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Moment Matching Model of Transmission Lines and Application to Interconnect Delay Estimation

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Abstract

In this paper, we present a method to estimate the signal delay in the interconnection modeled as a general resistor-transmission line-capacitor (R-T-C) network. The estimation is based on the propagation delay of the transmission lines and the moment matching techniques. We analyze the contribution of a transmission line to the moments of the network and provide a method to form a lumped moment matching model of the line. When the transmission lines are replaced by their p-th order moment matching models, the network is transformed into a lumped R-L-C network such that these two networks have exactly the same moments up to the order of p for each corresponding output node voltage . We also provide a recursive formula to compute the moments of the R-L-C network so that the moment matching techniques can be efficiently used in the delay estimation.

1 Introduction

In high speed electronic circuits, the delay due to interconnects is comparable to that of transistors and will be the dominant part of the entire system delay before long. A

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simple and accurate delay model for interconnects is essential to the design of VLSI systems.

The interconnects of a VLSI system on different level are modeled differently. The wires on a chip , on a printed circuit board and on an MCM are usually modeled as lumped RC lines, lossless transmission lines and lossy transmission lines, respectively [1]. It is expected that the volume of the products of MCM's will grow very rapidly, so that a good delay model for the interconnects in MCM's is in urgent need by the designers.

The model of interconnects on one layer of an MCM is made of floating resistors, grounded capacitors and lossy transmission lines. We now consider the case that the coupling among the transmission lines on the same layer and on the neighbouring layers is negligible, each transmission line consists of a floating wire and a ground wire and whose conductance per unit length is zero. Such a network is called an resistor-transmission line-capacitor network, or an R-T-C network for abbreviation. In most practical cases, the resistors and the floating wires of the transmission lines form a tree , the capacitors are connected between the nodes on the tree and the ground, and the input to the network is a voltage source connected between the root of the tree and the ground. Such a network is called an R-T-C tree. In some cases, there are loops among the resistors and the folating wires of the transmission lines. In this case, it is called an R-T-C mesh. R-T-C trees and meshes are called R-T-C networks or transmission line networks in this paper.

Many papers dealing with the analysis of transmission line networks have been published in recent years [2-8], and most of them provide time-domain simulation mthods for such networks. However, It is time consuming to estimate signal delay based on a time-domain simulation during the design process especially when an optimal design of interconnects is needed. A more efficient way to do delay estimation is to use the moment matching technique. After computaion of the moments of an output signal, the waveform of the signal can be approximated by using the moment matching technique and the delay can be estimated. The first moment of a node voltage is called its Elmore delay and is a good estimation of its signal delay when its waveform is monotonic or nearly monotonic. The formula of Elmore delay of RC tree and mesh networks is well known [9, 10] and is widely used in the optimal design of interconnects [12, 13]. However, there is no known simple formula for the Elmore delay of transmission line networks, not mentioning to those of higher order moments, which are needed for the dealy estimation when waveforms are nonmonotonic due to the reflection at the terminals of the transmssion lines.

In this paper, we study the computation of the Elmore delay and higher order moments of the node voltages of an R-T-C network. We first model a transmission line by n uniform sections of RLC network and transfer the R-T-C network into an R-L-C network. We provide simple formulas to compute the Elmore delay and higher order moments of the R-L-C networks. We let $n \rightarrow \infty$ to evaluate the contribution of a transmission line to the moments exactly. From the analytical results, we form a lumped moment matching model with a finite number of sections for the transmission line. When transmission lines in an R-T-C network T is replaced by their p-th order moment matching models, a p-th order moment matching R-L-C network $\hat{T}$ is formed such that the moments of the node voltages in $\hat{T}$ are exactly the same as those of the corresponding node voltages in T up to the order of p and the computation of the moments can be implemented by using the lumped R-L-C network. The delay estimation is then done by the computation of the propagation delay of the transmission lines and the rising delay based on the moment matching techniques.

We derive all the formulas from a typical transient process: the network is initially in zero-state and excited by a unit step voltage source. The results are the same in another typical transient process: the network is initially in an equilibrium state with all the capacitor voltages having the same value, all the branch currents equal to zero and the input terminal is connected to ground at $t = 0$.

This paper is organized as follows. In Sec.2, we review the definition of moments in both frequency and time domains. In Sec.3, we present a formula to compute Elmore delay in an R-T-C tree and an R-T-C mesh and provide simple first order moment matching models of a transmission line. In Sec.4, we provide a simple recursive formula to compute higher order moments of all the voltages in an R-L-C tree with linear time complexity and a general recursive formula for an R-T-C mesh. In Sec.5, we analyze the contribution of a line to the moments and present a p-th order moment matching model of a transmission line. In Sec.6, we provide a delay model of the R-T-C network. The comparisons between our method with other methods are givien in Sec.7.

2 Basic concept of moments

In this section, we review the definition of moments for the use in the following sections.

Let $v_{in}(t)$ be the input voltage of a linear network, $v_i(t)$ be one of its output voltages, $V_{in}(s)$ and $V_i(s)$ be the Laplace transform of $v_{in}(t)$ and $v_i(t)$ respectively; then, $H_i(s) = V_i(s)/V_{in}(s)$ is the transfer function and $h_i(t) = L^{-1}\{H_i(s)\}$ is the impulse response. Expand $H_i(s)$ in Taylor series in terms of s, we have

$$H_i(s) = H_i(0) + \sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^j H}{ds^j} \Big|_{s=0} s^j \tag{1}$$

and the j-th moment of $h_i(t)$ $(j \geq 0)$ is defined as

$$m_i^j = \frac{(-1)^j}{j!} H^{(j)}(0) \tag{2}$$

where $H^{(j)}(0) = d^j H(s)/ds^j \,|_{s=0}$. From this definition, we have

$$H_i(s) = \sum_{j=0}^{\infty} (-1)^j m_i^j = m_i^0 - m_i^1 s + m_i^2 s^2 - m_i^3 s^3 + \ldots \tag{3}$$

When $v_i(t)$ is a zero-state unit-step response, $h_i(t) = \dot{v}_i(t)$, and $H_i(s) = \int_0^\infty \dot{v}_i(t) e^{-st} dt$. Expand e^{-st} into Taylor series of s,

$$H_i(s) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} s^j \int_0^\infty t^j \dot{v}_i dt \tag{4}$$

and we have

$$m_i^j = \frac{1}{j!} \int_0^\infty t^j \dot{v}_i dt \tag{5}$$

Eqs. (2) and (5) are two equivalent definitions for the moments: one from frequency domain, and the other from time domain. When dealing with R-T-C trees, we use the definition from time domain; and when dealing with R-T-C meshes, we use the one from frequency domain. Although an R-T-C tree is a special case of an R-T-C mesh and the methods and algorithms for R-T-C meshes are more general than those for R-T-C trees, the analysis is simpler and clearer in concept and the algorithms are more efficient for the trees than for the meshes. Therefore, our discussion will follow the order of tree first and mesh next.

3 Elmore delay in R-T-C network

3.1 Elmore delay in R-T-C tree

In this section, we first talk about the Elmore delay in an R-L-C tree formed by replacing each transmission line with a large number of uniform RLC 2-ports, then present a simple Elmore delay model for a transmission line from which a first moment matching R-L-C tree can be formed to evaluate the Elmore delay of the original tree exactly and efficiently.

3.1.1 Elmore delay in R-L-C tree

Def.1

An R-L-C tree is a special kind of RLC network, where the resistors and inductors are floating from the ground and form a tree, the capacitors are connected between the nodes on the tree and the ground, and the input voltage source is connected between the root of the tree and the ground.

For simplicity, the root of an R-L-C tree T is named r, and the other non-grounded nodes are numbered from 1 to n_T consecutively. Let P_i be the path from root r to node i, and $A(i) = \{j|j \in P(i), j \neq i\}$ be the set of ancestors of node i. The nearest ancestor to node i is called its father and is denoted by $\bar{i}$, and node i is called a son node of node $\bar{i}$. Because of the tree structure, each non-root node has a father, and may have several sons. The set of son nodes of node i is denoted by $S(i)$. We designate node 1 to be the only one son node of the root, as if there are many son nodes of node r, the tree can be decomposed to some trees without interaction. Let $D(i) = \{j|i \in A(j)\}$ and $D(i)$ is defined as the set of descendents of node i. We denote $\hat{D}(i) = D(i) \bigcup \{i\}$. Let C_k be the capacitance connected to node k, R_k and L_k be the resistance and inductance of the branch between node $\bar{k}$ and k, and i_k be the current flowing from node $\bar{k}$ to node k. Then, from KCL, it is known that

$$i_k = \sum_{j \in \hat{D}(k)} C_j \dot{v}_j \tag{6}$$

For any pair of nodes i and k, let $P_{ik} = P_i \bigcap P_k$ be the common part of P_i and P_k and R_{ik} and L_{ik} be the total resistance and inductance on path P_{ik}. The component of the voltage drop from node r to node i contributed by the capacitance current $C_k \dot{v}_k$ is $R_{ik} C_k \dot{v}_k + L_{ik} C_k \ddot{v}_k$, and the voltage drop from the root r to node i is

$$v_r - v_i = \sum_k (R_{ik} C_k \dot{v}_k + L_{ik} C_k \ddot{v}_k) \tag{7}$$

where the sum is taken over all the nodes from 1 to n_T.

The Elmore delay of node voltage v_i is defined as its first moment, i.e.,

$$T_{Di} = \int_0^\infty t \dot{v}_i dt = \int_0^\infty (1 - v_i) dt \tag{8}$$

Note that $v_r = 1$ when $t > 0$. Substituting Eq.(7) to Eq.(8), we have

$$T_{Di} = \sum_k (R_{ik} C_k (v_k(\infty) - v_k(0)) + \sum_k (L_{ik} C_k (\dot{v}_k(\infty) - \dot{v}_k(0)) \tag{9}$$

According to the assumption that the circuit is in zero-initial state, so $v_k(0) = 0$ $\forall k$. When $t \to \infty$, $v_k \to 1$ and $\dot{v}_k \to 0$. For any node $k \neq 1$, as v_k, $v_{\bar{k}}$ and v_j of any of its son node j and the inductance currents are all zero at t=0, so the current $C_k \dot{v}_k$=0 at t=0. For node 1, if $L_1 = 0$, then $L_{i1} = L_1 = 0$ and the term $L_{i1} \dot{v}_1(0) = 0$; and if $L_1 \neq 0$, then from the initial zero state assumption, the current in L_1 is initially zero and so $\dot{v}_1(0) = 0$. Therefore, we have

$$T_{Di} = \sum_k R_{ik} C_k \tag{10}$$

Remarks.

1. The Elmore delay of an R-L-C tree T is the same as that of an RC tree $\hat{T}$ obtained by shorting all the inductances in T; i.e., the inductance plays no role in Elmore delay. The physical meaning of this result can be explained roughly as follows. Consider v_i of an R-L-C tree T and the corresponding $\hat{v}_i$ in the RC tree $\hat{T}$. The voltages of the inductances on path P_i contribute to the voltage drop $v_r - v_i$.

When $t \approx 0$ the currents through these inductances go up, the voltages on these inductances are positive which make $v_i(t) < \hat{v}_i(t)$. When $t \to \infty$, the currents through these inductances go down and their voltages become negative, which make $v_i(t) > \hat{v}_i(t)$. As Elmore delay can be regarded as an average of the signal delay, these two opposite factors compensate each other and the inductances do not have any effects on it.

2. The contribution of each capacitance C_k to T_{Di} is $R_{ik}C_k(v_k(\infty) - v_k(0)) = R_{ik}C_k$, i.e., the waveform of v_k has no effect on T_{Di}.

3. If the R-L-C tree is initially in equilibrium state with all the node voltages equal to 1 and all the currents equal to 0 and the input node r is connected to ground, then the Elmore delay of this discharging process is the same as expressed in Eq.(10).

4. In the interconnects of printed circuit boards, in order to match the transmission lines with their loads, some resistors may be connected between the flaoting RL tree and the ground. The moment computation of such networks is described in Appendix F.

3.1.2 Elmore delay model of transmission line

Now we consider the Elmore delay T_{Di} of v_i in an R-T-C tree. For a transmission line TL with total resistance R, inductance L and capacitance C, let its two floating terminals be t_1 and t_2 and let t_1 be an ancestor node of t_2 when TL is embedded in an R-T-C tree. We first model it by n sections of Γ- typed two port as shown in Fig.1, designate the internal nodes to be $1, 2, \ldots, n-1$, $t_2 = n$ and let $LL = \{1, 2, \ldots, n\}$. The contribution of such a line to T_{Di} is as follows.

1. Case 1. TL is not on path P_i. In this case, R has no contribution to R_{ik}, and the contribution of the capacitances is $R_{ik}C_1 + R_{ik}C_2 + \ldots + R_{ik}C_n = R_{ik}C$, i.e., the transmission line acts as a lumped capacitance C.

2. Case 2. TL is on path P_i. In this case, for any k such that TL is on P_{ik}, R is a part of R_{ik}. For any C_j connected to the subtree rooted at node t_2, the R plays a role as a lumped resistance. Now consider the contribution of the

capacitances in the model of the transmission line. Their contribution to T_{Di} becomes $\tau = \sum_{k \in LL} R_{ik} C_k$ with $R_{ik} = R_{t_1 t_1} + kR/n$, so that

$$\tau = \sum_{k \in LL} (R_{t_1 t_1} + \frac{k}{n} R) \frac{C}{n} = R_{t_1 t_1} C + \frac{n+1}{2n} RC \tag{11}$$

and $\tau \rightarrow R_{t_1 t_1} C + \frac{RC}{2}$ when $n \rightarrow \infty$. It can be seen from this formula that τ consists of two parts. For the first part $R_{t_1 t_1} C$, TL acts as a lumped capacitance C, and for the second part $RC/2$, it acts as a lumped resistance R connected between nodes t_1 and t_2 and a capacitance connected at node t_2.

Based on the above discussion, for Elmore delay in an R-T-C tree T, a transmission line with parameters R, L and C can be modeled as a simple Π -typed RC 2-port shown in Fig.2a or a T-typed RC 2-port shown in Fig.2b. Such a model is called the Elmore delay model or first-order moment matching model of the transmission line. By replacing each transmission line with its 1-st order moment matching model, an R-L-C tree is formed so that their corresponding node voltages have the same 1-st order moment as those of the original tree T. Such an R-L-C tree $\hat{T}$ is called a 1-st order moment matching tree of the original R-T-C tree T. Thus, the Elmore delay computation is performed by using $\hat{T}$ with an order of computation time $O(n)$ as will be described in the next section.

3.2 Elmore delay in R-T-C mesh

In this subsection, we generalize the results for an R-T-C tree to an R-T-C mesh. In contrast to the method used in the previous subsection, here we will start from the definition of moments in frequency domain rather than in time domain as we will not have the simple formula like Eq. (7).

3.2.1 Elmore delay in an R-L-C mesh

Def.2

An R-L-C mesh is another special kind of RLC network. The resistors and inductors are floating from the ground and form a mesh (i.e., there are loops among these elements), the capacitors are connected between the nodes in the mesh and the ground, and the input voltage source is connected between a node on the mesh and the ground.

As in the case of an R-L-C tree, let r be the source node. We denote N as the floating RL network with n_N nodes connected to capacitors as its terminals and r as its reference terminal. Thus, N is an $n_N + 1$-terminal RL network. Suppose its open circuit impedance matrix is $Z(s) = [Z_{ik}(s)]$. When N is connected to the capacitors and the voltage source, the input current entering node k is $-sC_kV_k(s)$. For each node i,

$$V_i(s) - V_r(s) = -\sum_k Z_{ik}(s)sC_kV_k(s) \tag{12}$$

and the transfer function $H_i(s) = V_i(s)/V_r(s)$ becomes

$$H_i(s) = 1 - \sum_k Z_{ik}(s)sC_kH_k(s) \tag{13}$$

Expanding $Z_{ik}(s)$ and $H_k(s)$ into Taylor series:

$$Z_{ik}(s) = Z_{ik}(0) + \sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^j Z_{ik}(s)}{ds^j} |_{s=0}\, s^j \tag{14}$$

and

$$H_k(s) = H_k(0) + \sum_{j=1}^{\infty} (-1)^j m_k^j s^j \tag{15}$$

from the above three equations, it can be seen that

$$T_{Di} = m_i^1 = \sum_k Z_{ik}(0)C_kH_k(0) \tag{16}$$

Note that when $s = 0$, N becomes a resistive $n_N + 1$-terminal network with open circuit matrix $R = [R_{ik}]$. Therefore, $Z_{ik}(0) = R_{ik}$. As N is a connected network, each element of R is finite. Also,

$$H_k(0) = \frac{V_k(0)}{V_r(0)} = \lim_{s\to 0} \frac{sV_k(s)}{sV_r(s)} = \lim_{t\to\infty} v_k(t) = 1 \tag{17}$$

Therefore, we have

$$T_{Di} = \sum_k R_{ik}C_k \tag{18}$$

Eq. (18) is the same as Eq. (10). However, now R_{ik} cannot be computed simply as in the case of an R-L-C tree, and it takes $O(n^3)$ time to compute all the R_{ik}'s in the general case. Also, it is worth mentioning that formula (18) is the same as in RC mesh case.

Remark.

Sometimes, there are floating capacitors in network N used as accelarate capacitors; meanwhile, there is at least one dc path from the source node to each other node in N. In this case, the open circuit resistance matrix R exists and Eq.(18) is still valid.

3.2.2 Elmore delay model of transmission line for R-T-C mesh

In this subsection, we will prove that the Elmore delay models for a transmission line derived in Sec.3.1.2 for R-T-C trees are also valid for R-T-C meshes.

As in the case of an R-T-C tree, we first consider the contribution of a transmission line TL to the Elmore delay T_{Di} in an R-T-C mesh. From the definition of R_{ik} it is known that for any pair of (i, k) such that they are not internal in the transmission line TL, the contribution of TL to R_{ik} is determined by its total resistance. Therefore, when TL is replaced by its Elmore delay model, all such R_{ik}'s remain unchanged. Now we consider the contribution of the capacitance of TL to T_{Di}. There are two cases as follows.

1. Case 1. TL is a cut of N.

   In this case, when TL is removed (open circuited) from N, N is divided into 2 separate parts N_1 and N_2. We assume that node r and node t_1 are in N_1 and node t_2 is in N_2. In the case that node i is in N_1 , for a node $k \in LL$ $R_{ik} = R_{it_1}$ and the total contribution of the line to T_{Di} is $\sum_{k \in LL} R_{ik}C_k = R_{it_1} \sum_{k \in LL} C/n = R_{it_1}C$. Therefore, TL acts as a lumped capacitance C. In the case that node i is in N_2, for each node $k \in LL$, let $R_k = kR/n$, then it is easy to show that

   $$R_{ik} = R_{t_1t_1} + R_k \tag{19}$$

   (See Appendix A.1 for the proof). Therefore, the contribution of the line to T_{Di} by the part of $\sum_{k \in LL} R_{ik}C_k$ is

$$\sum_{k=1}^{n}(R_{t_1t_1} + k\frac{R}{n})\frac{C}{n} = R_{t_1t_1}C + \frac{n+1}{2n}RC \tag{20}$$

which becomes $R_{t_1t_1}C + \frac{1}{2}RC$ when $n \to \infty$. This situation is the same as Case 2 of an R-T-C tree. As each transmission line in an R-T-C tree forms a cut for the network N, and it can easily be understood that the model applied to a transmission line in an R-T-C tree can be applied to a cut transmission line in an R-T-C mesh, too.

2. Case 2. TL is not a cut of N

   In this case, in order to evaluate the contribution of the line to T_{Di}, we derive a formula relating R_{it_1}, R_{it_2} and R_{ik} for the k-th internal node of the transmission line model $k \in LL$. Let $R_k = kR/n$ and $\bar{R}_k = (n-k)R/n$ be the total resistance of the model between nodes t_1, k and k, t_2 and $\hat{N}$ be the remaining part of N as shown in Fig.3. As the branch made of R and $\bar{R}$ do not form a cut in network N, $\hat{N}$ is connected and its open circuit resistance matrix $\hat{R} = [\hat{R}_{ik}]$ exists. Then, we have the following lemma for R_{ik}.

   Lemma 1.

$$R_{ik} = A_1(i) + B_1(i)R_k = A_2(i) + B_2(i)\bar{R}_k \tag{21}$$

   Let $\hat{R}_s = \hat{R}_{t_1t_1} + \hat{R}_{t_2t_2} - 2\hat{R}_{t_1t_2}$, then $A_1(i) = \hat{R}_{it_1} + (\hat{R}_{it_2} - \hat{R}_{it_1})(\hat{R}_{t_1t_1} - \hat{R}_{t_1t_2})/(\hat{R}_s + R)$, $B_1(i) = (\hat{R}_{it_2} - \hat{R}_{it_1})/(\hat{R}_s + R)$, $A_2(i) = \hat{R}_{it_2} + (\hat{R}_{it_1} - \hat{R}_{it_2})(\hat{R}_{t_2t_2} - \hat{R}_{t_1t_2})/(\hat{R}_s + R)$, and $B_2(i) = (\hat{R}_{it_1} - \hat{R}_{it_2})/(\hat{R}_s + R)$ (See Appendix A.2 for the proof). Note that the coefficients $A_1(i)$, $B_1(i)$, $A_2(i)$ and $B_2(i)$ are independent of node k. In the case $R_k = 0$, $R_{ik} = R_{it_1} = A_1(i)$ and in the case that $\bar{R}_k = 0$, $R_{ik} = R_{it_2} = A_2(i)$. Therefore, we have

$$R_{ik} = R_{it_1} + B_1(i)R_k = R_{it_2} + B_2(i)\bar{R}_k \tag{22}$$

   From Lemma 1 it can be seen that the contribution of the line to T_{Di} by the part of $\sum_{k\in LL} R_{ik}C_k$ is

$$\sum_{k=1}^{n}(R_{it_1} + kB_1(i)\frac{R}{n})\frac{C}{n} = R_{it_1}C + \frac{n+1}{2n}B_1(i)RC \tag{23}$$

which becomes $R_{it_1}C + \frac{1}{2}B_1(i)RC$ when $n \to \infty$. Also, the contribution can be expressed by $R_{it_2}C + \frac{1}{2}B_2(i)RC$. Now consider the case that the transmission line is replaced by its Elmore delay model. By applying Lemma 1, it can be seen that when the Π-typed model shown in Fig.2a is used, the contribution becomes $R_{it_1}C/2 + (R_{it_1} + B_1(i)R)C/2 = R_{it_1}C + \frac{1}{2}B_1(i)RC$; and when the T-typed model shown in Fig.2b is used, it becomes $(R_{it_1} + B_1(i)R/2)C = R_{it_1}C + \frac{1}{2}B_1(i)RC$. We can also prove that the contribution by either the model can be expressed as $R_{it_2}C + \frac{1}{2}B_2(i)RC$. This proves that the models shown in Fig.2 are also valid for the use in R-T-C meshes .

4 Computation of higher order moments

4.1 R-L-C tree case

In this section, we extend our method of moment computation from the 1-st order to a higher order. The method for the computation of m_i^p for a node voltage v_i on an R-L-C tree T is based on Theorem 1 with the proof shown in Appendix A.3.

Theorem 1.

The p-th order moment m_i^p of node voltage v_i can be expressed as

$$m_i^p = \sum_k (R_{ik}C_k m_k^{p-1} - L_{ik}C_k m_k^{p-2}) \tag{24}$$

where for any k, m_k^0 and m_k^{-1} are defined as 1 and 0, respectively.

Theorem 1 suggests a recursive formula to compute the moments from order 1 to an order p successively. From Eq. (24) we derive a formula relating the p-th order moment of a node k with the p-th order moment of its father node $\bar{k}$ as described in Theorem 2. Theorem 2 is useful not only in the derivation of a linear time complexity algorithm for the computation of all the p-th order moments of an R-L-C tree, but also in the derivation of a p-th order moment matching model as will be shown in the next section.

Theorem 2.

Let $\bar{k}$ be the father node of node k, R_k and L_k be the resistance and inductance of branch $(\bar{k}, k)$ and $C_{Tk}^p = \sum_{j \in D(k)} m_j^p C_j$, then

$$m_k^p = m_{\bar{k}}^p + R_k C_{Tk}^{p-1} - L_k C_{Tk}^{p-2} \quad (25)$$

The proof is shown in Appendix A.4.

From Theorem 2, we suggest a recursive algorithm to compute the p-th moments of all node voltages on an R-L-C tree with linear time complexity $O(n)$. The algorithm is similar to that given in [15] and consists of 2 recursive processes: $findC_T(k,p)$ used to find the $C_{T,k}^p$ from the leaves upwards to the root and $moment(k,p)$ used to find the moments from the root of the tree downwards to its leaves, which are described as follows.

Algorithm 1: $findC_T$

```
findC_T(k,p)
  { C^p_Tk = m^p_k C_k;
    if k is not a leaf node
      for each node j ∈ S(k) do
        C^p_Tk += findC_T(j,p);
    record C^p_Tk and return(C^p_Tk); }
  }
```

Algorithm 2: $moment(k,p)$

```
moment(k,p)
  {if k is the root
     m^p_k = 0;
   else
     m^p_k = m^p_k̄ + R_k C^(p-1)_Tk - L_k C^(p-2)_Tk;
```

```
record m_k^p;
if k is not a leaf
 for each j ∈ S(k) do
  moment(j,p);
 return; }
}
```

4.2 R-L-C mesh case

In the case of an R-T-C mesh, from Eqs.(13), (14), (15) and (2), we have

Theorem 3. The p-th moment of node voltage v_i in an R-L-C mesh can be expressed as

$$m_i^p = \sum_k C_k \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} R_{ik}^j m_k^{p-j-1} = \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} \sum_k C_k R_{ik}^j m_k^{p-j-1} \tag{26}$$

where $R_{ik}^j = d^j Z_{ik}(s)/ds^j \mid_{s=0}$.

This is a recursive formula for the $p-th$ moment m_i^p which is based on all of the moments from order 0 to order $p-1$. If we denote R_{ik}^1 by L_{ik}, then for $p = 2$, Eq.(26) becomes the same as Eq.(24). In the general case, for $p > 2$, Eq.(24) for an R-L-C tree is simpler than Eq.(26) for an R-L-C mesh . This is because in the tree case $Z_{ik}(s) = R_{ik} + sL_{ik}$ and $R_{ik}^j = 0$ for $j > 1$.

To use formula (26) , we need to compute the derivatives $R^j = d^j Z(s)/ds^j \mid_{s=0}$ of the matrix $Z(s)$, which can be done recursively as follows. Let $Y_n(s)$ be the nodal admittance matrix of network N with node r taken as a reference, then $Y_n Z = I$, where I is an $n \times n$ unit matrix. Differentiate both side of the above equation w.r.t. s, we have $Y_n^1 Z + Y_n Z^1 = 0$, so that

$$Z^1 = -ZY_n^1 Z \mid_{s=0} = -RY_n^1 \mid_{s=0} R \tag{27}$$

In the general case, we have the following formula to compute the k-th derivative of $Z(s)$:

$$R^k = d^k Z(s)/ds^k \mid_{s=0} = -R \sum_{j=0}^{k-1} C_k^j Y_n^{k-j} R^j \qquad (28)$$

where $C_k^j = k!/j!(k-j)!$.

5 Moment matching model of RLC transmission line

Def.3

An RLC line is a special kind of RLC tree such that each node has at most one son node. Note that an RLC line is a lumped circuit which is different from an RLC transmission line.

Def.4

An RLC line $\hat{TL}$ is called a p-th moment matching tree (mesh) model of an RLC transmission line TL iff for any R-T-C tree (mesh) where TL is embedded, when TL is replaced by $\hat{TL}$ and a new tree (mesh) $\hat{T}$ is formed, for each output node voltage v_i in T and its counterpart $\hat{v}_i$ in $\hat{T}$, their moments match each other up to the order of p.

Def.5

If each RLC transmission line in an R-T-C tree (mesh) T is replaced by its p-th moment matching tree (mesh) model, the resultant R-L-C tree (mesh) $\hat{T}$ is called a p-th moment matching model of T.

The purpose of finding a moment matching model of a transmission line is to form a moment matching model of an R-T-C tree (mesh) with a minimum number of nodes so that the computation of the moments of the node voltages can be done efficiently. Note that from Def.4, a p-th moment matching tree (mesh) model of a transmission line is valid in the use of any R-T-C tree (mesh) under any termination conditions.

5.1 R-T-C tree case

5.1.1 Contribution of a transmission line to moments

As in Section 2, we derive the moment matching model of a transmission line TL from its contribution to the moments m_i^p of any node voltage v_i on the R-T-C tree T where TL is in.

Case 1. TL is not on path P_i.

In this case, R and L has no contribution to R_{ik} and L_{ik} for any node k. For any node $k \in LL$, $R_{ik} = R_{it_1}$, $L_{ik} = L_{it_1}$ and the contribution of the capacitances to m_i^j is

$$\sum_{k=1}^{n} R_{ik} C_k m_k^{j-1} - \sum_{k=1}^{n} L_{ik} C_k m_k^{j-2} = [\frac{1}{n} \sum_{k=1}^{n} m_k^{j-1}] C R_{it_1} - [\frac{1}{n} \sum_{k=1}^{n} m_k^{j-2}] C L_{it_1} \quad (29)$$

We denote

$$U^j = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} m_k^j \quad (30)$$

and

$$V^j = C U^j \quad (31)$$

then

$$\lim_{n \to \infty} \sum_{k=1}^{n} (R_{ik} C_k m_k^{j-1} - L_{ik} C_k m_k^{j-2}) = R_{it_1} V^{j-1} - L_{it_1} V^{j-2} \quad (32)$$

and the function U^j characterizes the contribution of the capacitances in the transmission line TL to the moment of node voltage v_i. It can be seen that for j=0, $U^0 = 1$ and V^0 is simply equal to C, the total capacitance of the transmission line; and for $j \geq 1$, U^j is the average of the $j-th$ order moments of the voltages on the line.

Case 2. TL is on the path of P_i.

In this case, the whole node set of the R-L-C tree is divided into three subsects: $I_1 = \{k \mid P_{ki} \subseteq P_{t_1}\}$, $I_2 = \{k \mid P_{ki} \supset P_{t_2}\}$, and $I_3 = LL$. For any node $k \in I_1$, the transmission line TL has no contribution to R_{ik} and L_{ik}. For any node $k \in I_2$ and the term $R_{ik}C_k m_k^{j-1} - L_{ik}C_k m_k^{j-2}$, the parameters R and L of the line become a part of R_{ik} and L_{ik}, respectively, and the line plays a part as a lumped series connected branch with a resistance R and an inductance L. Now we consider the case that $k \in LL$. The total contribution of the line can be expressed as

$B_i^j = \sum_{k=1}^{n}(R_{ik}C_k m_k^{j-1} - L_{ik}C_k m_k^{j-2})$.

Note that $R_{ik} = R_{t_1t_1} + \frac{k}{n}R$ and $L_{ik} = L_{t_1t_1} + \frac{k}{n}L$, so that we have

$$B_i^j = R_{t_1t_1}\frac{C}{n}\sum_{k=1}^{n} m_k^{j-1} - L_{t_1t_1}\frac{C}{n}\sum_{k=1}^{n} m_k^{j-2} + \frac{C}{n^2}(R\sum_{k=1}^{n} km_k^{j-1} - L\sum_{k=1}^{n} km_k^{j-2}) \quad (33)$$

Let

$$W^j = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} km_k^j \quad (34)$$

and

$$X^j = CW^j \quad (35)$$

then

$$B_i^j = R_{t_1t_1}V^{j-1} - L_{t_1t_1}V^{j-2} + RX^{j-1} - LX^{j-2} \quad (36)$$

From Eqs. (32) and (36), it can be seen that U^j (V^j) and W^j (X^j)are the two functions characterizing the contribution of the capacitance of a transmission line TL to the moments of a R-T-C tree T where TL is in.

Example 1. We consider the contribution of an RLC transmission line to a second order moment m_i^2, which is described by the functions U^0, U^1, W^0 and W^1. It is easy to show that $U^0 = 1$. Let $m_{t_1}^1$ be the first order moment of v_{t_1} and $C_{Tl} = \sum_{k\in D(t_2)} C_k$, then U^1, W^0 and W^1 can be expressed as follows with the derivation shown in Appendix C.1.

$$U^1 = m_{t_1}^1 + \frac{1}{2}RC_{Tl} + \frac{1}{3}RC \quad (37)$$

$$W^0 = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{2} \tag{38}$$

and

$$W^1 = \frac{1}{2} m^1_{t_1} + \frac{1}{3} RC_{Tl} + \frac{5}{24} RC \tag{39}$$

5.1.2 Formation of moment matching model of transmission line

A p-th moment matching model of a transmission line TL is generally an RLC line shown in Fig.4. Let r be the number of sections in the model. In addition to the terminal nodes t_1 and t_2, there are internal nodes $s_1, s_2, \ldots, s_{r-1}$, and we denote $s_0 = t_1$ and $s_r = t_2$. The capacitance connected to node s_i is denoted by C_{s_i} and the resistance and inductance in the branch between s_{i-1} and s_i are denoted by R_{s_i} and L_{s_i}, respectively. From Sec.4 and Sec.5.1, it is known that the RLC line is a p-th order moment matching model of TL iff the following conditions are satisfied.

(a) Condition 1:

$$\sum_{i=1}^{r} R_{s_i} = R \tag{40}$$

(b) Condition 2:

$$\sum_{i=1}^{r} L_{s_i} = L \tag{41}$$

(c) Condition 3:

$$\sum_{i=0}^{r} C_{s_i} m^j_{s_i} = CU^j \tag{42}$$

for $j = 0, 1, \ldots, p-1$. Note that when $j = 0$, this implies $\sum_{i=0}^{r} C_{s_i} = C$.

(d) As described in Case 2 of Sec.5.1, the contribution of the total model to the moment m^j_k with $P_k \supset TL$ will be $B^j_i = \sum_{i=0}^{r}(R_{t_1 s_i} C_{s_i} m^{j-1}_{s_i} - L_{t_1 s_i} C_{s_i} m^{j-2}_{s_i})$. Note that $R_{t_1 s_i} = R_{t_1 t_1} + \sum_{k=1}^{i} R_{s_k}$ and $L_{t_1 s_i} = L_{t_1 t_1} + \sum_{k=1}^{i} L_{s_k}$, we have

$$B^j_i = R_{t_1 t_1} \sum_{i=0}^{r} C_{s_i} m^{j-1}_{s_i} - L_{t_1 t_1} \sum_{i=0}^{r} C_{s_i} m^{j-2}_{s_i}$$
$$+ \sum_{i=1}^{r} \sum_{k=1}^{i} R_{s_k} C_{s_i} m^{j-1}_{s_i} - \sum_{i=1}^{r} \sum_{k=1}^{i} L_{s_k} C_{s_i} m^{j-2}_{s_i}.$$

Compared this expression with the expression of B^j of Eq.(36) under condition 3, we have two more conditions:

Condition 4;

$$\sum_{i=1}^{r}\sum_{k=1}^{i} R_{s_k} C_{s_i} m_{s_i}^{j} = CRW^{j} \tag{43}$$

for $j = 0, 1, \ldots, p-1$, and

Condition 5:

$$\sum_{i=1}^{r}\sum_{k=1}^{i} L_{s_k} C_{s_i} m_{s_i}^{j} = CLW^{j} \tag{44}$$

for $j = 0, 1, \ldots, p-2$.

Let $\alpha_k = R_{s_k}/R$ for $k = 1, 2, \ldots, r$. Compared (44) with (43), it can be seen that if a set $A = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ is found to satisfy (40) and (43) for $j = 1, 2, \ldots, p-1$, we can always choose the parameters of the inductances such that $L_{s_k} = \alpha_k L$ ($k = 1, 2, \ldots, r$) to meet with the conditions (41) and (44) for $j = 1, 2, \ldots, p-2$. Therefore, we only need to consider conditions 1,3 and 4. Let $\beta_k = C_{sk}/C$ and $\alpha_{1i} = \sum_{k=1}^{i} \alpha_k$, then these conditions can be expressed in terms of the α's and β's as follows:

$$\sum_{i=1}^{r} \alpha_i = 1 \tag{45}$$

$$\sum_{i=0}^{r} \beta_i m_{s_i}^{j} = U^{j} \tag{46}$$

and

$$\sum_{i=1}^{r} \alpha_{1i} \beta_i m_{s_i}^{j} = W^{j} \tag{47}$$

for $j = 0, 1, \ldots, p-1$.

The above equations form the constraints to the parameters of a p-th moment matching model of an RLC transmission line. There are $2p+1$ equations related to these parameters. So, r, the number of sections of the model, is at least p. However, as can be seen from Example 1, for $j = 1$, each of the U^1 and W^1 consist of 3 terms with the first related to m_{t_1}, the second to C_{Tl} and the third to the parameters of the transmission line R and C. In order that the model be compatible to any terminations, the corresponding coefficients represented by the model should be the same as those of the original ones. Therefore, we have more than $2p+1$ constraints in the general case. In fact, we have found that the number of constraints is $3p$ and $r = \lfloor 3p/2 \rfloor$ in these cases. This conclusion is proved in Appendix E.

Example 2.

A second order moment matching tree model of a transmission line is shown in Fig.5a with the parameters $\alpha_1 = 0.20718$, $\alpha_2 = 0.61908$, $\alpha_3 = 0.17375$, $\beta_1 = 0.54051$, $\beta_2 = 0.45919$ and $\beta_3 = 3.9019e-09 \approx 0$. When the line is unloaded, the model can be simplified to that shown in Fig.5b with the parameters $\alpha_1 = 1/4$, $\alpha_2 = 3/4$, $\beta_0 = 0$, $\beta_1 = 2/3$ and $\beta_2 = 1/3$. The derivation of such models is described in Appendix C.2.

5.2 R-T-C mesh case

As in Sec.3.2.2, here we consider two cases for a transmission line TL with total resistance R, inductance L and capacitance C embedded in an R-T-C mesh. Note that in either case, as long as the total resistance and inductance of the model of TL remain R and L respectively, for any pair of nodes i and k not being the internal nodes of the model, Z_{ik} remains unchanged when TL is replaced by its model.

5.2.1 Case 1. TL is a cut of the resistor-transmission line network N

In Theorem 2, a relationship between the p-th moment of a node voltage v_k and that of its father node $v_{\bar{k}}$ is established. In an R-T-C mesh, there is no farther and son relationship between neighbour nodes. However, if a branch between two nodes $\bar{k}$ and k with an impedance $Z_k = R_k + sL_k$ from a cut of the R-L mesh N, we will get a similar result. In this case, let N_1 and N_2 be two subcircuits after Z_k is cut, and suppose that nodes r and $\bar{k}$ are in N_1 and node k is in N_2. Then, it can be shown that for each $i \in N_1$, $Z_{ik} = Z_{i\bar{k}}$, and for each $i \in N_2$, $Z_{ik} = Z_{i\bar{k}} + Z_k$. If we denote $C_{Tk}^p = \sum_{k \in N_2} C_k m_k^p$, then it is easy to show that Theorem 2 still holds.

From this result, it can be shown that the p-th moment matching tree model of the transmission line is valid for the use in a mesh. The proof will follow the same way as stated in Sec.3.2.2 and is omitted.

5.2.2 Case 2. TL is not a cut of N

In the case that TL does not form a cut, suppose that it is modeled by n sections RLC 2-port with parameters r_i, l_i and C_i, $i = 1, \ldots, n$, where $z_i = r_i + sl_i$ is connected between nodes $i-1$ (or t_1 when $i = 1$) and node i and C_i is connected between node i and the ground. As in Case 2. of Sec.3.2.2, for each node $k \in LL$ and $i \in \hat{N}$, we have

$$Z_{ik} = A_1(i) + B_1(i)Z_k = A_2(i) + B_2(i)\bar{Z}_k \tag{48}$$

where $Z_k = \sum_{j=1}^{k} z_j \equiv R_k + sL_k$ with $R_k = \sum_{j=1}^{k} r_j$ and $L_k = \sum_{j=1}^{k} l_j$, $\bar{Z}_k = Z - Z_k$ with $Z = Z_n = z_1 + z_2 + \ldots z_n$. Let $\hat{Z}_s = \hat{Z}_{t_1t_1} + \hat{Z}_{t_2t_2} - 2\hat{Z}_{t_1t_2}$, then $A_1(i) = \hat{Z}_{it_1} + (\hat{Z}_{it_2} - \hat{Z}_{it_1})(\hat{Z}_{t_1t_1} - \hat{Z}_{t_1t_2})/(\hat{Z}_s + Z)$,$B_1(i) = (\hat{Z}_{it_2} - \hat{Z}_{it_1})/(\hat{Z}_s + Z)$, $A_2(i) = \hat{Z}_{it_2} + (\hat{Z}_{it_1} - \hat{Z}_{it_2})(\hat{Z}_{t_2t_2} - \hat{Z}_{t_2t_1})/(\hat{Z}_s + Z)$ and $B_2(i) = (\hat{Z}_{it_1} - \hat{Z}_{it_2})/(\hat{Z}_s + Z)$. Note that coefficients A and B are functions of s. When $k = t_1$, $Z_k = 0$, $A_1(i) = Z_{it_1}$, so we have $Z_{ik} = Z_{it_1} + B_1(i)Z_k$. Also, when $k = t_2$, $\bar{Z}_k = 0$, $A_2(i) = Z_{it_2}$, and we have $Z_{ik} = Z_{it_2} + B_2(i)\bar{Z}_k$. Now we use the first expression. Note that $(B_1(i)Z_k)^{(j)} = B_1(i)^{(j)}R_k + jB_1(i)^{(j-1)}L_k$. Then, from Eq.(26), the contribution of TL to m_i^p is

$$B_i^p = \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} \sum_{k=1}^{n} R_{ik}^j m_k^{p-j-1} C_k =$$

$$= \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} R_{it_1}^j \sum_{k=1}^{n} m_k^{p-j-1} C_k +$$

$$+ \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} B_1^{(j)}(i) \sum_{k=1}^{n} m_k^{p-j-1} C_k R_k -$$

$$- \sum_{j=0}^{p-2} \frac{(-1)^j}{j!} B_1^{(j)}(i) \sum_{k=1}^{n} m_k^{p-j-1} C_k L_k$$

Substitute $R_k = kR/n$, $L_k = kL/n$, $C_k = C/n$ into the above expression and let $n \rightarrow \infty$, we have

$$B_i^p = \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} R_{it_1}^{(j)} V^{p-j-1} + \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} B_1^{(j)}(i) R X^{p-j-1} - \sum_{j=0}^{p-2} \frac{(-1)^j}{j!} B_1^{(j)}(i) L X^{p-j-2} \tag{49}$$

This equation is a generalization of Eq.(36). It can be seen from the equation that the functions U and W still characterize the contribution of TL to m_i^p.

In order to use the above equation to derive the moment metching model for the use in an R-T-C mesh, we need a formula relating m_i^j ($i \in LL$) to $m_{t_1}^j$ similar to Eq.(25) in the tree case. Let $P = 1/(Z + \hat{Z}_s)$, $Q = (\hat{Z}_{t_1t_1} - \hat{Z}_{t_1t_2})/(Z + \hat{Z}_s)$ and $C_{T\hat{N}}^p = \sum_{k \in \hat{N}} C_k \sum_{j=0}^{p} \frac{(-1)^j}{j!} B_1(k)^{(j)} m_k^{p-j}$, the formula is as follows with the derivation shown in Appendix B.

$$m_i^p = m_{t_1}^p + R_i C_{T\hat{N}}^{p-1} - L_i C_{T\hat{N}}^{p-2} + \sum_{k=1}^{i} C_k (R_k m_k^{p-1} - L_k m_k^{p-2}) +$$

$$+ R_i \sum_{k=i+1}^{n} C_k m_k^{p-1} - L_i \sum_{k=i+1}^{n} C_k m_k^{p-2} - \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} \sum_{k=1}^{n} C_k m_k^{p-j-1} *$$

$$*[P^{(j)} R_k R_i + j P^{(j-1)} (R_k L_i + R_i L_k) + \frac{j(j-1)}{2} P^{(j-2)} L_i L_k + Q^{(j)} R_i + j Q^{(j-1)} L_i] \tag{50}$$

This formula can be used for the model formed by a large number ($n \rightarrow \infty$) of uniform RLC sections or by a finite number of nonuniform RLC sections. In the first case, we have

$$m_i^p = m_{t_1}^p + \frac{i}{n}(R C_{T\hat{N}}^{p-1} - L C_{T\hat{N}}^{p-2}) + \frac{RC}{n^2}(\sum_{k=1}^{i} k m_k^{p-1} + i \sum_{k=i+1}^{n} m_k^{p-1}) -$$

$$- \frac{LC}{n^2}(\sum_{k=1}^{i} k m_k^{p-2} + i \sum_{k=i+1}^{n} m_k^{p-2}) - \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} \{ \frac{RC}{n^3} (P^{(j)} R + j P^{(j-1)} L) i \sum_{k=1}^{n} k m_k^{p-j-1} +$$

$$+\frac{LC}{n^3}j(P^{(j-1)}R+\frac{j-1}{2}P^{(j-2)}L)i\sum_{k=1}^{n}km_k^{p-j-1}+\frac{C}{n^2}(Q^{(j)}R+jQ^{(j-1)}L)i\sum_{k=1}^{n}m_k^{p-j-1}\} \tag{51}$$

Example 3.

Consider the case that $p = 1$. In this case, $C^0_{T\hat{N}} = \sum_{k\in\hat{N}} C_k \equiv C_{T\hat{N}}$ and

$$m_i^1 = m_{t_1}^1 + \frac{i}{n}RC_{T\hat{N}} + \frac{RC}{n^2}(ni - \frac{i^2}{2} + \frac{i}{2}) - \frac{i}{2n}R^2CP - \frac{i}{n}QRC$$

Let $n \to \infty$, then we have

$$U^1 = m_{t_1}^1 + \frac{1}{2}RC_{T\hat{N}} + \frac{1}{3}RC - \frac{1}{4}R^2CP - \frac{1}{2}QRC \tag{52}$$

and

$$W^1 = \frac{1}{2}m_{t_1}^1 + \frac{1}{3}RC_{T\hat{N}} + \frac{5}{24}RC - \frac{1}{6}R^2CP - \frac{1}{3}QRC \tag{53}$$

Compared these equations with Eqs.(37) and (39), it can be seen that the coefficients for the first three terms are correspondingly equal. Also, from the expression of m_i^1, it can be understood that in U^1 the coefficients of $-QRC$ are the same as that of $RC_{T\hat{N}}$ and the coefficient of $-R^2CP$ is just one half of it. A similar situation happens in the expression of W^1. This means that these coefficients do not set independent constraints and can be neglegted, and we can understand that the equations used to determine the parameters of the 2-nd order moment matching model for a mesh will be similar to those for a tree. However, in the case of a tree, node t_1 is an ancestor of node t_2 and we have seen that the model is dissymmetrical. In the case of a mesh, no difference can be told between nodes t_1 and t_2. Eq.(50) is derived w.r.t. node t_1, and a similar formula can be obtained w.r.t. node t_2. In order that the model be fitted with the constraints given by the two equations, the moment matching model should be symmetrical. It happens that the 1-st moment matching model is symmetrical, so it can be used in both a tree and a mesh; but for a higher order moment matching model, the one used in a mesh will be more complicated than that used in a tree.

Example 4. A 2-nd order moment matching mesh model.

The circuit of the model is a symmetric 2-port as shown in Fig.5c with $\alpha_1 = 0.24283$, $\alpha_2 = 0.25623$, $\beta_0 = 0.10275$, $\beta_1 = 0.26222$ and $\beta_2 = 0.26862$. The derivation is shown in Appendix C.

6 Delay model of R-T-C network

Let d_i be the signal delay of node voltage v_i w.r.t. the input voltage v_r. In an R-T-C network, d_i consists of two parts: the propagation delay d_{ip} and the rising delay d_{ir} as shown in Fig.6.

6.1 Propagation delay

The propagation delay d_{ip} of v_i is caused by the propagation time of the transmission lines in the network. For a transmission line with total inductance L and total capacitance C, its propagation time is $\tau = \sqrt{LC}$; i.e., any change of signal at t=0 at one port of the line will cause the change of signal at the other port only after $t = \tau$. Therefore, in an R-T-C network, when a step excitation is applied at the source terminal, it takes $\tau_i = d_{ip}$ to propagate to node i, and in the time period $t \in [0, d_{ip}]$ v_i stays at 0. When the path from node r to node i is relatively long, d_{ip} becomes the main part of the signal delay of v_i.

Given an R-T-C network T, The propagation delay d_{ip} for each node i can be found by using a graph $G(V, E)$. In the graph, vertex i in V corresponds to node i in T, and each edge $e_{ij} =< i, j >$ corresponds to a branch between nodes i and j in T. If this branch is made of a transmission line with a prapagation time τ_{ij}, the distance of edge e_{ij} is defined as τ_{ij}; otherwise, if it is made of a resistor, then its distance is defined as 0. Then, the propagation delay d_{ip} from the source node r to any node i is equal to the distance of the shortest path between vertices r and i which can be found by using the famous Dikstra algorithm [21].

6.2 Rising delay

To compute the rising delay d_{ir} of v_i, we use the moment matching technique.

Suppose that the transfer function of node i is $F_i(s) = V_i(s)/V_r(s)$ and its propagation delay is d_{ip}, then $F_i(s)$ can be expressed as $F_i(s) = H_i(s)e^{-d_{ip}s}$. Let $u_i(t) = L^{-1}(H_i(s)/s)$, then $v_i(t) = u_i(t - d_{ip})$ and the rising delay d_{ir} can be determined by using $u_i(t)$.

We find an approximation $\hat{u}_i(t)$ of $u_i(t)$ by approximating $H_i(s)$ by a rational function $\hat{H}_i(s)$ such that $\hat{H}_i(s)$ is a p-th order *Padé* approximation of $H_i(s)$ at s=0, i.e., their moments match from order 0 to order p. Given the moments m_{if}^j of function $F_i(s)$ for j=0 to p, from the definition, $H_i(s) = e^{d_{ip}s}F_i(s)$, and the moments m_i^j of $H_i(s)$ can be found by using the following formula:

$$m_i^j = \sum_{k=0}^{j} \frac{(-1)^k}{k!} m_{if}^{j-k} d_{ip}^k \tag{54}$$

Let $\hat{H}_i(s)$ be expressed as $B(s)/A(s) = (b_m s^m + b_{m-1}s^{m-1} \ldots + b_1 s + b_0)/(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0)$. We need $\hat{u}_i(0) = u_i(0) = 0$, which implies that $lim_{t\to 0}\hat{u}_i(t) = lim_{s\to\infty}(s\hat{U}_i(s)) = lim_{s\to\infty}\hat{H}(s) = 0$. Therefore, $m < n$. We choose $m = \lfloor (p-1)/2 \rfloor$ and $n = \lceil (p+1)/2 \rceil$; i.e., when $p = 2k+1$ is odd, $m = k$ and $n = k+1$, and when $p = 2k$ is even, $m = k-1$ and $n = k+1$. In order that $\hat{H}(s)$ be a p-th order *Padé* approximation of $H(s)$, the first $p+1$ coefficients of the Taylor series expansion of $\hat{H}_i(s)$ should be the same as those of $H_i(s)$, i.e.,

$$\frac{b_m s^m + b_{m-1}s^{m-1} \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = 1 - m_i^1 s + m_i^2 s^2 - \ldots + (-1)^p m_i^p s^p + \ldots$$

and we have

$$\begin{bmatrix} A & B \end{bmatrix} x = C \tag{55}$$

where

$$x = \begin{bmatrix} a_0 & a_1 & \ldots & a_{n-1} & b_0 & b_1 & \ldots & b_m \end{bmatrix}^t \tag{56}$$

A is a $(p+1) \times n$ submatrix such that

$$A = \begin{bmatrix} -1 & & & & & \\ m_i^1 & -1 & & & & \\ -m_i^2 & m_i^1 & -1 & & & \\ & & \cdots & \cdots & \cdots & \\ (-1)^n m_i^{n-1} & (-1)^{n-1} m_i^{n-2} & (-1)^{n-2} m_i^{n-3} & \cdots & m_i^1 & -1 \\ (-1)^{n+1} m_i^n & (-1)^n m_i^{n-1} & (-1)^{n-1} m_i^{n-2} & \cdots & -m_i^2 & m_i^1 \\ & & \cdots & \cdots & \cdots & \\ (-1)^{p+1} m_i^p & (-1)^p m_i^{p-1} & (-1)^{p-1} m_i^{p-2} & \cdots & (-1)^{p-n+3} m_i^{p-n+2} & (-1)^{p-n+2} m_i^{p-n+1} \end{bmatrix} \tag{57}$$

$$B = \begin{bmatrix} I_{(m+1)\times(m+1)} \\ 0_{n\times(m+1)} \end{bmatrix} \tag{58}$$

where $I_{(m+1)\times(m+1)}$ is a unit matrix, and $0_{n\times(m+1)}$ is a zero element matrix, and

$$C = \begin{bmatrix} 0 & \dots & 0 & 1 & -m_i^1 & m_i^2 & \dots & (-1)^{p-n} m_i^{p-n} \end{bmatrix}^t \tag{59}$$

Solving the above equations, we get the coefficients a's and b's. Let the n roots of $A(s)$ be denoted by $p_1, p_2, \dots, p_n$. Then, $\hat{U}_i(s) = \hat{H}_i(s)/s$ can be expressed by

$$\hat{U}_i = \frac{1}{s} + \sum_{j=1}^{n} \frac{k_j}{s - p_j} \tag{60}$$

where $k_j = B(p_j)/p_j A'(p_j)$, and $u_i(t)$ will be approximated by

$$\hat{u}_i(t) = 1 + \sum_{j=1}^{n} k_j e^{p_j t} \tag{61}$$

Given any voltage level $0 < V_{th} < 1$, the rising delay d_{ir} can be found by solving the equation

$$\sum_{j=1}^{n} k_j e^{p_j d_{ir}} = V_{th} - 1 \tag{62}$$

and the signal delay $d_i = d_{ip} + d_{ir}$.

We give three examples to illustrate our method. In these examples, V_{th} is set to 0.5, d_{ip} , d_{ir} and d_i are the propagation delay, rising delay and total delay of node voltage v_i computed by using our method, $d_{i,spice}$ is the delay obtained from the

	d_p	d_r	d	d_{spice}	δ
v_1	0.23461	0.11665	0.35126	0.33225	5.4%
v_2	0.35183	0.21733	0.52608	0.56916	7.5%
v_3	0.23459	0.11289	0.34748	0.35770	2.8%

Table 1: Delay Estimation of Example 6

SPICE simulation result and $\delta =\mid d_i/d_{i,spice} - 1 \mid$ is the relative error. From these examples, it can be seen that our model is efficient and accurate for practical use.

Example 5. For the output voltage v_1 in the interconnect circuit shown in Fig.8, by using the Elmore delay model, it is found that $d_{1p} = 6.5041ns$, $d_{ir} = 47.836ns$, $d_1 = 54.341ns$ while $d_{1,spice} = 54.165ns$, and $\delta = 0.32\%$.

Example 6. We compute the signal delay of the output voltages v_1, v_2 and v_3 of the interconnect circuit shown in Fig.9 by using a 3-rd order moment matching model. The results are listed in Table 1 with all the time in unit ns.

Example 7. This is an example of a transmission line mesh shown in Fig.10. The parameters of the vertical lines are: $R = 0.05\Omega/mm$, $L = 0.5025nH/mm$ and $C = 0.1552pF/mm$, and the parameters of the horizontal lines are: $R = 0.06\Omega/mm$, $L = 0.548nH/mm$ and $C = 0.1423pF/mm$. The length of each line is 30mm [18]. For the output voltage v_1, when an Elmore delay model is used, $d_{1p} = 0.79478ns$, $d_{1r} = 2.6733ns$, $d_1 = 3.4681ns$, $d_{1,spice} = 3.7566ns$ and $\delta = 7.7\%$.

7 Conclusion

In this paper, we present formulas and models to compute the moments of node voltages in a transmssion line network and a delay model based on the moment matching technique. The main contribution of this paper is the following.

(a) We provide a recursive formula to compute the moments of an R-L-C network. This formula is especially simple and efficient in the typical case of R-L-C tree networks. For an R-L-C tree with n folating nodes, the computation of the moments of all the node voltages from order 1 up to order p takes $O(pn)$ time. In the more general R-L-C mesh case, such a computation takes $O(pn^3)$ time. However, if in the floating network N the number of link branches l is much smaller than the number of tree branches n, a more efficient way can be used to compute the matrices R^j [14] so that the computaion can be done in $O(pln^2)$ time.

(b) We present a p-th order moment matching model of a transmission line embedded in transmission line networks. The model is made of RLC sections and is either dissymmetric when used in an R-T-C tree or symmetric when used in an R-T-C mesh. The necessary and sufficient conditions for such models are derived and the models from order 1 to order 3 are presented. In both the tree and the mesh case, the number of sections of the model $r = O(p)$ and $r = 1$ when $p = 1$. When the transmission lines are replaced by their p-th order moment matching models, a p-th order moment matching R-L-C network is formed such that the moments of the node voltages of the R-L-C network are exactly the same as the correspoding ones in the original R-T-C network. Then, the computation of the moments can be implemented by using the R-L-C network exactly and efficiently.

There are some other known methods to do moment computaion and delay estimation in transmission line networks. The comparisons between our method and these methods are as follows.

(a) Using a large number sections of RLC network as a model of a transmission line [16]. Such a model is not only time-consuming but also inaccurate.

(b) Using the model suggeted by [11] to compute the moments. The authors of [11] suggested a moment matching model for a transmission line under the condition that the output port of the line is open. Therefore, their model cannot be accurate for a transmission line embedded in any part of a transmission network. Even in the case that a transmission line is open-loaded, only an accurate 2nd order model is given there.

(c) A scattering parameter based mathod to compute the moments is provided in [17]. It takes $O(pn)$ time to compute the moments from order 1 to order p for one node voltage, and takes $O(pn^2)$ time to compute the moments for all the node voltages. Therefore, our method is more efficient than theirs. In another paper [18], the authors provide a method to extract the "time-of-flight" (the propagation delay) based on the operation on the scattering metrices of each part of the network, which takes much more time than our method.

(d) Our method is different from AWE [19, 20]. AWE uses the characteristic 2-port model of transmission lines and computes the moments of node voltages by recursively solving equations, while ours uses direct computation which is more efficient for the transmission line tree and mesh cases. If a moment matching R-L-C tree with n grounded capacitors and n floating inductors is formed by using our moment matching model and AWE is applied on that circuit, then AWE forms a set of state equations with 2n variables, and it is estimated that the computation of all the moments of all the node voltages is about 8 times slower than using our formulas.

Note that the methods and models presented in this paper can be applied not only to transmission line networks, but also to distributed RC line networks. Therefore, they can be efficiently used in the design of VLSI systems.

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Appendix

A Proof

A.1 Proof of Eq.(19)

In this case, network N looks like that shown in Fig.A1. As $R_{ik} = R_{ki}$, we input a unit current to terminal i. By KCL, $i_k = 1$. Therefore, $R_{ki} = v_k = v_{t_1} + v_{kt_1} = R_{it_1} + R_k$. Note that if the current is injected directly to node t_1, v_{t_1} remains unchanged. Therefore, $R_{it_1} = R_{t_1t_1}$ and Eq.(19) follows. □

A.2 Proof of Eq.(21)

Using the relation $R_{ik} = R_{ki}$, we input a unit current to terminal i as shown in Fig.A2a. Let v_s and $\hat{R}_s$ be the voltage and internal resistance of the Tevenin's equvalent 1-port looked from terminals t_1 and t_2 to the left. Then $v_s = \hat{R}_{it_1} - \hat{R}_{it_2}$. For $\hat{R}_s$, we form the circuit shown in Fig.A2b and split the unit current source as shown in Fig.A2c, then we have $\hat{R}_s = \hat{R}_{t_1t_1} - \hat{R}_{t_2t_1} - (\hat{R}_{t_1t_2} - \hat{R}_{t_2t_2}) = \hat{R}_{t_1t_1} + \hat{R}_{t_2t_2} - 2\hat{R}_{t_1t_2}$. Then, $I_{t_1t_2} = v_s/(\hat{R}_s + R)$ and $R_{ik} = R_{ki} = I_{t_1t_2}\bar{R}_k + v_{t_2} = v_{t_1} - R_k I_{t_1t_2}$ with $v_{t_2} = \hat{R}_{it_2} + I_{t_1t_2}(\hat{R}_{t_2t_2} - \hat{R}_{t_1t_2})$ and $v_{t_1} = \hat{R}_{it_1} + I_{t_1t_2}(\hat{R}_{t_1t_2} - \hat{R}_{t_1t_1})$. Then Eq.(21) follows from these equalities. □

A.3 Proof of Theorem 1

To prove Theorem 1, we first prove the following lemmas.

Lemma 2.

$$\lim_{t\to\infty} t^p(1 - v_i(t)) = 0$$

Proof. v_i can be expressed as

$$v_i(t) = 1 + \sum_k P_{i,k}(t)e^{-\delta_{i,k}t}$$

where $\delta_{i,k} > 0$ and $P_{i,k}(t)$ can be expressed as

$$P_{i,k}(t) = \sum_{j=0}^{J(i,k)} a_{i,k,j} t^j cos(\omega_{i,k} t + \phi_{i,k})$$

so the Lemma follows.

Lemma 3.

$$\lim_{t \to \infty} t^p \dot{v}_i(t) = 0$$

Proof.

$\dot{v}_i(t) = \sum_k (\dot{P}_{i,k}(t) - \delta_{i,k} P_{i,k}(t)) e^{-\delta_{i,k} t}$, and the Lemma follows.

Lemma 4.

$$m_i^{p-1} = -\frac{1}{p!} \int_0^\infty t^p \ddot{v}_i dt$$

Proof.

$$\int_0^\infty t^p \ddot{v}_i dt = \int_0^\infty t^p d\dot{v}_i = t^p \dot{v}_i \mid_0^\infty - \int_o^\infty p \dot{v}_i t^{p-1} dt = -p! m_i^{p-1}$$

Lemma 5.

The p-th moment of v_i is

$$m_i^p = \frac{1}{(p-1)!} \int_0^\infty t^{p-1} (1 - v_i(t)) dt$$

Proof. Let $M_i^p(\tau) = \int_0^\tau t^p \dot{v}_i(t) dt$, then $m_i^p = \lim_{\tau \to \infty} \frac{1}{p!} M_i^p(\tau)$. Now

$M_i^p(\tau) = \int_0^\tau t^p \dot{v}_i(t) dt = \int_0^\tau t^p dv_i(t) = \tau^p v_i(\tau) - p \int_0^\tau t^{p-1} v_i dt$

$= \tau^p v_i(\tau) - p \int_0^\tau t^{p-1} dt + p \int_0^\tau t^{p-1} dt - p \int_0^\tau t^{p-1} v_i dt$

$= \tau^p (v_i(\tau) - 1) + p \int_0^\tau t^{p-1} (1 - v_i(t)) dt$

Let $\tau \to \infty$, by Lemma 2,

$$M_i^p(\infty) = p \int_0^\infty t^{p-1} (1 - v_i(t)) dt$$

and

$$m_i^p = \frac{1}{p!} M_i^p(\infty) = \frac{1}{(p-1)!} \int_0^\infty t^{p-1}(1 - v_i(t))dt$$

Proof of Theorem 1.

From Eq.(7) and Lemma 5, we have

$$m_i^p = \frac{1}{(p-1)!} \sum_k (R_{ik} C_k \int_0^\infty t^{p-1} \dot{v}_k dt + L_{ik} C_k \int_0^\infty t^{p-1} \ddot{v}_k dt)$$

From the definition of the moment,

$$\frac{1}{(p-1)!} \int_0^\infty t^{p-1} \dot{v}_k dt = m_k^{p-1}$$

and from Lemma 4,

$$\frac{1}{(p-1)!} \int_0^\infty t^{p-1} \ddot{v}_k dt = -m_k^{p-2}$$

so the theorem holds. □

A.4 Proof of Theorem 2

Proof.

$$m_k^p = \sum_i (R_{ki} C_i m_i^{p-1} - L_{ki} C_i m_i^{p-2})$$

and

$$m_{\bar{k}}^p = \sum_i (R_{\bar{k}i} C_i m_i^{p-1} - L_{\bar{k}i} C_i m_i^{p-2})$$

Let the node set $I = \{1, 2, \ldots, n\}$ be divided into 2 subsets I_1 and I_2 such that $I_1 = \{i \mid P_{ki} = P_{\bar{k}i}\}$ and $I_2 = \{i \mid P_{ki} \supset P_{\bar{k}i}\}$. As $\bar{k}$ is the father node of node k, so that $I_2 = \hat{D}(k)$. Correspondingly, m_k^p and $m_{\bar{k}}^p$ are divided into two parts with respect to I_1 and I_2; i.e.,

$$m_{k,j}^p = \sum_{i \in I_j} (R_{ki} C_i m_i^{p-1} - L_{ki} C_i m_i^{p-2})$$

and

$$m^p_{\bar{k},j} = \sum_{i \in I_j} (R_{\bar{k}i} C_i m_i^{p-1} - L_{\bar{k}i} C_i m_i^{p-2})$$

for j=1 and 2. For each $i \in I_1$, as $P_{ki} = P_{\bar{k}i}$ so that $R_{ki} = R_{\bar{k}i}$, $L_{ki} = L_{\bar{k}i}$ and $m^p_{k,1} = m^p_{\bar{k},1}$. For each node $i \in I_2$, $R_{\bar{k}i} = R_{\bar{k}\bar{k}}$, $L_{\bar{k}i} = L_{\bar{k}\bar{k}}$, $R_{ki} = R_{\bar{k}\bar{k}} + R_k$, and $L_{ki} = L_{\bar{k}\bar{k}} + L_k$. Therefore,

$$m^p_{k,2} = m^p_{\bar{k},2} + \sum_{i \in \hat{D}(k)} (R_k C_i m_i^{p-1} - L_k C_i m_i^{p-2}) = m^p_{\bar{k},2} + R_k C_{Tk}^{p-1} - L_k C_{Tk}^{p-2}$$

and the theorem follows. □

B Derivation of Eq.(50)

The derivation of Eq.(50) is as follows.

For node t_1,

$$m^p_{t_1} = \sum_k C_k \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} R^j_{t_1 k} m_k^{p-j-1} = m^p_{t_1,1} + m^p_{t_1,2}$$

where $m^p_{t_1,1} = \sum_{k \in \hat{N}} C_k \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} R^j_{t_1 k} m_k^{p-j-1}$

and $m^p_{t_1,2} = \sum_{k \in LL} C_k \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} R^j_{t_1 k} m_k^{p-j-1}$. Similarly, for node $i \in LL$, $m^p_i = m^p_{i,1} + m^p_{i,2}$ with $m^p_{i,1} = \sum_{k \in \hat{N}} C_k \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} R^j_{ik} m_k^{p-j-1}$ and $m^p_{i,2} = \sum_{k \in LL} C_k \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} R^j_{ik} m_k^{p-j-1}$. Now we abbreviate $A_1(k)$ and $B_1(k)$ as A_k and B_k. Note that for each $k \in \hat{N}$, $Z_{ik} = Z_{t_1 k} + B_k(s) Z_i$ with $Z_i^0 = R_i$, $Z_i^1 = L_i$ and $Z_i^j = 0$ for $j > 1$. Then, we have

$$\begin{aligned} & m^p_{i,1} - m^p_{t_1,1} = \\ & = \sum_{k \in \hat{N}} C_k \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} B_k^j R_i m_k^{p-j-1} - \\ & - \sum_{k \in \hat{N}} C_k \sum_{j=1}^{p-1} \frac{(-1)^{j-1}}{(j-1)!} B_k^{j-1} L_i m_k^{p-j-1} = \\ & = R_i \sum_{k \in \hat{N}} C_k \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} B_k^j m_k^{p-j-1} - \\ & - L_i \sum_{k \in \hat{N}} C_k \sum_{j=0}^{p-2} \frac{(-1)^j}{j!} B_k^j m_k^{p-j-2} \end{aligned}$$

Let $C_{T\hat{N}}^{p-1} = \sum_{k \in \hat{N}} C_k \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} B_k^j m_k^{p-j-1}$, then we have

$$m_{i,1}^p - m_{t_1,1}^p = R_i C_{T\hat{N}}^{p-1} - L_i C_{T\hat{N}}^{p-2}$$

For $m_{i,2}^p - m_{t_1,2}^p$, we have

$$m_{i,2}^p - m_{t_1,2}^p = \sum_{k=1}^{n} C_k (\sum_{j=0}^{p-1} \frac{(-1)^j}{j!} m_k^{p-j-1} (Z_{ik} - Z_{t_1 k})^{(j)})$$

Note that in the above equation, $(Z_{ik} - Z_{t_1 k})^{(j)}$ is counted at $s = 0$. We now derive the formula for $Z_{ik} - Z_{t_1 k}$. We enter a unit current to node k and let the voltage at node t_1 and node i be V_{t_1} and V_i, respectively. Then, $Z_{ik} - Z_{t_1 k} = V_i - V_{t_1}$ as shown in Fig.A3. Let I_1 and I_2 be the currents flowing from node k to node t_1 and from node k to node t_2, from the equations $V_k = I_1(Z_k + \hat{Z}_{t_1 t_1}) + I_2 \hat{Z}_{t_1 t_2} = I_2(\bar{Z}_k + \hat{Z}_{t_2 t_2}) + I_1 \hat{Z}_{t_1 t_2})$ and $I_1 + I_2 = 1$, we have $I_1 = (Z - Z_k + \hat{Z}_{t_2 t_2} - \hat{Z}_{t_1 t_2})/(Z + \hat{Z}_s)$ and $I_2 = (Z_k + \hat{Z}_{t_1 t_1} - \hat{Z}_{t_1 t_2})/(Z + \hat{Z}_s)$. Then, when $k \le i$, $V_i - V_{t_1} = Z_k I_1 - (Z_i - Z_k) I_2 = Z_k - Z_k Z_i/(Z + \hat{Z}_s) - (\hat{Z}_{t_1 t_1} - \hat{Z}_{t_1 t_2})/(Z + \hat{Z}_s) Z_i$; and when $k > i$, $V_i - V_{t_1} = Z_i I_1 = Z_i(Z - Z_k + \hat{Z}_{t_2 t_2} - \hat{Z}_{t_1 t_2})/(Z + \hat{Z}_s) = Z_i - Z_k Z_i/(Z + \hat{Z}_s) - (\hat{Z}_{t_1 t_1} - \hat{Z}_{t_1 t_2})/(Z + \hat{Z}_s) Z_i$. Let $P = 1/(Z + \hat{Z}_s)$ and $Q = (\hat{Z}_{t_1 t_1} - \hat{Z}_{t_1 t_2})/(Z + \hat{Z}_s)$, then we have

$$Z_{ik} - Z_{t_1 k} = \begin{cases} Z_k - P Z_i Z_k - Q Z_i & k \le i \\ Z_i - P Z_i Z_k - Q Z_i & k > i \end{cases}$$

Note that $(P Z_i Z_k)^{(j)} = P^{(j)} R_i R_k + j P^{(j-1)} (R_i L_k + R_k L_i) + \frac{j(j-1)}{2} P^{(j-2)} L_i L_k$ and $(Q Z_i)^{(j)} = Q^{(j)} R_i + j Q^{(j-1)} L_i$ where $P^{(j)}$ and $Q^{(j)}$ are defined as 0 when $j < 0$. Therefore,

$$m_{i,2}^p - m_{t_1,2}^p = \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} \sum_{k=1}^{n} C_k m_k^{p-j-1} (Z_{ik} - Z_{t_1 k})^{(j)} =$$

$$\sum_{k=1}^{i} C_k (R_k m_k^{p-1} - L_k m_k^{(p-2)}) + R_i \sum_{k=i+1}^{n} C_k m_k^{p-1} - L_i \sum_{k=i+1}^{n} C_k m_k^{p-2} -$$

$$- \sum_{j=0}^{p-1} \frac{(-1)^j}{j!} \sum_{k=1}^{n} C_k m_k^{p-j-1} [P^{(j)} R_k R_i + j P^{(j-1)} (R_k L_i + R_i L_k) +$$

$$+\frac{j(j-1)}{2}P^{(j-2)}L_iL_k + Q^{(j)}R_i + jQ^{(j-1)}L_i]$$

and we have

$$m_i^p = m_{t_1}^p + R_iC_{T\hat{N}}^{p-1} - L_iC_{T\hat{N}}^{p-2} + \sum_{k=1}^{i} C_k(R_km_k^{p-1} - L_km_k^{(p-2)}) +$$

$$+R_i\sum_{k=i+1}^{n} C_km_k^{p-1} - L_i\sum_{k=i+1}^{n} C_km_k^{p-2} - \sum_{j=0}^{p-1}\frac{(-1)^j}{j!}\sum_{k=1}^{n} C_km_k^{p-j-1} *$$

$$*[P^{(j)}R_kR_i + jP^{(j-1)}(R_kL_i + R_iL_k + \frac{j(j-1)}{2}P^{(j-2)}L_iL_k + Q^{(j)}R_i + jQ^{(j-1)}L_i]$$

□

C Second order moment matching model of transmission line

We derive the second order moment matching model of a transmission line in the section.

C.1 Computation of U^1 and W^1

In order to compute U^1 and W^1, we first need to compute the first order moments m_k^1 for $k = 1$ to n.

Let $C_{Tl} = \sum_{k\in\hat{D}(t_2)} C_k$. From Theorem 2, we have

$m_1^1 = m_{t_1}^1 + \frac{R}{n}C_{T1} = m_{t_1}^1 + \frac{R}{n}(C + C_{Tl})$.

$m_2^1 = m_1^1 + \frac{R}{n}C_{T2}$

$= m_{t_1}^1 + \frac{R}{n}(C + C_{Tl}) + \frac{R}{n}(\frac{n-1}{n}C + C_{Tl})$

$= m_{t_1}^1 + \frac{R}{n}((1 + \frac{n-1}{n})C + 2C_{Tl})$

......

$$m_k^1 = m_{k-1}^1 + \frac{R}{n}C_{Tk}$$

$$= m_{t_1}^1 + \frac{R}{n}((1 + \frac{n-1}{n} + \ldots + \frac{n-k+1}{n})C + kC_{Tl})$$

$$= m_{t_1}^1 + \frac{R}{n}((k - k(k-1)/2n)C + kC_{Tl})$$

......

$$m_n^1 = m_{t_1}^1 + \frac{R}{n}((n - n(n-1)/2n)C + nC_{Tl})$$

Therefore, $\frac{1}{n}\sum_{k=1}^{n} m_k^1$

$$= m_{t_1}^1 + \frac{R}{n^2}\sum_{k=1}^{n} kC_{Tl} + \frac{RC}{n^2}(\sum_{k=1}^{n} k - \frac{1}{2n}(\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k))$$

$$= m_{t_1}^1 + \frac{n+1}{2n}RC_{Tl} + \frac{RC}{n^2}(\frac{n(n+1)}{2} - \frac{1}{2n}(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}))$$

Let $n \to \infty$,

$$U^1 = m_{t_1}^1 + \frac{1}{2}RC_{Tl} + \frac{1}{3}RC$$

Now we consider function W. According to the definition,

$$W^0 = \lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{2}$$

and

$$W^1 = \lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} km_k^1$$

where

$$km_k^1 = k(m_{t_1}^1 + \frac{R}{n}((k - k(k-1)/2n)C + kC_{Tl}))$$

Therefore, we have

$$W^1 = \frac{1}{2}m_{t_1}^1 + \frac{1}{3}RC_{Tl} + \frac{5}{24}RC$$

C.2 Dissymetric model

We derive a second order moment matching tree model of a transmission line as follows. According to what stated in Sec.5.1.2, it is known that the number of constraints is 6 and the number of sections in the model is $r = 3$. We set $\beta_0 = 0$ and let α_i and β_i for i from 1 to 3 be unknown parameters.

From Condition 1, we have

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \tag{63}$$

For j=0, from Condition 3, we have

$$\beta_1 + \beta_2 + \beta_3 = 1 \tag{64}$$

and from Condition 4,

$$\alpha_1\beta_1 + \alpha_{12}\beta_2 + \alpha_{13}\beta_3 = \frac{1}{2} \tag{65}$$

Now we consider Conditions 3 and 4 for j=1. Let C_{Tl} be expressed by by $\beta_l C$ and let $\beta_{k3} = \sum_{i=k}^{3} \beta_i$, then the first moments of the node voltages $m^1_{s_i}$ for $i = 1, 2, 3$ can be expressed as follows:

$$m^1_{s_1} = m^1_{t_1} + R_{s_1}(C_{s_1} + C_{s_2} + C_{s_3} + C_{Tl}) = m^1_{t_1} + \alpha_1\beta_l RC + \alpha_1\beta_{13}RC$$

$$m^1_{s_2} = m^1_{s_1} + R_{s_2}(C_{s_2} + C_{s_3} + C_{Tl})$$

$$= m^1_{t_1} + \alpha_{12}\beta_l RC + (\alpha_1\beta_{13} + \alpha_2\beta_{23})RC$$

and

$$m^1_{s_3} = m_{t_1} + \alpha_{13}\beta_l RC + (\alpha_1\beta_{13} + \alpha_2\beta_{23} + \alpha_3\beta_3)RC.$$

Therefore, from Condition 3,

$\beta_1 m^1_{s_1} + \beta_2 m^1_{s_2} + \beta_3 m^1_{s_3}$

$$= m_{t_1}^1 + (\alpha_1\beta_1 + \alpha_{12}\beta_2 + \alpha_{13}\beta_3)RC\beta_l$$

$$+(\alpha_1\beta_1\beta_{13} + \beta_2(\alpha_1\beta_{13} + \alpha_2\beta_{23}) + \beta_3(\alpha_1\beta_{13} + \alpha_2\beta_{23} + \alpha_3\beta_3))RC$$

.

Compared the above expression with Eq.(37) with the reference to Eq.(65), it can be seen that the coefficient of $m_{t_1}^1$ is 1 and the coefficient of $C_{Tl} = \beta_l C$ is $\frac{1}{2}R$. Therefore, we have one more constraint:

$$\alpha_1\beta_1\beta_{13} + \beta_2(\alpha_1\beta_{13} + \alpha_2\beta_{23}) + \beta_3(\alpha_1\beta_{13} + \alpha_2\beta_{23} + \alpha_3\beta_3) = \frac{1}{3} \tag{66}$$

which is equivalent to

$$\alpha_1\beta_{13}^2 + \alpha_2\beta_{23}^2 + \alpha_3\beta_3^2 = \frac{1}{3}$$

Now we consider Condition 4 for $j = 1$. We have

$\alpha_1\beta_1 m_{s_1}^1 + \alpha_{12}\beta_2 m_{s_2}^1 + \alpha_{13}\beta_3 m_{s_3}^1$

$$= (\alpha_1\beta_1 + \alpha_{12}\beta_2 + \alpha_{13}\beta_3)m_{t_1}^1$$

$$+(\alpha_1^2\beta_1 + \alpha_{12}^2\beta_2 + \alpha_{13}^2\beta_3)RC\beta_l + (\alpha_1^2\beta_1\beta_{13} + \alpha_{12}\beta_2$$

$$(\alpha_1\beta_{13} + \alpha_2\beta_{23}) + \alpha_{13}\beta_3(\alpha_1\beta_{13} + \alpha_2\beta_{23} + \alpha_3\beta_3))RC.$$

Compared with Eq.(39) and taking Eqs.(63), (64) and (65) into consideration, we have two more constraints:

$$\alpha_1^2\beta_1 + \alpha_{12}^2\beta_2 + \beta_3 = \frac{1}{3} \tag{67}$$

and

$$\alpha_1^2\beta_1 + \alpha_{12}\beta_2(\alpha_1 + \alpha_2\beta_{23}) + \beta_3/2 = \frac{5}{24} \tag{68}$$

Now we have a set of nonlinear equations. We use the function "fsolve" in MATLAB[22] to solve these equations and get the parameters $\alpha_1 = 0.20718$,

$\alpha_2 = 0.61908$, $\alpha_3 = 0.17375$, $\beta_1 = 0.54051$, $\beta_2 = 0.45919$ and $\beta_3 = 3.9019e-09 \approx 0$.

Remarks.

(a) From the above derivations, it can be seen that the coefficients of $\beta_l RC$ in U^1 is the same as the coefficient of $m_{t_1}^1$ in W^1 and the equations set by matching the corresponding coefficients of the model with these two coefficients are the same, as they can be both expressed by $\sum_{k=1}^{r} \alpha_{1k}\beta_k$. In fact, If we denote $\beta_l^j = \sum_{k\in D(t_2)} C_k m_k^j / C$, then the coefficient of $\beta_l^j RC$ in U^j and the coefficient of $m_{t_1}^j$ in W^j are both $\sum_{k=1}^{r} \alpha_{1k}\beta_k$.

(b) In this example, for the model circuit, U^1 and W^1 can be expressed as $U^1 = a_1 m_{t_1}^1 + b_1 RC\beta_l + c_1 RC$ and $W^1 = A_1 m_{t_1}^1 + B_1 RC\beta_l + C_1 RC$, where a_1, b_1, c_1, A_1, B_1 and C_1 are functions of the α's and β's. Compared with Eqs. (37) and (39), if we let the coefficients of the corresponding terms be equal, we have six equations with the one related to A_1 the same as that related to b_1, and it can be shown that these equations are just the same as those from Eq.(64) to Eq.(68). Therefore, another way to set up the constraints for a p-th order moment matching model is to get the expressions of U^{p-1} and W^{p-1} in terms of the α's and β's and the expressions in terms of the parameters of the line and let the coefficients of the correspoding terms be equal. The equations formed in this way in addition to the constraint $\sum_{i=1}^{r} \alpha_1 = 1$ form the whole set of constraints.

C.3 Open-ended model

When a transmission line is open-loaded, then Eq.(67) is missing and we have only 5 constraints. We may use a simplified model with only 2 sections as shown in Fig.5b. The parameters are determined by the following 5 equations:

$$\alpha_1 + \alpha_2 = 1$$

$$\beta_0 + \beta_1 + \beta_2 = 1$$

$$\alpha_1\beta_1 + \beta_2 = \frac{1}{2}$$

$$\alpha_1\beta_1 + \frac{1}{2}\beta_2 = \frac{1}{3}$$

and

$$\alpha_1^2\beta_1 + \frac{1}{2}\beta_2 = \frac{5}{24}$$

Solving these equations, we have $\alpha_1 = 1/4$, $\alpha_2 = 3/4$, $\beta_0 = 0$, $\beta_1 = 2/3$ and $\beta_2 = 1/3$.

C.4 Symmetric model

This is the model used in R-T-C meshes. We have seen that there are 6 constrints for a second order model, but we have also found that when the model is symmetric, when Condition 1 and Condition 2 for $j = 0$ are satisfied, Condition 3 for $j = 0$ is automatically satisfied. Therefore, we may used the model shown in Fig.5c with 5 unknown parameters. Let $\alpha_{12} = \alpha_1 + \alpha_2$, $\alpha_{13} = \alpha_{12} + \alpha_2$, $\alpha_{14} = \alpha_{13} + \alpha_1$, $\beta_{34} = \beta_1 + \beta_0$, $\beta_{24} = \beta_2 + \beta_{34}$ and $\beta_{14} = \beta_1 + \beta_{24}$, then the equations for the α's and β's are as follows:

$$\alpha_1 + \alpha_2 = 0.5$$

$$2(\beta_0 + \beta_1) + \beta_2 = 1$$

$$\alpha_1(\beta_{14}^2 + \beta_0^2) + \alpha_2(\beta_{24}^2 + \beta_{34}^2) = \frac{1}{3}$$

$$\alpha_1^2\beta_1 + \alpha_{12}^2\beta_2 + \alpha_{13}^2\beta_1 + \alpha_{14}^2\beta_0 = \frac{1}{3}$$

$$\alpha_1^2\beta_1\beta_{14} + \alpha_{12}\beta_2(\alpha_1\beta_{14} + \alpha_2\beta_{24}) + \alpha_{13}\beta_1(\alpha_1\beta_{14} + \alpha_2\beta_{24} + \alpha_2\beta_{34}) + \beta_0/2 = \frac{5}{24}$$

Solve these equations, we have $\alpha_1 = 0.24283$, $\alpha_2 = 0.25623$, $\beta_0 = 0.10275$, $\beta_1 = 0.26222$ and $\beta_2 = 0.26862$.

D Third order moment matching model of transmission line

We derive the third order moment matching model of a transmission line in this section.

D.1 Computation of U^2 and W^2

D.1.1 Computation of U^2

Let $C^j_{Tl} = \sum_{k\in \hat{D}(t_2)} C_k m^j_k$. From the definition,

$$U^2 = lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} m^2_k$$

where

$$m^2_k = m^2_{t_1} + \frac{R}{n}kC^1_{Tl} - \frac{L}{n}kC_{Tl} - \frac{L}{n}C[k(1+\frac{1}{2n}) - \frac{k^2}{2n}] + \frac{RC}{n^2}(m^1_1 + 2m^1_2 + \ldots + k(m^1_k + \ldots + m^1_n))$$

.

Let $U^2 = a_1 m^2_{t_1} + a_2 RC^1_{Tl} - a_3 LC_{Tl} - a_4 LC + a_5 RC$. Then

$$a_1 = lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} 1 = 1$$

$$a_2 = a_3 = lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n} = \frac{1}{2}$$

$$\begin{aligned} a_4 &= lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{n^2}[k(1+\frac{1}{2n}) - \frac{k^2}{2n}] \\ &= lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{n^2}[k - \frac{k^2}{2n}] \\ &= \frac{1}{2} - \frac{1}{2}\times\frac{1}{3} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} a_5 &= lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{n^2}(m^1_1 + 2m^1_2 + \ldots + k(m^1_k + \ldots + m^1_n)) \\ &\equiv a_{51} - a_{52} \end{aligned}$$

where

$$a_{51} = lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{n^2}(m_1^1 + 2m_2^1 + \ldots + nm_n^1) =$$

$$= W^1 = \frac{1}{2}m_{t_1}^1 + \frac{1}{3}RC_{Tl} + \frac{5}{24}RC$$

$$a_{52} = lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}(m_{k+1}^1 + 2m_{k+2}^1 + \ldots + (n-k)m_n^1)$$

Let the expression in the sum of above equation be b_k, i.e.,

$$b_k = m_{k+1}^1 + 2m_{k+2}^1 + \ldots + (n-k)m_n^1 = \sum_{j=k+1}^{n}(j-k)m_j^1.$$

Note that

$$m_j^1 = m_{t_1}^1 + j\frac{R}{n}C_{Tl} + \frac{RC}{n}[j(1+\frac{1}{2n}) - \frac{j^2}{2n}]$$

Let b_k be expressed by $c_{k1}m_{t_1}^1 + c_{k2}RC_{Tl} + c_{k3}RC$, then

$$c_{k1} = \sum_{j=k+1}^{n}(j-k) \approx \frac{1}{2}(n^2-k^2) - k(n-k) = \frac{1}{2}(n-k)^2,$$

$$c_{k2} = \sum_{j=k+1}^{n}\frac{(j-k)j}{n} \approx \frac{1}{n}[\frac{1}{3}(n^3-k^3) - \frac{k}{2}(n^2-k^2)]$$

and

$$c_{k3} = \sum_{j=k+1}^{n}\frac{1}{n}(j-k)[j(1+\frac{1}{2n}) - \frac{j^2}{2n}] \approx$$

$$\approx \sum_{j=k+1}^{n}\frac{1}{n}(j-k)[j-\frac{j^2}{2n}] \approx \frac{1}{n}[\frac{1}{3}(n^3-k^3) - \frac{1}{8n}(n^4-k^4) - \frac{k}{2}(n^2-k^2) + \frac{k}{6n}(n^3-k^3)].$$

Let a_{52} be expressed by $d_1m_{t_1}^1 + d_2RC_{Tl} + d_3RC$, then

$$d_1 = \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}c_{k1} = \frac{1}{2}\times\frac{1}{3} = \frac{1}{6},$$

$$d_2 = \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}c_{k2} = \frac{1}{3}(1-\frac{1}{4}) - \frac{1}{2}(\frac{1}{2}-\frac{1}{4}) = \frac{1}{8}$$

and

$$d_3 = \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} c_{k3} = \frac{1}{3}(1-\frac{1}{4}) - \frac{1}{8}(1-\frac{1}{5}) - \frac{1}{2}(\frac{1}{2}-\frac{1}{4}) + \frac{1}{6}(\frac{1}{2}-\frac{1}{5}) = \frac{3}{40}.$$

Therefore,

$$a_5 = a_{51} - a_{52} = \frac{1}{3}m^1_{t_1} + \frac{5}{24}RC_{Tl} + \frac{2}{15}RC,$$

and

$$U^2 = m^2_{t_1} + \frac{1}{2}RC^1_{Tl} + \frac{1}{3}RCm^1_{t_1} + \frac{5}{24}R^2CC_{Tl} + \frac{2}{15}(RC)^2 - \frac{1}{3}LC - \frac{1}{2}LC_{Tl}$$

D.1.2 Computation of W^2

Let $W^2 = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} km^2_k$ be expressed by

$$W^2 = e_1 m^2_{t_1} + e_2 RC^1_{Tl} - e_3 LC_{Tl} - e_4 LC + e_5 RC,$$

then

$$e_1 = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{2}$$

$$e_2 = e_3 = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{k^2}{n} = \frac{1}{3}$$

$$e_4 = \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} [k^2(1+\frac{1}{2n}) - \frac{k^3}{2n}] = \frac{1}{3} - \frac{1}{8} = \frac{5}{24}$$

and

$$e_5 = \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k(m^1_1 + 2m^1_2 + \ldots + k(m^1_k + \ldots + m^1_n)) \equiv e_{51} - e_{52}$$

where

$$e_{51} = \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k(m^1_1 + 2m^1_2 + \ldots + nm^1_n) = (\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} k)W^1 =$$

$$= \frac{1}{2}W^1 = \frac{1}{4}m^1_{t_1} + \frac{1}{6}RC_{Tl} + \frac{5}{48}RC.$$

$$e_{52} = \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k(m_{k+1}^1 + 2m_{k+2}^1 + (n-k)m_n^1) =$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k(c_{k1}m_{t_1}^1 + c_{k2}RC_{Tl} + c_{k3}RC) \equiv f_1 m_{t_1}^1 + f_2 RC_{Tl} + f_3 RC,$$

then

$$f_1 = \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} kc_{k1} = \frac{1}{2}(\frac{1}{2} - \frac{1}{4}) - (\frac{1}{3} - \frac{1}{4}) = \frac{1}{24},$$

$$f_2 = \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} kc_{k2} = \frac{1}{3}(\frac{1}{2} - \frac{1}{5}) - \frac{1}{2}(\frac{1}{3} - \frac{1}{5}) = \frac{1}{30},$$

and

$$f_3 = \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} kc_{k3} = \frac{1}{3}(\frac{1}{2} - \frac{1}{5}) - \frac{1}{8}(\frac{1}{2} - \frac{1}{6}) - \frac{1}{2}(\frac{1}{3} - \frac{1}{5}) + \frac{1}{6}(\frac{1}{3} - \frac{1}{6}) = \frac{7}{360}.$$

Then,

$$e_5 = (\frac{5}{24}m_{t_1}^1 + \frac{2}{15}RC_{Tk} + \frac{61}{720}RC)RC,$$

and

$$W^2 = \frac{1}{2}m_{t_1}^2 + \frac{1}{3}RC_{Tl}^1 + \frac{5}{24}m_{t_1}^1 RC + \frac{2}{15}R^2CC_{Tl} + \frac{61}{720}(RC)^2 - \frac{5}{24}LC - \frac{1}{3}LC_{Tl}.$$

Remarks.

(a) It can be seen from the expressions of U^2 and W^2, the coefficients of RC_{Tl}^1 and R^2CC_{Tl} in U^2 are the same as the coefficients of $m_{t_1}^2$ and $m_{t_1}^1 RC$ in W^2. The coefficient of RC_{Tl}^1 in U^2 and $m_{t_1}^2$ in W^2 are $a_2 = e_1 = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{2}$. The coefficient of R^2CC_{Tl} in U^2 can be expressed by $lim_{n\to\infty}\frac{1}{n^4}I_1$ and the coefficient of $m_{t_1}^1 RC$ in W^2 can be expressed by $lim_{n\to\infty}\frac{1}{n^4}I_2$, where

$$I_1 = \sum_{k=1}^{n}(\sum_{j=1}^{k} j^2 + \sum_{j=k+1}^{n} kj) \equiv \sum_{k=1}^{n}\sum_{j=1}^{n} p_{kj}$$

where

$$p_{kj} = \begin{cases} j^2 & j \le k \\ jk & j > k \end{cases}$$

and

$$I_2 = \sum_{k=1}^{n} k(\sum_{j=1}^{k} j + k(n-k)).$$

I_1 can be rewritten into the following form:

$$I_1 = \sum_{j=1}^{n}\sum_{k=1}^{n} p_{kj} = \sum_{j=1}^{n}(\sum_{k=1}^{j} jk + \sum_{k=j+1}^{n} j^2) = \sum_{j=1}^{n} j(\sum_{k=1}^{j} k + j(n-j))$$

Exchanging the symbols j and k, it can be seen that $I_1 = I_2$. Also, it can be seen in the next section that the constraints set by each corresponding pair of coefficients are the same. This observation is useful to prove that the total constraints for a p-th order moment matching model is 3p.

(b) It can be seen that the coefficients of the term $-LC_{Tl}$ and $-LC$ in U^2 are correspondingly the same as those of RC^1_{Tl} and $RCm^1_{t_1}$ and the constraints set by these coefficients are correspodingly the same. Similar situations happen in W^2, where the coefficients of $-LC_{Tl}$ and RC^1_{Tl} and coefficients of $-LC$ and $m^1_{t_1}RC$ are correspondingly the same. Therefore, there is no need to consider the constraints set by the terms with L. This coincides with what mentioned in Sec.5.1.2.

D.2 Dissymmetric model

The dissymmetric 3-rd order moment matching model is used for R-T-C trees and is shown in Fig.7a. There are 9 constraints to determine the α's and β's. The number of sections in the model is now $r = 4$. We have the following equations:

$$\alpha_{1r} = 1$$

$$\beta_{0r} = 1$$

$$\sum_{k=1}^{r} \alpha_k \beta_{kr} = \frac{1}{2}$$

$$\sum_{k=1}^{r} \alpha_k \beta_{kr}^2 = \frac{1}{3}$$

$$\sum_{k=1}^{r} \alpha_{1k}^2 \beta_k = \frac{1}{3}$$

Let $p_r = \alpha_{1r}\beta_r$ and $p_k = p_{k+1} + \alpha_{1k}\beta_k$ $(k = r-1, r-2, \ldots, 1)$, we have

$$\sum_{k=1}^{r} \alpha_k p_k \beta_{kr} = \frac{5}{24}$$

Let $q_1 = \alpha_1\beta_{1r}$ and $q_k = q_{k-1} + \alpha_k\beta_{kr}$ $(k = 2, \ldots, r)$, we have

$$\sum_{k=1}^{r} \beta_k q_k^2 = \frac{2}{15}$$

Let $a_r = \sum_{k=1}^{r} \alpha_{1k}^2\beta_k$ and $a_k = a_{k+1} - (\alpha_{1,k+1} - \alpha_{1k})\beta_{k+1}$ $(k = r-1, \ldots, 1)$, then we have the last two constraints

$$\sum_{k=1}^{r} \alpha_k\beta_k a_k = \frac{2}{15}$$

and

$$\sum_{k=1}^{r} \beta_k a_k q_k = \frac{61}{720}.$$

Solving these equations, we have $\alpha_1 = 0.13412$, $\alpha_2 = 0.35234$, $\alpha_3 = 0.31730$, $\alpha_4 = 0.19624$, $\beta_0 = 7.9144e-04$, $\beta_1 = 0.33288$, $\beta_2 = 0.29853$, $\beta_3 = 0.34972$ and $\beta_4 = 0.018081$.

D.3 Open-ended model

When the transmission line is open ended, the model is simplified as shown in Fig.7b with 3 sections. In the above equations, the 5-th and the 8-th are missing and now $r = 3$. Solving the equations, we have $\alpha_1 = 0.33197$, $\alpha_2 = 0.11001$, $\alpha_3 = 0.55823$, $\beta_0 = 0.12293$, $\beta_1 = 0.24794$, $\beta_2 = 0.37937$ and $\beta_3 = 0.24973$.

D.4 Symmetric model

This is the model used in R-T-C meshes. As in the case of the 2nd order model, the 3rd constraint is automatically satisfied by the 1st and 2nd constraints and the symmetric structure so that we have 8 constraints. The model is shown in Fig.7c with the number of sections $r = 7$. In order to use the equations given in Sec.D.2, for $k > 4$, let $\alpha_k = \alpha_{8-k}$ and $\beta_k = \beta_{7-k}$. Then we have $\alpha_1 = 0.15071$, $\alpha_2 = 0.11338$, $\alpha_3 = 0.15370$, $\alpha_4 = 0.15157$, $\beta_0 = 0.069663$, $\beta_1 = 0.13903$, $\beta_2 = 0.13980$ and $\beta_3 = 0.15802$.

E Number of constraints to p-th order moment matching model

In this section, we prove that the number of constraints to a p-th order moment matching model is 3p as stated in Sec.5.1.2.

This conclusion is based on the following observations. Please refer to the remarks given in Appedices C and D to help understanding the reasoning.

(a) As has been illustrated in Remark (b) of Appendix C, the constraints set by Condition 3(See Eq.(46) in Sec.5.1.2) for $j = 0, 1, \ldots, p-1$ is equivalent to the constraints set by matching the coefficients of the terms $m_{i_1}^{j}(RC)^{p-1-j}$ for $j = p-1, p-2, \ldots, 1$, the coefficients of the terms $RC_{Tl}^{j}(RC)^{p-2-j}$ for $j = p-2, p-1, \ldots, 0$ and the coefficient of the term $(RC)^{p-1}$. The number of these constraints is $2p-1$ in total.

(b) Similarly, the number of constraints set by Condition 4 is $2p-1$, too. As has been illustrated in Appendix D, the $4p-2$ constraints set by Conditions 3 and 4 are not totally independent. We have seen that in the 3-nd order model, the coefficients of RC_{Tl}^{1} and R^2CC_{Tl} in U^2 are correspondingly the same as those of $m_{i_1}^{2}$ and $m_{i_1}^{1}RC$ in W^2. In the general case, it can be shown that for a p-th order model, the coefficient of the term $RC_{Tl}^{i}(RC)^{p-2-i}$ in U^{p-1} is the same as the coefficient of the term $m_{i_1}^{i+1}(RC)^{p-2-i}$ in W^{p-1} for $i = 0, 1, \ldots, p-2$. Therefore, there are $p-1$ constraints set by U^{p-1} and W^{p-1} in common, and the number of independent constraints set by both Conditions 3 and 4 are reduced to $2(2p-1) - (p-1) = 3p-1$. In addition to Condition 1, the number of constraints is 3p. □

F Moment computation of R-L-C tree with leakage resistors

In this section, we consider the moment computaion of an R-L-C tree with leakage resistors. Such a network consists of a floating RL tree N, grounded capacitors and grounded resistors (leakage resistors) connected between the nodes of the RL tree and the ground. Let $Z_{ik} = R_{ik} + sL_{ik}$ be total impedance of the path P_{ik} as defined before. By the Substitution Theorem, each capacitor C_k is replaced by a current source $sC_kV_k(s)$. By the superposition theorem, each node voltage $V_i(s)$ is composed of two components: $V_{i,r}(s)$ caused by the source voltage $V_r(s)$ when all the capacitors are disconnected and $V_{i,c}(s)$ caused by the capacitance currents; i.e.,

$$V_i(s) = V_{i,r}(s) - \sum_k Z_{ik}(s)sC_kV_k(s) \tag{69}$$

Let $V_{ir}(s)/V_r(s)$ be denoted by $T_i(s)$ and $V_k(s)/V_r(s)$ by $H_k(s)$, then we have

$$H_i(s) = T_i(s) - \sum_k Z_{ik}(s)sC_kH_k(s) \tag{70}$$

Note that $T_i(s)$ is different from $H_i(s)$ in that $T_i(s)$ is the transfer function $V_{ir}(s)/V_r(s)$ when all the capacitances are disconnected to the floating RL network and only the leakage resistors are connected. If there are no leakage resistors, then $T_i(s) = 1$ and Eq.(70) becomes Eq.(13).

Let the $j-th$ moment of $T_i(s)$ and $H_i(s)$ be expressed by $m^j_{i,t}$ and m^j_i respectively. From Eq.(70), we have

$$m^p_i = m^p_{i,t} + \sum_k (R_{ik}C_km^{p-1}_k - L_{ik}C_km^{p-2}_k) \tag{71}$$

Note that when $p = 1$, the term $L_{ik}C_km^{p-2}_k$ is missing, and

$$m^0_k = H_k(0) = \lim_{t\to\infty} v_k(t) = v_k(\infty) \tag{72}$$

In the case there exist leakage resistors, $v_k(\infty) < 1$, and the Elmore delay of node voltage v_i can be expressed as

$$T_{Di} = m^1_{i,t} + \sum_k R_{ik}C_kv_k(\infty) \tag{73}$$

In order to use Eq.(71) to compute moments, we need to compute the moments of function $T_i(s)$, which can be done as follows.

Let $\hat{N}$ be the network consisting of the floating RL network N, the voltage source and the leakage resistors, NR be the node set where the leakage resistors are connected, and g_k be the conductance of the leakage resistor connected to node k, then we have

$$V_{ir}(s) = V_r(s) - \sum_{k \in NR} Z_{ik} g_k V_{kr}(s) \tag{74}$$

$$T_i(s) = 1 - \sum_{k \in NR} Z_{ik} g_k T_k(s) \tag{75}$$

$$m_{i,t}^0 = 1 - \sum_{k \in NR} R_{ik} g_k m_{k,t}^0 \tag{76}$$

and

$$m_{i,t}^j = - \sum_{k \in NR} R_{ik} g_k m_{k,t}^j + \sum_{k \in NR} L_{ik} g_k m_{k,t}^{j-1} \tag{77}$$

Let $NR = \{1, 2, \ldots, nr\}$ for simplicity. Let matrix $A = [a_{ik}]_{nr \times nr}$ be such that $a_{ik} = R_{ik} g_k$ for $i \neq k$ and $a_{ii} = R_{ii} g_i + 1$, vector $B^j = [b_i^j]_{nr}$ be such that when $j = 0$, $b_i^0 = 1 \ \forall i$ and when $j > 0$, $b_i^j = \sum_{k \in NR} L_{ik} g_k m_{k,t}^{j-1}$ and let $M_t^j = [m_{1,t}^j, m_{2,t}^j, \ldots, m_{nr,t}^j]^t$, From Eqs.(76) and (77), we have the following set of equations to solve $m_{i,t}^j$ from $j = 0$ to $j = p$ recursively.

$$AM_t^j = B^j \tag{78}$$

From Eqs.(71) it can be shown that the moment matching models derived so far are valid for the use of R-T-C network with leakage resistors. The reasoning is as follows. As the total resistance and inductance of each model reamin the same as those of the original transmission line, all the elements of the open circuit impedance matrix $Z = [z_{ik}]$ and all the functions $T_i(s)$ remain unchanged. For each moment m_i^p, the capacitance of a transmission line has no contribution to its first part $m_{i,t}^p$ and the contribution to its second part is the same as its model as proved in Sec.5.

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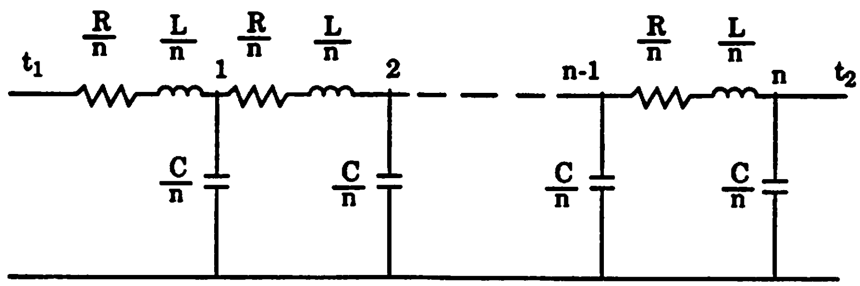


Fig.1 n section model of transmission line

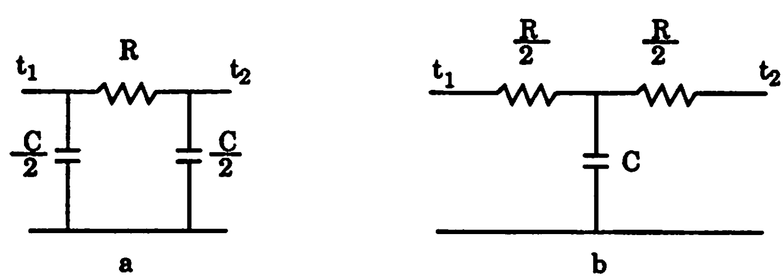


Fig.2 Elmore delay model

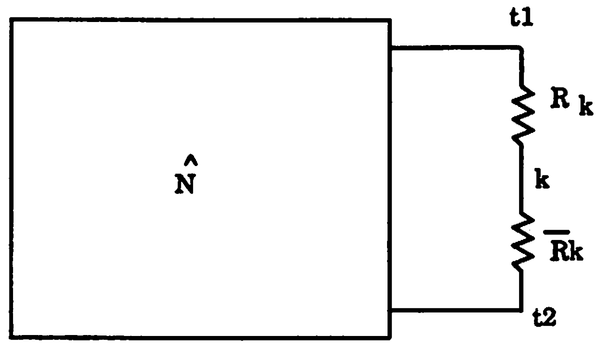


Fig.3 Derivation of Lemma 1

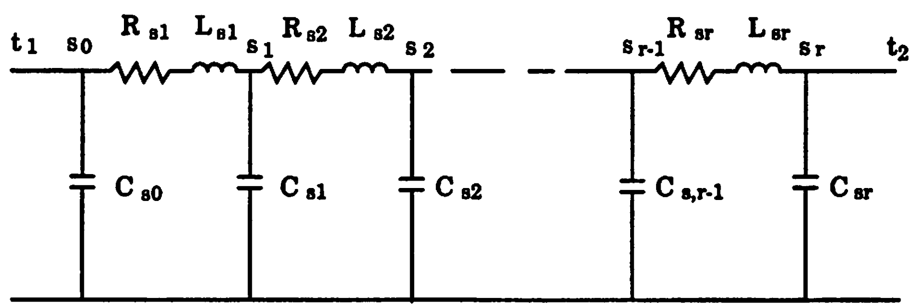
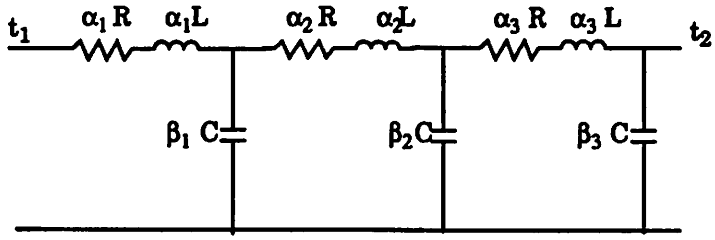
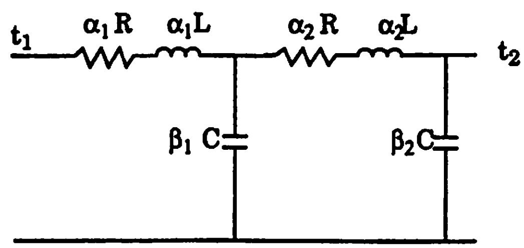


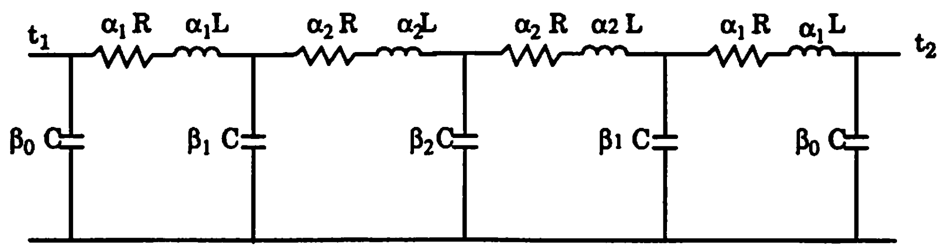
Fig.4 p-th moment matching model



a



b



c

Fig.5 2nd order moment matching model

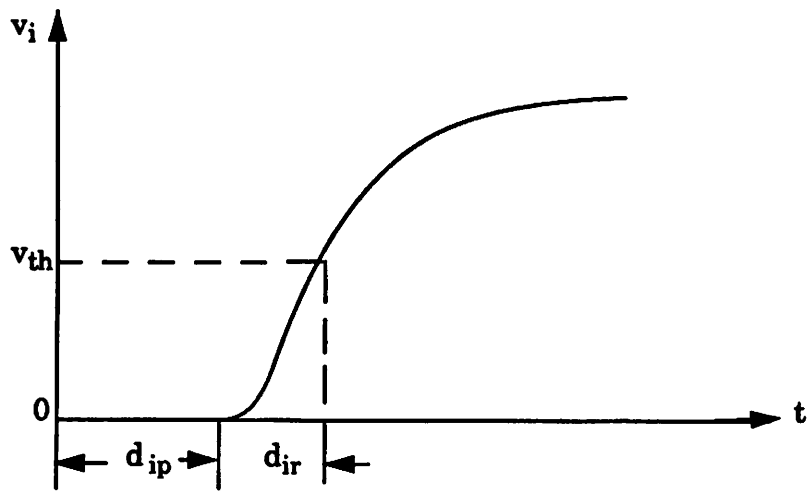


Fig.6 Propagation delay and rising delay

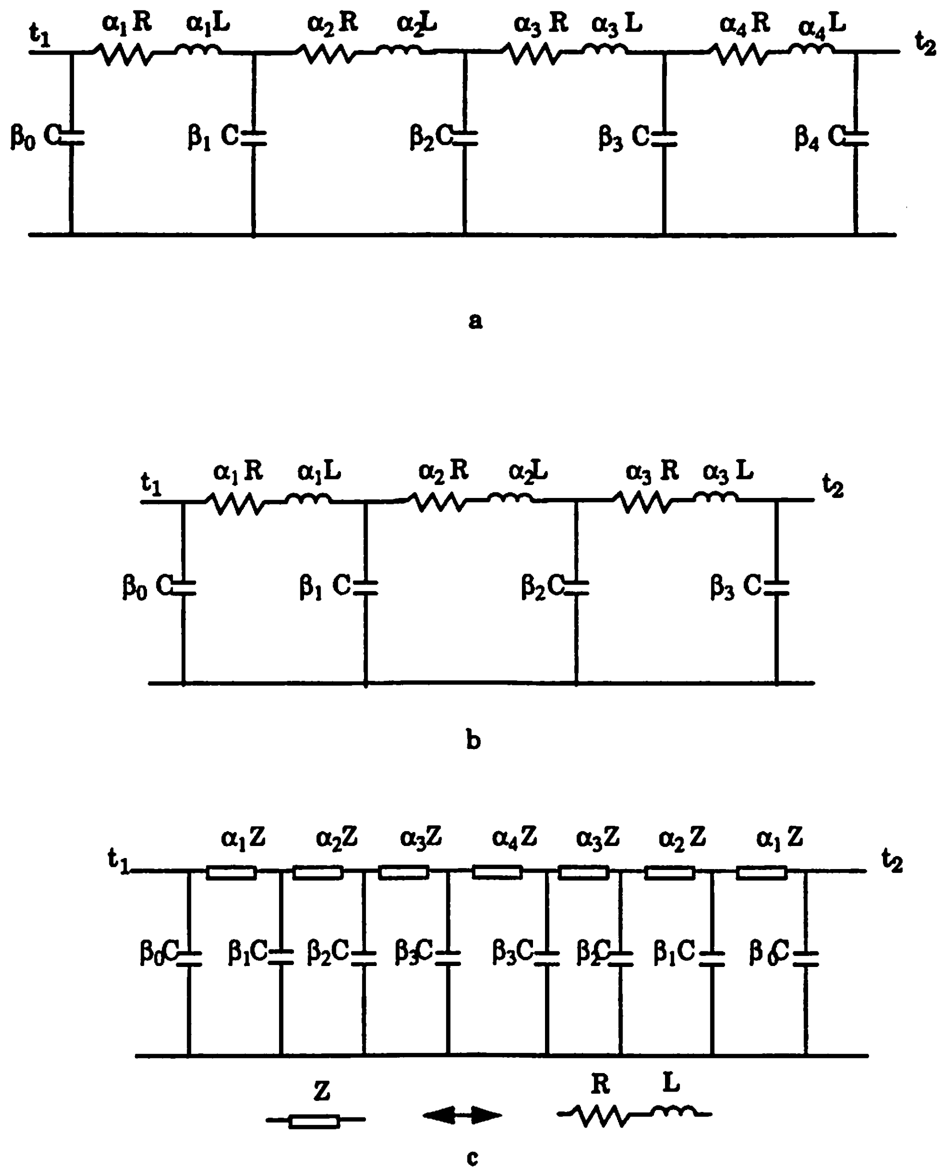
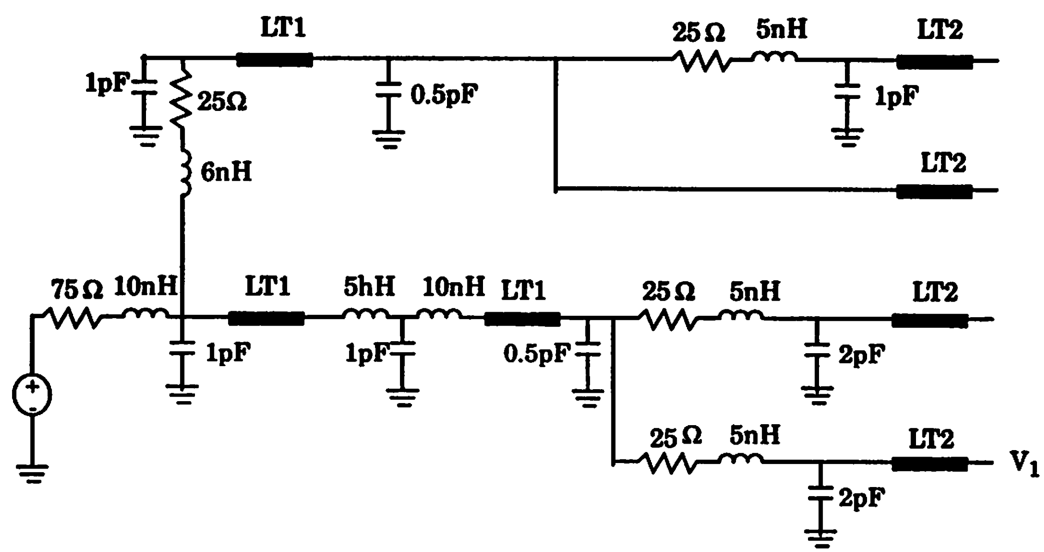
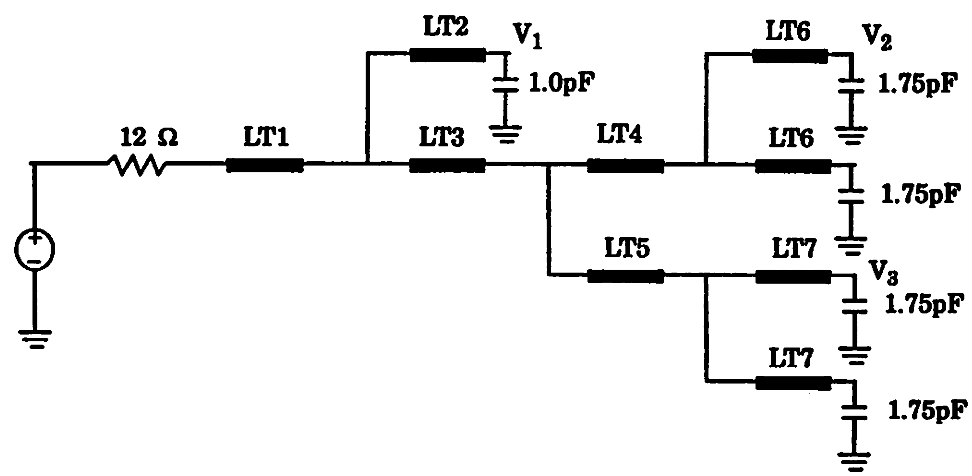


Fig.7 3rd order moment matching model



LT1: C=120pF L=40nH R=5Ω
LT2: C=150pF L=30nH R=5Ω

Fig.8 Circuit of Example 5



LT1: R=0.682Ω/cm, L=0.858nH/cm, C=1.395pF/cm, LEN=30mm
LT2: R=11.538Ω/cm, L=3.970nH/cm, C=0.283pF/cm, LEN=40mm
LT3: R=0.630Ω/cm, L=0.805nH/cm, C=1.395pF/cm, LEN=10mm
LT4: R=1.786Ω/cm, L=1.725nH/cm, C=0.651pF/cm, LEN=25mm
LT5: R=1.579Ω/cm, L=1.592nH/cm, C=0.705pF/cm, LEN=17.5mm
LT6: R=9.375Ω/cm, L=3.741nH/cm, C=0.300pF/cm, LEN=40mm
LT7: R=7.500Ω/cm, L=3.480nH/cm, C=0.323pF/cm, LEN=12.5mm

Fig.9 Cicuit of Example 6

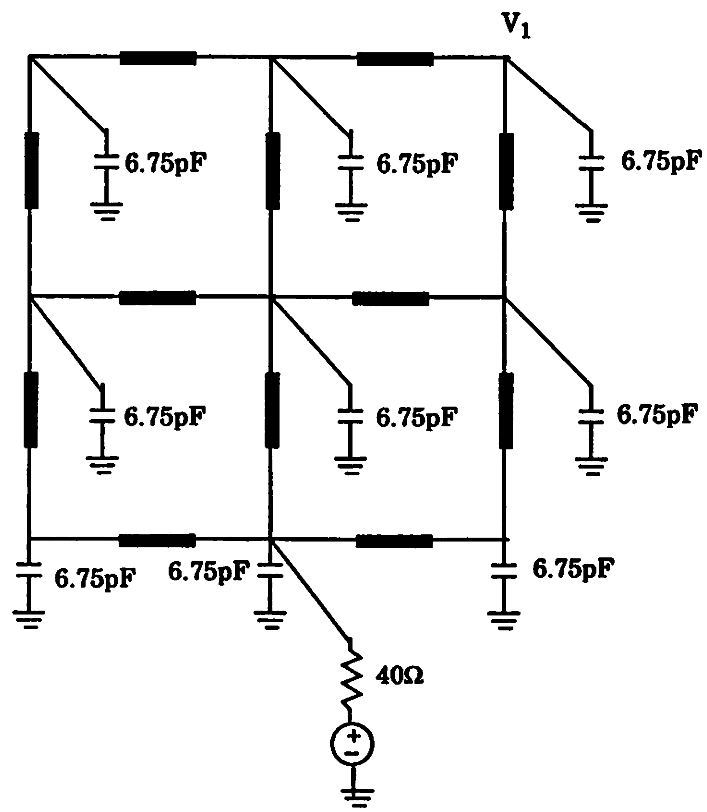


Fig.10 Circuit of Example 7

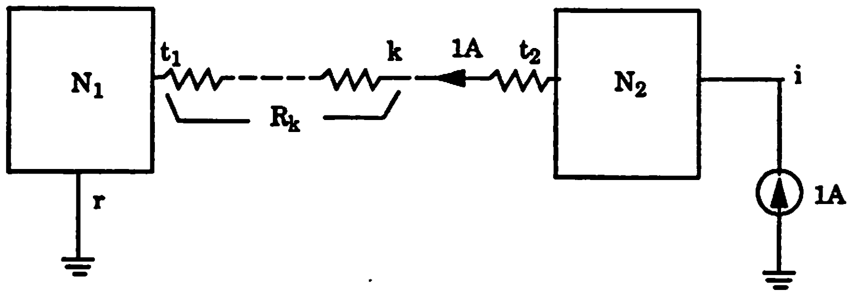


Fig.A1 Proof of Eq.(19)

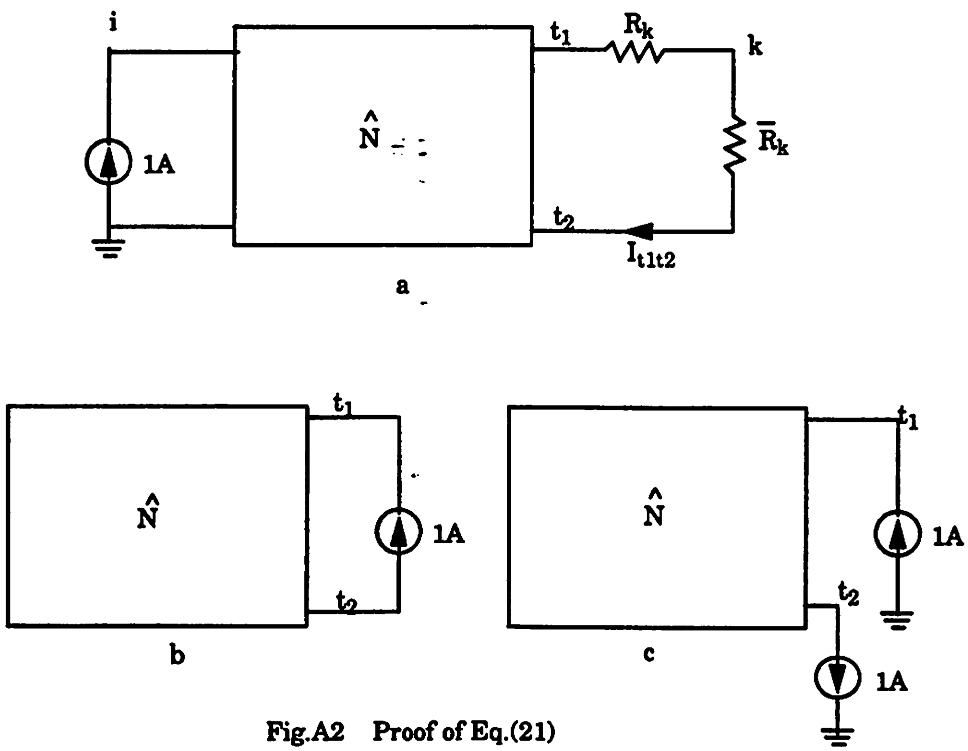


Fig.A2 Proof of Eq.(21)

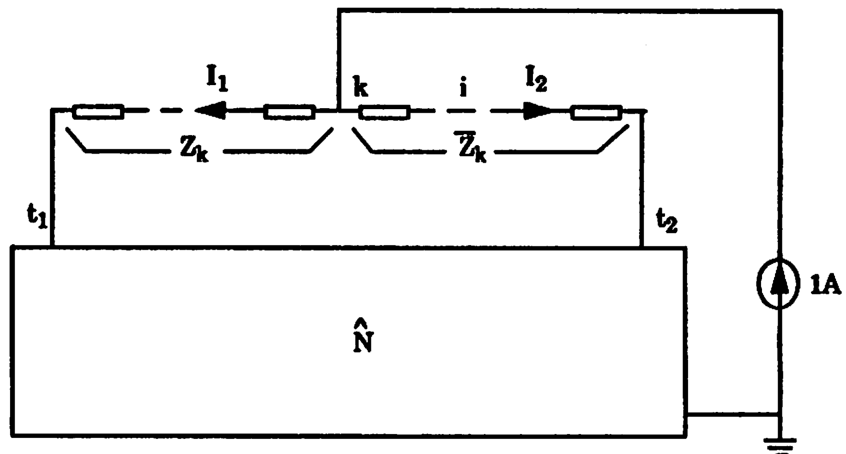


Fig.A3 Derivation of Eq.(50)