INDIRECT ADAPTIVE POLE-PLACEMENT CONTROL OF MIMO STOCHASTIC SYSTEMS: SELF-TUNING RESULTS

by

Karim Nassiri-Toussi and Wei Ren

Memorandum No. UCB/ERL M94/26

11 April 1994

INDIRECT ADAPTIVE POLE-PLACEMENT CONTROL OF MIMO STOCHASTIC SYSTEMS: SELF-TUNING RESULTS

by

Karim Nassiri-Toussi and Wei Ren

Memorandum No. UCB/ERL M94/26

11 April 1994

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

INDIRECT ADAPTIVE POLE-PLACEMENT CONTROL OF MIMO STOCHASTIC SYSTEMS: SELF-TUNING RESULTS

Karim Nassiri-Toussi† **Wei Ren** †

†EECS Dept., University of California at Berkeley, Berkeley, CA, U.S.A. 94720

Abstract. In this paper, we consider indirect adaptive pole-placement control (APPC) of linear multivariable stochastic systems. We propose a nonminimal but otherwise uniquely identifiable *overlapping* parameterization instead of the minimal representation used in current literature. Requiring less restrictive prior information than observability (and controllability) indices, this parameterization is more appropriate for multivariable ARMAX model representations. We will show that by using the Stochastic Gradient (SG) identification method, and with sufficient external excitation, the parameter estimates are strongly consistent for all initial conditions. Moreover, under more restrictive assumptions and minor modification, the MIMO adaptive pole-placement controller is globally asymptotically self-tuning even in the absence of external excitation. These represent the most complete study of stochastic multivariable APPC systems so far.

1. Introduction

Multivariable (MIMO) adaptive control systems have been the subject of study for over a decade, where the focus has been primarily on adaptive model reference and pole-placement control of deterministic systems.

The parameterization issue is a highly nontrivial problem for MIMO systems, contrary to the case of SISO systems. This issue was addressed in many early works, for example see [5] and [7]. In [6], an algorithm for direct adaptive pole-placement control of linear MIMO systems, using a nonminimal representation of the plant model, was proposed. This algorithm was further modified in [18] in order to guarantee uniqueness, and hence *identifiability*, of the nonminimal presentation. The algorithm presented in [1] also proposes a *similar identifiable* nonminimal representation appropriate for direct adaptive pole-placement control. In all these cases, prior knowledge of the observability and controllability indices is required in order to establish stability and parameter convergence.

However, since for most physical systems, these indices are not directly related to any physical property of the plant, this assumption is unlikely to be satisfied in most practical situations. The observability and controllability indices are uniquely determined for each linear model and as pointed out in [2], if the control designer's estimates of the observability indices are even slightly off their true values, the parameter estimates can be inconsistent even in open-loop condition.

In contrast, we propose, after [9], to use a nonminimal *overlapping*, but otherwise globally identifiable, parameterization of the model. As shown in [9], although this parameterization requires prior knowledge of a certain multiindex too, its advantage over the canonical parameterizations is that the corresponding multiindex is not uniquely determined for each plant model. In fact, any such arbitrarily chosen multiindex corresponds to a parameterization of the plant model, for almost every model of the compatible order. Therefore, by knowing only the minimal order (McMillan degree) of the plant, we may choose any compatible multiindex and be certain that the corresponding parameterization holds for at least an arbitrarily close approximation of the true plant model. This naturally requires far less information about the dynamics of the plant in practice.

In addition, the suggested parameterization is compatible with the ARMAX representation of the plant and can be written in the predictor-equation form, and thus, is more appropriate for indirect adaptive control applications. One may compare this to the complexity of parameterizations in [6, 7].

Regarding the issues of stability and parameter convergence, almost all of the available results deal with the deterministic case: By using a *hybrid control*[1] approach, [4] for the SISO case, and [18] for the MIMO case, show that with sufficient persistency of excitation, the proposed direct APPC algorithms are stable and that the parameter estimates are convergent.

In [1], bypassing the issue of global stability, the authors concentrate on the necessary and sufficient frequency domain conditions for persistency of excitation of the regression vector for the direct APPC algorithm. They show that in the MIMO case, the necessary and the sufficient number of frequency components in the spectrum of the reference signal are in general different.

All these results require sufficiently persistent external excitation to prove stability of the system. Only in [10], where a complex modification of the direct APPC algorithm is used,

[1] The controller parameters are updated at T intervals for the continuous-time case, and with a slower rate for the discrete-time case

global stability can be guaranteed without conditions on the reference signal.

As to indirect APPC algorithms, it has been known for a long time that under certain estimation methods, the indirect APPC algorithm, deterministic or stochastic, is globally stable for every reference signal if no pole-zero cancellation occurs in the estimated model. However, the only result, which considers asymptotic self-tuning of the indirect APPC algorithm in the absence of external excitation in the stochastic case, is that of [16] for the SISO case.

In this paper, we extend those results to the MIMO case, using the overlapping parameterization that we explained before. The prior information that we require is the minimal order of the plant, and knowledge of a convex compact set in the parameter space, to which the true parameter matrix belongs and every point of which corresponds to a minimal input-output transfer function. We then introduce an indirect APPC scheme, based on the Stochastic Gradient (SG) algorithm with parameter projections, and prove that the corresponding adaptive control system is globally stable and that with sufficiently persistent external excitation, the parameter estimates are consistent.

Next, we increase the amount of required information by assuming prior knowledge of a convex compact set in the parameter space, to which the true parameter matrix belongs and for every point of which the corresponding pole-placement control design is uniquely determined. Then, by a slight modification of the original pole-placement control design, we can show that the resulting SG-based indirect APPC algorithm is globally stable and asymptotically self-tuning even in the absence of external excitation. By asymptotically self-tuning, we mean that the control input u converges to the desired control u^*, the control input that one would obtain if the true parameters were known, in the mean-square sense. In other words,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N} \|u_k - u_k^*\|^2 = 0 \qquad \text{almost surely.}$$

In fact, we prove a stronger result in this case; that the controller transfer function indeed converges to its desired value even in the absence of external excitation.

To decrease the restrictive amount of necessary prior information, we suggest applying the switching scheme of [13]. We conjecture that the stochastic adaptive system is still globally stable under a variation of this scheme, *provided that sufficiently persistent external excitation exists.*

The organization of the paper is as follows: In the next section, the problem setup and the pole-placement controller designs are discussed. In section 3, the indirect adaptive control equations are given and the overlapping parameterizations are presented. In section 4, we present the stability and self-tuning property results concerning the proposed *indirect* APPC algorithms.

Notations: We shall use the following notations in the remainder of this paper:

- We define the vector space $\mathcal{M}_p^2$ as the space of $\mathcal{R}^p$-valued sequences $\{h_k\}$ for which $\limsup_{k\to\infty} \frac{1}{N}\sum_{k=0}^{N} \|h_k\|^2 < \infty$. and let the subspace $\mathcal{M}_{0_p}^2 \subset \mathcal{M}_p^2$ be the set of $\mathcal{R}^p$-valued sequences $\{h_k\}$ such that $\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N} \|h_k\|^2 = 0$.
  It is clear that $\mathcal{M}^2$ and $\mathcal{M}_0^2$ are vector spaces, *i.e.* $h_k + g_k$ is in $\mathcal{M}^2$ (or $\mathcal{M}_0^2$) if h_k and g_k are in $\mathcal{M}^2$ (or $\mathcal{M}_0^2$).
- For any given left Matrix Fraction Description (MFD) of a transfer function matrix such as $H = A^{-1}B$, we denote the corresponding right MFD by $H = B_R A_R^{-1}$. Similarly, for any given right MFD such as $T = CD^{-1}$, we denote the corresponding left MFD by $T = D_L^{-1} C_L$.
  When there is no confusion, we write the polynomial matrix $A(q^{-1})$ simply as A.
- Let $\|A\|_F := (\text{Trace} A^T A)^{\frac{1}{2}} = \left(\sum_{i,j} A_{ij}^2\right)^{\frac{1}{2}}$ denote the Frobenius norm for any matrix A. Similarly, for a polynomial matrix $A(z) = A^0 + A^1 z^1 + \ldots + A^n z^n$, we define the norm $\|A\|_F^2 := \left(\|A^0\|_F^2 + \ldots + \|A^n\|_F^2\right)$.
- For any polynomial $a(z)$, ∂a denotes the degree of a, while for the polynomial matrix $A(z)$, ∂A is the maximum degree of elements of A. $\partial_{r_i} A$ and $\partial_{c_i} A$ denote the maximum degree of the i-th row and the i-th column of A, respectively.
- Estimate of every quantity will be specified by adding a "hat" to its notation, while addition of a "bar" denotes a (potential) limit point of the estimate sequence, e.g. $\widehat{A}_k$ is the estimate of the polynomial matrix A at step k and $\overline{A}$ is a (potential) limit point of $\widehat{A}_k$.

2. The Problem Setup

2.1 The Plant

Let the plant subject to control be modeled by the following equation:

$$y_k = \mathbf{H}(q)u_k + \mathbf{G}(q)w_k, \qquad \text{for } k \geq 0, \tag{1}$$

where q is the forward shift operator. We assume that the output y and noise w are p-dimensional, while the control input u is of dimension m, and that all the processes are adapted to the increasing family of σ-fields $\mathcal{F}_k$ in the underlying probability space $(\Omega, P, \mathcal{F})$.

For a physical system, $\mathbf{H}$ is strictly proper, *i.e.* $\lim_{q\to\infty} \mathbf{H}(q) = 0$, while with little loss of generality, we can take $\mathbf{G}$ to be a proper transfer function such that $\lim_{q\to\infty} \mathbf{G}(q) = I_{p\times p}$. By considering the composite transfer function $[\mathbf{H}(q)\ ;\ \mathbf{G}(q)]$, we find the following *irreducible* left MFD of $[\mathbf{H} \;\; \mathbf{G}]$:

$$[\mathbf{H}(q) \;\; \mathbf{G}(q)] = D^{-1}[N \;\; N'],$$

With no loss of generality, we can also assume that D is row-reduced (see [11]). Furthermore, $\partial_{r_i} N < \partial_{r_i} D$ and $\partial_{r_i} N' \leq \partial_{r_i} D$. Hence, there exist polynomial matrices A, B, C in term of q^{-1}, the backward shift operator, such that

$$A(q^{-1})y_k = B(q^{-1})u_k \; + \; C(q^{-1})w_k, \tag{2}$$

where

$$A = I + A^1 q^{-1} + \ldots + A^\nu q^{-\nu}, \qquad B = B^1 q^{-d} + \ldots + B^\nu q^{-\nu},$$
$$C = I + C^1 q^{-1} + \ldots + C^\nu q^{-\nu} \qquad \text{and} \qquad \nu = \max_i \partial_{r_i} D.$$

We make the following assumptions on the above system:

A1) *w_k is a $\mathcal{R}^p$-valued $\{\mathcal{F}_k\}$-martingale difference process satisfying for all $k \geq 0$ and some $\sigma > 2$,*

$$E[w_k w_k^T \mid \mathcal{F}_{k-1}] = \Sigma_w \Sigma_w^T > 0, \qquad \sup_k E[\|w_k\|^\sigma \mid \mathcal{F}_{k-1}] < \infty. \tag{3}$$

The finite-moment assumption ensures that w_k is ergodic up to the second order, cf. [14], i.e. $\lim_{N\to\infty} \sum_{k=0}^{N} w_k = 0$, $\lim_{N\to\infty} \sum_{k=O}^{N} w_k w_{k-j}^T = \delta_j \Sigma_w \Sigma_w^T$ almost surely. Note that knowledge of Σ_w is not assumed throughout this paper.
Moreover, we let the sequence $\{x_0, w_0, w_1, \ldots\}$ have a probability measure absolutely continuous with respect to the Lebesgue measure, where x_0 is the initial state of (1).

A2) *$C(q^{-1})$ is a stable polynomial matrix, i.e.* $\det C(z) \neq 0$ *for all $|z| \leq 1$. Furthermore, $C(q^{-1})$ is Strictly Positive Real (SPR), i.e. $C(e^{j\omega}) + C^T(e^{-j\omega}) > 0$ for all $\omega \in \mathcal{R}$.*

A3) $\mathbf{H}(q)$ *and* $[\mathbf{H}(q) \; ; \; \mathbf{G}(q)]$ *have identical McMillan degrees. Thus, since $D^{-1}[N \; ; \; N']$, and consequently $A^{-1}[B \; ; \; C]$, are irreducible MFDs, A and B are left coprime.*

2.2. The Pole-Placement Controller

The controller is given by the following equation:

$$u = -S(q^{-1})R(q^{-1})^{-1} y + \mathbf{T}(q)\overline{y}, \tag{4a}$$

or equivalently,

$$R_L(q^{-1})u_k = -S_L(q^{-1})y_k + \mathbf{T}'(q)\overline{y}_k \tag{4b}$$

where the polynomials R, S and transfer function $\mathbf{T}$, or the polynomials R_L, S_L and transfer function $\mathbf{T}'$, are determined by a pole-placement control design and $R_L{}^{-1} S_L = SR^{-1}$.

$\overline{y}$ is the $\mathcal{R}^p$-valued "deterministic" reference input satisfying the following assumption:

A4) *$\overline{y}$ is either deterministic ($\mathcal{F}_0$-measurable) or independent of w. Furthermore, it is uniformly bounded and second-order ergodic,*
i.e. $1/N\sum_{k=k_0}^{k_0+N}\overline{y}_k$ *and* $1/N\sum_{k=k_0}^{k_0+N}\overline{y}_k\overline{y}_{k-m}{}^T$ *have k_0–independent limits as $N\to\infty$ for every finite m.*

For the above controller and system description (2), it is easy to find that the closed-loop system satisfies the following equations:

$$\begin{cases} y = B_R(R_L A_R + S_L B_R)^{-1}T(q)\overline{y} \,+\, R(AR+BS)^{-1}Cw, \\ u = A_R(R_L A_R + S_L B_R)^{-1}T(q)\overline{y} - S(AR+BS)^{-1}Cw, \end{cases} \tag{5}$$

where $B_R A_R{}^{-1} = A^{-1}B$ is the right MFD of $\mathbf{H}$. Without loss of generality, we can assume that A_R is also monic and has the same invariant polynomial as A.

Equations (5) determine what the pole-ploacement control design should be. Typically, the pole-placement control objective is to have $B(R_L A_R + S_L B_R)^{-1}\mathbf{T}(q) = BM^{-1}$, where $M(q^{-1})$ is a prescribed monic stable polynomial matrix. One can achieve this by letting $R_L A_R + S_L B_R = M$ and $\mathbf{T} = I$, and therefore, computing (R_L, S_L), in term of the right MFD of the system.

However, to identify the system, it is the representation (2), corresponding to the left MFD of the plant transfer functions, that can readily be transformed into a predictor equation and hence, be written as a linear regression model suitable for identification. This suggests that we solve the Diophantine equation $AR+BS=M$ instead to find R and S in terms of A and B polynomial matrices.

But the proof of the following lemma indicates that for a general polynomial matrix M, $AR+BS=M$ does not necessarily mean that $R_L A_R + S_L B_R = M$ for some left MFD of SR^{-1}.

Lemma 1. *Let $A^{-1}B = B_R A_R{}^{-1}$ be irreducible MFDs. Then, there exist solutions (R,S) and (R_L, S_L) to the following Diophantine equations:*

$$AR \,+\, BS = M, \qquad R_L A_R \,+\, S_L B_R = M, \tag{6}$$

such that $S_L R = R_L S$.

Proof. Since $(A,\ B)$ and $(A_R,\ B_R)$ are left and right coprime pairs, the Bezout identity holds, *i.e.* there exist polynomial matrices X_L, Y_L, X_R and Y_R such that (cf. [11])

$$\begin{cases} X_L A_R + Y_L B_R = I \\ A X_R + B Y_R = I \end{cases} \quad \text{and} \quad \begin{bmatrix} X_L & Y_L \\ -B & A \end{bmatrix}\begin{bmatrix} A_R & -Y_R \\ B_R & X_R \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{7}$$

Using the above equations, one can show that the classes of solutions of Diophantine equations (6) are the following sets for any X_L, Y_L, X_R, Y_R satisfying (7):

$$\begin{aligned} &\{(R = X_R\,M + B_R\,Q,\ S = Y_R\,M - A_R\,Q), \quad \forall Q \text{ poly. matrix}\} \\ &\{(R_L = M\,X_L + Q\,B,\ S_L = M\,Y_L - Q\,A), \quad \forall Q \text{ poly. matrix}\} \end{aligned} \tag{8}$$

Let Q and Q' be matrices for which $MQ = Q'M$ and pick the solution pairs (R, S) and (R_L, S_L) corresponding to the Q and Q' matrices, respectively. Then, for these solutions, we have

$$\begin{aligned} S_L\,R - R_L\,S &= [M \quad Q'] \begin{bmatrix} X_L & Y_L \\ B & -A \end{bmatrix} \begin{bmatrix} -Y_R & A_R \\ X_R & B_R \end{bmatrix} \begin{bmatrix} M \\ Q \end{bmatrix} \\ &= [M \quad Q'] \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} M \\ Q \end{bmatrix} = MQ - Q'M = 0. \end{aligned}$$

In particular one may choose, $Q = Q' = 0$ or $Q = Q' = c\,I$.

On the other hand, for an arbitrary polynomial matrix Q and general M, there may not exist any polynomial matrix Q' such that $MQ = Q'M$. This means that for the solution pair (R, S) corresponding to Q, there may not exist any left MFD such that (6) is satisfied.

□

Remarks.

⋆**2.1.** If $M = \alpha\,I$ for some polynomial $\alpha(q^{-1})$, then every Q commutes with M. Hence, *corresponding to each solution (R, S) of $A\,R + B\,S = \alpha\,I$, there exists a solution (R_L, S_L) of $R_L A_R + S_L B_R = \alpha\,I$ such that $S_L R = R_L S$.*
We also note that in most applications, one is only concerned with the poles of M^{-1}, or more precisely, the monic invariant polynomial of M, $\lambda_M(q^{-1})$. Thus, in such cases one may take M to be of the simple form $\alpha\,I$.

⋆**2.2.** Similar to the scalar case, one can choose the degrees of the polynomial matrix S (S_L), or R (R_L), such that the solution is unique. (8) shows that S is the remainder of the right division of $Y_R M$ by A_R, while S_L is the remainder of the left division of MY_L by A. Thus, there exists a unique solution $(R,\ S)$ of $A\,R + B\,S = M$ such that ${A_R}^{-1}S$ is strictly proper. Similarly, the solution of $R_L\,A_R + S_L\,B_R = M$ is unique if we require $S_L A^{-1}$ to be strictly proper. In particular, *if A_R (A) is row (column)-reduced, the solutions are uniquely determined provided that we have $\partial_{r_i} S < \partial_{r_i} A_R$ $(\partial_{c_i} S_L < \partial_{c_i} A)$*, cf. [11].
Note that in general, these unique solutions of equations (6) do not satisfy ${R_L}^{-1} S_L = SR^{-1}$. However, as mentioned before, the $M = \alpha\,I$ case is an exception.

By considering the above arguments, we propose the following pole-placement control designs, which assign a unique pair of controller polynomials (R, S) to every set of polynomial matrices A, B, C such that A and B are left coprime:

$$u_k^* := -S\zeta_k + \lambda_{A^*}(q^{-1})z_k, \qquad \text{where} \qquad R\zeta_k = y_k \tag{9}$$

and the controller polynomials R and S and the process z are given by the following mappings:

$$\mathbf{PP1}: \qquad (R,S) = \mathcal{K}_{PP1}(A,B) \qquad \textit{and} \qquad A^* z_k = \alpha(q^{-1})\overline{y}_k \tag{10}$$

where

$$\mathcal{K}_{PP1}(A,B) = \arg\min\{\|R\|^2 + \|S\|^2 \ \mid \ A\,R + B\,S = (\alpha\,\lambda_{A^*})\,I, \qquad \partial S < \nu\}$$

or

$$\mathbf{PP2}: \qquad (R,S) = \mathcal{K}_{PP2}(A,B,C) \qquad \textit{and} \qquad A^* z_k = \overline{y}_k \tag{11}$$

where

$$\mathcal{K}_{PP2}(A,B,C) = (R,S) \qquad \ni \qquad A\,R + B\,S = \lambda_{A^*}C \qquad \textit{and} \qquad \partial_{r_i}S < \partial_{r_i}A_R,$$

for some irreducible right MFD $B_R{A_R}^{-1} = A^{-1}B$ *such that* A_R *is row-reduced. Here,* $A^*(q^{-1})$ *is the prescribed stable monic polynomial matrix,* λ_{A^*} *denotes the invariant polynomial of* A^* *and* α *is a stable monic observer polynomial chosen by the control designer. Note that* R *is always monic.*

By remark 2.2, we know that the mapping $\mathcal{K}_{PP2}$ is well-defined (solution of the Diophantine equation is unique). On the other hand, note that the set $\{(R,S) \mid A\,R + B\,S = (\alpha\,\lambda_{A^*})\,I,\ \ \partial S < \nu\}$ is nonempty and hence, the mapping $\mathcal{K}_{PP1}$ is also well-defined. In particular, $\mathcal{K}_{PP2}(A,B,\alpha\lambda_{A^*}I)$ belongs to the above-mentioned set.

It is easy to verify that the closed-loop system descriptions for the above controller designs are given respectively by

$$\begin{cases} y = R\,B\,A^{*-1}\,\overline{y} \ + \ (\alpha\,\lambda_{A^*})^{-1}R\,C\,w, & \text{for PP1} \\ y = R\,C^{-1}B\,A^{*-1}\overline{y} \ + \ {\lambda_{A^*}}^{-1}R\,w, & \text{for PP2.} \end{cases} \tag{12}$$

As we shall explain in the next section, the PP1 design requires less prior information, but does not necessarily result in asymptotic self-tuning. On the other hand, the PP2 design, requiring additional information about the plant, induces an asymptotically self-tuning APPC.

3. The Indirect APPC Algorithms

3.1. The Overlapping Parameterization and Indirect APPC Algorithms

In this paper, we use the overlapping parameterization introduced in [9] to design the indirect APPC algorithm. We formally define this parameterization by using the following result:

Theorem A. **Left & right MFD overlapping parameterizations:** (After [9]) *Consider a $p \times m$ MIMO linear time-invariant system of McMillan degree n. Then, there exist* nonunique *multiindices $\{\nu_1, \ldots, \nu_p\}$ and $\{\mu_1, \ldots, \mu_m\}$ and* unique irreducible *left and right MFDs $A^{-1}B = B_R {A_R}^{-1}$, describing the system, such that*

$$\nu_i \geq 0, \quad \mu_i \geq 0, \qquad \sum_i \nu_i = n, \qquad \sum_i \mu_i = n, \tag{13a}$$

$$\partial_{c_i} A \leq \nu_i, \qquad \partial B \leq \max_i \nu_i \qquad \text{and} \qquad \partial_{r_i} A_R \leq \mu_i, \qquad \partial B_R \leq \max_i \mu_i. \tag{13b}$$

We call these parameterizations left and right MFD overlapping parameterizations and the corresponding multiindices the left and right MFD overlapping multiindices, respectively.

Moreover, if $\mathcal{L}_n$ is the $(p+m)n$ manifold of $p \times m$ MIMO linear time-invariant systems of McMillan degree n, and
$\mathcal{L}_{n;\, \nu_1,\ldots,\nu_p;\, \mu_1,\ldots,\mu_m}$ is the submanifold of n-th order systems with left and right MFD overlapping multiindices $\{\nu_i\}$ and $\{\mu_i\}$ respectively, then

$$\mathcal{L}_{n;\, \nu_1,\ldots,\nu_p;\, \mu_1,\ldots,\mu_m} \quad \text{is dense in} \quad \mathcal{L}_n.$$

□

Thus, for every multivariable LTI system of McMillan degree n, we may choose multiindices $\{\nu_i\}$ and $\{\mu_i\}$ arbitrarily as long as (13a) is satisfied. Then, these multiindices are at least the left and right MFD overlapping multiindices of an arbitrarily close approximation of the plant. This means that the left and right overlapping multiindices can be guessed as long as an accurate estimate of n, the McMillan degree of the system, exists.

Therefore, we have the following assumption regarding the required prior information:

P1) *n : the Smith-McMillan degree of the transfer function $\mathbf{H}(q)$ (or equivalently $[\mathbf{H}(q)\,;\ \mathbf{G}(q)]$) is known. Without loss of generality, we also assume that one left MFD overlapping multiindex of $[\mathbf{H}(q)\,;\ \mathbf{G}(q)]$ and one right MFD overlapping multiindex of $\mathbf{H}(q)$, denoted respectively by $\{\nu_1, \ldots, \nu_p\}$ and $\{\mu_1, \ldots, \mu_m\}$ and satisfying (13a-b), are known a priori.*

Using the above parameterization, we can write (2) in the following regression model:

$$y_k = \theta^{*T} \phi_k^{\epsilon} + w_k, \tag{14}$$

where

$$\theta^{*T} = [-A^1_{(1)}, \ldots, -A^{\nu_1}_{(1)}, \ldots, -A^1_{(p)}, \ldots, -A^{\nu_p}_{(p)}, B^1, \ldots, B^{\vartheta}, C^1, \ldots, C^{\vartheta}]_{p \times (n+\nu(m+p))},$$

$$\phi_k^{\epsilon\,T} = [y^T_{1_{k-1}}, \ldots, y^T_{1_{k-\nu_1}}, \ldots, y^T_{p_{k-1}}, \ldots, y^T_{p_{k-\nu_p}}, u^T_{k-1}, \ldots, u^T_{k-\vartheta}, w^T_{k-1}, \ldots, w^T_{k-\vartheta}]_{1 \times (n+2\nu(m+p))},$$

$\nu = \max_i \nu_i$, $X_{(i)}$ is the i-th column of X and x_{i_k} the i-th element of x_k. As explained before, this parameterization is unique and furthermore, A and B are left coprime by **A3**.

Henceforth, we use the notation θ^* for both the parameter matrix and the corresponding set of polynomial matrices (A, B, C).

Remark $\star$3.1.

Note that the above parameterization is overparameterized, because it is a well-known fact that the set of $p' \times m'$ linear time-invariant systems of McMillan degree n forms a $(p' + m')n$ manifold and hence, can be parameterized by a minimum of $(p' + m')n$ parameters, cf. [3]. For system (1), this gives a minimum number of $n(2p + m)$ parameters. On the other hand, since $\nu \geq \frac{n}{p}$, the number of parameters used here $p(n + \nu(m + p)) \geq n(2p + m)$ and is equal only if $\nu = \frac{n}{p}$.
An implication of this fact is that except when $\nu_i = \frac{n}{p}$ for all i, every neighborhood of θ^* in $\mathcal{R}^{(n+\nu(m+p))\times p}$ includes systems with McMillan degrees greater than n.

The indirect, or *certainty-equivalence*, adaptive pole-placement controller is then given by

$$u_k^* := -\widehat{S}_k \zeta_k + \lambda_{A^*}(q^{-1}) z_k, \qquad \text{where} \qquad \widehat{R}_k \zeta_k = y_k, \tag{15a}$$

where by using (10) or (11),

$$(\widehat{R}_k, \widehat{S}_k) = \mathcal{K}_{PP1}(\widehat{A}_k, \widehat{B}_k) \qquad \textit{or} \qquad (\widehat{R}_k, \widehat{S}_k) = \mathcal{K}_{PP2}(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k) \tag{15b}$$

and z is given by the pole-placement control design laws (10) or (11). $\hat{\theta}_k$ is of course the estimate of the parameter matrix θ^*, which we shall identify with the corresponding set of time-dependent polynomial matrices $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k)$. We also note that the absolute continuity assumption in **A1** ensures that at every finite $k > 0$, $(\widehat{A}_k, \widehat{B}_k)$ is a left coprime pair with probability 1 and $\widehat{R}_k$ and $\widehat{S}_k$ are well-defined. See [12].

Remark $\star$3.2

Solving the Diophantine equations $\overline{A}\,\overline{R} + \overline{B}\,\overline{S} = \alpha \lambda_{A^*} I$ or $\overline{A}\,\overline{R} + \overline{B}\,\overline{S} = \lambda_{A^*} \overline{C}$ for $\overline{R}$ and $\overline{S}$ is equivalent to solving the following set of linear equations:

$$\Pi_{(\overline{A},\overline{B})} \cdot Z_{(\overline{R},\overline{S})} = \Xi(\bar{\theta}), \tag{16a}$$

where Π is the generalized Sylvester's matrix depending on the coefficients of $\overline{A}$ and $\overline{B}$, Z is the unknown vector of coefficients of $\overline{R}$ and $\overline{S}$ and $\Xi(\bar{\theta})$ is the vector of coefficients of the RHS of the Diophantine equation, in a certain order. Furthermore, with the parameterization (14), the dimension of Π is

$$\begin{cases} (p^2 \max\{\eta, \nu + n - 1\}) \times (p^2 \max\{\eta, \nu + n - 1\} + (m - 1)pn) & \text{for PP1,} \\ (p^2 \max\{\eta, \nu + \mu - 1\}) \times (p^2 \max\{\eta, \nu + \mu - 1\}) & \text{for PP2.} \end{cases} \tag{16b}$$

where $\mu = \max_i \mu_i$ and η, degree of the RHS of the Diophantine equation, is equal to $\partial\alpha + \partial\lambda_{A^*}$ for the PP1, and $\partial\lambda_{A^*} + \nu$ for the PP2 design.
Here, we have assumed that $\partial_{r_i} A_R = \mu_i$ for all i, in the definition of $\mathcal{K}_{PP_2}$ mapping. This is true if and only if $\partial \det A = \partial \det A_R = n$ or in other words, the transfer function $[\mathbf{H} \;\; \mathbf{G}]$ has no poles at zero. This, for example, is always true if the discrete-time system is the discretization of a physical continuous-time plant. In that case, A and A_R have to be column and row-reduced respectively and $\partial_{c_i} A_R = \nu_i$ for all i, as well. By using this assumption in the PP2 case, the following claim holds:

Lemma 2. *There exists a bounded open neighborhood of θ^*, $\Theta^o \subseteq \mathcal{R}^{(n+\nu(m+p))\times p}$, such that* $\inf_{\bar{\theta}\in\Theta^o} \left|\det\left(\Pi_{(\overline{A},\overline{B})}\Pi^T_{(\overline{A},\overline{B})}\right)\right| > 0$, *where $\bar{\theta}$ corresponds to the set of polynomial matrices $(\overline{A}, \overline{B}, \overline{C})$.*
Hence, the mapping $\mathcal{K}_{PP_1}$, or $\mathcal{K}_{PP_2}$, is uniformly continuous over Θ^o.

Proof. A and B being left coprime means that $\Pi_{(A,B)}$ is of full row rank for the PP1 case. For the PP2 design, the assumption made in remark 3.2 implies that $\Pi_{(A,B)}$ is nonsingular. Therefore, there exists a bounded neighborhood of θ^*, Θ^o, where $\Pi_{(\overline{A},\overline{B})}\Pi^T_{(\overline{A},\overline{B})}$ is nonsingular on the closure. Noting that

$$\begin{cases} Z_{(\overline{A},\overline{B})} = \Pi^T_{(\overline{A},\overline{B})}\left(\Pi_{(\overline{A},\overline{B})}\Pi^T_{(\overline{A},\overline{B})}\right)^{-1}\Xi & \text{for PP1} \quad \text{and} \\ Z_{(\overline{A},\overline{B})} = \Pi^{-1}_{(\overline{A},\overline{B})}\Xi(\bar{\theta}) & \text{for PP2,} \end{cases} \tag{16c}$$

where $\Pi_{(\overline{A},\overline{B})}$ and $\Xi(\bar{\theta})$ are continuous functions of $\bar{\theta}$, it is clear that $\mathcal{K}_{PP_1}$, or $\mathcal{K}_{PP_2}$, are uniformly continuous over Θ^o. □

3.2 The SG Identification Algorithm and Potential Self-Tuning

In this paper, we consider the Stochastic Gradient (SG) algorithm for estimation of the plant parameter matrix θ^*. The SG estimate-update equation with a posteriori prediction error, corresponding to the linear regression equation (14), is then given by

$$\hat{\theta}_k = \hat{\theta}_{k-1} + r_k^{-1}\phi_k(y_k^T - \phi_k^T\hat{\theta}_{k-1}), \tag{17}$$

where $r_k := \bar{r} + \sum_{i=0}^k \phi_i^T\phi_i$ for some $\bar{r} > 0$ and

$$\phi_k := \phi_k^\epsilon \Big|_{w_k \to \widehat{w}_k}, \qquad \widehat{w}_k = y_k - \hat{\theta}_k^T\phi_k.$$

Notation:

Let $u(\bar{\theta})$ and $y(\bar{\theta})$ be the input and output of the plant under certainty-equivalence (indirect) adaptive control if the parameter estimates are constant or "frozen" at $\bar{\theta}$. Thus by (9), we must have

$$u_k(\bar{\theta}) = -\overline{D}\zeta_k(\bar{\theta}) + \lambda_{A^*} z_k, \qquad \overline{R}\zeta_k(\bar{\theta}) = y_k(\bar{\theta}), \tag{18}$$

where $(\overline{R}, \overline{S})$ are controller polynomial matrices computed from $\bar{\theta}$ and z is obtained from $\overline{y}$ by (10), or (11). Clearly, $u^* := u(\theta^*)$ and $y^* := y(\theta^*)$ are the "desired" input and output which one would obtain if the parameter estimates were fixed at θ^* the true parameter matrix.

Then, it is well-known that if the adaptive system is stable and $\bar{\theta} \equiv (\overline{A}, \overline{B}, \overline{C})$ is a limit point of the SG estimate $\hat{\theta}_k$, $\bar{\theta}$ satisfies the following

$$\overline{A} y_k(\bar{\theta}) - \overline{B} u_k(\bar{\theta}) - \overline{C} w_k(\bar{\theta}) = 0 \qquad \text{a.s.} \tag{19}$$

Let $\mathbf{H}^c(q, \theta^*, \bar{\theta})$ be the transfer function matrix for the closed-loop system consisting of the plant and the time-invariant controller given by (18), *i.e.* $\begin{bmatrix} y(\bar{\theta}) \\ u(\bar{\theta}) \end{bmatrix} = \mathbf{H}^c(q, \theta^*, \bar{\theta}) \begin{bmatrix} z \\ w \end{bmatrix}$. Then, (18) and (19) and definition of $u(\bar{\theta})$ and $y(\bar{\theta})$ imply that

$$\mathbf{H}^c(q, \theta^*, \bar{\theta}) \begin{bmatrix} z \\ w \end{bmatrix} = \mathbf{H}^c(q, \bar{\theta}, \bar{\theta}) \begin{bmatrix} z \\ w \end{bmatrix}. \tag{20}$$

A1 and **A4**, that is independence of $\overline{y}$, and hence z, from w and infinite order of persistency of excitation of w, then imply that

$$\mathbf{H}^c_w(q, \theta^*, \bar{\theta}) = \mathbf{H}^c_w(q, \bar{\theta}, \bar{\theta}) \qquad \text{a.s.} \qquad \text{for every process } z \text{ (or } \overline{y}).$$

By using (2), (5), (18) and (19) and the fact that R is monic, it is easy to verify that the above identity is equivalent to

$$(A\overline{R} + B\overline{S})^{-1} C = (\overline{A}\,\overline{R} + \overline{B}\,\overline{S})^{-1} \overline{C} \tag{21}$$

for all external excitations. For the PP1 control design, we only get

$$(A\overline{C} - \overline{A}C)\overline{R} + (B\overline{C} - \overline{B}C)\overline{S} = 0,$$

while for the PP2 design, (21) gives

$$(A\overline{R} + B\overline{S}) = \lambda_{A^*} C = AR + BS \qquad \Rightarrow \qquad \overline{R} = R, \quad \overline{S} = S.$$

This is of course due to the fact that the PP2 design sets the degrees of the S polynomial matrix such that $AR + BS = M$ has a unique solution for all M. But, this is not true in general for the PP1 design.

We conclude from the above discussion that if the adaptive system is stable and the parameter estimates converge, then the PP2 controller polynomials always converge to their desired values. In particular, u and y converge to u^* and y^*. We call an adaptive control algorithm with such a property, *potentially self-tuning*.

This is equivalent to saying that PP2 is potentially self-tuning regardless of the external excitation, while PP1 is not in general.
Subsequently, we prove that with appropriate projection of the SG estimates, only PP2 is asymptotically self-tuning regardless of the external excitation. Nonetheless we should add that with sufficiently persistent external excitation, the SG estimates are strongly consistent for both control designs, which also implies asymptotical self-tuning.

4. The Stability and Asymptotic Self-Tuning Results

It is known, see [17], that the SG-based adaptive control algorithm under either of PP1 or PP2 designs is stable if the parameter estimates are restricted to a bounded subset of the parameter space Θ^o, where the corresponding controller mapping is uniformly continuous over the closure of Θ^o, and $\det(\overline{A}\,\overline{R}+\overline{B}\,\overline{S})(q^{-1})$, the characteristic polynomial of $\mathbf{H}^c(q,\bar\theta,\bar\theta)$, is *uniformly stable* over Θ^o.[2]

If such a set exists, we can then modify the SG algorithm (17) so that the estimates are projected into this set. But we also want to have finite number of projections so that the asymptotic properties of the SG algorithm remain unaffected by the projections. We can achieve this property if there exists a priorly known compact convex set $\Theta \subset \Theta^o$ such that $\theta^* \in \Theta$. Lemma 2 implies that such a set exists. Therefore, the following assumptions are feasible:

P2) *For the* **PP1** *design:*
There exists a priorly known compact convex set $\Theta \subset \mathcal{R}^{(n+\nu(m+p))\times p}$ *such that* $\theta^* \in \Theta$ *and for every* $\bar\theta \in \Theta$, $\overline{A}$ *and* $\overline{B}$ *are left coprime.*

By lemma 2, $\mathcal{K}_{PP_1}$ is uniformly continuous over this set. Furthermore, $\overline{A}\,\overline{R}+\overline{B}\,\overline{S} = \alpha\lambda_{A^*}I$ is constant over Θ.
The above assumption can be very restrictive: The set of points for which $\overline{A}$ and $\overline{B}$ are not left coprime divides the parameter space into disconnected regions. Therefore, the above assumption means that one should know to which region θ^* belongs and have a good initial estimate of θ^*.

The next assumption is stronger:

P3) *For the* **PP2** *design:* $[\mathbf{H}(q)\ \ \mathbf{G}(q)]$ *has no poles at zero, and there exists a priorly known compact convex set* $\Theta \subset \mathcal{R}^{(n+\nu(m+p))\times p}$ *such that* $\theta^* \in \Theta$ *and for every* $\bar\theta \in \Theta$, $\inf_{\bar\theta\in\Theta}\left|\det(\Pi_{(\overline{A},\overline{B})})\right| > 0$, *where* Π *is introduced in (16a-b), and* $\overline{C}(z) \neq 0$ *for all* $|z| \leq 1+\tau$ *and some* $\tau > 0$.

[2] We say a polynomial $X(q^{-1},\bar\theta)$ is uniformly stable over Θ, if for every $\bar\theta\in\Theta$, $X(z,\bar\theta)\neq 0$ for all $|z|\leq 1+\tau$ for some $\tau>0$.

Lemma 2 implies that $\mathcal{K}_{PP2}$ is uniformly continuous over Θ, as well. Furthermore, $\overline{A}\,\overline{R} + \overline{B}\,\overline{S} = \lambda_{A^*}\overline{C}$ is uniformly continuous over Θ.

Assuming that the compact convex set Θ is given, one can always find an open neighborhood of Θ, namely Θ^o, where the same properties are satisfied. We then propose the following scheme where the SG algorithm has been modified by projecting the estimates into the set Θ whenever they exit the set Θ^o:

$$\widehat{\hat{\theta}}_k := \hat{\theta}_{k-1} + r_k^{-1}\phi_k(y_k^T - \hat{\theta}_{k-1}^T\phi_k) \qquad \text{and}$$
$$\hat{\theta}_k = \begin{cases} \widehat{\hat{\theta}}_k, & \text{if } \widehat{\hat{\theta}}_k \in \Theta^o \\ \arg\min\limits_{\theta\in\Theta} \|\theta - \widehat{\hat{\theta}}_k\|, & \text{if } \widehat{\hat{\theta}}_k \notin \Theta^o, \end{cases} \tag{22}$$

where $\widehat{w}_k = y_k - \widehat{\hat{\theta}}_k^T\phi_k$. Clearly, $\hat{\theta}_k$ given by the above modified algorithm will always belong to Θ^o. We have the following result regarding the number of projections:

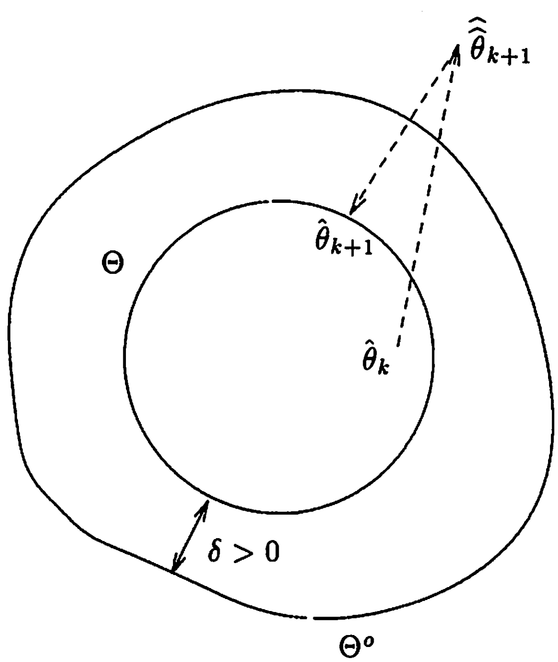

Fig. 1: The Projection Scheme (22)

Lemma 3. *Let $\delta > 0$ be the minimum distance between $\partial\Theta^o$, the boundary of Θ^o, and Θ. Then, if assumptions* **A1** *and* **A2** *are satisfied, there exists an integer-valued finite random variable $k_0 \geq 0$ such that $\hat{\theta}_k = \widehat{\hat{\theta}}_k$ for all $k \geq k_0$, i.e. there exist only a finite number of projections almost surely.*
In particular, all the asymptotic properties of the SG algorithm hold for the modified algorithm too.

Proof. See the appendix. $\square$

This means that we can ignore the projections in our asymptotic analysis of APPC algorithms. Thus, we have the following propositions:

Proposition 1. *Consider the discrete-time stochastic MIMO plant model (1) with description (2). Let assumptions* **A1-A4** *and* **P1-P2** *be satisfied and apply the indirect APPC algorithm (15a) based on the controller mapping $\mathcal{K}_{PP_1}$ (10), parameterization (14) and the modified SG algorithm (22) projecting the parameter estimates into an appropriate set Θ^o .*
Then, the following statements are true:

i) Stability: the adaptive system is stable in the mean-square sense, i.e. $u \in \mathcal{M}^2_m$ and $y \in \mathcal{M}^2_p$ almost surely, or

$$\begin{cases} \sup\limits_N \frac{1}{N}\sum_{k=0}^N \|y_k\|^2 < \infty \quad \text{a.s.} \\ \sup\limits_N \frac{1}{N}\sum_{k=0}^N \|u_k\|^2 < \infty \quad \text{a.s.} \end{cases} \tag{23a}$$

ii) Convergence of the prediction error in the mean-square sense:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N} \|y_k - \hat{\theta}_k^T\phi_k\|^2 = \|\Sigma_w\|_F^2 \qquad \text{a.s.} \tag{23b}$$

Moreover, if $\overline{y}$ is of sufficiently high order of excitation, in particular, if $\overline{y}$ is a martingale difference process independent of w and satisfying similar properties as in **A1**, *we have*

iii) Strong Consistency: $\widehat{A}_k \to A$, $\widehat{B}_k \to B$, $\widehat{C}_k \to C$ as $k \to \infty$, almost surely. In particular, the adaptive controller is asymptotically self-tuning, i.e. $u - u^* \in \mathcal{M}^2_{0_m}$ and $y - y^* \in \mathcal{M}^2_{0_p}$ almost surely.

Proof. See the appendix. $\square$

Remark 4.1. The strong consistency also holds if $\overline{y}$ is of finite but sufficient order of excitation. However, in this case $\overline{y}$ must be persistently exciting in the strong sense, *i.e.* there exist an integer $N > 0$ and a real number $\beta > 0$ such that

$$\sum_{k=k_0}^{k_0+N} \overline{Y}_k \overline{Y}_k^T > \beta I \qquad \forall k_0 \geq 0, \qquad \text{where } \overline{Y}_k^T = [\overline{y}_k^T, \ldots, \overline{y}_{k-\sigma+1}^T] \tag{24}$$

and σ is the order of excitation.

Proposition 2. *Consider the adaptive control system of proposition 1, where the APPC algorithm is now based on the controller mapping $\mathcal{K}_{PP_2}$ (11), and assumption* **P3**, *instead of* **P2**, *is satisfied.*

Then in addition to i) and ii) of proposition 1, the following result is obtained, where $\overline{y}$ is any process satisfying **A4** *regardless of its order of persistency of excitation:*

iii) Self-tuning: $\|R - \widehat{R}_k\|_F \in \mathcal{M}_0^2$, $\|S - \widehat{S}_k\|_F \in \mathcal{M}_0^2$, $y_k - y_k^* \in \mathcal{M}_{0_p}^2$ *and* $u_k - u_k^* \in \mathcal{M}_{0_m}^2$, *where* $(R, S) = \mathcal{K}_{PP_2}(A, B, C)$ *are the ideal controller polynomial matrices.*

Also, $\hat{\theta}_k \to \Omega$ *almost surely as* $k \to \infty$, *where for every* $\hat{\theta} \equiv (\widehat{A}, \widehat{B}, \widehat{C}) \in \Omega$, $\mathcal{K}_{PP_2}(\widehat{A}, \widehat{B}, \widehat{C}) = \mathcal{K}_{PP_2}(A, B, C)$.

Switching Approach: As mentioned before, assumptions **P2** and **P3** are in general restrictive. One approach to weaken these assumptions is the hysteresis switching technique introduced in [13] for deterministic systems. By using this approach, less prior information will be required:

Suppose there exists priorly known constants $M > 0$ and $\delta > 0$ such that $\|\theta^*\| \le M$ and $\left|\det(\Pi_{(A,B)}\Pi_{(A,B)}^T)\right| \ge \delta$. ($\Pi_{(A,B)}$ was introduced in §3.1, (16a).)
Then, there exists a finite number, say l, compact convex sets $\Theta_1, \ldots, \Theta_l$ such that

$$\begin{cases} \left\{ \bar{\theta} \mid \|\bar{\theta}\| \le M \quad \& \quad \left|\det(\Pi_{(\overline{A},\overline{B})}\Pi_{(\overline{A},\overline{B})}^T)\right| \ge \delta \right\} \subset \bigcup_{i=1}^l \Theta_i \qquad \text{and} \\ \bigcup_{i=1}^l \Theta_i \subset \left\{ \bar{\theta} \mid \left|\det(\Pi_{(\overline{A},\overline{B})}^T)\right| \ge \frac{\delta}{2} \right\}. \end{cases}$$

Clearly, $\theta^* \in \bigcup_{i=1}^l \Theta_i$. Without loss of generality, let $\theta^* \in \Theta_1$. Moreover, there exist open neighborhoods of $\{\Theta_i\}$, namely $\{\Theta_i^o\}$, such that $\bigcup_{i=1}^l \Theta_i^o \subset \{\left|\det(\Pi_{(\overline{A},\overline{B})}^T)\right| \ge \frac{\delta}{4}\}$.

Then, the switching technique of [13] suggests running l modified SG algorithms such as (22), for each Θ_i. Let $\{\hat{\theta}_k^i\}$ denote the resulting l SG estimates. Then, at each step k, we let

$$(\widehat{R}_k, \widehat{S}_k) = \mathcal{K}_{PP_1}(\widehat{A}_k^{i_k}, \widehat{B}_k^{i_k}),$$

where $(\widehat{A}_k^i, \widehat{B}_k^i, \widehat{C}_k^i) \equiv \hat{\theta}_k^i$ and

$$i_k = \arg\min f(\hat{\theta}^i, k)$$

for some cost function $f(\cdot,\cdot)$. For instance, we suggest

$$f(\hat{\theta}^i, N) = \frac{1}{r_N} \sum_{k=0}^{N} \|y_k - \hat{\theta}_{k-1}^{i\,T} \phi_k\|^2.$$

Note that since $n \ge 1$, $\phi_k^T = [y_{k-1}, \ldots]$. Therefore, $r_N \ge \sum_{k=0}^N \|y_{k-1}\|^2$, where $\liminf_{N\to\infty} \frac{1}{N} \sum_{k=0}^N \|y_k\|^2 \ge \|\Sigma_w\|_F^2$. In other words, $r_N \to \infty$ as $N \to \infty$.

Then, comparing $f(\hat{\theta}^i, N)$ and $f(\hat{\theta}^1, N)$, and using **A1**, uniform boundedness of the estimates and certain martingale convergence results mentioned in the appendix, we get

$$\limsup_{N\to\infty} (f(\hat{\theta}^i, N) - f(\hat{\theta}^1, N)) \geq$$
$$\limsup_{N\to\infty} \left(\frac{1}{r_N} \sum_{k=0}^{N} \|y_k - \hat{\theta}_k^{i\,T} \phi_k - w_k\|^2 - \frac{1}{r_N} \sum_{k=0}^{N} \|y_k - \hat{\theta}_k^{1\,T} \phi_k - w_k\|^2 \right). \tag{25}$$

Since $\theta^* \in \Theta_1$, the number of projections on this set is almost surely finite and thus, the asymptotic properties of the SG algorithm hold for $\hat{\theta}^1$. This means that the second term in the RHS of (25) goes to zero as $N \to \infty$. On the other hand, *if the external excitation is sufficiently persistent*, and θ^* does not belong to the closure of Θ_i^o, *i.e.* θ^* has a strictly positive distance from the set Θ_i^o, we conjecture that

$$\frac{1}{r_N} \sum_{k=0}^{N} \|y_k - \hat{\theta}_k^{i\,T} \phi_k - w_k\|^2$$

is greater than some positive constant infinitely often for almost every sample path.
If this claim is true, then for almost each sample path, there exists a finite time $\overline{k}$ such that after $\overline{k}$, the adaptive control algorithm switches forever to $\hat{\theta}^j$ estimates for a j such that $\theta^* \in \Theta_j$. In other words, $(\widehat{R}_k, \widehat{S}_k) = \mathcal{K}_{PP1}(\widehat{A}_k^j, \widehat{B}_k^j)$ for all $k \geq \overline{k}$. Now, on Θ_j, the asymptotic properties of the SG algorithm hold and thus, the same results as in proposition 1 and 2 are obtained. Identical arguments can also be used for $\mathcal{K}_{PP2}$.

5. Conclusions

In this paper, we proposed an indirect APPC algorithm for discrete-time MIMO linear stochastic systems based on overlapping parameterizations. This parameterization is globally identifiable, requires less prior information, and is more appropriate for ARMAX representations, than that of the existing pole-placement algorithms.
Assuming that the parameter estimates are projected into a desirable set, we proved stability of the SG-based APPC algorithm and showed that with sufficient amount of persistent external excitation, the parameter estimates are strongly consistent. We also showed that a slightly modified APPC algorithm is asymptotically self-tuning even in the absence of external excitation.
To weaken the projection assumption, we suggested a switching technique. Although we did not present a formal proof, we conjecture that the APPC algorithm will be stable under the proposed switching scheme. The drawback of this method is that it requires parallel estimation. It would be desirable to find more practical alternatives that can be used in stochastic adaptive control.

References

[1] M. De Mathelin , and M. Bodson, "Multivariable Adaptive Control: Identifiable Parameterizations and Parameter Convergence,", 1994. To be published in IEEE Trans. Aut. Control.

[2] P. E. Caines, and J. Rissanen, "Maximum Likelihood Estimation of Parameters in Multivariable Gaussian Stochastic Processes," *Caines-Rissanen74*, vol. IF-20, 1974, pp. 102.

[3] P. E. Caines, *Linear Stochastic Systems*, John Wiley & Sons, 1988.

[4] H. Elliot, R. Cristi, and M. Das, "Global Stability of adaptive Pole Placement Algorithms," *IEEE Trans. Aut. Control*, vol. AC-30, no. 4, 1985, pp. 348-356.

[5] H. Elliot, and W. A. Wolovich, "A Parameter Adaptive Control Structure for Linear Multivariable Systems," *IEEE Trans. Aut. Control*, vol. AC-27, no. 2, 1982, pp. 340-352.

[6] H. Elliot, W. A. Wolovich, and M. Das, "Arbitrary Adaptive Pole-Placement for Linear Multivariable Systems," *IEEE Trans. Aut. Control*, vol. AC-29, no. 3, 1984, pp. 221-229.

[7] H. Elliot, and W. A. Wolovich, "Parameterization Issues in Multivariable Adaptive Control," *Automatica*, vol. 20, no. 5, 1984, pp. 533-545.

[8] J. J. Fuchs, "Explicit Self-Tuning Methods," *IEE Proc., Pt. D*, vol. 127, no. 6, 1980, pp. 259-264.

[9] M. Gevers, and V. Wertz, "Uniquely Identifiable State-Space and ARMA Parameterizations for Multivariable Linear Systems," *Automatica*, vol. 20, no. 3, 1984, pp. 333-347.

[10] D. Janecki, "Stability Analysis of Elliot's Direct Adaptive Pole Placement," *Systems & Control Letters*, no. 11, 1988, pp. 19-26.

[11] T. Kailath, *Linear Systems*, Prentice-Hall, 1980.

[12] S. P. Meyn, and P. E. Caines, "The Zero Divisor Problem of Multivariable Stochastic Adaptive Control," *Systems & Control Letters*, vol. 6, 1985, pp. 235-238.

[13] A. S. Morse, D. Q. Mayne, and G. C. Goodwin, "Applications of Hysteresis Switching in Parameter Adaptive Control," *IEEE Trans. Aut. Control*, vol. AC-37, no. 9, 1992, pp. 1343-1354.

[14] J. Neveu, *Discrete Parameter Martingales*, North-Holland, 1975.

[15] W. Ren, and P. R. Kumar, "Stochastic Adaptive System Theory: Recent Advances and a Reappraisal," *Foundations of Adaptive Control*, edited by P. V. Kokotovic, Springer-Verlag, 1991, pp. 269-307.

[16] W. Ren, "The Self-Tuning of Stochastic Adaptive Pole-Placement Controllers," *Proc. of the 1993 IEEE Conference on Decision and Control, San Antonio, TX*, 1993, pp. 1581.

[17] C. Samson, and J.-J. Fuchs, "Discrete Adaptive Regulation of Not Necessarily Mini-

mum Phase Systems," *IEE Proc., Pt. D*, vol. 128, 1981, pp. 101-108.
[18] Y. Willner, M. Heymann, and M. Bodson, "Parameter Convergence and Uniqueness in Nonminimal Parameterizations for Multivariable Adaptive Control," *SIAM J. Control & Optimization*, vol. 30, no. 4, 1992, pp. 800-815.

Appendix:

The Asymptotic Properties of The SG Algorithm:

The following asymptotic properties hold for the SG algorithm. For a complete account see [15].

Theorem B. *Consider the regression model* $y_k = \theta^{*T}\phi_k^\epsilon + w_k$ *and assume that* **A1** *and* **A2** *are satisfied.* $\hat{\theta}_k$*, the SG estimate of* θ^* *given by (17), satisfies the following properties, where we have defined* $\widetilde{\theta}_k := \theta^* - \hat{\theta}_k$*:*

SG1) $\|\widetilde{\theta}_k\|$ *converges as* $k \to \infty$, *hence,* $\sup_k \|\hat{\theta}_k\| < \infty$ *almost surely.*

SG2) $\sum_{k=0}^{\infty} \|\hat{\theta}_k - \hat{\theta}_{k-m}\| < \infty$, *for every finite* m *almost surely.*

SG3) $\sum_{k=0}^{N} \|w_k - \widehat{w}_k\|^2 = o(r_N) + O(1)$ *almost surely.*

SG4)
$$\begin{cases} \sum_{k=0}^{N} \|\widetilde{\theta}_{k-m}^T \phi_k\|^2 = o(r_N) + O(1) & \text{for all finite } m, \\ \sum_{k=0}^{N} \|\widetilde{\theta}_{k-m}^T \phi_k^\epsilon\|^2 = o(r_N) + O(1) & \text{for all finite } m, \quad \text{and} \\ \sum_{k=0}^{N} \|\widetilde{\theta}_N^T \phi_k^\epsilon\|^2 = o(r_N) + O(1) & \text{almost surely.} \end{cases}$$

Analysis Tools:

In the subsequent proofs, we frequently use the following tools in addition to the properties of the SG algorithm. The reader can easily verify **T1**, **T2** and (C1). For a proof of **T3**, (C2) and **T4**, see for example [14]. **T5** was shown in [8].

Facts. *Let* π_k *be any sequence of* $\mathcal{R}^p$*-valued vectors and* $X_k(q^{-1})$ *and* $Y_k(q^{-1})$ *be any time-dependent polynomial matrices such that*
$\sup_k(\|X_k\|_F + \|Y_k\|_F) < \infty \qquad X_k - X_{k-1} \to 0$ *as* $k \to \infty$. *Then,*

T1)
$$\sum_{k=0}^{N} \|X_k \pi_k\|^2 = O\left(\sum_{k=0}^{N} \|\pi_k\|^2\right),$$

T2)
$$\sum_{k=0}^{N} \|X_k(Y_k \pi_k)\|^2 = \sum_{k=0}^{N} \|(XY)_k \pi_k\|^2 + o\left(\sum_{k=0}^{N} \|\pi_k\|^2\right) + O(1) \qquad \text{and}$$

$$\sum_{k=0}^{N} \|X_k \pi_k\|^2 = \sum_{k=0}^{N} \|X_{k-m} \pi_k\|^2 + o\left(\sum_{k=0}^{N} \|\pi_k\|^2\right) + O(1) \qquad \forall m \text{ finite}, \tag{C1}$$

where $(XY)_k$ is the product of X and Y treated as time-invariant polynomial matrices. In particular, if $\pi_k \in \mathcal{M}_p^2$ $(\mathcal{M}_{0_p}^2)$, then $X_k\pi_k \in \mathcal{M}_p^2$ $(\mathcal{M}_{0_p}^2)$.

Next, let $\{\pi_k\}$ be an $\mathcal{R}^p$-valued $\{\mathcal{F}_k\}$-martingale difference sequence satisfying condition similar to **A1** *and $\{h_k\}$ be an $\mathcal{R}^p$-valued $\{\mathcal{F}_{k-1}\}$-adapted random process (or a process independent of the process $\{\pi_k\}$).*
Then, $\pi_k \in \mathcal{M}_p^2$ and $\sum_{k=0}^N h_k^T \pi_k = o\left(\sum_{k=0}^N \|h_k\|^2\right) + O(1)$ almost surely. Therefore,

$$\textbf{T3)} \qquad \sum_{k=0}^{N} \|h_k + \pi_k\|^2 = (1 + o(1)) \sum_{k=0}^{N} \|h_k\|^2 + O(N) \qquad \text{a.s.}$$

Furthermore, if $\sup_k \|h_k\| < \infty$ almost surely,

$$h_k^T \pi_k \in \mathcal{M}_0^2 \qquad \Longleftrightarrow \qquad h_k \in \mathcal{M}_{0_p}^2 \qquad (C2)$$

(C2) and **T3** *imply the following, where we let $X_k(q^{-1})$ and $Y_k(q^{-1})$ be $\mathcal{F}_k$-measurable time-dependent polynomial matrices such that*
$\sup_k(\|X_k\|_F + \|Y_k\|_F) < \infty$ *and* $\|X_k - X_{k-1}\|_F + \|Y_k - Y_{k-1}\|_F \to 0$ *as* $k \to \infty$ *a.s.:*
If $\{h_k\}$ is an $\mathcal{F}_{k-n^*}$-adapted process with $n^* > \partial Y_k$ (or a process independent of $\{\pi_k\}$) such that $h_k \in \mathcal{M}^2$ almost surely, we have
T4)

$$X_k h_k + Y_k \pi_k \in \mathcal{M}_{0_p}^2 \qquad \Longleftrightarrow \qquad \begin{cases} X_k h_k \in \mathcal{M}_{0_p}^2, \\ Y_k \pi_k \in \mathcal{M}_{0_p}^2, \end{cases} \qquad \Longleftrightarrow \qquad \|Y_k\|_F \in \mathcal{M}_0^2 \qquad \text{a.s.}$$

In particular, if $Y_k = Y$ for all k, $Y\pi_k \in \mathcal{M}_{0_p}^2$ almost surely implies $Y = 0$.

(To prove **T4**, we use (C1) to approximate X_k and Y_k with X_{k-n^*} and Y_{k-n^*}, with the error being in $\mathcal{M}_0^2$. Then, by using **T1**, **T3** to show that the cross terms are in $\mathcal{M}_0^2$ almost surely and (C2), the statement is proved.)

And finally, we have the following assertion:

T5) *Any time-varying finite-dimensional system $\{x_{k+1} = \Phi_k x_k + W_k u_k, \ y_k = H_k x_k\}$, that satisfies the following 3 conditions, is $\mathcal{L}_2$-stable, i.e.*

$$\sum_{k=0}^N \|y_k\|^2 = O\left(\sum_{k=0}^N \|u_k\|^2\right) \text{ for any sequence } \{u_k\} \in \mathcal{M}^2\text{:}$$

1. $\sup_k (\|\Phi_k\| + \|W_k\| + \|H_k\|) < \infty$.
2. $\sup_k (\|\Phi_k - \Phi_{k-1}\| + \|W_k - W_{k-1}\| + \|H_k - H_{k-1}\|) \to 0$ *as* $k \to \infty$.
3. *Φ_k is uniformly stable with respect to k, which is equivalent to saying that every eigenvalue of Φ_k, for all k, is inside a disk of radius λ^* for some $\lambda^* < 1$.*

□

Proof of Lemma 3:

Let $\widetilde{\widetilde{\theta}}_k = \theta^* - \widehat{\widehat{\theta}}_k$. To prove the lemma, it is sufficient to establish the following claim:

Claim. *Let $\widehat{\widehat{\theta}} \in \Theta^{oc}$, the complement of Θ^o, and $\hat{\theta} := \arg\min_{\theta\in\Theta} \|\theta - \widehat{\widehat{\theta}}\|$. Then,*

$$\|\hat{\theta} - \theta^*\|^2 \leq \|\widehat{\widehat{\theta}} - \theta^*\|^2 - \delta,$$

where δ is the minimum distance of the boundary of Θ^o and Θ.

Proof. With no loss of generality we assume $\theta^* = 0$. Since $\widehat{\widehat{\theta}} \notin \Theta^o$ and $\Theta \subset \Theta^o$, we have $\|\widehat{\widehat{\theta}} - \hat{\theta}\| \geq \delta$. Also by definition of $\hat{\theta}$, $\|\widehat{\widehat{\theta}} - \hat{\theta}\| \leq \|\widehat{\widehat{\theta}}\|$. This implies that

$$\lambda = \frac{\langle \widehat{\widehat{\theta}}, \hat{\theta}\rangle}{\|\hat{\theta}\|^2} \geq \frac{1}{2} \qquad \text{and} \qquad 1 - \lambda = 1 - \frac{\langle \widehat{\widehat{\theta}}, \hat{\theta}\rangle}{\|\hat{\theta}\|^2} = \frac{\langle \hat{\theta} - \widehat{\widehat{\theta}}, \hat{\theta}\rangle}{\|\hat{\theta}\|^2} \leq \frac{1}{2}.$$

We claim that $1 - \lambda \leq 0$. Suppose not, then $0 < \lambda < 1$ and $\lambda\hat{\theta} + (1-\lambda)\theta^* = \lambda\hat{\theta} \in \Theta$, because Θ is convex. But $\langle \widehat{\widehat{\theta}} - \lambda\hat{\theta}, \hat{\theta}\rangle = 0$. Thus by the orthogonality principle,

$$\|\lambda\hat{\theta} - \widehat{\widehat{\theta}}\| \leq \|\hat{\theta} - \widehat{\widehat{\theta}}\|.$$

Since $\hat{\theta}$ has the minimum distance to $\widehat{\widehat{\theta}}$ in Θ, one must have $\lambda = 1$. Hence, $\lambda \geq 1$. Now,

$$\|\widehat{\widehat{\theta}}\|^2 = \|\hat{\theta}\|^2 + \|\widehat{\widehat{\theta}} - \hat{\theta}\|^2 + 2\langle \widehat{\widehat{\theta}} - \hat{\theta}, \hat{\theta}\rangle = \|\hat{\theta}\|^2 + \|\widehat{\widehat{\theta}} - \hat{\theta}\|^2 + 2(1-\lambda)\|\hat{\theta}\|^2 \leq \|\hat{\theta}\|^2 + \delta$$

as intended. □

Thus, $\|\widetilde{\theta}_k\|^2 \leq \|\widetilde{\widetilde{\theta}}_k\|^2 - 1\!\!1_{\{\widetilde{\theta}_k \neq \widetilde{\widetilde{\theta}}_k\}}\delta$ for all k, where $1\!\!1_Q$ is the indicator function for the set Q. Then, it is easy to see that

$$\sum_{k=0}^{N} \|\widetilde{\theta}_k\|^2 + N_k\delta \leq \sum_{k=0}^{N} \|\widetilde{\widetilde{\theta}}_k\|^2$$

where N_k is the number of projections up to time k. Substituting for $\widetilde{\widetilde{\theta}}_k$ from (22) and applying the same arguments which are used to prove the SG algorithm properties (see [15] for details), in particular **SG1**, one can show that there exists a finite random variable $M > 0$ such that $\|\widetilde{\theta}_k\| \leq M$ and $N_k \leq M$ for all k. This completes our proof. □

Proof of Proposition 1:

First, note that by lemma 3, we may ignore the effect of projections in our asymptotic analysis except that we have $\hat{\theta}_k \in \Theta^o$ for all k.

Θ^o is chosen such that $\mathcal{K}_{PP_1}$ is uniformly continuous over its closure. Thus by **SG1** and **SG2**, we know that every estimated or controller parameter x_k, satisfies the following:

$$\sup_k |x_k| < \infty \qquad \text{and} \qquad |x_k - x_{k-1}| \to 0 \quad \text{as} \quad k \to \infty.$$

This means that we may now apply the **T1-T5** tools.

To show *i)*, it suffices to show that $\limsup\limits_{N\to\infty} \frac{r_N}{N} < \infty$. This is trivially true if $\sup_N r_N < \infty$, therefore, with no loss of generality, we assume $\lim\limits_{N\to\infty} r_N = \infty$. Thus from SG 4, we get

$$\sum_{k=0}^{N} \left\| \widehat{A}_k y_k - \widehat{B}_k u_k - \widehat{C}_k w_k \right\|^2 = o(r_N) \qquad \text{a.s.}$$

Substituting y_k and u_k with $u_k = -\widehat{S}_k\zeta_k + z_k$ and $y_k = \widehat{R}_k\zeta_k$, where $A^* z_k = (\alpha\lambda_{A^*})\overline{y}_k$ by (15a) and (10), and using **T2** gives us

$$\sum_{k=0}^{N} \Big\| \underbrace{(\widehat{A}\widehat{R} + \widehat{B}\widehat{S})_k}_{\alpha\lambda_{A^*}} \zeta_k - \widehat{B}_k z_k - \widehat{C}_k w_k \Big\|^2 = o(r_N) + o\left(\sum_{k=0}^{N} \|\zeta_k\|^2 \right) \qquad \text{a.s.} \tag{C3}$$

Since $\alpha\lambda_{A^*}$ and A^* are stable and w, $\overline{y}$ and hence, z are in $\mathcal{M}^2$ almost surely, **SG1** implies

$$\sum_{k=0}^{N} \|\zeta_k\|^2 = O\left(\sum_{k=0}^{N} \|(\alpha\lambda_{A^*})\zeta_k\|^2 \right) = o(r_N) + O(N) \qquad \text{a.s.}$$

Considering that the coefficients of $\widehat{R}_k$ and $\widehat{S}_k$ are almost surely uniformly bounded with respect to k, we also have $\sum_{k=0}^{N} \|u_k\|^2 = O\left(\sum_{k=0}^{N} \|\zeta_k\|^2\right) + O(N)$ and $\sum_{k=0}^{N} \|y_k\|^2 = O\left(\sum_{k=0}^{N} \|\zeta_k\|^2\right)$. But $r_N \le \overline{r} + K\sum_{k=0}^{N} \left\|[y_k^T \; u_k^T \; \widehat{w}_k^T]\right\|^2$ for some positive integer, where $\sum_{k=0}^{N} \|\widehat{w}_k\|^2 = o(N)$ by **SG3**. Therefore,

$$r_N = o(r_N) + O(N) \qquad \Longrightarrow \qquad \frac{r_N}{N} = O(1) \qquad \text{a.s.}$$

Of immediate implications of this result are that ζ, y and u are all in $\mathcal{M}^2$ almost surely, hence *i)* is proved. Considering that $y_k - \hat{\theta}_k^T\phi_k = \tilde{\theta}_k^T\phi_k^\epsilon + \hat{\theta}_k(\phi_k^\epsilon - \phi_k) - w_k$, where the first two terms are $\mathcal{F}_{k-1}$-measurable and in $\mathcal{M}_{0_p}^2$ by **SG1**, **SG3** and **SG4**, we are also able to infer *ii)* from **T4**.

Next, let $\overline{y}$ be a martingale difference sequence satisfying conditions similar to **A1**. **SG3** and the fact that $\frac{r_N}{N} = O(1)$ almost surely imply that

$$\sum_{k=0}^{N} \left\| \tilde{\theta}_N^T \phi_k^\epsilon \right\|^2 = \sum_{k=0}^{N} \left\| \widehat{A}_N y_k - \widehat{B}_N u_k - \widehat{C}_N w_k \right\|^2 = o(N) \qquad \text{a.s.}$$

On the other hand, consider the plant model (2). It is clear that (2) is equivalent to

$$\det A_R y_k = B_R \mathrm{Adj}(A_R)\, u_k + C_R \mathrm{Adj}(A_R)\, w_k + e_k, \tag{C4}$$

where $\mathrm{Adj}(X) = \det X\, X^{-1}$ for any matrix X and e here represents the effect of initial conditions. Since the system is stable in the sense of statement *i)*, we deduce that $e_k \in \mathcal{M}_p^2$ almost surely. Thus, multiplying $\widetilde{\theta}_N^T \phi_k^\epsilon$ by $\det A_R$ and substituting for y from (C4) gives

$$\sum_{k=0}^{N} \left\| \left(\widehat{A}_N B_R - \widehat{B}_N A_R\right) \mathrm{Adj}(A_R) u_k + \left(\widehat{A}_N C_R - \widehat{C}_N A_R\right) \mathrm{Adj}(A_R) w_k + e_k' \right\|^2 = o(N) \tag{C5}$$

almost surely, where e_k' is due to initial conditions and is in $\mathcal{M}_p^2$ almost surely. But $u_k = -\widehat{S}_k \zeta_k + z_k$, where $(\alpha\lambda_{A^*})\zeta_k - \widehat{B}_k z_k - \widehat{C}_k w_k \in \mathcal{M}_{0_p}^2$ almost surely by (C3). Multiplying the argument of square-sum in (C5) by $\alpha\lambda_{A^*}$ and using these equalities to substitute for $(\alpha\lambda_{A^*})u_k$, we arrive at

$$\sum_{k=0}^{N} \Big\| \left(\widehat{A}_N B_R - \widehat{B}_N A_R\right) \mathrm{Adj}(A_R)(\alpha\lambda_{A^*} I - \widehat{S}_k \widehat{B}_k)\, z_k +$$
$$\left[\alpha\lambda_{A^*} \left(\widehat{A}_N C_R - \widehat{C}_N A_R\right) \mathrm{Adj}(A_R) - \left(\widehat{A}_N B_R - \widehat{B}_N A_R\right) \mathrm{Adj}(A_R) \widehat{S}_k \widehat{C}_k\right] w_k + e_k'' \Big\|^2 = o(N)$$

almost surely, where e'' is due to initial conditions and is in $\mathcal{M}_p^2$ almost surely. By independence of e'', $\overline{y}$, and hence, z from w, and **T4'**, an extension of **T4** mentioned following this proof, the above is equivalent to

$$\sum_{k=0}^{N} \left\| \left(\widehat{A}_N B_R - \widehat{B}_N A_R\right) \mathrm{Adj}(A_R)(\alpha\lambda_{A^*} I - \widehat{S}_k \widehat{B}_k)\, z_k \right\|^2 = o(N) \tag{C6}$$
$$\sum_{k=0}^{N} \left\| \alpha\lambda_{A^*} \left(\widehat{A}_N C_R - \widehat{C}_N A_R\right) \mathrm{Adj}(A_R) - \left(\widehat{A}_N B_R - \widehat{B}_N A_R\right) \mathrm{Adj}(A_R) \widehat{S}_k \widehat{C}_k \right\|^2 = o(N)$$

almost surely. Note that $A^* z_k = (\alpha\lambda_{A^*})\overline{y}_k$ indicates that $\det A^*\, z_k = (\alpha\lambda_{A^*})\mathrm{Adj}(A^*)\overline{y}_k$. Therefore, if we multiply the argument of square-sum in the first equality of (C6) by $\det A^*$, substitute for $\det A^*\, z_k$ and use **T4'**, with the martingale difference process in this case being $\overline{y}$, we get

$$\sum_{k=0}^{N} \left\| \left(\widehat{A}_N B_R - \widehat{B}_N A_R\right) \mathrm{Adj}(A_R)(\alpha\lambda_{A^*} I - \widehat{S}_k \widehat{B}_k) \right\|^2 = o(N) \qquad \text{a.s.} \tag{C7}$$

Since $\mathrm{Adj}(A_R)(\alpha\lambda_{A^*} I - \widehat{S}_k \widehat{B}_k)$ is a monic polynomial matrix, one can easily show that (C7) implies that $\lim_{N\to\infty}(\widehat{A}_N B_R - \widehat{B}_N A_R) = 0$ or $\lim_{N\to\infty} \widehat{A}_N^{-1} \widehat{B}_N = B_R A_R^{-1} = A^{-1}B$. Now, as mentioned in §3.1, the parameterization (14) is globally identifiable at θ^*, thus

$\lim_{N\to\infty}\widehat{A}_N = A$ and $\lim_{N\to\infty}\widehat{B}_N = B$. Applying this result to the second equality of (C6) would result in $\lim_{N\to\infty}\widehat{C}_N = AC_R {A_R}^{-1} = C$.
Because $\hat{\theta}_N \to \theta^*$, self-tuning is trivially achieved, that is

$$(\widehat{R}_k, \widehat{S}_k) = \mathcal{K}_{PP1}(\widehat{A}_k, \widehat{B}_k) \quad \longrightarrow \quad \mathcal{K}_{PP1}(A, B) = (R, S) \qquad \text{as } k\to\infty \qquad \text{a.s..}$$

Using the above and stability of $A\,R + B\,S = \alpha\lambda_{A^*}$, we also get $u - u^* \in \mathcal{M}^2_{0_m}$ and $y - y^* \in \mathcal{M}^2_{0_p}$ almost surely. □

In the above proof, we used the following extension of **T4**:
Let $\{\pi_k\}$ and $\{h_k\}$ be the same processes defined for **T4** *and X^1_k, X^2_k, Y^1_k and Y^2_k be time-dependent polynomial matrices satisfying similar conditions. Then,*
T4′)

$$\sum_{k=0}^{N} \left\|X^1_N X^2_k h_k + Y^1_N Y^2_k \pi_k\right\|^2 = o(N) \qquad \Longleftrightarrow \qquad \begin{cases} \sum_{k=0}^{N} \left\|X^1_N X^2_k h_k\right\|^2 = o(N) \\ \sum_{k=0}^{N} \left\|Y^1_N Y^2_k \pi_k\right\|^2 = o(N) \end{cases},$$

$$\text{where} \qquad \sum_{k=0}^{N} \left\|Y^1_N Y^2_k \pi_k\right\|^2 = o(N) \qquad \Longleftrightarrow \qquad \sum_{k=0}^{N} \left\|Y^1_N Y^2_k\right\|^2_F = o(N) \qquad \text{a.s.}$$

To establish this fact, we use a combination of proof of (C2) and the following lemma:
Lemma 4. *Let $\{\pi_k\}$ be an $\mathcal{R}^p$-valued $\{\mathcal{F}_k\}$-martingale difference sequence satisfying condition similar to* **A1** *and $\{x_N\}$ be an $\mathcal{R}^p$-valued $\{\mathcal{F}_N\}$-adapted random process such that x_N is uniformly bounded almost surely. Then,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N} (x_N^T \pi_k)^2 = 0 \qquad \Longleftrightarrow \qquad \lim_{N\to\infty} x_N = 0 \qquad \text{a.s.}$$

Proof. Suppose $x_N \not\to 0$ as $N\to\infty$ with probability $p > 0$. Then, there exists $\epsilon > 0$ such that for all N, there exists $N' \geq N$ such that $\|x_{N'}\| \geq \epsilon$.
On the other hand, $\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N}\pi_k\pi_k^T = \Sigma_\pi\Sigma_\pi^T > 0$ almost surely, meaning that there exists $N_0 > 0$ such that for all $N \geq N_0$

$$x^T\left(\Sigma_\pi\Sigma_\pi^T - \frac{1}{2}\lambda_{\min}(\Sigma_\pi\Sigma_\pi^T) I\right)x \leq x^T\Big(\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N}\pi_k\pi_k^T\Big)x, \qquad \forall x\in\mathcal{R}^p$$

almost surely, where $\lambda_{\min}(R)$ denotes the minimum eigenvalue of matrix R. Combining the above two statements implies that for all $N \geq N_0$, there exists $N' \geq N$ such that

$$\begin{aligned} \frac{1}{N'}\sum_{k=0}^{N'}(x_{N'}^T\pi_k)^2 &= \|\Sigma_\pi^T x_{N'}\|^2 + x_{N'}^T\Big(\frac{1}{N'}\sum_{k=0}^{N'}\pi_k\pi_k^T - \Sigma_\pi\Sigma_\pi^T\Big)x_{N'} \\ &\geq \lambda_{\min}(\Sigma_\pi\Sigma_\pi^T)\|x_{N'}\|^2 - \frac{1}{2}\lambda_{\min}(\Sigma_\pi\Sigma_\pi^T)\|x_{N'}\|^2 \geq \frac{1}{2}\lambda_{\min}(\Sigma_\pi\Sigma_\pi^T)\epsilon^2 > 0 \end{aligned}$$

with probability p. This means that $\frac{1}{N}\sum_{k=0}^{N}(x_N^T\pi_k)^2 \not\to 0$ as $N \to \infty$ with at least probability $p > 0$, clearly a contradiction. $\square$

Proof of Proposition 2:

Proofs of statements *i)* and *ii)* are similar to those stated for proposition 1. Again, ignoring the projections in our asymptotic analysis and by using the fact that $\mathcal{K}_{PP_2}$ is uniformly continuous over the closure of Θ^o, **SG1** and **SG2**, we conclude that $\widehat{C}_k$ is uniformly stable with respect to k and

$$\sup_k |x_k| < \infty \qquad \text{and} \qquad |x_k - x_{k-1}| \to 0 \quad \text{as} \quad k \to \infty$$

for every estimated or controller parameter x_k. Therefore, **T1-T5** are again applicable. A similar procedure as in the proof of proposition 1 then gives us

$$\sum_{k=0}^{N}\Big\|\underbrace{(\widehat{A}_k\widehat{R}_k + \widehat{B}_k\widehat{S}_k)}_{\lambda_{A^*}\widehat{C}_k}\zeta_k - \widehat{B}_k z_k - \widehat{C}_k w_k\Big\|^2 = o(r_N) + o\left(\sum_{k=0}^{N}\|\zeta_k\|^2\right) \qquad \text{a.s.,} \qquad (C8)$$

where $A^* z_k = \lambda_{A^*}\overline{y}_k$ and by stability of A^* and **A4**, $z_k \in \mathcal{M}^2$.

Hence, $\sum_{k=0}^{N}\left\|\lambda_{A^*}\widehat{C}_k\zeta_k\right\|^2 = o(r_N) + o\left(\sum_{k=0}^{N}\|\zeta_k\|^2\right) + O(N)$ by **T1** and **A1**.

Considering that $\widehat{C}_k$ is uniformly stable with respect to k, $\|\widehat{C}_k - \widehat{C}_{k-1}\| \to 0$ as $k \to \infty$ by **SG2** and $\sup_k \|\widehat{C}_k\| < \infty$ by **SG1**, we may use **T5** to conclude that $\sum_{k=0}^{N}\|\zeta_k\|^2 = o(r_N) + O(N)$. Then, using arguments identical to those used in the proof of proposition 1, we find that $\frac{r_N}{N} = O(1)$ and that ζ, y and u are all in $\mathcal{M}^2$ almost surely. Therefore, the system is mean-square stable and *i)* is true. *ii)* is proved in the same way as in proposition 1.

Regarding iii), we note that (C8) now becomes $\lambda_{A^*}\widehat{C}_k\zeta_k - \widehat{B}_k u_k - \widehat{C}_k w_k \in \mathcal{M}_0^2$ almost surely. Multiplying this by $\mathrm{Adj}(\widehat{C}_k)$ and using **SG1**, **T1**, **T2** and the fact that ζ, w and z are in $\mathcal{M}^2$ almost surely, we obtain

$$\lambda_{A^*} \det \widehat{C}_k\zeta_k - \mathrm{Adj}(\widehat{C}_k)\widehat{B}_k z_k - \det \widehat{C}_k w_k \in \mathcal{M}_0^2 \qquad \text{a.s.} \qquad (C9)$$

On the other hand, we have $Ay_k - Bu_k - Cw_k = 0$. By using the controller equation (15a) to substitute for y_k and u_k, this equation becomes

$$(A\widehat{R}_k + B\widehat{S}_k)\zeta_k - Bu_k - Cw_k = 0. \qquad (C10)$$

Multiplying (C9) by $A\widehat{R}_k + B\widehat{S}_k$ and (C10) by $-\lambda_{A^*} \det \widehat{C}_k$ and adding the two, using **T1**, **T2** and the fact ζ, w and z are in $\mathcal{M}^2$ to simplify the result, we find

$$\Big[(A\widehat{R} + B\widehat{S})\mathrm{Adj}(\widehat{C})\widehat{B} - \lambda_{A^*} \det \widehat{C}\, B\Big]_k SA_k z_k +$$
$$\Big[(A\widehat{R} + B\widehat{S}) \det \widehat{C} - \lambda_{A^*} \det \widehat{C}\, C\Big]_k w_k \in \mathcal{M}_0^2 \qquad \text{a.s.}$$

Then, by independence of w and $\overline{y}$, hence w and z, and using **T4**, we can conclude that $(A\widehat{R}_k + B\widehat{S}_k - \lambda_{A^*} C) \det \widehat{C}_k \in \mathcal{M}_0^2$ almost surely. Since $\det \widehat{C}_k$ is monic, this is equivalent to

$$\left\| A(\widehat{R}_k - R) + B(\widehat{B}_k - B) \right\|_F \in \mathcal{M}_0^2 \qquad \text{a.s.} \tag{C11}$$

Note that $\|X_k\|_F \in \mathcal{M}_0^2$ for a time-dependent polynomial matrix X_k, if and only if Frobenius norms of all the coefficients of X_k are in $\mathcal{M}_0^2$ almost surely. Clearly, one can write $A(\widehat{R}_k - R) + B(\widehat{B}_k - B) = [1\, q^{-1} \ldots]\Pi_{(A,B)} \widetilde{Z}_k$ where elements of the matrix Π are the elements of coefficients of A and B polynomial matrices, while $\widetilde{Z}_k$ is a vector of elements of coefficients $\widehat{R}_k - R$ and $\widehat{S}_k - S$. Thus, (C11) is equivalent to saying that $\Pi \widetilde{Z}_k \in \mathcal{M}_0^2$. But on Θ^o, $\Pi_{(A,B)}$ is nonsingular. Therefore, $\left\|\widetilde{Z}_k\right\|^2 \leq L \left\|\Pi \widetilde{Z}_k\right\|^2$ for some $L > 0$ and consequently, $\widetilde{Z}_k \in \mathcal{M}_0^2$, *i.e.* $\left\|R - \widehat{R}_k\right\|_F \in \mathcal{M}_0^2$ and $\left\|S - \widehat{S}_k\right\|_F \in \mathcal{M}_0^2$ almost surely. Now, let ζ^* be the process satisfying $y_k^* = R\zeta_k^*$ and $u_k^* = -S\zeta_k^* + z_k$, where y^* and u^* satisfy $Ay_k^* - Bu_k^* - Cw_k = 0$. Thus, we have

$$\lambda_{A^*} C\zeta_k^* - Bz_k - Cw_k = 0. \tag{C12}$$

Subtracting (C10) from the above equation yields

$$\left(A(R - \widehat{R}_k) + B(S - \widehat{S}_k)\right)\zeta_k \; + \; (AR + BS)(\zeta_k^* - \zeta_k) = 0.$$

Since $AR + BS = \lambda_{A^*} C$ is stable, the above equality implies

$$\begin{aligned}
\sum_{k=0}^{N} \|\zeta_k - \zeta_k^*\|^2 &= O\left(\sum_{k=0}^{N} \left\| \left(A(R - \widehat{R}_k) + B(S - \widehat{S}_k)\right)\zeta_k \right\|^2\right) = o(N) \\
&= O\left(\sum_{k=0}^{N} \left\| \left(A(R - \widehat{R}_k) + B(S - \widehat{S}_k)\right)\lambda_{A^*} \det \widehat{C}_k \zeta_k \right\|^2\right) \qquad \text{a.s.,}
\end{aligned}$$

where we have used **T5** and the fact that $\det \widehat{C}_k$ is uniformly stable. By using (C9), the above equation can be written as

$$\begin{aligned}
\sum_{k=0}^{N} \|\zeta_k - \zeta_k^*\|^2 = O\Big(\sum_{k=0}^{N} \|(A(R - \widehat{R}_k) + B(S - \widehat{S}_k))\mathrm{Adj}(\widehat{C}_k)\widehat{B}_k z_k\|^2\Big) + \\
O\Big(\sum_{k=0}^{N} \|(A(R - \widehat{R}_k) + B(S - \widehat{S}_k)) \det \widehat{C}_k w_k\|^2\Big) \; + \; o(N) \qquad \text{a.s.,}
\end{aligned}$$

which by **T4** and the fact that $\overline{y}$ and hence, z are uniformly bounded, means that $\zeta_k - \zeta_k^* \in \mathcal{M}_{0_p}^2$ almost surely. Now, $y_k - y_k^* = \widehat{R}_k(\zeta_k - \zeta_k^*) + (\widehat{R}_k - R)\zeta_k^*$ and $u_k - u_k^* = \widehat{S}_k(\zeta_k^* - \zeta_k) +$

$(S-\widehat{S}_k)\zeta_k^*$, where the parameters are almost surely uniformly bounded by **SG1**. Since $\lambda_{A^*}\det C$ is stable, $\sum_{k=0}^N \left\|(\widehat{R}_k - R)\zeta_k^*\right\|^2 = O\left(\sum_{k=0}^N \left\|(\widehat{R}_k - R)\lambda_{A^*}\det C\zeta_k^*\right\|^2\right)$, where

$$(\widehat{R}_k - R)\lambda_{A^*}\det C\zeta_k^* = (\widehat{R}_k - R)\mathrm{Adj}(C)Bz_k + (\widehat{R}_k - R)\det Cw_k \in \mathcal{M}_{0_p}^2 \qquad \text{a.s.}$$

by (C7), **T4** and the fact that z is uniformly bounded. The conclusion is that $y_k - y_k^* \in \mathcal{M}_{0_p}^2$ almost surely. Similarly, $u_k - u_k^* \in \mathcal{M}_{0_m}^2$ almost surely.
Using these estimates and the fact that by **SG4**,

$$\sum_{k=0}^N \left\|\widetilde{\theta}_N^T\phi_k^\epsilon\right\|^2 = \sum_{k=0}^N \left\|\widehat{A}_N y_k - \widehat{B}_N u_k - \widehat{C}_N w_k\right\|^2 = o(r_N) + O(1) = o(N) \qquad \text{a.s.,}$$

we arrive at

$$\sum_{k=0}^N \left\|\widehat{A}_N y_k^* - \widehat{B}_N u_k^* - \widehat{C}_N w_k\right\|^2 \le 3\Big(\sum_{k=0}^N \left\|\widehat{A}_N(y_k^* - y_k)\right\|^2 + \sum_{k=0}^N \left\|\widehat{B}_N(u_k^* - u_k)\right\|^2$$
$$+\sum_{k=0}^N \left\|\widehat{A}_N y_k - \widehat{B}_N u_k - \widehat{C}_N w_k\right\|^2\Big) = o(N) \qquad \text{a.s.,}$$

which is equivalent to saying that

$$\sum_{k=0}^N \left\|\widetilde{\theta}_N^T\phi_k^*\right\|^2 = o(N) \qquad \Longrightarrow \qquad \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^N \phi_k^*\phi_k^{*T}\widetilde{\theta}_N = 0 \qquad \text{a.s.}$$

Now, since y^* and u^* correspond to a stable linear time-invariant closed-loop system driven by second-order ergodic inputs $\overline{y}$ and w, $\frac{1}{N}\sum_{k=0}^N \phi_k^*\phi_k^{*T}$ converges to a finite limit Γ almost surely. Hence, the above equality suggests that $\widetilde{\theta}_N$ converges to $\Omega' = \mathcal{N}(\Gamma)$ almost surely as $N \to \infty$. Let $\widetilde{\theta} = \theta^* - \hat{\theta}$ be an arbitrary point in Ω'. Then, $\Gamma\widetilde{\theta} = 0$ or equivalently, $\widetilde{\theta}^T\phi_k^* = \widehat{A}y_k^* - \widehat{B}u_k^* - \widehat{C}w_k \in \mathcal{M}_{0_p}^2$, which after substituting for y_k^* and u_k^* becomes

$$(\widehat{A}R + \widehat{B}S)\zeta_k^* - \widehat{B}z_k - \widehat{C}w_k \in \mathcal{M}_{0_p}^2 \qquad \text{a.s..} \qquad (C13)$$

Multiplying (C13) by $\lambda_{A^*}\det C$ and (C12) by $-(\widehat{A}R + \widehat{B}S)\mathrm{Adj}(C)$ and adding the two results in

$$\left[(\widehat{A}\,R + \widehat{B}\,S)\mathrm{Adj}(C)B - \lambda_{A^*}\det C\widehat{B}\right]z_k + \left[(\widehat{A}\,R + \widehat{B}\,S) - \lambda_{A^*}\widehat{C}\right]\det Cw_k \in \mathcal{M}_0^2$$

almost surely. **T4** then implies that $\widehat{A}R + \widehat{B}S = \lambda_{A^*}\widehat{C}$. Since $\hat{\theta} \in \Theta^o$ due to projections, this Diophantine equation has a unique solution, *i.e.*

$$\widehat{R} = R \qquad \text{and} \qquad \widehat{S} = S \qquad \Longrightarrow \qquad \mathcal{K}_{PP_2}(\widehat{A},\widehat{B},\widehat{C}) = \mathcal{K}_{PP_2}(A,B,C).$$

$\square$