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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Moment Models of General Transmission Lines with Application to Interconnect Analysis

Qingjian Yu *and Ernest S.Kuh

Electronics Research Lab, U.C.Berkeley

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Abstract

In this paper, we present new moment models for uniform, nonuniform and coupled transmission lines. The moment model of a line is simply based on the relationships between the two port currents (KCL) and the two port voltages (KVL) of the line. The parameters of the model depend on the mean values of the voltage moments and the weighted voltage moments of the line. Simple formulas are given to compute these mean values efficiently. By using such models and moment matching techniques, interconnects modeled as transmission line networks can be efficiently simulated.

1 Introduction

With the rapid increase of the signal frequency and decrease of the feature sizes in high speed electronic circuits, interconnects play increasingly important roles. Not only the signal delay due to interconnects is often significantly larger than the transistor delay, but also the transmission line effect such as reflection, dispersion and crosstalk may cause false action of the circuits.

*On leave from Nanjing University of Science and Technology

The interconnects of a VLSI system on different level are modeled differently. The wires on a chip , on a printed circuit board and on an MCM are usually modeled as lumped or distributed RC lines, lossless transmission lines and lossy transmission lines, respectively [1].

Many papers dealing with the analysis of interconnect networks have been published in recent years [2-8,19-26]. The asymptotic waveform evaluation and other moment matching techniques have recently proven useful in the analysis of interconnects. To make use of such techniques efficiently, we need good models for interconnect networks, especially for transmission line networks.

In the early days, interconnects were modeled as lumped or distributed RC networks. [9] gave the formula for the first moment(Elmore delay) of a node voltage in an RC tree, which was then used in an RC mesh [10]. AWE [19] extended the moment computation method to the general RLC network based on a state variable approach. RICE [20] presented moment model of capacitors and inductors and provided methods to compute moments by analyzing resistive moment model circuits, especially it improved AWE by exploiting the treelike structure of most interconnect circuits. However, in RICE, transmission lines are modeled as a large number of RLGC sections, which is neither exact nor efficient in computation. Moment computation models of transmission lines have been presented in [23, 24, 26]. In [23], the moment models are formed either by recursively solving second order differential equations or by computing matrix exponentials (even for a single line) which is not very efficient in computation. In [24], the moment model is based on the transmission matrix (ABCD matrix) of transmission lines. In order to use such a model, moments of port currents of transmission lines must be included as unknown variables. In [26], a method called reciprocal expansion (REX) is introduced which finds the moments of the reciprocal of a transfer function of an interconnect instead of the moments of the transfer function itself. In REX, the formation of the moment model is based on recursive integration with the time complexity $O(p^4)$ for a pth order model, which is cost for high order computation. Also, this method is good for interconnects modeled as transmission line trees, but its extension to more general interconnect topology has not been known.

In our recent paper [27], we presented a moment matching model for RLC transmission lines. The model is a lumped RLC line. When each transmission line is replaced by its p-th order moment matching model, the resultant circuit has exactly the same moment as that of the original circuit up to the order of p for each output node voltage. In this paper, we will extend this method to develop new moment models for general RLGC transmission lines, including single uniform and nonuniform lines and coupled line systems. Starting from the telegrapher's equations of a transmission line, we form a model of a transmission line based on the relationships between the two port currents (KCL) and the two port voltages (KVL) of the line. The parameters of the model depend on the mean values of voltage moments and weighted voltage moments of the line. Simple formulas are given to compute these mean values efficiently. For uniform lines, the model is exact, and for nonuniform lines, it can be as accurate as needed. The model can be used for any transmission line network, and is especially efficient for RLC transmission line tree networks. Also, the model can be used for distributed RC lines as a special case.

This paper is organized as follows. In Sec.2, we review the moment model of a lumped circuit. In Sec.3, we derive a new moment model for a single uniform transmission line, and we extend the result to a single nonuniform line and coupled lines in Sec.4 and 5, respectively. In Sec.6, we present an efficient recursive algorithm for moment computation of RLC transmission line tree networks. Experimental results and conclusions are given in Sec.7.

2 Moment model of lumped circuits

The method of moment computation proposed in this paper is based on the moment model of circuits. We first review moment model of lumped circuits in this section.

2.1 KCL and KVL of moment model of circuit

Given a linear circuit N, let $V(s)$ and $I(s)$ be the Laplace transform of its branch voltage and current vectors, respectively. As far as moment computation is concerned, the input signal (either a voltage or a current) is set as $\delta(t)$. Expand $V(s)$ and $I(s)$

into Taylor series,

$$V(s) = V^0 - V^1 s + V^2 s^2 + \ldots + (-1)^p V^p s^p + \ldots \tag{1}$$

and

$$I(s) = I^0 - I^1 s + I^2 s^2 + \ldots + (-1)^p I^p s^p + \ldots \tag{2}$$

Then, V^p and I^p are called the p-th order voltage and current moment vector, respectively. The circuit N^p induced from the circuit N for which the branch voltage and current vectors are V^p and I^p is called a p-th order moment model of N.

Let A and B be the incidence matrix and the fundamental loop matrix of the graph induced from the circuit N. Then, the KCL and KVL of the circuit can be expressed as

$$AI(s) = 0 \tag{3}$$

and

$$BV(s) = 0 \tag{4}$$

Substituting Eq.(2) to Eq.(3) and Eq.(1) to Eq.(4), then for each $p \geq 0$, we have

$$AI^p = 0 \tag{5}$$

and

$$BV^p = 0 \tag{6}$$

From the above two equations it can be seen that N^p and N have the same circuit topology.

2.2 Moment model of two terminal elements

A circuit model describing the relationship between the voltage moment and current moment of an element is called its moment model . A model relating a p-th order voltage moment with j-th order current moments ($j \leq p$) or relating a p-th order current moment with j-th order voltage moments ($j \leq p$) is called a p-th order moment model .

For 2-terminal elements R, L, C and independent source, their moment models are as follows:

1. For a resistor R, $V^p = RI^p$; i.e., the p-th order moment model of a resistor is R itself as shown in Fig.1.1.

2. For a capacitance C, $I^p = -CV^{p-1}$. This means that the moment model is a current source with its direction opposite to that of the capacitance voltage and its value determined by C and V^{p-1}. Such a model is shown in Fig.1.2. In the most practical cases, moment computation is implemented recursively from low order up to high orders. In this case, the p-th order moment model for a capacitance is an independent current source.

3. For an inductance L, $V^p = -LI^{p-1}$. The p-th order moment model is a voltage source as shown in Fig.1.3.

4. For an independent voltage (current) source, $V(s) = 1$ ($I(s) = 1$) and its model is just a voltage (current) source with the value equal to 1 for the 0-th order moment and the value equal to 0 for higher order moments.

2.3 Moment model of dependent sources

For the four types of dependent sources: VCCS (voltage-controlled current source), VCVS (voltage-controlled voltage source), CCCS (current-controlled current source) and CCVS (current-controlled voltage source), if their parameters are constant, then their p-th order moment models are the same as themselves.

2.4 Moment model of a lumped circuit

For a given lumped circuit, replace each element by its p-th order moment model , the p-th order moment model of the circuit will be formed. It is a resistive circuit. By analyzing the circuit, all the p-th moments of the node voltages and branch currents can be found. After that, the p+1 -th order moment model of the circuit can be formulated. Thus, the moment computation can be implemented recursively from order 0 to any order needed.

In the next three sections, we will derive moment models for single uniform transmission lines, single nonuniform transmission lines and coupled transmission lines. By using such models and the models of lumped elements, moment models of interconnects made of lumped and distributed elements can be formed and moment computation can be implemented by using these models.

3 Moment model of single uniform transmission line

3.1 T-typed moment model of single uniform transmission line

We first consider a single uniform RLGC transmission line TL. Let r, l, g, c and d be its resistance, inductance, conductance, capacitance per unit length and the length, and $R = rd$, $L = ld$, $G = gd$ and $C = cd$ be its total resistance, inductance, conductance and capacitance. Let $V(x, s)$ and $I(x, s)$ be its line voltage and line current at coordinate x, where $x = 0$ and $x = d$ correspond to the two ends of the line. The telegrapher's equations of the line are as follows:

$$\frac{dV(x,s)}{dx} = -rI(x,s) - slI(x,s) \tag{7}$$

$$\frac{dI(x,s)}{dx} = -gV(x,s) - scV(x,s) \tag{8}$$

Note that these two equations are in fact the KVL and KCL equations for an infinitesimal section of the line at coordinate x. Let

$$V(x,s) = V^0(x) - V^1(x)s + V^2(x)s^2 + \ldots + (-1)^p V^p(x)s^p + \ldots \tag{9}$$

and

$$I(x,s) = I^0(x) - I^1(x)s + I^2(x)s^2 + \ldots + (-1)^p I^p(x)s^p + \ldots \tag{10}$$

Substituting Eqs.(9) and (10) to Eqs.(7) and (8), and letting the coefficients of s^p $(p = 0, 1, 2, \ldots)$ on both side of the equations be equal, we have

$$\frac{dV^p(x)}{dx} = -rI^p(x) + lI^{p-1}(x) \tag{11}$$

$$\frac{dI^p(x)}{dx} = -gV^p(x) + cV^{p-1}(x) \tag{12}$$

Now we derive an equation relating $I^p(d)$ with $I^p(0)$. Integrating both sides of Eq.(12) from d to x, we have

$$I^p(x) - I^p(d) = -g\int_d^x V^p(y)dy + c\int_d^x V^{p-1}(y)dy \tag{13}$$

and

$$I^p(0) = I^p(d) + g\int_0^d V^p(y)dy - c\int_0^d V^{p-1}(x)dx \tag{14}$$

We define

$$U^p = \frac{1}{d}\int_0^d V^p(x)dx \quad . \tag{15}$$

where U^p is the mean of the p-th order voltage moments $V^p(x)$ along the line and is called the p-th order mean for simplicity. Then, we have

$$I^p(0) = I^p(d) + GU^p - CU^{p-1} \tag{16}$$

Note that the difference $I^p(0) - I^p(d) = GU^p - CU^{p-1}$. The first term GU^p represents the total p-th order current moment flowing from the line to ground through the conductance of the line, and the second term $-CU^{p-1}$ represent the total p-th order current moment toward the line through the capacitance of the line. These two terms are characterized by the total conductance G with the p-th order mean U^p, and the total capacitance C with the p-1 the order mean U^{p-1}, respectively.

Next, we derive an equation relating $V^p(d)$ with $V^p(0)$. Integrate both sides of Eq.(11) from 0 to x. By using Eq.(13), we have

$$V^p(x) - V^p(0) = -rxI^p(d) + lxI^{p-1}(d) + rg\int_0^x\int_d^y V^p(z)dzdy$$
$$-rc\int_0^x\int_d^y V^{p-1}(z)dzdy - lg\int_0^x\int_d^y V^{p-1}(z)dzdy + lc\int_0^x\int_d^y V^{p-2}(z)dzdy \tag{17}$$

where V^{-1} is defined as 0 when $p=1$. Especially, when $x=d$, we have

$$V^p(d) - V^p(0) = -RI^p(d) + LI^{p-1}(d) + rg\int_0^d\int_d^x V^p(y)dydx$$
$$-rc\int_0^d\int_d^x V^{p-1}(y)dydx - lg\int_0^d\int_d^x V^{p-1}(y)dydx + lc\int_0^d\int_d^x V^{p-2}(y)dydx \tag{18}$$

Now we transform the above double integrals to single ones. Let $Q^j(x) = \int_d^x V^j(y)dy$. Then, in the above equation, $\int_0^d\int_d^x V^j(y)dydx = \int_0^d Q^j(x)dx = xQ^j(x)\mid_{x=0}^{x=d} - \int_0^d x\frac{dQ^j(x)}{dx}dx$ $= -\int_0^d xV^j(x)dx$. Therefore, we have

$$V^p(d) - V^p(0) = -RI^p(d) + LI^{p-1}(d)$$

$$- rg\int_0^d xV^p(y)dx + rc\int_0^d xV^{p-1}(y)dx + lg\int_0^d xV^{p-1}(y)dx - lc\int_0^d xV^{p-2}(x)dx \quad (19)$$

We define

$$W^p = \frac{1}{d^2}\int_0^d xV^p(y)dx \quad (20)$$

where W^p is the mean value of weighted p-th order voltage moments along a line with the weight equal to the relative distance x/d. W^p is called a p-th order x-mean for simplicity. Then, we have

$$V^p(d) = V^p(0) - RI^p(d) + LI^{p-1}(d) - RGW^p + RCW^{p-1} + LGW^{p-1} - LCW^{p-2} \quad (21)$$

From the above equation, it can be seen that the difference of the voltage moments $V^p(d) - V^p(0)$ consists of two parts. The first part $-RI^p(d) + LI^{p-1}(d)$ represents the contribution of the load current $I(d)$. As $I(d)$ can be regarded as a current component flowing through the whole line, its effect is the same as if it passed through a lumped RL branch. The second part can be divided into two subparts. The first subpart $-RGW^p + LGW^{p-1}$ represents the voltage drop caused by the conductance currents, and the second subpart $RCW^{p-1} - LCW^{p-2}$ represents the voltage drop caused by capacitance currents. For this part, the x-mean W characterizes the contribution of the conductive and capacitive currents. The weight x/d is introduced in W because the current flowing through a conductance or a capacitance at position x only causes a voltage drop in the region [0,x].

Let $E_s^p = LI^{p-1}(d) + RCW^{p-1} + LGW^{p-1} - LCW^{p-2}$. Then Eq.(21) can be rewritten as follows:

$$V^p(d) = V^p(0) - RI^p(d) - RGW^p + E_s^p \quad (22)$$

From Eqs.(16) and (22), a p-th order T-typed moment model of a transmission line can be derived as shown in Fig.2a. In this model, $V^p(0)$ and $V^p(d)$ are regarded as the port voltages, and $I^p(0)$ and $I^p(d)$ are regarded as port currents. The current source CU^{p-1} and voltage source E_s^p are independent sources, and the current source GU^p and RGW^p are dependent sources. We will make the last two sources explicitly dependent on the p-th order port voltages as will be shown in the next subsection.

In most practical cases, when the operating frequency of a circuit is not very high, the dielectric loss in a transmission line is negligible compared with the resistance loss, and the leakage current through the dielectric medium is much smaller than that of the distributed capacitances during the most part of a transient response. In such cases, the g parameter can be set to 0 and the transmission line becomes an RLC line. Then , Eq.(16) is simplified to

$$I^p(0) = I^p(d) - CU^{p-1} \tag{23}$$

and Eq.(22) to

$$V^p(d) = V^p(0) - RI^p(d) + E_s^p \tag{24}$$

where $E_s^p = LI^{p-1}(d) + RCW^{p-1} - LCW^{p-2}$, and the moment model is simplified to that shown in Fig.2b. This T-typed model is equivalent to the π typed model as shown in Fig.2c, where $J_{0s}^p = CU^{p-1} - E_s^p/R$ and $J_{ds}^p = E_s^p/R$.

3.2 Computation of mean values U and W

From what mentioned in the last section, the moment model of a transmission line depends on two mean values U and W of some order. In this section, we present formulas for the computation of U^p and W^p.

From the ABCD matrix of the transmission line, we have

$$\begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = \begin{bmatrix} ch\theta x & -Z_0 sh\theta x \\ -\frac{1}{Z_0} sh\theta x & ch\theta x \end{bmatrix} \begin{bmatrix} V(0,s) \\ I(0,s) \end{bmatrix} \tag{25}$$

where $\theta = \sqrt{(sc+g)(sl+r)}$ and $Z_0 = \sqrt{(sl+r)/(sc+g)}$. Let $x = d$ and $\gamma = \theta d$, we have

$$I(0,s) = \frac{ch\gamma V(0,s) - V(d,s)}{Z_0 sh\gamma} \tag{26}$$

$$V(x,s) = \frac{sh\theta(d-x)V(0,s) + sh\theta x V(d,s)}{sh\gamma} \tag{27}$$

Define

$$U(s) = \frac{1}{d}\int_0^d V(x,s)dx \tag{28}$$

and

$$W(s) = \frac{1}{d^2}\int_0^d xV(x,s)dx \tag{29}$$

$U(s)$ and $W(s)$ are the mean of the voltage $V(x,s)$ and weighted voltage $\frac{x}{d}V(x,s)$ along the line, and are called the mean and x-mean functions, respectively. It is easy to show that $U(s) = \sum_{j=0}^{\infty}(-1)^j U^j s^j$ and $W(s) = \sum_{j=0}^{\infty}(-1)^j W^j s^j$, i.e., U^p and W^p are the p-th order moments of $U(s)$ and $W(s)$, respectively, and we can find U^p and W^p from $U(s)$ and $W(s)$.

Substituting Eq.(27) to Eqs.(28) and (29), respectively, we have

$$U(s) = f(s)(V(0,s) + V(d,s)) \tag{30}$$

where

$$f(s) = (ch\gamma - 1)/\gamma sh\gamma \tag{31}$$

and

$$W(s) = h_1(s)V(0,s) + h_2(s)V(d,s) \tag{32}$$

where

$$h_1(s) = (sh\gamma - \gamma)/\gamma^2 sh\gamma \tag{33}$$

and

$$h_2(s) = (\gamma ch\gamma - sh\gamma)/\gamma^2 sh\gamma \tag{34}$$

Let $V(0,s) = \sum_{p=0}^{\infty}(-1)^p V^p(0)s^p$, $V(d,s) = \sum_{p=0}^{\infty}(-1)^p V^p(d)s^p$, $f(s) = \sum_{k=0}^{\infty} f_k s^k$, $h_1(s) = \sum_{k=0}^{\infty} h_{1k}s^k$, and $h_2(s) = \sum_{k=0}^{\infty} h_{2k}s^k$. Then, from Eqs.(30) and (32), we have

$$U^p = \sum_{j=0}^{p}(-1)^j f_j(V^{p-j}(0) + V^{p-j}(d)) \tag{35}$$

and

$$W^p = \sum_{j=0}^{p}(-1)^j(h_{1j}V^{p-j}(0) + h_{2j}V^{p-j}(d)) \tag{36}$$

The method for the computation of f_j, h_{1j} and h_{2j} is shown in Appendix A.

Eqs.(35) and (36) can be rewritten in the following form:

$$U^p = f_0(V^p(0) + V^p(d)) + U_s^p \tag{37}$$

where $U_s^p = \sum_{j=1}^{p}(-1)^j f_j(V^{p-j}(0) + V^{p-j}(d))$ only depends on $V^i(0)$ and $V^i(d)$ with $i < p$, and

$$W^p = h_{10}V^p(0) + h_{20}V^p(d) + W_s^p \tag{38}$$

where $W_s^p = \sum_{j=1}^{p}(-1)^j(h_{1j}V^{p-j}(0) + h_{2j}V^{p-j}(d))$ is also independent of $V^p(0)$ and $V^p(d)$. From the above two equations, the p-th order moment model of an RLGC transmission line can be reformed as shown in Fig.3a or b. The parameters of the model in Fig.3b are as follows: $G_0 = G(f_0 - h_{10})$, $G_{0d} = G(f_0 - h_{20})$, $J^p_{0s} = CU^{p-1} - GU^p_s - E^p_s/R + GW^p_s$, $G_d = Gh_{20}$, $G_{d0} = Gh_{10}$ and $J^p_{ds} = E^p_s/R - GW^p_s$. Note that in this model, the parameters of the resistors and dependent sources are constant w.r.t. the order p.

4 Moment model of nonuniform transmission lines

In this section, we extend the moment model of uniform transmission lines to nonuniform transmission lines. We will follow the same way as we have done for the derivation of the moment model of a uniform line.

4.1 Equations of voltage and current moments

Consider a nonuniform line with $r(x)$, $l(x)$, $c(x)$ and $g(x)$ being the resistance, inductance, capacitance and conductance per unit length at coordinate x where $x = 0$ and $x = d$ correspond to the near and far end of the line, respectively. Then, the telegrapher's equation of the line can be written as

$$\frac{dV(x,s)}{dx} = -r(x)I(x,s) - sl(x)I(x,s) \tag{39}$$

$$\frac{dI(x,s)}{dx} = -g(x)V(x,s) - sc(x)V(x,s) \tag{40}$$

From the above two equations, we have

$$\frac{dV^p(x)}{dx} = -r(x)I^p(x) + l(x)I^{p-1}(x) \tag{41}$$

and

$$\frac{dI^p(x)}{dx} = -g(x)V^p(x) + c(x)V^{p-1}(x) \tag{42}$$

Integrating both sides of Eq.(42) from d to x, we have

$$I^p(x) - I^p(d) = -\int_d^x g(x)V^p(x)dx + \int_d^x c(x)V^{p-1}(x)dx \tag{43}$$

and

$$I^p(0) - I^p(d) = \int_0^d g(x)V^p(x)dx - \int_0^d c(x)V^{p-1}(x)dx \tag{44}$$

As can be imagined, now $\int_0^d g(x)V^p(x)dx$ and $-\int_0^d c(x)V^{p-1}(x)dx$ represent the contribution of the conductance and capacitance currents to the difference $I^p(0) - I^p(d)$, respectively. Now let $G = \int_0^d g(x)dx$ and $C = \int_0^d c(x)dx$ be the total conductance and total capacitance of the line. We define

$$U_g^p = \frac{1}{G}\int_0^d g(x)V^p(x)dx \tag{45}$$

and

$$U_c^p = \frac{1}{C}\int_0^d c(x)V^p(x)dx \quad . \tag{46}$$

U_g^p and U_c^p are the mean values of weighted p-th order voltage moment $E^p(x)$, with the first one being weighted by $g(x)$ and the second one being weighted by $c(x)$. They are called the p-th order g-mean and c-mean, respectively. By using the above definitions, Eq.(44) can be expressed as follows:

$$I^p(0) = I^p(d) - GU_g^p + CU_c^{p-1} \tag{47}$$

Eq.(47) is similar to Eq.(16) except that now U_g^p and U_c^{p-1} replace U^p and U^{p-1}, respectively. In the case that $c(x)/C = g(x)/G$, $U_c^p = U_g^p$. The physical meaning of Eq.(47) is similar to that of Eq.(16).

Let $R = \int_0^d r(x)dx$ and $L = \int_0^d l(x)dx$ be the total resistance and inductance of the line. Integrating both sides of Eq.(41) and substituting Eq.(43) to it, we have

$$V^p(d) - V^p(0) = -RI^p(d) + LI^{p-1}(d) + \int_0^d r(x)\int_d^x g(y)V^p(y)dydx$$
$$-\int_0^d r(x)\int_d^x c(y)V^{p-1}(y)dydx - \int_0^d l(x)\int_d^x g(y)V^{p-1}(y)dydx + \int_0^d l(x)\int_d^x c(y)V^{p-2}(y)dydx \tag{48}$$

Now we transform the four double integrals in the above equation into single integrals. Take $I_{rg} = \int_0^d r(x)\int_d^x g(y)V^p(y)dydx$ as an example. Let $r_t(x) = \int_0^x r(y)dy$ be the total resistance in the interval $[0, x]$ and denote $Q(x) = \int_d^x g(y)V^p(y)dy$. Then, $I_{rg} = \int_0^d r(x)\int_d^x g(y)V^p(y)dydx = \int_0^R Q(x)dr_t(x) = Q(x)r_t(x)\ |_{x=0}^{x=d} - \int_0^d r_t(x)g(x)V^p(x)dx = -\int_0^d r_t(x)g(x)V^p(x)dx$. Let $l_t(x) = \int_0^x l(y)dy$. We define

$$W_{rg}^p = \frac{1}{RG}\int_0^d r_t(x)g(x)V^p(x)dx \tag{49}$$

$$W_{rc}^p = \frac{1}{RC}\int_0^d r_t(x)c(x)V^p(x)dx \tag{50}$$

$$W_{lg}^p = \frac{1}{LG}\int_0^d l_t(x)g(x)V^p(x)dx \tag{51}$$

and

$$W_{lc}^p = \frac{1}{LC}\int_0^d l_t(x)c(x)V^p(x)dx \tag{52}$$

Then, Eq.(48) becomes

$$V^p(d) = V^p(0) - RI^p(d) + LI^{p-1}(d) - RGW_{rg}^p + RCW_{rc}^{p-1} + LGW_{lg}^{p-1} - LCW_{lc}^{p-2} \tag{53}$$

Eq.(53) is similar to Eq.(21) except that now we have four $W's$ instead of one. In the special case that $r(x)/R = l(x)/L$ and $c(x)/C = g(x)/G$, these four $W's$ become the same. W_{rg}^p, W_{rc}^p, W_{lg}^p and W_{lc}^p are mean values of the weighed p-th order voltage moments with the weights being $r_t(x)g(x)$, $r_t(x)c(x)$, $l_t(x)g(x)$ and $l_t(x)c(x)$, and are called the p-th order rg, rc, lg and lc - mean, respectively. From Eqs.(47) and (53), a p-th order moment model for the transmission line can be derived as shown in Fig.4. This model is similar to that shown in Fig.2, except that now $E_s^p = LJ^{p-1}(d) + RCW_{rc}^{p-1} + LGW_{lg}^{p-1} - LCW_{lc}^{p-2}$.

4.2 Computation of the values U and W

In order to derive formulas for the values U and W, we use Taylor series expansion for $r(x)$, $l(x)$, $g(\dot{x})$ and $c(x)$ such that $r(x) = \sum_{n=0}^{\infty} r_n x^n$, etc., and use Taylor series expansion for any two variable function $F(x,s)$ as $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f_{nk}s^k x^n$. Starting from the telegrapher's equations, we can find a formula relating $E(x,s)$ with $E(0,s)$ and $E(d,s)$ as follows with the derivation shown in Appendix B:

$$V(x,s) = P(x,s)V(0,s) + Q(x,s)V(d,s) \tag{54}$$

Expanding $P(x,s)$ and $Q(x,s)$ in Taylor series of x and s and truncating the infinite series for x by a finite number N: $P(x,s) = \sum_{n=0}^{N}\sum_{k=0}^{\infty} P_{nk}x^n s^k$ and $Q(x,s) = \sum_{n=0}^{N}\sum_{k=0}^{\infty} Q_{nk}x^n s^k$, we have

$$U_g^p = \sum_{j=0}^{p}(-1)^j(f_{g1j}V^{p-j}(0) + f_{g2j}V^{p-j}(d)) \tag{55}$$

where $f_{g1j} = \sum_{n=0}^{N}\frac{1}{n+1}\sum_{k=0}^{N} P_{kj}g_{n-k}d^{n+1}$ and $f_{g2j} = \sum_{n=0}^{N}\frac{1}{n+1}\sum_{k=0}^{N} Q_{kj}g_{n-k}d^{n+1}$.

The formulas for other U and W values are similar and are omitted for simplicity.

Let $W^p_{rg} = \sum_{j=0}^{p}(-1)^j(h_{rg1j}V^{p-j}(0) + h_{rg2j}V^{p-j}(d))$, then the p-th order moment model of a nonuniform line shown in Fig.4a can be transformed into that shown in Fig.3b with the parameters $G_0 = G(f_{g10} - h_{rg10})$, $G_{0d} = G(f_{g20} - h_{rg20})$, $J^p_{0s} = CU^{p-1}_c - GU^p_{gs} - E^p_s/R + GW^p_{rgs}$, $G_d = Gh_{rg20}$, $G_{d0} = Gh_{rg10}$ and $J^p_{ds} = E^p_s/R - GW^p_{rgs}$, where $U^p_{gs} = \sum_{j=1}^{p}(-1)^j(f_{g1j}V^{p-j}(0) + f_{g2j}V^{p-j}(d))$ and $W^p_{rgs} = \sum_{j=1}^{p}(-1)^j(h_{rg1j}V^{p-j}(0) + h_{rg2j}V^{p-j}(d))$.

5 Moment model of coupled transmission lines

Now we extend our results to coupled transmission lines. Here we only deal with the uniform line case, and the extension to the nonuniform line case can be done by following the same way as stated in the last section.

The extension can simply be done by redefining the parameters r, l, g, c, R, L, G and C as matrices, and V, I, U and W as vectors. All the equations in Sec.3.1 are still valid with the above interpretation. The moment model can be formed by using these equations. For the i-th line coupled with lines from number 1 to n, we have

$$I^p_i(0) = I^p_i(d) + \sum_{k=1}^{n}(G_{ik}U^p_k - C_{ik}U^{p-1}_k) \tag{56}$$

and

$$V^p_i(d) = V^p_i(0) - \sum_{k=1}^{n}(R_{ik}I^p_k(d) + R_{ik}\sum_{j=1}^{n}G_{kj}W^p_j) + E^p_{si} \tag{57}$$

where

$$E^p_{si} = \sum_{k=1}^{n}\{L_{ik}I^{p-1}_k(d) + \sum_{j=1}^{n}[(R_{ik}C_{kj} + L_{ik}G_{kj})W^{p-1}_j - L_{ik}C_{kj}W^{p-2}_j]\} \tag{58}$$

A moment model for the case $n = 2$ is shown in Fig.5 for illustration.

Now we discuss the computation of mean values U and W. We first consider 2 typical cases.

5.1 Case 1.

This is the case with the typical structure that the transmission line system is made of microstrips with the same size and the same separation between two adjacent lines.

It is also assumed that the resistance and conductance matrices of the system are diagonal, and the coupling effects are significant only between immediately adjacent lines so that its inductance matrix and capacitance matrix are tridiagonal and Toeplitz matrices and there exists a constant transformation matrix to diagonalize both the matrices simultaneously [25].

Suppose that in matrix L, $l_{ii} = l$ for all i, $l_{ij} = l_m$ for $|i-j| \leq 1$ and $i \neq j$ and $l_{ij} = 0$ for $|i-j| > 1$. Similarly, in matrix C, $c_{ii} = c$ for all i, $c_{ij} = c_m$ for $|i-j| \leq 1$ and $i \neq j$ and $c_{ij} = 0$ for $|i-j| > 1$. Let $\mu_i = -2cos(i\pi/(n+1))$, $\phi_0(\mu) = 1$, $\phi_1(\mu) = \mu$, and $\phi_j(\mu) = \mu\phi_{j-1}(\mu) - \phi_{j-2}(\mu)$ for $j > 1$. Let $\delta_j^2 = \sum_{i=1}^{n}(\phi_{i-1}(\mu_j))^2$. Then, there exists a matrix $P = [p_{ij}]$ with $p_{ij} = \phi_{i-1}(\mu_j)/\delta_j$ such that $P^{-1} = P^t$, $l' = P^t l P$ and $c' = P^t c P$ are diagonal matrices, $r = P^t r P$ and $g = P^t g P$. Using the transformation $V = PE$ and $I = PJ$, we will have the following decoupled system:

$$\frac{dE(x,s)}{dx} = -rJ(x,s) - sl'J(x,s) \tag{59}$$

$$\frac{dJ(x,s)}{dx} = -gE(x,s) - sc'E(x,s) \tag{60}$$

From $V = PE$ we have $E(0,s) = P^t V(0,s)$, $E(d,s) = P^t V(d,s)$, $E^p(0) = P^t V^p(0)$ and $E^p(d) = P^t V^p(d)$. Let the p-th order U and W values of the i-th transmission line be U_i^p and W_i^p, and those of the j-th decoupled transmission line be U_{dj}^p and W_{dj}^p. U_{dj}^p and W_{dj}^p can be found by using formulas (35) and (36), then U_i^p and W_i^p can be found by using the following equations:

$$U_i^p = \sum_{j=1}^{n} p_{ij} U_{dj}^p \tag{61}$$

and

$$W_i^p = \sum_{j=1}^{n} p_{ij} W_{dj}^p \tag{62}$$

5.2 Case 2.

In this case all the lines are RLC lines, i.e., their conductance matrix g is a zero matrix. Eq.(13) is now simplified to

$$I^p(x) = I^p(d) + c\int_d^x V^{p-1}(y)dy \tag{63}$$

and Eq.(17) to

$$V^p(x) = V^p(0) + (-rI^p(d) + lI^{p-1}(d))x + rcA^{p-1}(x) - lcA^{p-2}(x) \tag{64}$$

where

$$A^j(x) = -\int_0^x \int_d^y V^j(z)dzdy \tag{65}$$

Let

$$X^j = \frac{1}{d^3}\int_0^d A^j(x)dx \tag{66}$$

and

$$Z^j = \frac{1}{d^4}\int_0^d xA^j(x)dx \tag{67}$$

Then, we have

$$U^p = V^p(0) + \frac{1}{2}(-RI^p(d) + LI^{p-1}(d)) + RCX^{p-1} - LCX^{p-2} \tag{68}$$

and

$$W^p = \frac{1}{2}V^p(0) + \frac{1}{3}(-RI^p(d) + LI^{p-1}(d)) + RCZ^{p-1} - LCZ^{p-2} \tag{69}$$

Let $I^0(d)$ be the 0-th order current moment vector of the line system. As the conductance matrix is zero, so $V^0(x) = V^0(0) - rI^0(d)x$ and we can find

$$U^0 = V^0(0) - \frac{1}{2}RI^0(d)$$

and

$$W^0 = \frac{1}{2}V^0(0) - \frac{1}{3}RI^0(d)$$

Starting from $j = 0$ and using Eqs.(64) - (67), we can find X^j and Z^j recursively. It can be shown that X^j and Z^j are characterized by a coefficient array C as shown in Table 1.

From Eqs.(64) - (67), it can be understood that X^p and Z^p are polynomials of variables $V^k(0)$ and $I^k(d)$ with $k = 0 - p$. The coefficient of each term is the product of some c_j and R, L and C. For example,

$$X^0 = c_1V^0(0) + c_2(-RI^0(d))$$

$$Z^0 = c_2V^0(0) + c_3(-RI^0(d))$$

c_1	c_2	c_3	c_4	c_5	c_6
$\frac{1}{3}$	$\frac{5}{24}$	$\frac{2}{15}$	$\frac{61}{720}$	$\frac{17}{315}$	$\frac{277}{8064}$
c_7	c_8	c_9	c_{10}	c_{11}	c_{12}
$\frac{62}{2835}$	$\frac{50521}{3628800}$	$\frac{1382}{155925}$	$\frac{540553}{95800320}$	$\frac{21844}{6081075}$	$\frac{598082943}{261534873600}$

Table 1: Coefficient Array C

We define an operator $shift()$. For a term $A = c_i P$ where P is independent of c_i, $shift(A) = c_{i+2}P$; and for a term $A = B + C$, $shift(A) = shift(B) + shift(C)$. Then, from Eqs.(64) - (67), we have the following recursive formulas:

$$X^{p+1} = c_1 V^{p+1}(0) + c_2(-RI^{p+1}(d) + LI^p(d)) + RCshift(X^p) - LCshift(X^{p-1}) \quad (70)$$

and

$$Z^{p+1} = c_2 V^{p+1}(0) + c_3(-RI^{p+1}(d) + LI^p(d)) + RCshift(Z^p) - LCshift(Z^{p-1}) \quad (71)$$

Example 1

From

$$X^0 = c_1 V^0(0) + c_2(-RI^0(d))$$

$$Z^0 = c_2 V^0(0) + c_3(-RI^0(d))$$

we have

$$X^1 = c_1 V^1(0) + c_2(-RI^1(d) + LI^0(d)) + RCshift(X^0)$$

$$= c_1 V^1(0) + c_2(-RI^1(d) + LI^0(d)) + RC(c_3 V^0(0) + c_4(-RI^0(d)))$$

$$Z^1 = c_2 V^1(0) + c_3(-RI^1(d) + LI^0(d)) + RCshift(Z^0)$$

$$= c_2 V^1(0) + c_3(-RI^1(d) + LI^0(d)) + RC(c_4 V^0(0) + c_5(-RI^0(d)))$$

$$X^2 = c_1 V^2(0) + c_2(-RI^2(d) + LI^1(d)) + RCshift(X^1) - LCshift(X^0)$$

$$= c_1 V^2(0) + c_3(-RI^2(d) + LI^1(d)) + RC[c_3 V^1(0) + c_4(-RI^1(d) + LI^0(d)) + RC(c_5 V^0(0) + c_6(-RI^0(d)))]$$

$$-LC[c_3 V^0(0) + c_4(-RI^0(d))]$$

$$Z^2 = c_2 V^2(0) + c_3(-RI^2(d) + LI^1(d)) + RCshift(Z^1) - LCshift(Z^0)$$

$$= c_2V^2(0)+c_3(-RI^2(d)+LI^1(d))+RC[c_4V^1(0)+c_5(-RI^1(d)+LI^0(d))+RC(c_6V^0(0)+c_7(-RI^0(d)))]$$

$$-LC[c_5V^0(0) + c_6(-RI^0(d))]$$

and so on.

In the case that there are no leakage resistors in the network, $I^0(d) = \mathbf{0}$ and $V^0(0) = \mathbf{e}$ being a unit vector. In the general case, $I^0(d)$ and $V^0(0)$ can be found by replacing each transmission line with a resistance and then analyzing the circuit. After analyzing the k-th order moment model of the original network, $V^k(0)$ and $I^k(d)$ are known, X^k, Z^k, U^k and W^k can be computed and the $k+1$-th order moment model of the circuit can be formed. Note that in this case moment models of the coupled transmission line system can be formed and the moment computation can be implemented without decoupling the system.

5.3 The General Case

Now we consider the most general case. From Eqs.(1) and (2), we have

$$\frac{d^2V(x,s)}{dx^2} = (sl+r)(sc+g)V(x,s) \tag{72}$$

Let $\Lambda^2(s) = diag(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2)$ and T be the eigenvalue and eigenvector matrix of matrix $(sl+r)(sc+g)$, respectively. Let $V(x,s) = T(s)E(x,s)$. Then, it can be derived that

$$E(x,s) = sh^{-1}\Lambda d(sh\Lambda(d-x)E(0,s) + sh\Lambda xE(d,s)) \tag{73}$$

and

$$V(x,s) = Tsh^{-1}\Lambda d(sh\Lambda(d-x)T^{-1}V(0,s) + sh\Lambda xT^{-1}V(d,s)) \tag{74}$$

Let $\Gamma = \Lambda d$.Then, the functions $U(s)$ and $W(s)$ as defined in Sec.3 can be expressed as follows:

$$U(s) = F(s)(V(0,s) + V(d,s)) \tag{75}$$

where

$$F(s) = T(ch\Gamma - 1)\Gamma^{-1}sh^{-1}\Gamma T^{-1} \tag{76}$$

and

$$W(s) = H_1(s)V(0,s) + H_2(s)V(d,s) \tag{77}$$

where

$$H_1(s) = T(sh\Gamma - \Gamma)\Gamma^{-2}sh^{-1}\Gamma T^{-1} \tag{78}$$

and

$$H_2(s) = T(\Gamma ch\Gamma - sh\Gamma)\Gamma^{-2}sh^{-1}\Gamma T^{-1} \tag{79}$$

Let $V(0,s) = \sum_{p=0}^{\infty}(-1)^p V^p(0)s^p$, $V(d,s) = \sum_{p=0}^{\infty}(-1)^p V^p(d)s^p$, $F(s) = \sum_{k=0}^{\infty} F_k s^k$, $H_1(s) = \sum_{k=0}^{\infty} H_{1k}s^k$, and $H_2(s) = \sum_{k=0}^{\infty} H_{2k}s^k$. Then, from Eqs.(75) and (77), we have

$$U^p = \sum_{j=0}^{p}(-1)^j F_j(V^{p-j}(0) + V^{p-j}(d)) \tag{80}$$

and

$$W^p = \sum_{j=0}^{p}(-1)^j (H_{1j}V^{p-j}(0) + H_{2j}V^{p-j}(d)) \tag{81}$$

In order to use the above equations to compute U^p and W^p, we need to compute the matrices F_j, H_{1j} and H_{2j}. The formulas for these coefficients are given in Appendix C.

6 Moment computation of RLC transmission line tree networks

In the previous three sections, we derived moment models for single uniform and nonuniform lines and coupled lines. By using such models and moment models of lumped elements, moment models for orders from low to high can be formed and moment computation can be done by analyzing these models for which any circuit analysis techniques can be used.

Now we consider the moment computation of a typical network: an RLC transmission line tree network. Such a network consists of resistors, inductors, RLC transmission lines and lumped capacitors. Each transmission line consists of a floating wire and a ground wire. The resistors, inductors, and the floating wires form a tree, and a voltage source is applied to the root of the tree. The capacitors are connected between the nodes on the tree and the ground. Such a network is of our special interest because it is a model of most practical interconnects and the computation of its moments can be done with extreme efficiency.

We use the following notations to describe the circuit. We denote the lumped capacitance connected between node k and the ground by C_{kk}. For each node k on the tree, let $D(k)$ be the set of nodes in the subtree rooted at node k except node k itself. let $S(k)$ be the nodes in $D(k)$ and adjacent to node k. $S(k)$ is called the set of son nodes of node k and k is called the father node of any node $j \in S(k)$. We denote the father node of node k by $\bar{k}$ and the branch connected between nodes $\bar{k}$ and k by b_k. b_k either consists of a series of resistance R_k and inductance L_k, or consists of a transmission line denoted by TL_k. In the later case, the total resistance, capacitance, inductance, p-th order U and W values of TL_k are denoted by R_k, L_k, C_k, U_k^p and W_k^p, respectively.

The 0-th order moment model are formed simply by replace each capacitor with an open circuit, each inductor with a short circuit, and each transmission line with a resistor. As all the resistors in the model are floating, all the 0-th order voltage moments are 1 and all the 0-th order current moments are 0, and it can easily be shown that for each transmission line, $U^0 = 1$ and $W^0 = 0.5$.

Now consider the p-th order moment computation with $p \geq 1$. We replace each element in the network by its p-th order moment model and form a p-th order moment model of the network. Note that in such a case, the model is still a tree. Let I_k^p be the current in branch b_k entering node k. From KCL, we have

$$I_k^p = -C_{kk}V_k^{p-1} + \sum_{j \in S(k)} (I_j^p - C_jU_j^{p-1}) \tag{82}$$

where C_{kk} is the lumped capacitance connected to node k and C_j is the total capacitance of transmission line TL_j.

From Eq.(82), we have a recursive algorithm to compute I_k^p for all tree branches from the leaves to the root as follows.

Algorithm 1: $findI(k,p)$

$findI(k,p)$

{ $I_k^p = -C_{kk}V_k^{p-1}$;

if b_k is a transmission line

find U_k^{p-1};

else $U_k^{p-1} = 0$;

```
if k is not a leaf node
  for each node j ∈ S(k) do
    I_k^p += findI(j,p);
return(I_k^p − C_k U_k^{p−1});
}
```

After using the above algorithm to compute all the p-th order moments of the tree branch currents, the p-th ($p \geq 1$) order moments of the node voltages can be computed recursively from the root to the leaves by using the following algorithm.

Algorithm 2: moment(k,p)

```
moment(k,p)
 {if k is the root
    V_k^p = 0;
  else
   { V_k^p = V_k^p − R_k I_k^p + L_k I_k^{p−1};
    if branch b_k is a transmission line TL_k
     {find W_k^{p−1};
      V_k^p += R_k C_k W_k^{p−1} − L_k C_k W_k^{p−2}; }
   }
  if k is not a leaf
    for each j ∈ S(k) do
      moment(j,p);
    return;
  }
```

The time complexity of these algorithm is linear to the number of nodes n of the network. From Eq.(35) and Eq.(36) it can be seen that the computation of the function U_k^{p-1} takes $O(p)$ time. Therefore, the time complexity of this algorithm is $O(np)$. By using the above algorithms recursively to compute the moments from order 1 to p takes time $O(np^2)$.

The above algorithms can be applied to tree networks made of single uniform and/or nonuniform transmission lines. For a tree network made of coupled transmission lines, a similar algorithms can be written. The main changes from the above algorithms are as follows. 1. All the mean values $U's$ and $W's$ of transmission lines are computed first; 2. In the "return" statement of $FindI(k,p)$, $C_k U_k^{p-1}$ should be replaced by $\sum_j C_{kj} U_j^{p-1}$ where j is taken over all the transmission lines coupled with TL_k including k; 3.In moment(k,p), for a transmission line TL_k coupled with other lines, $R_k I_k^p$ should be replaced by $\sum_j R_{kj} I_j^p$, $L_k I_k^{p-1}$ by $\sum_j L_{kj} I_j^{p-1}$, $R_k C_k W_k^{p-1}$ by $\sum_j R_{kj} \sum_i C_{ji} W_i^{p-1}$, and $L_k C_k W_k^{p-2}$ by $\sum_j L_{kj} \sum_i C_{ji} W_i^{p-2}$.

7 Experiments and Conclusions

7.1 Experiments

We will show three examples of using the moment matching technique to find the time domain response. The input to the three circuits is a unit ramp function with a rising time of 0.1ns. We use our moment model to compute the moments of output node voltages. After extracting a flight time, we use *Padé* approximation to find their rational approximations and obtain the time domain response by inverse Laplace transform. For each example, SPICE simulation is done and the results are shown in the figures with a postfix "s" for comparison. As SPICE cannot handle RLGC lines, each uniform RLGC line is modeled by 50 identical lumped RLGC sections, and each nonuniform line is modeled by 50 different RLGC sections. It can be seen from the figures that the moment matching method is quite accurate for the simulation of interconnects.

Example 2. This circuit is shown in Fig.6 which contains 7 uniform lossy transmission lines. An 8-th order moment matching is used to find the output voltage "**vout**", which is compared with the SPICE simulation result "**vouts**" as shown in Fig.7.

Example 3. This circuit is formed by replacing TL_7 in Fig.6 by a parabolic line as shown in Fig.8. The line parameters are: $r = r_0(1+\alpha x)^2$, $l = l_0(1+\alpha x)^2$, $c = c_0(1+\alpha x)^{-2}$ and $g = g_0(1+\alpha x)^{-2}$, where $r_0 = 75\Omega/m$, $l_0 = 100nH/m$, $c_0 = 150pF/m$, $g_0 = 0.01S/m$, $\alpha = 20.0$ and $LEN = 0.02m$. The output voltage "**vout1**" is found by using an 8-th order moment matching, which is compared with the SPICE simulation result "**vouts**" as shown in Fig.9.

Example 4. This circuit is shown in Fig.10 which contains two coupled line systems. The length of each line is 0.1m. The parameters of the first systems are:

$$l = \begin{bmatrix} 494.6 & 63.3 \\ 63.3 & 494.6 \end{bmatrix} nH/m$$

$$C = \begin{bmatrix} 62.8 & -4.9 \\ -4.9 & 62.8 \end{bmatrix} pF/m$$

$$R = \begin{bmatrix} 75 & \\ & 75 \end{bmatrix} \Omega/m.$$

The parameters of the second system are:

$$l = \begin{bmatrix} 494.6 & 63.3 & & \\ 63.3 & 494.6 & 63.3 & \\ & 63.3 & 494.6 & 63.3 \\ & & 63.3 & 494.6 \end{bmatrix} nH/m$$

$$C = \begin{bmatrix} 62.8 & -4.9 & & \\ -4.9 & 62.8 & -4.9 & \\ & -4.9 & 62.8 & -4.9 \\ & & -4.9 & 62.8 \end{bmatrix} pF/m$$

$$R = \begin{bmatrix} 75 & & & \\ & 75 & & \\ & & 75 & \\ & & & 75 \end{bmatrix} \Omega/m.$$

The waveforms **v7** and **v9** are found by using a 5-th order and an 11-th order moment matching and are compared with the SPICE simulation results **v7s** and **v9s**, respectively. They are shown in Fig.11 and 12, respectively.

The CPU time by using moment matching techniques in each of the above three examples is less than 1/60 second. The CPU time for SPICE simulation is 37.6s, 38.54s and 17.346s for Example 1, 2 and 3, respectively. The moment matching technique runs three order of magnitude faster than SPICE.

7.2 Conclusion

We have presented new moment models for transmission lines. These models are directly derived from telegrapher's equations. Their parameters are based on the mean values of the voltage moments and weighted voltage moments along the lines, which can be efficiently computed. For uniform lines, these models are exact; and for nonuniform lines, the model can be made as accurate as needed. These models can be used in any transmission line networks. As the models have the T-typed or π-typed circuit structure, when an interconnect is of the tree structure, its moment model circuit is of the same type. Therefore, these models are especially well suited for the use of moment computation algorithm for RLC transmission line tree networks. Meanwhile, all the models can be applied to distributed RC lines as a special case.

Our moment model of transmission lines is different from other known models. Compared with the lumped model made of a large number of RLGC sections, ours is more accurate and efficient. Compared with the models suggested in [23], which are formed either by recursively solving second order differential equations or by the computation of exponential matrix functions, the formation of out model is simpler and faster. When using the models suggested in [24], two port currents of transmission lines are introduced in circuit equations. In contrast, by using our model, for single lines and couple line system with diagonal resistance matrices, no port currents are needed; and for lines with resistive coupling, only one port current (the far-end port current, See Fig.5) is introduced for each line. Therefore, our model results in fewer unknown variables for the circuit equations in the general case. Compared with REX [25], the formation of a p-th order model in REX takes $O(p^4)$ time, while the formation of our model takes $O(p^2)$ time.

Appendix

A Computation of f_j, h_{1j} and h_{2j}

Let $P(s)$ be any function of $f(s)$, $h_1(s)$ or $h_2(s)$. Then, $P(s)$ can be expressed as

$$P(s) = \frac{\sum_{n=0}^{\infty} a_n \gamma^{2n}}{\sum_{n=0}^{\infty} b_n \gamma^{2n}} = \sum_{n=0}^{\infty} c_n \gamma^{2n} \tag{83}$$

where $b_0 = 1$. For $P(s) = f(s)$, $a_n = 1/(2(n+1))!$, $b_n = 1/(2n+1)!$; for $P(s) = h_1(s)$, $a_n = 1/(2n+3)!$, $b_n = a_{n-1}$ $(n \geq 1)$; and for $P(s) = h_2(s)$, $a_n = 2(n+1)/(2n+3)!$ and $b_n = 1/(2n+1)!$. The coefficients $c_n's$ $(n = 0, 1, \ldots)$ can be computed from the known $a_n's$ and $b_n's$ by using the following formula starting from $n = 0$:

$$c_n = a_n - \sum_{k=0}^{n-1} c_k b_{n-k} \tag{84}$$

Let $\gamma^{2n} = (a_2 s^2 + a_1 s + a_0)^n$ be expressed as $\sum_{k=0}^{2n} \alpha_j^n s^j$. Then, $\alpha_0^0 = 1$, $\alpha_j^0 = 0$ for $j > 0$; and the coefficients of α_j^n with $n \geq 1$ can be computed by using the following recursive formula. Let $m = min(j, 2)$, then

$$\alpha_j^n = \sum_{i=0}^{m} \alpha_{j-i}^{n-1} a_i \tag{85}$$

Let $k = \lfloor \frac{j+1}{2} \rfloor$. Then, $P(s)$ can be expressed as $\sum_{j=0}^{\infty} p_j s^j$ with

$$p_j = \sum_{n=k}^{\infty} c_n \alpha_j^n \tag{86}$$

and $p_j \approx \sum_{n=k}^{N} c_n \alpha_j^n$ where N is so chosen that the error due to the truncation is small enough.

B Derivation of Eq.(54)

We differentiate both sides of Eq.(41) w.r.t. x, and have

$$\frac{d^2 V(x,s)}{dx^2} + a(x,s)\frac{dV(x,s)}{dx} + b(x,s)V(x,s) = 0 \tag{87}$$

where

$$a(x,s) = -\frac{r'(x) + sl'(x)}{r(x) + sl(x)} \tag{88}$$

and

$$b(x,s) = -Z(x,s)Y(x,s) = -(r(x) + sl(x))(g(x) + sc(x)) \tag{89}$$

$a(x,s)$ and $b(x,s)$ can be expressed as

$$a(x,s) = \sum_{n=0}^{\infty} a_n(s)x^n = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} a_{nj}s^jx^n \tag{90}$$

and

$$b(x,s) = \sum_{n=0}^{\infty} b_n(s)x^n = \sum_{n=0}^{\infty}\sum_{j=0}^{2} b_{nj}s^jx^n \tag{91}$$

The coefficients a_{nj} and b_{nj} can be expressed as follows.

Let $r(x) = \sum_{n=0}^{\infty} r_nx^n$, $l(x) = \sum_{n=0}^{\infty} l_nx^n$, $c(x) = \sum_{n=0}^{\infty} c_nx^n$, and $g(x) = \sum_{n=0}^{\infty} g_nx^n$. Then

$$b_n(s) = b_{n0} + b_{n1}s + b_{n2}s^2 \tag{92}$$

where $b_{n0} = \sum_{k=0}^{n} r_kg_{n-k}$, $b_{n1} = \sum_{k=0}^{n}[r_kc_{n-k} + g_kl_{n-k}]$ and $b_{n2} = \sum_{k=0}^{n} c_kl_{n-k}$. From the definition of $a(x,s)$, we have

$$a(x,s) = -\frac{\sum_{n=1}^{\infty} n(r_n + sl_n)x^{n-1}}{\sum_{n=0}^{\infty}(r_n + sl_n)x^n} = \sum_{n=0}^{\infty} a_n(s)x^n \tag{93}$$

or

$$-\sum_{n=1}^{\infty} n(r_n + sl_n)x^{n-1} = \sum_{n=0}^{\infty}(r_n + sl_n)x^n \sum_{n=0}^{\infty} a_n(s)x^n \tag{94}$$

From Eqs.(88) and (94), we have

$$a_0(s) = -\frac{r_1 + sl_1}{r_0 + sl_0} \tag{95}$$

and

$$a_n(s) = -\frac{(n+1)(r_{n+1} + sl_{n+1}) + \sum_{k=1}^{n}(r_k + sl_k)a_{n-k}(s)}{r_0 + sl_0} \tag{96}$$

and $a_n(s)$ can be computed recursively from n=0 to any order.

From Eqs.(95) and (96), it can be seen that $a_n(s)$ can be expressed as $\sum_{j=0}^{\infty} a_{nj}s^j$. From Eq.(95), it can be derived that $a_{00} = -r_1/r_0$ and

$$a_{0j} = \frac{(-1)^{j+1}}{r_0}(\frac{l_0}{r_0})^{j-1}(r_1\frac{l_0}{r_0} - l_1) \tag{97}$$

For $n > 0$,

$$a_{n0} = -\frac{1}{r_0}[(n+1)r_{n+1} + \sum_{k=1}^{n} r_ka_{n-k,0}] \tag{98}$$

and

$$a_{nk} = -\frac{1}{r_0}[(-1)^k(n+1)(\frac{l_0}{r_0})^{k-1}(r_{n+1}\frac{l_0}{r_0} - l_{n+1})$$

$$+\sum_{i=1}^{n}\sum_{j=0}^{k}(r_i a_{n-i,j}+l_i a_{n-i,j-1})(-1)^{k-j}(\frac{l_0}{r_0})^{k-j}] \tag{99}$$

In the special case that $l(x)/r(x) = const$, $a_{n0} = a_n(s)$.

Suppose that the boundary conditions to Eqs.(41) and (42) are known as $V(0,s)$ and $I(0,s)$. Then, $\frac{dV(x,s)}{dx}\mid_{x=0}= -(r(0)+sl(0))I(0,s)$. Let the solution $V(x,s)$ to Eq.(87) be

$$V(x,s)=\sum_{n=0}^{\infty}V_n(s)x^n \tag{100}$$

then $V_0(s) = V(0,s)$ and $V_1(s) = \frac{dV(x,s)}{dx}\mid_{x=0}= -(r(0)+sl(0))\acute{I}(0,s)$. Substituting Eq.(100) to Eq.(87), we have

$$\sum_{k=0}^{\infty}(k+1)(k+2)V_{k+2}(s)x^k+\sum_{n=0}^{\infty}a_nx^n\sum_{k=0}^{\infty}(k+1)V_{k+1}(s)x^k+\sum_{n=0}^{\infty}b_nx^n\sum_{k=0}^{\infty}V_k(s)x^k=0 \tag{101}$$

and

$$V_{k+2}(s)=-\frac{\sum_{i=0}^{k}[(i+1)V_{i+1}(s)a_{k-i}(s)+V_i(s)b_{k-i}(s)]}{(k+1)(k+2)} \tag{102}$$

$V_i(s)$ can be expressed as $\alpha_i(s)V(0,s)+\beta_i(s)I(0,s)$. For example, $\alpha_0 = 1$, $\beta_0 = 0$, $\alpha_1 = 0$ and $\beta_1 = -Z(0,s)$. From Eq.(102), we have

$$\alpha_{k+2}(s)=-\frac{\sum_{i=0}^{k}[(i+1)\alpha_{i+1}(s)a_{k-i}(s)+\alpha_i(s)b_{k-i}(s)]}{(k+1)(k+2)} \tag{103}$$

and

$$\beta_{k+2}(s)=-\frac{\sum_{i=0}^{k}[(i+1)\beta_{i+1}(s)a_{k-i}(s)+\beta_i(s)b_{k-i}(s)]}{(k+1)(k+2)} \tag{104}$$

Now we truncate the infinite series in Eq.(100) to a finite series with N+1 terms, then $V(x,s)\approx\sum_{n=0}^{N}V_n(s)x^n \equiv A(x,s)V(0,s)+B(x,s)I(0,s)$ with $A(x,s)=\sum_{n=0}^{N}\alpha_n(s)x^n$ and $B(x,s)=\sum_{n=0}^{N}\beta_n(s)x^n$.

From what mentioned above, α_n and β_n can be expressed as $\sum_{k=0}^{\infty}\alpha_{nk}s^k$ and $\sum_{k=0}^{\infty}\beta_{nk}s^k$. It has been shown that α_0, β_0, and α_1 are constant, $\beta_{10} = -r(0)$ and $\beta_{11} = -l(0)$. From Eqs.(103) and (104), it can be derived that

$$\alpha_{k+2,j}=-\frac{\sum_{i=0}^{k}[(i+1)\sum_{p=0}^{j}\alpha_{i+1,p}a_{k-i,j-p}+\sum_{p=0}^{2}\alpha_{i,j-p}b_{k-i,p}]}{(k+1)(k+2)} \tag{105}$$

and

$$\beta_{k+2,j}=-\frac{\sum_{i=0}^{k}[(i+1)\sum_{p=0}^{j}\beta_{i+1,p}a_{k-i,j-p}+\sum_{p=0}^{2}\beta_{i,j-p}b_{k-i,p}]}{(k+1)(k+2)} \tag{106}$$

Then, $A(x,s)$ and $B(x,s)$ can be expressed as $\sum_{k=0}^{\infty} A_k(x)s^k$ and $\sum_{k=0}^{\infty} B_k(x)s^k$, where $A_k = \sum_{n=0}^{N} \alpha_{nk}x^n$ and $B_k = \sum_{n=0}^{N} \beta_{nk}x^n$.

Now we have the equation

$$V(x,s) = A(x,s)V(0,s) + B(x,s)I(0,s) \tag{107}$$

and

$$V(d,s) = A(d,s)V(0,s) + B(d,s)I(0,s) \tag{108}$$

From the above equations, we have

$$V(x,s) = P(x,s)V(0,s) + Q(x,s)V(d,s) \tag{109}$$

where $P(x,s) = A(x,s) - \frac{A(d,s)}{B(d,s)}B(x,s)$ and $Q(x,s) = \frac{B(x,s)}{B(d,s)}$. The expansion of $P(x,s)$ and $Q(x,s)$ into Taylor series can be done similarly and is omitted.

C Computation of U and W for coupled transmission lines

C.1 Eigenvalues and eigenvectors of matrix $M = (r+sl)(g+sc)$

We first present formulas for the eigenvalues and eigenvectors of the matrix $M = (r+sl)(g+sc)$.

Let $Q = \Lambda^2$ be the eigenvalue matrix of matrix M and T be the matrix of its eigenvectors, then we have $MT = TQ$. Let $M = M_0 + M_1 s + M_2 s^2$ where $M_0 = rg$, $M_1 = rc + lg$ and $M_2 = lc$, and $Q = \sum_{k=0}^{\infty} Q_k s^k$ and $T = \sum_{k=0}^{\infty} T_k s^k$. Then,

$$M_0 T_0 = T_0 Q_0 \tag{110}$$

and Q_0 and T_0 can be found by using any algorithm of eigenvalue problems. Also,

$$M_0 T_1 + M_1 T_0 = T_0 Q_1 + T_1 Q_0 \tag{111}$$

and

$$M_0 T_k + M_1 T_{k-1} + M_2 T_{k-2} = T_k Q_0 + T_0 Q_k + \sum_{j=1}^{k-1} T_j Q_{k-j} \tag{112}$$

let $T_k = [T_{k1}, T_{k2}, \ldots, T_{kn}]$, $Q_j = diag\{q_{j1}, q_{j2}, \ldots, q_{jn}\}$, $A = \sum_{j=1}^{k-1} T_j Q_{k-j} - M_1 T_{k-1} - M_2 T_{k-2} \equiv [A_1, A_2, \ldots, A_n]$ where T_{kj} and A_j are the j-th column vector of matrices T_k and A, respectively. Then, from Eq.(112), we have

$$(M_0 - q_{0j} I) T_{kj} = q_{kj} T_{0j} + A_j \tag{113}$$

There are $n+1$ unknowns in the above n equations. By an additional constraint $\| T_{kj} \|_2 = 1$, we can solve for q_{kj} and T_{kj}.

As $\Lambda^2 = Q$, we have $\Lambda_0^2 = Q_0$ and $2\Lambda_0 \Lambda_k = Q_k - \sum_{j=1}^{k-1} \Lambda_j \Lambda_{k-j}$, and we have $\Lambda_0 = \sqrt{Q_0}$ and

$$\Lambda_k = \frac{1}{2} \Lambda_0^{-1} (Q_k - \sum_{j=1}^{k-1} \Lambda_j \Lambda_{k-j}) \tag{114}$$

C.2 Taylor series expansion of $F(s)$, $H_1(s)$ and $H_2(s)$

C.2.1 Taylor series expansion of an inverse matrix

Suppose that matrix $A(s) = \sum_{k=0}^{\infty} A_k s^k$ and $B_s = A^{-1}(s) = \sum_{k=0}^{\infty} B_k s^k$. From $B(s)A(s) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} B_j A_{k-j} = I$, we have $B_0 = A_0^{-1}$ and

$$B_k = -A_0^{-1} \sum_{j=0}^{k-1} B_j A_{k-j} \tag{115}$$

Then, B_k can be found recursively from $k = 0$ up to any $k = p$.

This method can be used to find the Taylor series expansion of $T^{-1}(s)$, Γ^{-1}, Γ^{-2} and $sh^{-1}\Gamma$. For the last three matrices, the computation should be done for diagonal elements only.

C.2.2 Taylor series expansion of a matrix product

The matrix functions $F(s)$, $H_1(s)$ and $H_2(s)$ are products of 5 matrices. Except for the first and last factors T and T^{-1}, the other three factors are diagonal matrices whose product is also a diagonal matrix so that they can be expressed as $P = ABC$ in the general case, where B is a diagonal matrix. Assume that $A(s) = \sum_{k=0}^{\infty} A_k s^k$ and similar expansions exist for matrices B and C. Then, we have $T_0 = A_0 B_0 C_0$; and for $p \geq 1$,

$$T_p = \sum_{i+j+k=p} A_i B_j C_k = \sum_{i=0}^{p} A_i \sum_{j=0}^{p-i} B_j C_{p-i-j} \tag{116}$$

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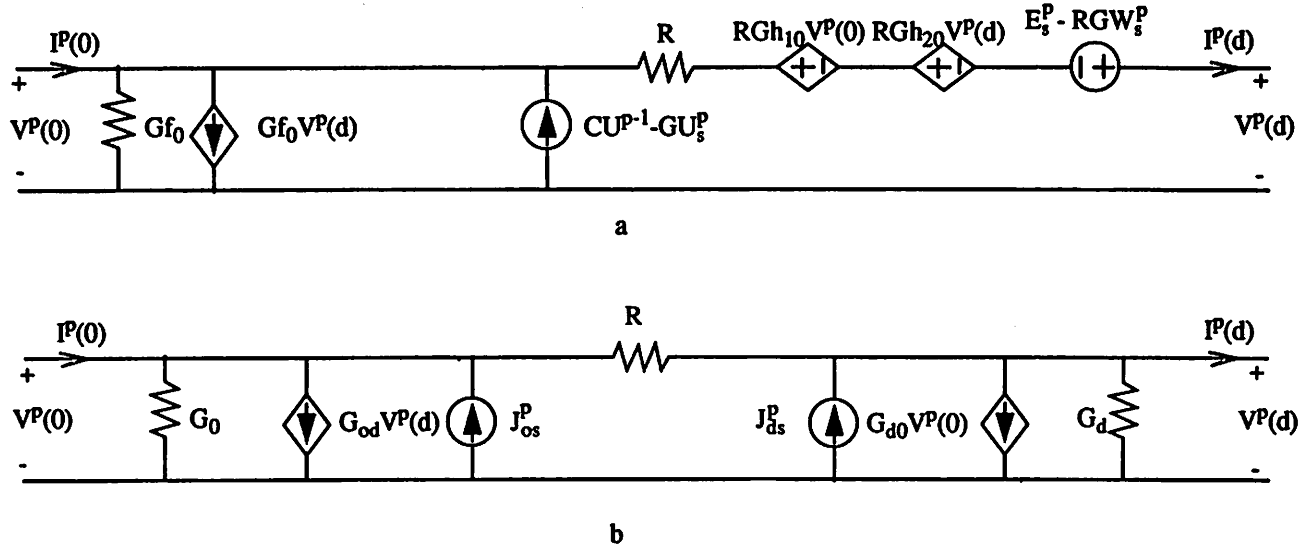


Fig.3 p-th Order Moment Model

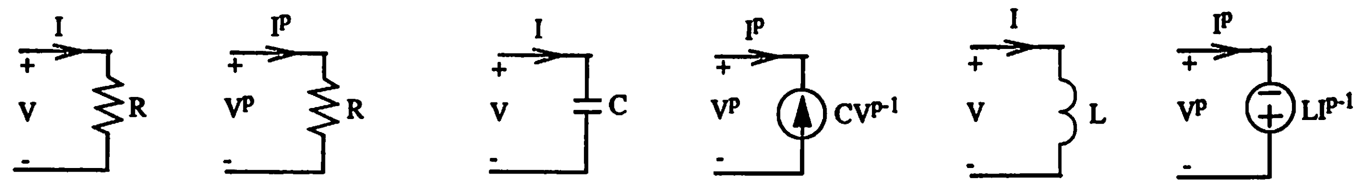
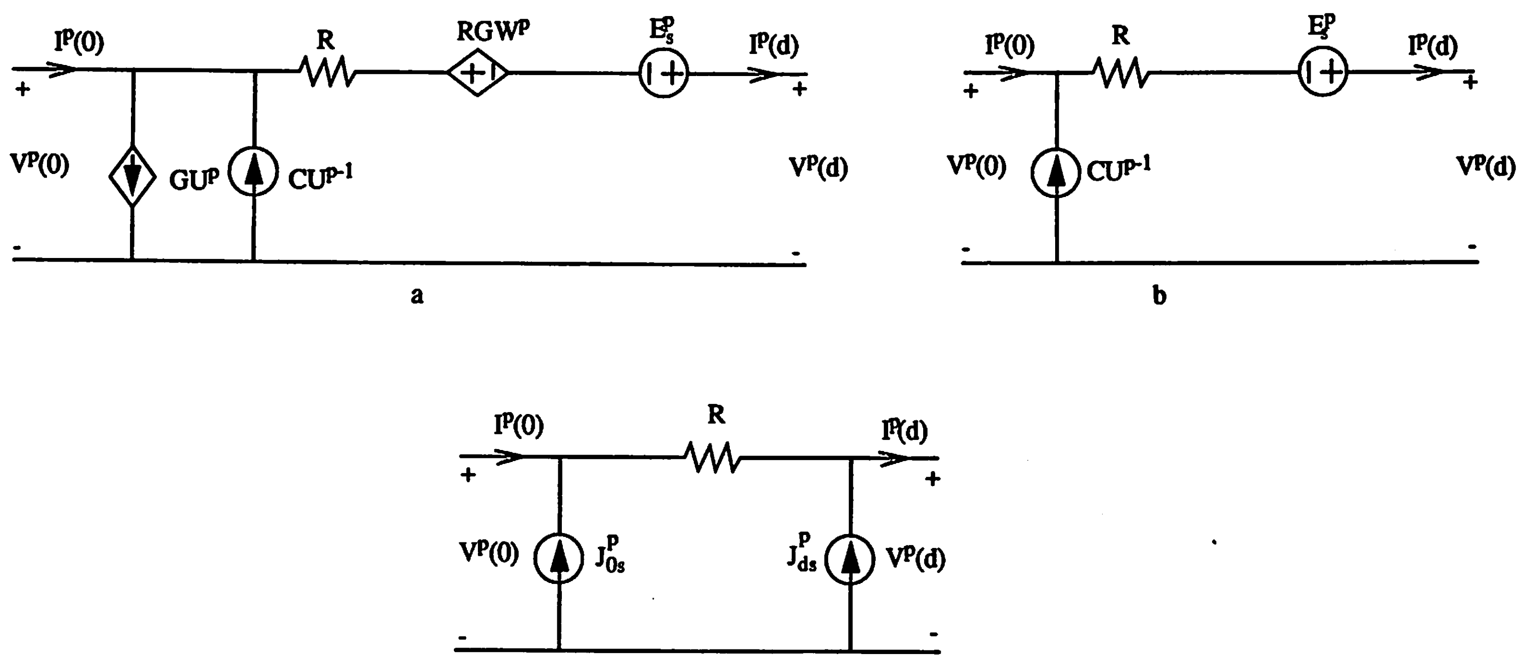


Fig.1.1 Fig.1.2 Fig.1.3

Fig.1 Moment Model of R, L and C



c

Fig.2 Moment Model of Uniform Transmission Line

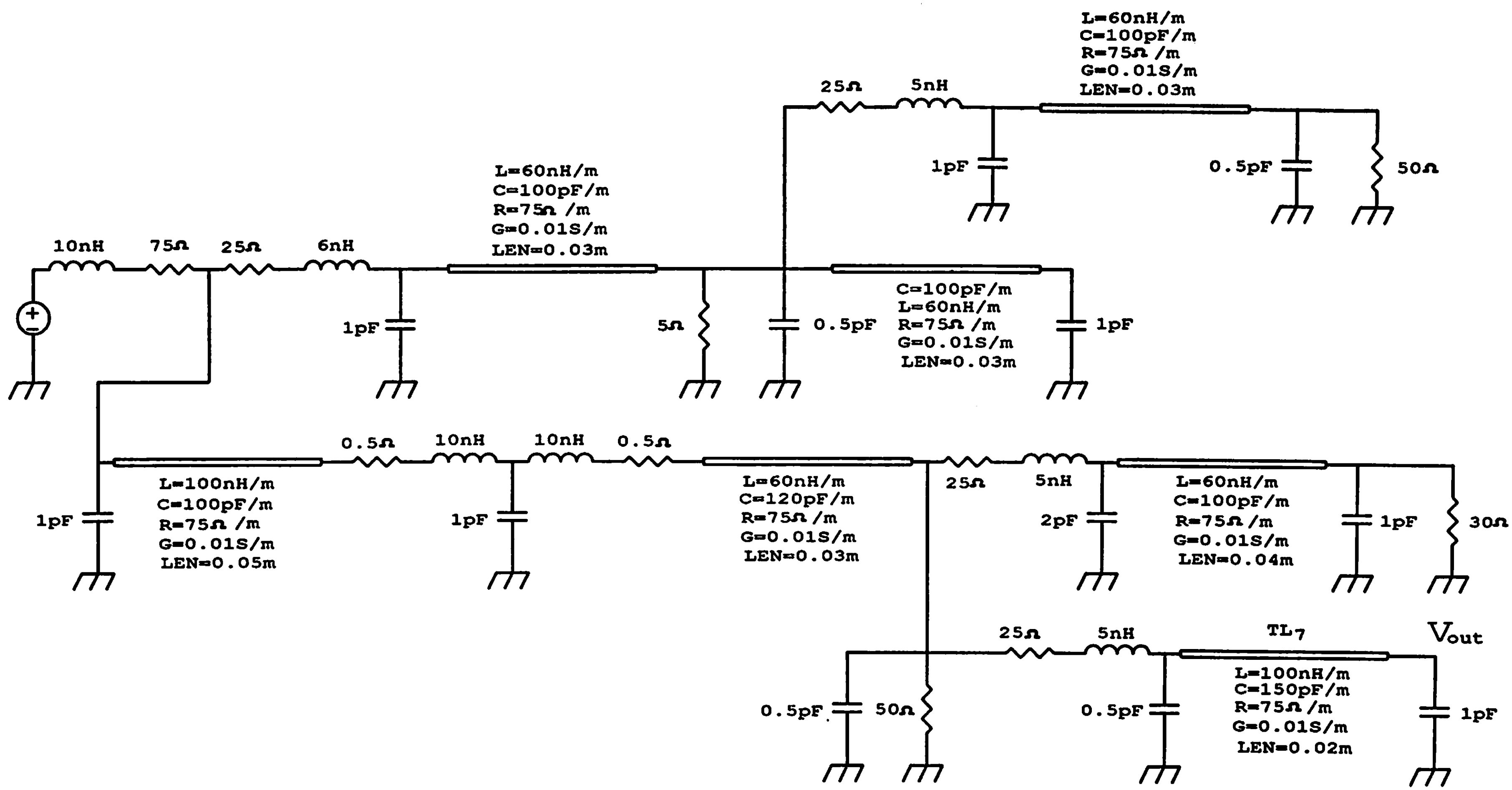


Fig.6 Circuit for Example 1

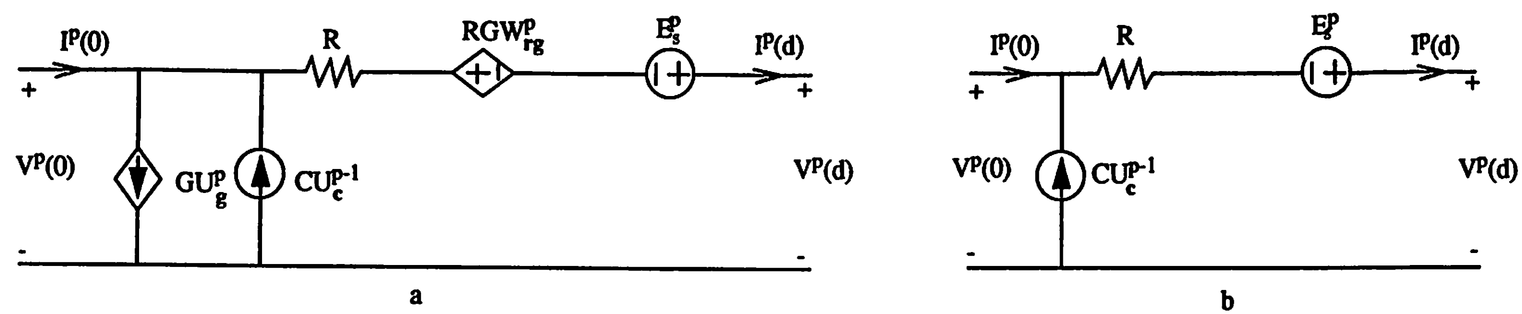
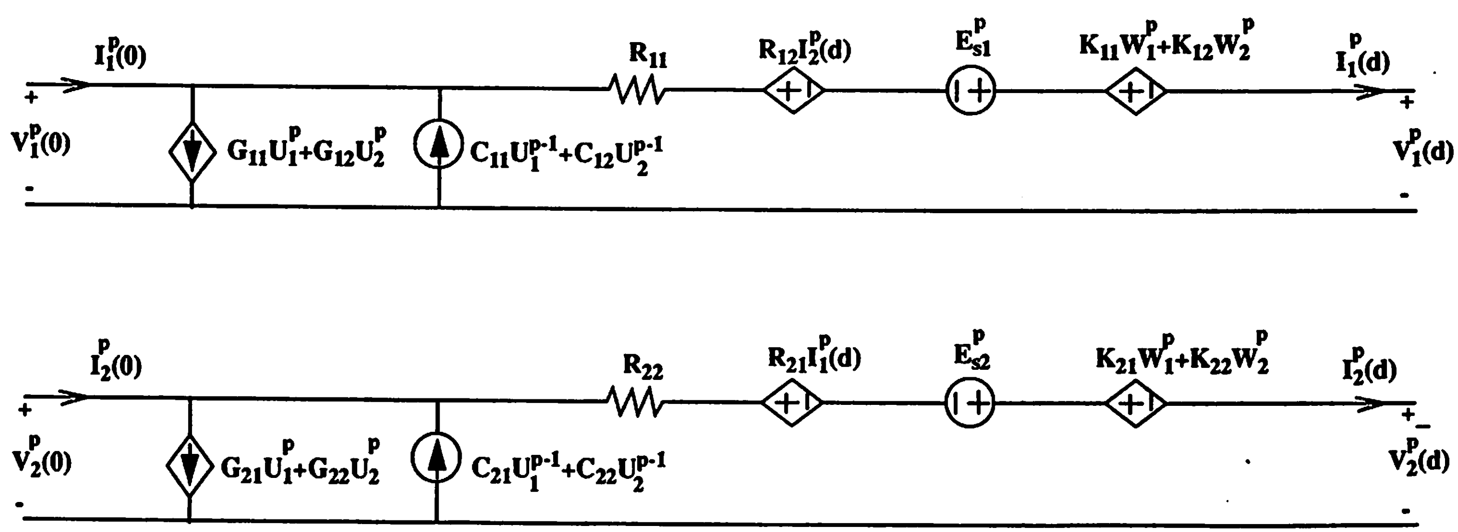


Fig.4 Moment Model of Nonuniform Line



$$K_{ij} = R_{i1}G_{1j} + R_{i2}G_{2j}$$

Fig.5 Moment Model of Two Coupled Lines

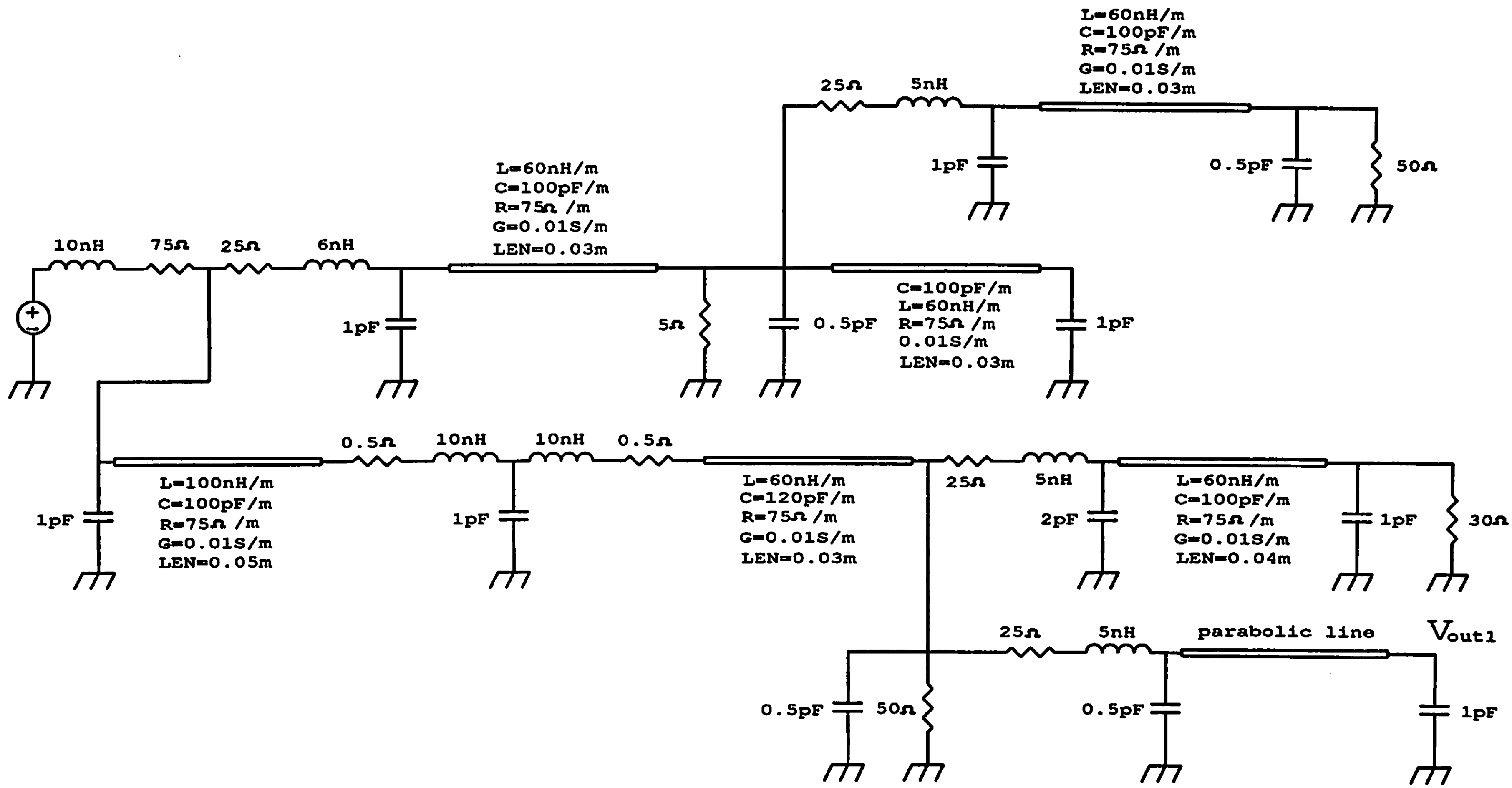
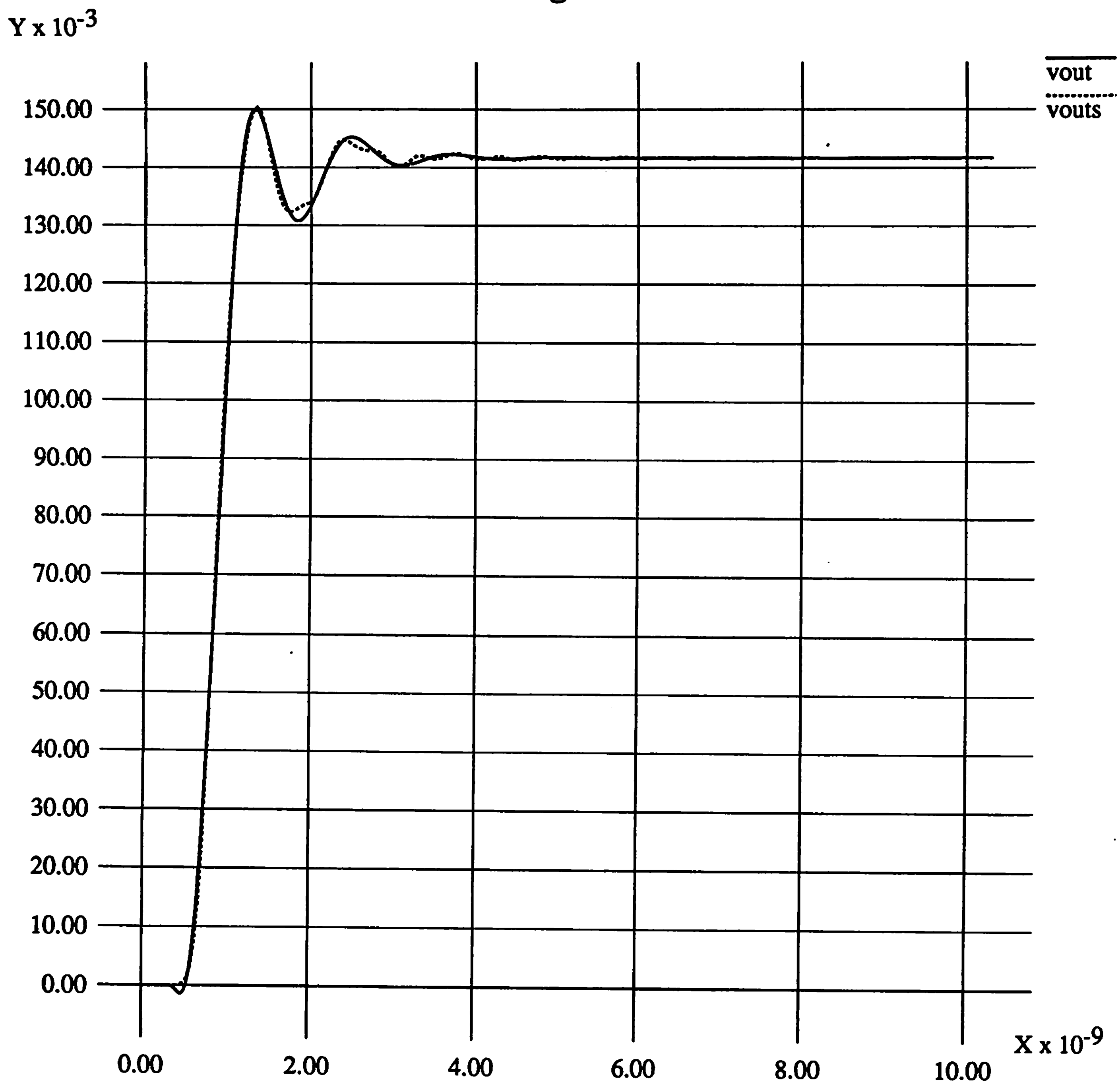


Fig.8 Circuit for Example 2

Fig.7


Y x 10-3
150.00
140.00
130.00
120.00
110.00
100.00
90.00
80.00
70.00
60.00
50.00
40.00
30.00
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10.00
X x 10-9
vout
vouts

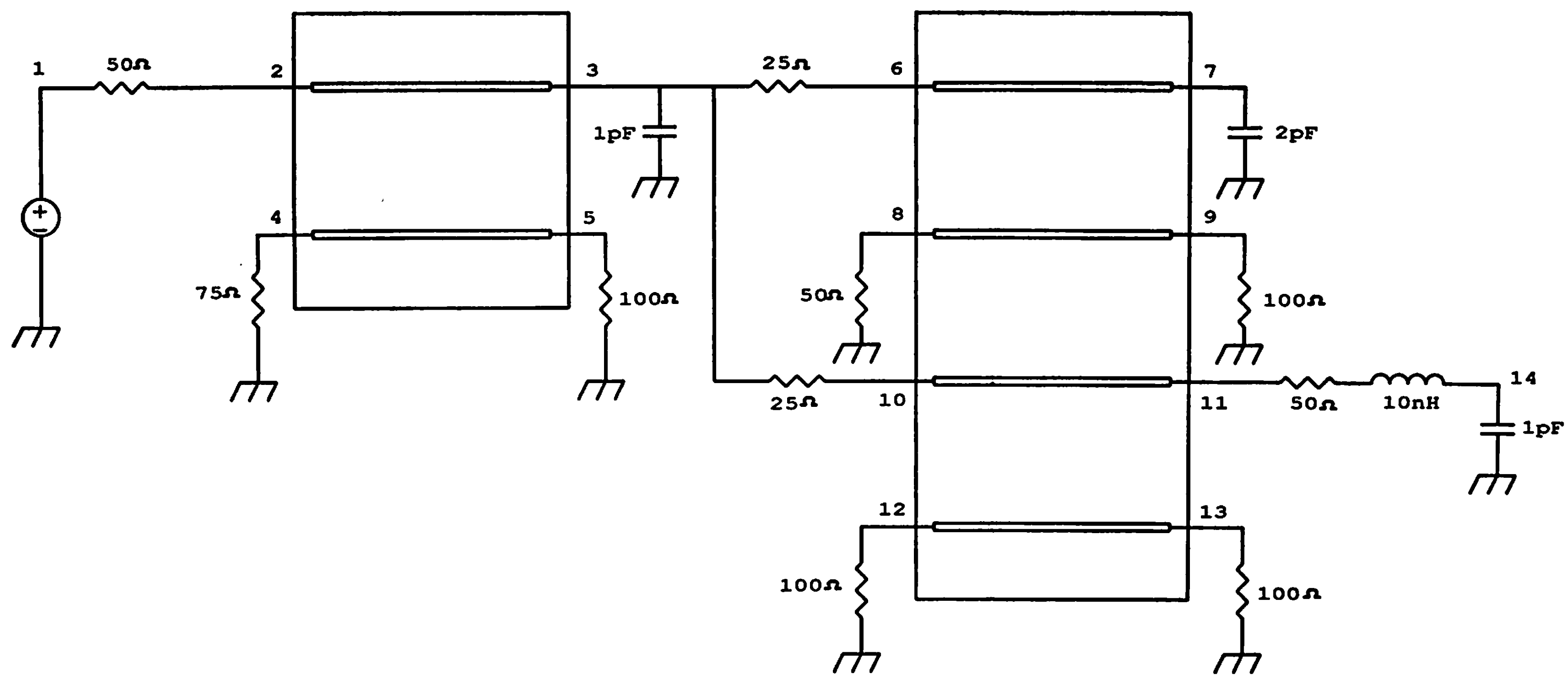


Fig.10 Circuit for Example 3

Fig.9

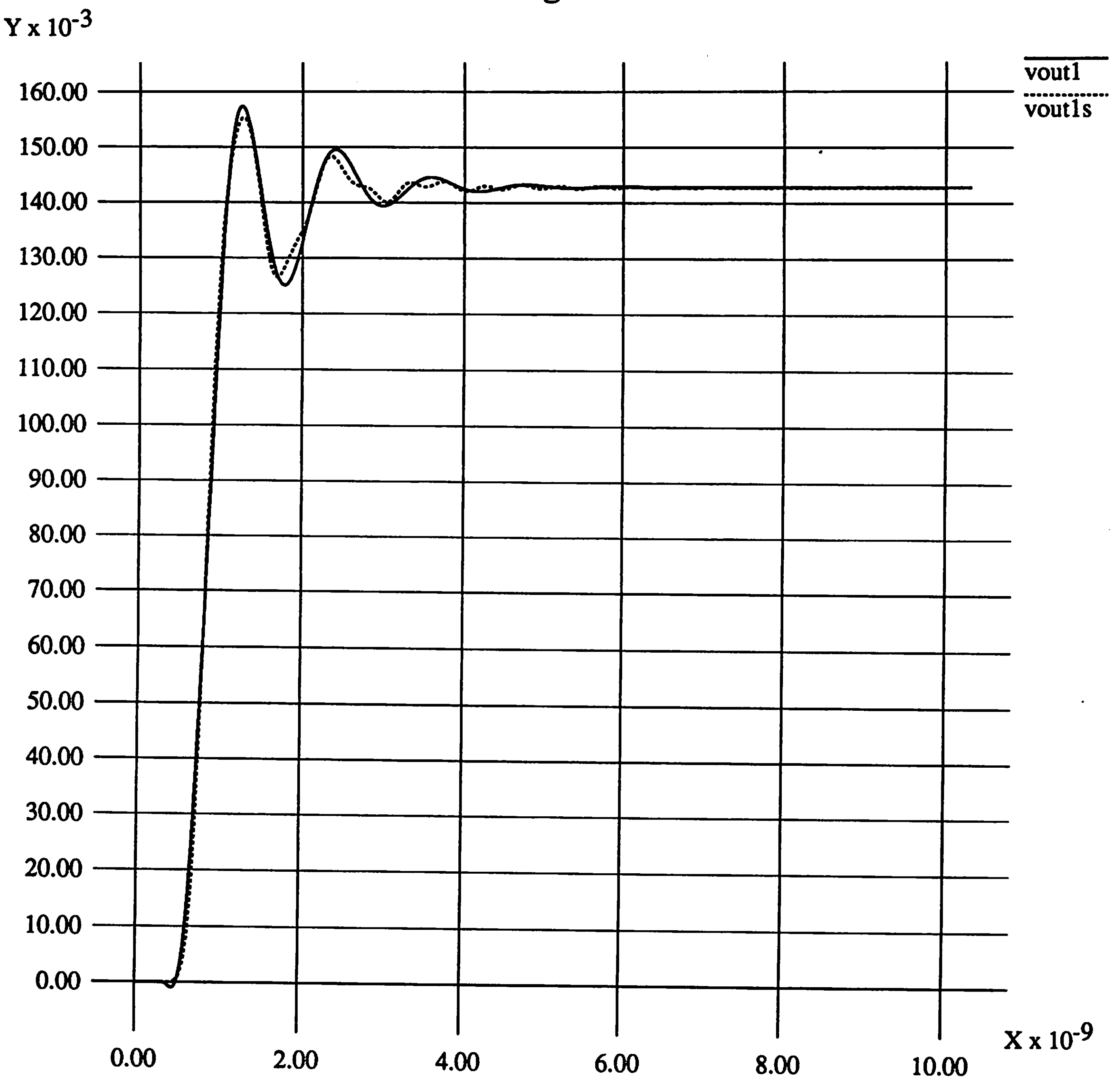
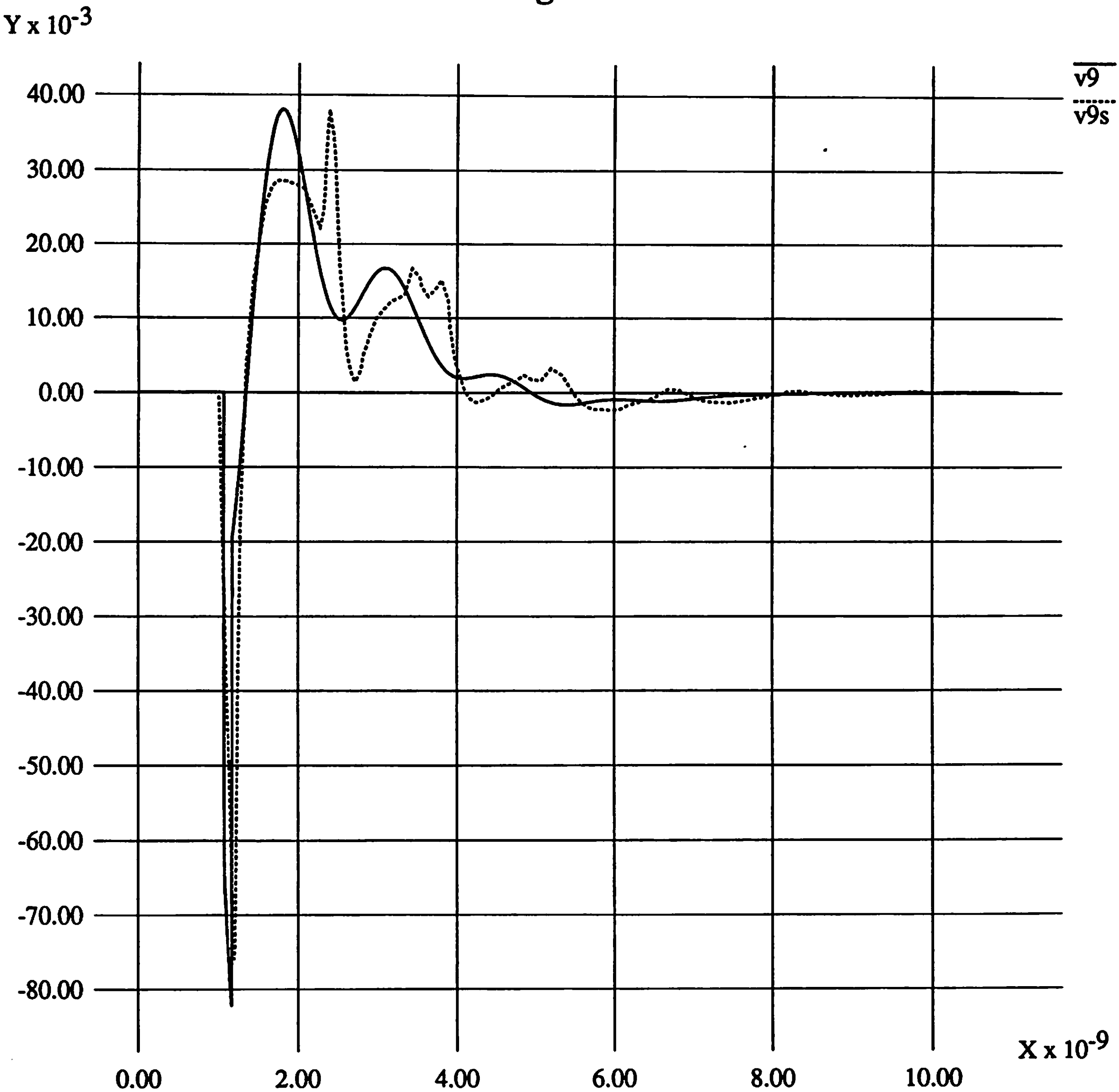
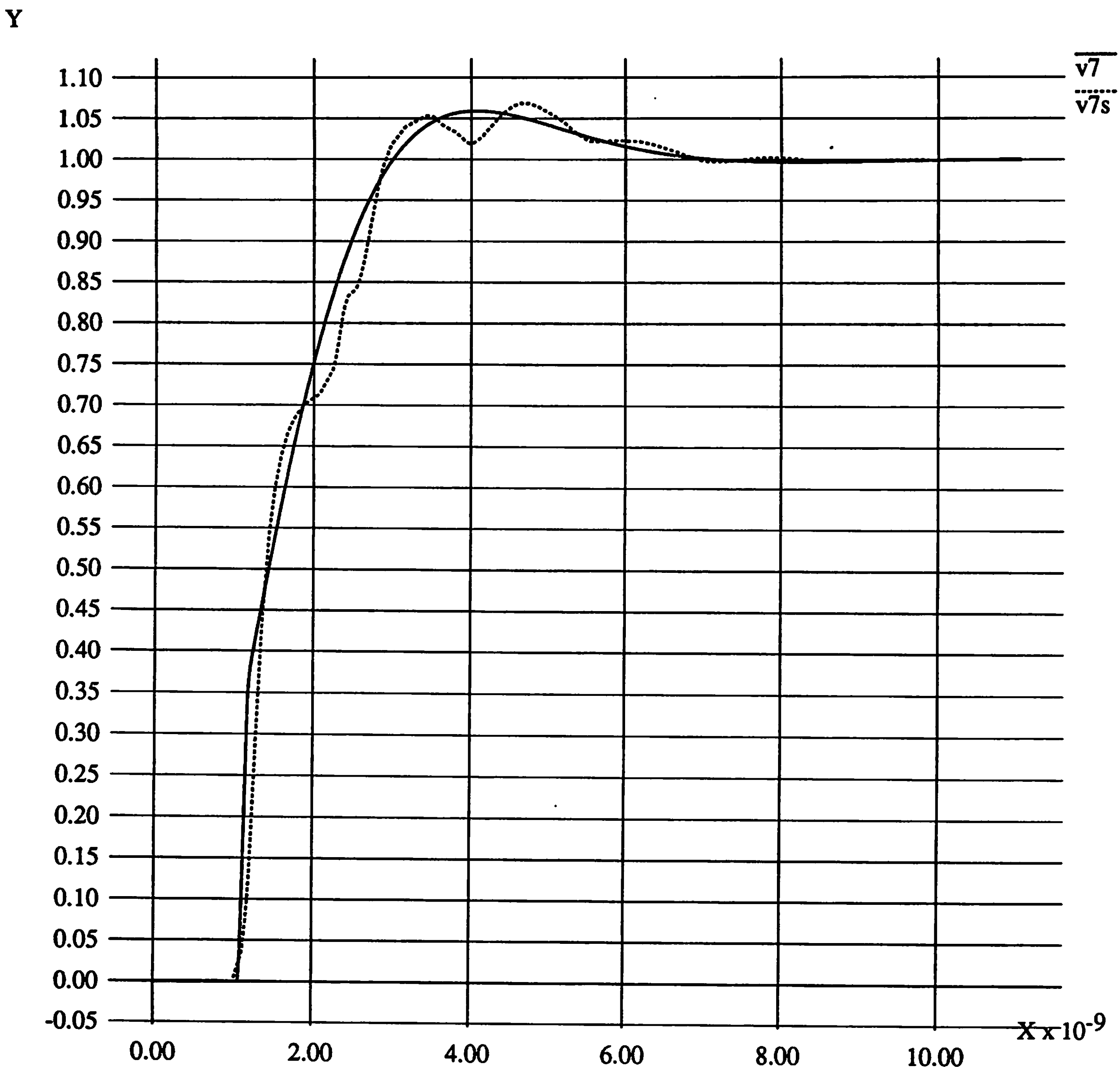

Y x 10-3
160.00
150.00
140.00
130.00
120.00
110.00
100.00
90.00
80.00
70.00
60.00
50.00
40.00
30.00
20.00
10.00
0.00
0.00
2.00
4.00
6.00
8.00
10.00
X x 10-9
vout1
vout1s

Fig.12



Y x 10-3
40.00
30.00
20.00
10.00
0.00
-10.00
-20.00
-30.00
-40.00
-50.00
-60.00
-70.00
-80.00
v9
v9s
X x 10-9
0.00
2.00
4.00
6.00
8.00
10.00

Fig.11



Y
v7
v7s
1.10
1.05
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
0.00
2.00
4.00
6.00
8.00
10.00
X x 10^{-9}