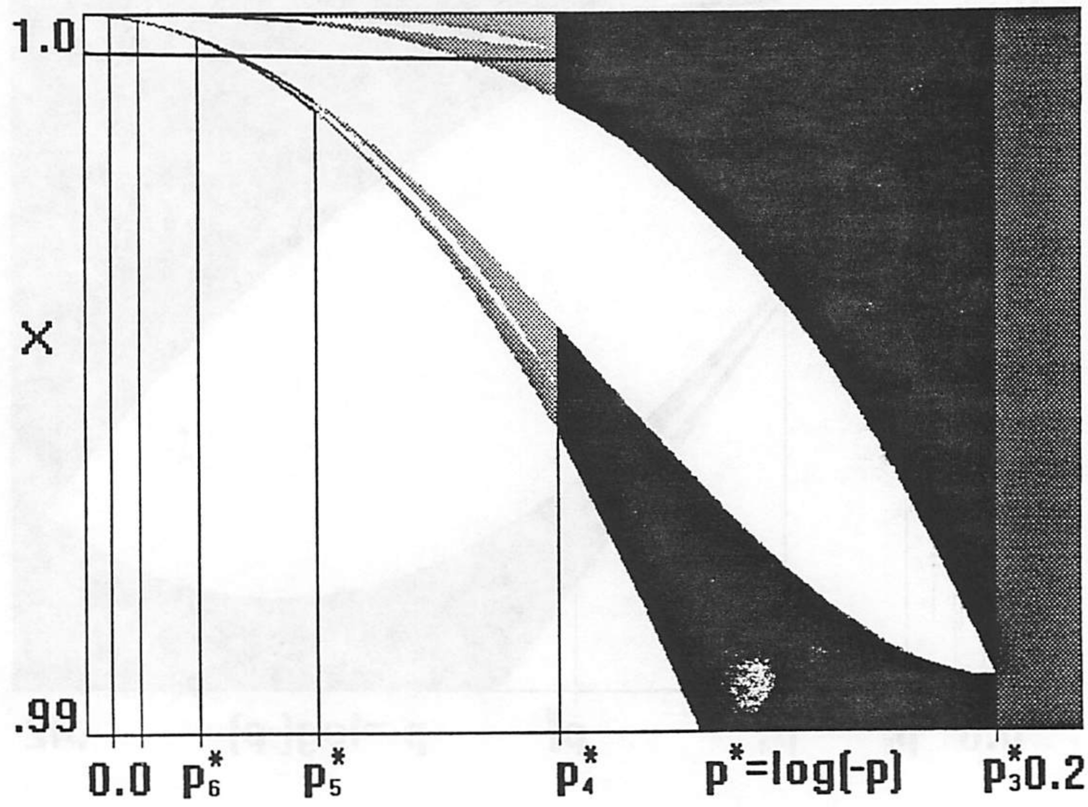

Fig. 15(c)

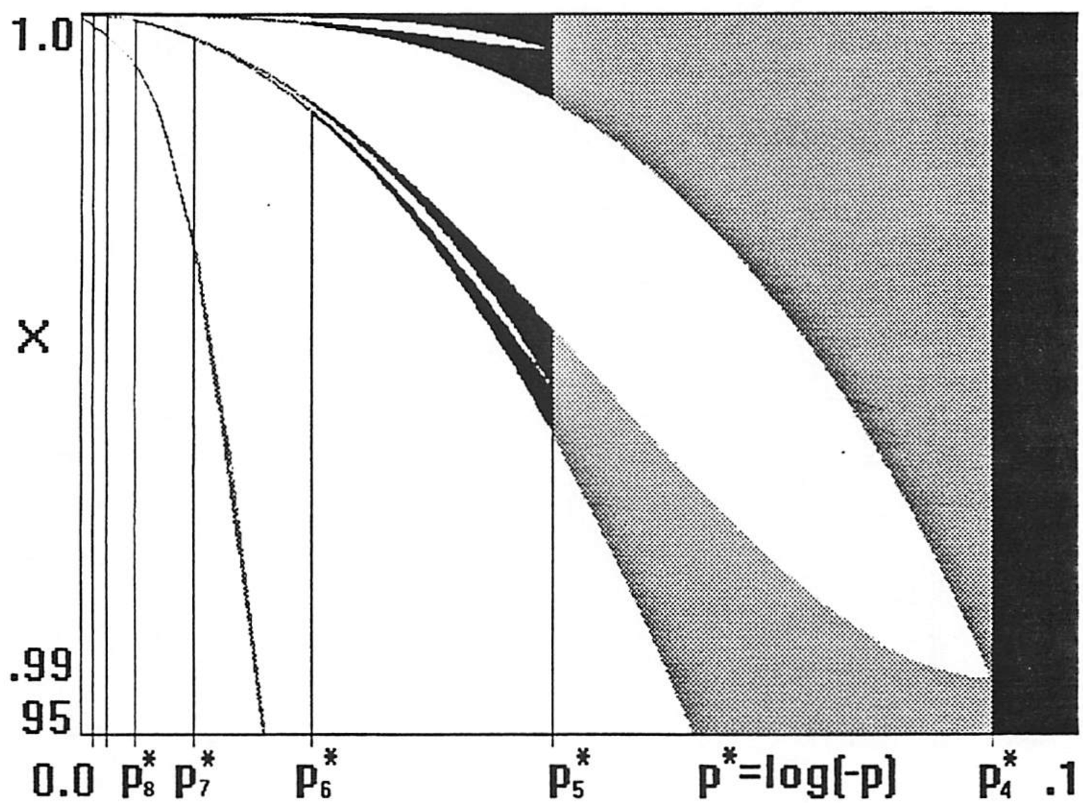

Fig. 15(d)