Relative Perturbation Theory: (II) Eigenspace Variations *

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Abstract

In this paper, we consider how eigenspaces of a Hermitian matrix A change when it is perturbed to $\widetilde{A} = D^* A D$ and how singular values of a (nonsquare) matrix B change when it is perturbed to $\widetilde{B} = D_1^* B D_2$, where D, D_1 and D_2 are assumed to be close to identity matrices of suitable dimensions, or either D_1 or D_2 close to some unitary matrix. We have been able to generalize well-known Davis-Kahan $\sin\theta$ theorems. As applications, we obtained bounds for perturbations of graded matrices.

We will follow the notation introduced in Li [7].

1 Introduction

Let A and $\widetilde{A}$ be two $n \times n$ Hermitian matrices with eigendecompositions

$$A = U\Lambda U^* = (U_1, U_2)\begin{pmatrix} \Lambda_1 & \\ & \Lambda_2 \end{pmatrix}\begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix}, \tag{1.1}$$

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$$\widetilde{A} = \widetilde{U}\widetilde{\Lambda}\widetilde{U}^* = (\widetilde{U}_1, \widetilde{U}_2)\begin{pmatrix} \widetilde{\Lambda}_1 & \\ & \widetilde{\Lambda}_2 \end{pmatrix}\begin{pmatrix} \widetilde{U}_1^* \\ \widetilde{U}_2^* \end{pmatrix}, \tag{1.2}$$

where U, $\widetilde{U} \in \mathbb{U}_n$, U_1, $\widetilde{U}_1 \in \mathbb{C}^{n\times k}$ $(1 \le k < n)$ and

$$\Lambda_1 = \mathrm{diag}(\lambda_1, \cdots, \lambda_k), \quad \Lambda_2 = \mathrm{diag}(\lambda_{k+1}, \cdots, \lambda_n), \tag{1.3}$$

$$\widetilde{\Lambda}_1 = \mathrm{diag}(\widetilde{\lambda}_1, \cdots, \widetilde{\lambda}_k), \quad \widetilde{\Lambda}_2 = \mathrm{diag}(\widetilde{\lambda}_{k+1}, \cdots, \widetilde{\lambda}_n). \tag{1.4}$$

Suppose now that A and $\widetilde{A}$ are *close*. The question is: *How close are the eigenspaces spanned by U_i and $\widetilde{U}_i$?* This question has been well answered by four celebrated theorems so-called $\sin\theta$, $\tan\theta$, $\sin 2\theta$ and $\tan 2\theta$ due to Davis and Kahan [2] for arbitrary perturbations in the sense that the perturbations to A can be made arbitrary as long as $\widetilde{A} - A$ is kept small. This paper, on the other hand, will address the following question: *How close are the eigenspaces spanned by U_i and $\widetilde{U}_i$ under the assumption that $\widetilde{A} = D^*AD$ for some D close to I?* A similar question for singular value decompositions will be answered also. We will deal with perturbations of the following kinds:

- **Eigenvalue problems:**
  1. A and $\widetilde{A} = D^*AD$ for Hermitian case, where D is nonsingular and close to I or more generally to a unitary matrix;
  2. $H = D^*AD$ and $\widetilde{H} = D^*\widetilde{A}D$ for graded nonnegative Hermitian case, where it is assumed that A and $\widetilde{A}$ are nonsingular and often that D is a highly graded diagonal matrix (this assumption is not necessary to our theorems below).
- **Singular value problems:**
  1. B and $\widetilde{B} = D_1^*BD_2$, where D_1 and D_2 are nonsingular and close to I or more generally to two unitary matrices;
  2. $G = BD$ and $\widetilde{G} = \widetilde{B}D$ for graded case, where it is assumed that B and $\widetilde{B}$ are nonsingular and often that D is a highly graded diagonal matrix (this assumption is not necessary to our theorems below).

Recently, Eisenstat and Ipsen [4] launched an attack towards the above mentioned perturbations except graded cases. We will give a brief comparison among their results and ours.

Notation: For $X \in \mathbb{C}^{n\times k}$, $\mathcal{R}(X)$ denotes the subspace spanned by the column vectors of X.

Let both X, $Y \in \mathbb{C}^{n\times k}$ $(n > k)$ have full column rank k, and define the angle matrix $\Theta(X, Y)$ between X and Y as

$$\Theta(X, Y) \stackrel{\mathrm{def}}{=} \arccos((X^*X)^{-\frac{1}{2}}X^*Y(Y^*Y)^{-1}Y^*X(X^*X)^{-\frac{1}{2}})^{-\frac{1}{2}}.$$

The *canonical angles* between the subspace $\mathcal{X} = \mathcal{R}(X)$ and $\mathcal{Y} = \mathcal{R}(Y)$ are defined to be the singular values of the Hermitian matrix $\Theta(X, Y)$. The following lemma is well-known. For a proof of it, the reader is referred to, e.g, Li [6, Lemma 2.1].

Lemma 1.1 *Suppose that $\widetilde{Y} = (Y, Y_1) \in \mathbb{C}^{n\times n}$ is a nonsingular matrix with*

$$\widetilde{Y}^{-1} = \begin{pmatrix} S^* \\ S_1^* \end{pmatrix}, \quad S \in \mathbb{C}^{n\times k}.$$

Then for any unitarily invariant norm $|||\cdot|||$

$$|||\sin\Theta(X, Y)||| = \left|\left|\left|(S_1^* S_1)^{-\frac{1}{2}} S_1^* X (X^* X)^{-\frac{1}{2}}\right|\right|\right|.$$

Taking $X = U_1$ and $\widetilde{X} = \widetilde{U}_1$ (Ref. (1.1) and (1.2)), one has

$$\begin{aligned} \Theta(U_1, \widetilde{U}_1) &= \arccos(U_1^* \widetilde{U}_1 \widetilde{U}_1^* U_1)^{-\frac{1}{2}}, \\ \left|\left|\left|\sin\Theta(U_1, \widetilde{U}_1)\right|\right|\right| &= \left|\left|\left|\widetilde{U}_2^* U_1\right|\right|\right|. \end{aligned} \tag{1.5}$$

For more discussions on angles between subspaces, the reader is referred to Davis and Kahan [2] and Stewart and Sun [9, Chapters I and II].

This paper is organized as follows. We briefly review Davis-Kahan $\sin\theta$ theorems and their generalizations—Wedin $\sin\theta$ theorems—to singular value decompositions in §2. In §3, we first define four indispensable relative gap functions. We present our $\sin\theta$ theorems for A and $\widetilde{A} = D^*AD$ in §3.2, those for graded nonnegative Hermitian matrices in §3.3. Theorems for B and $\widetilde{B} = D_1^* B D_2$ and for graded matrices are given in §3.4 and in §3.5 respectively. We discuss how to bound from below relative gaps, for example, between Λ_1 and $\widetilde{\Lambda}_2$ by relative gaps between Λ_1 and Λ_2 in §4. A word will be said regarding Eisenstat-Ipsen's Theorems in §5. Detailed proofs are saved for §§6, 7, 8 and 9.

2 Known Theorems

Let A and $\widetilde{A}$ be two Hermitian matrices whose eigendecompositions are given by (1.1) and (1.2). Define

$$R \stackrel{\text{def}}{=} \widetilde{A}U_1 - U_1\Lambda_1 = (\widetilde{A} - A)U_1. \tag{2.1}$$

In matrix language, The following two theorems are the matrix versions of two $\sin\theta$ theorems proved by Davis and Kahan [2].

Theorem 2.1 (Davis-Kahan) *If*

$$\delta \stackrel{\text{def}}{=} \min_{1\le i\le k, 1\le j\le n-k} |\lambda_i - \widetilde{\lambda}_{k+j}| > 0,$$

then

$$\|\sin\Theta(U_1, \widetilde{U}_1)\|_F \le \frac{\|R\|_F}{\delta} = \frac{\|(\widetilde{A} - A)U_1\|_F}{\delta}. \tag{2.2}$$

In this theorem, the spectrum of Λ_1 and that of $\widetilde{\Lambda}_2$ are only required to be disjoint. In the next theorem, they are required, more strongly, to be well-separated.

Theorem 2.2 (Davis-Kahan) *Assume there is an interval $[\alpha, \beta]$ and a $\underline{\delta} > 0$ such that the spectrum of Λ_1 lies entirely in $[\alpha, \beta]$ while that of $\widetilde{\Lambda}_2$ lies entirely outside of $(\alpha - \underline{\delta}, \beta + \underline{\delta})$ (or such that the spectrum of $\widetilde{\Lambda}_2$ lies entirely in $[\alpha, \beta]$ while that of Λ_1 lies entirely outside of $(\alpha - \underline{\delta}, \beta + \underline{\delta})$). Then for any unitarily invariant norm $|||\cdot|||$*

$$\left|\left|\left|\sin\Theta(U_1, \widetilde{U}_1)\right|\right|\right| \le \frac{|||R|||}{\underline{\delta}} = \frac{\left|\left|\left|(\widetilde{A} - A)U_1\right|\right|\right|}{\underline{\delta}}. \tag{2.3}$$

Now, let us consider the perturbations of Singular Value Decompositions (SVD). Let B and $\widetilde{B}$ be two $m \times n$ ($m \ge n$) complex matrices with SVDs

$$B = U\Sigma V^* = (U_1, U_2)\begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix}, \tag{2.4}$$

$$\widetilde{B} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^* = (\widetilde{U}_1, \widetilde{U}_2)\begin{pmatrix} \widetilde{\Sigma}_1 & 0 \\ 0 & \widetilde{\Sigma}_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{V}_1^* \\ \widetilde{V}_2^* \end{pmatrix}, \tag{2.5}$$

where $U, \widetilde{U} \in \mathbb{U}_m$, $V, \widetilde{V} \in \mathbb{U}_n$, $U_1, \widetilde{U}_1 \in \mathbb{C}^{m\times k}$, $V_1, \widetilde{V}_1 \in \mathbb{C}^{n\times k}$ ($1 \le k < n$) and

$$\Sigma_1 = \operatorname{diag}(\sigma_1, \cdots, \sigma_k), \quad \Sigma_2 = \operatorname{diag}(\sigma_{k+1}, \cdots, \sigma_n), \tag{2.6}$$

$$\widetilde{\Sigma}_1 = \operatorname{diag}(\widetilde{\sigma}_1, \cdots, \widetilde{\sigma}_k), \quad \widetilde{\Sigma}_2 = \operatorname{diag}(\widetilde{\sigma}_{k+1}, \cdots, \widetilde{\sigma}_n). \tag{2.7}$$

Define residuals

$$R \stackrel{\text{def}}{=} \widetilde{B}V_1 - U_1\Sigma_1 = (\widetilde{B} - B)V_1 \quad \text{and} \quad S \stackrel{\text{def}}{=} \widetilde{B}^* U_1 - V_1\Sigma_1 = (\widetilde{B}^* - B^*)U_1. \tag{2.8}$$

Wedin [10] showed the following two theorems:

Theorem 2.3 (Wedin) *If*

$$\delta \stackrel{\text{def}}{=} \min\left\{\min_{1\le i\le k, 1\le j\le n-k} |\sigma_i - \widetilde{\sigma}_{k+j}|, \min_{1\le i\le k} \sigma_i\right\} > 0,$$

then

$$\begin{aligned} &\sqrt{\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2 + \|\sin\Theta(V_1,\widetilde{V}_1)\|_F^2} \\ \le\quad & \frac{\sqrt{\|R\|_F^2 + \|S\|_F^2}}{\delta} = \frac{\sqrt{\|(\widetilde{B}-B)V_1\|_F^2 + \|(\widetilde{B}^*-B^*)U_1\|_F^2}}{\delta}. \end{aligned} \tag{2.9}$$

Similarly to above Davis-Kahan theorems, by imposing further restrictions, we will have

Theorem 2.4 (Wedin) *If there exist $\alpha > 0$ and $\underline{\delta} > 0$ such that*

$$\min_{1\le i\le k} \sigma_i \ge \alpha + \underline{\delta} \quad \text{and} \quad \max_{1\le j\le n-k} \widetilde{\sigma}_{k+j} \le \alpha,$$

then for any unitarily invariant norm $|||\cdot|||$

$$\begin{aligned} &\max\left\{\left|\left|\left|\sin\Theta(U_1,\widetilde{U}_1)\right|\right|\right|, \left|\left|\left|\sin\Theta(V_1,\widetilde{V}_1)\right|\right|\right|\right\} \\ \le\quad & \frac{\max\{|||R|||, |||S|||\}}{\underline{\delta}} = \frac{\max\left\{\left|\left|\left|(\widetilde{B}-B)V_1\right|\right|\right|, \left|\left|\left|(\widetilde{B}^*-B^*)U_1\right|\right|\right|\right\}}{\underline{\delta}}. \end{aligned} \tag{2.10}$$

3 Statement of Theorems

3.1 Relative Gap Functions

For two square diagonal matrices

$$D = \mathrm{diag}(d_1, d_1, \cdots, d_k),\ \widetilde{D} = \mathrm{diag}(\widetilde{d}_1, \widetilde{d}_1, \cdots, \widetilde{d}_\ell)$$

and an integer $m \geq k + \ell$, the functions RelGap_2 and $\widetilde{\mathrm{RelGap}}$ are defined as:

$$\mathrm{RelGap}_2(D, \widetilde{D}; m) \stackrel{\mathrm{def}}{=} \begin{cases} \min\left\{ \min\limits_{1\leq i\leq k, 1\leq j\leq \ell} \mathrm{RelDist}_2(d_i, \widetilde{d}_j)\ , \right. \\ \qquad\qquad \left. \min\limits_{1\leq i\leq k} \mathrm{RelDist}_2(d_i, 0) \right\}, & \text{if } m > k+\ell, \\ \min\limits_{1\leq i\leq k, 1\leq j\leq \ell} \mathrm{RelDist}_2(d_i, \widetilde{d}_j), & \text{if } m = k+\ell, \end{cases}$$

$$\mathrm{RelGap}_2(D, \widetilde{D}) \stackrel{\mathrm{def}}{=} \mathrm{RelGap}_2(D, \widetilde{D}; k+\ell),$$

and

$$\widetilde{\mathrm{RelGap}}(D, \widetilde{D}; m) \stackrel{\mathrm{def}}{=} \begin{cases} \min\left\{ \min\limits_{1\leq i\leq k, 1\leq j\leq \ell} \widetilde{\mathrm{RelDist}}(d_i, \widetilde{d}_j)\ , \right. \\ \qquad\qquad \left. \min\limits_{1\leq i\leq k} \widetilde{\mathrm{RelDist}}(d_i, 0) \right\}, & \text{if } m > k+\ell, \\ \min\limits_{1\leq i\leq k, 1\leq j\leq \ell} \widetilde{\mathrm{RelDist}}(d_i, \widetilde{d}_j), & \text{if } m = k+\ell, \end{cases}$$

$$\widetilde{\mathrm{RelGap}}(D, \widetilde{D}) \stackrel{\mathrm{def}}{=} \widetilde{\mathrm{RelGap}}(D, \widetilde{D}; k+\ell).$$

From these definitions, we can see that

$$\mathrm{RelGap}_2(D, \widetilde{D}) = \mathrm{RelGap}_2(\widetilde{D}, D) \quad \text{and} \quad \widetilde{\mathrm{RelGap}}(D, \widetilde{D}) = \widetilde{\mathrm{RelGap}}(\widetilde{D}, D),$$

but generally for $m > k + \ell$ we do not have

$$\mathrm{RelGap}_2(D, \widetilde{D}; m) = \mathrm{RelGap}_2(\widetilde{D}, D; m) \text{ and } \widetilde{\mathrm{RelGap}}(D, \widetilde{D}; m) = \widetilde{\mathrm{RelGap}}(\widetilde{D}, D; m).$$

Our next two relative gap functions[1] $\underline{\mathrm{RelGap}}_p$ (for some $1 \leq p \leq \infty$) and $\underline{\widetilde{\mathrm{RelGap}}}$ take, besides two square diagonal matrix arguments, two more nonnegative numbers as arguments. They have a very narrow domain. To be specific, we say $\underline{\mathrm{RelGap}}_p(D, \widetilde{D}; \alpha, \beta)$ and $\underline{\widetilde{\mathrm{RelGap}}}(D, \widetilde{D}; \alpha, \beta)$ are *definable* if $0 \leq \alpha < \beta$ and if either

$$\|D\|_2 = \max_{1\leq i\leq k} |d_i| \leq \alpha < \beta \leq \min_{1\leq j\leq \ell} |\widetilde{d}_j| = \|\widetilde{D}^{-1}\|_2^{-1},$$

[1]We intentionally put underline to indicate that there are intervals containing the spectra of D and $\widetilde{D}$.

or

$$\|D^{-1}\|_2^{-1} = \min_{1\le i\le k} |d_i| \ge \beta > \alpha \ge \max_{1\le j\le \ell} |\widetilde{d}_j| = \|\widetilde{D}\|_2$$

holds, and $\underline{\mathrm{RelGap}}_p(D_1, \widetilde{D}_2; \alpha, \beta)$ and $\underline{\widetilde{\mathrm{RelGap}}}(D, \widetilde{D}; \alpha, \beta)$ are defined to be

$$\begin{aligned} \underline{\mathrm{RelGap}}_p(D, \widetilde{D}; \alpha, \beta) &\stackrel{\mathrm{def}}{=} \mathrm{RelDist}_p(\alpha, \beta) = \frac{\beta - \alpha}{\sqrt[p]{\alpha^p + \beta^p}}, \\ \underline{\widetilde{\mathrm{RelGap}}}(D, \widetilde{D}; \alpha, \beta) &\stackrel{\mathrm{def}}{=} \widetilde{\mathrm{RelDist}}(\alpha, \beta) = \frac{\beta - \alpha}{\sqrt{\alpha\beta}}. \end{aligned}$$

We caution the reader to notice the inherent difference imposed on the separation between D and $\widetilde{D}$ associated with the four above defined relative gap functions RelGap_2, $\widetilde{\mathrm{RelGap}}$, $\underline{\mathrm{RelGap}}_p$ and $\underline{\widetilde{\mathrm{RelGap}}}$. It can be proved that

Proposition 3.1 *If* $\underline{\mathrm{RelGap}}_p(D_1, \widetilde{D}_2; \alpha, \beta)$ *and* $\underline{\widetilde{\mathrm{RelGap}}}(D, \widetilde{D}; \alpha, \beta)$ *exist, then*

$$\begin{aligned} \min_{1\le i\le k,\, 1\le j\le \ell} \mathrm{RelDist}_p(d_i, \widetilde{d}_j) &\ge \mathrm{RelDist}_p(\alpha, \beta), \\ \min_{1\le i\le k,\, 1\le j\le \ell} \widetilde{\mathrm{RelDist}}(d_i, \widetilde{d}_j) &\ge \widetilde{\mathrm{RelDist}}(\alpha, \beta). \end{aligned}$$

Using the fact that $\mathrm{RelDist}_p(\xi, \zeta) \le 2^{-1/p}\, \widetilde{\mathrm{RelDist}}(\xi, \zeta)$ for ξ, $\zeta \in \mathbb{C}$, one can show

Proposition 3.2

$$\begin{aligned} \mathrm{RelGap}_2(D, \widetilde{D}; m) &\le \frac{1}{\sqrt{2}} \widetilde{\mathrm{RelGap}}(D, \widetilde{D}; m), \\ \mathrm{RelGap}_p(D, \widetilde{D}; \alpha, \beta) &\le \frac{1}{2^{1/p}} \widetilde{\mathrm{RelGap}}(D, \widetilde{D}; \alpha, \beta), \end{aligned}$$

provided $\mathrm{RelGap}_p(D, \widetilde{D}; \alpha, \beta)$ *exists.*

3.2 Eigenspace Variations

Let A and $\widetilde{A} = D^* A D$ be two Hermitian matrices whose eigendecompositions are given by (1.1) and (1.2), where D is a nonsingular square matrix close to I. Theorem 3.1 below handles the case when eigenvalues of $\widetilde{\Lambda}_2$ and Λ_1 are not so well-separated, but disjoint.

Theorem 3.1 *If* $\mathrm{RelGap}_2 \equiv \mathrm{RelGap}_2(\Lambda_1, \widetilde{\Lambda}_2) > 0$*, then*

$$\|\sin\Theta(U_1, \widetilde{U}_1)\|_F \le \frac{\sqrt{\|(I - D^{-1})U_1\|_F^2 + \|(I - D^*)U_1\|_F^2}}{\mathrm{RelGap}_2}. \qquad (3.1)$$

By imposing a stronger condition on the separation between the spectra of $\widetilde{\Lambda}_2$ and Λ_1, we have the following bound on any unitarily invariant norm of $\sin\Theta(U_1, \widetilde{U}_1)$.

Theorem 3.2 *Assume that there exist $\alpha \geq 0$ and $\delta > 0$ such that the spectrum of Λ_1 lies entirely in $[-\alpha, \alpha]$ while that of $\widetilde{\Lambda}_2$ lies entirely outside $(-\alpha-\delta, \alpha+\delta)$ (or such that the spectrum of Λ_1 lies entirely outside $(-\alpha-\delta, \alpha+\delta)$ while that of $\widetilde{\Lambda}_2$ lies entirely in $[-\alpha, \alpha]$). Then for any unitarily invariant norm $|||\cdot|||$*

$$\left|\left|\left|\sin\Theta(U_1, \widetilde{U}_1)\right|\right|\right| \leq \frac{\sqrt[q]{|||(I-D^{-1})U_1|||^q + |||(I-D^*)U_1|||^q}}{\underline{\mathrm{RelGap}}_p}, \tag{3.2}$$

where $\underline{\mathrm{RelGap}}_p \equiv \underline{\mathrm{RelGap}}_p(\Lambda_1, \widetilde{\Lambda}_2; \alpha, \alpha+\delta)$.

3.3 Eigenspace Variations for Graded Matrices

Consider a graded Hermitian matrix $H = D^*AD \in \mathbb{C}^{n\times n}$, where A is an $n\times n$ positive definite matrix. (Thus H is nonnegative definite.) Perturb H to $\widetilde{H} = D^*\widetilde{A}D$ in such a way that

$$\|A^{-1}\|_2\|\Delta A\|_2 < 1,$$

where

$$\Delta A \stackrel{\mathrm{def}}{=} \widetilde{A} - A.$$

The question we want to solve is by how much eigenspaces of H are changed. Write the eigendecompositions of H and $\widetilde{H}$ as

$$\begin{aligned} H &= U\Lambda U^* = (U_1, U_2)\begin{pmatrix} \Lambda_1 & \\ & \Lambda_2 \end{pmatrix}\begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix}, & (3.3)\\ \widetilde{H} &= \widetilde{U}\widetilde{\Lambda}\widetilde{U}^* = (\widetilde{U}_1, \widetilde{U}_2)\begin{pmatrix} \widetilde{\Lambda}_1 & \\ & \widetilde{\Lambda}_2 \end{pmatrix}\begin{pmatrix} \widetilde{U}_1^* \\ \widetilde{U}_2^* \end{pmatrix}, & (3.4)\end{aligned}$$

where U, $\widetilde{U} \in \mathbb{U}_n$, U_1, $\widetilde{U}_1 \in \mathbb{C}^{n\times k}$ $(1 \leq k < n)$ and Λ_i's and $\widetilde{\Lambda}_j$'s are as defined in (1.3) and (1.4). In the present case, all $\lambda_i \geq 0$ and all $\widetilde{\lambda}_j \geq 0$. We will prove

Theorem 3.3 *Assume $\|A^{-1}\|_2\|\Delta A\|_2 < 1$. If $\widetilde{\mathrm{RelGap}} \equiv \widetilde{\mathrm{RelGap}}(\Lambda_1, \widetilde{\Lambda}_2) > 0$, then*

$$\begin{aligned} &\|\sin\Theta(U_1, \widetilde{U}_1)\|_F \\ &\quad\leq \frac{\|(I + A^{-1/2}(\Delta A)A^{-1/2})^{1/2} - (I + A^{-1/2}(\Delta A)A^{-1/2})^{-1/2}\|_F}{\widetilde{\mathrm{RelGap}}} \quad (3.5)\\ &\quad\leq \frac{\|A^{-1}\|_2}{\sqrt{1 - \|A^{-1}\|_2\|\Delta A\|_2}}\frac{\|\Delta A\|_F}{\widetilde{\mathrm{RelGap}}}. \end{aligned}$$

Theorem 3.4 *Assume $\|A^{-1}\|_2\|\Delta A\|_2 < 1$, and assume that there exist $\alpha > 0$ and $\delta > 0$ such that*

$$\max_{1\le i\le k} \lambda_i \le \alpha \quad \textit{and} \quad \min_{1\le j\le n-k} \widetilde{\lambda}_{k+j} \ge \alpha+\delta$$

or

$$\min_{1\le i\le k} \lambda_i \ge \alpha+\delta \quad \textit{and} \quad \max_{1\le j\le n-k} \widetilde{\lambda}_{k+j} \le \alpha.$$

Then for any unitarily invariant norm $|||\cdot|||$

$$\begin{aligned}
&\left|\left|\left|\sin\Theta(U_1,\widetilde{U}_1)\right|\right|\right| \\
&\le \frac{|||(I+A^{-1/2}(\Delta A)A^{-1/2})^{1/2}-(I+A^{-1/2}(\Delta A)A^{-1/2})^{-1/2}|||}{\underline{\widetilde{\text{RelGap}}}} \qquad (3.6)\\
&\le \frac{\|A^{-1}\|_2}{\sqrt{1-\|A^{-1}\|_2\|\Delta A\|_2}}\frac{|||\Delta A|||}{\underline{\widetilde{\text{RelGap}}}},
\end{aligned}$$

where $\underline{\widetilde{\text{RelGap}}} \equiv \underline{\widetilde{\text{RelGap}}}(\Lambda_1,\widetilde{\Lambda}_2;\alpha,\alpha+\delta)$.

3.4 Singular Space Variations

Let B and $\widetilde{B} = D_1^* B D_2$ be two $m\times n$ (complex) matrices whose SVDs are given by (2.4) and (2.5), where $m\ge n$, D_1 and D_2 are two nonsingular matrices close to identities. The following two theorems concern singular space perturbations.

Theorem 3.5 *If* $\text{RelGap}_2 \equiv \text{RelGap}_2(\Sigma_1,\widetilde{\Sigma}_2;m) > 0$*, then*

$$\begin{aligned}
&\sqrt{\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2+\|\sin\Theta(V_1,\widetilde{V}_1)\|_F^2}\\
&\quad\le\frac{\sqrt{\|(I-D_1^*)U_1\|_F^2+\|(I-D_1^{-1})U_1\|_F^2+\|(I-D_2^*)V_1\|_F^2+\|(I-D_2^{-1})V_1\|_F^2}}{\text{RelGap}_2}.
\end{aligned} \qquad (3.7)$$

Similarly to Theorem 3.2, by imposing a stronger condition on the separation between the spectra of $\widetilde{\Sigma}_2$ and Σ_1, we have the following bound on any unitarily invariant norm of $\sin\Theta(U_1,\widetilde{U}_1)$ and $\sin\Theta(V_1,\widetilde{V}_1)$.

Theorem 3.6 *If there exist $\alpha>0$ and $\delta>0$ such that*

$$\min_{1\le i\le k}\sigma_i \ge \alpha+\delta \quad \textit{and} \quad \max_{1\le j\le n-k}\widetilde{\sigma}_{k+j}\le\alpha,$$

then for any unitarily invariant norm $|||\cdot|||$

$$\max\left\{\left|\left|\left|\sin\Theta(U_1,\widetilde{U}_1)\right|\right|\right|, \left|\left|\left|\sin\Theta(V_1,\widetilde{V}_1)\right|\right|\right|\right\}$$

$$\leq \frac{1}{\underline{\mathrm{RelGap}}_p}\max\left\{\sqrt[q]{\left|\left|\left|(I-D_2^{-1})V_1\right|\right|\right|^q+|||(D_1^*-I)U_1|||^q},\right.$$

$$\left.\sqrt[q]{\left|\left|\left|(I-D_1^{-1})U_1\right|\right|\right|^q+|||(D_2^*-I)V_1|||^q}\right\}, \tag{3.8}$$

$$\left|\left|\left|\begin{pmatrix}\sin\Theta(U_1,\widetilde{U}_1) & \\ & \sin\Theta(V_1,\widetilde{V}_1)\end{pmatrix}\right|\right|\right|$$

$$\leq\frac{\sqrt[q]{\left|\left|\left|\begin{pmatrix}(I-D_1^{-1})U_1 & \\ & (I-D_2^{-1})V_1\end{pmatrix}\right|\right|\right|^q+\left|\left|\left|\begin{pmatrix}(I-D_1^{*})U_1 & \\ & (I-D_2^{*})V_1\end{pmatrix}\right|\right|\right|^q}}{\underline{\mathrm{RelGap}}_p}, \tag{3.9}$$

where $\underline{\mathrm{RelGap}}_p \equiv \underline{\mathrm{RelGap}}_p(\Sigma_1,\widetilde{\Sigma}_2;\alpha,\alpha+\delta)$.

In both Theorems 3.5 and 3.6, we assumed that both D_1 and D_2 are close to identity matrices. But, intuitively D_2 should not affect U_1 much as long as it is close to a unitary matrix. The most extremal case is when $D_2\in\mathbb{U}_n$. In what follows, we are going to explore this intuition. We will prove

Theorem 3.7 *If*[2]

$$\mathrm{RelGap}_2\equiv\mathrm{RelGap}_2(\Sigma_1,\widetilde{\Sigma}_2;m)>\frac{1}{2\sqrt{2}}\max\{\|D_1^*-D_1^{-1}\|_2,\,\|D_2^*-D_2^{-1}\|_2\}, \tag{3.10}$$

then

$$\left\|\sin\Theta(U_1,\widetilde{U}_1)\right\|_F \leq \frac{\sqrt{\|(I-D_1^{-1})U_1\|_F^2+\|(I-D_1^*)U_1\|_F^2}}{\mathrm{RelGap}_2-\epsilon_2}+\frac{\|D_2^*-D_2^{-1}\|_F}{2\left[\widetilde{\mathrm{RelGap}}-\widetilde{\epsilon}_1\right]}, \tag{3.11}$$

$$\left\|\sin\Theta(V_1,\widetilde{V}_1)\right\|_F \leq \frac{\|D_1^*-D_1^{-1}\|_F}{2\left[\widetilde{\mathrm{RelGap}}-\widetilde{\epsilon}_2\right]}+\frac{\sqrt{\|(I-D_2^{-1})V_1\|_F^2+\|(I-D_2^*)V_1\|_F^2}}{\mathrm{RelGap}_2-\epsilon_1}, \tag{3.12}$$

[2]This implies, by Proposition 3.2,

$$\widetilde{\mathrm{RelGap}}\equiv\widetilde{\mathrm{RelGap}}(\Sigma_1,\widetilde{\Sigma}_2;m)>\frac{1}{2}\max\{\|D_1^*-D_1^{-1}\|_2,\,\|D_2^*-D_2^{-1}\|_2\}.$$

where $\widetilde{\text{RelGap}} \equiv \widetilde{\text{RelGap}}(\Sigma_1, \widetilde{\Sigma}_2; m)$ *and*

$$\widetilde{\epsilon}_1 = \frac{1}{2}\|D_1^* - D_1^{-1}\|_2, \qquad \widetilde{\epsilon}_2 = \frac{1}{2}\|D_2^* - D_2^{-1}\|_2,$$
$$\epsilon_1 = \frac{1}{2\sqrt{2}}\|D_1^* - D_1^{-1}\|_2, \qquad \epsilon_2 = \frac{1}{2\sqrt{2}}\|D_2^* - D_2^{-1}\|_2.$$

The inequality (3.11) clearly says that D_2 contributes to $\sin\Theta(U_1, \widetilde{U}_1)$ with its departure from *some* unitary matrix, and similar for (3.12).

Remark. When one of the D_1 and D_2 is I, the assumption (3.10) becomes unnecessary in order for the conclusion of Theorem 3.7 to be true. In fact, if either D_1 or D_2 is I, we only need to assume $\text{RelGap}_2(\Sigma_1, \widetilde{\Sigma}_2; m) > 0$ which in turn insure $\widetilde{\text{RelGap}}(\Sigma_1, \widetilde{\Sigma}_2; m) > 0$.

Theorem 3.8 *Suppose that there exist* $\alpha > 0$ *and* $\delta > 0$ *such that*

$$\min_{1\le i\le k} \sigma_i \ge \alpha + \delta \quad \textit{and} \quad \max_{1\le j\le n-k} \widetilde{\sigma}_{k+j} \le \alpha.$$

Assume that[3]

$$\text{RelDist}_p(\alpha, \alpha+\delta) > \frac{1}{2^{1+1/p}} \max\{\|D_1^* - D_1^{-1}\|_2, \|D_2^* - D_2^{-1}\|_2\}. \tag{3.13}$$

Then for any unitarily invariant norm $|||\cdot|||$

$$\left|\left|\left|\sin\Theta(U_1, \widetilde{U}_1)\right|\right|\right| \le \frac{\sqrt[q]{|||(I - D_1^{-1})U_1|||^q + |||(I - D_1^*)U_1|||^q}}{\underline{\text{RelGap}}_p - \epsilon_2} + \frac{|||D_2^* - D_2^{-1}|||}{2\left[\widetilde{\underline{\text{RelGap}}} - \widetilde{\epsilon}_1\right]}, \tag{3.14}$$

$$\left|\left|\left|\sin\Theta(V_1, \widetilde{V}_1)\right|\right|\right| \le \frac{|||D_1^* - D_1^{-1}|||}{2\left[\widetilde{\underline{\text{RelGap}}} - \widetilde{\epsilon}_2\right]} + \frac{\sqrt[q]{|||(I - D_2^{-1})V_1|||^q + |||(I - D_2^*)V_1|||^q}}{\underline{\text{RelGap}}_p - \epsilon_1}, \tag{3.15}$$

where $\underline{\text{RelGap}}_p \equiv \underline{\text{RelGap}}_p(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \alpha+\delta)$, $\widetilde{\underline{\text{RelGap}}} \equiv \widetilde{\underline{\text{RelGap}}}(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \alpha+\delta)$, *and*

$$\widetilde{\epsilon}_1 = \frac{1}{2}\|D_1^* - D_1^{-1}\|_2, \qquad \widetilde{\epsilon}_2 = \frac{1}{2}\|D_2^* - D_2^{-1}\|_2,$$
$$\epsilon_1 = \frac{1}{2^{1+1/p}}\|D_1^* - D_1^{-1}\|_2, \qquad \epsilon_2 = \frac{1}{2^{1+1/p}}\|D_2^* - D_2^{-1}\|_2.$$

[3] This implies

$$\widetilde{\text{RelDist}}(\alpha, \alpha+\delta) > \frac{1}{2}\max\{\|D_1^* - D_1^{-1}\|_2, \|D_2^* - D_2^{-1}\|_2\}.$$

Remark. If either D_1 or D_2 is I, we only need to assume $\mathrm{RelGap}_2(\Sigma_1, \widetilde{\Sigma}_2; m) > 0$ which in turn insure $\widetilde{\mathrm{RelGap}}(\Sigma_1, \widetilde{\Sigma}_2; m) > 0$, instead of (3.13).

Now, Let's briefly mention a possible application of Theorems 3.5, 3.6, 3.7 and 3.8. It has something to do with *deflation* in computing the singular value systems of a bidiagonal matrix. Taking account of the remarks we have made, we get

Corollary 3.1 *Assume $D_1 = I$ and D_2 takes the form*

$$D_2 = \begin{pmatrix} I & X \\ & I \end{pmatrix},$$

where X is a matrix of suitable dimensions. If $\mathrm{RelGap}_2 \equiv \mathrm{RelGap}_2(\Sigma_1, \widetilde{\Sigma}_2; m) > 0$, then

$$\begin{aligned} \left\| \sin\Theta(U_1, \widetilde{U}_1) \right\|_F &\leq \frac{\|X\|_F}{\sqrt{2}\,\widetilde{\mathrm{RelGap}}}, \\ \sqrt{\left\| \sin\Theta(U_1, \widetilde{U}_1) \right\|_F^2 + \left\| \sin\Theta(V_1, \widetilde{V}_1) \right\|_F^2} &\leq \frac{\sqrt{2}\|X\|_F}{\mathrm{RelGap}_2}, \end{aligned}$$

where $\widetilde{\mathrm{RelGap}} \equiv \widetilde{\mathrm{RelGap}}(\Sigma_1, \widetilde{\Sigma}_2; m)$.

Corollary 3.2 *Assume $D_2 = I$ and D_1 takes the form*

$$D_1 = \begin{pmatrix} I & X \\ & I \end{pmatrix},$$

where X is a matrix of suitable dimensions. If $\mathrm{RelGap}_2 \equiv \mathrm{RelGap}_2(\Sigma_1, \widetilde{\Sigma}_2; m) > 0$, then

$$\begin{aligned} \left\| \sin\Theta(V_1, \widetilde{V}_1) \right\|_F &\leq \frac{\|X\|_F}{\sqrt{2}\,\widetilde{\mathrm{RelGap}}}, \\ \sqrt{\left\| \sin\Theta(U_1, \widetilde{U}_1) \right\|_F^2 + \left\| \sin\Theta(V_1, \widetilde{V}_1) \right\|_F^2} &\leq \frac{\sqrt{2}\|X\|_F}{\mathrm{RelGap}_2}, \end{aligned}$$

where $\widetilde{\mathrm{RelGap}} \equiv \widetilde{\mathrm{RelGap}}(\Sigma_1, \widetilde{\Sigma}_2; m)$.

Corollary 3.3 *Assume $D_1 = I$ and D_2 takes the form*

$$D_2 = \begin{pmatrix} I & X \\ & I \end{pmatrix},$$

where X is a matrix of suitable dimensions. Suppose that there exist $\alpha > 0$ and $\delta > 0$ such that

$$\min_{1 \leq i \leq k} \sigma_i \geq \alpha + \delta \quad \textit{and} \quad \max_{1 \leq j \leq n-k} \widetilde{\sigma}_{k+j} \leq \alpha.$$

Then

$$\left\|\left|\sin\Theta(U_1,\widetilde{U}_1)\right|\right\| \leq \frac{\|X\|}{\underline{\widetilde{\mathrm{RelGap}}}},$$
$$\left\|\left|\sin\Theta(V_1,\widetilde{V}_1)\right|\right\| \leq \frac{2^{1/q}\,\|X\|}{\underline{\mathrm{RelGap}}_p},$$

where $\underline{\mathrm{RelGap}}_p \equiv \underline{\mathrm{RelGap}}_p(\Sigma_1,\widetilde{\Sigma}_2;\alpha,\alpha+\delta)$, $\underline{\widetilde{\mathrm{RelGap}}} \equiv \underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1,\widetilde{\Sigma}_2;\alpha,\alpha+\delta)$.

Corollary 3.4 *Assume $D_2 = I$ and D_1 takes the form*

$$D_1 = \begin{pmatrix} I & X \\ & I \end{pmatrix},$$

where X is a matrix of suitable dimensions. Suppose that there exist $\alpha > 0$ and $\delta > 0$ such that

$$\min_{1\leq i\leq k}\sigma_i \geq \alpha+\delta \quad \textit{and} \quad \max_{1\leq j\leq n-k}\widetilde{\sigma}_{k+j} \leq \alpha.$$

Then

$$\left\|\left|\sin\Theta(U_1,\widetilde{U}_1)\right|\right\| \leq \frac{2^{1/q}\,\|X\|}{\underline{\mathrm{RelGap}}_p},$$
$$\left\|\left|\sin\Theta(V_1,\widetilde{V}_1)\right|\right\| \leq \frac{\|X\|}{\underline{\widetilde{\mathrm{RelGap}}}},$$

where $\underline{\mathrm{RelGap}}_p \equiv \underline{\mathrm{RelGap}}_p(\Sigma_1,\widetilde{\Sigma}_2;\alpha,\alpha+\delta)$, $\underline{\widetilde{\mathrm{RelGap}}} \equiv \underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1,\widetilde{\Sigma}_2;\alpha,\alpha+\delta)$.

3.5 Singular Space Variations for Graded Matrices

Now, we consider perturbations of a graded singular value problem: $G = BD \in \mathbb{C}^{n\times n}$ and $\widetilde{G} = \widetilde{B}D \in \mathbb{C}^{n\times n}$, where it is assumed that B is nonsingular and D is an $n\times n$ matrix close to I. Set

$$\Delta B \stackrel{\mathrm{def}}{=} \widetilde{B} - B.$$

If $\|(\Delta B)B^{-1}\|_2 < 1$, then $\widetilde{B} = B + \Delta B = [I + (\Delta B)B^{-1}]B$ is nonsingular also. Let G and $\widetilde{G}$ having the following SVDs:

$$G = U\Sigma V^* = (U_1,U_2)\begin{pmatrix}\Sigma_1 & \\ & \Sigma_2\end{pmatrix}\begin{pmatrix}V_1^* \\ V_2^*\end{pmatrix}, \tag{3.16}$$
$$\widetilde{G} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^* = (\widetilde{U}_1,\widetilde{U}_2)\begin{pmatrix}\widetilde{\Sigma}_1 & \\ & \widetilde{\Sigma}_2\end{pmatrix}\begin{pmatrix}\widetilde{V}_1^* \\ \widetilde{V}_2^*\end{pmatrix}, \tag{3.17}$$

where U, V, $\widetilde{U}$, $\widetilde{V} \in \mathbb{U}_n$, U_1, V_1, $\widetilde{U}_1$, $\widetilde{V}_1 \in \mathbb{C}^{n\times k}$ and Σ_i's and $\widetilde{\Sigma}_j$'s are given as in (2.6) and (2.7). Write

$$\widetilde{G} = \widetilde{B}D = [I + (\Delta B)B^{-1}]BD.$$

So applying Theorems 3.5, 3.6, 3.7 and 3.8 to G and $\widetilde{G}$ with $D_1^* = I + (\Delta B)B^{-1}$ and $D_2 = I$ gives the following two theorems on graded singular value problems.

Theorem 3.9 *Assume* $\|(\Delta B)B^{-1}\|_2 < 1$. *If* $\mathrm{RelGap}_2 \equiv \mathrm{RelGap}_2(\Sigma_1, \widetilde{\Sigma}_2) > 0$, *then*

$$\begin{aligned}
&\sqrt{\left\|\sin\Theta(U_1,\widetilde{U}_1)\right\|_F^2 + \left\|\sin\Theta(V_1,\widetilde{V}_1)\right\|_F^2} \\
&\quad\le \frac{\sqrt{\|(\Delta B)B^{-1}U_1\|_F^2 + \|[I + B^{-*}(\Delta B)^*]^{-1}B^{-*}(\Delta B)^*U_1\|_F^2}}{\mathrm{RelGap}_2} \qquad (3.18)\\
&\quad\le \|B^{-1}\|_2\sqrt{1 + \frac{1}{(1-\|B^{-1}\|_2\|\Delta B\|_2)^2}}\,\frac{\|\Delta B\|_F}{\mathrm{RelGap}_2},
\end{aligned}$$

$$\begin{aligned}
&\left\|\sin\Theta(V_1,\widetilde{V}_1)\right\|_F \\
&\quad\le \frac{\|I + (\Delta B)B^{-1} - (I + (\Delta B)B^{-1})^{-*}\|_F}{2\,\widetilde{\mathrm{RelGap}}(\Sigma_1,\widetilde{\Sigma}_2)} \qquad (3.19)\\
&\quad\le \left(\frac{\|(\Delta B)B^{-1} + B^{-*}(\Delta B)^*\|_F}{\|(\Delta B)B^{-1}\|_F} + \frac{\|(\Delta B)B^{-1}\|_2}{1-\|(\Delta B)B^{-1}\|_2}\right)\frac{\|(\Delta B)B^{-1}\|_F}{2\,\widetilde{\mathrm{RelGap}}(\Sigma_1,\widetilde{\Sigma}_2)}\\
&\quad\le \left(1 + \frac{1}{1-\|B^{-1}\|_2\|\Delta B\|_2}\right)\frac{\|B^{-1}\|_2\|\Delta B\|_F}{2\,\widetilde{\mathrm{RelGap}}(\Sigma_1,\widetilde{\Sigma}_2)}.
\end{aligned}$$

Theorem 3.10 *Assume* $\|(\Delta B)B^{-1}\|_2 < 1$. *If there exist* $\alpha > 0$ *and* $\delta > 0$ *such that*

$$\min_{1\le i\le k}\sigma_i \ge \alpha + \delta \quad \textit{and} \quad \max_{1\le j\le n-k}\widetilde{\sigma}_{k+j} \le \alpha$$

or, the other way around, i.e.,

$$\max_{1\le i\le k}\sigma_i \le \alpha \quad \textit{and} \quad \min_{1\le j\le n-k}\widetilde{\sigma}_{k+j} \ge \alpha + \delta,$$

then for any unitarily invariant norm $|||\cdot|||$

$$\begin{aligned}
&\max\left\{\left|\left|\left|\sin\Theta(U_1,\widetilde{U}_1)\right|\right|\right|, \left|\left|\left|\sin\Theta(V_1,\widetilde{V}_1)\right|\right|\right|\right\} \\
&\quad\le \frac{\max\left\{|||(\Delta B)B^{-1}U_1|||, |||[I + B^{-*}(\Delta B)^*]^{-1}B^{-*}(\Delta B)^*U_1|||\right\}}{\mathrm{RelGap}_\infty} \qquad (3.20)\\
&\quad\le \frac{\|B^{-1}\|_2}{1-\|B^{-1}\|_2\|\Delta B\|_2}\,\frac{|||\Delta B|||}{\mathrm{RelGap}_\infty},
\end{aligned}$$

$$
\begin{aligned}
&\left|\!\left|\!\left|\begin{pmatrix} \sin\Theta(U_1,\widetilde{U}_1) & \\ & \sin\Theta(V_1,\widetilde{V}_1)\end{pmatrix}\right|\!\right|\!\right| \\
&\quad\le \frac{\sqrt[q]{|\!|\!|(\Delta B)B^{-1}U_1|\!|\!|^q + |\!|\!|[I+B^{-*}(\Delta B)^*]^{-1}B^{-*}(\Delta B)^*U_1|\!|\!|^q}}{\underline{\mathrm{RelGap}}_p} \qquad (3.21)\\
&\quad\le \|B^{-1}\|_2\sqrt[q]{1+\frac{1}{(1-\|B^{-1}\|_2\|\Delta B\|_2)^q}}\;\frac{|\!|\!|\Delta B|\!|\!|}{\underline{\mathrm{RelGap}}_p},\\
&\left|\!\left|\!\left|\sin\Theta(V_1,\widetilde{V}_1)\right|\!\right|\!\right| \\
&\quad\le \frac{|\!|\!|I+(\Delta B)B^{-1}-(I+(\Delta B)B^{-1})^{-1}|\!|\!|}{2\,\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1,\widetilde{\Sigma}_2)} \qquad (3.22)\\
&\quad\le \left(\frac{|\!|\!|(\Delta B)B^{-1}+B^{-*}(\Delta B)^*|\!|\!|}{|\!|\!|(\Delta B)B^{-1}|\!|\!|}+\frac{\|(\Delta B)B^{-1}\|_2}{1-\|(\Delta B)B^{-1}\|_2}\right)\frac{|\!|\!|(\Delta B)B^{-1}|\!|\!|}{2\,\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1,\widetilde{\Sigma}_2)}\\
&\quad\le \left(1+\frac{1}{1-\|B^{-1}\|_2\|\Delta B\|_2}\right)\frac{\|B^{-1}\|_2\,|\!|\!|\Delta B|\!|\!|}{2\,\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1,\widetilde{\Sigma}_2)}.
\end{aligned}
$$

where $\underline{\mathrm{RelGap}}_p \equiv \underline{\mathrm{RelGap}}_p(\Sigma_1,\widetilde{\Sigma}_2;\alpha,\alpha+\delta)$, *(and thus* $\mathrm{RelGap}_\infty = \delta/(\alpha+\delta)$*).*

Remark. The inequalities (3.19) and (3.22) may provide much tighter bounds than (3.18) and (3.20), especially when $I+(\Delta B)B^{-1}$ is very close to a skew Hermitian matrix.

4 More on Relative Gaps

In theorems of §3, relative gaps RelGap_2, $\widetilde{\mathrm{RelGap}}$, $\underline{\mathrm{RelGap}}_p$ and $\underline{\widetilde{\mathrm{RelGap}}}$ play an indispensable role. Those gaps are imposed on either between Λ_1 and $\widetilde{\Lambda}_2$ or between Σ_1 and $\widetilde{\Sigma}_2$. In some applications, it may be more convenient to have theorems where only positive relative gaps between Λ_1 and Λ_2 or between Σ_1 and Σ_2 are assumed. Based on results of Ostrowski [5, pp. 224–225], Barlow and Demmel [1], Demmel and Veselić [3], Mathias [8], and Li [7], theorems in §3 can be modified to accommodate this need. In what follows, we list inequalities for how to bound relative gaps between Λ_1 and $\widetilde{\Lambda}_2$ or between Σ_1 and $\widetilde{\Sigma}_2$ from below for each theorem by their corresponding relative gaps between Λ_1 and Λ_2 or between Σ_1 and Σ_2. The derivations of these inequalities depends on the fact that [7]

1. $\mathrm{RelDist}_1$, $\mathrm{RelDist}_2$ and $\mathrm{RelDist}_\infty$ are metrics on $\mathbb{R}$;
2. $\mathrm{RelDist}_p$ is a metric on $\mathbb{R}_{\geq 0}$; and
3. $\widetilde{\mathrm{RelDist}}$ is a generalized metric on $\mathbb{R}_{\geq 0}$.

For Theorem 3.1: If $\|I - D^* D\|_2 < \mathrm{RelGap}_2(\Lambda_1, \Lambda_2)$,

$$\mathrm{RelGap}_2(\Lambda_1, \widetilde{\Lambda}_2) \geq \mathrm{RelGap}_2(\Lambda_1, \Lambda_2) - \|I - D^* D\|_2.$$

For Theorem 3.2: Assume that for some $0 \leq \alpha_1 < \beta_1$, $\underline{\mathrm{RelGap}}_p(\Lambda_1, \Lambda_2; \alpha_1, \beta_1)$ is definable. If $\|I - D^* D\|_2 < \underline{\mathrm{RelGap}}_p(\Lambda_1, \Lambda_2; \alpha_1, \beta_1)$, then $\underline{\mathrm{RelGap}}_p(\Lambda_1, \widetilde{\Lambda}_2; \alpha, \beta)$ is definable for some $0 \leq \alpha < \beta$ and

$$\underline{\mathrm{RelGap}}_p(\Lambda_1, \widetilde{\Lambda}_2; \alpha, \beta) \geq \underline{\mathrm{RelGap}}_p(\Lambda_1, \Lambda_2; \alpha_1, \beta_1) - \|I - D^* D\|_2.$$

For Theorem 3.3: If $\epsilon < \widetilde{\mathrm{RelGap}}(\Lambda_1, \Lambda_2)$,

$$\widetilde{\mathrm{RelGap}}(\Lambda_1, \widetilde{\Lambda}_2) \geq \widetilde{\mathrm{RelGap}}(\Lambda_1, \Lambda_2) - \epsilon,$$

where $\epsilon = \frac{\|A^{-1}\|_2 \|\Delta A\|_2}{\sqrt{1 - \|A^{-1}\|_2 \|\Delta A\|_2}}$.

For Theorem 3.4: Assume that for some $0 \leq \alpha_1 < \beta_1$, $\underline{\widetilde{\mathrm{RelGap}}}(\Lambda_1, \Lambda_2; \alpha_1, \beta_1)$ is definable. If $\epsilon < \underline{\widetilde{\mathrm{RelGap}}}(\Lambda_1, \Lambda_2; \alpha_1, \beta_1)$, then $\underline{\widetilde{\mathrm{RelGap}}}(\Lambda_1, \widetilde{\Lambda}_2; \alpha, \beta)$ is definable for some $0 \leq \alpha < \beta$ and

$$\underline{\widetilde{\mathrm{RelGap}}}(\Lambda_1, \widetilde{\Lambda}_2; \alpha, \beta) \geq \underline{\widetilde{\mathrm{RelGap}}}(\Lambda_1, \Lambda_2; \alpha_1, \beta_1) - \epsilon,$$

where $\epsilon = \frac{\|A^{-1}\|_2 \|\Delta A\|_2}{\sqrt{1 - \|A^{-1}\|_2 \|\Delta A\|_2}}$.

For Theorem 3.5: Let

$$\epsilon = \max\{|1 - \sigma_{\min}(D_1)\sigma_{\min}(D_2)|,\ |1 - \sigma_{\max}(D_1)\sigma_{\max}(D_2)|\}.$$

If $\epsilon < \mathrm{RelGap}_2(\Sigma_1, \Sigma_2; m)$, then

$$\mathrm{RelGap}_2(\Sigma_1, \widetilde{\Sigma}_2; m) \geq \mathrm{RelGap}_2(\Sigma_1, \Sigma_2; m) - \epsilon.$$

For Theorems 3.6: Let

$$\epsilon = \max\{|1 - \sigma_{\min}(D_1)\sigma_{\min}(D_2)|,\ |1 - \sigma_{\max}(D_1)\sigma_{\max}(D_2)|\}.$$

Assume that for some $0 \leq \alpha_1 < \beta_1$, $\underline{\mathrm{RelGap}}_p(\Sigma_1, \Sigma_2; \alpha_1, \beta_1)$ is definable. If $\epsilon < \underline{\mathrm{RelGap}}_p(\Sigma_1, \Sigma_2; \alpha_1, \beta_1)$ then $\underline{\mathrm{RelGap}}_p(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta)$ is definable for some $0 \leq \alpha < \beta$ and

$$\underline{\mathrm{RelGap}}_p(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta) \geq \underline{\mathrm{RelGap}}_p(\Sigma_1, \Sigma_2; \alpha_1, \beta_1) - \epsilon.$$

For Theorems 3.7: Let ϵ_1, ϵ_2, $\widetilde{\epsilon}_1$ and $\widetilde{\epsilon}_2$ be as defined in the theorem. If

$$\widetilde{\epsilon}_1 + \widetilde{\epsilon}_2 + \max\{\widetilde{\epsilon}_1,\ \widetilde{\epsilon}_2\} < \widetilde{\mathrm{RelGap}}(\Sigma_1, \Sigma_2; m),$$

then

$$\begin{aligned}
\mathrm{RelGap}_2(\Sigma_1, \widetilde{\Sigma}_2; m) &\geq \mathrm{RelGap}_2(\Sigma_1, \Sigma_2; m) - \epsilon_1 - \epsilon_2, \\
\widetilde{\mathrm{RelGap}}(\Sigma_1, \widetilde{\Sigma}_2; m) &\geq \widetilde{\mathrm{RelGap}}(\Sigma_1, \Sigma_2; m) - \widetilde{\epsilon}_1 - \widetilde{\epsilon}_2.
\end{aligned}$$

For Theorems 3.8: Let ϵ_1, ϵ_2, $\widetilde{\epsilon}_1$ and $\widetilde{\epsilon}_2$ be as defined in the theorem. Assume that for some $0 \leq \alpha_1 < \beta_1$, $\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \Sigma_2; \alpha_1, \beta_1)$ is definable. If

$$\widetilde{\epsilon}_1 + \widetilde{\epsilon}_2 + \max\{\widetilde{\epsilon}_1,\ \widetilde{\epsilon}_2\} < \underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \Sigma_2; \alpha_1, \beta_1),$$

then $\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta)$ is definable for some $0 \leq \alpha < \beta$ and

$$\begin{aligned}
\underline{\mathrm{RelGap}}_p(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta) &\geq \underline{\mathrm{RelGap}}_p(\Sigma_1, \Sigma_2; \alpha_1, \beta_1) - \epsilon_1 - \epsilon_2, \\
\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta) &\geq \underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \Sigma_2; \alpha_1, \beta_1) - \widetilde{\epsilon}_1 - \widetilde{\epsilon}_2.
\end{aligned}$$

For Theorem 3.9: Let $\epsilon = \|B^{-1}\|_2 \|\Delta B\|_2$ and

$$\widetilde{\epsilon} = \left(1 + \frac{1}{1 - \|B^{-1}\|_2 \|\Delta B\|_2}\right) \frac{\|B^{-1}\|_2 \|\Delta B\|_2}{2}.$$

If $\epsilon < \mathrm{RelGap}_2(\Sigma_1, \Sigma_2)$ and, $\widetilde{\epsilon} < \widetilde{\mathrm{RelGap}}(\Sigma_1, \Sigma_2)$,

$$\begin{aligned}
\mathrm{RelGap}_2(\Sigma_1, \widetilde{\Sigma}_2) &\geq \mathrm{RelGap}_2(\Sigma_1, \Sigma_2) - \epsilon, \\
\widetilde{\mathrm{RelGap}}(\Sigma_1, \widetilde{\Sigma}_2) &\geq \widetilde{\mathrm{RelGap}}(\Sigma_1, \Sigma_2) - \widetilde{\epsilon}.
\end{aligned}$$

For Theorem 3.10: Let $\epsilon = \|B^{-1}\|_2 \|\Delta B\|_2$ and

$$\widetilde{\epsilon} = \left(1 + \frac{1}{1 - \|B^{-1}\|_2 \|\Delta B\|_2}\right) \frac{\|B^{-1}\|_2 \|\Delta B\|_2}{2}.$$

Assume that for some $0 \le \alpha_1 < \beta_1$, $\underline{\mathrm{RelGap}}_p(\Sigma_1, \Sigma_2; \alpha_1, \beta_1)$ and $\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \Sigma_2; \alpha_1, \beta_1)$ are definable. If $\epsilon < \underline{\mathrm{RelGap}}_p(\Sigma_1, \Sigma_2; \alpha_1, \beta_1)$ and $\widetilde{\epsilon} < \underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \Sigma_2; \alpha_1, \beta_1)$, then $\underline{\mathrm{RelGap}}_p(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta)$ and $\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta)$ are definable for some $0 \le \alpha < \beta$ and

$$\begin{aligned}
\underline{\mathrm{RelGap}}_p(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta) &\ge \underline{\mathrm{RelGap}}_p(\Sigma_1, \Sigma_2; \alpha_1, \beta_1) - \epsilon, \\
\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta) &\ge \underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \Sigma_2; \alpha_1, \beta_1) - \widetilde{\epsilon}.
\end{aligned}$$

5 A Word on Eisenstat-Ipsen's Theorems

Eisenstat and Ipsen [4] have obtained a few bounds on eigenspace variations for A and $\widetilde{A} = D^* A D$ and on singular space variations for B and $= \widetilde{B} = D_1^* B D_2$. Here we are not going to state their results for subspaces of dimension $k > 1$ since their results contain a factor of $\sqrt{k}$ which makes their results uncompetitive to ours.

Regarding to the problem related to Theorem 3.1, Eisenstat and Ipsen [4] tried to bound the angle θ_j between $\widetilde{u}_j$ $(1 \le j \le k)$ and the $\mathcal{R}(U_1)$, where $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_k$ are the columns of $\widetilde{U}_1$. They showed

$$\sin\theta_j \le \frac{\|I - D^{-*}D^{-1}\|_2}{\eta_j} + \|I - D\|_2, \tag{5.1}$$

where

$$\eta_j \stackrel{\text{def}}{=} \begin{cases} \min\limits_{k+1 \le i \le n} \frac{|\lambda_i - \widetilde{\lambda}_j|}{|\widetilde{\lambda}_j|}, & \text{if } \widetilde{\lambda}_j \ne 0, \\ \infty & \text{otherwise.} \end{cases}$$

The inequality (5.1) does provide a nice bound. Generally,

$$\sin\theta_j \le \|\sin\Theta(U_1, \widetilde{U}_1)\|_2$$

and all of them may be strict. To make a fair comparison to our inequality (3.1), let's consider the case $k = 1$. One infers from our inequality (3.1) that[4]

$$\sin\theta_1 \le \frac{\sqrt{\|I - D^{-1}\|_2^2 + \|I - D^*\|_2^2}}{\min\limits_{2 \le i \le n} \mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_1)}. \tag{5.2}$$

At the first sight, one might think (5.1) is potentially sharper because η_1 may be much larger than $\min\limits_{2 \le i \le n} \mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_1)$. This is totally wrong! Yes, η_1 may be very large, but this is shadowed by the extra term $\|I - D\|_2$ in (5.1) which stays no matter how large η_1 is. As a matter of fact[5], we should really think of

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[4]By treating A and $\widetilde{A}$ symmetrically, one can see Theorem 3.1 remains valid with $\mathrm{RelGap}_2(\Lambda_1, \widetilde{\Lambda}_2)$ replaced by $\mathrm{RelGap}_2(\widetilde{\Lambda}_1, \Lambda_2)$.

[5]Going through our proof of Theorem 3.1, we can see that when $k = 1$ and $\lambda_1 = 0$ we have

$$\sin\theta_1 \le \|I - D\|_2.$$

In fact, besides (6.1) one also has

$$\widetilde{\Lambda}_1 \widetilde{U}_1^* U_2 - \widetilde{U}_1^* U_2 \Lambda_2 = \widetilde{\Lambda}_1 \widetilde{U}_1^* (D^{-1} - I) U_2 + \widetilde{U}_1^* (I - D^*) U_2 \Lambda_2.$$

When $\widetilde{\Lambda}_1 = 0$, this equation reduces to

$$-\widetilde{U}_1^* U_2 = \widetilde{U}_1^* (I - D^*) U_2,$$

provided $\lambda_j \ne 0$ for $j = k+1, \cdots, n$.

(5.1) as (at least for tiny $\|I-D\|_2$)

$$\sin\theta_1 \le \frac{\|I-D^{-*}D^{-1}\|_2 + \|I-D\|_2}{\min\limits_{2\le i\le n} \mathrm{RelDist}_\infty(\lambda_i, \widetilde{\lambda}_1)}. \tag{5.3}$$

Well, still we cannot say definitely which one of (5.2) and (5.3) is always sharper than the other. However asymptotically, (5.3) is less sharp by a factor $3/2$. In fact, if $I-D$ is very tiny, then

$$\begin{aligned}
\min_{2\le i\le n} \mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_1) &\approx \frac{1}{\sqrt{2}} \min_{2\le i\le n} \mathrm{RelDist}_\infty(\lambda_i, \widetilde{\lambda}_1), \\
\sqrt{\|I-D^{-1}\|_2^2 + \|I-D^*\|_2^2} &\approx \sqrt{2}\|I-D\|_2, \\
\|I-D^{-*}D^{-1}\|_2 + \|I-D\|_2 &\approx 3\|I-D\|_2.
\end{aligned}$$

So asymptotically, the inequalities (5.2) and (5.3) read, respectively,

$$\begin{aligned}
\sin\theta_1 &\le \frac{2\|I-D\|_2}{\min\limits_{2\le i\le n} \mathrm{RelDist}_\infty(\lambda_i, \widetilde{\lambda}_1)} + O(\|I-D\|_2^2), \\
\sin\theta_1 &\le \frac{3\|I-D\|_2}{\min\limits_{2\le i\le n} \mathrm{RelDist}_\infty(\lambda_i, \widetilde{\lambda}_1)} + O(\|I-D\|_2^2).
\end{aligned}$$

Eisenstat and Ipsen [4] treated singular value problems in a very similar way.

6 Proofs of Theorems 3.1 and 3.2

Let R be as defined in (2.1). Notice

$$\begin{aligned}
\widetilde{U}_2^* R &= \widetilde{U}_2^* \widetilde{A} U_1 - \widetilde{U}_2^* U_1 \Lambda_1 \\
&= \widetilde{\Lambda}_2 \widetilde{U}_2^* U_1 - \widetilde{U}_2^* U_1 \Lambda_1, \\
\widetilde{U}_2^* R &= \widetilde{U}_2^* (\widetilde{A} - A) U_1 \\
&= \widetilde{U}_2^* \left[A - D^* A + D^* A - D^* A D \right] U_1 \\
&= \widetilde{U}_2^* \left[(I - D^*) A + D^* A D (D^{-1} - I) \right] U_1 \\
&= \widetilde{\Lambda}_2 \widetilde{U}_2^* (D^{-1} - I) U_1 + \widetilde{U}_2^* (I - D^*) U_1 \Lambda_1.
\end{aligned}$$

Thus, we have

$$\widetilde{\Lambda}_2 \widetilde{U}_2^* U_1 - \widetilde{U}_2^* U_1 \Lambda_1 = \widetilde{\Lambda}_2 \widetilde{U}_2^* (D^{-1} - I) U_1 + \widetilde{U}_2^* (I - D^*) U_1 \Lambda_1. \tag{6.1}$$

Proof of Theorem 3.1: Set

$$\begin{aligned}
Q &= \widetilde{U}_2^* U_1 = (q_{ij}), \\
E &= \widetilde{U}_2^* (D^{-1} - I) U_1 = (e_{ij}), \\
\widetilde{E} &= \widetilde{U}_2^* (I - D^*) U_1 = (\widetilde{e}_{ij}).
\end{aligned}$$

Then the equation (6.1) reads $\widetilde{\Lambda}_2 Q - Q \Lambda_1 = \widetilde{\Lambda}_2 E + \widetilde{E} \Lambda_1$, or componentwisely $\widetilde{\lambda}_{k+i} q_{ij} - q_{ij} \lambda_j = \widetilde{\lambda}_{k+i} e_{ij} + \widetilde{e}_{ij} \lambda_j$, so

$$|(\widetilde{\lambda}_{k+i} - \lambda_j) q_{ij}|^2 \leq (\widetilde{\lambda}_{k+i}^2 + \lambda_j^2)(|e_{ij}|^2 + |\widetilde{e}_{ij}|^2),$$

which yields, under the conditions of Theorem 3.1, that

$$|q_{ij}|^2 \leq \frac{|e_{ij}|^2 + |\widetilde{e}_{ij}|^2}{\mathrm{RelDist}_2^2(\widetilde{\lambda}_{k+i}, \lambda_j)} \leq \frac{|e_{ij}|^2 + |\widetilde{e}_{ij}|^2}{\mathrm{RelGap}_2^2}.$$

So

$$\begin{aligned}
\|Q\|_F^2 &= \sum_{i=1}^{n-k} \sum_{j=1}^{k} |q_{ij}|^2 \leq \frac{\sum_{i=1}^{n-k} \sum_{j=1}^{k} (|e_{ij}|^2 + |\widetilde{e}_{ij}|^2)}{\mathrm{RelGap}_2^2} \\
&= \frac{\|\widetilde{U}_2^* (D^{-1} - I) U_1\|_F^2 + \|\widetilde{U}_2^* (I - D^*) U_1\|_F^2}{\mathrm{RelGap}_2^2} \\
&\leq \frac{1}{\mathrm{RelGap}_2^2} \left(\|(D^{-1} - I) U_1\|_F^2 + \|(I - D^*) U_1\|_F^2 \right).
\end{aligned}$$

This completes the proof of Theorem 3.1, since $\|\sin \Theta(U_1, \widetilde{U}_1)\|_F = \|Q\|_F$. ∎

Proof of Theorem 3.2: It follows from the perturbation equation (6.1) that

$$\widetilde{U}_2^* U_1 - \widetilde{\Lambda}_2^{-1}\widetilde{U}_2^* U_1 \Lambda_1 = \widetilde{U}_2^*(D^{-1} - I)U_1 + \widetilde{\Lambda}_2^{-1}\widetilde{U}_2^*(I - D^*)U_1\Lambda_1. \tag{6.2}$$

By the assumptions of the theorem, we know that $\|\Lambda_1\|_2 \leq \alpha$ and $\|\widetilde{\Lambda}_2^{-1}\|_2 \leq \frac{1}{\alpha+\delta}$. Therefore

$$\begin{aligned}
&\left(1 - \frac{\alpha}{\alpha+\delta}\right) \left|\left|\left| \widetilde{U}_2^* U_1 \right|\right|\right| \\
\leq\ & \left|\left|\left| \widetilde{U}_2^* U_1 \right|\right|\right| - \frac{1}{\alpha+\delta} \left|\left|\left| \widetilde{U}_2^* U_1 \right|\right|\right| \alpha \\
\leq\ & \left|\left|\left| \widetilde{U}_2^* U_1 \right|\right|\right| - \|\widetilde{\Lambda}_2^{-1}\|_2 \left|\left|\left| \widetilde{U}_2^* U_1 \right|\right|\right| \|\Lambda_1\|_2 \\
\leq\ & \left|\left|\left| \widetilde{U}_2^* U_1 \right|\right|\right| - \left|\left|\left| \widetilde{\Lambda}_2^{-1}\widetilde{U}_2^* U_1 \Lambda_1 \right|\right|\right| \\
\leq\ & \left|\left|\left| \widetilde{U}_2^* U_1 - \widetilde{\Lambda}_2^{-1}\widetilde{U}_2^* U_1 \Lambda_1 \right|\right|\right| \\
=\ & \left|\left|\left| \widetilde{U}_2^*(D^{-1} - I)U_1 + \widetilde{\Lambda}_2^{-1}\widetilde{U}_2^*(I - D^*)U_1\Lambda_1 \right|\right|\right| \qquad \text{(by (6.2))} \\
\leq\ & |||(D^{-1} - I)U_1||| + \frac{\alpha}{\alpha+\delta} |||(I - D^*)U_1||| \\
\leq\ & \sqrt[p]{1 + \frac{\alpha^p}{(\alpha+\delta)^p}} \sqrt[q]{|||(D^{-1} - I)U_1|||^q + |||(I - D^*)U_1|||^q}
\end{aligned}$$

from which the inequality (3.2) follows. ∎

7 Proofs of Theorems 3.5 and 3.6

Let R and S be defined as in (2.8).

7.1 The Square Case: $m = n$

When $m = n$, the SVDs (2.4) and (2.5) read

$$\begin{aligned} B &= U\Sigma V^* = (U_1, U_2)\begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix}\begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix}, \\ \widetilde{B} &= \widetilde{U}\widetilde{\Sigma}\widetilde{V}^* = (\widetilde{U}_1, \widetilde{U}_2)\begin{pmatrix} \widetilde{\Sigma}_1 & \\ & \widetilde{\Sigma}_2 \end{pmatrix}\begin{pmatrix} \widetilde{V}_1^* \\ \widetilde{V}_2^* \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \widetilde{U}_2^* R &= \widetilde{U}_2^* \widetilde{B} V_1 - \widetilde{U}_2^* U_1 \Sigma_1 \\ &= \widetilde{\Sigma}_2 \widetilde{V}_2^* V_1 - \widetilde{U}_2^* U_1 \Sigma_1, \\ \widetilde{U}_2^* R &= \widetilde{U}_2^* (\widetilde{B} - B) V_1 = \widetilde{U}_2^* (D_1^* B D_2 - D_1^* B + D_1^* B - B) V_1 \\ &= \widetilde{U}_2^* \left[\widetilde{B}(I - D_2^{-1}) + (D_1^* - I) B \right] V_1 \\ &= \widetilde{\Sigma}_2 \widetilde{V}_2^* (I - D_2^{-1}) V_1 + \widetilde{U}_2^* (D_1^* - I) U_1 \Sigma_1. \end{aligned}$$

On the other hand, for the residual S, we have

$$\begin{aligned} \widetilde{V}_2^* S &= \widetilde{V}_2^* \widetilde{B}^* U_1 - \widetilde{V}_2^* V_1 \Sigma_1, \\ &= \widetilde{\Sigma}_2 \widetilde{U}_2^* U_1 - \widetilde{V}_2^* V_1 \Sigma_1. \\ \widetilde{V}_2^* S &= \widetilde{V}_2^* (\widetilde{B}^* - B^*) U_1 = \widetilde{V}_2^* (D_2^* B^* D_1 - D_2^* B^* + D_2^* B^* - B^*) U_1 \\ &= \widetilde{V}_2^* \left[\widetilde{B}^*(I - D_1^{-1}) + (D_2^* - I) B^* \right] U_1 \\ &= \widetilde{\Sigma}_2 \widetilde{U}_2^* (I - D_1^{-1}) U_1 + \widetilde{V}_2^* (D_2^* - I) V_1 \Sigma_1. \end{aligned}$$

Therefore, we obtained two perturbation equations as follows.

$$\widetilde{\Sigma}_2 \widetilde{V}_2^* V_1 - \widetilde{U}_2^* U_1 \Sigma_1 = \widetilde{\Sigma}_2 \widetilde{V}_2^* (I - D_2^{-1}) V_1 + \widetilde{U}_2^* (D_1^* - I) U_1 \Sigma_1, \quad (7.1)$$

$$\widetilde{\Sigma}_2 \widetilde{U}_2^* U_1 - \widetilde{V}_2^* V_1 \Sigma_1 = \widetilde{\Sigma}_2 \widetilde{U}_2^* (I - D_1^{-1}) U_1 + \widetilde{V}_2^* (D_2^* - I) V_1 \Sigma_1. \quad (7.2)$$

Set

$$\begin{aligned} &Q = \widetilde{U}_2^* U_1 = (q_{ij}), && \widetilde{Q} = \widetilde{V}_2^* V_1 = (\widetilde{q}_{ij}), \\ &E = \widetilde{V}_2^* (I - D_2^{-1}) V_1 = (e_{ij}), && \widetilde{E} = \widetilde{U}_2^* (D_1^* - I) U_1 = (\widetilde{e}_{ij}), \\ &F = \widetilde{U}_2^* (I - D_1^{-1}) U_1 = (f_{ij}), && \widetilde{F} = \widetilde{V}_2^* (D_2^* - I) V_1 = (\widetilde{f}_{ij}). \end{aligned} \quad (7.3)$$

It is easy to see that for any unitarily invariant norm $|||\cdot|||$ (ref. (1.5))

$$\left|\left|\left|\sin\Theta(U_1,\widetilde{U}_1)\right|\right|\right| = |||Q|||\,, \qquad \left|\left|\left|\sin\Theta(V_1,\widetilde{V}_1)\right|\right|\right| = \left|\left|\left|\widetilde{Q}\right|\right|\right|\,, \tag{7.4}$$

$$|||E||| \le \left|\left|\left|(I-D_2^{-1})V_1\right|\right|\right|\,, \qquad \left|\left|\left|\widetilde{E}\right|\right|\right| \le |||(D_1^*-I)U_1|||\,, \tag{7.5}$$

$$|||F||| \le \left|\left|\left|(I-D_1^{-1})U_1\right|\right|\right|\,, \qquad \left|\left|\left|\widetilde{F}\right|\right|\right| \le |||(D_2^*-I)V_1|||\,. \tag{7.6}$$

Proof of Theorem 3.5: Componentwisely, with the assignments of (7.3), the equations (7.1) and (7.2) read

$$\begin{aligned} \widetilde{\sigma}_{k+i}\widetilde{q}_{ij} - q_{ij}\sigma_j &= \widetilde{\sigma}_{k+i}e_{ij} + \widetilde{e}_{ij}\sigma_j\,, \\ \widetilde{\sigma}_{k+i}q_{ij} - \widetilde{q}_{ij}\sigma_j &= \widetilde{\sigma}_{k+i}f_{ij} + \widetilde{f}_{ij}\sigma_j\,, \end{aligned}$$

where $1 \le i \le n-k$, $1 \le j \le k$, which yields

$$\begin{aligned} |\widetilde{\sigma}_{k+i}\widetilde{q}_{ij} - q_{ij}\sigma_j|^2 &\le (\widetilde{\sigma}_{k+i}^2 + \sigma_j^2)(|e_{ij}|^2 + |\widetilde{e}_{ij}|^2) \\ &\Rightarrow |e_{ij}|^2 + |\widetilde{e}_{ij}|^2 \ge \frac{|\widetilde{\sigma}_{k+i}\widetilde{q}_{ij} - q_{ij}\sigma_j|^2}{\widetilde{\sigma}_{k+i}^2 + \sigma_j^2}, \\ |\widetilde{\sigma}_{k+i}q_{ij} - \widetilde{q}_{ij}\sigma_j|^2 &\le (\widetilde{\sigma}_{k+i}^2 + \sigma_j^2)(|f_{ij}|^2 + |\widetilde{f}_{ij}|^2) \\ &\Rightarrow |f_{ij}|^2 + |\widetilde{f}_{ij}|^2 \ge \frac{|\widetilde{\sigma}_{k+i}q_{ij} - \widetilde{q}_{ij}\sigma_j|^2}{\widetilde{\sigma}_{k+i}^2 + \sigma_j^2}. \end{aligned}$$

Summing on i and j for $i = 1, 2, \cdots, n-k$ and for $j = 1, 2, \cdots, k$, we get

$$\|E\|_F^2 + \|\widetilde{E}\|_F^2 + \|F\|_F^2 + \|\widetilde{F}\|_F^2 \ge \sum_{i=1}^{n-k}\sum_{j=1}^{k} \frac{|\widetilde{\sigma}_{k+i}\widetilde{q}_{ij} - q_{ij}\sigma_j|^2 + |\widetilde{\sigma}_{k+i}q_{ij} - \widetilde{q}_{ij}\sigma_j|^2}{\widetilde{\sigma}_{k+i}^2 + \sigma_j^2}.$$

Since

$$\begin{aligned} &|\widetilde{\sigma}_{k+i}\widetilde{q}_{ij} - q_{ij}\sigma_j|^2 + |\widetilde{\sigma}_{k+i}q_{ij} - \widetilde{q}_{ij}\sigma_j|^2 \\ &\quad= \widetilde{\sigma}_{k+i}^2|\widetilde{q}_{ij}|^2 + |q_{ij}|^2\sigma_j^2 - 2\Re(\widetilde{\sigma}_{k+i}\widetilde{q}_{ij}\bar{q}_{ij}\sigma_j) \\ &\qquad+ \widetilde{\sigma}_{k+i}^2|q_{ij}|^2 + |\widetilde{q}_{ij}|^2\sigma_j^2 - 2\Re(\widetilde{\sigma}_{k+i}\bar{q}_{ij}\widetilde{q}_{ij}\sigma_j) \\ &\quad\ge (\widetilde{\sigma}_{k+i} - \sigma_j)^2(|q_{ij}|^2 + |\widetilde{q}_{ij}|^2), \end{aligned}$$

where $\Re(\cdot)$ takes the real part of a complex number, so

$$\begin{aligned} \|E\|_F^2 + \|\widetilde{E}\|_F^2 + \|F\|_F^2 + \|\widetilde{F}\|_F^2 &\ge \sum_{i=1}^{n-k}\sum_{j=1}^{k} \frac{(\widetilde{\sigma}_{k+i} - \sigma_j)^2}{\widetilde{\sigma}_{k+i}^2 + \sigma_j^2}(|q_{ij}|^2 + |\widetilde{q}_{ij}|^2) \\ &\ge \mathrm{RelGap}_2^2(\|Q\|_F^2 + \|\widetilde{Q}\|_F^2), \end{aligned}$$

which, together with the equations (7.4), (7.5) and (7.6) for $\|\!|\cdot|\!\| = \|\cdot\|_F$, complete the proof of the theorem. ∎

Proof of Theorem 3.6: It follows from the equations (7.1) and (7.2) that

$$\begin{aligned}
\widetilde{\Sigma}_2\widetilde{V}_2^*V_1\Sigma_1^{-1} - \widetilde{U}_2^*U_1 &= \widetilde{\Sigma}_2\widetilde{V}_2^*(I-D_2^{-1})V_1\Sigma_1^{-1} + \widetilde{U}_2^*(D_1^*-I)U_1, && (7.7)\\
\widetilde{\Sigma}_2\widetilde{U}_2^*U_1\Sigma_1^{-1} - \widetilde{V}_2^*V_1 &= \widetilde{\Sigma}_2\widetilde{U}_2^*(I-D_1^{-1})U_1\Sigma_1^{-1} + \widetilde{V}_2^*(D_2^*-I)V_1. && (7.8)
\end{aligned}$$

To prove the inequality (3.8), we consider first the case $\left\|\!\left|\widetilde{U}_2^*U_1\right|\!\right\| \geq \left\|\!\left|\widetilde{V}_2^*V_1\right|\!\right\|$. By the assumptions of the theorem, we know that $\|\widetilde{\Sigma}_2 \leq \alpha$ and $\|\Sigma_1^{-1}\|_2 \leq \frac{1}{\alpha+\delta}$. Now it follows from (7.7) that

$$\begin{aligned}
&\left(1-\frac{\alpha}{\alpha+\delta}\right)\left\|\!\left|\widetilde{U}_2^*U_1\right|\!\right\| \\
&= \left\|\!\left|\widetilde{U}_2^*U_1\right|\!\right\| - \alpha\left\|\!\left|\widetilde{U}_2^*U_1\right|\!\right\|\frac{1}{\alpha+\delta} \\
&\leq \left\|\!\left|\widetilde{U}_2^*U_1\right|\!\right\| - \alpha\left\|\!\left|\widetilde{V}_2^*V_1\right|\!\right\|\frac{1}{\alpha+\delta} \\
&\leq \left\|\!\left|\widetilde{U}_2^*U_1\right|\!\right\| - \|\widetilde{\Sigma}_2\|_2\left\|\!\left|\widetilde{V}_2^*V_1\right|\!\right\|\|\Sigma_1^{-1}\|_2 \\
&\leq \left\|\!\left|\widetilde{U}_2^*U_1\right|\!\right\| - \left\|\!\left|\widetilde{\Sigma}_2\widetilde{V}_2^*V_1\Sigma_1^{-1}\right|\!\right\| \\
&\leq \left\|\!\left|\widetilde{\Sigma}_2\widetilde{V}_2^*V_1\Sigma_1^{-1} - \widetilde{U}_2^*U_1\right|\!\right\| \\
&= \left\|\!\left|\widetilde{\Sigma}_2\widetilde{V}_2^*(I-D_2^{-1})V_1\Sigma_1^{-1} + \widetilde{U}_2^*(D_1^*-I)U_1\right|\!\right\| \qquad \text{(by (7.7))} \\
&\leq \left\|\!\left|\widetilde{\Sigma}_2\widetilde{V}_2^*(I-D_2^{-1})V_1\Sigma_1^{-1}\right|\!\right\| + \left\|\!\left|\widetilde{U}_2^*(D_1^*-I)U_1\right|\!\right\| \\
&\leq \alpha\left\|\!\left|\widetilde{V}_2^*(I-D_2^{-1})V_1\right|\!\right\|\frac{1}{\alpha+\delta} + \left\|\!\left|\widetilde{U}_2^*(D_1^*-I)U_1\right|\!\right\| \\
&\leq \frac{\alpha}{\alpha+\delta}\|\!|(I-D_2^{-1})V_1|\!\| + \|\!|(D_1^*-I)U_1|\!\| \\
&\leq \sqrt[p]{1+\frac{\alpha^p}{(\alpha+\delta)^p}}\sqrt[q]{\|\!|(I-D_2^{-1})V_1|\!\|^q + \|\!|(D_1^*-I)U_1|\!\|^q},
\end{aligned}$$

which produces that if $\left\|\!\left|\widetilde{U}_2^*U_1\right|\!\right\| \geq \left\|\!\left|\widetilde{V}_2^*V_1\right|\!\right\|$,

$$\left\|\!\left|\widetilde{U}_2^*U_1\right|\!\right\| \leq \frac{\sqrt[q]{\|\!|(I-D_2^{-1})V_1|\!\|^q + \|\!|(D_1^*-I)U_1|\!\|^q}}{\underline{\mathrm{RelGap}}_p}. \tag{7.9}$$

Similarly, if $\left\|\!\left|\widetilde{U}_2^*U_1\right|\!\right\| < \left\|\!\left|\widetilde{V}_2^*V_1\right|\!\right\|$, one can prove, using (7.8), that

$$\left\|\!\left|\widetilde{V}_2^*V_1\right|\!\right\| \leq \frac{\sqrt[q]{\|\!|(I-D_1^{-1})U_1|\!\|^q + \|\!|(D_2^*-I)V_1|\!\|^q}}{\underline{\mathrm{RelGap}}_p}. \tag{7.10}$$

The inequality (3.8) is now a simple consequence of (7.9), (7.10) and (7.4).

The inequality (3.9) is left unproved. Let the assignments of (7.3) hold. The equations (7.1) and (7.2) are equivalent to the following single matrix equation with dimensions doubled.

$$\begin{pmatrix} & \widetilde{\Sigma}_2 \\ \widetilde{\Sigma}_2 & \end{pmatrix}\begin{pmatrix} \widetilde{Q} & \\ & Q \end{pmatrix} - \begin{pmatrix} \widetilde{Q} & \\ & Q \end{pmatrix}\begin{pmatrix} & \Sigma_1 \\ \Sigma_1 & \end{pmatrix}$$
$$= \begin{pmatrix} & \widetilde{\Sigma}_2 \\ \widetilde{\Sigma}_2 & \end{pmatrix}\begin{pmatrix} F & \\ & E \end{pmatrix} + \begin{pmatrix} \widetilde{E} & \\ & \widetilde{F} \end{pmatrix}\begin{pmatrix} & \Sigma_1 \\ \Sigma_1 & \end{pmatrix}.$$

On post-multiplying by $\begin{pmatrix} & \Sigma_1 \\ \Sigma_1 & \end{pmatrix}^{-1}$, we have

$$\begin{pmatrix} & \widetilde{\Sigma}_2 \\ \widetilde{\Sigma}_2 & \end{pmatrix}\begin{pmatrix} \widetilde{Q} & \\ & Q \end{pmatrix}\begin{pmatrix} & \Sigma_1 \\ \Sigma_1 & \end{pmatrix}^{-1} - \begin{pmatrix} \widetilde{Q} & \\ & Q \end{pmatrix}$$
$$= \begin{pmatrix} & \widetilde{\Sigma}_2 \\ \widetilde{\Sigma}_2 & \end{pmatrix}\begin{pmatrix} F & \\ & E \end{pmatrix}\begin{pmatrix} & \Sigma_1 \\ \Sigma_1 & \end{pmatrix}^{-1} + \begin{pmatrix} \widetilde{E} & \\ & \widetilde{F} \end{pmatrix}.$$

Under the conditions of Theorem 3.6, it is easy to verify that

$$\left\| \begin{pmatrix} & \widetilde{\Sigma}_2 \\ \widetilde{\Sigma}_2 & \end{pmatrix} \right\|_2 \le \alpha, \quad \left\| \begin{pmatrix} & \Sigma_1 \\ \Sigma_1 & \end{pmatrix}^{-1} \right\|_2 \le \frac{1}{\alpha+\delta}.$$

Applying the similar trick we used so far, we can get

$$\left|\!\left|\!\left| \begin{pmatrix} \widetilde{Q} & \\ & Q \end{pmatrix} \right|\!\right|\!\right| \le \frac{1}{\underline{\mathrm{RelGap}}_p} \sqrt[q]{\left|\!\left|\!\left| \begin{pmatrix} F & \\ & E \end{pmatrix} \right|\!\right|\!\right|^q + \left|\!\left|\!\left| \begin{pmatrix} \widetilde{E} & \\ & \widetilde{F} \end{pmatrix} \right|\!\right|\!\right|^q}. \qquad (7.11)$$

Since $\widetilde{Q}$ and $\sin\Theta(V_1, \widetilde{V}_1)$ have the same singular values and so do Q and $\sin\Theta(U_1, \widetilde{U}_1)$ (ref. (7.4)),

$$\left|\!\left|\!\left| \begin{pmatrix} \widetilde{Q} & \\ & Q \end{pmatrix} \right|\!\right|\!\right| = \left|\!\left|\!\left| \begin{pmatrix} \sin\Theta(U_1, \widetilde{U}_1) & \\ & \sin\Theta(V_1, \widetilde{V}_1) \end{pmatrix} \right|\!\right|\!\right|. \qquad (7.12)$$

Note also

$$\begin{pmatrix} F & \\ & E \end{pmatrix} = \begin{pmatrix} \widetilde{U}_2^* & \\ & \widetilde{V}_2^* \end{pmatrix}\begin{pmatrix} (I - D_1^{-1})U_1 & \\ & (I - D_2^{-1})V_1 \end{pmatrix},$$
$$\begin{pmatrix} \widetilde{E} & \\ & \widetilde{F} \end{pmatrix} = \begin{pmatrix} \widetilde{U}_2^* & \\ & \widetilde{V}_2^* \end{pmatrix}\begin{pmatrix} (D_1^* - I)U_1 & \\ & (D_2^* - I)V_1 \end{pmatrix}.$$

Thus, one has

$$\left\|\!\left|\begin{pmatrix} F & \\ & E \end{pmatrix}\right|\!\right\| \le \left\|\!\left|\begin{pmatrix} (I-D_1^{-1})U_1 & \\ & (I-D_2^{-1})V_1 \end{pmatrix}\right|\!\right\|, \tag{7.13}$$

$$\left\|\!\left|\begin{pmatrix} \widetilde{E} & \\ & \widetilde{F} \end{pmatrix}\right|\!\right\| \le \left\|\!\left|\begin{pmatrix} (D_1^*-I)U_1 & \\ & (D_2^*-I)V_1 \end{pmatrix}\right|\!\right\|. \tag{7.14}$$

The inequality (3.9) now follows from (7.11), (7.12), (7.13) and (7.14). ∎

7.2 The Non-Square Case: $m > n$

Augment B and $\widetilde{B}$ by a zero block $0_{m,m-n}$ as follows.

$$B_{\mathrm{a}} = (B, 0_{m,m-n}) \quad \text{and} \quad \widetilde{B}_{\mathrm{a}} = (\widetilde{B}, 0_{m,m-n}).$$

Since $\widetilde{B} = D_1^* B D_2$, we get

$$\widetilde{B}_{\mathrm{a}} = D_1^* B_{\mathrm{a}} \begin{pmatrix} D_2 & \\ & I_{m-n} \end{pmatrix} \stackrel{\text{def}}{=} D_1^* B_{\mathrm{a}} D_{\mathrm{a}2}.$$

From the SVDs (2.4) and (2.5) of B and $\widetilde{B}$, one can calculate the SVDs of B_{a} and $\widetilde{B}_{\mathrm{a}}$:

$$B_{\mathrm{a}} = U\Sigma_{\mathrm{a}}V_{\mathrm{a}}^* = (U_1, U_2)\begin{pmatrix} \Sigma_1 & \\ & \Sigma_{\mathrm{a}2} \end{pmatrix}\begin{pmatrix} V_{\mathrm{a}1}^* \\ V_{\mathrm{a}2}^* \end{pmatrix}, \tag{7.15}$$

$$\widetilde{B}_{\mathrm{a}} = \widetilde{U}\widetilde{\Sigma}_{\mathrm{a}}\widetilde{V}_{\mathrm{a}}^* = (\widetilde{U}_1, \widetilde{U}_2)\begin{pmatrix} \widetilde{\Sigma}_1 & \\ & \widetilde{\Sigma}_{\mathrm{a}2} \end{pmatrix}\begin{pmatrix} \widetilde{V}_{\mathrm{a}1}^* \\ \widetilde{V}_{\mathrm{a}2}^* \end{pmatrix}, \tag{7.16}$$

where

$$\Sigma_{\mathrm{a}2} = \begin{pmatrix} \Sigma_2 & \\ & 0_{m-n,m-n} \end{pmatrix}, V_{\mathrm{a}1} = \begin{pmatrix} V_1 \\ 0_{m-n,k} \end{pmatrix}, V_{\mathrm{a}2} = \begin{pmatrix} V_2 & \\ & I_{m-n} \end{pmatrix},$$

similarly for $\widetilde{\Sigma}_{\mathrm{a}2}$, $\widetilde{V}_{\mathrm{a}1}$ and $\widetilde{V}_{\mathrm{a}2}$. The following fact is easy to establish

$$\left\|\!\left|\sin\Theta(V_{\mathrm{a}1}, \widetilde{V}_{\mathrm{a}1})\right|\!\right\| = \left\|\!\left|\sin\Theta(V_1, \widetilde{V}_1)\right|\!\right\|.$$

Applying the square case of Theorems 3.5 and 3.6 to $m \times m$ matrices B_{a} and $\widetilde{B}_{\mathrm{a}}$ defined above will complete the proofs. ∎

8 Proofs of Theorems 3.7 and 3.8

We have seen in §7 how to deal with the nonsquare case by converting it to the square case. So here we will only give proofs for the square case:

$$m = n.$$

Let $\widehat{B} = D_1^* B$ and $\check{B} = BD_2$ and their SVDs be

$$\begin{aligned} \widehat{B} &= \widehat{U}\widehat{\Sigma}\widehat{V}^* = (\widehat{U}_1, \widehat{U}_2)\begin{pmatrix} \widehat{\Sigma}_1 & \\ & \widehat{\Sigma}_2 \end{pmatrix}\begin{pmatrix} \widehat{V}_1^* \\ \widehat{V}_2^* \end{pmatrix}, \\ \check{B} &= \check{U}\check{\Sigma}\check{V}^* = (\check{U}_1, \check{U}_2)\begin{pmatrix} \check{\Sigma}_1 & \\ & \check{\Sigma}_2 \end{pmatrix}\begin{pmatrix} \check{V}_1^* \\ \check{V}_2^* \end{pmatrix}, \end{aligned}$$

where $\widetilde{U}$, $\check{U} \in \mathbb{U}_n$, $\widetilde{V}$, $\check{V} \in \mathbb{U}_n$, U_1, $\check{U}_1 \in \mathbb{C}^{n\times k}$, V_1, $\check{V}_1 \in \mathbb{C}^{n\times k}$ and

$$\begin{aligned} \widehat{\Sigma}_1 &= \mathrm{diag}(\widehat{\sigma}_1, \cdots, \widehat{\sigma}_k), \quad \widehat{\Sigma}_2 = \mathrm{diag}(\widehat{\sigma}_{k+1}, \cdots, \widehat{\sigma}_n), \\ \check{\Sigma}_1 &= \mathrm{diag}(\check{\sigma}_1, \cdots, \check{\sigma}_k), \quad \check{\Sigma}_2 = \mathrm{diag}(\check{\sigma}_{k+1}, \cdots, \check{\sigma}_n). \end{aligned}$$

Because the gap assumptions made in Theorem 3.7 and Theorem 3.8, we can make [7]

$$\begin{aligned} &\widetilde{\mathrm{RelDist}}(\sigma_i, \widehat{\sigma}_i) \le \frac{1}{2}\|D_1^* - D_1^{-1}\|_2, \quad \widetilde{\mathrm{RelDist}}(\widehat{\sigma}_i, \widetilde{\sigma}_i) \le \frac{1}{2}\|D_2^* - D_2^{-1}\|_2, \\ &\widetilde{\mathrm{RelDist}}(\sigma_i, \check{\sigma}_i) \le \frac{1}{2}\|D_2^* - D_2^{-1}\|_2, \quad \widetilde{\mathrm{RelDist}}(\check{\sigma}_i, \widetilde{\sigma}_i) \le \frac{1}{2}\|D_1^* - D_1^{-1}\|_2. \end{aligned}$$

By the fact $\mathrm{RelDist}_p(\xi, \zeta) \le 2^{-1/p}\widetilde{\mathrm{RelDist}}(\xi, \zeta)$, the above four inequalities imply

$$\begin{aligned} &\mathrm{RelDist}_p(\sigma_i, \widehat{\sigma}_i) \le \frac{1}{2^{1+1/p}}\|D_1^* - D_1^{-1}\|_2, \quad \mathrm{RelDist}_p(\widehat{\sigma}_i, \widetilde{\sigma}_i) \le \frac{1}{2^{1+1/p}}\|D_2^* - D_2^{-1}\|_2, \\ &\mathrm{RelDist}_p(\sigma_i, \check{\sigma}_i) \le \frac{1}{2^{1+1/p}}\|D_2^* - D_2^{-1}\|_2, \quad \mathrm{RelDist}_p(\check{\sigma}_i, \widetilde{\sigma}_i) \le \frac{1}{2^{1+1/p}}\|D_1^* - D_1^{-1}\|_2. \end{aligned}$$

Consider $\widehat{B}$ and $\widetilde{B} = \widehat{B}D_2$. We have

$$\widehat{B}\widehat{B}^* = \widehat{U}\widehat{\Sigma}\widehat{\Sigma}^*\widehat{U}^* = (\widehat{U}_1, \widehat{U}_2)\begin{pmatrix} \widehat{\Sigma}_1^2 & \\ & \widehat{\Sigma}_2^2 \end{pmatrix}\begin{pmatrix} \widehat{U}_1^* \\ \widehat{U}_2^* \end{pmatrix}, \tag{8.1}$$

$$\widetilde{B}\widetilde{B}^* = \widetilde{U}\widetilde{\Sigma}\widetilde{\Sigma}^*\widetilde{U}^* = (\widetilde{U}_1, \widetilde{U}_2)\begin{pmatrix} \widetilde{\Sigma}_1^2 & \\ & \widetilde{\Sigma}_2^2 \end{pmatrix}\begin{pmatrix} \widetilde{U}_1^* \\ \widetilde{U}_2^* \end{pmatrix}. \tag{8.2}$$

Notice that

$$\begin{aligned} \widetilde{B}\widetilde{B}^* - \widehat{B}\widehat{B}^* &= \widetilde{B}D_2^*\widehat{B}^* - \widetilde{B}D_2^{-1}\widehat{B}^* \\ &= \widetilde{B}(D_2^* - D_2^{-1})\widehat{B}^*, \\ \widetilde{U}^*(\widetilde{B}\widetilde{B}^* - \widehat{B}\widehat{B}^*)\widehat{U} &= \widetilde{\Sigma}\widetilde{\Sigma}^*\widetilde{U}^*\widehat{U} - \widetilde{U}^*\widehat{U}\widehat{\Sigma}\widehat{\Sigma}^*, \\ \widetilde{U}^*\widetilde{B}(D_2^* - D_2^{-1})\widehat{B}^*\widehat{U} &= \widetilde{\Sigma}\widetilde{V}^*(D_2^* - D_2^{-1})\widehat{V}\widehat{\Sigma}^*. \end{aligned}$$

Thus, we have $\widetilde{\Sigma}^2\widetilde{U}^*\widehat{U} - \widetilde{U}^*\widehat{U}\widehat{\Sigma}^2 = \widetilde{\Sigma}\widetilde{V}^*(D_2^* - D_2^{-1})\widehat{V}\widehat{\Sigma}$ which gives

$$\widetilde{\Sigma}_2^2\widetilde{U}_2^*\widehat{U}_1 - \widetilde{U}_2^*\widehat{U}_1\widehat{\Sigma}_1^2 = \widetilde{\Sigma}_2\widetilde{V}_2^*(D_2^* - D_2^{-1})\widehat{V}_1\widehat{\Sigma}_1. \tag{8.3}$$

Consider now $\check{B}$ and $\widetilde{B} = D_1^*\check{B}$. We have

$$\begin{aligned} \check{B}^*\check{B} &= \check{V}\check{\Sigma}^*\check{\Sigma}\check{V}^* = (\check{V}_1, \check{V}_2)\begin{pmatrix} \check{\Sigma}_1^2 & \\ & \check{\Sigma}_2^2 \end{pmatrix}\begin{pmatrix} \check{V}_1^* \\ \check{V}_2^* \end{pmatrix}, && (8.4) \\ \widetilde{B}^*\widetilde{B} &= \widetilde{V}\widetilde{\Sigma}^*\widetilde{\Sigma}\widetilde{V}^* = (\widetilde{V}_1, \widetilde{V}_2)\begin{pmatrix} \widetilde{\Sigma}_1^2 & \\ & \widetilde{\Sigma}_2^2 \end{pmatrix}\begin{pmatrix} \widetilde{V}_1^* \\ \widetilde{V}_2^* \end{pmatrix}. && (8.5) \end{aligned}$$

Notice that

$$\begin{aligned} \widetilde{B}^*\widetilde{B} - \check{B}^*\check{B} &= \widetilde{B}^*D_1^*\check{B} - \widetilde{B}^*D_1^{-1}\check{B} \\ &= \widetilde{B}^*(D_1^* - D_1^{-1})\check{B}, \\ \widetilde{V}^*(\widetilde{B}^*\widetilde{B} - \check{B}^*\check{B})\check{V} &= \widetilde{\Sigma}^*\widetilde{\Sigma}\widetilde{V}^*\check{V} - \widetilde{V}^*\check{V}\check{\Sigma}^*\check{\Sigma}, \\ \widetilde{V}^*\widetilde{B}^*(D_1^* - D_1^{-1})\check{B}\check{V} &= \widetilde{\Sigma}^*\widetilde{U}^*(D_1^* - D_1^{-1})\check{U}\check{\Sigma}. \end{aligned}$$

Thus, we have $\widetilde{\Sigma}^2\widetilde{V}^*\check{V} - \widetilde{V}^*\check{V}\check{\Sigma}^2 = \widetilde{\Sigma}\widetilde{U}^*(D_1^* - D_1^{-1})\check{U}\check{\Sigma}$ which gives

$$\widetilde{\Sigma}_2^2\widetilde{V}_2^*\check{V}_1 - \widetilde{V}_2^*\check{V}_1\check{\Sigma}_1^2 = \widetilde{\Sigma}_2\widetilde{U}_2^*(D_1^* - D_1^{-1})\check{U}_1\check{\Sigma}_1. \tag{8.6}$$

Two other useful eigendecompostions are

$$\begin{aligned} BB^* &= U\Sigma\Sigma^*U^* = (U_1, U_2)\begin{pmatrix} \Sigma_1^2 & \\ & \Sigma_2^2 \end{pmatrix}\begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix}, && (8.7) \\ B^*B &= V\Sigma^*\Sigma V^* = (V_1, V_2)\begin{pmatrix} \Sigma_1^2 & \\ & \Sigma_2^2 \end{pmatrix}\begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix}. && (8.8) \end{aligned}$$

Proof of Theorems 3.7: It follows from (8.3) and (8.6) that

$$\begin{aligned} \left\|\sin\Theta(\widehat{U}_1, \widetilde{U}_1)\right\|_F = \left\|\widetilde{U}_2^*\widehat{U}_1\right\|_F &\le \frac{\left\|\widetilde{V}_2^*(D_2^* - D_2^{-1})\widehat{V}_1\right\|_F}{\widetilde{\mathrm{RelGap}}(\widehat{\Sigma}_1^2, \widetilde{\Sigma}_2^2)}, \\ \left\|\sin\Theta(\check{V}_1, \widetilde{V}_1)\right\|_F = \left\|\widetilde{V}_2^*\check{V}_1\right\|_F &\le \frac{\left\|\widetilde{U}_2^*(D_1^* - D_1^{-1})\check{U}_1\right\|_F}{\widetilde{\mathrm{RelGap}}(\check{\Sigma}_1^2, \widetilde{\Sigma}_2^2)}. \end{aligned}$$

On the other hand, applying Theorem 3.1 to BB^* and $\widehat{B}\widehat{B}^* = D_1^*BB^*D_1$ leads to (ref. (8.7) and (8.1))

$$\left\|\sin\Theta(U_1, \widehat{U}_1)\right\|_F \le \frac{\sqrt{\|(I - D_1^{-1})U_1\|_F^2 + \|(I - D_1^*)U_1\|_F^2}}{\mathrm{RelGap}_2(\Sigma_1^2, \widehat{\Sigma}_2^2)};$$

Applying Theorem 3.1 to B^*B and $\check{B}^*\check{B} = D_2^* B^* B D_2$ leads to (ref. (8.8) and (8.4))

$$\left\|\sin\Theta(V_1, \check{V}_1)\right\|_F \le \frac{\sqrt{\|(I - D_2^{-1})V_1\|_F^2 + \|(I - D_2^*)V_1\|_F^2}}{\mathrm{RelGap}_2(\Sigma_1^2, \check{\Sigma}_2^2)}.$$

Notice that [7]

$$\begin{aligned}
\widetilde{\mathrm{RelGap}}(\widehat{\Sigma}_1^2, \widetilde{\Sigma}_2^2) &\ge 2\,\widetilde{\mathrm{RelGap}}(\widehat{\Sigma}_1, \widetilde{\Sigma}_2) \ge 2\left[\widetilde{\mathrm{RelGap}}(\Sigma_1, \widetilde{\Sigma}_2) - \widetilde{\epsilon}_1\right],\\
\widetilde{\mathrm{RelGap}}(\check{\Sigma}_1^2, \widetilde{\Sigma}_2^2) &\ge 2\,\widetilde{\mathrm{RelGap}}(\check{\Sigma}_1, \widetilde{\Sigma}_2) \ge 2\left[\widetilde{\mathrm{RelGap}}(\Sigma_1, \widetilde{\Sigma}_2) - \widetilde{\epsilon}_2\right],\\
\mathrm{RelGap}_2(\Sigma_1^2, \widehat{\Sigma}_2^2) &\ge \mathrm{RelGap}_2(\Sigma_1, \widehat{\Sigma}_2) \ge \mathrm{RelGap}_2(\Sigma_1, \widetilde{\Sigma}_2) - \epsilon_2,\\
\mathrm{RelGap}_2(\Sigma_1^2, \check{\Sigma}_2^2) &\ge \mathrm{RelGap}_2(\Sigma_1, \check{\Sigma}_2) \ge \mathrm{RelGap}_2(\Sigma_1, \widetilde{\Sigma}_2) - \epsilon_1,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\epsilon}_1 &= \frac{1}{2}\|D_1^* - D_1^{-1}\|_2, &\qquad \widetilde{\epsilon}_2 &= \frac{1}{2}\|D_2^* - D_2^{-1}\|_2,\\
\epsilon_1 &= \frac{1}{2\sqrt{2}}\|D_1^* - D_1^{-1}\|_2, &\qquad \epsilon_2 &= \frac{1}{2\sqrt{2}}\|D_2^* - D_2^{-1}\|_2.
\end{aligned}$$

The proof will be completed by employing [9]

$$\begin{aligned}
\left\|\sin\Theta(U_1, \widetilde{U}_1)\right\|_F &\le \left\|\sin\Theta(U_1, \widehat{U}_1)\right\|_F + \left\|\sin\Theta(\widehat{U}_1, \widetilde{U}_1)\right\|_F,\\
\left\|\sin\Theta(V_1, \widetilde{V}_1)\right\|_F &\le \left\|\sin\Theta(V_1, \check{V}_1)\right\|_F + \left\|\sin\Theta(\check{V}_1, \widetilde{V}_1)\right\|_F.
\end{aligned}$$

$\blacksquare$

Proof of Theorems 3.8: Denote $\beta = \alpha + \delta$. Let $\widehat{\alpha}$ and $\check{\alpha}$ be the largest positive numbers so that

$$\widetilde{\mathrm{RelDist}}(\alpha, \widehat{\alpha}) \le \frac{1}{2}\|D_2^* - D_2^{-1}\|_2 \quad \text{and} \quad \widetilde{\mathrm{RelDist}}(\alpha, \check{\alpha}) \le \frac{1}{2}\|D_1^* - D_1^{-1}\|_2$$

which guarantee that

$$\|\widehat{\Sigma}_2\|_2 \le \widehat{\alpha} \quad \text{and} \quad \|\check{\Sigma}_2\|_2 \le \check{\alpha},$$

and

$$\mathrm{RelDist}_p(\alpha, \widehat{\alpha}) \le \frac{1}{2^{1+1/p}}\|D_2^* - D_2^{-1}\|_2 \quad \text{and} \quad \mathrm{RelDist}_p(\alpha, \check{\alpha}) \le \frac{1}{2^{1+1/p}}\|D_1^* - D_1^{-1}\|_2;$$

and let $\widehat{\beta}$ and $\check{\beta}$ be the smallest numbers so that

$$\widetilde{\mathrm{RelDist}}(\beta, \widehat{\beta}) \le \frac{1}{2}\|D_1^* - D_1^{-1}\|_2 \quad \text{and} \quad \widetilde{\mathrm{RelDist}}(\beta, \check{\beta}) \le \frac{1}{2}\|D_2^* - D_2^{-1}\|_2$$

which guarantee that

$$\|\widehat{\Sigma}_1^{-1}\|_2^{-1} \geq \widehat{\beta} \quad \text{and} \quad \|\check{\Sigma}_1^{-1}\|_2^{-1} \geq \check{\beta}$$

and

$$\mathrm{RelDist}_p(\beta, \widehat{\beta}) \leq \frac{1}{2^{1+1/p}} \|D_1^* - D_1^{-1}\|_2 \quad \text{and} \quad \mathrm{RelDist}_p(\beta, \check{\beta}) \leq \frac{1}{2^{1+1/p}} \|D_2^* - D_2^{-1}\|_2.$$

Because of our gap assumptions, $\min\{\widehat{\beta}, \check{\beta}\} > \alpha$ and $\beta > \max\{\widehat{\alpha}, \check{\alpha}\}$.

It follows from (8.3) and (8.6) that

$$\begin{aligned}
\left|\!\left|\!\left|\sin\Theta(\widehat{U}_1, \widetilde{U}_1)\right|\!\right|\!\right| = \left|\!\left|\!\left|\widetilde{U}_2^* \widehat{U}_1\right|\!\right|\!\right| &\leq \frac{\left|\!\left|\!\left|\widetilde{V}_2^*(D_2^* - D_2^{-1})\widehat{V}_1\right|\!\right|\!\right|}{\underline{\widetilde{\mathrm{RelGap}}}(\widehat{\Sigma}_1^2, \widetilde{\Sigma}_2^2; \alpha^2, \widehat{\beta}^2)}, \\
\left|\!\left|\!\left|\sin\Theta(\check{V}_1, \widetilde{V}_1)\right|\!\right|\!\right| = \left|\!\left|\!\left|\widetilde{V}_2^* \check{V}_1\right|\!\right|\!\right| &\leq \frac{\left|\!\left|\!\left|\widetilde{U}_2^*(D_1^* - D_1^{-1})\check{U}_1\right|\!\right|\!\right|}{\underline{\widetilde{\mathrm{RelGap}}}(\check{\Sigma}_1^2, \widetilde{\Sigma}_2^2; \alpha^2, \check{\beta}^2)}.
\end{aligned}$$

On the other hand, applying Theorem 3.2 to BB^* and $\widehat{B}\widehat{B}^* = D_1^* BB^* D_1$ leads to (ref. (8.7) and (8.1))

$$\left|\!\left|\!\left|\sin\Theta(U_1, \widehat{U}_1)\right|\!\right|\!\right| \leq \frac{\sqrt[q]{\left|\!\left|\!\left|(I - D_1^{-1})U_1\right|\!\right|\!\right|^q + \left|\!\left|\!\left|(I - D_1^*)U_1\right|\!\right|\!\right|^q}}{\underline{\mathrm{RelGap}}_p(\Sigma_1^2, \widehat{\Sigma}_2^2; \widehat{\alpha}^2, \beta^2)};$$

Applying Theorem 3.1 to B^*B and $\check{B}^*\check{B} = D_2^* B^* B D_2$ leads to (ref. (8.8) and (8.4))

$$\left|\!\left|\!\left|\sin\Theta(V_1, \check{V}_1)\right|\!\right|\!\right| \leq \frac{\sqrt[q]{\left|\!\left|\!\left|(I - D_2^{-1})V_1\right|\!\right|\!\right|^q + \left|\!\left|\!\left|(I - D_2^*)V_1\right|\!\right|\!\right|^q}}{\underline{\mathrm{RelGap}}_2(\Sigma_1^2, \check{\Sigma}_2^2; \check{\alpha}^2, \beta^2)}.$$

Notice that [7]

$$\begin{aligned}
\underline{\widetilde{\mathrm{RelGap}}}(\widehat{\Sigma}_1^2, \widetilde{\Sigma}_2^2; \alpha^2, \widehat{\beta}^2) &\geq 2\,\underline{\widetilde{\mathrm{RelGap}}}(\widehat{\Sigma}_1, \widetilde{\Sigma}_2; \alpha, \widehat{\beta}) \geq 2\left[\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta) - \widetilde{\epsilon}_1\right], \\
\underline{\widetilde{\mathrm{RelGap}}}(\check{\Sigma}_1^2, \widetilde{\Sigma}_2^2; \alpha^2, \check{\beta}^2) &\geq 2\,\underline{\widetilde{\mathrm{RelGap}}}(\check{\Sigma}_1, \widetilde{\Sigma}_2; \alpha, \check{\beta}) \geq 2\left[\underline{\widetilde{\mathrm{RelGap}}}(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta) - \widetilde{\epsilon}_2\right], \\
\underline{\mathrm{RelGap}}_p(\Sigma_1^2, \widehat{\Sigma}_2^2; \widehat{\alpha}^2, \beta^2) &\geq \underline{\mathrm{RelGap}}_p(\Sigma_1, \widehat{\Sigma}_2; \widehat{\alpha}, \beta) \geq \underline{\mathrm{RelGap}}_p(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta) - \epsilon_2, \\
\underline{\mathrm{RelGap}}_p(\Sigma_1^2, \check{\Sigma}_2^2; \check{\alpha}^2, \beta^2) &\geq \underline{\mathrm{RelGap}}_p(\Sigma_1, \check{\Sigma}_2; \check{\alpha}, \beta) \geq \underline{\mathrm{RelGap}}_p(\Sigma_1, \widetilde{\Sigma}_2; \alpha, \beta) - \epsilon_1,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\epsilon}_1 &= \frac{1}{2}\|D_1^* - D_1^{-1}\|_2, & \widetilde{\epsilon}_2 &= \frac{1}{2}\|D_2^* - D_2^{-1}\|_2, \\
\epsilon_1 &= \frac{1}{2^{1+1/p}}\|D_1^* - D_1^{-1}\|_2, & \epsilon_2 &= \frac{1}{2^{1+1/p}}\|D_2^* - D_2^{-1}\|_2.
\end{aligned}$$

The proof will be completed by employing [9]

$$\begin{aligned}
\left|\!\left|\!\left|\sin\Theta(U_1,\widetilde{U}_1)\right|\!\right|\!\right| &\leq \left|\!\left|\!\left|\sin\Theta(U_1,\widehat{U}_1)\right|\!\right|\!\right| + \left|\!\left|\!\left|\sin\Theta(\widehat{U}_1,\widetilde{U}_1)\right|\!\right|\!\right|, \\
\left|\!\left|\!\left|\sin\Theta(V_1,\widetilde{V}_1)\right|\!\right|\!\right| &\leq \left|\!\left|\!\left|\sin\Theta(V_1,\check{V}_1)\right|\!\right|\!\right| + \left|\!\left|\!\left|\sin\Theta(\check{V}_1,\widetilde{V}_1)\right|\!\right|\!\right|.
\end{aligned}$$

∎

9 Proofs of Theorems 3.3 and 3.4

Notice that

$$\begin{aligned}
H &= D^* A D = (A^{1/2} D)^* \, A^{1/2} D \stackrel{\mathrm{def}}{=} B B^*, \\
\widetilde{H} &= D^* A^{1/2} \big(I + A^{-1/2}(\Delta A) A^{-1/2}\big) A^{1/2} D \\
&= \left(\big(I + A^{-1/2}(\Delta A) A^{-1/2}\big)^{1/2} A^{1/2} D \right)^* \, \big(I + A^{-1/2}(\Delta A) A^{-1/2}\big)^{1/2} A^{1/2} D \\
&\stackrel{\mathrm{def}}{=} \widetilde{B} \widetilde{B}^*,
\end{aligned}$$

where

$$\begin{aligned}
B &= D^* A^{1/2}, \\
\widetilde{B} &= D^* A^{1/2} \big(I + A^{-1/2}(\Delta A) A^{-1/2}\big)^{1/2} \stackrel{\mathrm{def}}{=} D^* A^{1/2} D_2 = B D_2.
\end{aligned}$$

Given the eigendecompositions of H and $\widetilde{H}$ as in (3.3) and (3.4), one easily see that B and $\widetilde{B}$ admit the following SVDs.

$$\begin{aligned}
B &= U \Lambda^{1/2} V^* = (U_1, U_2) \begin{pmatrix} \Lambda_1^{1/2} & \\ & \Lambda_2^{1/2} \end{pmatrix} \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix}, \\
\widetilde{B} &= \widetilde{U} \widetilde{\Lambda}^{1/2} \widetilde{V}^* = (\widetilde{U}_1, \widetilde{U}_2) \begin{pmatrix} \widetilde{\Lambda}_1^{1/2} & \\ & \widetilde{\Lambda}_2^{1/2} \end{pmatrix} \begin{pmatrix} \widetilde{V}_1^* \\ \widetilde{V}_2^* \end{pmatrix},
\end{aligned}$$

where U, $\widetilde{U}$ are the same as in (3.3) and (3.4), V_1, $\widetilde{V}_1 \in \mathbb{C}^{n \times k}$. Notice that

$$\begin{aligned}
\widetilde{H} - H = \widetilde{B} \widetilde{B}^* - B B^* &= \widetilde{B} D_2^* B^* - \widetilde{B} D_2^{-1} B^* \\
&= \widetilde{B} (D_2^* - D_2^{-1}) B^*, \\
\widetilde{U}^* (\widetilde{H} - H) U &= \widetilde{\Lambda} \widetilde{U}^* U - \widetilde{U}^* U \Lambda, \\
\widetilde{U}^* \widetilde{B} (D_2^* - D_2^{-1}) B^* U &= \widetilde{\Lambda}^{1/2} \widetilde{V}^* (D_2^* - D_2^{-1}) V \Lambda^{1/2}.
\end{aligned}$$

Thus, $\widetilde{\Lambda} \widetilde{U}^* U - \widetilde{U}^* U \Lambda = \widetilde{\Lambda}^{1/2} \widetilde{V}^* (D_2^* - D_2^{-1}) V \Lambda^{1/2}$ which yields

$$\widetilde{\Lambda}_2 \widetilde{U}_2^* U_1 - \widetilde{U}_2^* U_1 \Lambda_1 = \widetilde{\Lambda}_2^{1/2} \widetilde{V}_2^* (D_2^* - D_2^{-1}) V_1 \Lambda_1^{1/2}. \tag{9.1}$$

The following inequality will be very useful in the rest of our proofs.

$$\begin{aligned}
&\left\| \widetilde{V}_2^* (D_2^* - D_2^{-1}) V_1 \right\| \le \| D_2^* - D_2^{-1} \| \\
&= \left\| (I + A^{-1/2}(\Delta A) A^{-1/2})^{1/2} - (I + A^{-1/2}(\Delta A) A^{-1/2})^{-1/2} \right\| \\
&\le \|(I + A^{-1/2}(\Delta A) A^{-1/2})^{-1/2}\|_2 \left\| A^{-1/2} (\Delta A) A^{-1/2} \right\| \\
&\le \frac{\|A^{-1}\|_2 \, \|\Delta A\|}{\sqrt{1 - \|A^{-1}\|_2 \|\Delta A\|_2}}.
\end{aligned}$$

Proof of Theorem 3.3: It follow from (9.1) that

$$\left\|\sin\Theta(U_1,\widetilde{U}_1)\right\|_F = \left\|\widetilde{U}_2^* U_1\right\|_F \le \frac{\|\widetilde{V}_2^*(D_2^* - D_2^{-1})V_1\|_F}{\widetilde{\mathrm{RelGap}}},$$

as required. ∎

Proof of Theorem 3.4: It follow from (9.1) that

$$\left|\!\left|\!\left|\sin\Theta(U_1,\widetilde{U}_1)\right|\!\right|\!\right| = \left|\!\left|\!\left|\widetilde{U}_2^* U_1\right|\!\right|\!\right| \le \frac{\left|\!\left|\!\left|\widetilde{V}_2^*(D_2^* - D_2^{-1})V_1\right|\!\right|\!\right|}{\underline{\widetilde{\mathrm{RelGap}}}},$$

as required. ∎

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References

[1] J. Barlow and J. Demmel. Computing accurate eigensystems of scaled diagonally dominant matrices. *SIAM Journal on Numerical Analysis*, 27:762–791, 1990.

[2] C. Davis and W. Kahan. The rotation of eigenvectors by a perturbation. III. *SIAM Journal on Numerical Analysis*, 7:1–46, 1970.

[3] J. Demmel and K. Veselić. Jacobi's method is more accurate than QR. *SIAM Journal on Matrix Analysis and Applications*, 13(4):1204–1245, 1992.

[4] S. C. Eisenstat and I. C. F. Ipsen. Relative perturbation bounds for eigenspaces and singular vector subspaces. In J. G. Lewis, editor, *Proceedings of the Fifth SIAM Conference on Applied Linear Algebra*, pages 62–66, Philadelphia, 1994. SIAM Publications.

[5] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, 1985.

[6] R.-C. Li. On perturbations of matrix pencils with real spectra. *Mathematics of Computation*, 62:231–265, 1994.

[7] R.-C. Li. Relative perturbation theory: (i) eigenvalue variations. Technical Report UCB//CSD-94-855, Computer Science Divison, Department of EECS, University of California at Berkeley, 1994.

[8] R. Mathias. Spectral perturbation bounds for graded positive definite matrices. Manuscript, Department of Mathematics, College of Willian & Mary, 1994.

[9] G. W. Stewart and J.-G. Sun. *Matrix Perturbation Theory.* Academic Press, Boston, 1990.

[10] P.-Å. Wedin. Perturbation bounds in connection with singular value decomposition. *BIT*, 12:99–111, 1972.