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**ON THE GLOBAL DEGREE OF NONHOLONOMY
OF A CAR WITH n TRAILERS**

by

O. J. Sjørdalen

Memorandum No. UCB/ERL M93/39

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Abstract

An upper bound of the global degree of nonholonomy is found for a car with n trailers. This bound grows exponentially as a function of n similarly to the Fibonacci numbers. The bound is hence lower than previous upper bounds. Controllability is also shown for the kinematic model considered.

1 Introduction

A car with n trailers is a nonholonomic system due to the rolling constraints of the wheels. The configuration of the system is given by two position coordinates and $n + 1$ angles, whereas there are only two inputs, namely one tangential velocity and one angular velocity. Thus, the system has two degrees of freedom. The study of a car with n trailers has attracted much attention recently and has involved tools from nonlinear control theory and differential geometry. An important concept for such systems is the degree of nonholonomy which expresses the level of Lie-bracketing needed to span the tangent space at each configuration. This degree thus expresses how “controllable” the system is.

A kinematic model for a car with n trailers was presented by [3]. Controllability for this model was proven and the (global) degree of nonholonomy was shown to be bounded by 2^{n+1} . In [6], a coordinate transformation and a feedback transformation of the inputs were proposed which converted locally the kinematics to a chained form. The conversion was local in all the orientations but global in the position. Since the degree of nonholonomy of a $n + 3$ -dimensional chained form is $n + 2$, this conversion showed that the degree of nonholonomy of a car with n trailers is $n + 2$ when there are no right angles between the trailers. In [7], another set of coordinates were used to convert the system to a chained

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form. This conversion which was global also in the orientation of the last trailer, showed that the degree of nonholonomy was independent of the orientation of the last trailer.

In this paper, an exponential bound lower than 2^{n+1} is found for the global degree of nonholonomy of a car with n trailers.

The paper is organized as follows: Mathematical preliminaries are presented in Section 2. The kinematic model of a car with n trailers is presented in Section 3 and controllability of this system is shown in Section 4. Bounds on the degree of nonholonomy are given in Section 5.

2 Mathematical Preliminaries

In this section some definitions and results which are useful in the analysis of the degree of nonholonomy will be presented. Definitions 1–5 are taken from [5] and [2].

Definition 1 *Let f be a smooth vector field on a smooth manifold M and λ a smooth real-valued function on M . The Lie derivative of λ along f is a function $L_f \lambda : M \rightarrow \mathbb{R}$ defined as*

$$(L_f \lambda)(p) = (f(p))(\lambda)$$

Definition 2 *For f and g any (smooth) vector fields on M , we define a new vector field, denoted as $[f, g]$ and called the Lie bracket of f and g by setting*

$$([f, g](p))(\lambda) = (L_f L_g \lambda)(p) - (L_g L_f \lambda)(p)$$

where λ is a smooth function on M .

$[f, g]$ is skew commutative, bilinear over \mathbb{R} and satisfies the Jacobi identity. If the two vector fields f and g both are defined on an open subset U of \mathbb{R}^n , then

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

at each x in U . Let $V^\infty(M)$ denote the set of vector fields on a manifold M considered as a module over the ring $C^\infty(M)$ of C^∞ real valued functions on M . $V^\infty(M)$ with the product $[f, g]$ is a Lie algebra.

Consider the nonlinear driftless system

$$\dot{x} = \sum_{j=1}^m g_j(x) u_j \tag{1}$$

where $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ are local coordinates for a smooth state space manifold M , and $u = [u_1, \dots, u_m]^T \in U \subset \mathbb{R}^m$. The vector fields g_1, \dots, g_m are smooth on M .

Definition 3 *System (1) is controllable if*

$$\forall x_1, x_2 \in M \exists T < \infty \exists u : [0, T] \rightarrow U \mid x(T, 0, x_1, u) = x_2$$

Definition 4 *The Control Lie Algebra \mathcal{L} for (1) is the smallest subalgebra of $V^\infty(M)$ that contains g_1, \dots, g_m .*

The vector fields which are elements in \mathcal{L} span the accessibility distribution.

Definition 5 *The accessibility distribution L of (1) is given by*

$$L(x) = \text{span} \{v(x) \mid v \in \mathcal{L}\}, \quad x \in M$$

L is the involutive closure of $\Delta \triangleq \text{span}\{g_1, \dots, g_m\}$.

Theorem 1 *Consider System (1). Assume that the state space manifold M is connected and that*

$$\dim L(x) = n, \quad \forall x \in M \quad (2)$$

Then the system is controllable.

Proof: See [5] p. 83.

□

A stronger concept than controllability is given by the following definition, [4],

Definition 6 *System (1) is well-controllable if there exists a basis of n vector fields in the accessibility distribution L such that the determinant of the basis is constant for all $x \in M$.*

Let the distribution Δ be given by $\Delta(x) \triangleq \text{span}\{g_1(x), \dots, g_m(x)\}$ for $x \in M$. For every point $x \in M$ we construct a chain

$$\Delta(x) = \Delta_1(x) \subset \Delta_2(x) \subset \dots \quad (3)$$

of linear spaces in a tangent space $T_x M$ defining $\Delta_i(x)$ as a linear envelope of all the values of vector fields that can be represented by Lie brackets of length $\leq i$ of admissible vector fields, [1]. This means that

$$\Delta_2 = \Delta_1 + [\Delta_1, \Delta_1], \dots, \Delta_i = \Delta_{i-1} + [\Delta_{i-1}, \Delta_1] \quad (4)$$

We can show that if u and v are vector fields such that $u \in \Delta_i$ and $v \in \Delta_j$ then

$$\begin{aligned} f(x)u &\in \Delta_i, \quad f(x) \in C^\infty \\ [u, v] &\in \Delta_{i+j} \\ u + v &\in \Delta_{\max\{i, j\}} \end{aligned}$$

Motivated by this, we introduce the length of a vector field as follows:

Definition 7 Let u, v , and w be vector fields in L generated by the distribution Δ and let $f(x)$ and $g(x)$ be smooth functions. The length of a vector field $l(u)$ is defined such that

$$l(u) = 1, \quad u \in \Delta \quad (5)$$

$$l(u) = l(v) + l(w), \quad u = [f(x)v, g(x)w] \quad (6)$$

$$l(u) = \max\{l(v), l(w)\}, \quad u = f(x)v + g(x)w \quad (7)$$

We see from this definition that $l(u)$ depends on how u is generated from Δ . This length function has the property that

$$l(u) = k \Rightarrow u \in \Delta_k$$

where k is a constant.

By a growth vector of a distribution Δ at a point x we mean a sequence of integers $\{n_i(x)\}$, where $n_i(x) = \dim D_i(x)$. The distribution is regular if

$$\forall i \forall x \in M \exists k_i \quad n_i(x) = k_i$$

where k_i is a constant.

The distribution Δ (3) is **completely nonholonomic** if for some i_0 , $\Delta_i = TM$ for all $i \geq i_0$ where TM is the tangent bundle. The degree of nonholonomy is defined as follows, [1],

Definition 8 Let Δ (3) be a completely nonholonomic distribution. The degree of nonholonomy of Δ , $d(\Delta)$, is

$$d(\Delta) = \min\{i_0 \mid \forall i \geq i_0, \Delta_i = TM\}$$

where TM is the tangent bundle.

Note that the manifold M considered can be restricted to a small neighborhood around each configuration. In general, the degree of nonholonomy changes with the manifold considered. The **global degree of nonholonomy** is the degree when M is the whole configuration space, i.e. the maximum degree of nonholonomy over all configurations, if it exists. Although the degree of nonholonomy is well-defined at each configuration, the existence of a global maximum is not guaranteed. In this paper, an upper bound for the global degree of nonholonomy is derived.

The following lemma follows readily from Definitions 7 and 8:

Lemma 1 Assume that there exist a constant k and n vector fields v_1, \dots, v_n such that for all $i \in \{1, \dots, n\}$, $l(v_i) \leq k$ and

$$\forall x \in M \mid \text{span}\{v_1(x), \dots, v_n(x)\} = T_x M$$

where $T_x M$ is the tangent space at the configuration x . Then the degree of nonholonomy $d(\Delta) \leq k$.

3 Kinematic Model

A car in this context is represented by two driving wheels connected by an axle. A kinematic model of a car with two degrees of freedom pulling n trailers is here given by

$$\begin{aligned}\dot{\theta}_0 &= \omega \\ \dot{\theta}_i &= \frac{1}{L_i} \sin(\theta_{i-1} - \theta_i) v_{i-1} \quad i = 1, \dots, n \\ \dot{y} &= \sin \theta_n v_n \\ \dot{x} &= \cos \theta_n v_n\end{aligned} \quad (8)$$

where (x, y) is the absolute position of the center of the axle between the two wheels of the last trailer.

θ_i is the orientation angle of trailer i with respect to the x -axis, with $i \in \{1, \dots, n\}$. θ_0 is the orientation angle of the pulling car with respect to the x -axis.

L_i is the distance from the wheels of trailer i to the wheels of trailer $i - 1$, where $i \in \{2, \dots, n\}$. L_1 is then the distance from the wheels of trailer 1 to the wheels of the car.

We denote

$$\begin{aligned}\alpha_i &= \theta_i - \theta_{i+1}, \quad i \in \{0, \dots, n-1\} \\ \alpha_n &= \theta_n\end{aligned} \quad (9)$$

v_0 is the tangential velocity of the car and is an input to the system. The other input is the angular velocity of the car, ω . The tangential velocity of trailer i , v_i , is given by

$$v_i = \cos(\theta_{i-1} - \theta_i) v_{i-1} = \prod_{j=0}^{i-1} \cos \alpha_j \quad v_0 = C_0^i(\Theta_i) v_0 \quad (10)$$

where $i \in \{1, \dots, n\}$ and

$$\begin{aligned}C_0^i(\Theta_i) &\triangleq \prod_{j=0}^{i-1} \cos \alpha_j \\ \Theta_i &\triangleq [\theta_0, \dots, \theta_i]^T\end{aligned}$$

The two input vector fields are then given by

$$\mathbf{V}_0 = [0, \frac{1}{L_1} \sin \alpha_0, \frac{1}{L_2} \sin \alpha_1 C_0^1, \dots, \frac{1}{L_n} \sin \alpha_{n-1} C_0^{n-1}, \sin \alpha_n C_0^n, \cos \alpha_n C_0^n]^T \quad (11)$$

$$\mathbf{\Omega}_0 = [1, 0, \dots, 0]^T \quad (12)$$

An illustration of the system is presented in *Fig. 1*.

4 Controllability

Laumond proved that a car with n trailers is controllable where the kinematic equations were given in terms of $\alpha_i = \theta_i - \theta_{i+1}$ and (x_0, y_0) where (x_0, y_0) is the position of the pulling car, [3]. In this section controllability will be shown along the same lines for the kinematic model (8) where also general distances L_i between the trailers are included.

We introduce the following vector fields

$$Y_0 = [\Omega_0, V_0] \quad (13)$$

$$V_{i+1} = \cos \alpha_i V_i - \sin \alpha_i Y_i \quad (14)$$

$$\Omega_{i+1} = (\sin \alpha_i V_i + \cos \alpha_i Y_i) L_{i+1} \quad (15)$$

$$Y_{i+1} = [\Omega_{i+1}, V_{i+1}] \quad (16)$$

where $i \in \{0, \dots, n-1\}$.

Theorem 2 *Let the vector fields V_i , Ω_i , and Y_i be iteratively defined for $i \in \{0, \dots, n\}$ by (11)–(12) and (13)–(16). These vector fields have then the following structure*

$$V_i = [\mathbf{o}_{i+1}, \mathbf{v}_i^n]^T \quad (17)$$

$$\Omega_i = [\mathbf{o}_i, 1, \mathbf{o}_{n-i+2}]^T \quad (18)$$

$$Y_i = [\mathbf{o}_{i+1}, \mathbf{y}_i^n]^T \quad (19)$$

where

$$\mathbf{o}_i \triangleq [0, 0, \dots, 0], \quad \dim \mathbf{o}_i = i \quad (20)$$

$$\mathbf{v}_i^n \triangleq \left[\frac{1}{L_{i+1}} \sin \alpha_i, \cos \alpha_i \mathbf{v}_{i+1}^n \right], \quad \mathbf{v}_{n+1}^n \triangleq 1, \quad L_{n+1} \triangleq 1 \quad (21)$$

$$\mathbf{y}_i^n \triangleq \left[\frac{1}{L_{i+1}} \cos \alpha_i, -\sin \alpha_i \mathbf{v}_{i+1}^n \right] \quad (22)$$

for $i \in \{0, \dots, n\}$.

Proof: The proof will be given by induction. Assume that the vector fields V_i , Ω_i , and Y_i are given by (17)–(19) for an $i \in \{0, \dots, n-1\}$. We find from (14) and (17)–(19) that

$$\begin{aligned} V_{i+1} &= \cos \alpha_i V_i - \sin \alpha_i Y_i \\ &= [\mathbf{o}_{i+1}, \cos \alpha_i \mathbf{v}_i^n - \sin \alpha_i \mathbf{y}_i^n]^T \\ &= [\mathbf{o}_{i+1}, \frac{1}{L_{i+1}} (\cos \alpha_i \sin \alpha_i - \sin \alpha_i \cos \alpha_i), \cos^2 \alpha_i \mathbf{v}_{i+1}^n + \sin^2 \alpha_i \mathbf{v}_{i+1}^n]^T \\ &= [\mathbf{o}_{i+2}, \mathbf{v}_{i+1}^n]^T \end{aligned} \quad (23)$$

$$(24)$$

$$\Omega_{i+1} = (\sin \alpha_i V_i + \cos \alpha_i Y_i) L_{i+1} \quad (25)$$

$$= (\sin \alpha_i [\mathbf{o}_{i+1}, \mathbf{v}_i^n]^T + \cos \alpha_i [\mathbf{o}_{i+1}, \mathbf{y}_i^n]^T) L_{i+1} \quad (26)$$

$$= [\mathbf{o}_{i+1}, 1, \mathbf{o}_{n-i}]^T \quad (27)$$

$$(28)$$

$$\mathbf{Y}_{i+1} = [\boldsymbol{\Omega}_{i+1}, \mathbf{V}_{i+1}] = \frac{\partial \mathbf{V}_{i+1}}{\partial q} \boldsymbol{\Omega}_{i+1} - \frac{\partial \boldsymbol{\Omega}_{i+1}}{\partial q} \mathbf{V}_{i+1} = \frac{\partial \mathbf{V}_{i+1}}{\partial \theta_{i+1}}$$

where $q \triangleq [\theta_0, \theta_1, \dots, \theta_n, x, y]^T$. We find from (9), (23), and (21)

$$\begin{aligned} \mathbf{Y}_{i+1} &= \frac{\partial \mathbf{V}_{i+1}}{\partial \alpha_{i+1}} \frac{\partial \alpha_{i+1}}{\partial \theta_{i+1}} + \frac{\partial \mathbf{V}_{i+1}}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial \theta_{i+1}} \\ &= \frac{\partial \mathbf{V}_{i+1}}{\partial \alpha_{i+1}} - \frac{\partial \mathbf{V}_{i+1}}{\partial \alpha_i} \\ &= [\mathbf{o}_{i+2}, \frac{1}{L_{i+2}} \cos \alpha_{i+1}, -\sin \alpha_{i+1} \mathbf{v}_{i+2}^n]^T \\ &= [\mathbf{o}_{i+2}, \mathbf{y}_{i+1}^n]^T \end{aligned}$$

We find from (11) and (12) that

$$\mathbf{Y}_0 = [\boldsymbol{\Omega}_0, \mathbf{V}_0] = \frac{\partial \mathbf{V}_0}{\partial \theta_0} = [0, \mathbf{y}_0^n]^T$$

Hence, (19) is satisfied for $i = 0$. The proof is then completed by noting from (11) and (12) that (17) and (18) are satisfied for $i = 0$.

□

We see from this theorem that the vector fields \mathbf{V}_i and $\boldsymbol{\Omega}_i$ as defined by (13)–(16) have the same structure with respect to the sub-trailer system consisting of trailers i through n as the input vector fields \mathbf{V}_0 and $\boldsymbol{\Omega}_0$ have with respect to the complete system. Therefore, using the inputs to e.g. generate a motion in $\boldsymbol{\Omega}_i$ direction makes trailer i turn.

Laumond has already shown that a car with n trailers is controllable, [3]. The following theorem states the same result for a kinematic model which is given by Eq. (8).

Theorem 3 *The kinematic model of a car with n trailers as given by (8)–(12) is controllable.*

Proof: From (17)–(19) and (21)–(22) we have

$$\det [\boldsymbol{\Omega}_0, \dots, \boldsymbol{\Omega}_n, \mathbf{Y}_n, \mathbf{V}_n] = \frac{1}{L_{i+1}} (\cos^2 \alpha_n + \sin^2 \alpha_n) = \frac{1}{L_{i+1}} > 0$$

Therefore,

$$\text{span} \{ \boldsymbol{\Omega}_0, \dots, \boldsymbol{\Omega}_n, \mathbf{Y}_n, \mathbf{V}_n \} = \mathbb{R}^{n+3}, \quad \forall q \in \mathbb{R}^{n+3}$$

From the construction of $\boldsymbol{\Omega}_i$, \mathbf{Y}_n , and \mathbf{V}_n , Eqs. (13)–(16), it follows that the system is controllable according to Definition 1.

□

Remark: Since $\det [\boldsymbol{\Omega}_0, \dots, \boldsymbol{\Omega}_n, \mathbf{Y}_n, \mathbf{V}_n] = \frac{1}{L_{i+1}}$ everywhere the kinematic model (8)–(12) is well-controllable, Definition 6, as also shown by Laumond, [3].

5 Degree of Nonholonomy

In this section we will find an upper bound on the degree of nonholonomy of a car with n trailers.

Let the Control Lie Algebra \mathcal{L} be generated by the input vectors Ω_0 and V_0 . Let the distribution Δ be given by $\Delta \triangleq \text{span}\{\Omega_0, V_0\}$. We note that with V_i , Ω_i , and Y_i as given by (13)–(16), we have

$$l(Y_i) = l(\Omega_i) + l(V_i)$$

where $l(\cdot)$ is defined in Definition 7. Therefore,

$$l(Y_i) > l(\Omega_i), \quad l(Y_i) > l(V_i) \quad (29)$$

Since $l(\Omega_i) = \max\{l(Y_{i-1}), l(V_{i-1})\} = l(Y_{i-1})$ then

$$l(\Omega_i) > l(\Omega_{i-1})$$

Lemma 1 then implies that the degree of nonholonomy is bounded by

$$d(\Delta) \leq \max\{l(\Omega_0), \dots, l(\Omega_n), l(Y_n), l(V_n)\} = l(Y_n) \quad (30)$$

From (5)–(7), (13)–(16) and (29) we get

$$\begin{aligned} l(Y_i) &= l(\Omega_i) + l(V_i) \\ &= \max\{l(V_{i-1}), l(Y_{i-1})\} + \max\{l(V_{i-1}), l(Y_{i-1})\} = 2l(Y_{i-1}) \end{aligned}$$

Since $l(Y_0) = 2$, $l(Y_i) = 2^{i+1}$. Eq. (30) then gives the following upper bound on the degree of nonholonomy:

$$d(\Delta) \leq 2^{n+1}$$

This result was first found by Laumond, [3]. In the following this upper bound will be reduced.

The following lemma will be useful to find another expression for Y_i , (19), involving lower degree of Lie bracketing.

Lemma 2 *Given the C^∞ vector fields $F(x)$, $G(x)$, and the C^∞ functions $f(x)$, $g(x)$. Then*

$$[fF, gG] = f(L_G g)F - g(L_F f)G + fg[F, G]$$

Proof: This is found from Definition 2 by direct calculation. □

Theorem 4 *Let the vector field Y_{i+1} be given by (14)–(16). Then this vector field can be expressed as follows:*

$$Y_{i+1} = L_{i+1}[Y_i, V_i] + \sin \alpha_i V_i + \cos \alpha_i Y_i \quad (31)$$

Proof: The vector field \mathbf{Y}_{i+1} is defined by Eq. (16):

$$\mathbf{Y}_{i+1} = [\mathcal{N}_{i+1}, \mathbf{V}_{i+1}]$$

The definitions of \mathbf{V}_{i+1} and \mathcal{N}_{i+1} , (14) and (15), give

$$\begin{aligned} \mathbf{Y}_{i+1} &= L_{i+1}[\sin \alpha_i \mathbf{V}_i + \cos \alpha_i \mathbf{Y}_i, \cos \alpha_i \mathbf{V}_i - \sin \alpha_i \mathbf{Y}_i] \\ &= L_{i+1}[\sin \alpha_i \mathbf{V}_i, \cos \alpha_i \mathbf{V}_i] - L_{i+1}[\sin \alpha_i \mathbf{V}_i, \sin \alpha_i \mathbf{Y}_i] + \\ &\quad L_{i+1}[\cos \alpha_i \mathbf{Y}_i, \cos \alpha_i \mathbf{V}_i] - L_{i+1}[\cos \alpha_i \mathbf{Y}_i, \sin \alpha_i \mathbf{Y}_i] \end{aligned} \quad (32)$$

We note from (9) and (1) that

$$L_{\mathbf{V}_i} \sin \alpha_i = [\mathbf{o}_i, -\sin \alpha_i, \sin \alpha_i, \mathbf{o}_{n-i+1}] \mathbf{V}_i$$

Using Theorem 2 and Lemma 2 then gives

$$\begin{aligned} [\sin \alpha_i \mathbf{V}_i, \cos \alpha_i \mathbf{V}_i] &= \sin \alpha_i([\mathbf{o}_i, -\sin \alpha_i, \sin \alpha_i, \mathbf{o}_{n-i+1}][\mathbf{o}_{i+1}, \mathbf{v}_i^n]^T) \mathbf{V}_i - \\ &\quad \cos \alpha_i([\mathbf{o}_i, \cos \alpha_i, -\cos \alpha_i, \mathbf{o}_{n-i+1}][\mathbf{o}_{i+1}, \mathbf{v}_i^n]^T) \mathbf{V}_i \\ &= \frac{1}{L_{i+1}} \sin \alpha_i \mathbf{V}_i \end{aligned} \quad (33)$$

$$\begin{aligned} [\sin \alpha_i \mathbf{Y}_i, \sin \alpha_i \mathbf{V}_i] &= \sin \alpha_i([\mathbf{o}_i, \cos \alpha_i, -\cos \alpha_i, \mathbf{o}_{n-i+1}][\mathbf{o}_{i+1}, \mathbf{v}_i^n]^T) \mathbf{Y}_i - \\ &\quad \sin \alpha_i([\mathbf{o}_i, \cos \alpha_i, -\cos \alpha_i, \mathbf{o}_{n-i+1}][\mathbf{o}_{i+1}, \mathbf{y}_i^n]^T) \mathbf{V}_i - \\ &\quad [\mathbf{V}_i, \mathbf{Y}_i] \sin^2 \alpha_i \\ &= \frac{1}{L_{i+1}}(\sin \alpha_i \cos^2 \alpha_i \mathbf{V}_i - \sin^2 \alpha_i \cos \alpha_i \mathbf{Y}_i) + \sin^2 \alpha_i [\mathbf{Y}_i, \mathbf{V}_i] \end{aligned} \quad (34)$$

$$\begin{aligned} [\cos \alpha_i \mathbf{Y}_i, \cos \alpha_i \mathbf{V}_i] &= \cos \alpha_i([\mathbf{o}_i, -\sin \alpha_i, \sin \alpha_i, \mathbf{o}_{n-i+1}][\mathbf{o}_{i+1}, \mathbf{v}_i^n]^T) \mathbf{Y}_i - \\ &\quad \cos \alpha_i([\mathbf{o}_i, -\sin \alpha_i, \sin \alpha_i, \mathbf{o}_{n-i+1}][\mathbf{o}_{i+1}, \mathbf{y}_i^n]^T) \mathbf{V}_i + \\ &\quad [\mathbf{Y}_i, \mathbf{V}_i] \cos^2 \alpha_i \\ &= \frac{1}{L_{i+1}}(-\sin \alpha_i \cos^2 \alpha_i \mathbf{V}_i + \sin^2 \alpha_i \cos \alpha_i \mathbf{Y}_i) + \cos^2 \alpha_i [\mathbf{Y}_i, \mathbf{V}_i] \end{aligned} \quad (35)$$

$$\begin{aligned} [\sin \alpha_i \mathbf{Y}_i, \cos \alpha_i \mathbf{Y}_i] &= \sin \alpha_i([\mathbf{o}_i, -\sin \alpha_i, \sin \alpha_i, \mathbf{o}_{n-i+1}][\mathbf{o}_{i+1}, \mathbf{y}_i^n]^T) \mathbf{Y}_i - \\ &\quad \cos \alpha_i([\mathbf{o}_i, \cos \alpha_i, -\cos \alpha_i, \mathbf{o}_{n-i+1}][\mathbf{o}_{i+1}, \mathbf{y}_i^n]^T) \mathbf{Y}_i \\ &= \frac{1}{L_{i+1}} \cos \alpha_i \mathbf{Y}_i \end{aligned} \quad (36)$$

Eqs. (32) and (33)–(36) imply

$$\mathbf{Y}_{i+1} = L_{i+1}[\mathbf{Y}_i, \mathbf{V}_i] + \sin \alpha_i \mathbf{V}_i + \cos \alpha_i \mathbf{Y}_i$$

□

From Eq. (15) we see that this expression for Y_{i+1} can alternatively be written

$$Y_{i+1} = L_{i+1}[Y_i, V_i] + \frac{1}{L_{i+1}}\Omega_{i+1}$$

This theorem says that the vector field Y_{i+1} can be given by (31) as an alternative to (16). This alternative expression for Y_{i+1} reduces the levels of Lie bracketing and leads to the following theorem:

Theorem 5 *The degree of nonholonomy $d(\Delta)$ for System (8) with input vectors V_0 and Ω_0 is bounded by*

$$d(\Delta) \leq F(n+3) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+3} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+3} \right]$$

where $F(i)$ is the i^{th} Fibonacci number, i.e.

$$F(0) = 0, \quad F(1) = 1, \quad F(i+2) = F(i+1) + F(i), \quad i \geq 0$$

Proof:

Eq. (31) implies that

$$l(Y_{i+1}) = \max\{l(Y_i) + l(V_i), l(V_i), l(Y_i)\} = l(Y_i) + l(V_i) \quad (37)$$

Since

$$l(V_{i+1}) = l(\Omega_{i+1}) = \max\{l(V_i), l(Y_i)\} \quad (38)$$

then

$$l(Y_{i+1}) > l(V_{i+1}), \quad l(Y_{i+1}) > l(\Omega_{i+1}) \quad (39)$$

Eqs. (38)–(39) imply that $l(\Omega_{i+1}) > l(\Omega_i)$. Therefore,

$$\max\{l(\Omega_0), \dots, l(\Omega_n), l(Y_n), l(V_n)\} = l(Y_n)$$

The degree of nonholonomy is hence bounded by

$$d(\Delta) \leq l(Y_n) \quad (40)$$

Eqs. (38) and (39) imply that

$$l(V_{i+1}) = l(Y_i)$$

From Eq. (37) we then have

$$l(Y_{i+2}) = l(Y_{i+1}) + l(Y_i) \quad (41)$$

From the definition of Y_0 (13) we have that $l(Y_0) = 2$. Since $l(V_0) = 1$, Eq. (37) implies that $l(Y_1) = 3$. From Eq. (41) we then see that

$$l(Y_i) = F(i+3), \quad i \in \{0, \dots, n\}$$

From Eq. (40) we can then conclude

$$d(\Delta) \leq F(n+3)$$

where the explicit expression for $F(n+3)$ is a standard result.

□

6 Conclusions

In this paper we have shown that the global degree of nonholonomy of a car with n trailers is bounded by the $n + 3$ rd Fibonacci number. This means that the worst case bound grows exponentially. The rate is however lower than for previous bounds. The exact degree of nonholonomy at each configuration will in general be lower than this upper bound. The research on the exact degree of nonholonomy at each configuration is actually in progress. This analysis reveals a certain structure of the Lie products generated by the input vectors which can contribute to the understanding of the control problem of such systems.

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References

- [1] V. Gershkovich and A. Vershik. Nonholonomic manifolds and nilpotent analysis. *Journal of Geometry and Physics*, 5(3):407–452, 1988.
- [2] A. Isidori. *Nonlinear Control Systems*. Springer-Verlag, 1989.
- [3] J.-P. Laumond. Controllability of a multibody mobile robot. In *ICAR'91*, Pisa, Italy, June 1991.
- [4] J.-P. Laumond. Singularities and topological aspects in nonholonomic motion planning. In Z. X. Li and J. Canny, editors, *Progress in Nonholonomic Motion Planning*. Kluwer Academic Publisher, 1992.
- [5] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer-Verlag, 1990.
- [6] O. J. Sørдалen. Conversion of the kinematics of a car with n trailers into a chained form. In *Proc. 1993 IEEE International Conference on Robotics and Automation*, Atlanta, Georgia, May 1993. IEEE.
- [7] D. Tilbury, R. Murray, and S. Sastry. Trajectory generation for the n -trailer problem using goursat normal form. Technical Report UCB/ERL M93/12, University of California, Berkeley, February 1993.

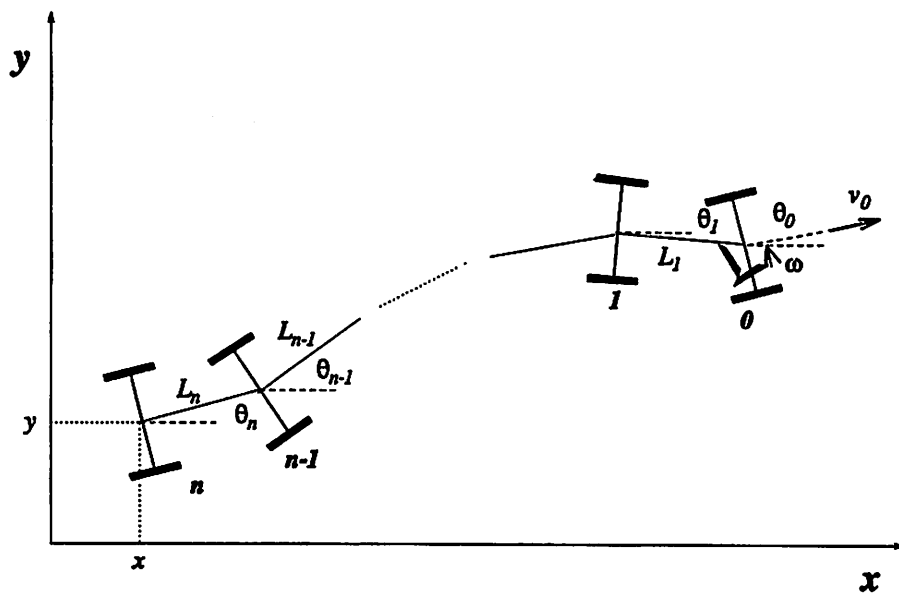


Figure 1: Model of a car with n trailers.