UNCERTAINTY STRUCTURES IN ADAPTIVE AND ROBUST STABILIZATION

by

Gevorg Nahapetian and Wei Ren

Memorandum No. UCB/ERL M94/23

11 April 1994

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Let us now consider a more general class of controllers: dynamic controllers which may even depend on s, i. e. a gain scheduling dynamic controller of the form

$$u = h(x, x_c),$$
$$\dot{x}_c = f(x, x_c, s). \tag{6}$$

with $x_c \in R^{n_c}$. Consider the closed loop system

$$\dot{z} = \tilde{A}(s) z + \tilde{B}(s) p(z, s), \tag{7}$$

where

$$z^T = \begin{bmatrix} x^T & x_c^T \end{bmatrix}, \tilde{A}(s) = \begin{bmatrix} A(s) & 0 \\ 0 & 0 \end{bmatrix}, \tilde{B}(s) = \begin{bmatrix} B(s) & 0 \\ 0 & I \end{bmatrix}, \text{ and } p(z, s) = \begin{bmatrix} h(x, x_c) \\ f(x, x_c, s) \end{bmatrix}.$$

Lemma 2.

Suppose we are given Σ_1, and there exist an integer $n_c \geq 0$, continuous functions f: $R^{n+n_c+n_s} \to R^{n_c}$ and h: $R^{n+n_c} \to R^m$, and a positive definite matrix $P \in R^{n+n_c \times n+n_c}$ such that

$$L(z, t) := z^T (\tilde{A}^T(s) P + P\tilde{A}(s)) z + 2z^T P\tilde{B}(s) p(z, s) \leq -\alpha \|x\|^2 \tag{8}$$

for some $\alpha > 0$, $\forall s \in S$ and $\forall (z, t) \in R^{n+n_c} \times R_+$, then $x(t) \to 0$.

Proof.

It is clear that $\|z\|$ is bounded. The fact that $x(t) \to 0$ follows from LaSalle's Invariance theorem [27].

Definition 2.

We say that Σ_1 is robustly quadratically stabilizable by dynamic controller (RQSDC), if Σ_1 satisfies condition (8) of Lemma 2.

Note that RQSDC as defined here is slightly more relaxed than quadratic stabilizability introduced in [19] in two ways: our compensator may depend on parameters, and the Lyapunov derivative $L(z, t)$ is bounded by the norm of the state of the plant rather than the full state, which also includes the state of the controller. However in next section we will show RQSDC is equivalent to RQSSC, hence is equivalent to the earlier definition in [19].

The following definition is natural and does not necessarily require the existence of parameter-independent Lyapunov function.

Definition 3.

Σ_1 is said to be robustly stabilizable by linear control (RSLC) if there exists a state feedback of the form $u = Kx$ such that $Re(\sigma(A(s) + B(s)K)) < 0 \quad \forall s \in S$. Where σ denotes spectrum, and Re denotes real part.

2.2 Adaptive stabilization

Before we proceed to adaptive stabilization let us first introduce the relevant notions of pointwise stabilizability, controllability invariance, and control Lyapunov function.

Definition 4.

Σ_1 is said to be pointwise stabilizable if there exist continuous mappings $P(\cdot): R^{n_s} \to R^{n \times n}$ and $u_n: R^{n+n_s} \to R^m$, such that $P(s)$ is positive definite $\forall s \in S$ and we have the following property

$$L(x, s, t) = x^T(A^T(s)P(s) + P(s)A(s))x + x^TP(s)B(s)u_n(x, s) \leq -\alpha\|x\|^2 \qquad (9)$$

for some $\alpha > 0$. If $P(s) = P = const$ we say that Σ_1 admits control Lyapunov function (CLF) $V = x^TPx$ [25, 29].

Note the difference between admitting CLF and RQSSC, in the latter case, the control law does not depend on the parameter s.

Definition 5 [20, 28].

Σ_1 is said to be controllability invariant if the pair $(A(s), B(s))$ is controllable for any fixed value $s \in S$.

Again, note the difference between RSLC and controllability invariance, where the latter implies the existence of *a parameter-dependent* control $u = K(s)x$ which stabilizes the system $\forall s \in S$.

We now consider adaptive stabilization, starting with adaptive quadratic stabilization. Then we describe certainty equivalence adaptive stabilization [12].

Definition 6.

Σ_1 is said to be adaptively quadratically stabilizable (AQS) if there exist C^1 functions $d\colon R^{n+n_s} \to R^{n_s}$ and $u_n\colon R^{n+s} \to R^m$, and a constant positive definite matrix $P \in R^{n \times n}$, such that the augmented Lyapunov function $W(x, \hat{s}) = x^T P x + (\hat{s} - s)^T (\hat{s} - s)$ under the adaptive control law $u = u_n(x, \hat{s})$ and parameter estimator $\dot{\hat{s}} = d(x, \hat{s})$ satisfies

$$\dot{W}(x, \hat{s}) = x^T (A^T(s) P + PA(s)) x + x^T PB(s) u_n(x, \hat{s}) + 2\dot{\hat{s}}^T (\hat{s} - s) \le -\alpha \|x\|^2. \quad (10)$$

From LaSalle's Theorem, it is clear that with the above adaptive control, we have $x(t) \to 0$ and $\|\hat{s}\|$ is bounded.

Finally, there is the well known certainty equivalence adaptive control. In [11, 12] it was shown that as long as model (1) is pointwise stabilizable, then it can be stabilized by certainty equivalence adaptive controller using suitable identifier.

3. RQSDC $\equiv$ RQSSC $\equiv$ RQS

In this section we establish the equivalence between robust quadratic stabilization by dynamic controller and robust quadratic stabilization by static control. While it is clear that RQSDC implies RQSSC, the fact that the reverse is true is somewhat surprising. The following theorem is an extension of Theorem 3.2 of [25].

Theorem 1.

The following statements are equivalent for the uncertain system Σ_1.

(i) Σ_1 is RQSSC;

(ii) Σ_1 is RQSDC.

The following lemma is needed for the proof of the theorem.

Lemma 3.

Consider the uncertain system Σ_1, and the closed loop system (7), and suppose condition (8) of Lemma 2 is satisfied with some positive definite matrix P. Then there exists a control such that condition (8) of Lemma 2 is satisfied with a block diagonal $\tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$, with $\tilde{P}_1 \in R^{n \times n}$ and $\tilde{P}_2 \in R^{n_c \times n_c}$.

Proof of Lemma 3.

Let $P = \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix}$ with $P_1 \in R^{n \times n}$, $P_2 \in R^{n_c \times n_c}$ and $P_3 \in R^{n \times n_c}$. Now consider a linear transformation $z = T\tilde{z}$ of (7) with $T = \begin{bmatrix} I_n & 0 \\ -P_2^{-1}P^T{}_3 & I_{n_c} \end{bmatrix}$, where I_n and I_{n_c} are $n \times n$ and $n_c \times n_c$ identity matrices respectively. hen it is obvious that the matrix $\tilde{P} = T^T PT$ is positive definite, and by choice of T it is easy to verify that $\tilde{P}$ is block diagonal. Finally, let us consider the transformed system with state $\tilde{z} = \begin{bmatrix} x^T & \tilde{x}^T_c \end{bmatrix}^T$, and let $M = P_2^{-1}P^T{}_3$, then the transformed system will be described by

$$\begin{aligned} \dot{x} &= A(s)x + B(s)h(x, -Mx + \tilde{x}_c), \\ \dot{\tilde{x}}_c &= M\dot{x} + \dot{x}_c = MA(s)x + B(s)h(x, -Mx + \tilde{x}_c) + f(x, -Mx + \tilde{x}_c) = \tilde{f}(x, \tilde{x}_c, s). \end{aligned} \tag{11}$$

with $u = h(x, x_c) = h(x, -Mx + \tilde{x}_c) = \tilde{h}(x, \tilde{x}_c)$. Now let us consider Lyapunov function $V = \tilde{z}^T\tilde{P}\tilde{z} = z^T Pz$ then

$$L(\tilde{z}, t) = L(z, t) \le -\alpha\|x\|^2, \tag{12}$$

and hence condition (8) of Lemma 2 is satisfied with the block diagonal matrix $\tilde{P}$. □

Proof of Theorem 1.

It is trivial that (i) implies (ii). Also (ii) implies (i) for the case when the dimension of controller is equal to zero. So we need only to show that (ii) implies (i) for the case when the dimension of the dynamic controller is greater than zero.

In a view of Lemma 3 we need to consider only the case when the matrix P in definition of RQSDC is block diagonal.

Suppose Σ_1 is RQSDC with $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$, i. e.,

$$z^T (\tilde{A}^T (s) P + P\tilde{A} (s)) z + 2z^T P\tilde{B} (s) p (z, s) \le -\alpha \|x\|^2, \tag{13}$$

where all terms are as in (7).

Let $S = P^{-1} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$ and $z = Sw$, then (13) will become

$$w^T (S\tilde{A}^T (s) + \tilde{A} (s) S) w + 2w^T \begin{bmatrix} B (s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} h (x, x_c) \\ f (x, x_c, s) \end{bmatrix} \le -\alpha \|x\|^2, \tag{14}$$

$$w^T \left(\begin{bmatrix} A (s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} + \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} A^T (s) & 0 \\ 0 & 0 \end{bmatrix} \right) w + 2w^T \begin{bmatrix} B (s) h (x, x_c) \\ f (x, x_c, s) \end{bmatrix} \le -\alpha \|x\|^2. \tag{15}$$

This must hold for all w. Let $w = \begin{bmatrix} r \\ 0 \end{bmatrix}$, then (15) will become

$$r^T (A (s) S_1 + S_1 A^T (s)) r + r^T B (s) h (x, 0) \le -\alpha \|x\|^2. \tag{16}$$

Setting $S_1 r = x$ gives us

$$x^T (A^T (s) P + PA (s)) x + 2x^T PB (s) h (x, 0) \le -\alpha \|x\|^2, \tag{17}$$

which shows that Σ_1 is RQSSC. □

In the sequel, we will refer both RQSSC and RQSDC simply as RQS.

4. Robust Quadratic Stabilization, Adaptive Quadratic Stabilization and the Existence of Control Lyapunov Function

In this section we examine the inter-relationships among robust quadratic stabilizability (RQS), adaptive quadratic stabilizability (AQS) and the existence of control Lyapunov function (CLF).

First, we establish relationship between the existence of CLF and AQS. We show that for Σ_1 they are equivalent given that Σ_1 is linearly parametrized. This result is given in the following theorem.

Theorem 2.

Consider linearly parametrized uncertain system Σ_1. Then the following statements are equivalent:

(i) Σ_1 admits CLF;

(ii) Σ_1 is AQS.

Proof of Theorem 2.

First, we show by construction, following [1], that for linearly parametrized Σ_1, the existence of control Lyapunov function implies AQS.

Let us define the following augmented Lyapunov function W,

$$W(x, \hat{s}) = V(x) + (\hat{s}-s)^T(\hat{s}-s) = x^TPx + (\hat{s}-s)^T(\hat{s}-s), \tag{18}$$

where $\hat{s}(t)$ will be the "estimate" of s.

The time derivative of W along (1) with control $u_n(x, \hat{s})$ is

$$\begin{aligned}\dot{W} = x^T(A^T(s)P + PA(s))x + x^TPB(s)u_n(x, \hat{s}) + 2\dot{\hat{s}}^T(\hat{s}-s) = \\ x^T(A^T(\hat{s})P + PA(\hat{s}))x + x^TPB(\hat{s})u_n(x, s) + \\ x^T(A^T(s-\hat{s})P + PA(s-\hat{s}))x + x^TPB(s)u_n(x, \hat{s}) + 2\dot{\hat{s}}^T(\hat{s}-s).\end{aligned} \tag{19}$$

Hence,

$$\begin{aligned}\dot{W} \le -\alpha\|x\|^2 + 2(s-\hat{s})^T \Delta\tilde{A}^T(x) P x + 2x^T P \Delta\tilde{B}(u_n(x,\hat{s}))(s-\hat{s}) + 2\dot{\hat{s}}^T(\hat{s}-s) \le \\ -\alpha\|x\|^2 + 2(x^T P \Delta\tilde{A}(x) + x^T P \Delta\tilde{B}(u_n(x,\hat{s})) - \dot{\hat{s}}^T)(s-\hat{s}).\end{aligned} \tag{20}$$

So $\dot{W} \le -\alpha\|x\|^2$ if we set

$$\dot{\hat{s}} = (x^T P \Delta\tilde{A}(x) + x^T P \Delta\tilde{B}(u_n(x,\hat{s})))^T. \tag{21}$$

So we can see that for linearly parametrized Σ_1, the existence of CLF is a sufficient condition for AQS.

Let us show now that (ii) implies (i). Suppose Σ_1 is AQS, then when $\hat{s} = s$ we get that

$$W(x,\hat{s}) = x^T P x + (\hat{s}-s)^T(\hat{s}-s) = x^T P x = V(x), \tag{22}$$

and

$$\begin{aligned}\dot{W} = x^T(A^T(s)P + PA(s))x + x^T PB(s)u_n(x,\hat{s}) + 2\dot{\hat{s}}^T(\hat{s}-s) = \\ x^T(A^T(s)P + PA(s))x + x^T PB(s)u_n(x,s) \le -\alpha\|x\|^2,\end{aligned} \tag{23}$$

which shows that Σ_1 admits CLF. □

We now compare the set of uncertain systems which is RQS and the set which admits CLF. In general the latter is larger than the former. However we show that for uncertain system Σ_2 with independent uncertain parameters in matrices A and B, the two are actually equivalent under some compactness and convexity conditions. The same issue is considered when control is restricted to be linear.

Theorem 3.

Consider the uncertain system Σ_2 defined in (2). Let the sets Π and Q be compact, and $\beta := \{B(q) : q \in Q\}$ be convex. Then the following statements are equivalent:

(i) Σ_2 admits CLF;

(ii) Σ_2 is RQS.

The following lemma due to Barmish [15] is needed for the proof of Theorem 2.

Lemma 4.

An uncertain system Σ_2 with compact uncertainty parameter sets Π and Q is RQS if and only if there exists an $n \times n$ positive definite matrix S such that

$$x^T (A(p) S + SA^T(p)) x < 0, \tag{24}$$

for all pairs $(x, p) \in N \times \Pi$ with $x \neq 0$ where
$N: = \{x \in R^n : B^T x = 0 \text{ for some } B \in conv \{B(q) : q \in Q\}\}$ and *conv* stands for convex hull.

Proof of Theorem 3.

It is clear that (ii) implies (i).

In order to show that (i) implies (ii), we will show that (i) implies (24), since (24) is a necessary and sufficient condition for RQS.

Suppose that Σ_2 admits CLF, and let P, $u_n(x, p, q)$, and α satisfy (9). We now show (24) is satisfied with $S = P^{-1}$. Suppose not, i.e., suppose that there exists a pair $(\hat{x}, \hat{p}) \in N \times \Pi$, such that $\hat{x} \neq 0$ and

$$\hat{x}^T (A(\hat{p}) S + SA^T(\hat{p})) \hat{x} \geq 0. \tag{25}$$

Let $\hat{y} = P^{-1}\hat{x}$, then

$$\hat{y}^T (A^T(\hat{p}) P + PA(\hat{p})) \hat{y} \geq 0. \tag{26}$$

Note that $\hat{y} \in N_P$, where N_P is the null space of $\tilde{B}^T P$ for some $\tilde{B} \in conv\{B(q) : q \in Q\}$. Since β is convex, $\beta = conv\{B(q) : q \in Q\}$. Therefore, there is a $\hat{q} \in Q$, such that $B(\hat{q}) = \tilde{B}$ and $B^T(\hat{q}) P\hat{y} = 0$. Hence for parameters $(\hat{p}, \hat{q})$, we have $L(\hat{y}, \hat{p}, \hat{q}, t) = \hat{y}^T (A^T(\hat{p}) P + PA(\hat{p})) \hat{y}$, which is non-negative by (25). This contradicts (9). □

Based on Theorem 2 and Theorem 3 we can state the following result.

Theorem 4.

Consider linearly parametrized uncertain system Σ_2 defined in (2). Let the sets Π and Q be compact, and $\beta := \{B(q) : q \in Q\}$ be convex. Then the following statements are equivalent:

(i) Σ_2 admits CLF;

(ii) Σ_2 is RQS;

(iii) Σ_2 is AQS.

Yet another class of uncertain systems for which AQS and RQS are equivalent under nonlinear as well as linear control has been shown by Rotea and Khargonekar [25]. However, the connection between AQS and the existence of control Lyapunov function is not recognized there.

Consider the following uncertain linear system Σ_{nb} with norm bounded uncertainty

$$\Sigma_{nb}: \quad \dot{x}(t) = Ax(t) + Bu(t) + D\Delta(t)\,[E_1 x(t) + E_2 u(t)], \tag{27}$$

where the real matrices A, B, D, E_1, and E_2, are known and of appropriate dimensions. The real uncertainty matrix Δ is assumed to belong to the norm bounded set,

$$U := \{\Delta \in R^{k \times p} : \|\Delta\| \le 1\}, \tag{28}$$

where $\|\cdot\|$ is the spectral norm.

Recognizing the equivalence between AQS and the existence of control Lyapunov function, we have the following theorem from [25].

Theorem 5.

Consider system Σ_{nb}. Then the following statements are equivalent:

(i) Σ_{nb} is AQS;

(ii) Σ_{nb} is RQS;

(iii) Σ_{nb} is RQS via linear control.

When control is restricted to be linear, we have the following theorem

Theorem 6.

Let Σ_B denote Σ_2 in the case when B is a constant matrix of full column rank. Then the following statements are equivalent.

(i) Σ_B admits CLF with control $u_n(x, s)$ being a linear function of x, i.e. $u_n(x, s) = K(s)x$;

(ii) Σ_B is robustly quadratically stabilizable (RQS) by linear control.

The following lemma due to Barmish and Hollot [24] is needed for the proof of the above theorem.

Lemma 5.

Consider an uncertain system Σ_B. Let Π be compact, and Θ be any $n \times (n-m)$ orthonormal matrix whose range space equals the null space of B^T. Then the system Σ_B is RQS via linear control if and only if there exist a positive definite matrix S, such that

$$\Theta^T (A(p)S + SA^T(p))\Theta < 0, \forall p \in \Pi. \tag{29}$$

Proof of Theorem 6.

Again, it is trivial that (ii) implies (i).

In order to show that (i) implies (ii), we will show that (i) implies (29) with $S = P^{-1}$. Let us prove by contradiction. Suppose there is a y and $\hat{p}$ such that

$$y^T\Theta^T (A(\hat{p})S + SA^T(\hat{p}))\Theta y \geq 0. \tag{30}$$

Let $x = P^{-1}\Theta y$, then $x^T(PA(\hat{p}) + A^T(\hat{p})P)x \geq 0$.

Since $\Theta y \in N(B^T)$, $B^T Px = B^T\Theta y = 0$. Hence

$$L(x, \hat{p}, t) = x^T(PA(\hat{p}) + A^T(\hat{p})P)x \geq 0, \tag{31}$$

contradicting (9). □

In this section we have shown the equivalence of AQS, RQS and the existence of CLF function for uncertain system Σ_2. For more general uncertain system Σ_1, it is found that the equivalence of AQS and RQS does not hold in general. However, AQS and the existence of CLF remain equivalent for linearly parametrized Σ_1. This gives AQS an open loop characterization.

5. A Necessary and Sufficient Condition for Adaptive Stabilization Based on Parameter-Dependent Lyapunov Function

In this section we develope necessary and sufficient condition for adaptive stabilization based on parameter-dependent Lyapunov function. This section builds on [1] which gives sufficient condition for such design to be possible.

Consider linearly parametrized system Σ_1 and assume it is pointwise stabilizable. Consider the augmented Lyapunov function

$$W = x^T P(\hat{s}) x + (\hat{s} - s)^T (\hat{s} - s), \tag{32}$$

where $\hat{s}$ is the parameter estimate to be specified later. Let us compute the time derivative of (32) along Σ_1.

$$\begin{aligned}
\dot{W} &= x^T (A^T(s) P(\hat{s}) + P(\hat{s}) A(s)) x + x^T P(\hat{s}) B(s) u + 2(\hat{s} - s)^T \dot{\hat{s}} + \\
&+ x^T \frac{\partial}{\partial s} [P(\hat{s}) x] \dot{\hat{s}} = 2x^T A_0^T P(\hat{s}) x + 2x^T P(\hat{s}) \Delta\tilde{A}(x) s + \\
&+ 2x^T P(\hat{s}) B_0 u + 2x^T P(\hat{s}) \Delta\tilde{B}(u) s + 2(\hat{s} - s)^T \dot{\hat{s}} + x^T \frac{\partial}{\partial s} [P(\hat{s}) x] \dot{\hat{s}} = \\
&= 2x^T A_0^T P(\hat{s}) x + 2x^T P(\hat{s}) \Delta\tilde{A}(x) \hat{s} + 2x^T P(\hat{s}) B_0 u + 2x^T P(\hat{s}) \Delta\tilde{B}(u) \hat{s} + \\
&2x^T P(\hat{s}) \Delta\tilde{A}(x) (s - \hat{s}) + 2x^T P(\hat{s}) \Delta\tilde{B}(u) (s - \hat{s}) + 2(\hat{s} - s)^T \dot{\hat{s}} + x^T \frac{\partial}{\partial s} [P(\hat{s}) x] \dot{\hat{s}} = \\
&= x^T (A^T(\hat{s}) P(\hat{s}) + P(\hat{s}) A(\hat{s})) x + x^T P(\hat{s}) B(\hat{s}) u + x^T \frac{\partial}{\partial s} [P(\hat{s}) x] \dot{\hat{s}} + \\
&+ 2\left[x^T P(\hat{s}) \Delta\tilde{A}(x) + x^T P(\hat{s}) \Delta\tilde{B}(u) - \dot{\hat{s}}^T\right] (s - \hat{s}).
\end{aligned} \tag{33}$$

Since parameter s is unknown, the only way to eliminate the last term above is by choosing the parameter estimator to be

$$\dot{\hat{s}} = [x^T P(\hat{s}) \Delta\tilde{A}(x) + x^T P(\hat{s}) \Delta\tilde{B}(u)]^T = \Delta\tilde{A}^T(x) P(\hat{s}) x + \Delta\tilde{B}^T(u) P(\hat{s}) x. \quad (34)$$

Then (33) becomes

$$\begin{aligned} \dot{W} &= x^T (A^T(\hat{s}) P(\hat{s}) + P(\hat{s}) A(\hat{s})) x + x^T P(\hat{s}) B(\hat{s}) u + \\ &+ x^T \frac{\partial}{\partial s}[P(\hat{s}) x] \left[\Delta\tilde{A}^T(x) P(\hat{s}) x + \Delta\tilde{B}^T(u) P(\hat{s}) x\right] = \\ &= x^T (A^T(\hat{s}) P(\hat{s}) + P(\hat{s}) A(\hat{s})) x + x^T \frac{\partial}{\partial s}[P(\hat{s}) x] \Delta\tilde{A}^T(x) P(\hat{s}) x + \\ &+ 2x^T P(\hat{s}) \left[B(\hat{s}) u + \Delta\tilde{B}(u) (x^T \frac{\partial}{\partial s}[P(\hat{s}) x])\right] = \\ &= x^T (A^T(\hat{s}) P(\hat{s}) + P(\hat{s}) A(\hat{s})) x + x^T \frac{\partial}{\partial s}[P(\hat{s}) x] \Delta\tilde{A}^T(x) P(\hat{s}) x + \\ &+ 2x^T P(\hat{s}) \left[B_0 u + \Delta\tilde{B}(u) \hat{s} + \Delta\tilde{B}(u) (x^T \frac{\partial}{\partial s}[P(\hat{s}) x])\right]. \end{aligned} \quad (35)$$

Using (3) we get

$$\begin{aligned} \dot{W} &= x^T (A^T(\hat{s}) P(\hat{s}) + P(\hat{s}) A(\hat{s})) x + x^T \frac{\partial}{\partial s}[P(\hat{s}) x] \Delta\tilde{A}^T(x) P(\hat{s}) x + \\ &+ 2x^T P(\hat{s}) B(\hat{s} + x^T \frac{\partial}{\partial s}[P(\hat{s}) x]) u. \end{aligned} \quad (36)$$

Let

$$\bar{B}(x, \hat{s}) = B(\hat{s} + x^T \frac{\partial}{\partial s}[P(\hat{s}) x]). \quad (37)$$

Therefore we can see that we can not influence the sign of (36) if

$$x^T P(\hat{s}) B(\hat{s} + x^T \frac{\partial}{\partial s}[P(\hat{s}) x]) = 0. \quad (38)$$

When this happens, the following lemma shows that it is necessary to have both $x^T (A^T(\hat{s}) P(\hat{s}) + P(\hat{s}) A(\hat{s})) x < 0$ and $x^T \frac{\partial}{\partial s}[P(\hat{s}) x] \Delta\tilde{A}^T(x) P(\hat{s}) x < 0$, for $\dot{W}$ to be negative.

Lemma 6.

When (38) holds, the necessary and sufficient conditions for (36) to be negative $\forall x \neq 0$ are the following:

(i) $x^T (A^T(\hat{s}) P(\hat{s}) + P(\hat{s}) A(\hat{s})) x < 0$;

(ii) $x^T\frac{\partial}{\partial s}[P(\hat{s})x]\Delta\tilde{A}^T(x)P(\hat{s})x<0.$

Proof of Lemma 6.

The sufficiency of the above statement is trivial.

To prove necessity we notice that the first term is quadratic in x, and the second is of fourth power of x, since $\Delta\tilde{A}(\cdot)$ is linear in x. Now suppose the necessity is false, i. e.

$$x^T(A^T(\hat{s})P(\hat{s})+P(\hat{s})A(\hat{s}))x = a>0, \quad x^T\frac{\partial}{\partial s}[P(\hat{s})x]\Delta\tilde{A}^T(x)P(\hat{s})x = b<0,$$

and $a+b<0$. Then for $\tilde{x} = \alpha x$, $x\neq 0$ (38) still holds and (36) will become

$$\dot{W}(\tilde{x}) = \alpha^2 a+\alpha^4 b = \alpha^2(a+\alpha^2 b).$$

Clearly for sufficiently small α, $\dot{W}(\tilde{x})>0$, hence contradiction. □

Based on this observation we can now prove the following theorem, which is the main result of this section.

Theorem 7.

Consider linearly parametrized uncertain system Σ_1, then it can be adaptively stabilized using parameter-dependent Lyapunov function (32) if and only if there exist continuous mappings $P(\cdot): R^{n_s}\rightarrow R^{n\times n}$ and $u_n: R^{n+n_s}\rightarrow R^m$, such that $P(\hat{s})$ is positive definite $\forall\hat{s}\in S$ and we have following properties

(i) $$x^T(A^T(\hat{s})P(\hat{s})+P(\hat{s})A(\hat{s}))x+2x^TP(\hat{s})B\left(\hat{s}+x^T\frac{\partial}{\partial s}[P(\hat{s})x]\right)u_n(x,\hat{s})\leq \leq-\alpha\|x\|^2$$

for some $\alpha>0$;

and

(ii) $x^T\frac{\partial}{\partial s}[P(\hat{s})x]\Delta\tilde{A}^T(x)P(\hat{s})x<0$, for $\forall\hat{s}\in S$ and $\forall x\neq 0$ such that

$$x^T P(\hat{s}) B(\hat{s} + x^T \frac{\partial}{\partial s}[P(\hat{s}) x]) = 0.$$

Proof of Theorem 7.

Necessity.

The necessity of (ii) follows from Lemma 6. Condition (i) implies when $x^T P(\hat{s}) B(\hat{s} + x^T \frac{\partial}{\partial s}[P(\hat{s}) x]) = 0$, we have $x^T (A^T(\hat{s}) P(\hat{s}) + P(\hat{s}) A(\hat{s})) x < 0$, hence the necessity of (i).

Sufficiency.

Since the case when $x^T P(\hat{s}) B(\hat{s} + x^T \frac{\partial}{\partial s}[P(\hat{s}) x]) = 0$ is already taken care of in a view of conditions (i) and (ii). We consider the case when $x^T P(\hat{s}) B(\hat{s} + x^T \frac{\partial}{\partial s}[P(\hat{s}) x]) \neq 0$.

A control which achieves condition (i) can be chosen to be the minimum effort control i.e.

$$\begin{aligned} u = min \{ \|u_n\| \ | x^T P(\hat{s}) B(\hat{s} + x^T \frac{\partial}{\partial s}[P(\hat{s}) x]) u_n &= -\alpha \|x\|^2 - \\ -x^T (A^T(\hat{s}) P(\hat{s}) + P(\hat{s}) A(\hat{s})) x - x^T \frac{\partial}{\partial s}[P(\hat{s}) x] \Delta \tilde{A}^T(x) P(\hat{s}) x \} . \end{aligned} \tag{39}$$

Hence we have that $\dot{W} \leq -\alpha \|x\|^2$, $\forall x$ and so by LaSalle's Invariance Theorem [27], $x(t) \to 0$, and $\|\hat{s}\|$ is bounded. So Σ_1 is adaptively stabilized based on parameter-dependent Lyapunov function (32). □

Note that (i) implies pointwise stabilizability condition (9).

6. Uncertainty Structures

In this section we explore in greater detail the uncertainty structures allowed by various robust and adaptive designs outlined in Section 2. We apply the robust stabilization results of [16, 17, 20, 26] to adaptive stabilizability and point out interesting hierarchy of uncertainty structures. We limit ourselves to single input systems whose entries of $(A, b$ [1]$)$ matrices vary independently. As shown in [16, 17, 20, 26], the class of this type of uncertain systems which possesses controllability invariance, and yet has the minimal number

1. We use lower case to emphasize the single input case.

of sign invariant entries is a subset of the so-called standard form, denoted by Σ_S is defined below.

Definition 7.

A single input uncertain linear system $\{A(s),\ b(s)\}$ is said to be in standard form Σ_S if the entries of $\{A(s),\ b(s)\}$ are independent of one another and the $n \times (n+1)$ associated matrix $M(s)$ defined as

$$M(s) \doteq [A(s)\ b(s)] = \{m_{ij}(s)\} \tag{40}$$

has the following property: $m_{ii+1}(s)$ is a sign-invariant function of s for each $i,\ 1 \le i \le n$.

Definition 8.

An uncertain structure in Σ_S is a set of uncertain systems in Σ_S whose uncertain entries have the same locations.

Definition 9.

An uncertainty structure is said to be RQS (or AQS etc.) if and only if every uncertain system in the set is RQS (or AQS etc.).

In Σ_S, the class of the systems which satisfies the antisymmetric stepwise configuration [16] (denoted by Σ_{AS}) or the generalized antisymmetric stepwise (GAS) configuration (Σ_{GAS}) is of special interest. For ease of reference, we define Σ_{AS} below. The definition of Σ_{GAS} is somewhat involved and we refer to [17, 20, 26].

Definition 10. Antisymmetric stepwise configuration [16].

We say Σ_S has antisymmetric stepwise (AS) configuration if its associated matrix $M(s)$ satisfies following condition.

If $k \ge h+2$ and $m_{hk}(s)$ is not identically zero, then $m_{uv}(s) \equiv 0$ for all $u \ge v$, $u \le k-1$, and $v \le h+1$.

The following results shown in [16, 17, 20, 26] are relevant.

Theorem 8.

(i) An uncertainty structure with bounded uncertainties in Σ_S is RQS if and only if it is in Σ_{AS}.

(ii) An uncertainty structure with bounded uncertainties in Σ_S is robustly stabilizable by linear control if and only if it is in Σ_{GAS}.

(iii) An uncertainty structure in Σ_S possesses controllability invariance if and only if it is in Σ_{GAS}.

It is particularly revealing to note that for uncertain systems in standard form, the following interesting hierarchy is true.

Strict matching condition [23] $\subset$ extended matching condition [9] $\subset$

pure feedback form [3] $\subset$ AS configuration [16] $\subset$ GAS configuration [17, 26].

This hierarchy is illustrated for third order systems in terms of their associated matrices M in Figure 1.

Since AQS is equivalent to RQS when restricted to Σ_2 and when linear parametrization and some compactness and convexity conditions hold, as shown in Section 4 we have the following theorem.

Theorem 9.

An uncertainty structure with bounded uncertainties in Σ_S is AQS if and only if it is in Σ_{AS}.

Proof of Theorem 9.

We first note that $\Sigma_S \subset \Sigma_2$ and linear parametrization and the convexity condition holds automatically. Hence every uncertainty structure in Σ_S which is AQS (therefore RQS by Theorem 4) has to be in Σ_{AS} by Theorem 8. Conversely every uncertain system with bounded uncertainties in Σ_{AS} is RQS, hence AQS. □

$$\begin{bmatrix} 0 & \theta & 0 & 0 \\ 0 & 0 & \theta & 0 \\ \ast & \ast & \ast & \theta \end{bmatrix}$$

Strict matching condition structure.

$$\begin{bmatrix} 0 & \theta & 0 & 0 \\ \ast & \ast & \theta & 0 \\ \ast & \ast & \ast & \theta \end{bmatrix}$$

Extended matching condition structure.

$$\begin{bmatrix} \ast & \theta & 0 & 0 \\ \ast & \ast & \theta & 0 \\ \ast & \ast & \ast & \theta \end{bmatrix}$$

Pure feedback form structure.

$$\begin{bmatrix} \ast & \theta & 0 & 0 \\ \ast & \ast & \theta & 0 \\ \ast & \ast & \ast & \theta \end{bmatrix} \begin{bmatrix} 0 & \theta & \ast & \ast \\ 0 & 0 & \theta & \ast \\ 0 & 0 & 0 & \theta \end{bmatrix} \begin{bmatrix} 0 & \theta & \ast & 0 \\ 0 & 0 & \theta & 0 \\ \ast & \ast & \ast & \theta \end{bmatrix} \begin{bmatrix} 0 & \theta & \ast & \ast \\ 0 & 0 & \theta & 0 \\ 0 & 0 & \ast & \theta \end{bmatrix}$$

Antisymmetric stepwise configuration structures.

$$\begin{bmatrix} \ast & \theta & 0 & 0 \\ \ast & \ast & \theta & 0 \\ \ast & \ast & \ast & \theta \end{bmatrix} \begin{bmatrix} 0 & \theta & \ast & \ast \\ 0 & 0 & \theta & \ast \\ 0 & 0 & 0 & \theta \end{bmatrix} \begin{bmatrix} 0 & \theta & \ast & 0 \\ 0 & 0 & \theta & 0 \\ \ast & \ast & \ast & \theta \end{bmatrix} \begin{bmatrix} 0 & \theta & \ast & \ast \\ 0 & 0 & \theta & 0 \\ 0 & 0 & \ast & \theta \end{bmatrix} \begin{bmatrix} 0 & \theta & 0 & \ast \\ 0 & \ast & \theta & 0 \\ 0 & 0 & 0 & \theta \end{bmatrix}$$

Generalized antisymmetric stepwise configuration structures.

Here θ denotes sign invariant entry and "*" denotes "do not care" entry.

FIGURE 1. Uncertainty structures for third order system.

As shown in [12], for certainty equivalence adaptive control to stabilize an uncertain system, it is enough to have pointwise stabilizability, hence we have the following theorem.

Theorem 10.

Every uncertain system in Σ_{GAS} can be stabilized by certainty equivalence adaptive control.

7. Concluding Remarks

In this paper, we have attempted to give a full picture of the inter-relationships among several approaches to robust and adaptive stabilization for uncertain continuous time linear time invariant systems using state feedback. In this process, we have noted a revealing hierarchy of uncertainty structures.

The inter-relationships and the hierarchy are summarized in Figure 2. It is interesting to note that adaptive stabilization based on parameter-independent Lyapunov function can accommodate uncertainty structures larger than the pure feedback form. This suggests that it may be possible to adaptively stabilize nonlinear systems with uncertainty structures broader than the pure feedback form considered in [2, 3].

Besides differences in allowable uncertainty structures, other differences between robust and adaptive stabilization include that the robust approaches may require larger control gains, and that on the other hand robust quadratic stabilization allows arbitrarily time varying parameters.

It is worthwhile pointing out that many results here *can not* be extended to discrete time systems. It is not too difficult to see that unlike continuous time systems, the set of discrete time uncertain systems which are RQS or AQS or RQLC are substantially smaller than the sets which possesses controllability invariance or which can be stabilized by certainty equivalence adaptive control. This is the reason why results of [1-10, 13-25] can not be easily extended to the discrete time systems.

Much further works remain along the direction of this paper. These includes tracking and disturbance rejection problems, output feedback and nonlinear systems.

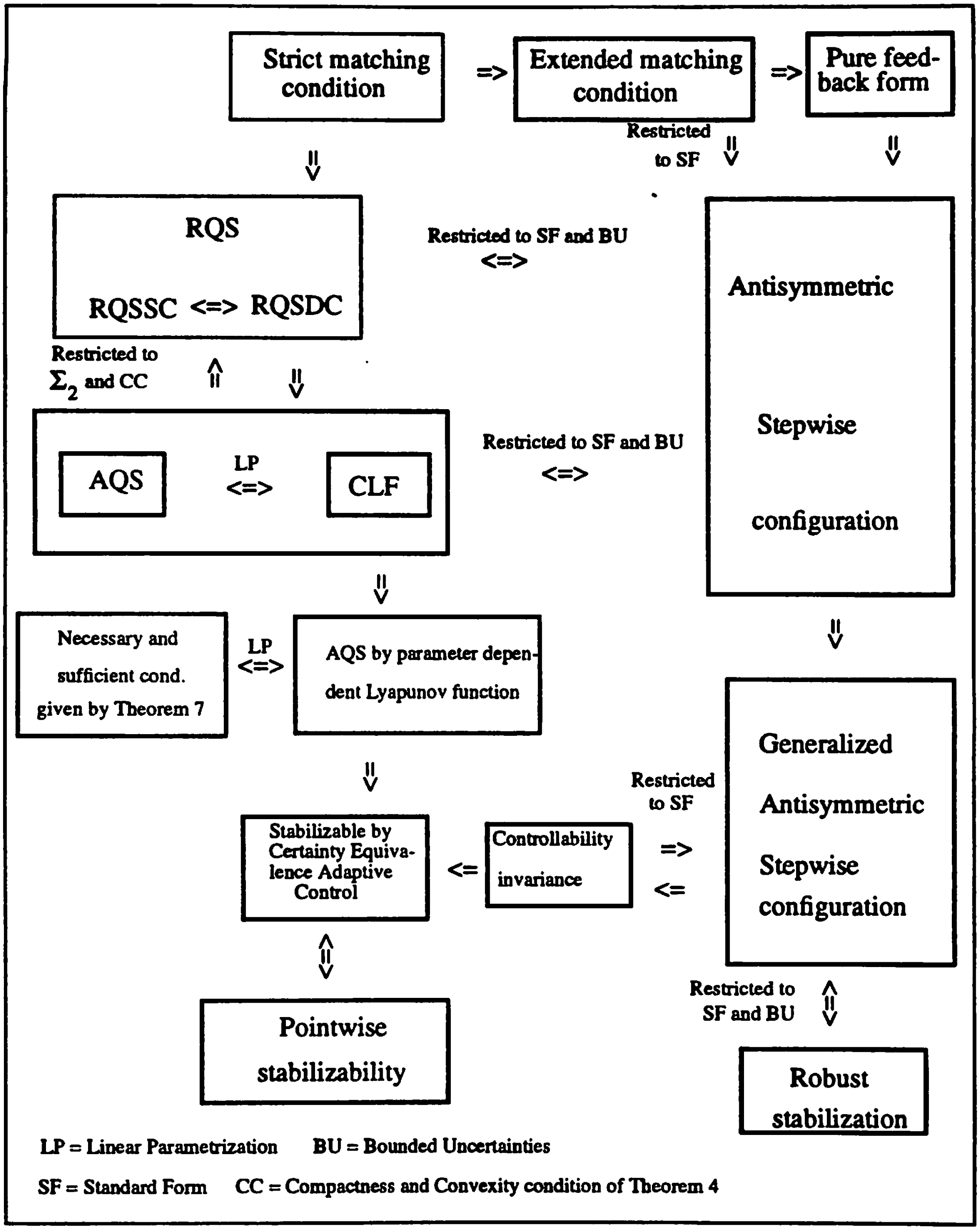

FIGURE 2. Summary of Inter-relationships.

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