Copyright © 1993, by the author(s).
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

STABILIZATION OF MULTIPLE INPUT CHAINED FORM CONTROL SYSTEMS

by

G. C. Walsh and L. G. Bushnell

Memorandum No. UCB/ERL M93/43

7 June 1993

STABILIZATION OF MULTIPLE INPUT CHAINED FORM CONTROL SYSTEMS

by

G. C. Walsh and L. G. Bushnell

Memorandum No. UCB/ERL M93/43

7 June 1993

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Stabilization of Multiple Input

Chained Form Control Systems[1]

G. C. Walsh and L.G. Bushnell

Intelligent Machines and Robotics Laboratory

University of California

Berkeley, CA 94720

June 7, 1993

[1]Research supported in part by NSF under grant IRI90-14490

Abstract

This paper gives a control law for stabilizing multiple input chained form control systems. This extends an earlier result [10] on stabilizing the above class of systems which have two inputs. Along the way, we construct a transformation from general chained form systems with multiple generators to a power form. A control law which stabilizes the origin of a three-input control system that models the kinematics of a fire truck is simulated, confirming the theoretical results.

Chapter 1

Introduction

We are interested in stabilizing control laws for control systems of the form

$$\dot{\xi} = g_0(\xi)u_0 + g_1(\xi)u_1 + \cdots + g_m(\xi)u_m \tag{1.1}$$

arising from systems with nonholonomic, or non-integrable, linear velocity constraints where ξ in a open set $U \subset \mathbf{R}^n$ is the state of the system, $u(\xi, t) \in \mathbf{R}$ are the inputs and g_i are smooth linearly independent vector fields.

A special case of the two-input problem was presented in [4]. In previous work [10] we proposed a control law motivated by [8, 11] which stabilized the control system to a point in the state space for a special class of two-input nonholonomic systems that could be transformed, by a coordinate transformation and state feedback, to a two-input single-chain single-generator chained form:

$$
\begin{aligned}
\dot{x}_0^0 &= v_0 \\
\dot{x}_1^0 &= v_1 \\
\dot{x}_{10}^1 &= x_1^0 v_0 \\
\dot{x}_{10}^2 &= x_{10}^1 v_0 \\
&\vdots \\
\dot{x}_{10}^{n-2} &= x_{10}^{n-3} v_0 \,.
\end{aligned}
$$

It was then shown that there exists a coordinate transformation to the power form

$$
\begin{aligned}
\dot{y}_0 &= v_0 \\
\dot{y}_1 &= v_1 \\
\dot{z}_{10}^1 &= y_0 v_1 \\
\dot{z}_{10}^2 &= \frac{1}{2}(y_0)^2 v_1 \\
&\vdots \\
\dot{z}_{10}^{n-2} &= \frac{1}{(n-2)!}(y_0)^{n-2} v_1
\end{aligned}
$$

whose origin was stabilized. This paper is motivated by the desire to extend this work to the general (m+1)-input, m-chain, single-generator case.

The extended problem can be stated as follows: given a nonlinear control system of the form in equation (1.1), find a control law $u(\xi, t)$ which makes the origin globally asymptotically stable.

The major contributions of this paper are the presentation of the transformation from a (m+1)-input, m(m+1)-chain, (m+1)-generator chained form system into a power form and the proof of a stabilizing control law that makes the origin of a (m+1)-input, m-chain, single-generator power form control system globally asymptotically stable.

The outline of this paper is as follows: in Section 2, the transformations to chained form and power form are given for a (m+1)-input, m-chain, single-generator case along with the power form transformation from a general chained form system. In Section 3, a stabilizing control law is given for (m+1)-input, m-chain, single-generator control systems in power form. In Section 4, a three-input example, the fire truck, is used to illustrate the theoretic results. In Section 5, conclusions are made.

Chapter 2

Transformation to Power Form

We are interested in stabilizing the origin of control systems represented in power form. To transform the original system in equation (1.1) into power form, we present a two-step algorithm. The first step is to transform the (m+1)-input original system into a m-chain single-generator chained form as described in [2]. The second step is to transform this chained form system into a power form. We choose not to prove the one-step transformation from the original system to power form since the first transformation exists and is proven in the literature and the second transformation is straight forward once the system is in chained form. We also give the transformation from a system in general chained form that has m+1 inputs, m(m+1) chains and m+1 generators to power form.

We suggest Chapter 1 of [6] for reference on distributions and Lie brackets.

2.1 Original System to Chained Form

We now present sufficient conditions under which the (m+1)-input system in equation (1.1) may be transformed into a m-chain, single-generator chained form.

The general idea is to construct m+2 distributions using the vector fields $g_0, g_1, \ldots, g_m$ and various Lie brackets. We then check the conditions of full rank on the distribution that contains all of the vector fields

$$\begin{gathered} g_0, g_1, \ldots, g_m, \\ \mathrm{ad}_{g_0} g_1, \mathrm{ad}_{g_0} g_2, \ldots, \mathrm{ad}_{g_0} g_m, \\ \mathrm{ad}_{g_0}^2 g_1, \mathrm{ad}_{g_0}^2 g_2, \ldots, \mathrm{ad}_{g_0}^2 g_m, \ldots, \\ \mathrm{ad}_{g_0}^{n_1} g_1, \mathrm{ad}_{g_0}^{n_2} g_2, \ldots, \mathrm{ad}_{g_0}^{n_m} g_m \end{gathered}$$

where $\mathrm{ad}_{g_0}^k g_i$ is the k^{th}-order Lie bracket $[g_0, [g_0, [\ldots k\ times \ldots [g_0, g_i]]]]$. The m+1 other distributions, Δ_i, defined as in the proposition below need to be checked for involutivity, i.e., $f, g \in \Delta_i \Rightarrow [f, g] \in \Delta_i$. The following proposition is proven in [2].

Proposition 1 [2]**Converting (m+1)-input Systems to m-Chain, Single-Generator Chained Form**

Given the system

$$\dot{\xi} = g_0(\xi)u_0 + \cdots + g_m(\xi)u_m ,$$

with $\xi \in U \subset \mathbf{R}^n$, $u_i(\xi, t) \in \mathbf{R}$ and the g_i smooth linearly independent vector fields. Construct the following distributions:

$$\begin{aligned}
\Delta_0 &:= \mathrm{span}\{g_0, \mathrm{ad}_{g_0}^{r_j} g_j;\ r_j = 0, \ldots, n_j;\ j = 1, \ldots, m\} \\
\Delta_1 &:= \mathrm{span}\{\mathrm{ad}_{g_0}^{r_j} g_j;\ r_j = 0, \ldots, n_j;\ j = 1, \ldots, m\} \\
\Delta_2 &:= \mathrm{span}\{\mathrm{ad}_{g_0}^{r_j} g_j;\ r_1 = 0, \ldots, n_1 - 1;\ r_j = 0, \ldots, n_j\ \ j = 2, \ldots, m\} \\
\Delta_3 &:= \mathrm{span}\{\mathrm{ad}_{g_0}^{r_j} g_j;\ r_1 = 0, \ldots, n_1 - 1;\ r_2 = 0, \ldots, n_2 - 1;\ r_j = 0, \ldots, n_j\ \ j = 3, \ldots, m\} \\
&\ \ \vdots \\
\Delta_{m+1} &:= \mathrm{span}\{\mathrm{ad}_{g_0}^{r_j} g_j;\ r_j = 0, \ldots, n_j - 1;\ j = 1, \ldots, m\}
\end{aligned}$$

where $\sum_{j=1}^{m} n_j + m + 1 = n$.

If first we have that $\Delta_0(\xi) = \mathbf{R}^n$ *and second* Δ_j *for* $1 \leq j \leq m+1$ *are involutive, then there exists a local feedback transformation*

$$\begin{aligned} x &= \Phi(\xi) \\ u &= \beta(\xi)v \end{aligned}$$

such that system is in m-chain, single-generator chained form:

$$\begin{array}{llll} \dot{x}_0^0 = v_0 & & & \\ \dot{x}_1^0 = v_1 & \dot{x}_2^0 = v_2 & \cdots & \dot{x}_m^0 = v_m \\ \dot{x}_{10}^1 = x_1^0 v_0 & \dot{x}_{20}^1 = x_2^0 v_0 & \cdots & \dot{x}_{m0}^1 = x_m^0 v_0 \\ \vdots & \vdots & \ddots & \vdots \\ \dot{x}_{10}^{n_1} = x_{10}^{n_1-1} v_0 & \dot{x}_{20}^{n_2} = x_{20}^{n_2-1} v_0 & \cdots & \dot{x}_{m0}^{n_m} = x_{m0}^{n_m-1} v_0 \,. \end{array}$$

The notation uses the convention that x_{ji}^k is the state for the k^{th} level of chain j using state i as the generator. The state x_j^0 is the top of chain j.

2.2 Chained Form to Power Form

Given a general chained form system with m+1 inputs, m(m+1) chains and m+1 generators:

$$\begin{aligned}
\dot{x}_j^0 &= v_j \quad 0 \le j \le m \\
\dot{x}_{ji}^1 &= x_j^0 v_i \quad j > i \ \textit{and}\ x_{ij}^1 := x_i^0 x_j^0 - x_{ji}^1 \\
\dot{x}_{ji}^k &= x_{ji}^{k-1} v_i \quad 1 \le k \le n_j,\ \ 0 \le j, i \le m;\ j \ne i\ ,
\end{aligned}$$

the global transformation to power form is given as follows:

$$\begin{aligned}
y_j &= x_j^0 \quad 0 \le j \le m \\
z_{ji}^k &= (-1)^k x_{ji}^k + \sum_{n=0}^{k-1} (-1)^n \frac{1}{(k-n)!} \left(x_i^0\right)^{k-n} x_{ji}^n \\
& \quad 1 \le k \le n_j,\ \ 1 \le j \le m,\ \ 0 \le i \le m\ . \qquad (2.1)
\end{aligned}$$

We identify $x_{ji}^0 = x_j^0$ as the tops of the chains. Using this transformation, the general chained form equation is converted to the power form dynamics:

$$\begin{aligned}
\dot{y}_j &= v_j \quad 0 \le j \le m \\
\dot{z}_{ji}^k &= \frac{1}{k!} (y_i)^k v_j \quad 1 \le k \le n_j,\ \ 1 \le j \le m,\ \ 0 \le i \le m\ . \qquad (2.2)
\end{aligned}$$

In this paper, we will be using the single-generator version of the power form. To find this form, we use the following transformation on the single-generator chained form system

$$
\begin{aligned}
y_j &= x_j^0 \quad 0 \le j \le m \\
z_{j0}^k &= (-1)^k x_{j0}^k + \sum_{n=0}^{k-1} (-1)^n \frac{1}{(k-n)!} \left(x_0^0\right)^{k-n} x_{j0}^n \quad 1 \le k \le n_j, \;\; 1 \le j
\end{aligned}
\tag{2.3}
$$

which yields the single-generator power form:

$$
\begin{aligned}
\dot{y}_j &= v_j \quad 0 \le j \le m \\
\dot{z}_{j0}^k &= \frac{1}{k!} (y_0)^k v_j \quad 1 \le k \le n_j, \;\; 1 \le j \le m \,.
\end{aligned}
\tag{2.4}
$$

Chapter 3

Stabilization

We generalize the control law given in [10] to the multiple input case. The simplicity of the law is bought at the expense of the complexity of the proof.

Proposition 2 *Given an $(m+1)$-input, m-chain, single-generator control system whose evolution in time is given by:*

$$\begin{aligned} \dot{y}_j &= v_j \text{ for } 0 \leq j \leq m \\ \dot{z}_{j0}^k &= \frac{1}{k!}(y_0)^k v_j \text{ for } 1 \leq k \leq n_j \text{ and for } 0 \leq j \leq m, \end{aligned} \tag{3.1}$$

The origin $(y, z) = 0$ is locally asymptotically stable under the action of the controls:

$$
\begin{aligned}
v_0 &= -y_0 + \rho(z)\,(\cos(t) - \sin(t)) \\
v_j &= -y_j + \sum_{h=1}^{n_j} c_j^h z_{j0}^h \cos(ht) \text{ for } 1 \le j \le m
\end{aligned}
\tag{3.2}
$$

where $\rho(z) = \sum_{j=1}^{m} \sum_{k=1}^{n_j} \left(z_{j0}^k\right)^2$ and each $c_j^h < 0$.

Proof: The outline of the proof is as follows: first we find an approximation to the resulting time–varying center manifold and then we examine the dynamics on this reduced space. After an averaging transformation on this reduced space we may utilize the direct method of Lyapunov to conclude that the origin of the system under the inputs (3.2) is locally asymptotically stable. We will try to follow closely [10] whenever possible.

We will now solve for the closed–loop dynamics of the system (2.4). The dynamics under the controls (3.2) are described by (for $1 \le j \le m$):

$$
\begin{aligned}
\dot{y}_0 &= -y_0 + \rho(z)\,(\cos(t) - \sin(t)) \\
\dot{y}_j &= -y_j + \sum_{h=1}^{n_j} c_j^h z_{j0}^h \cos(ht) \\
\dot{z}_{h0}^k &= \frac{1}{k!}\,(y_0)^k \left(-y_j + \sum_{h=1}^{n_j} c_j^h z_{j0}^h \cos(ht)\right) \text{ for } 1 \le k \le n_j
\end{aligned}
\tag{3.3}
$$

Notice that the time–varying terms in (3.3) are linear in the state; therefore we cannot directly apply center manifold theory. In order to apply this theory, we will first transform the system into one which has higher–order time–varying terms. We start with the coordinate y_0.

$$\tilde{y}_0 = y_0 - \rho(z)\cos(t)$$

Now we solve for the dynamics of $\tilde{y}_0$:

$$\begin{aligned}
\dot{\tilde{y}}_0 &= -y_0 + \rho(z)(\cos(t) - \sin(t)) + \rho(z)\sin(t) - \left(\frac{d}{dt}\rho(z)\right)\cos(t) \\
&= -\left(y_0 - \rho(z)\cos(t)\right) - \left(\frac{d}{dt}\rho(z)\right)\cos(t) \\
&= -\tilde{y}_0 - g_0(y,z,t)
\end{aligned}$$

with the function $g_0(y,z,t)$ being

$$\begin{aligned}
g_0(y,z,t) &= \left(\frac{d}{dt}\rho(z)\right)\cos(t) \\
&= 2\sum_{j=1}^{m}\sum_{k=1}^{n_j} z_{j0}^k \frac{1}{k!}(y_0)^k \left(-y_j + \sum_{h=1}^{n_j} c_j^h z_{j0}^h \cos(ht)\right)\cos(t)\ .
\end{aligned}$$

Note that $g_0(y,z,t)$ is at least cubic in the state (y,z). This fact will aid us in estimating the time–varying center manifold. Now we will transform the other y_j coordinates.

$$\tilde{y}_j = y_j + \sum_{h=1}^{n_j} z_{j0}^h \left(\frac{-hc_j^h}{1+h^2} \sin(ht) + \frac{-c_j^h}{1+h^2} \cos(ht) \right) \text{ for } 1 \le j \le m$$

Once again, we solve for the dynamics of $\tilde{y}_j$:

$$\begin{aligned}
\dot{\tilde{y}}_j &= -y_j + \sum_{h=1}^{n_j} c_j^h z_{j0}^h \cos(ht) + \\
&\quad \sum_{h=1}^{n_j} z_{j0}^h \left(h \frac{-hc_j^h}{1+h^2} \cos(ht) - h \frac{-c_j^h}{1+h^2} \sin(ht) \right) + g_j(y,z,t) \\
&= -\left(y_j + \sum_{h=1}^{n_j} z_{j0}^h \left(\frac{-hc_j^h}{1+h^2} \sin(ht) - \left(h \frac{-hc_j^h}{1+h^2} + c_j^h \right) \cos(ht) \right) \right) + g_j(y,z,t) \\
&= -\left(y_j + \sum_{h=1}^{n_j} z_{j0}^h \left(\frac{-hc_j^h}{1+h^2} \sin(ht) + \frac{-c_j^h}{1+h^2} \cos(ht) \right) \right) + g_j(y,z,t) \\
&= -\tilde{y} + g_j(y,z,t)
\end{aligned}$$

where the term $g_j(y,z,t)$ is given by

$$g_j(y,z,t) = \sum_{h=1}^{n_j} \frac{1}{h!} (y_0)^h \left(-y_j + \sum_{i=1}^{n_j} c_j^i z_{j0}^i \cos(it) \right) \left(\frac{-hc_j^h}{1+h^2} \sin(ht) + \frac{-c_j^h}{1+h^2} \cos(ht) \right) . \tag{3.4}$$

Again note that $g_j(y,z,t)$ is at least quadratic in the state (y,z). All of the $g_j(y,z,t)$ may be rewritten in terms of $\tilde{y}$, with no corresponding loss of order. Call these functions $\tilde{g}_j(\tilde{y}, z, t)$.

We will now introduce a generalization of the center manifold theorem which appears in [10]. The following statement uses the notation of [3] so that $f'(0,0,w)$ refers to the partial derivative of f with respect to all variables and evaluated at $(y,z,w) = (0,0,w)$.

Lemma 1 ("Time–varying" Center Manifold) *Consider the system:*

$$\begin{aligned} \dot{y} &= By + g(y,z,w) \\ \dot{z} &= Az + f(y,z,w) \\ \dot{w} &= Sw \end{aligned} \tag{3.5}$$

with $y \in \mathbf{R}^{m+1}$, $z \in \mathbf{R}^{n-m-1}$, and $w \in \mathbf{R}^p$ and where the eigenvalues of A and S have zero real part. The functions f, g and h are C^2 with $f(0,0,w) = 0$, $f'(0,0,w) = 0$, $g(0,0,w) = 0$ and $g'(0,0,w) = 0$. Then, given $M > 0$, there exists a center manifold for (3.5), $y = h(z,w)$ for $\|w\| < M$, $\|z\| < \delta(M)$, for some $\delta > 0$ and dependent on M, where $h(z,w)$ is C^2 and $h(0,w) = 0$ and $h'(0,w) = 0$.

To apply this result to our system, create the vector w, with $w_1^j = \cos(jt)$ and $w_2^j = \sin(jt)$. Note that the dimension of w, given by p, is then twice the maximum n_j for $1 \leq j \leq m$. The differential equation describing the evolution of this vector may be written as $\dot{w} = Sw$, with the eigenvalues of S having zero real part. We can then substitute an element of the vector w for each of the time–varying terms in the equations. In this form, we may apply the "Time-varying" Center Manifold theorem. The time–varying center manifold is then given by (with the time dependencies resubstituted):

$$\begin{aligned} \tilde{y}_0 &= h_0(z,t) \\ \tilde{y}_j &= h_j(z,t) \text{ for } 1 \leq j \leq m \; . \end{aligned}$$

Since $g_0(y,z,t)$ is at least cubic in the state (y,x) it may be verified that by calculation that $h_0''(0,t) = 0$. We will check the stability of $z = 0$ by solving for the dynamics of z on the center manifold. It is useful to note that

$$\begin{aligned} v_0 &= -y_0 + \rho(z)(\cos(t) - \sin(t)) \\ &= -\tilde{y}_0 - \rho(z)\sin(t) \; . \end{aligned}$$

And for the cases $1 \leq j \leq m$, we find

$$\begin{aligned} v_j &= -y_j + \sum_{h=1}^{n_j} c_j^h z_{j0}^h \cos(ht) \\ &= -\tilde{y}_j + \sum_{h=1}^{n_j} z_{j0}^h \left(\frac{-hc_j^h}{1+h^2} \sin(ht) + \left(\frac{-c_j^h}{1+h^2} + c_j^h \right) \cos(ht) \right) \\ &= -\tilde{y}_j + \sum_{h=1}^{n_j} \frac{-hc_j^h}{1+h^2} z_{j0}^h \left(\sin(ht) - h\cos(ht)\right) . \end{aligned}$$

The dynamics on the reduced space are then given by:

$$\dot{z}_{j0}^k = \frac{1}{k!} \left(h_0(z,t) + \rho(z)\cos(t)\right)^k \left(-h_j(z,t) + \sum_{h=1}^{n_j} \frac{-hc_j^h}{1+h^2} z_{j0}^h \left(\sin(ht) - h\cos(ht)\right) \right)$$

and because of the higher order nature of the h_i's, we may write that

$$\dot{z}_{j0}^k = f_{j0}^k(z,t) = \frac{1}{k!}\rho(z)^k \cos^k(t) \left(\sum_{h=1}^{n_j} \frac{-hc_j^h}{1+h^2} z_{j0}^h \left(\sin(ht) - h\cos(ht)\right) \right) + O(2j+2) . \tag{3.6}$$

Induction may be used to verify that:

$$\cos^k(t) = \sum_{i=1}^{\ell} \alpha_{ki} \cos([k - 2(i-1)]t)$$

where $\alpha_{ki} > 0$ and $\ell = \frac{k}{2}$ if k is even and $\frac{k+1}{2}$ if k is odd. As we do not have exponential stability of the averaged system nor high frequency sinusoidal terms [7] we cannot apply averaging to the terms of equation (3.6) and conclude stability. Alternatively, one might notice that each chain (denoted by the index j) is decoupled from the effects of the others except through the positive semi-definite term $\rho(z)$. We would like to apply the results of for the single chain case repeatedly to argue local stability.

Lemma 2 ("Averaging" Transformation) *Consider the time–varying nonlinear system*

$$\dot{x} = f(x,t) \tag{3.7}$$

where f is of period T, C^r, and where the i^{th} entry of the vector f satisfies $f_i = O(x)^{2k+1}$, with $k \geq 1$. Then there exists a C^r local change of coordinates $x = y + \Psi(y,t)$ under which (3.7) becomes

$$\dot{y} = \bar{f}(y) + \hat{f}(y,t)$$

where $\bar{f}$ is the time average of f and $\hat{f}_i(y,t) = O(y)^{2k+2}$ and of period T.

Remark: Lemma 2 closely resembles lemma 2.2 of [10], except now there is no dependence between the position and the order of an element f_i of the vector f. Note that if we permute the vector x and hence the vectors f and y, the order of the element f_i of the vector f of the previous result will no longer necessarily of order $2i+1$. In fact, if the element f_i were of order $2k+1$, the previous lemma would have asserted $\bar{f}_i$ to be order $2k+1$ and $\tilde{f}_i$ to be order $2k+2$. The order of $\tilde{f}_i$ and $\bar{f}$ is then determined by the order of f_i. However, a formal proof is still required.

Proof: The proof closely resembles a result described in [5]. We will divide $f(x,t)$ into its time average, given by $\bar{f}(x)$, and the remainder $\tilde{f}(x,t)$. We make the coordinate change:

$$x = y + \Psi(y,t)$$

with $Psi(y,t)$ specified later. We now solve for the dynamics of y.

$$(I + D_y\Psi)\,\dot{y} + \frac{\partial\Psi}{\partial t} = \dot{x} = \bar{f}(y+\Psi) + \tilde{f}(y+\Psi,t)$$
$$\dot{y} = (I + D_y\Psi)^{-1}\left(\bar{f}(y+\Psi) + \tilde{f}(y+\Psi,t) - \frac{\partial\Psi}{\partial t}\right)$$

Set $\frac{\partial\Psi}{\partial t} = \tilde{f}(y,t)$. As $\tilde{f}(y,t)$ has zero mean, Ψ is a bounded function of time. Furthermore, this implies that Ψ is of higher order, thus the coordinate change is valid locally. Expanding the terms, we see:

$$\dot{y} = (I + D_y\Psi)^{-1}\left(\bar{f}(y) + f(y+\Psi,t) - f(y,t)\right)$$
$$= \left(I - D_y\Psi + O(\|D_y\Psi\|^2)\right)\left(\bar{f} + D_y f\Psi + O(\|\Psi\|^2)\right)$$
$$= \bar{f}(y) + \hat{f}(y,t)$$

Now we will check the order of $\hat{f}_i$.

$$\hat{f} \quad = \quad D_y f\Psi - D_y\Psi\left(\bar{f}(y) + f_i(y+\Psi,t) - f_i(y,t)\right) \tag{3.8}$$

As the order of the element f_i is $2k+1$, the elements of the corresponding row of $D_y f$ are of order $2k$. The lowest order in Ψ, as it is the time integral of $\tilde{f}$, is 3 from the case $k = 1$. We conclude that the first term is of order $2k+3$. The same argument holds for the second term, as the element Ψ_i is of order $2k+1$, the elements of the corresponding row of $D_y\Psi$ is of order $2k$. The product is then at least of order $2k+3$. □.

With lemma 2 in mind we will define the vectors $z, f(z,t) \in \mathbf{R}^{n-m-1}$.

$$\begin{aligned} z &= \left(z_{10}^{1}, z_{10}^{2} \quad \cdots \quad z_{10}^{n_1}, z_{20}^{1} \quad \cdots \quad z_{m0}^{n_m}\right)^T \\ f(z,t) &= \left(f_{10}^{1}, f_{10}^{2} \quad \cdots \quad f_{10}^{n_1}, f_{20}^{1} \quad \cdots \quad f_{m0}^{n_m}\right)^T \end{aligned} \tag{3.9}$$

Note that by equation (3.6) z satisfies:

$$\dot{z} = f(z,t) \tag{3.10}$$

By lemma 2 here exists an locally valid averaging transformation. In spirit with the earlier notation, we will write the transformations maps $z \to \bar{z}$.

$$\dot{\bar{z}}_{j0}^{k} = \bar{f}_{j0}^{k}(\bar{z}) + \hat{f}_{j0}^{k}(\bar{z},t)$$

Further, $\bar{f}_{j0}^{k}(\bar{z})$ is the time average of $f_{j0}^{k}(\bar{z},t)$ and $\hat{f}_{j0}^{k}(\bar{z},t)$ is of order $2k+2$. Now we will recall the case specific Lyapunov result from [10]:

Lemma 3 (Case Specific Lyapunov Result) *Consider the time–varying nonlinear system*

$$\dot{y} = \bar{f}(y) + \tilde{f}(y,t) \tag{3.11}$$

where $y \in \mathbf{R}^n$*. If*

$$\|\tilde{f}(y,t)\| \leq \beta_i \|y\|^{2(1+i)}$$

for all y *in some open neighborhood of the origin and*

$$\bar{f}(y) \quad = \quad A\psi(y)$$

where A *is a square lower triangular matrix with* $a_{ii} < 0$ *for* $i = 1, \ldots, n$ *and*

$$\psi_i(y) \quad = \quad y_i \|y\|^{2i}$$

then the origin of (3.11) is locally asymptotically stable.

The same kind of permutation argument can now be applied to Lemma 3 as was applied to Lemma 2, except in this case it would be wise to preserve the order within each chain in order to retain the the lower block diagonal structure of the matrix A. For our case, matrix A will then have the block diagonal form (recall that the chains are decoupled except for the function

$\rho(z))$:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & A_m \end{bmatrix}$$

where $A_j \in \mathbf{R}^{n_j \times n_j}$ has a lower block diagonal form. The diagonal terms of the matrix A_j are denoted by a_j^{kk} and are given by,

$$a_j^{kk} = -\frac{1}{2k!}\frac{k^2}{1+k^2}\alpha_{1k}c_j^k$$

similar to those in [10]. As $\alpha_{1k}c_j^k < 0$, we may conclude local asymptotic stability.

Remark 1: Corollary 2.1 of [10] may be easily extended to the multiple input case. We may therefore conclude that the controls (3.2) will locally asymptotically stabilize the chained form system as well the power form.

Remark 2: As the chains are decoupled, we may can make the stability global by using saturation functions as described in [10]. The example will incorporate this strategy. The controls with saturation functions incorporated are given by:

$$
\begin{aligned}
v_0 &= -y_0 + \sigma(\rho(z))(\cos(t) - \sin(t)) \\
v_j &= -y_j + \sum_{h=1}^{n_h} c_j^h \sigma(z_{j0}^h)\cos(ht) \text{ for } 1 \le j \le m
\end{aligned}
\tag{3.12}
$$

with $c_j^h < 0$ and with $\sigma : \mathbf{R} \to \mathbf{R}$ C^3 and nondecreasing. The function is the identity map near the origin but never greater in magnitude than some $\epsilon > 0$. Provided this ϵ is small enough, the origin will be globally asymptotically stable.

Remark 3: These control laws may be adapted in order to regulate a class of mechanical systems. It has been shown in the literature [1, 9] that a large number of mechanical systems with nonholonomic velocity contraits may be written as:

$$
\begin{aligned}
\dot{\xi} &= \sum_{i=1}^{m} g_i(\xi)u_i \\
\dot{u} &= v
\end{aligned}
$$

The first equation describes what is sometimes refered to as the *kinematic* system. We have developed controls laws $u_d(\xi, t)$ which render $\xi = 0$ globally asymptotically stable, solving the *kinematic* point stability problem for a class of mechanical systems. Since $u_d(\xi,t)$ is a C^1 function of ξ and t we we may apply the following control law in order to render the point $(\xi,u)=(0,0)$ globally asymptotically stable,

$$v(\xi,u,t) \;=\; -(u-u_d(\xi,t))+(\frac{\partial u(\xi,t)}{\partial \xi}g(\xi)u+\frac{\partial u(\xi,t)}{\partial t})$$

To demonstrate that this control law stabilizes $(0,0)$, examine the dynamics of the error of the input $e=u-u_d(\xi,t)$.

$$\begin{aligned} \dot{e} &= \dot{u}-\dot{u}(\xi,t) \\ &= -e \end{aligned}$$

Note that if we view the state of the system as (e,ξ), the equations will look like $\dot{\xi}=g(\xi)u_d(\xi,t)+g(\xi)e$. The linearly stable coordinates will then contain not only the terms due to $u_d(\xi,t)$ as before but also the entire vector e. In the center manifold, $e=0$. Consequently the the equations for the dynamics on the center manifold are exactly the same as in the *kinematic* case thus we may conlude stability.

Chapter 4

An Example

The stabilization method presented in this paper will be illustrated by a simple three-input example. The example we use is the fire truck, or tiller truck shown in Figure (4.1). The kinematic equations were derived in [2] to have the following form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi}_0 \\ \dot{\theta}_0 \\ \dot{\phi}_1 \\ \dot{\theta}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \tan\theta_0 \\ 0 \\ \frac{\tan\phi_0}{l_0\cos\theta_0} \\ 0 \\ \frac{-\sin(\phi_1-\theta_0+\theta_1)}{l_1\cos\phi_1\cos\theta_0} \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_3 \,.$$

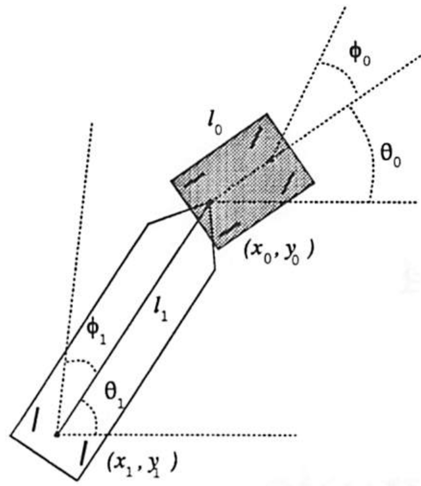


Figure 4.1: Configuration Space of the Fire Truck

The system has six states which are the Cartesian location of the center of the rear axle of the cab, (x, y); the steering angle of the front wheels relative to the cab's orientation, ϕ_0; the absolute cab orientation with respect to the horizontal axis of the inertial frame, θ_0; the steering angle of the rear wheels with respect to the trailer body, ϕ_1; and the absolute trailer orientation, θ_1. The constants l_0 and l_1 correspond to physical parameters of the system. The three inputs u_1, u_2 and u_3 correspond to the driving velocity, the cab's steering velocity and the trailer's steering velocity, respectively.

The coordinate transformation [2],

$$x_0^0 = x$$

$$
\begin{aligned}
x_1^0 &= \frac{\tan\phi_0}{l_0\cos^3\theta_0} \\
x_2^0 &= \frac{-\sin(\phi_1-\theta_0+\theta_1)}{l_1\cos\phi_1\cos\theta_0} \\
x_{10}^1 &= \tan\theta_0 \\
x_{20}^1 &= \theta_1 \\
x_{10}^2 &= y
\end{aligned}
$$

is used to put the system into a two-chained single-generator form:

$$
\begin{array}{lll}
\dot{x}_0^0 = v_0 & \dot{x}_1^0 = v_1 & \dot{x}_2^0 = v_2 \\
 & \dot{x}_{10}^1 = x_1^0 v_0 & \dot{x}_{20}^1 = x_2^0 v_0 \\
 & \dot{x}_{10}^2 = x_{10}^1 v_0 & .
\end{array}
\tag{4.1}
$$

Using equation (2.3), we calculate the coordinate transformation

$$
\begin{aligned}
y_0 &= x_0^0 \\
y_1 &= x_1^0 \\
y_2 &= x_2^0 \\
z_{10}^1 &= -x_{10}^1 + x_0^0 x_1^0 \\
z_{10}^2 &= x_{10}^2 - x_0^0 x_1^0 + \frac{1}{2}(x_0^0)^2 x_1^0
\end{aligned}
$$

$$z_{20}^1 = -x_{20}^1 + x_0^0 x_2^0$$

to give us the power form

$$\begin{aligned} \dot{y}_0 = v_0 \quad \dot{y}_1 &= v_1 \qquad \dot{y}_2 = v_2 \\ \dot{z}_{10}^1 &= y_0 v_1 \qquad \dot{z}_{20}^1 = y_0 v_2 \\ \dot{z}_{10}^2 &= \tfrac{1}{2}(y_0)^2 v_1 \quad . \end{aligned} \tag{4.2}$$

We now present the simulation of the fire truck system. The simulation was performed on the system in power form with the states being stabilized to the origin from a given initial point by using the following control laws:

$$\begin{aligned} v_0 &= -y_0 + \rho_0 \,(\cos t - \sin t) \\ v_1 &= -y_1 + c_1\, \rho_1 \, \cos t + c_2\, \rho_2 \, \cos 2t \\ v_2 &= -y_2 + c_3\, \rho_3 \, \cos t\,. \end{aligned} \tag{4.3}$$

The c_1, c_2, c_3 are constants and $\rho_1, \rho_2, \rho_3, \rho_4$ are saturation functions designed to yield global stabilization results and defined as follows:

$$\rho_0 = \left(\sigma\left(\sqrt{(z_{10}^1)^2 + (z_{20}^1)^2 + (z_{10}^2)^2}\right)\right)^2$$

$$\begin{aligned}
\rho_1 &= \sigma(z_{10}^1) \\
\rho_2 &= \sigma(z_{10}^2) \\
\rho_3 &= \sigma(z_{20}^1)\,.
\end{aligned}$$

where $\sigma(z)$ is a saturation function linear between $(-\epsilon, \epsilon)$.

The coordinates in power form were then transformed back into the original coordinates for analysis and a movie animation. Then the inverse coordinate transformation from power form to the original coordinates is as follows.

$$\begin{aligned}
x &= y_0 \\
y &= z_{10}^2 + y_0 y_1 - \frac{1}{2}(y_0)^2 y_1 \\
\phi_0 &= \tan^{-1}\left(l_0\, y_1 \cos^3(\tan^{-1}(y_0 y_1 - z_{10}^1))\right) \\
\theta_0 &= \tan^{-1}\left(y_0 y_1 - z_{10}^1\right) \\
\phi_1 &= \frac{1}{\cos\left(y_0 y_2 - z_{10}^1 - \tan^{-1}(y_0 y_1 - z_{10}^1)\right)} \\
&\quad \times \tan^{-1}\left(l_1 y_2 \cos\left(\tan^{-1}(y_0 y_1 - z_{10}^1)\right) + \sin\left(y_0 y_2 - z_{10}^1 - \tan^{-1}(y_0 y_1 - z_{10}^1)\right)\right) \\
\theta_1 &= y_0 y_2 - z_{10}^1 \qquad (4.4)
\end{aligned}$$

Figure 4.2 shows nine frames from a movie of the simulation results for

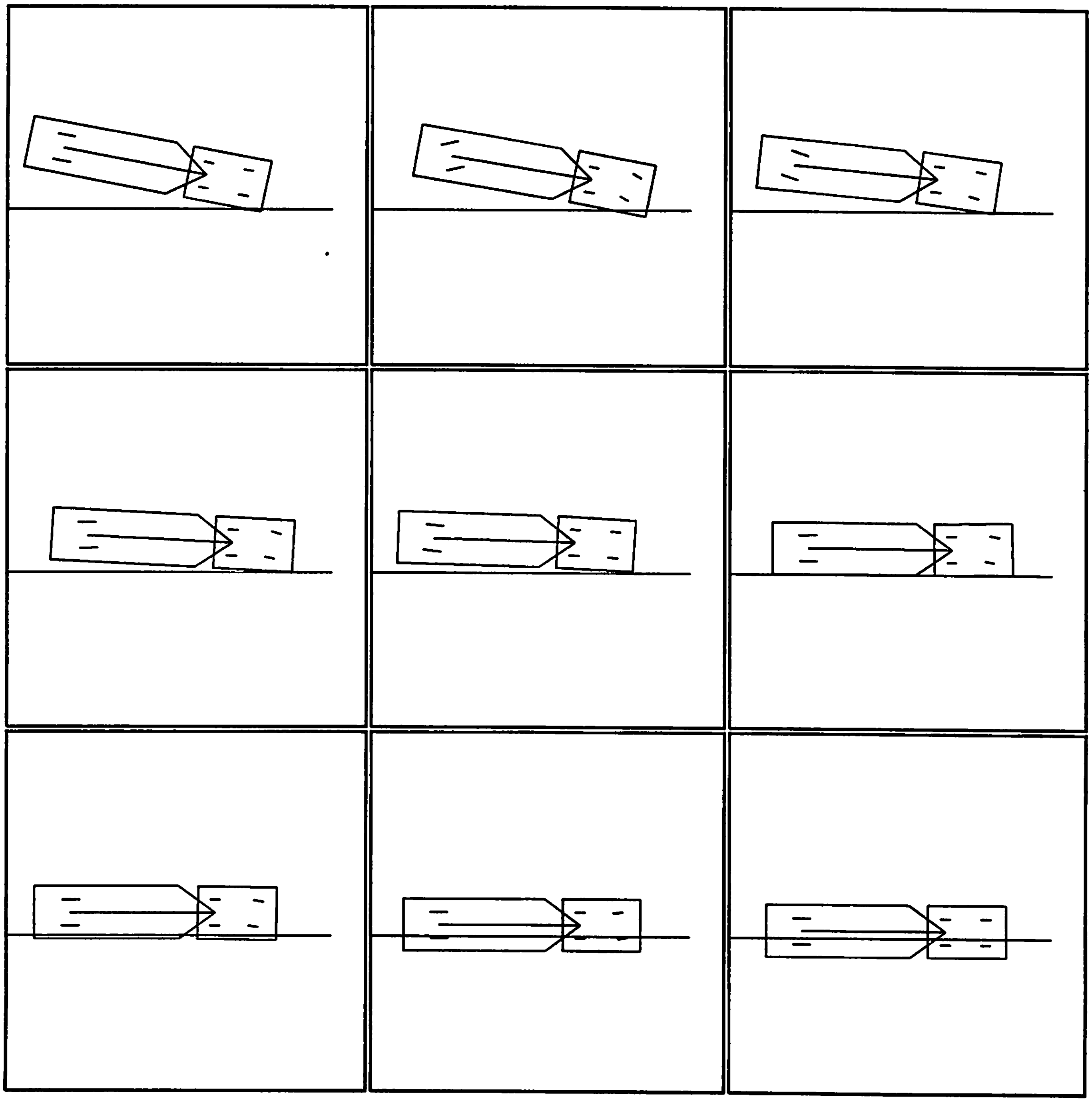


Figure 4.2: Movie of Parallel Parking Trajectory

a parallel parking maneuver, starting from an arbitrary initial point that illustrates the control law. The fire truck is being stabilized to the origin. The $y = 0$ line is shown in the plots and we see that after 8000 iterations, the fire truck is heading towards the origin.

Figure 4.3 shows the x-y plot for the same maneuver. In this plot we see the Lissajous figures nicely.

Figure 4.4 shows the trajectories of the body angles and y position of the center of the rear axle of the cab for the same parallel parking maneuver. We can see the success of the stabilization after 8000 iterations in the y position plot.

Figure 4.5 shows the saturation functions used in equation (4.3). For the parallel parking trajectory, we chose $\epsilon = 0.5$.

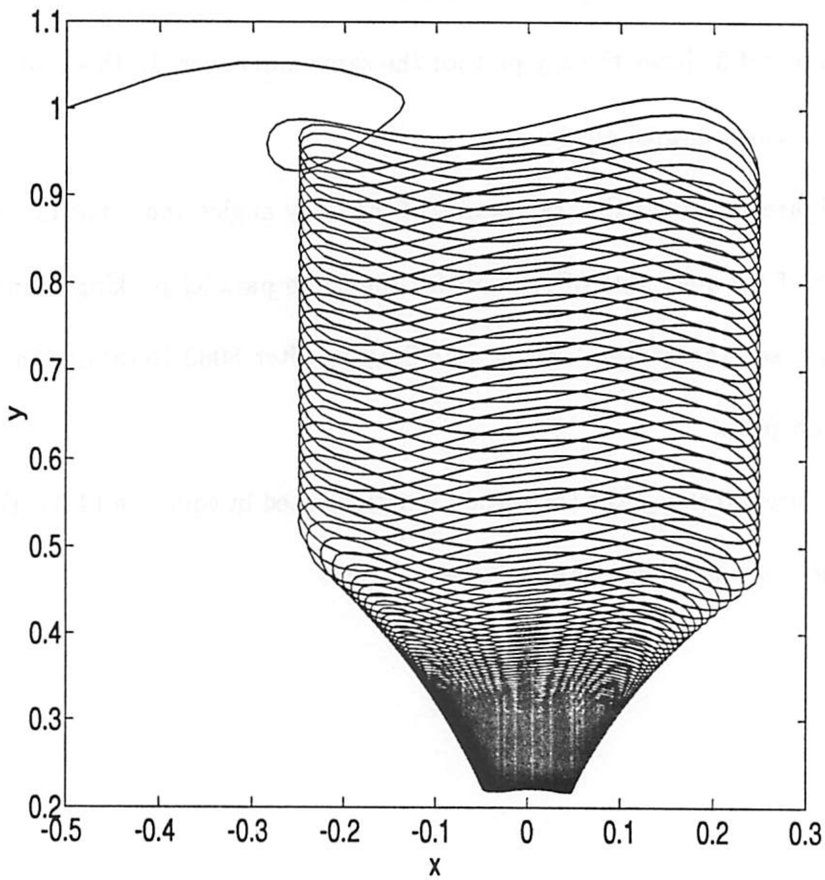
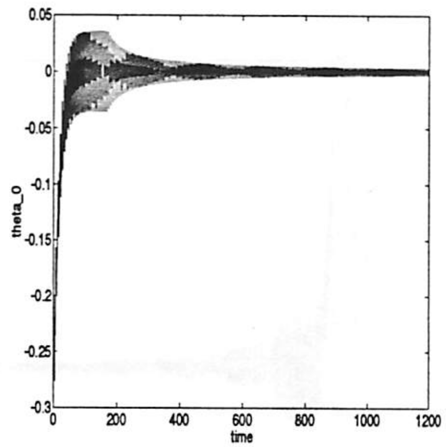
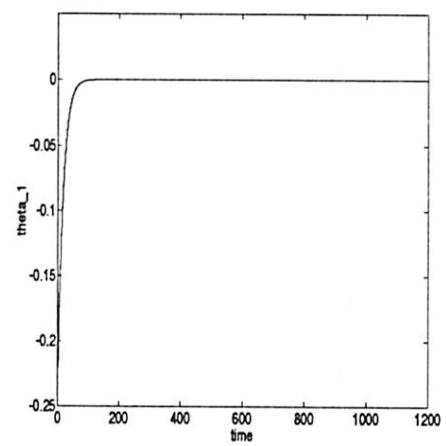


Figure 4.3: X-Y Trajectory of the Parallel Parking Maneuver



θ_0 vs. t



θ_1 vs. t

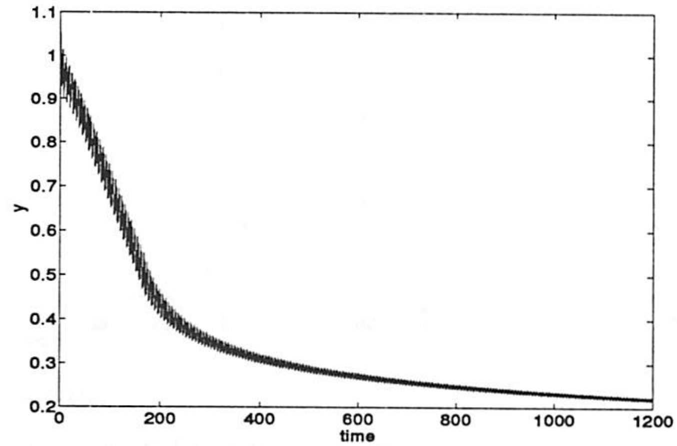
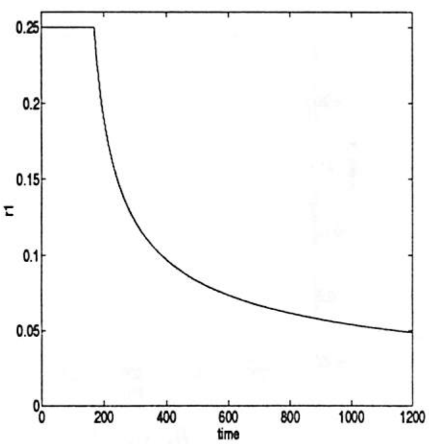
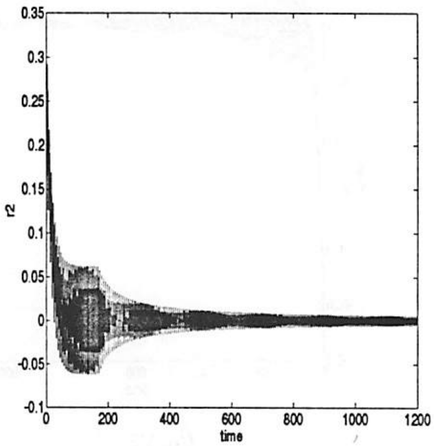


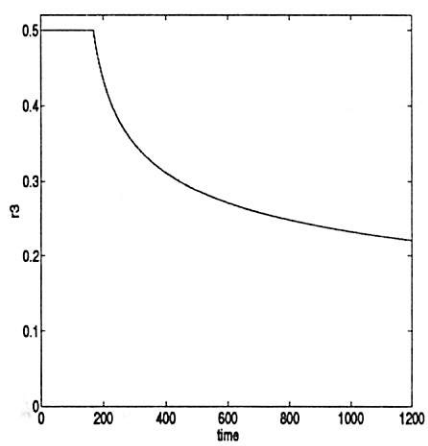
Figure 4.4: Trajectories of the Parallel Parking Maneuver



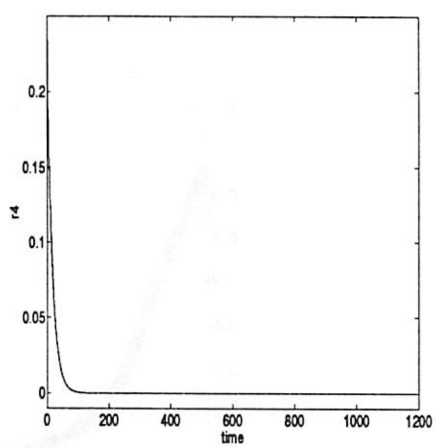
ρ_1 vs. t



ρ_2 vs. t



ρ_3 vs. t



ρ_4 vs. t

Figure 4.5: Saturation Functions for Parallel Parking Maneuver

Chapter 5

Conclusion

In summary, our contributions are the presentation of a transformation from most general chained form to power form and a control law for a class of multiple input nonholonomic control systems without drift. In addition, we illustrated our results with an example from the literature, solving incidentally the parallel parking problem.

It is important to notice that the control laws presented here are smooth and, thus, are easily adapted to dynamical control systems. Furthermore, the convergence in the coordinate directions can be modified by changing the coefficients of the control law.

Possible future work includes finding a control law that asymptotically stabilizes to the origin an (m+1)-generator control system. We are also looking to enlarge the class of nonholonomic control systems which we may transform into chained form.

Acknowledgements

We are indebted to S.S. Sastry for suggesting the investigation of control laws for multiple input nonholonomic control systems.

Bibliography

[1] M. Reyhanoglu A. Bloch and N. H. McClamroch. Control and stabilization of nonholonomic caplygin dynamic systems. In *IEEE Conference on Decision and Control*, pages 1127 – 1132, Brighton, England, 1991.

[2] L. Bushnell, D. Tilbury, and S. S. Sastry. Steering three-input chained form nonholonomic systems using sinusoids: The firetruck example. In *European Controls Confernce*, 1993. Also as Electrocic Research Laboratory memo M92/107.

[3] J. Carr. *Applications of Center Manifold Theory.* Springer-Verlag, New York, 1981.

[4] C. Canudis de Witt and O. J. Sordalen. Exponential stabilization of mobile wheeled robots with nonholonomic constraints. In *IEEE Conference on Decision and Control*, pages 692 – 697, 1991.

[5] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer-Verlag, New York, 1983.

[6] A. Isidori. *Nonlinear Control Systems.* Springer-Verlag, second edition, 1989.

[7] G. Lafferriere and H. J. Sussmann. Motion planning for controllable systems without drift. In *IEEE International Conference on Robotics and Automation*, pages 1148 - 1153, 1991.

[8] R. Murray and S. Sastry. Steering nonholonomic systems in chained form. In *IEEE Conference on Decision and Control*, pages 1121 -1126, 1991.

[9] Ju. I. Neimark and N. A. Fufaev. *Dynamics of Nonholonomic Systems.* American Mathematical Society, Providence, Rhode Island, 1972. Volume 33 in Translations of Mathematical Monographs.

[10] A. Teel, R. Murray, and G. Walsh. Nonholonomic control systems: From steering to stabilization with sinusoids. In *IEEE Conference on Decision and Control*, pages 1603 - 1609, 1992. Also as Electronic Research Laboratory memo M92/28.

[11] D. Tilbury, J-P. Laumond, R. Murray, S. Sastry, , and G. Walsh. Steering car-like robots with trailers using sinusoids. In *IEEE International Conference on Robotics and Automation*, 1992.