TIME DOMAIN NON-MONTE CARLO NOISE SIMULATION FOR NONLINEAR DYNAMIC CIRCUITS WITH ARBITRARY EXCITATIONS

by

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Abstract

A new, *time-domain*, *non-Monte Carlo* method for computer simulation of *electrical noise* in *nonlinear dynamic circuits* with *arbitrary excitations* is presented. This time-domain noise simulation method is based on the results from the theory of *stochastic differential equations*. The noise simulation method is general in the sense that any *nonlinear dynamic circuit* with *any kind of excitation*, which can be simulated by the *transient analysis* routine in a *circuit simulator*, can be simulated by our noise simulator in time-domain to produce the noise variances and covariances of circuit variables as a function of time, provided that noise models for the devices in the circuit are available. Noise *correlations* between circuit variables at *different time points* can also be calculated. Previous work on computer simulation of noise in integrated circuits is reviewed with comparisons to our method. Shot, thermal and flicker noise models for integrated-circuit devices, in the context of our time-domain noise simulation method, are described. This noise simulation method was implemented in a circuit simulator, namely SPICE. Using this implementation, two examples of noise simulation (a MOSFET ring-oscillator and a BJT active mixer) are given.

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Chapter 1

Introduction

This paper presents a new, *time-domain*, *non-Monte Carlo* method for computer simulation of *electrical noise* in *nonlinear dynamic circuits* with *arbitrary excitations*. This time-domain noise simulation method is based on the results from the theory of *stochastic differential equations*. The noise phenomena considered in this work are caused by the small current and voltage fluctuations that are generated within the integrated-circuit devices themselves. Noise caused by the extraneous pickup of man-made signals is excluded. The existence of noise is basically due to the fact that electrical charge is not continuous but is carried in discrete amounts equal to the electron charge. Electrical noise is associated with fundamental processes in integrated-circuit devices [1].

The importance of noise simulation, probably, does not need elucidation. Noise represents a lower limit to the size of electrical signal that can be amplified by a circuit without significant deterioration in signal quantity. It also results in an upper limit to the useful gain of an amplifier, because if the gain is increased without limit the output stages of the circuit will eventually begin to cut off or saturate on the amplified noise from the input stages [1]. The influence of noise on the performance is not limited to amplifier circuits. For instance, active integrated mixer circuits, which are widely used for down conversion in UHF and microwave receivers, add noise to their output. It is desirable to be able to pre-

dict the noise performance of a given mixer design [2,3]. Most of the time, amplifier circuits operate in small-signal conditions, that is, the operating point of the circuit does not change. For analysis and simulation, the amplifier circuit with a fixed operating-point can be modeled as a linear time-invariant network by making use of the small-signal models of the integrated-circuit devices. On the other hand, for a mixer circuit, the presence of a large local-oscillator signal causes substantial change in the active devices' operating points over time. So, a linear time-invariant network model is not accurate for a mixer circuit. There are many other kinds of circuits which do not operate in small-signal conditions, such as a voltage-controlled-oscillator (VCO) composed of delay cells. Noise simulation of these circuits requires a method which can handle nonlinear dynamic circuits with arbitrary excitations.

The three important types of noise in integrated circuits are *shot* noise, *thermal* noise and *flicker* noise which will all be considered in this work. Shot noise and thermal noise are related directly to physical parameters; empirical data are not required for their characterization. On the other hand, flicker noise requires empirical data for its prediction.

In Chapter 2 below, previous work on computer simulation of noise in integrated circuits is reviewed with comparisons to our method. In Chapter 3, shot, thermal and flicker noise models for integrated-circuit devices, in the context of our time-domain noise simulation method, are described. Chapter 4 describes our time-domain, non-Monte Carlo method for computer simulation of electrical noise in nonlinear dynamic circuits with arbitrary excitations. In Chapter 5, the implementation of the noise simulation method, in the context of a nodal-analysis circuit simulation program (SPICE), is described. Two examples of noise simulation, using the implementation described in Chapter 5, are presented in Chapter 6. Conclusions and future work are stated in Chapter 7. Background material on stochastic differential equations is given in the Appendix.

Chapter 2

Previous Work

Previous work on noise simulation can be classified according to two specifications:

• Domain of simulation. (Time domain, frequency domain or a mixture of both)

• The type of circuit the method can handle. (e.g. linear time-invariant circuits)

2.1 Frequency Domain Methods

2.1.1 Noise Analysis and Simulation for Linear Time-Invariant or Small-Signal Equivalent Circuits

The electrical noise sources in passive elements and integrated-circuit devices have been investigated extensively. Small-signal equivalent circuits, including noise, for many integrated-circuit components have been constructed [1]. The noise performance of a circuit can be analyzed in terms of these small-signal equivalent circuits by considering each of the uncorrelated noise sources in turn and separately computing its contribution at the output. Consider a noise current source with a one-sided spectral density $S(f)$. In a small bandwidth Δf, the mean-square value of the noise current is given by $\overline{i^2} = S(f)\Delta f$. The

noise current in bandwidth Δf can be represented approximately by a sinusoidal current generator with rms value $i = \sqrt{S(f)\Delta f}$. If the noise current in bandwidth Δf is now applied as an input signal to a linear time-invariant circuit, its effect can be calculated by substituting the sinusoidal generator and performing circuit analysis in frequency domain in the usual fashion. When the circuit response to the sinusoid is calculated, the mean-square value of the output sinusoid gives the mean-square value of the output noise in bandwidth Δf. Thus, network noise calculations reduce to familiar sinusoidal circuit analysis calculations. The only difference occurs when multiple noise sources are applied, as is always the case in practical circuits. Each noise source is then represented by a separate sinusoidal generator and the output contribution of each one is separately calculated. The total output noise in bandwidth Δf is calculated as a mean-square value by adding the individual mean-square contributions from each output sinusoid. This depends, however, on the original noise sources being independent. This requirement is satisfied when the noise sources in a device are modeled such that all the noise sources arise from separate mechanisms (i.e. they are independent). This analysis is done for a range of frequencies. The contributions from this range of frequencies are summed at the output [1].

For simple circuits the above procedure leads to expressions for total output noise, which enable the designer to assess the relative importance of the various noise sources, and thus to optimize noise performance. For a complicated circuit, the large number of noise sources and circuit complexity completely preclude hand calculation. In fact, even machine computation of the noise contributions from all sources can be time consuming. The way to avoid excessive computation in noise simulation is to attempt to isolate only the most significant noise sources, and to consider only their contribution to the noise output. Aside from the obvious loss of accuracy which arises with such an approach, there is also a reliance on the intuition of the designer, which, for complicated circuits, may not always be correct [4].

Fortunately, an extremely efficient computational technique based on the interreciprocal adjoint network concept, was proposed [4][5]. This technique calculates the noise contribution from an arbitrarily large number of noise sources at a given frequency with little more computer time than is normally required for a single noise source. The noise analysis in SPICE is based on this method.

Unfortunately, this method is only applicable to *linear time-invariant* circuits (e.g. the small-signal equivalent circuits corresponding to circuits with fixed operating points). It is not appropriate for noise simulation of circuits with changing bias conditions (e.g. a mixer circuit as discussed in Chapter 1), or circuits which are not meant to operate in small-signal conditions (e.g. a ring-oscillator composed of MOSFET inverter delay cells).

2.1.2 Noise Simulation for Nonlinear Circuits with a Periodic Large Signal Excitation

[2,3] and [6] present noise analysis techniques for nonlinear circuits with a periodic large signal excitation (e.g., mixer circuits). The noise analysis for a nonlinear circuit with a periodic large signal excitation reduces to the analysis of a *linear periodically time-varying* circuit with *cyclostationary* [2,3][6] noise sources. This is arrived by a first-order Taylor's expansion of the circuit equations around the *periodic steady-state* solution of the circuit without the noise sources and the small-signal excitations. This Taylor's approximation is similar to the one we will present in Section 4.1. The noise analysis methods described in [2,3] and [6] use frequency-domain methods based on Fourier transforms to analyze the linear periodically time-varying circuit with cyclostationary noise sources obtained by the Taylor's approximation. Even though the implementations in [2,3] and [6] are rather different, both of the techniques are based on manipulating *impulse responses* and *transfer functions* for a linear periodically time-varying system, and *spectral densities* for cyclostationary noise sources. These noise analysis techniques are applicable to only a limited class of nonlinear circuits with two excitations, where one of the excitations is large and periodic and the other is small (e.g., mixer circuits, switched capacitor circuits).

2.2 Time Domain Methods

2.2.1 Monte Carlo Noise Simulation for Nonlinear Dynamic Circuits

The previous work on noise simulation in time-domain is restricted to techniques which employ the Monte Carlo method [7]. In this method, current sources, which repre-

sent the equivalent noise sources of the devices, are introduced in each noisy component. Then the simulator performs many transient simulations of the circuit while the noise sources generate noise. Between two transient simulations, parameters of the noise generators are changed in order to get a different trajectory for the circuit. A transient simulation is also run without the noise sources. After all these simulations, at each time step, the rms noise value is calculated by statistical analysis. The noise current generators are supposed to represent the physical noise sources of the devices. There are several schemes to generate noise, but most of these schemes "disturb" the normal functioning of a simulator. It is stated in [7] that random characteristics, introduced by the noise signals, are disruptive of the integration of differential equations with stochastic terms. To circumvent this problem, the noise generation method that has been used creates "continuous" and "fully deterministic signals". The generated noise signals are the sum of a fixed number of sinusoids. Magnitudes, frequencies and phases of the sinusoids are used to control the characteristics of the noise sources. Frequencies and magnitudes are selected so as to approximate a desired power spectral density. Phases are changed to obtain different "sample paths" for the noise sources. If n is a noise process, then $n(t)$ is a random variable for fixed t. A "sample path" of n, denoted by $n_S(t)$, is a deterministic function of t consisting of observation values for n at every t.

This method has several drawbacks. First of all, noise generators are only approximations to modeled physical noise sources of a device even if the random number generators are assumed to be ideal: Power spectral densities are approximated with discrete steps in frequency. Moreover, pseudo-random number generators often do not generate a large sequence of independent numbers, but reuse old random numbers instead. This becomes a problem if a circuit with many noise sources is simulated. This is usually the case, because every device has several noise sources associated with its model.

In this method, the same circuit is simulated many times by obtaining "different" sample paths for each noise source. Then a statistical analysis is carried out to calculate averages and variances over these many simulations. The noise content in a waveform will be much smaller when compared with the magnitude of the waveform itself. As a result, the waveforms obtained for different sample paths of noise generators will be very close to each other. It is known that, in a simulator, these waveforms are only numerical approxi-

mations to the actual waveforms, therefore they contain *numerical noise*. The rms value of noise is calculated by taking a difference of these waveforms. That is, two large numbers, which have uncertainty in them, are being subtracted from each other. Consequently, the rms noise calculated with this method, in fact, includes the *noise* generated by the numerical algorithms. This furthermore degrades the accuracy of the results obtained by this method.

This method has one advantage when compared with the frequency domain methods discussed in Section 2.1: It is not restricted to linear time-invariant, or to nonlinear circuits with a large signal periodic excitation. In theory, it is applicable to the general class of nonlinear dynamic circuits with any kind of excitation.

2.2.2 Our Method - Time-Domain non-Monte Carlo Noise Simulation for Nonlinear Dynamic Circuits with Arbitrary Excitations

Our method, unlike the frequency domain methods, is not restricted to linear time-invariant or nonlinear circuits with a large signal periodic excitation. Our time-domain noise simulation method is general in the sense that, any *nonlinear dynamic circuit* with *any kind of excitation*, which can be simulated by the *transient analysis* routine in a *circuit simulator*, can be simulated by our noise simulator in time-domain to produce the noise variances and covariances of circuit variables as a function of time, provided that noise models for the devices in the circuit are available. The noise models for integrated-circuit devices, in the context of our noise simulator, are described in Chapter 3.

Our time-domain noise simulation method is based on the results from the theory of stochastic differential equations. There are no pseudo-random number generators involved in the simulation, therefore the problems associated with them do not exist.

The simulation of the average waveforms (without noise in the circuit) and the simulation of noise are separated, even though they are done concurrently. Thus, the *numerical noise* problem that arises in Monte Carlo methods is avoided.

Our method is capable of calculating *variances and covariances* (that is, the *covariance matrix*) for the noise content in the node voltages and other circuit variables in a cir-

cuit as a *function of time*. Furthermore, *correlations* between circuit variables at *different time points* can also be calculated.

Finally, the implementation of our method fits naturally into a circuit simulator (such as SPICE) which is capable of doing time-domain transient simulations. Noise simulation is done along with the transient simulation over the time interval specified by the user.

Chapter 3

Noise Models

The electrical noise sources in passive elements and integrated-circuit devices have been investigated extensively, and appropriate models have been derived [1][8]. Traditionally, these noise models are presented as *stationary* noise sources in the small-signal equivalent (at an operating point) circuits of the devices [1]. In this chapter, we describe the *adaptation* of these noise models for use in our time-domain noise simulation method. In our method, the noise sources are inserted in the large-signal models of the integrated-circuit devices and they are, in general, *nonstationary*. In Section 3.1, the adaptation of *shot, thermal* and *flicker* noise models for resistors and junction diodes will be described. The noise models for these two simple devices are representative of noise models for all other integrated-circuit devices such as BJTs and MOSFETs, because all kinds of noise we consider (shot, thermal and flicker noise) exist in these devices. The noise source models we use in our method are adapted from [1] and [8].

As it will become clear in Chapter 4, our noise simulation method requires that noise sources are *white*. The thermal and shot noise sources are modeled as white noise sources, hence they can be directly included in the noise simulation. However, the flicker noise sources can *not* be included in the noise simulation as they are. The inclusion of *flicker* noise sources into the noise simulation method will be described in Section 3.2.

3.1 Shot, Thermal and Flicker Noise Models

All the noise sources in the devices are modeled as uncorrelated *current* sources, because they are to be employed in the noise simulation in the context of an MNA circuit simulator, namely SPICE.

3.1.1 Resistors

Monolithic and thin-film resistors display thermal noise. The thermal noise in a resistor can be modeled by a white Gaussian noise current source with *intensity*

$$IN^R_{thermal} = \sqrt{2kT\frac{1}{R}} \tag{3.1}$$

where k is Boltzmann's constant, T is the absolute temperature in Kelvins and R is the resistance. The thermal noise source associated with a resistor is a *stationary* white noise process, assuming that the resistance value is a constant as a function of time. The *intensity* of a stationary white Gaussian noise process is equal to the square root of the power spectral density. For a stationary white Gaussian noise process, the power spectral density (a function of frequency) is a constant on the entire real axis. This current noise source is included in the circuit between the positive and negative nodes of the resistor as seen in Fig. 3.1.
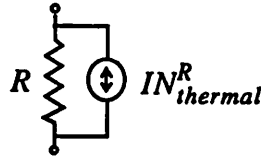
Figure 3.1:   Resistor Model with Noise Sources

The noise current source symbol in Fig. 3.1 has a double-ended arrow to represent the fact that it has no polarity.

3.1.2 Junction Diodes

The series resistance r_s, in the model of a junction diode (Fig. 3.2), is a physical resistor due to the resistivity of silicon, hence it exhibits thermal noise. The thermal noise in r_s can be modeled by a white Gaussian noise current source with *intensity*

$$IN_{thermal}^{r_s} = \sqrt{2kT\frac{1}{r_s}} \tag{3.2}$$

The *pn*-junction exhibits shot noise which is associated with the current flow through the diode. The *intensity* of the shot noise current, which is white Gaussian, is given by

$$IN_{shot}^{D}(t) = \sqrt{qI_D(t)} \tag{3.3}$$

where q is the electronic charge ($1.6 \times 10^{-19}C$), and $I_D(t)$ is the *noiseless* diode current. Note that, in this case, *intensity* is a function of time, hence this white noise source is not *stationary*. The square of the time-varying intensity for a *nonstationary* white noise source as above can be thought to be the *time-varying* power spectral density, which is a *constant* (as a function of frequency) on the entire real axis. During nonlinear operation the current through the diode shows variations as a function of time, so does the intensity. The noise-less diode current $I_D(t)$ is obtained by a transient simulation of the circuit without the noise sources. $I_D(t)$ is calculated using the values of the circuit variables (i.e. the diode terminal voltages) from this transient simulation. Then, the intensity given by (3.3) is calculated using the values of $I_D(t)$. Details will be given in the description of the noise simulation method in Chapter 4.

(3.3) is given in [1] to be valid for a constant current flow. The variations in the diode current in any practical circuit are much *slower* when compared with the *shot noise generating mechanisms* in a junction, which validates (3.3) for a time-varying current. This fact is also valid for *thermal noise generating mechanisms*. (3.1) can also be used for a *time-varying* resistor. In this case, the variations in a resistance in any practical circuit are much slower when compared with the thermal noise generating mechanisms. This makes it possible to model shot noise associated with a time-varying current and thermal noise associated with a time-varying resistance as a nonstationary white Gaussian noise, as given by (3.1) and (3.3).

The flicker noise source in a diode is modeled by a *nonstationary* noise process which has a *time-varying* power spectral density given by (see Fig. 3.2)

$$S_{flicker}^{D}(f, t) = KF\frac{I_D(t)^a}{f} \tag{3.4}$$

where KF is a constant for a particular device, a is a constant in the range 0.5 to 2 and f is the frequency. This noise source can not be included in the noise simulation directly, because it is not white (i.e. the *time-varying* power spectral density is *not* a constant as a function of frequency). A way of synthesizing this source from white noise sources will be discussed in Section 3.2.
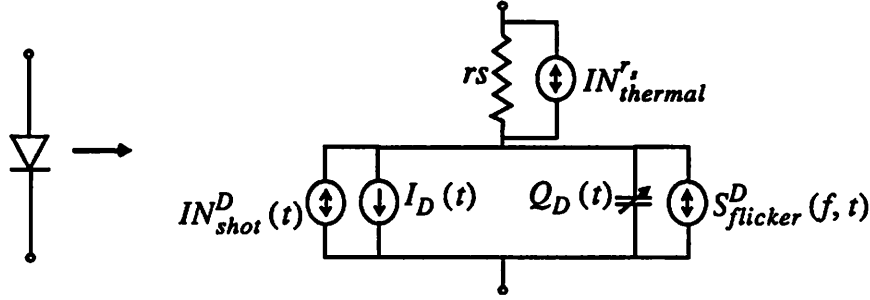
Figure 3.2: Diode Model with Noise Sources

3.1.3 BJTs

The base resistance r_b, the collector resistance r_c and the emitter resistance r_e in a BJT model are actual physical resistors and thus have thermal noise. The thermal noise in these resistors are modeled by white noise current sources with *intensities* (see Fig. 3.3)

$$IN^{r_b}_{thermal} = \sqrt{2kT\frac{1}{r_b}} \qquad IN^{r_c}_{thermal} = \sqrt{2kT\frac{1}{r_c}} \qquad IN^{r_e}_{thermal} = \sqrt{2kT\frac{1}{r_e}} \qquad (3.5)$$

The collector current $I_c(t)$ shows *full shot noise* which is modeled by a white noise source with time-varying intensity given by (see Fig. 3.3)

$$IN^{I_c}_{shot}(t) = \sqrt{qI_c(t)} \qquad (3.6)$$

The base current $I_b(t)$ also shows *full shot noise* which is uncorrelated with the collector current shot noise. It is modeled by a white noise source with time-varying intensity given by (see Fig. 3.3)

$$IN^{I_b}_{shot}(t) = \sqrt{qI_b(t)} \qquad (3.7)$$

Flicker noise in a BJT has been found experimentally to be represented by a current noise source across the internal base-emitter junction [1]. It is modeled by a *nonstationary* noise process which has a *time-varying* power spectral density given by (see Fig. 3.3)

$$S_{flicker}^{BJT}(f, t) = KF\frac{I_b(t)^a}{f} \tag{3.8}$$

Since all these noise sources arise from separate, independent physical mechanisms, they are *uncorrelated* with each other.
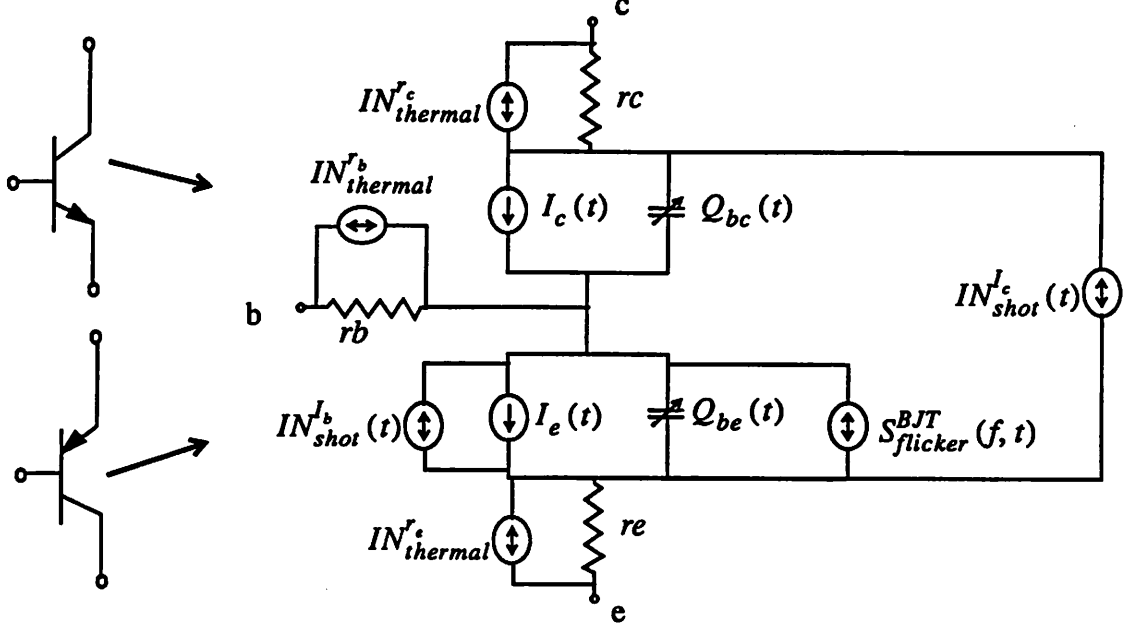
Figure 3.3: BJT Model with Noise Sources

3.1.4 MOSFETs

The drain resistance r_d and the source resistance r_s in a MOSFET model are actual physical resistors and thus have thermal noise. The thermal noise in these resistors are modeled by white noise current sources with *intensities* (see Fig. 3.4)

$$IN_{thermal}^{r_d} = \sqrt{2kT\frac{1}{r_d}} \qquad IN_{thermal}^{r_s} = \sqrt{2kT\frac{1}{r_s}} \tag{3.9}$$

The channel material in a MOSFET is *resistive,* hence it exhibits *thermal* noise. This noise source is modeled by a white noise source with time-varying intensity given by (see Fig. 3.4) [8]

$$IN_{thermal}^{channel}(t) = \cfrac{\sqrt{2kT\frac{2}{3}g_m(t)} \qquad \text{if in the saturation region}}{\sqrt{2kT\beta(t)(V_{gs}(t)-V_{th}(t))\frac{2}{3}\frac{1-(1-u)^3}{u(2-u)}} \quad \text{if in the linear region}} \qquad (3.10)$$

where

$$u = \frac{V_{ds}(t)}{V_{gs}(t)-V_{th}(t)} \qquad \beta(t) = \mu c_{ox}(1+\lambda V_{ds}(t))$$

$$g_m(t) = \beta(t)(V_{gs}(t)-V_{th}(t)) \qquad\qquad\qquad (3.11)$$
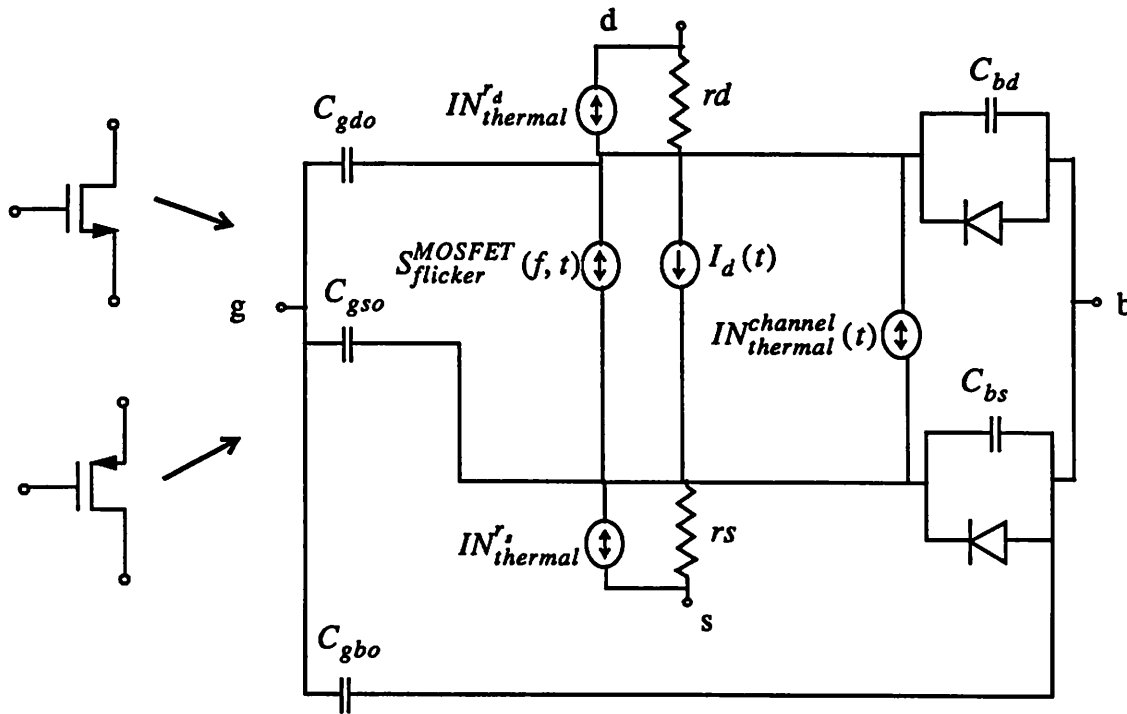
Figure 3.4: MOSFET Model with Noise Sources

Flicker noise in a MOSFET is also found experimentally to be represented by a internal drain-internal source current generator [1]. It is modeled by a *nonstationary* noise process which has a *time-varying* power spectral density given by (see Fig. 3.4)

$$S_{flicker}^{MOSFET}(f,t) = KF\frac{I_d(t)^a}{f} \qquad\qquad (3.12)$$

3.2 Inclusion of Flicker Noise Sources into Simulation

In our noise simulation method, only *white* noise sources are allowed. Flicker noise sources have a power spectral density which is *not* a constant as a function of frequency. The natural way to include flicker noise sources into simulation is, somehow, to synthesize them using white noise sources. A promising approach for $1/f$ noise generation is to use the summation of *Lorentzian spectra* which is defined by (3.13) [9]. This approach has been used in instrumentation to generate continuous-time $1/f$ noise over a specified range of frequencies. It has been shown that a constant distribution of 1.4 poles per decade gives a $1/f$ spectrum with less than 1% error [9]. A sum of N Lorentzian spectra is given by

$$ S(f) = \frac{2\sigma^2}{\pi} \sum_{h=1}^{N} \frac{\varphi_h}{\varphi_h^2 + f^2} \tag{3.13} $$

where φ_hs designate the pole-frequencies and f is the frequency. It has been shown in [9] that $N = 20$ poles uniformly distributed over 14 decades are sufficient to generate $1/f$ noise over 10 decades with a maximum error less than 1%.

Each Lorentzian spectrum in the summation in (3.13) can be easily obtained by using the *thermal* noise generator of a resistor R_h connected in parallel to a capacitance $C_h = C$, and their sum can be achieved by putting, as shown in Fig. 3.5, N of such $R_h - C_h$ groups in series [9]. For a choice of $N = 15$, with $C = 32nF$, $R_1 = 5 \times 10^8 \Omega$ and $R_{h+1} = R_h/5$ we obtain the spectra shown in Fig. 3.6 at the output of the circuit given in Fig. 3.5. An ideal $1/f$ spectra is also shown in Fig. 3.6.
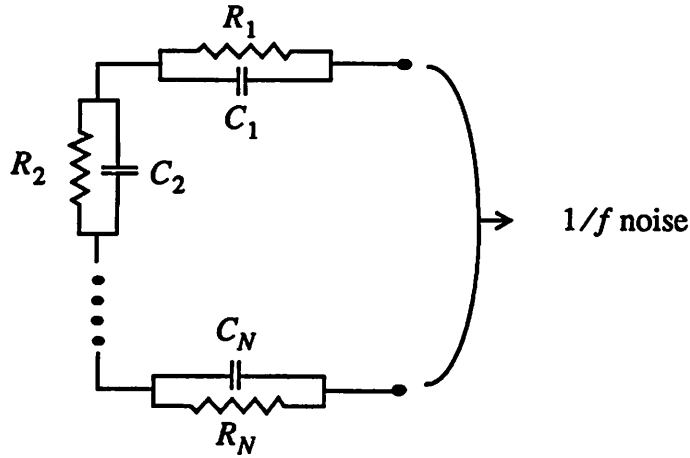
Figure 3.5: $1/f$ Noise Synthesizing Circuit
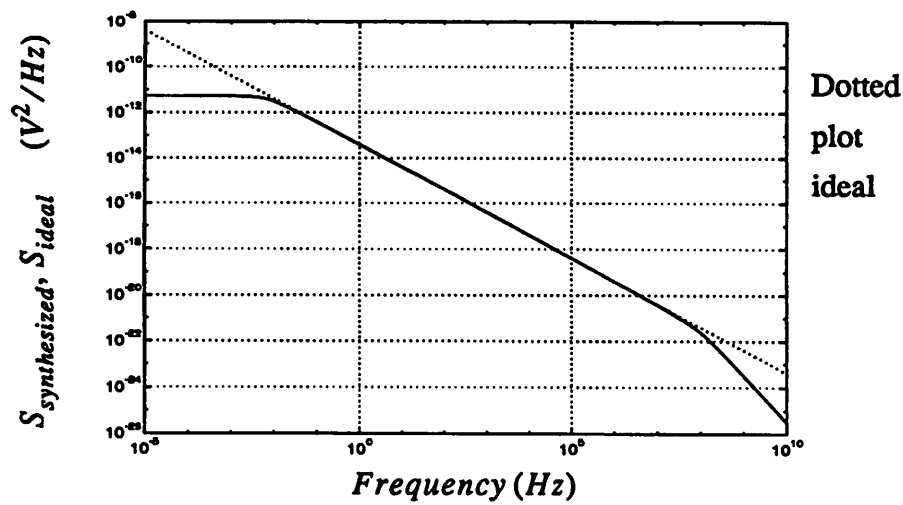
Figure 3.6: The Spectra at the Output of the $1/f$ Noise Synthesizing Circuit
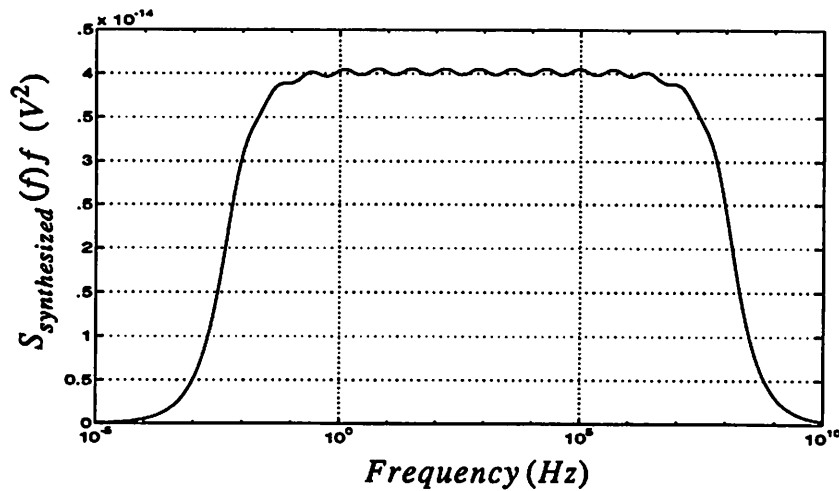
Figure 3.7: Frequency Range of "Good" Approximation to $1/f$ Noise

In Fig. 3.7, we plot $S(f)f$ as a function of frequency for the $1/f$ noise synthesizing circuit, in which the frequency range of "good" approximation to $1/f$ noise is seen explicitly. By adding more poles ($R_h - C_h$ groups) to the circuit, the "good" approximation frequency range can be extended.

In the noise simulation, a flicker noise source in the model of an integrated-circuit device is built by using the circuit in Fig. 3.5 with an ideal *voltage-controlled current*

source. This is illustrated in Fig. 3.8. The voltage-controlled current source is connected between the two nodes of a device where the flicker noise source is modeled.

The spectral density of the $1/f$ noise obtained from the circuit in Fig. 3.5 is approximately (a good approximation in the desired frequency range)

$$S(f) = \frac{2\sigma^2}{\pi}\frac{1}{f} \tag{3.14}$$

where $\sigma^2 = kT/(2C)$. This spectral density is time-invariant. The flicker noise models given in Section 3.1 require time-varying spectral densities. This is achieved by having a time-varying *transconductance* ($g(t)$) for the voltage-controlled current source in Fig. 3.8. For instance, for a diode, we require that the flicker noise source spectral density is in the form given by (3.4). This is assured with

$$g(t) = \sqrt{\frac{\pi KFI_D(t)^a}{2\sigma^2}} \tag{3.15}$$

One disadvantage of synthesizing flicker noise sources with this method is the increased CPU time for noise simulation: Every flicker noise source synthesizing circuit (Fig. 3.5) introduces extra nodes in the circuit.
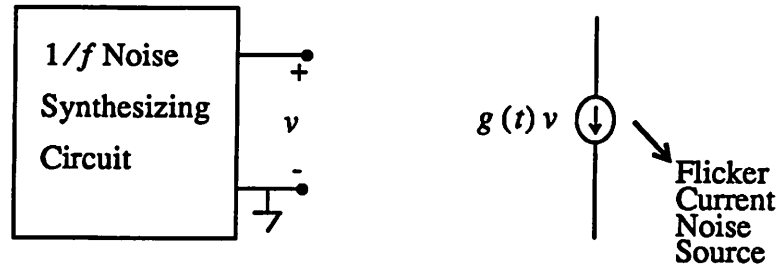
Figure 3.8: Flicker Current Noise Source Synthesis for Simulation

Chapter 4

Time-Domain non-Monte Carlo Noise Simulation for Nonlinear Dynamic Circuits with Arbitrary Excitations

The noise simulation method will be described assuming that modified nodal analysis (MNA) [10] is used for the formulation of circuit equations. MNA is the method for circuit equation formulation in most of the circuit simulators (such as SPICE) available. Translation of the noise simulation method into other ways of circuit equation formulation is straightforward.

4.1 Derivation of the Stochastic Differential Equation for Noise from MNA Formulation of the Nonlinear Circuit Equations

The MNA equations for any circuit, *without the noise sources*, can be written compactly as

$$F(\dot{x}, x, t) = 0 \qquad x(0) = x_0 \qquad (4.1)$$

where x is the vector of the circuit variables with dimension n, $\dot{x}$ is the time derivative of x, t is time and F is mapping x, $\dot{x}$ and t into a vector of real numbers of dimension n. It is obvious that $x = x(t)$ and $\dot{x} = \dot{x}(t)$. The time dependence of x and $\dot{x}$ will not be written explicitly for notational simplicity. In MNA, the circuit variables consist of node voltages and branch currents for some elements (e.g. inductors and voltage sources). The circuit equations consist of the node equations (KCL) and branch equations of the elements for which branch currents are included in the circuit variables vector. Under some rather mild conditions (which are satisfied by well modeled circuits) on the continuity and differentiability of F, it can be proven that there exists a unique solution to (4.1) assuming that a fixed initial value $x(0) = x_0$ is given [10]. Let x_s be the solution to (4.1). The *transient analysis* in circuit simulators solves for x_s using numerical methods for solving ordinary differential equations (ODEs) [10]. The initial value vector $x(0) = x_0$ is obtained by a dc solution of the circuit before the transient simulation is started. For a circuit, there may be several different dc solutions.

The first-order Taylor's expansion of F around x_s is expressed as

$$F(\dot{x}, x, t) \cong F(\dot{x}_s, x_s, t) + \frac{\partial}{\partial x}F(\dot{x}, x, t)\bigg|_{\substack{x = x_s \\ \dot{x} = \dot{x}_s}} (x - x_s) + \frac{\partial}{\partial \dot{x}}F(\dot{x}, x, t)\bigg|_{\substack{x = x_s \\ \dot{x} = \dot{x}_s}} (\dot{x} - \dot{x}_s) \quad (4.2)$$

which will be used later.

If the noise sources are included in the circuit, the MNA formulation of the circuit equations can be written as

$$F(\dot{x}, x, t) + B(x, t)v = 0 \qquad x(0) = x_0 + x_{noise, 0} \qquad (4.3)$$

where $B(x, t)$ is an $n \times p$ matrix, the entries of which are a function of x, and v is a vector of p white Gaussian stochastic processes. A one-dimensional Gaussian white noise is a stationary Gaussian process $\xi(t)$, for $-\infty < t < \infty$, with mean $\varepsilon[\xi(t)] = 0$ and a constant spectral density on the entire real axis. The covariance function of $\xi(t)$ is given by $\varepsilon[\xi(s)\xi(t + s)] = \delta(t)$, where δ is Dirac's delta function [11]. The white Gaussian noise $\xi(t)$ is a very useful mathematical idealization for describing random influences that fluctuate rapidly and hence are virtually uncorrelated for different instants of time. A white Gaussian noise model is appropriate for *thermal* and *shot* noise in integrated circuits [1]. Flicker noise sources are taken care of in the way described in Section 3.2. v in (4.3)

is simply a combination of p independent one-dimensional white Gaussian noise processes as defined above. These noise processes actually correspond to the noise sources which are included in the models of integrated-circuit devices. Since the noise models for the integrated-circuit devices are to be employed here in the context of an MNA circuit simulator (SPICE), noise sources in the devices are all modeled as uncorrelated *current* sources. Details about noise models were given in Chapter 3.

$B(x, t)$, in (4.3), contains the *intensities*, as described in Chapter 3, for the white noise sources in v. The *intensities* for these noise sources are, in general, a function of time. For instance, in nonlinear operation, the current flowing through a device may show variations. Shot noise in the device is associated with this direct current flow, and the intensity of shot noise is a function of this current. In fact, because of intensity variations, this noise source is not *stationary*. Thus, the *nonstationarity* of the noise sources in the circuit are captured in $B(x, t)$. Naturally, every current noise source is connected between two nodes, one of which may be the ground node. Every column in $B(x, t)$ corresponds to a noise source in v. Every column has either one or two nonzero entries, depending on whether, one of the nodes which the corresponding current noise source is connected to, is ground. The rows of $B(x, t)$ correspond to either a node equation (KCL) or a branch equation. There are no nonzero entries in the rows which correspond to the branch equations.

(4.3) is a system of nonlinear stochastic differential equations (SDEs). The solution processes for this kind of SDEs, where the forcing is an irregular stochastic process such as white noise, have nondifferentiable sample paths [12]. They require fundamentally different and complex methods of analysis and numerical solution [12]. (Background material on stochastic differential equations can be found in the Appendix). Fortunately, some characteristics of our problem help us simplify the numerical solution of (4.3): The noise content in the signals in any useful circuit is, almost always, much smaller when compared with the signal itself.

Let x_{sn} be the solution of (4.3). x_{sn} is not deterministic, since it is the solution of the circuit equations with the noise sources included, and satisfies

$$F(\dot{x}_{sn}, x_{sn}, t) + B(x_{sn}, t)v = 0 \qquad x_{sn}(0) = x_0 + x_{noise, 0} \qquad (4.4)$$

where x_0 is deterministic, and $x_{noise,0}$ is a vector of n zero-mean random variables. We use (4.2) in (4.4) to approximate $F(\dot{x}_{sn}, x_{sn}, t)$, and we obtain

$$F(\dot{x}_s, x_s, t) + \frac{\partial}{\partial x}F(\dot{x}, x, t)\bigg|_{\substack{x=x_s \\ \dot{x}=\dot{x}_s}} (x_{sn} - x_s) + \frac{\partial}{\partial \dot{x}}F(\dot{x}, x, t)\bigg|_{\substack{x=x_s \\ \dot{x}=\dot{x}_s}} (\dot{x}_{sn} - \dot{x}_s) + B(x_{sn}, t)v \cong 0 \tag{4.5}$$

$$x_{sn}(0) = x_0 + x_{noise,0}$$

Defining

$$x_{noise} = x_{sn} - x_s \tag{4.6}$$

x_{noise} is, actually, the difference between the solutions of the circuit equations, with and without the noise sources. In other words, x_{noise} is the *noise content* in x_{sn}. x_{noise} is much smaller when compared with x_s, which validates the above approximation.

For notational simplicity, define

$$A(t) = \frac{\partial}{\partial x}F(\dot{x}, x, t)\bigg|_{\substack{x=x_s \\ \dot{x}=\dot{x}_s}} \qquad C(t) = \frac{\partial}{\partial \dot{x}}F(\dot{x}, x, t)\bigg|_{\substack{x=x_s \\ \dot{x}=\dot{x}_s}} \tag{4.7}$$

where $A(t)$ and $C(t)$ are $n \times n$ matrices with time-dependent entries.

Furthermore, we approximate

$$B(x_{sn}, t) \cong B(x_s, t) \tag{4.8}$$

and define

$$B(t) = B(x_s, t) \tag{4.9}$$

If (4.6), (4.7), (4.8) and (4.9) are substituted in (4.5) we obtain

$$F(\dot{x}_s, x_s, t) + A(t)x_{noise} + C(t)\dot{x}_{noise} + B(t)v \cong 0$$
$$x_{noise}(0) = x_0 + x_{noise,0} - x_s(0) \tag{4.10}$$

Since x_s is the solution of (4.1) we have

$$F(\dot{x}_s, x_s, t) = 0 \qquad x_s(0) = x_0 \tag{4.11}$$

and if we substitute (4.11) in (4.10), we obtain

$$A(t)x_{noise} + C(t)\dot{x}_{noise} + B(t)v = 0 \qquad x_{noise}(0) = x_{noise,0} \tag{4.12}$$

(4.12) is a *linear SDE* [11] in x_{noise} with time-varying coefficients. $A(t)$, $B(t)$ and $C(t)$ are functions of x_s, and they do not depend on x_{noise}. The solution of this equation will be discussed in the next four sections. An intuitive explanation for (4.12) follows.

The numerical solution of (4.1), transient analysis of a circuit in time-domain, can be viewed as solving for the *changing operating point* of the circuit as a function of time. With this perspective, the approximations made (the first-order Taylor's expansion of $F(\dot{x}_{sn}, x_{sn}, t)$ around x_s in (4.4), and the approximation expressed by (4.8)), in the derivation of (4.12), can be interpreted as *linearizing* the nonlinear circuit at its *current* time operating point and inserting the noise current sources into this linear network. Linearization of the nonlinear circuit at every time point can, alternatively, be expressed as changing the parameters of the linearized circuit at every time point, thus obtaining a *linear time-varying* network. The intensities for the noise current sources are calculated using the information in the *current* operating point of the circuit.

4.2 Transformation of the Stochastic Differential Equation for Noise into State-Equation Form

To make use of some of the results from the theory of SDEs, (4.12) will be put into the form

$$\dot{y} = E(t)y + F(t)v \qquad y(0) = y_0 \qquad (4.13)$$

If $C(t)$ is a full-rank matrix, this can be easily done by premultiplying both sides of (4.12) by the inverse of $C(t)$. However, this is not true in general; $C(t)$ may have zero columns. For instance, if a circuit variable is a node voltage, and if this node does not have any capacitors connected to it in the circuit, then all of the entries in the column of $C(t)$ corresponding to this circuit variable will be zero for all t. At this point, we should note that the zero-nonzero structure of $A(t)$, $B(t)$ and $C(t)$ is independent of t. This can be easily seen by considering the interpretation given at the end of the last section. At every time point, the structure of the *linearized* circuit does not change, but the parameters of the circuit (e.g. resistor, capacitor values) do change, creating a *linear time-varying* network. So, as explained above, some of the columns of $C(t)$ are structurally zero, independent of

t. If we reorder the variables in x_{noise} in such a way that the zero columns of $C(t)$ are grouped at the right-hand side of the matrix, (4.12) becomes

$$\begin{bmatrix} A_1(t) & A_2(t) \end{bmatrix} \begin{bmatrix} x^1_{noise} \\ x^2_{noise} \end{bmatrix} + \begin{bmatrix} C_1(t) & 0 \end{bmatrix} \begin{bmatrix} \dot{x}^1_{noise} \\ \dot{x}^2_{noise} \end{bmatrix} + B(t)v = 0$$

$$\begin{bmatrix} x^1_{noise}(0) \\ x^2_{noise}(0) \end{bmatrix} = \begin{bmatrix} x^1_{noise,0} \\ x^2_{noise,0} \end{bmatrix} \tag{4.14}$$

where $A_1(t)$ and $C_1(t)$ are $n \times m$, $A_2(t)$ is $n \times k$, x^1_{noise} is an m-dimensional vector, x^2_{noise} is a k-dimensional vector, m is the number of nonzero columns in $C(t)$ and k is the number of zero columns. Naturally, $n = m + k$. If we expand (4.14), we obtain

$$A_1(t) x^1_{noise} + A_2(t) x^2_{noise} + C_1(t) \dot{x}^1_{noise} + B(t)v = 0$$

$$x^1_{noise}(0) = x^1_{noise,0} \qquad x^2_{noise}(0) = x^2_{noise,0} \tag{4.15}$$

Next, we apply *elementary operations* to (4.15) such that $C_1(t)$ is replaced by its *reduced row echelon form*. Any of the following operations on the set of n equations in (4.15) is called an *elementary operation*

1. Interchanging the order of any two equations.

2. Multiplying the both sides of an equation by a nonzero scalar.

3. Adding a multiple of any equation to another equation.

After these operations, (4.15) turns into

$$\bar{A}_1(t) x^1_{noise} + \bar{A}_2(t) x^2_{noise} + \bar{C}_1(t) \dot{x}^1_{noise} + \bar{B}(t)v = 0$$

$$x^1_{noise}(0) = x^1_{noise,0} \qquad x^2_{noise}(0) = x^2_{noise,0} \tag{4.16}$$

where $\bar{C}_1(t)$ is in its *reduced row echelon form*

$$\bar{C}_1(t) = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{4.17}$$

I_m is the m-dimensional identity matrix. Now, we partition the equations in (4.16) as shown below

$$\begin{bmatrix} \bar{A}_{11}(t) \\ \bar{A}_{12}(t) \end{bmatrix} x_{noise}^1 + \begin{bmatrix} \bar{A}_{21}(t) \\ \bar{A}_{22}(t) \end{bmatrix} x_{noise}^2 + \begin{bmatrix} I_m \\ 0 \end{bmatrix} \dot{x}_{noise}^1 + \begin{bmatrix} \bar{B}_1(t) \\ \bar{B}_2(t) \end{bmatrix} v = 0$$

(4.18)

$$x_{noise}^1(0) = x_{noise,0}^1 \qquad x_{noise}^2(0) = x_{noise,0}^2$$

into two groups, consisting of m and k equations. If the two groups of equations are written separately, we obtain

$$\bar{A}_{11}(t) x_{noise}^1 + \bar{A}_{21}(t) x_{noise}^2 + \dot{x}_{noise}^1 + \bar{B}_1(t) v = 0$$

$$x_{noise}^1(0) = x_{noise,0}^1 \qquad x_{noise}^2(0) = x_{noise,0}^2$$

(4.19)

$$\bar{A}_{12}(t) x_{noise}^1 + \bar{A}_{22}(t) x_{noise}^2 + \bar{B}_2(t) v = 0$$

(4.20)

We solve for x_{noise}^2 in (4.20) to get

$$x_{noise}^2 = -(\bar{A}_{22}(t)^{-1}) \bar{A}_{12}(t) x_{noise}^1 - (\bar{A}_{22}(t)^{-1}) \bar{B}_2(t) v$$

(4.21)

The above step assumes that $\bar{A}_{22}(t)$ $(k \times k)$ is nonsingular. Nonsingularity of $\bar{A}_{22}(t)$ means that the variables in x_{noise} which appear *without* time-derivatives in the equations can be expressed in terms of the variables which appear *with* derivatives. In other words, for every variable appearing in the equations without a derivative, there exists at least one equation, without time-derivatives, containing this variable. This condition is always satisfied if the variable is a node voltage. If a node voltage appears without time-derivatives in the equations, this means that there are no capacitors connected to this node. Then the KCL equation for this node, which does not contain derivatives, is the equation which is used to express this node voltage in terms of the other variables. On the other hand, this condition is not always satisfied for voltage source currents, which always appear without derivatives in the equations. The only equations a voltage source current can appear in, are the equations corresponding to the nodes the voltage source is connected to. If capacitors are connected to these nodes, then the only equations containing the voltage source current have derivatives in them, hence they can not be used to express the voltage source current in terms of the other variables, which means that $\bar{A}_{22}(t)$ is singular. Note that this problem arises only if both nodes of the voltage source are connected to capacitors, or one of them is connected to a capacitor and the other is ground. This problem can be taken care of by eliminating the voltage source current variable and substituting the branch equation for

this voltage source in the circuit equations in (4.12). At this point, we will assume that $\bar{A}_{22}(t)$ is nonsingular.

Define

$$D_1(t) = -(\bar{A}_{22}(t)^{-1})\bar{A}_{12}(t) \qquad D_2(t) = -(\bar{A}_{22}(t)^{-1})\bar{B}_2(t) \qquad (4.22)$$

Equation (4.21) also suggests that the initial values $x^1_{noise,0}$ and $x^2_{noise,0}$ can not be independent of each other. Indeed, we require

$$x^2_{noise,0} = D_1(0)x^1_{noise,0} + D_2(0)v(0) \qquad (4.23)$$

where we have used (4.22) in (4.21) with $t = 0$.

Next, we use (4.21) and (4.22) in (4.19) to get

$$\bar{A}_{11}(t)x^1_{noise} + \bar{A}_{21}(t)D_1(t)x^1_{noise} + \dot{x}^1_{noise} + \bar{B}_1(t)v + \bar{A}_{21}(t)D_2(t)v = 0$$
$$x^1_{noise}(0) = x^1_{noise,0} \qquad (4.24)$$

and hence

$$\dot{x}^1_{noise} = -(\bar{A}_{11}(t) + \bar{A}_{21}(t)D_1(t))x^1_{noise} - (\bar{B}_1(t) + \bar{A}_{21}(t)D_2(t))v$$
$$x^1_{noise}(0) = x^1_{noise,0} \qquad (4.25)$$

Defining

$$E(t) = -(\bar{A}_{11}(t) + \bar{A}_{21}(t)D_1(t)) \qquad F(t) = -(\bar{B}_1(t) + \bar{A}_{21}(t)D_2(t)) \qquad (4.26)$$

and using (4.26) in (4.25) results in

$$\dot{x}^1_{noise} = E(t)x^1_{noise} + F(t)v \qquad x^1_{noise}(0) = x^1_{noise,0} \qquad (4.27)$$

4.3 Solution of the Stochastic Differential Equation for Noise

The SDE for noise, which was derived in the last two sections, is given by

$$\dot{x}^1_{noise} = E(t)x^1_{noise} + F(t)v \qquad x^1_{noise}(0) = x^1_{noise,0} \qquad (4.28)$$

$$x^2_{noise} = D_1(t) x^1_{noise} + D_2(t) v$$

$$x^2_{noise,0} = D_1(0) x^1_{noise,0} + D_2(0) v(0)$$

(4.29)

with

$$\begin{bmatrix} x^1_{noise} \\ x^2_{noise} \end{bmatrix} = x_{noise}(\text{reordered})$$

(4.30)

(4.28) is a linear differential equation where the forcing is an irregular stochastic process which is *white noise*. A mathematically rigorous treatment of equations of this type requires a new theory. In 1951, Ito defined the *Ito* or *stochastic* integral and in doing so put the theory of SDEs on a solid foundation [11]. (4.28) is written symbolically as a linear SDE, but it is interpreted as an integral equation with *Ito* or *Stratonovich* stochastic integrals [11]. The solution of (4.28) obtained by the *Stratonovich* interpretation is equal to the one obtained by the *Ito* interpretation, because it is a *linear SDE in the narrow sense* [11]. A detailed explanation of Ito and Stratonovich stochastic integrals and stochastic differential equations can be found in [11], [12] and [13]. Some background material on SDEs can also be found in the Appendix. In the Appendix, a short introduction to SDEs is given, and moreover, some results from the theory of SDEs, which will be used in the development of the noise simulation method, are summarized. In the following development, we state and use these results from the theory of SDEs.

(4.28) is often written in the form

$$dx^1_{noise} = E(t) x^1_{noise} dt + F(t) dw \qquad x^1_{noise}(0) = x^1_{noise,0}$$

(4.31)

where w is a vector of p independent one-dimensional Wiener processes. A p-dimensional Wiener process can be defined as a process with independent and stationary, $N(0, (t_1 - t_2) I_p)$ -distributed increments $w(t_1) - w(t_2)$, with initial value $w(0) = 0$. Here, $N(Mean, Cov)$ denotes the p-dimensional normal distribution with expectation vector *Mean* and covariance matrix *Cov* [11]. A Wiener process can be thought to be the "integral" of a white noise, or, alternatively, white noise is the "derivative" of a Wiener process in the sense of coincidence of the covariance functionals [11]. In our case, we have

$$w(t) = \int_0^t v(\tau)\, d\tau \qquad v(t) = \dot{w}(t) \qquad (4.32)$$

As with ordinary differential equations, the general solution of a linear SDE can be found explicitly. The method of solution also involves an integrating factor or, equivalently, a fundamental solution of an associated homogeneous differential equation. The solution of (4.28) is given by

$$x_{noise}^1(t) = \phi(t, t_0) x_{noise}^1(t_0) + \int_{t_0}^t \phi(t, \tau) F(\tau)\, dw(\tau) \qquad (4.33)$$

where $\phi(t, \tau)$ is the matrix determined as a function of t by the homogeneous differential equation

$$\frac{d\phi}{dt} = E(t)\phi \qquad \phi(\tau, \tau) = I_m \qquad (4.34)$$

(4.33) involves an *Ito* integral as opposed to a Riemann integral [11]. The integral in (4.33) can not be interpreted as an ordinary Riemann integral, because almost all sample functions of $w(t)$ are of unbounded variation. Ito's definition of the stochastic integral includes the ordinary Riemann integral as a special case [11]. If the functions $E(t)$ and $F(t)$ are "measurable" and bounded on the time interval of interest, there exists a unique solution for every initial value $x_{noise}^1(t_0)$ [11]. We are interested in the case where

$$x_{noise}^1(0) = x_{noise, 0}^1 \qquad (4.35)$$

In our problem, it is sufficient to find the *probabilistic characteristics* of x_{noise}^1 as a function of t. In other words, we would like to determine the *mean* and the *covariance matrix* of x_{noise}^1 as a function of time in the time interval desired. If x_{noise}^1 is a *Gaussian* stochastic process, then it is *completely* characterized by its mean and covariance function as a function of time. Further explanation on this topic will be given in Section 4.5. If we substitute (4.35) in (4.33) with $t_0 = 0$ we obtain

$$x_{noise}^1(t) = \phi(t, 0) x_{noise, 0}^1 + \int_0^t \phi(t, \tau) F(\tau)\, dw(\tau) \qquad (4.36)$$

If we take the expectation of both sides of (4.36) we get the mean of x_{noise}^1 which is a function of t. Considering that $\varepsilon[v(t)] = 0$ and $\varepsilon[x_{noise,0}^1] = 0$, we get

$$m^1(t) = \varepsilon[x_{noise}^1(t)] = 0 \tag{4.37}$$

Next, we would like to determine the covariance matrix of the components of x_{noise}^1 as a function of t, which is given by

$$K^1(t) = \varepsilon[x_{noise}^1(t)x_{noise}^1(t)^T] \tag{4.38}$$

since mean is zero as given by (4.37). Consider

$$dx_{noise}^1 x_{noise}^1{}^T = x_{noise}^1 dx_{noise}^1{}^T + (dx_{noise}^1)x_{noise}^1{}^T + F(t)F(t)^T dt \tag{4.39}$$

Notice that there is an extra term in (4.39) which would not be there if we were using ordinary calculus instead of *stochastic*, or *Ito* calculus. This equation is obtained from *Ito's Theorem* [11] using (4.31). We use (4.31) to expand (4.39) and obtain

$$\begin{aligned} dx_{noise}^1 x_{noise}^1{}^T = {} & (E(t)x_{noise}^1 x_{noise}^1{}^T + x_{noise}^1 x_{noise}^1{}^T E(t)^T + F(t)F(t)^T)\,dt \\ & + x_{noise}^1(F(t)\,dw)^T + (F(t)\,dw)x_{noise}^1{}^T \end{aligned} \tag{4.40}$$

If we take the expectation of both sides of this equation, noting that x_{noise}^1 and dw are uncorrelated and using (4.38), we get

$$\dot{K}^1(t) = E(t)K^1(t) + K^1(t)E(t)^T + F(t)F(t)^T \tag{4.41}$$

where $K^1(t)$ is the unique symmetric nonnegative-definite solution of the matrix equation (4.41) with the initial value $K^1(0) = \varepsilon[x_{noise,0}^1(x_{noise,0}^1)^T] = K_0^1$. Calculation of the initial value K_0^1 will be described in Section 4.4. The differential equation for $K^1(t) = K^1(t)^T$, (4.41), satisfies the Lipschitz and boundedness conditions in the time interval of interest, so that a unique solution exists [11]. (4.41) represents (in view of symmetry of $K^1(t)$) a system of $m(m+1)/2$ linear differential equations. (4.41) can be solved for $K^1(t)$ using a numerical method (such as Backward Euler) for the solution of ODEs.

$K^1(t)$ represents the noise covariance matrix of circuit variables as a function of time. So, the information about the noise variances of circuit variables, or the noise correlations between circuit variables at a given time point are contained in $K^1(t)$. In some

problems, one might be interested in the noise correlations of circuit variables at different time points, which can be expressed as

$$K^1(t_1, t_2) = \varepsilon\, [x_{noise}^1(t_1)\, x_{noise}^1(t_2)^T]$$ (4.42)

In a similar way to the derivation of (4.41), one can derive

$$\frac{\partial}{\partial t_2} K^1(t_1, t_2) = K^1(t_1, t_2)\, E(t_2)^T$$ (4.43)

with the initial condition $K^1(t_1, t_1) = K^1(t_1)$ [13]. Integrating (4.43) at various values of t_1, one can obtain a number of sections of the covariance function $K^1(t_1, t_2)$ at $t_2 > t_1$. Then, $K^1(t_1, t_2)$ at $t_2 < t_1$ is determined by

$$K^1(t_1, t_2) = K^1(t_2, t_1)^T$$ (4.44)

4.4 Calculation of the Initial Value for the Linear ODE for the Covariance Matrix of the Components of x_{noise}^1

In the last section, we have derived a linear ODE, (4.41), for the covariance matrix of x_{noise}^1. In order to be able to solve (4.41), we need to know the initial value K_0^1. We set K_0^1 to the solution of the following matrix equation in P

$$E(0)\,P + P\,E(0)^T + F(0)\,F(0)^T = 0$$ (4.45)

The matrix equation (4.45) has a symmetric nonnegative-definite solution P, if the equation $\dot{z} = E(0)\,z$ is asymptotically stable (that is, if all the eigenvalues of $E(0)$ have negative real parts) [11]. (4.45) represents (in view of symmetry of P) a system of $m(m+1)/2$ linear equations.

4.4.1 Comparison of the Initial Value Calculation with SPICE Noise Simulation for Linear Time-Invariant Circuits

In general, for a nonlinear dynamic circuit with arbitrary excitations, $E(t)$ and $F(t)$ have time-varying entries. On the other hand, for the special class of *linear time-*

invariant circuits, or for *nonlinear dynamic circuits with dc excitations*, $E(t)$ and $F(t)$ do not change with t, that is,

$$E(t) = E(0) \qquad F(t) = F(0) \qquad \text{for} \qquad t \geq 0 \qquad (4.46)$$

In this case, with the initial value K_0^1 being set to the solution of (4.45), the solution of (4.41) is given by

$$K^1(t) = K_0^1 \qquad \text{for} \qquad t \geq 0 \qquad (4.47)$$

From (4.47), we conclude that noise in a linear time-invariant, or a nonlinear dynamic circuit with dc excitations, is *stationary* (in the wide sense), that is, the covariance matrix $K^1(t)$ for noise variables is a *constant* function of time. Note that the mean of the noise variables is always zero for all kinds of circuits, as given by (4.37).

As shown above, the time-domain noise simulation for a linear time-invariant, or a nonlinear dynamic circuit with dc excitations, reduces to solving the linear equation system (4.45). This can be compared with the frequency domain noise simulation currently implemented in SPICE which works for small-signal equivalent, that is, linear time-invariant circuits. Solving (4.45) is equivalent to calculating *the total integrated noise for all the circuit variables over the frequency range from 0 to* ∞ (that is, the noise variances for all the circuit variables) in SPICE noise simulation. In fact, the solution of (4.45) provides more information than the total integrated noise for all the circuit variables over the frequency range from 0 to ∞. By solving (4.45) we also obtain the *noise covariances* of all circuit variables. Calculating the noise covariance between two circuit variables in SPICE noise simulation requires the calculation of total integrated noise over the frequency range from 0 to ∞ for the difference of the two circuit variables. This needs to be done for every circuit variable pair to calculate all covariances.

4.5 The Condition for x_{noise}^1 to be a Gaussian Process

It is important to determine whether the noise in the circuit (solution of the SDE given by (4.28)) is a Gaussian noise process or not, because a Gaussian process is completely characterized by its mean and covariance function as a function of time, and we have a means of calculating the mean and covariance function for the solution of (4.28).

The solution of (4.28) is a *Gaussian* stochastic process if and only if the initial value $x^1_{noise, 0}$ is normally distributed or constant [11]. Up to this point, we have characterized the initial value $x^1_{noise, 0}$ as being an m-dimensional vector of zero-mean random variables with the covariance matrix given by the solution of (4.45). Here, we restrict $x^1_{noise, 0}$ to be a vector of zero-mean normally distributed random variables with the covariance matrix given by the solution of (4.45). So, $x^1_{noise, 0}$ is completely characterized by its covariance matrix. With this restriction on the initial value $x^1_{noise, 0}$, x^1_{noise} (solution of (4.28)) is a *Gaussian* stochastic process, *nonstationary* in general, and it is completely characterized by its mean (given by (4.37)) and covariance function (given as the solution of (4.41) and (4.43) as a function of time). For linear time-invariant, or nonlinear dynamic circuits with dc excitations, x^1_{noise} is a *stationary* (in the strict sense) *Gaussian* process, completely characterized by the covariance matrix which is a constant function of time as given by the solution of (4.45).

Chapter 5

Implementation of the Noise Simulation Method in SPICE

The noise simulation method described in Chapter 4, along with the noise models described in Chapter 3, was implemented inside the circuit simulator SPICE3 (version 3f4) [14]. Time-domain noise simulation is done along with the *transient simulation* in SPICE3 in the time interval specified by the user. A detailed explanation of the numerical methods employed in the simulator will not be given here, since they all have been discussed extensively in the literature.

The transient simulation in SPICE3 solves for x_s, which is the solution of (4.1), using numerical methods for solving ordinary algebraic-differential equations (e.g., trapezoidal method, backward differentiation methods). (4.1) is repeated below

$$F(\dot{x}, x, t) = 0 \qquad x(0) = x_0 \tag{5.1}$$

The initial value vector $x(0) = x_0$ in (5.1) is obtained by a dc solution of the circuit before the transient simulation is started. The numerical methods for solving (5.1) subdivide the time interval $[0,T]$, in which the transient simulation is to be performed, into a finite set of distinct points:

$$t_0 = 0, t_R = T, t_{r+1} = t_r + h_{r+1} \qquad r = 0, 1, ..., R. \tag{5.2}$$

where h_{r+1}s are the time steps. At each time point t_r, the numerical methods compute an "approximation" $x_s [r]$ of the exact solution $x_s (t_r)$ [10].

The noise simulation (solution of (4.41)) is done *concurrently* with the transient simulation. (4.41) represents a system of $m (m + 1) / 2$ linear differential equations, which is repeated below

$$\dot{K}^1 (t) = E (t) K^1 (t) + K^1 (t) E (t)^T + F (t) F (t)^T$$
$$K^1 (0) = \varepsilon [x^1_{noise, 0} (x^1_{noise, 0})^T] = K^1_0 \tag{5.3}$$

We currently use the *Backward Euler* scheme to discretize these equations in time. The time steps (given by (5.2)) chosen by the transient simulation routines in SPICE3 are also used to discretize (5.3).

At each time point t_r, after the transient simulation routines have calculated $x_s [r]$, the matrices $A [r] \cong A (t_r)$, $C [r] \cong C (t_r)$ and $B [r] \cong B (t_r)$, as defined by (4.7) and (4.9), are calculated using the values in $x_s [r]$. These matrices are stored in *sparse* matrix data structures. The routines for loading these matrices have been written for each device. The routines for loading $B [r]$ contain the noise models for the devices, which are described in Chapter 3. Then all the operations described in Section 4.2 are performed to calculate $E [r] \cong E (t_r)$ and $F [r] \cong F (t_r)$ from $A [r]$, $C [r]$ and $B [r]$. The numerical operations actually done somewhat differ from what has been described in Section 4.2 because of efficiency reasons. All of these operations are performed using *sparse* matrix data structures and routines. Then, $E [r]$ and $F [r]$ are used to calculate $K^1 [r] \cong K^1 (t_r)$ in the discretized solution of (5.3) with the Backward Euler scheme. This last operation requires the solution of $m (m + 1) / 2$ simultaneous linear equations, because Backward Euler is an *implicit* method [10]. Here, m is, roughly, the number of nodes in a circuit, to which a capacitor is connected. Simulations have shown that, for larger circuits, the CPU time spent for this last operation at a time point *heavily* dominates the CPU time required by the other operations. Most of the CPU time is used for solving systems of linear equations. We currently use a general-purpose direct sparse matrix solver to solve systems of linear equations. The computational cost of noise simulation is still high for large-scale

circuits, which will be reduced by using a more efficient sparse matrix solver that is *tuned* for our problem.

The operations described in the above paragraph are performed at every time point from $r = 0$ to $r = R$. Upon completion, $x_s[r] \cong x_s(t_r)$, $r = 0, ..., R$ contains the *mean* waveforms for the circuit variables as a function of time, which is the usual SPICE transient simulation output. And $K^1[r] \cong K^1(t_r)$, $r = 0, ..., R$ contains the waveforms for the covariance matrix of the noise contents in the circuit variables, as defined by (4.38) as a function of time, which is the noise simulation output.

Chapter 6

Noise Simulation Examples

In this chapter, we present two examples of noise simulation using the implementation discussed in Chapter 5. In particular, noise simulations for a MOSFET ring-oscillator circuit and a BJT active mixer circuit will be presented. For both of these circuits, we have included only the shot and thermal noise sources in the simulation. One reason for this is that flicker noise has little effect on the noise performance of these circuits. Secondly, including the flicker noise sources into simulation increases the simulation time because of the extra nodes created for flicker noise source synthesis.

6.1 MOSFET Ring-Oscillator
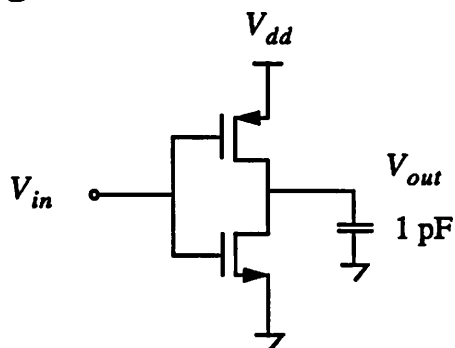
Figure 6.1: MOSFET Inverter

Three of the MOSFET inverters shown in Fig. 6.1 were connected in a ring-oscillator configuration and a noise simulation was done. In Fig. 6.2, the *mean* and *noise variance* of one of the taps of this ring-oscillator can be seen. As seen in Fig. 6.2, the noise at one of the taps of the ring-oscillator is *nonstationary*, that is, the noise variance is *not* a constant as a function of time. The noise variance is *highest* during low-to-high and high-to-low *transitions* of the tap voltage.

Figure 6.2:   Noise Simulation for the MOSFET Ring-oscillator - Tap Voltage

Ring-oscillator based VCOs and delay-lines are used in many phase/delay-locked systems such as clock generators, data synchronization and clock recovery circuits. Phase noise/jitter is a major concern in the design of such systems. Behavioral models which

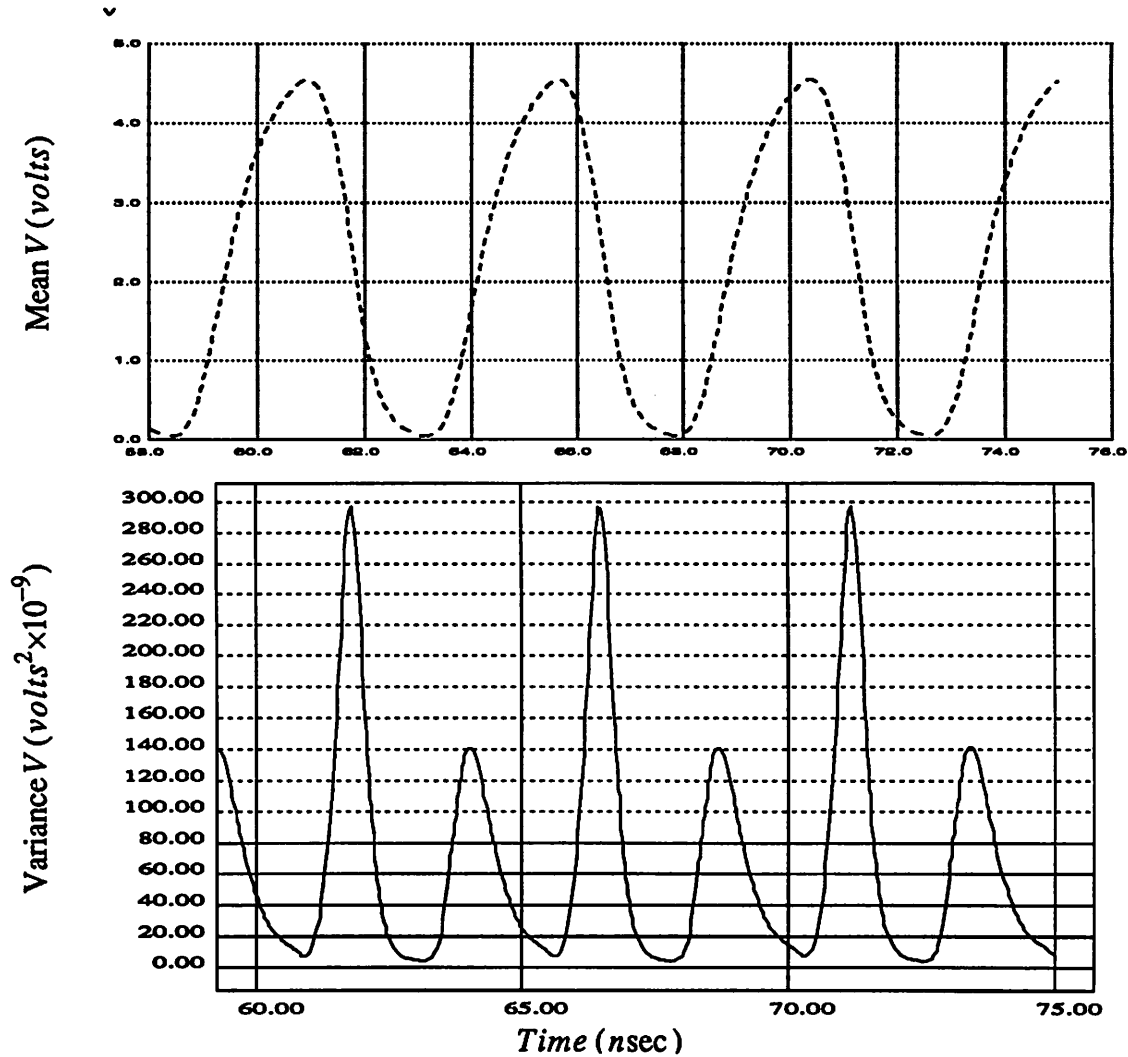capture noise effects, and behavioral simulation is used to predict phase noise/jitter in the design of these systems [15]. Behavioral phase noise simulation requires *noise parameter extraction techniques* which are used to extract behavioral model noise parameters from a transistor-level description of the circuit. Our transistor-level noise simulator can be used for simulating ring-oscillator VCOs and delay-lines to obtain the timing jitter at the output of a delay cell as well as the correlations between the jitters at the outputs of the delay cells. This information is then used in behavioral simulation to predict the phase noise performance of phase/delay-locked systems [15].

6.2 BJT Active Mixer

The BJT active mixer circuit was obtained from industry sources. It contains 14 BJTs, 21 resistors, 5 capacitors, and 18 parasitic capacitors connected between some of the nodes and ground. The LO (local oscillator) input is a sine-wave at 1.75 GHz with an amplitude of 178 mV. The RF input is a sine-wave at 2 GHz with an amplitude of 31.6 mV. Thus, the IF frequency is 250 MHz. Power supply voltage is $V_{cc} = 5V$. Flicker noise sources are not included in the simulation, because flicker noise is rarely a factor at RF and microwave frequencies [2].

This circuit was simulated to calculate the noise variance at the output, and the result is shown in Fig. 6.3, where the noise variance is shown as a function of time. This waveform is periodic with a period of 4 nsecs. (IF frequency is 250 MHz.)

The noise at the output of this circuit is *not* stationary, because the signals applied to the circuit are large enough to change the operating point. The noise analysis of this circuit by assuming a small-signal equivalent circuit around a fixed operating point does not give correct results. Such an analysis would predict the noise at the output as stationary, i.e. a constant noise variance as a function of time.

The noise performance of a mixer circuit is commonly characterized by its *noise figure* which can be defined by [1]

$$NF = \frac{\text{total output noise}}{\text{that part of the output noise due to the source resistance}} \qquad (6.1)$$
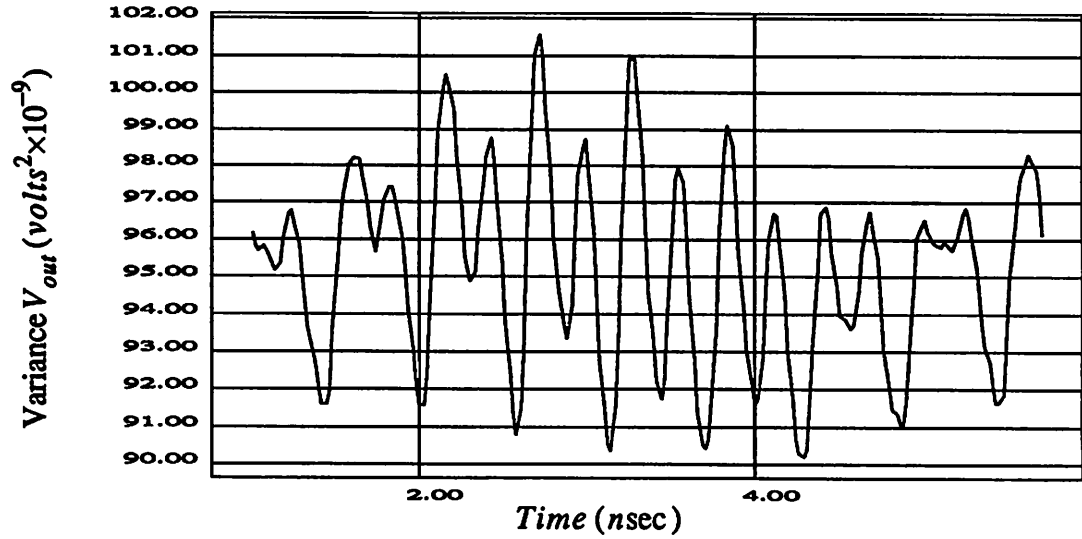
Figure 6.3: BJT Active Mixer - Noise Variance at the Output

This definition is intended for circuits in small-signal operation. For such circuits, noise figure is a scalar quantity. In our case, the noise at the output of the mixer circuit changes as a function of time over one period. We can generalize the noise figure definition such that noise figure is a quantity that is a function of time. For the mixer circuit we have simulated, the noise figure turns out to be a periodic function of time. To calculate the noise figure as defined, we simulate the mixer circuit again to calculate the noise variance at the output with all the noise sources turned off except the noise source for the source resistance $RS_{RF} = 50\Omega$ at the RF port. In this case, the noise variance at the output comes out to be as shown in Fig. 6.4. This waveform is also periodic with a period of 4 nsec.

Then we can calculate the noise figure as below, and the result is shown in Fig. 6.5.

$$NF(t) = 10\log\left(\frac{\text{Total Noise Variance } V_{out}(t)}{\text{Noise Variance } V_{out}(t) \text{ due to the source resistance}}\right) \quad (6.2)$$

As observed in Fig. 6.5, the maximum and minimum value of the noise figure over one period differs by over 4 dB.

In measuring noise figure for actual mixer circuits, the measurement equipment gives a single number instead of a periodic waveform as a function of time for noise figure. In the above discussion, we have designated noise figure to be a quantity which is a

periodic function of time. We can calculate an average for noise figure which will correspond to the measurement noise figure:

$$AverageNF = \frac{1}{T} \int\limits_{t_0}^{t_0+T} NF\,(\tau)\,d\tau \qquad (6.3)$$

where T is the period of the noise figure waveform. For the mixer circuit we have simulated, the average noise figure is calculated to be 20.14 dB, which was obtained by calculating the average over the waveform in Fig. 6.5.
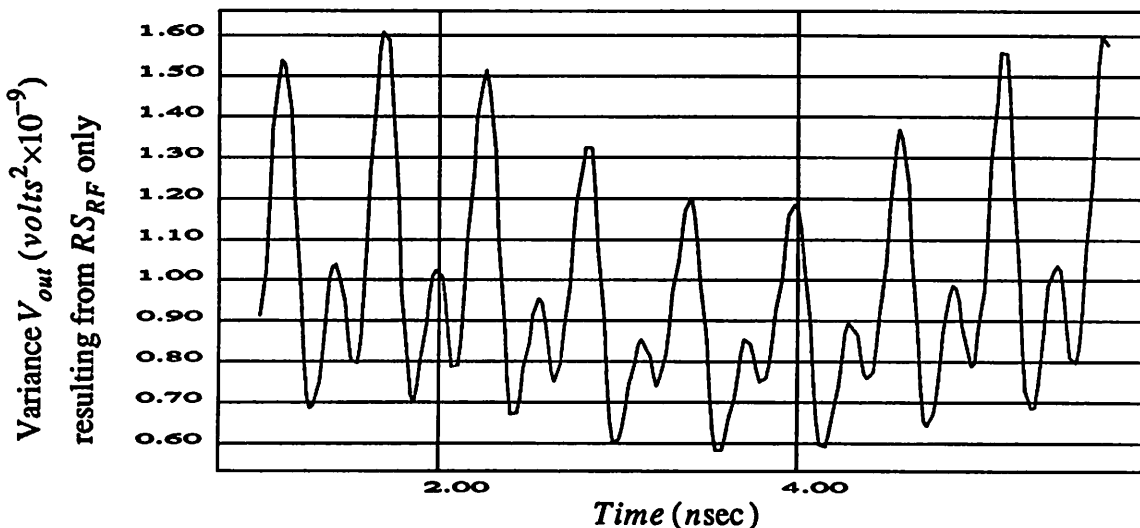
Figure 6.4: BJT Active Mixer - Noise Variance at the Output resulting from RS_{RF} only
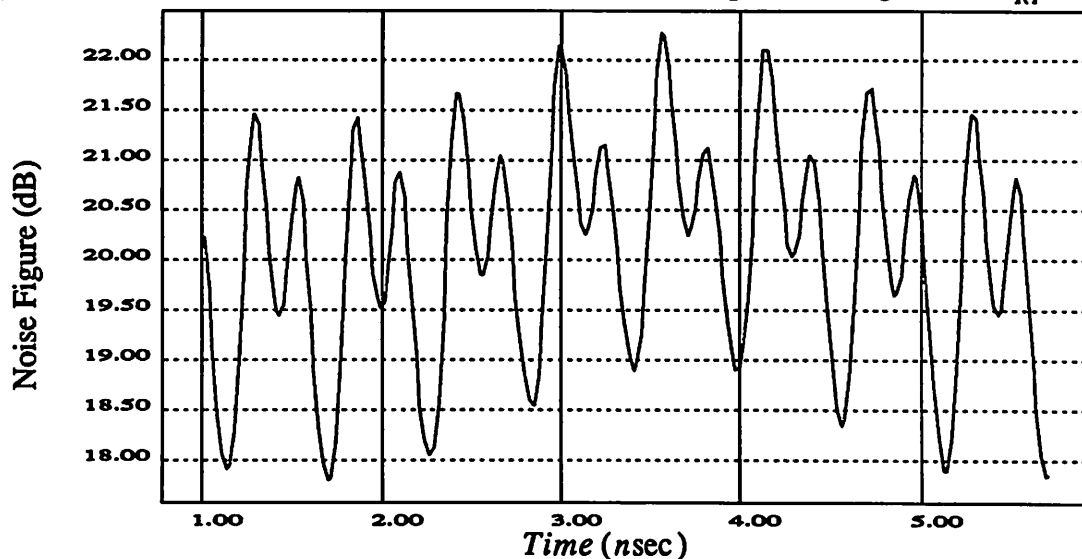
Figure 6.5: BJT Active Mixer - Noise Figure

This BJT mixer circuit has 65 nodes (including the internal nodes for BJTs) which are connected to capacitors. The noise simulation requires the solution of 2145 ($65 \times 66/2$) simultaneous linear equations at every time point, as it was explained in Chapter 5. The simulation (with 250 time points) took approximately a day and a half (human time) on a DECstation 5000/125 with our current implementation. We are in the process of making modifications in the numerical algorithms which are expected to reduce the computational cost of noise simulation.

Chapter 7

Conclusions and Future Work

We have presented a new, *time-domain*, *non-Monte Carlo* method for computer simulation of *electrical noise* in *nonlinear dynamic circuits* with *arbitrary excitations*. Previous work on computer simulation of noise in integrated circuits was reviewed with comparisons to our method. Shot, thermal and flicker noise models for integrated-circuit devices, in the context of our noise simulation method, were presented. This noise simulation method was implemented in a circuit simulator (SPICE). Using this implementation, two examples of noise simulation (a MOSFET ring-oscillator and a BJT active mixer) were given.

We plan to compare the results from this noise simulator with noise measurements on actual circuits as part of the future work. We will include other device models in the noise simulation. We also plan to work on the numerical methods used in the noise simulator to make it more efficient. We will be using our transistor-level noise simulator in the top-down constraint-driven design of a clock generator circuit for a RAMDAC. The noise simulator will be used to extract *noise parameters* in the *behavioral modeling* of phase/delay-locked loops [15].

Appendix

Background on Stochastic Differential Equations

A detailed theoretical and practical treatment of stochastic integrals and stochastic differential equations can be found in [11], [12] and [13]. In this Appendix, only a short introduction to SDEs is given. The results from the theory of SDEs, which were used in the development of the noise simulation method in Chapter 4, are also summarized. Most of the material to be presented in this Appendix is summarized from [11].

A.1 Introduction

Differential equations for random functions (stochastic processes) arise in the investigation of numerous physics and engineering problems. They are usually of one of the following two fundamentally different types.

On the one hand, certain functions, coefficients, parameters, and boundary or initial values in classical differential equation problems can be random. Simple examples are

$$\dot{X}_t = A(t)X_t + B(t) \qquad X_{t_0} = c \qquad (A.1)$$

with random functions $A(t)$ and $B(t)$ as coefficients and with random initial value c, or

$$\dot{X}_t = f(t, X_t, \eta_t) \qquad X_{t_0} = c \qquad (A.2)$$

with the random function η_t, the random initial value c, and the fixed function f (all the functions are scalar). If these random functions have certain regularity properties, one can consider the above-mentioned problems simply as a family of classical problems for the individual sample functions and treat them with the classical methods of the theory of differential equations.

The situation is quite different if random "functions" of the so-called "white noise" type appear in what is written formally as an ordinary differential equation, for example, the "function" ξ_t in the equation

$$\dot{X}_t = f(t, X_t) + G(t, X_t)\xi_t, \qquad X_{t_0} = c \tag{A.3}$$

This "white noise" is conceived as a stationary Gaussian stochastic process with mean value zero and a constant spectral density on the entire real axis. Such a process does not exist in the conventional sense, since it would have to have the Dirac delta function as covariance function, and hence an infinite variance (and independent values at all points). Nonetheless, the "white noise" ξ_t is a very useful mathematical idealization for describing random influences that fluctuate rapidly and hence virtually uncorrelated for different instants of time.

Such equations were first treated in 1908 by Langevin in the study of the Brownian motion of a particle in a fluid. If X_t is a component of the velocity, at an instant t, of a free particle that performs a Brownian motion, Langevin's equation is

$$\dot{X}_t = -\alpha X_t + \sigma \xi_t, \qquad \alpha > 0, \sigma \qquad \text{constants} \tag{A.4}$$

Here, $-\alpha X_t$ is the systematic part of the influence of the surrounding medium due to dynamic friction. The constant α is found from Stoke's law to be $\alpha = 6\pi a \eta / m$, where a is the radius of the (spherical) particle, m is its mass, and η is the viscosity of the surrounding fluid. On the other hand, the term $\sigma \xi_t$ represents the force exerted on the particle by the molecular collisions. Since under normal conditions the particle uniformly undergoes about 10^{21} molecular collisions per second from all directions, $\sigma \xi_t$ is indeed a rapidly varying fluctuational term, which can be idealized as "white noise." If we normalize ξ_t so that its covariance is the delta function, then $\sigma^2 = 2\alpha kT / m$ (where k is Boltzmann's constant and T is the absolute temperature of the surrounding fluid). The same equation (A.4) arises formally for the current in an electric circuit. This time, ξ_t represents

the thermal noise. Of course, (A.4) is a special case of equation (A.3), the right-hand member of which is decomposed as the sum of a systematic part f and a fluctuational $G\xi_t$.

In model (A.4) of Brownian motion, one can calculate explicitly the probability distributions of X_t even though ξ_t is not a random function in the usual sense. As a matter of fact, every process X_t with these distributions (Ornstein-Uhlenbeck process) has sample functions that, with probability 1, are nondifferentiable, so that (A.4) and, more generally, (A.3) cannot be regarded as ordinary differential equations.

For a mathematically rigorous treatment of equations of type (A.3), a new theory is necessary. It turns out that, whereas "white noise" is only a generalized stochastic process, the indefinite integral

$$W_t = \int_0^t \xi_s ds \tag{A.5}$$

can nonetheless be identified with the Wiener process. This is a Gaussian stochastic process with continuous (but nowhere differentiable) sample functions, with mean $\varepsilon[W_t] = 0$ and with covariance $\varepsilon[W_t W_s] = \min(t, s)$.

If we write (A.5) symbolically as

$$dW_t = \xi_t dt \tag{A.6}$$

(A.3) can be put in the differential form

$$dX_t = f(t, X_t) dt + G(t, X_t) dW_t \qquad X_{t_0} = c \tag{A.7}$$

This is a stochastic differential equation (Ito's) for the process X_t. It should be understood as an abbreviation for the integral equation

$$X_t = c + \int_{t_0}^t f(s, X_s) ds + \int_{t_0}^t G(s, X_s) dW_s \tag{A.8}$$

Since the sample functions of W_t are with probability 1 continuous though not of unbounded variation in any interval, the second integral in (A.8) cannot, even for smooth G, be regarded in general as an ordinary Riemann-Stieltjes integral with respect to the sample functions of W_t, because the value depends on the intermediate points in the approximating sums. In 1951, Ito defined integrals of the form

$$Y_t = \int_{t_0}^{t} G(s)\, dW_s \tag{A.9}$$

for a broad class of so-called nonanticipating functionals G of the Wiener process W_t and in doing so put the theory of stochastic differential equations on a solid foundation. This theory has its peculiarities. For example, the solution of the equation

$$dX_t = X_t dW_t \qquad X_0 = 1 \tag{A.10}$$

is not e^{W_t}, but

$$X_t = e^{W_t - t/2} \tag{A.11}$$

which does not derive by purely formal calculation according to the classical rules. It turns out that the solution of a stochastic differential equation of the form (A.7) is a Markov process with continuous sample functions-in fact, a diffusion process. Conversely, every (smooth) diffusion process is the solution of a stochastic differential equation of the form (A.7) where f and G^2 are respectively the coefficients of drift and diffusion.

For diffusion processes, there exist effective methods for calculating transitional and finite-dimensional distributions and distributions of many functionals. These methods belong to the so-called analytical or indirect of probability methods which deal not with the timewise development of the state X_t, but, for example with the timewise development of transition probabilities $P(X_t \in B | X_s = x)$.

In contrast, the calculus of stochastic differential equations belongs to the probabilistic or direct methods, since with them, we are concerned wit the random variable X_t and its variation. An equation of the form (A.7) or (A.8) represents a construction rule (albeit in general a complicated one) with which one can construct the trajectories of X_t from the trajectories of a Wiener process W_t and an initial value c.

In the development of the theory for SDEs, first, the definition of the *stochastic integral*, or *Ito's integral* is developed for integrals of the form given by (A.9). Then, *stochastic differentials* (as given by (A.7)) are defined based on the definition of the *stochastic integral*. The theory for stochastic differential equations (existence and uniqueness of solutions, properties of the solutions, etc.) is developed based on *Ito's Theorem* concerning stochastic differentials. We will give the statement of Ito's Theorem and an example of its

application in Section A.2. The development of the definition of stochastic integral and stochastic differentials, as well as the proofs for theorems, will be omitted here, because we are interested in the practical results obtained from the theory rather than the technicalities of the development and proofs.

The functions f and G in the stochastic differential equation given by (A.7) are in general nonlinear functions of the state X_t of the system. Just as with ordinary differential equations, a much more complete theory can be developed in the stochastic case when the coefficient functions $f(t, x)$ and $G(t, x)$ are linear functions of x, especially when G is independent of x. If $f(t, x)$ is a linear function of x and G is independent of x, then the stochastic differential equation given by (A.7) is said to be *linear in the narrow sense*. Fortunately, the stochastic differential equation for noise, which was derived in Chapter 4 and given by (4.28), is an SDE which is linear in the narrow sense. Actually, (4.28) is a special case of a linear SDE in the narrow sense: For (4.28), $f(t, x)$ is an *homogeneous* linear function of x. In Section (A.3), we concentrate on linear stochastic differential equations in the narrow sense, and summarize the practical results of the theory which were stated and used in the development of the noise simulation method in Chapter 4.

A.2 Ito's Theorem on Stochastic Differentials

Ito's Theorem in its most general form:

Ito's Theorem

Let $u = u(t, x)$ denote a continuous function defined on $[t_0, T] \times R^d$ with values in R^k and with the continuous partial derivatives (k-vectors)

$$\frac{\partial}{\partial t} u(t, x) = u_t$$

$$\frac{\partial}{\partial x_i} u(t, x) = u_{x_i} \qquad x = (x_1, \ldots, x_d)'$$

$$\frac{\partial^2}{\partial x_i \partial x_j} u(t, x) = u_{x_i x_j} \qquad i, j \leq d$$

(A.12)

If the d-dimensional stochastic process X_t is defined on $[t_0, T]$ by the stochastic differential

$$dX_t = f(t)\,dt + G(t)\,dW_t \qquad W_t \rightarrow m-\text{dimensional} \tag{A.13}$$

then the k-dimensional process

$$Y_t = u(t, X_t) \tag{A.14}$$

defined on $[t_0, T]$ with initial value $Y_{t_0} = u(0, X_{t_0})$ also possesses a stochastic differential with respect to the *same Wiener process* W_t, and we have

$$dY_t = \left(u_t(t, X_t) + u_x(t, X_t)f(t) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} u_{x_i x_j}(t, X_t)\,(G(t)\,G(t)')_{ij} \right) dt \\ + u_x(t, X_t)\,G(t)\,dW_t \tag{A.15}$$

Here, $u_x = (u_{x_1}, \ldots, u_{x_d})$ is a $k \times d$ matrix and $u_{x_i x_j}$ is a k-dimensional column vector. The double summation in (A.15) can also be written as follows:

$$\sum_{i=1}^{d}\sum_{j=1}^{d} u_{x_i x_j}(GG')_{ij} = tr(u_{xx}GG') = tr(GG'u_{xx}) \tag{A.16}$$

where $u_{xx} = (u_{x_i x_j})$ is a $d \times d$ matrix whose elements are k-vectors. Then, (A.15) takes the form

$$dY_t = u_t dt + u_x dX_t + \frac{1}{2} tr(GG'u_{xx})\,dt \tag{A.17}$$

The noteworthy is the *extra* differential term in (A.17) which is formed from the second derivatives $u_{x_i x_j}$.

Example using Ito's Theorem

A special case of Ito's Theorem for $k = m = 1$ yields in the case

$$u = x_1 x_2 \tag{A.18}$$

the following result: If

$$\begin{aligned} dX_1(t) &= f_1(t)\,dt + G_1(t)\,dW_t \\ dX_2(t) &= f_2(t)\,dt + G_2(t)\,dW_t \end{aligned} \tag{A.19}$$

then

$$\begin{aligned} d(X_1(t)X_2(t)) &= X_1(t)\,dX_2(t) + X_2(t)\,dX_1(t) + G_1(t)\,G_2(t)\,dt \\ &= (X_1 f_2 + X_2 f_1 + G_1 G_2)\,dt + (X_1 G_2 + X_2 G_1)\,dW_t \end{aligned} \tag{A.20}$$

This is the rule for integration of stochastic integrals by parts. In comparison with the corresponding formulas for ordinary integrals or differentials, there is the extra term

$$G_1 G_2 (dW)^2 = G_1 G_2 dt \qquad \text{(A.21)}$$

A.3 Linear Stochastic Differential Equations in the Narrow Sense

In this section, we shall investigate those equations that are obtained from a deterministic linear system

$$\dot{X}_t = A(t)X_t + a(t) \qquad \text{(A.22)}$$

(where $A(t)$ is a $d \times d$ matrix, and X_t and $a(t)$ are vectors with components in R^d) by the addition of a fluctuational term

$$B(t)\xi_t \qquad \text{(A.23)}$$

(where $B(t)$ is $d \times m$ matrix and ξ_t is an m-dimensional white noise) that is independent of the state of the system; that is we shall investigate equations of the form

$$dX_t = (A(t)X_t + a(t)) dt + B(t) dW_t \qquad \text{(A.24)}$$

If the functions $A(t)$, $a(t)$, and $B(t)$ are measurable and bounded on $[t_0, T]$ (as we shall assume to be the case in what follows), there exists, for every initial value $X_{t_0} = c$ a unique solution.

Now, we review a few familiar items regarding deterministic linear systems $(B(t) = 0)$.

The matrix $\Phi(t) = \Phi(t, t_0)$ of solutions of the homogeneous equation

$$\dot{X}_t = A(t)X_t \qquad \text{(A.25)}$$

with unit vectors $c = e_i$ in the x_i-direction as initial value, in other words, the solution of the matrix equation

$$\dot{\Phi}(t) = A(t)\Phi(t) \qquad \Phi(t_0) = I \qquad \text{(A.26)}$$

is called the *fundamental matrix* of the system

$$\dot{X}_t = A(t)X_t + a(t) \tag{A.27}$$

The solution with initial value $X_{t_0} = c$ can be represented with the aid of $\Phi(t)$ in the following form:

$$X_t = \Phi(t)\left(c + \int_{t_0}^{t} \Phi(s)^{-1} a(s)\, ds\right) \tag{A.28}$$

If, for example, $A(t) = A$ is independent of t, then

$$\Phi(t) = e^{A(t-t_0)} = \sum_{n=0}^{\infty} A^n(t-t_0)^n/n! \tag{A.29}$$

Therefore,

$$X_t = e^{A(t-t_0)}c + \int_{t_0}^{t} e^{A(t-s)} a(s)\, ds \tag{A.30}$$

With this knowledge, we can now easily determine the solution of the "nonhomogeneous" equation (A.24):

Theorem A.3.1. The linear (in the narrow sense) stochastic differential equation

$$dX_t = (A(t)X_t + a(t))\, dt + B(t)\, dW_t, \qquad X_{t_0} = c \tag{A.31}$$

has on $[t_0, T]$ the solution

$$X_t = \Phi(t)\left(c + \int_{t_0}^{t} \Phi(s)^{-1} a(s)\, ds + \int_{t_0}^{t} \Phi(s)^{-1} B(s)\, dW_s\right) \tag{A.32}$$

Here, $\Phi(t)$ is the fundamental matrix of the deterministic equation $\dot{X}_t = A(t)X_t$.

Proof A.3.1. Proof is based on Ito's Theorem.

We mention in particular the following special cases:

Corollary A.3.2. If the matrix $A(t) = A$ in (A.24) is independent of t, then

$$X_t = e^{A(t-t_0)}c + \int_{t_0}^{t} e^{A(t-s)}(a(s)\, ds + B(s)\, dW_s) \tag{A.33}$$

Corollary A.3.3. For $d = 1$ (but m arbitrary),

$$\Phi(t) = \exp\left(\int_{t_0}^{t} A(s)\, ds\right) \tag{A.34}$$

and hence

$$X_t = \exp\left(\int_{t_0}^{t} A(s)\, ds\right)\left(c + \int_{t_0}^{t}\exp\left(-\int_{t_0}^{s} A(u)\, du\right)(a(s)\, ds + B(s)\, dW_s)\right) \tag{A.35}$$

The solution X_t has moments of second order if $\varepsilon[|c|^2] < \infty$. In the case of linear SDEs in the narrow sense, the first two moments of X_t can easily be calculated from the explicit form of the solution:

Theorem A.3.4. For the solution X_t of the linear stochastic differential equation

$$dX_t = (A(t)X_t + a(t))\, dt + B(t)\, dW_t \qquad X_{t_0} = c \tag{A.36}$$

we have, under the assumption $\varepsilon[|c|^2] < \infty$,

$$m_t = \varepsilon[X_t] = \Phi(t)\left(\varepsilon[c] + \int_{t_0}^{t}\Phi(s)^{-1}a(s)\, ds\right) \tag{A.37}$$

Therefore, m_t is the solution of the deterministic linear differential equation

$$\dot{m}_t = A(t)m_t + a(t) \qquad m_{t_0} = \varepsilon[c] \tag{A.38}$$

Next,

$$K(s,t) = \varepsilon[(X_s - \varepsilon[X_s])(X_t - \varepsilon[X_t])']$$

$$K(s,t) = \Phi(s)\left(\begin{array}{c}\varepsilon[(c-\varepsilon[c])(c-\varepsilon[c])'] + \\[4pt] \displaystyle\int_{t_0}^{\min(s,t)}\Phi(u)^{-1}B(u)B(u)'(\Phi(u)^{-1})'\, du\end{array}\right)\Phi(t)' \tag{A.39}$$

In particular, the covariance matrix of the components of X_t

$$K(t) = K(t,t) = \varepsilon[(X_t - \varepsilon[X_t])(X_t - \varepsilon[X_t])'] \tag{A.40}$$

is the unique symmetric nonnegative-definite solution of the matrix equation

$$\dot{K}(t) = A(t)K(t) + K(t)A(t)' + B(t)B(t)' \tag{A.41}$$

with the initial value $K(t_0) = \varepsilon[(c - \varepsilon[c])(c - \varepsilon[c])']$. The differential equation for $K(t) = K(t)'$, (A.41), satisfies the Lipschitz and boundedness conditions on $[t_0, T]$, so that a unique solution exists. (A.41) therefore represents (in view of symmetry of K) a system of $d(d+1)/2$ linear ordinary differential equations.

Now, we present an important special case:

Theorem A.3.5. The solution (A.32) of the linear equation

$$dX_t = (A(t)X_t + a(t))dt + B(t)dW_t, \qquad X_{t_0} = c \qquad (A.42)$$

is a *Gaussian* stochastic process X_t if and only if c is normally distributed or constant. The mean value m_t and the covariance matrix $\varepsilon[(X_s - m_s)(X_t - m_t)']$ are given in Theorem A.3.4. The process X_t has independent increments if and only if c is a constant or $A(t) = 0$ (that is, $\Phi(t) = I$).

Now that we know the process is Gaussian in the case of normally distributed c, the question arises as to when it is *stationary*. A necessary and sufficient condition for this is

$$m_t = \text{const}$$
$$K(s, t) = \bar{K}(s - t) \qquad (A.43)$$

These conditions are certainly satisfied if

$$\varepsilon[c] = 0$$
$$a(t) = 0 \qquad (A.44)$$

(in this case, $m_t = 0$) and

$$A(t) = A$$
$$B(t) = B \qquad (A.45)$$

(that is, the original equation is autonomous and the solution X_t exists on $[t_0, \infty)$). Furthermore, by virtue of (A.39),

$$A\bar{K}(0) + \bar{K}(0)A' = -BB' \qquad (A.46)$$

and

$$\bar{K}(0) = \varepsilon[cc'] \qquad (A.47)$$

The matrix equation (A.46) has a nonnegative-definite solution $\bar{K}(0)$, namely,

$$\bar{K}(0) = \int_0^\infty e^{At} BB' e^{A't} dt \qquad (A.48)$$

if the deterministic equation $\dot{X}_t = AX_t$ is asymptotically stable (that is, if all the eigenvalues of A have negative real parts). Furthermore, from formula (A.39) we get for $t = s$

$$e^{-A(t-t_0)} \bar{K}(0) e^{-A'(t-t_0)} = \bar{K}(0) + \int_{t_0}^t e^{-A(s-t_0)} BB' e^{-A'(s-t_0)} ds \qquad (A.49)$$

Therefore, for general $t, s \geq t_0$,

$$\bar{K}(s-t) = K(s,t) = \begin{cases} e^{A(s-t)} \bar{K}(0) & s \geq t \\ \bar{K}(0) e^{A'(t-s)} & s \leq t \end{cases} \qquad (A.50)$$

We write this result as

Theorem A.3.6. The solution of the equation

$$dX_t = (A(t)X_t + a(t)) dt + B(t) dW_t, \qquad X_{t_0} = c \qquad (A.51)$$

is a *stationary Gaussian process* if $A(t) = A$, $a(t) = 0$, $B(t) = B$, the eigenvalues of A have negative real parts, and c is $N(0, K)$ -distributed, where K is the solution

$$K = \int_0^\infty e^{At} BB' e^{A't} dt \qquad (A.52)$$

of the equation $AK + KA' = -BB'$. Then, for the process X_t,

$$\varepsilon[X_t] = 0 \qquad (A.53)$$

and

$$\varepsilon[X_s X_t'] = \begin{cases} e^{A(s-t)} K & s \geq t \geq t_0 \\ K e^{A'(t-s)} & t \geq s \geq t_0 \end{cases} \qquad (A.54)$$

Obviously, under the above conditions, the process X_t is stationary in the wide sense with the above first and second moments even when c is not normally distributed but $\varepsilon[c] = 0$ and $\varepsilon[cc'] = K$.

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