EXTERIOR DIFFERENTIAL SYSTEMS AND NONHOLONOMIC MOTION PLANNING

by

Dawn Marie Tilbury

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Abstract

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Dawn Marie Tilbury

Doctor of Philosophy in
Electrical Engineering and Computer Sciences
University of California at Berkeley
Professor S. Shankar Sastry, Chair

This dissertation addresses the problem of motion planning for systems with velocity constraints which restrict the system velocities but do not reduce the reachable configuration space. Such constraints are *nonholonomic.* Traditional motion planning methods assume that a robot can move in any direction. These plans cannot be followed by nonholonomic systems, such as car-like mobile robots which cannot move sideways.

The theory of exterior differential (particularly Pfaffian) systems is presented as an analysis method for nonholonomic systems. The Goursat normal form for Pfaffian systems of codimension two is shown to be equivalent to the two-input chained form for nonholonomic control systems. Since the N-trailer system satisfies the conditions for conversion into Goursat form, it can be put into chained form. Several methods are presented for steering systems in chained form, using sinusoidal, piecewise constant, or polynomial inputs. This method is used to find feasible paths for a car towing n trailers.

For Pfaffian systems with codimension greater than two, an *extended Goursat normal form* is defined, and necessary and sufficient conditions for converting systems into this form are given. This form is equivalent to the multi-input chained form for nonholonomic control systems, and the steering methods which were developed for the two-input chained form are generalized to the multi-input case.

A generalization of the N-trailer system which has more than two inputs is the multi-steering trailer system. Some configurations of this system (including the fire truck) can be converted into extended Goursat normal form and thus steered using one of the methods described above. For the configurations of this system which do not satisfy the conditions for conversion, there exists a *prolongation* of the Pfaffian system which can be converted into extended Goursat normal form. Integral curves for the prolonged system can be projected down onto the original system to give feasible paths. Necessary and sufficient conditions for converting systems into extended Goursat normal form using a particular type of prolongation are also presented.

Finally, some of the results for converting nonholonomic systems into Goursat normal form are specialized to give conditions for linearizing control systems.

This dissertation is dedicated to the memory of my father

Michael Tilbury

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Chapter 1

Introduction

In the past few years there has been a great deal of interest in the generation of motion planning algorithms for mobile robots or carts with nonholonomic or non-integrable velocity constraints, such as those coming from the kinematics of the drive mechanisms of the carts. A recent collection of papers [26] brings together some of the important results in nonholonomic motion planning. This work represents somewhat of a departure from the more traditional robot motion planning techniques (see for example [8, 22, 52]), which focused on understanding the complexity of the computational effort associated with planning collision-free trajectories for robots (with no constraints on their instantaneous velocities). Unfortunately the motion plans arising from these more traditional methods often required sideways motion of robot carts with wheels, and thus were not appropriate for most mobile robots [23, 24].

In this dissertation, the motion planning problem for several systems of car-like mobile robots and trailers is considered and solved. The "canonical" example of a car pulling n trailers [25, 35] is examined and converted into a canonical form for which the solution to the steering problem is straightforward. These results are then extended to systems with more than one steering wheel, such as a fire truck [7] and a multi-steering trailer system [58].

The nonholonomic constraints for these systems arise from constraining each pair of wheels to roll without slipping. Strictly speaking, if an axle has a differential that keeps the pair of wheels rolling without slipping, then each wheel turns a different amount in accordance with a simple geometric relationship called the Alexander-Maddocks condition [2]. Because the angle of each axle can be determined from the path taken by a point at the center of the axle and the geometry involved, in this dissertation, the differentials will be neglected and the wheels on each axle will be assumed to be parallel.

The system of a car with n trailers has been viewed as a canonical example because each trailer adds one dimension to the state space of the system (representing its angle with respect to the inertial frame) and one nonholonomic constraint. Regardless of the number of trailers attached, the general system always has two degrees of freedom, corresponding to the driving and steering directions of the front car. It has been shown that every point in the state space is reachable, i.e. that the system is completely controllable [25]. One of the questions that this dissertation answers is the one of constructive controllability; explicit open loop controls for steering the car with n trailers from an initial to a final position are given.

An early paper by Murray and Sastry [36] (see also [37]) studied motion planning for nonholonomic systems, and focused attention on a specific class of systems in "chained form":

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1. \end{aligned}$$

This class of systems was inspired by some early work of Brockett [4] on optimal control of "canonical systems." Murray and Sastry [37] gave sufficient conditions for converting systems into chained form as well as an algorithm (using sinusoids at integrally related frequencies) for steering chained form systems. The theory was used to transform the front-wheel drive car, a car with one trailer, and a hopping robot into chained form, and to find feasible trajectories for these systems using the sinusoidal steering algorithm. However, the car with two trailers did not fit the sufficient conditions and was left an open problem.

This dissertation represents somewhat of a departure from most of the previous work on motion planning for mobile robots. Traditionally, a nonholonomic system has been defined by a distribution Δ on the tangent space to the configuration manifold of the system, specifying the directions in which the system is allowed to move at every point. If a basis for this distribution Δ is given by the m vector fields $\{g_1, \ldots, g_m\}$, then the tangent vector to any feasible path for the system must be a linear combination of these vector fields. In this manner, the nonholonomic system defines a control system,

$$\dot{x} = \sum_{i=1}^{m} g_i u_i$$

where x are local coordinates on the configuration manifold and the functions u_i are called the inputs or controls. It should be noted that such systems are *drift-free;* that is, if all the inputs are zero then the system does not move. Every point in the configuration space can be considered an equilibrium point in that sense. Also, because of the drift-free property and the fact that there are fewer inputs than states, the linearization of such a control system about any (equilibrium) point is not controllable. Most of the standard control techniques rely on linearization methods and thus cannot be applied to such systems.

Equivalently, a nonholonomic system can be defined by a codistribution on the cotangent space to the configuration manifold, specifying the directions that the system is not allowed to move:

$$I = \{\alpha^1, \ldots, \alpha^{n-m}\}$$

This codistribution generates a *Pfaffian system,* and can thus be analyzed using the theory of exterior differential systems. This formulation is the dual of that described above in the sense that the codistribution I annihilates the distribution Δ, that is $I = \Delta^\perp$ or, in coordinates, $\alpha^i(g_j) = 0$ for all i, j. In the context of motion planning for systems with nonholonomic velocity constraints, this is in some sense a very natural framework since a basis for I can be written in coordinates by taking each $\alpha^i = 0$ as the i^{th} nonholonomic constraint, for example, in the N-trailer system, that the wheels on the i^{th} axle roll without slipping. A paper by Murray [40] was instrumental in bringing the theory of exterior differential systems to the attention of the nonholonomic motion planning community.

Such exterior differential systems and their properties were first studied by Pfaff in the early 1800's; see [3] for a historical overview. The path planning problem for a mobile robot towing multiple trailers can be formulated as the problem of finding integral curves for the corresponding Pfaffian system. Classical results on exterior differential systems by Goursat, Engel, Cartan, and others on classification and canonical forms can be used to find such integral curves. Most of the relevant results in exterior differential systems and canonical forms for Pfaffian systems are presented in an abbreviated fashion in the monograph by Bryant et al. [3]. Chapter 2 of this dissertation contains a very brief summary of some of the necessary mathematical tools. Chapter 3, taken largely from the paper by Tilbury, Murray, and Sastry [55], presents the Pfaff, Engel, and Goursat normal forms for Pfaffian systems of codimension two, along with the example system of a car towing multiple trailers. The N-trailer system is formulated as a Pfaffian system, and it is shown how this system can be transformed into the Goursat normal form. Since the Goursat normal form for Pfaffian systems is the dual of the chained form for nonholonomic control systems, the steering methods developed for chained form systems can be applied to the N-trailer system. Several such methods are described in Section 3.4, and sample paths for the N-trailer system, such as parallel-parking and backing into a loading dock, are also presented.

Trailer systems with more than two inputs, such as the fire truck, were first studied in conjunction with Linda Bushnell [7], and it was shown how the kinematic equations could be converted into chained form and then steered using sinusoids. The generalization of both the fire truck and the n-trailer system is the multi-steering trailer system, first described and analyzed by Tilbury, Sørdalen, Bushnell, and Sastry [58]. This system consists of a mobile robot towing multiple trailers, several of which are steerable. In the original paper, the kinematic equations of the system were defined and it was shown how the control system could be converted into a multi-input chained form system. The transformation required that a number of states be added to the system in a dynamic state feedback, interpreted in the context of this system as "virtual trailers." A chain of these virtual trailers was added in front of each steerable axle, the virtual chains diverging from the physical chain of trailers. The methods proposed for steering two-input chained form systems in [55] were extended to multi-input chained form systems in [58]; these methods will also be presented in this dissertation in Section 4.5. Once a path is found using one of these methods for the extended system, the path can be projected down to the original system; the information about the trajectories of the virtual trailers is not needed and can be discarded. One example of a parallel-parking type maneuver for such a system is shown in Section 4.6.

Since the fire truck could be exactly converted into chained form, it was not known if this dynamic state feedback was necessary for the general multi-steering trailer system to be converted into multi-input chained form. Only sufficient conditions exist for converting control systems into chained forms, and thus the answer could not be found using that theory but would rather come from the theory of exterior differential systems. Gardner and Shadwick [20] cast the problem of exactly linearizing control systems into the exterior differential systems framework. Murray [39] generalized their result to give necessary and sufficient conditions for conversion to what he called the "extended Goursat normal form," but only for the case when one of the towers is longer than the others. The complete necessary and sufficient conditions for conversion to extended Goursat normal form can be found in [56] and also in Section 4.2 of this dissertation.

The first piece of the answer to the question of the necessity of dynamic state feedback for the multi-steering trailer system was given by Tilbury and Sastry [56] where it was shown that a particular example of the multi-steering trailer system did not satisfy the conditions for conversion into extended Goursat normal form without allowing for dynamic state feedback, or prolongation in the language of exterior differential systems. Necessary and sufficient conditions for converting systems into extended Goursat normal form using a special type of prolongation, namely prolongation by differentiation, were also given. The complete answer to the question can be found in another paper by Tilbury and Sastry [57], and also in Chapter 4 of this dissertation, where the general multi-steering system is analyzed using the techniques of exterior differential systems. It is shown exactly which arrangements of trailers and steering wheels satisfy the conditions for conversion into extended Goursat form (or equivalently, multi-input chained form), and for those that do not, what is the minimal dimension of prolongation that is necessary to achieve a transformation into extended Goursat normal form (so that paths can be found using one of the methods of Section 4.5).

The Brunovsky normal form for a linear control system is a special case of the extended Goursat normal form. Thus, the problem of exactly linearizing a control system can be recast as the problem of converting the corresponding Pfaffian system into the particular extended Goursat normal form that is required (the coordinate corresponding to time is not allowed to be transformed); this investigation was begun by Gardner and Shadwick [20]. The problem of linearization after prolongation (or dynamic state feedback) was addressed from the exterior differential systems point of view in the Ph.D. dissertation of Sluis [46]. The paper by Tilbury and Sastry [56] continues this line of work. In particular, the problem of linearization by time-scaling is presented and solved, and necessary and sufficient conditions for linearizing control systems using dynamic extension are given. These results are summarized in Chapter 5 of this dissertation. The final chapter of this dissertation contains some conclusions and a brief discussion of open problems.

Chapter 2

Mathematical Preliminaries

This chapter contains a very brief overview of some of the definitions and theorems on Pfaffian systems that will be used in this dissertation. Although the theory of exterior differential systems is powerful enough to analyze systems of partial differential equations, only a small subset of this theory, that restricted to Pfaffian systems or systems of ordinary differential equations, will be needed in this dissertation. The interested reader is encouraged to consult the monograph by Bryant *et al.* [3] for more details on Pfaffian and exterior differential systems. Much of the material on exterior algebra can be found in the books by Flanders [15] and Munkres [38].

2.1 Exterior algebra and the exterior derivative

Let V be a real vector space of dimension n. The *exterior* or *wedge product* of two vectors is called a two-vector. For any $\alpha, \beta \in V$, their wedge product is denoted $\alpha \wedge \beta$. Using this product, a new vector space $\Lambda^2(V)$ is defined as the space spanned by all two-vectors on V. The wedge product satisfies the following properties:

$$\alpha \wedge \beta = -\beta \wedge \alpha$$
$$\alpha \wedge \alpha = 0$$
$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$$
$$a\alpha \wedge (b\beta + c\gamma) = (ab)\alpha \wedge \beta + (ac)\alpha \wedge \gamma$$

for all $\alpha, \beta, \gamma \in V$, and $a, b, c \in \mathbb{R}$. If $\alpha^1, \ldots, \alpha^n$ is a basis for V, then $\{\alpha^i \wedge \alpha^j : i < j\}$ is a basis for $\Lambda^2(V)$. The dimension of $\Lambda^2(V)$ is $\binom{n}{2}$.

Higher-order vectors are defined similarly using the wedge product. The space of k-vectors is denoted by $\Lambda^k(V)$, has a basis given by

$$\{\alpha^{i_1} \wedge \alpha^{i_2} \wedge \cdots \wedge \alpha^{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

and has dimension $\binom{n}{k}$. For completeness, let $\Lambda^0(V) = \mathbb{R}$ and $\Lambda^1(V) = V$.

The wedge product gives a simple method for testing linear dependence of vectors.

Proposition 1 *The vectors $v_1, \ldots, v_p \in V$ are linearly dependent if and only if*

$$v_1 \wedge \cdots \wedge v_p = 0$$

Thus, $\Lambda^k(V)$ is the zero vector space for $k > n$.

An element λ of the exterior or Grassman algebra over V

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V)$$

can be uniquely expressed as the sum of its components

$$\lambda = \lambda_0 + \lambda_1 + \cdots + \lambda_n \qquad \lambda_i \in \Lambda^i(V)$$

Now, consider a differentiable manifold M of dimension n and its tangent bundle TM. The bundle $\Lambda(TM)$ is constructed by attaching the exterior algebra of T_xM to every point $x \in M$,

$$\Lambda(T_xM) = \Lambda^0(T_xM) \oplus \Lambda^1(T_xM) \oplus \Lambda^2(T_xM) \oplus \cdots \oplus \Lambda^n(T_xM)$$

The bundle $\Lambda(TM)$ has $\Lambda^k(TM)$ as sub-bundles. A *section* of the bundle

$$\Lambda^k(TM) = \bigcup_{x \in M} \Lambda^k(T_xM)$$

over M is called a k-vector field on M.

The bundle $\Lambda(T^*M)$ is constructed similarly, using the cotangent bundle, and a section of the bundle $\Lambda^k(T^*M)$ is called an *exterior differential form of degree k* or simply a *k-form.* The notation $\Omega^k(M)$ will be used to mean the module (over the ring of smooth functions) of all smooth sections of $\Lambda^k(T^*M)$, in particular, $\Omega^1(M)$ is the module of all one-forms on M, and $\Omega(M) = \bigoplus \Omega^k(M)$ is the module of forms on M.

For local coordinates $x = (x_1, \ldots, x_n)$ on M, a local basis for the tangent space to M at x, or T_xM, is:

$$\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}.$$

Its dual basis for the cotangent space T_x^*M is denoted by

$$\{dx_1, \ldots, dx_n\}$$

where the dx_i are defined by the following relations:

$$dx_j\left(\frac{\partial}{\partial x_i}\right) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In terms of these local coordinates, a p-form ω can be written as

$$\omega = \sum w_{i_1 \ldots i_p}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \qquad i_1 < i_2 < \cdots < i_p$$

where the coefficient functions $w_{i_1 \ldots i_p}(x)$ are smooth functions on M.

Theorem 2 (Exterior Derivative) *There exists a unique linear map*

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

which satisfies the following properties:

1. *For* $f \in \Omega^0(M)$,
$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n,$$
relative to a local coordinate chart, or the usual differential.

2. *For* $\alpha \in \Omega^r(M), \beta \in \Omega^s(M)$,
$$d(\alpha + \beta) = d\alpha + d\beta$$
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta$$

3. $d(d\omega) = 0$ *for all* $\omega \in \Omega(M)$.

A proof of this theorem can be found in Flanders [15] and Munkres [38]. This operator d is called the *exterior derivative* and it is an intrinsic operator, that is, it is independent of the choice of coordinates.

2.2 Velocity constraints

For the systems considered in this dissertation, the configuration space of the system will be an n-dimensional manifold M with local coordinates $q = (q_1, \dots, q_n)$. The non-slipping conditions on the wheels will give rise to linear constraints on the system velocity, which can be written (in the most general way) as

$$A(q)\dot{q} = 0 \tag{2.1}$$

The matrix $A(q)$ has as elements $a_{ij}(q)$ which are smooth functions on M. The velocity of the system is constrained to be in the null space of the matrix $A(q)$.

If the system follows a trajectory $q(t)$ through the configuration space, the velocity vector $\dot{q}$ is in the tangent space to M at q, and can be represented in local coordinates by the expression

$$v(q) = v_1 \frac{\partial}{\partial q_1} + \dots + v_n \frac{\partial}{\partial q_n}$$

For the velocity of the system to satisfy the constraint condition (2.1) at every q, the velocity vector $v(q)$ must be annihilated by the one-forms α^i which correspond to the rows of $A(q)$,

$$\alpha^i = a_{i1}(q)dq_1 + \dots + a_{in}(q)dq_n$$

that is,

$$\alpha^i \cdot v(q) = 0$$

for $i = 1, \dots, s$ (where s is the number of rows of the matrix $A(q)$). Throughout this dissertation, the one-forms α^i will be referred to as the velocity constraints.

Consider first the case of a single velocity constraint, call it α. If this one-form α constrains the system velocity to be everywhere tangent to a submanifold $M' \subset M$, then the configuration of the system will always lie on this submanifold. In this case, the velocity constraint α is equivalent to a position constraint $f(q) = c_0$, where $M' = f^{-1}(c_0)$ (assuming that c_0 is a regular value of f). The function f provides a *foliation* of the manifold M into submanifolds; each submanifold $M^i = f^{-1}(c_i)$ is called a leaf of the foliation. If the initial configuration of the system lies on the leaf M', and the system velocity is always tangent to M', then the configuration of the system will lie on the submanifold M' for all time. The constraint α is then said to be *holonomic*. In this case, the entire configuration space M is not reachable by the system. In general, the dimension of the reachable configuration space is reduced by one for each holonomic constraint.

A holonomic velocity constraint can be either the exact differential of some function, $\alpha = df$, or a scaled version of an exact differential, $\alpha = \gamma df$, for some function γ. In the first case $d\alpha = d(df) = 0$ and α is said to be *exact*. The second case gives, using the definition of the exterior derivative,

$$d\alpha = d\gamma \wedge df + \gamma d(df)$$
$$d\alpha \wedge \alpha = d\gamma \wedge df \wedge \gamma df = 0$$

Such an α is said to be *integrable*, with $1/\gamma$ being the integrating factor.

If, on the other hand,

$$d\alpha \wedge \alpha \neq 0$$

the velocity constraint α is said to be *nonholonomic*. A system subjected to a single nonholonomic velocity constraint can reach every point in its configuration space (in the absence of obstacles, of course). That is, although the directions which the system is allowed to move are constrained at every point, given any two points in the configuration space, there exists a path which connects these two points and satisfies the velocity constraint.

The definitions of holonomic and nonholonomic are often given in the language of distributions. A single one-form α defines an $n-1$ dimensional distribution Δ such that α annihilates every vector field in Δ. Let $\{g_1, \ldots, g_{n-1}\}$ be a basis for Δ; necessarily $\alpha(g_i) = 0$. If the distribution Δ is *involutive*, that is, if the Lie bracket of any two vector fields in Δ also lies in Δ,

$$[g_i, g_j] \in \Delta \qquad \text{for all } i, j$$

then the distribution Δ (or equivalently the constraint α) is said to be holonomic. If there exists some Lie bracket $[g_i, g_j]$ which is not in the span of Δ, then the distribution (or the constraint) is said to be nonholonomic.

The relationship between the Lie bracket and the exterior derivative is given by the following lemma, sometimes referred to as "Cartan's magic formula."

Lemma 3 *Let α be a one-form on M and let X and Y be smooth vector fields on M. Then*

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]).$$

Proof. It suffices to show that the lemma is true for a basis element, and hence for $\alpha = \gamma df$.

On the one hand,

$$\begin{aligned} d\alpha(X,Y) &= (d\gamma \wedge df)(X,Y) \\ &= d\gamma(X)\, df(Y) - d\gamma(Y)\, df(X) \\ &= X(\gamma)\, Y(f) - Y(\gamma)\, X(f). \end{aligned}$$

Furthermore,

$$\begin{aligned} &X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]) \\ &\quad = X(\gamma Y(f)) - Y(\gamma X(f)) - \gamma(XY(f) - YX(f)) \\ &\quad = X(\gamma)Y(f) - Y(\gamma)X(f), \end{aligned}$$

and the lemma is proved. □

2.3 Pfaffian systems

Now consider s linearly independent velocity constraints, each represented by a one-form on an n-dimensional manifold M. The collection of these one-forms defines a codistribution I on M, that is

$$I = \{\alpha^1, \ldots, \alpha^s\}$$

Although the one-forms α^i are not uniquely defined, and will have different expressions depending on which coordinates are chosen, the codistribution which they span is intrinsic.

Definition 1 (Pfaffian Systems) *On a manifold M of dimension n, a* Pfaffian system *is a codistribution I of one-forms spanned by $\{\alpha^1, \ldots, \alpha^s\}$.*

The dimension of the Pfaffian system is defined to be s, the number of independent one-forms which generate it. The codimension *of I is $n - s$. A* complement *to the Pfaffian system is a collection of $n - s$ one-forms $\{\mu_1, \ldots, \mu_{n-s}\}$ which are independent of I.*

An integral curve *of a Pfaffian system is a curve $c : \mathbb{R} \to M$, the tangent vector to which is annihilated by I:*

$$\alpha^i \cdot \dot{c} = 0$$

A Pfaffian system with independence condition *is a Pfaffian system I together with a one-form τ on M which is not allowed to vanish on integral curves, that is $\tau \cdot \dot{c} \neq 0$ for any integral curve c.*

For a codistribution $I = \{\alpha^1, \dots, \alpha^s\}$, two k-forms η and ζ are said to be *congruent* modulo I if

$$\eta = \zeta + \sum_{i=1}^{s} \theta^i \wedge \alpha^i$$

for some forms θ^i in $\Omega^{k-1}(M)$. Congruence will be represented by the symbol $\equiv$.

In general, a Pfaffian system can be generated by some holonomic and some non-holonomic constraints. Since the holonomic constraints reduce the reachable configuration space, for the purposes of analysis it is useful to eliminate them and to consider the reduced system on a lower-dimensional configuration space which is completely reachable.

Consider all the one-forms in I which are integrable modulo the entire codistribution:

$$\{\alpha : d\alpha = 0 \mod I\}$$

If this set is the entire codistribution I, then I is integrable; that is I is equivalent to $\{df_1, \dots, df_s\}$ for some functions f_i, and the set of constraints is said to be completely holonomic. If the dimension of this set is less than the dimension of I, then it is itself a Pfaffian system, and one can consider all the constraints which are integrable modulo this new system. This process of recursively taking the set of constraints which are integrable modulo the current system is formalized in the definition of the derived flag.

Definition 2 (Derived Flag) *Consider a Pfaffian system $I = \{\alpha^1, \dots, \alpha^s\}$ on $\mathbb{R}^n$.*

The first derived system *of I is the set of all constraints in I which are integrable modulo the system:*

$$I^{(1)} = \{\alpha \in I : d\alpha \equiv 0 \mod I\}$$

The derived flag *is the sequence of nested codistributions,*

$$I = I^{(0)} \supset I^{(1)} \supset \cdots \supset I^{(N)}$$

defined by $I^{(0)} = I$ and $I^{(k+1)} = (I^{(k)})^{(1)}$. If the dimension of $I^{(k)}$ is not well-defined for all k, then the derived flag of the system is not defined.

Let $U \subset \mathbb{R}^n$ be an open set on which the derived flag is defined, that is, the dimension of each $I^{(i)}$ is constant on U. Consider the Pfaffian system I on U. Then, there exists some finite integer N for which $I^{(N+1)} = I^{(N)}$; this system is called the *bottom derived system.* Since $d\alpha \equiv 0 \mod I^{(N)}$ for all $\alpha \in I^{(N)}$, the bottom derived system is integrable. That is, there there exist functions $f_1, \ldots, f_q$ such that $I^{(N)} = \{df_1, \ldots, df_q\}$. Integral curves of I are then constrained to lie on level surfaces of f.

A basis of one-forms α^j for I is said to be *adapted to the derived flag* if a basis for each derived system $I^{(i)}$ can be taken to be some subset of the α^j's. Given any basis for I, it is straightforward to construct an adapted basis by computing the derived flag, choosing a basis for the last nontrivial derived system, and moving up the derived flag, adding new elements to the basis to complete the basis for each derived system.

Chow's theorem states that the configuration space is completely reachable if and only if the bottom derived system is empty.

Theorem 4 (Chow [11]) *Let $I = \{\alpha^1, \ldots, \alpha^s\}$ represent a set of constraints on $\mathbb{R}^n$. Then, there exists a path $q(t)$ between any two points satisfying $\alpha^i \cdot \dot{q} = 0$ for all i if and only if there exists an N such that $I^{(N)} = \{0\}$.*

Let $\Delta_0 = \{g_1, \ldots, g_{n-s}\}$ be the distribution which is annihilated by I. Recursively define a filtration of distributions $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \cdots$ by the relationship

$$\begin{aligned} \Delta_{i+1} &= \Delta_i \cup [\Delta_i, \Delta_i] \\ &= \{g_i, [g_j, g_k] : g_i, g_j, g_k \in \Delta_i\} \end{aligned}$$

that is, Δ_{i+1} is the distribution which contains all vector fields in Δ_i as well as all Lie brackets of vector fields in Δ_i. Using Lemma 3, it is straightforward to show that Δ_i is precisely the distribution annihilated by the i^{th} derived system $I^{(i)}$. Thus, an equivalent statement of Chow's theorem is that there exists a path $q(t)$ between any two points such that $\dot{q}(t)$ is in the span of $\{g_1, \ldots, g_{n-s}\}$ if and only if there exists an N such that $\Delta_N = TM$.

A variant of the familiar Frobenius theorem will be used in this dissertation and is stated here for reference.

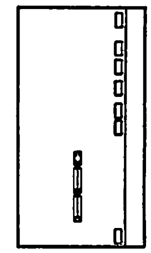
Theorem 5 (Frobenius [3]) *Let $\{\alpha^1, \ldots, \alpha^p\}$ be a set of linearly independent one-forms, and $f_1, \ldots, f_q$ a set of functions whose differentials are linearly independent of each other and of the α^i's. If*

$$d\alpha^i \wedge \alpha^1 \wedge \cdots \wedge \alpha^p \wedge df_1 \wedge \cdots \wedge df_q = 0$$

for $i = 1, \ldots, p$, then there exist coordinate functions $z_1, \ldots, z_p$ and coefficient functions a_{ij}, b_{ij} such that the one-forms α^i can be written as:

$$\alpha^i = \sum_{j=1}^{p} a_{ij} dz_j + \sum_{j=1}^{q} b_{ij} df_j$$

The proof follows from the standard Frobenius theorem and the fact that the codistribution $\{\alpha^1, \ldots, \alpha^p, df_1, \ldots, df_q\}$ is integrable.



Chapter 3

Goursat Normal Forms and the N-trailer system

In this chapter, the problem of finding feasible paths for the system of a car-like mobile robot towing n trailers will be considered and solved. Most of the results presented in this chapter, with the exception of Section 3.3, originally appeared in [55]. The analysis and coordinate transformations presented for the systems with kingpin hitches appear for the first time in this dissertation.

Each axle of the car and trailer system gives a constraint on the system velocity, representing that the wheels are only allowed to roll in the direction they are pointing and may not slip sideways. Neither do they slip in the direction that they are rolling; however, the angle of each wheel about its axle will not be modeled. The set of velocity constraints generates a Pfaffian system, and thus feasible paths for the mobile robot are the same as integral curves for the Pfaffian system.

Paths for the n-trailer system are found by converting the Pfaffian system into a normal form, called the Goursat normal form, and doing the planning in these new coordinates. Although the coordinate transformation into Goursat form is only guaranteed to exist locally, it will be shown that in many practical examples, the transformation is defined on most of the configuration space and is thus useful for planning many types of trajectories. Three different methods for finding trajectories for systems in Goursat form will be discussed in final section of this chapter. A movie animation of a parallel-parking trajectory found in this manner can be seen in the upper right-hand corner of the pages of this chapter.

3.1 Normal forms for Pfaffian systems

One way to find integral curves for Pfaffian systems is to transform the system into a normal form. If such a normal form can be found, then there is a one-to-one correspondence between the integral curves of the normal form system and the original system.

3.1.1 Pfaff's and Engel's normal forms

The simplest type of normal form for a nonholonomic system involves finding a normal form for a single constraint on $\mathbb{R}^n$.

Theorem 6 (Pfaff's theorem) *Suppose α is a one-form on $\mathbb{R}^n$ which satisfies $(d\alpha)^{r+1} \wedge \alpha = 0$, $(d\alpha)^r \wedge \alpha \neq 0$. Then there exist local coordinates z such that*

$$\alpha = dz_1 + z_2 dz_3 + z_4 dz_5 + \cdots + z_{2r} dz_{2r+1}.$$

Note that the case $r = 0$ is a special case of the Frobenius theorem. In the case $r = 1$, the proof [3] shows that there exist two functions f_1 and f_2 which satisfy the following partial differential equations:

$$\begin{aligned} d\alpha \wedge \alpha \wedge df_1 &= 0 \\ \alpha \wedge df_1 \wedge df_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} \alpha \wedge df_1 &\neq 0 \\ df_1 \wedge df_2 &\neq 0. \end{aligned} \tag{3.1}$$

Once suitable functions f_1 and f_2, have been found, α can be scaled such that

$$\alpha = df_2 + g df_1$$

The Pfaff theorem guarantees that there exists a solution f_1, f_2 to the equations (3.1) (it need not be unique).

In the case of a single constraint in $\mathbb{R}^3$, Pfaff's theorem shows that if the constraint is non-integrable then the corresponding control system can be written in chained form. This follows because if α is not integrable then $d\alpha \wedge \alpha \neq 0$ but $(d\alpha)^2 \wedge \alpha = 0$ by a dimension count. Therefore, Theorem 6 can be applied to conclude that

$$\alpha = dz_3 - z_2 dz_1.$$

A basis for the distribution annihilated by such a one-form α (on $\mathbb{R}^3$) is given by the two vector fields

$$g_1 = \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_3} \qquad g_2 = \frac{\partial}{\partial z_2},$$

which is the chained form in $\mathbb{R}^3$.

Engel's theorem applies to the case of two non-integrable constraints in $\mathbb{R}^4$, and also results in a chained form.

Theorem 7 (Engel's theorem [3]) *Let I be a two-dimensional codistribution on $\mathbb{R}^4$ with* $\dim I^{(1)} = 1$ *and* $\dim I^{(2)} = 0$. *Then there exist local coordinates such that*

$$I = \{dz_4 - z_3 dz_1, dz_3 - z_2 dz_1\}. \tag{3.2}$$

Proof. Choose a basis $I = \{\alpha^1, \alpha^2\}$ which is adapted to the derived flag, that is $\alpha^1 \in I^{(1)}$. It follows that $d\alpha^1 \wedge \alpha^1 \neq 0$ (since $I^{(2)} = \{0\}$) and $(d\alpha^1)^2 \wedge \alpha^1 = 0$ (by dimension count). Hence Pfaff's theorem can be used to find two functions f_1 and f_2 which satisfy (3.1). Defining the coordinates $z_4 = f_2$ and $z_1 = f_1$, the constraint α^1 can be scaled so that $\alpha^1 = dz_4 - z_3 dz_1$.

To determine the final coordinate z_2, the structure of α^2 can be used. Since $\alpha^1 \in I^{(1)}$, it follows that its exterior derivative is equal to zero modulo the system I, or $d\alpha^1 \wedge \alpha^1 \wedge \alpha^2 = 0$. But $d\alpha^1 \wedge \alpha^1 = -dz_3 \wedge dz_1 \wedge dz_4$ and hence

$$\alpha^2 = a\, dz_3 + b\, dz_1 + c\, dz_4$$

Note that the dz_4 term can be eliminated since α^2 is only defined mod α^1,

$$\alpha^2 \equiv a\, dz_3 + b\, dz_1 \mod \alpha^1$$

If either a or b is zero, the assumptions on the dimensions of $I^{(1)}$ and $I^{(2)}$ are violated. Also, α^2 can be scaled by any function, yielding:

$$\frac{1}{a}\alpha^2 \equiv dz_3 + \frac{b}{a} dz_1 \mod \alpha^1 \tag{3.3}$$

A choice of $z_2 = -b/a$ will give a basis for the codistribution which is in Engel's normal form:

$$\begin{aligned} \bar{\alpha}_1 &= \alpha_1 && = dz_4 - z_3 dz_1 \\ \bar{\alpha}_2 &= \frac{1}{a}\alpha_2 + \lambda \alpha_1 && = dz_3 - z_2 dz_1 \end{aligned}$$

where λ is chosen such that equation (3.3) becomes an equality. □

A basis for the distribution annihilated by Engel's system is given by the two vector fields

$$g_1 = \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_3} + z_3 \frac{\partial}{\partial z_4} \qquad g_2 = \frac{\partial}{\partial z_2}$$

which represent the chained form in $\mathbb{R}^4$.

3.1.2 Goursat normal form

The Goursat normal form applies to Pfaffian systems of codimension two on arbitrary dimensional manifolds.

Definition 3 (Goursat normal form) *A codimension two Pfaffian system I on $\mathbb{R}^n$ with generators of the form*

$$I = \{dz_n - z_{n-1}dz_1, \dots, dz_3 - z_2dz_1\} \tag{3.4}$$

is said to be in Goursat normal form.

A basis for the annihilated by the Goursat system is given by the two vector fields

$$g_1 = \frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_3} + \cdots + z_{n-1}\frac{\partial}{\partial z_n} \qquad g_2 = \frac{\partial}{\partial z_2}$$

which is precisely the chained form in $\mathbb{R}^n$.

Defining the constraints of the Goursat normal form as

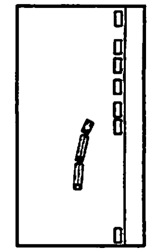
$$\begin{aligned} \alpha^1 &= dz_n - z_{n-1}dz_1 \\ &\vdots \\ \alpha^{n-2} &= dz_3 - z_2dz_1 \end{aligned}$$

the derived flag of I has the structure

$$\begin{aligned} I^{(0)} &= \{\alpha^1, \alpha^2, \cdots, \alpha^{n-3}, \alpha^{n-2}\} \\ I^{(1)} &= \{\alpha^1, \alpha^2, \cdots, \alpha^{n-3}\} \\ &\vdots \\ I^{(n-4)} &= \{\alpha^1, \alpha^2\} \\ I^{(n-3)} &= \{\alpha^1\} \\ I^{(n-2)} &= \{0\} \end{aligned} \tag{3.5}$$

Integral curves of the system are unconstrained in their z_1, z_n coordinates alone. Once $z_1(t), z_n(t)$ are specified as functions of some parameter t, the other coordinates $z_i(t)$ are determined. The following classical theorem gives necessary and sufficient conditions for converting a codimension two Pfaffian system into Goursat normal form:

Theorem 8 (Goursat Normal Form [3]) *A Pfaffian system I of codimension two on $\mathbb{R}^n$ has a set of generators which are in Goursat normal form if and only if there exists a basis set of forms $\{\alpha^1, \ldots, \alpha^{n-2}\}$ for I and a one-form π satisfying the congruences:*



$$\begin{aligned} d\alpha^i &\equiv -\alpha^{i+1} \wedge \pi \mod \alpha^1, \ldots, \alpha^i \quad i = 1, \ldots, n-3 \\ d\alpha^{n-2} &\not\equiv 0 \qquad\qquad \mod I \end{aligned} \tag{3.6}$$

The proof can be found in [3] and will not be given here. Note that any set of generating one-forms which satisfy the congruences (3.6) are adapted to the derived flag, which will have the structure of (3.5). An algorithm is presented here which can be used to convert Pfaffian systems into Goursat form.

Algorithm 1 *Converting Systems into Goursat form [55].* For a Pfaffian system $I = \{\omega^1, \ldots, \omega^s\}$ on a manifold M of dimension n with $s = n - 2$,

1. Construct a basis $I = \{\alpha^1, \ldots, \alpha^s\}$ which is adapted to the derived flag. Check the Goursat congruences to ensure that they are satisfied for some π. The candidate one-form π is easily found by noting that it must satisfy the first congruence, $d\alpha^1 \equiv -\alpha^2 \wedge \pi \mod \alpha^1$. Such a π can always be found, and is determined uniquely mod α^1, α^2. Checking whether this π will satisfy the other congruences is a matter of algebra, and can be tedious in practice.

2. It follows from the congruences that α^1 satisfies Pfaff's theorem for $r = 1$, and thus two functions f_1 and f_2 as defined in (3.1) can be found. Define the coordinates $z_1 = f_1$ and $z_n = f_2$, and scale α^1 such that

$$\alpha^1 = dz_n - z_{n-1} dz_1 \ .$$

3. The remaining coordinates are determined by simple differentiation. Given z_i, the next coordinate z_{i-1} is determined by algebraically solving the equation

$$\alpha^{n-i+1} \equiv dz_i - z_{i-1} dz_1 \mod \alpha^1, \ldots, \alpha^{n-i+1}.$$

The proof of Goursat's theorem essentially shows that this equation always has a solution.

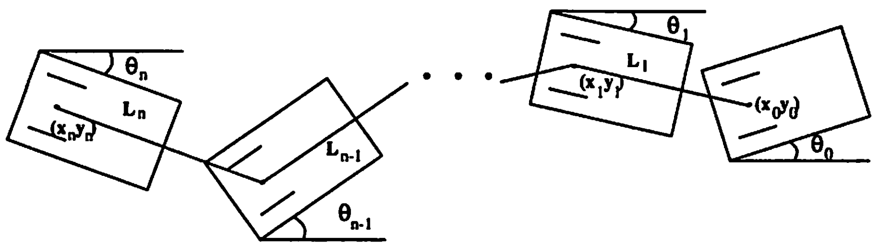


Figure 3.1: The mobile robot Hilare with n trailers.

3.2 The N-trailer Pfaffian system

The system of a mobile robot towing n trailers can be represented as a Pfaffian system: the velocity of the system is constrained in n directions corresponding the the n axles of wheels. A basis for this constraint codistribution (or equivalently, the Pfaffian system) is found by writing down the rolling without slipping conditions for all n axles.

In this section, the configuration space and the velocity constraints for the system will be defined. After checking the conditions of Theorem 8, it will be shown how Algorithm 1 can be used to convert the kinematic constraints into Goursat and chained forms.

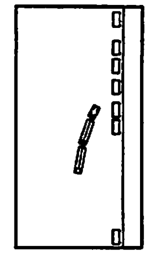
3.2.1 The system of rolling constraints and its derived flag

Consider a single-axle mobile robot such as Hilare[1] with n trailers attached, as sketched in Figure 3.1. Each trailer is attached to the body in front of it by a rigid bar, and the rear set of wheels of each body is constrained to roll without slipping. The trailers are assumed to be identical, but to have possibly different link lengths L_i. The x, y coordinates of the midpoint between the two wheels on the i^{th} axle are referred to as (x^i, y^i) and the hitch angles (all measured with respect to the horizontal) are given by θ^i. The connections between the bodies give rise to the following relations:

$$\begin{aligned} x^{i-1} &= x^i + L_i \cos\theta^i \\ y^{i-1} &= y^i + L_i \sin\theta^i \end{aligned} \qquad i = 1, 2, \ldots, n. \tag{3.7}$$

Obviously, the entire space parameterized by all the coordinates $(x^0, y^0, \theta^0, \ldots, x^n, y^n, \theta^n) \in \mathbb{R}^{2n+2} \times (S^1)^{n+1}$ is not reachable. Taking into account the connection relations (3.7), any one of the Cartesian positions x^i, y^i together with all the hitch angles $\theta^0, \ldots, \theta^n$ will completely

[1]The Hilare family of mobile robots resides at LAAS in Toulouse, see for example [10, 18].



represent the configuration of the system. The configuration space is thus $M = \mathbb{R}^2 \times (S^1)^{n+1}$ and has dimension $n+3$. In any neighborhood, the configuration space can be parameterized by $\mathbb{R}^{n+3}$.

The velocity constraints on the system arise from constraining the wheels of the robot and trailers to roll without slipping; the velocity of each body in the direction perpendicular to its wheels must be zero. Each pair of rear wheels is modeled as a single wheel at the midpoint of the axle. The non-slipping conditions are given, in terms of coordinates,

$$\sin\theta^i \dot{x}^i - \cos\theta^i \dot{y}^i = 0. \tag{3.8}$$

Equation (3.8) models the fact that the velocity perpendicular to each pair of wheels is zero. Following the discussion of Section 2.2, this velocity constraint can be written as a one-form,

$$\alpha^i = \sin\theta^i dx^i - \cos\theta^i dy^i \quad i = 0, \ldots, n \tag{3.9}$$

The one-forms $\alpha^0, \alpha^1, \ldots, \alpha^n$ represent the constraints that the wheels of the zeroth trailer (i.e. the cab), the first trailer, ..., the n^{th} trailer, respectively roll without slipping. The Pfaffian system corresponding to this mobile robot system is generated by the codistribution spanned by all of the rolling without slipping constraints:

$$I = \{\alpha^0, \ldots, \alpha^n\} \tag{3.10}$$

and has dimension $n+1$ on a manifold of dimension $n+3$.

Before finding the derived flag associated with I, it is useful to investigate some properties of the constraints and their exterior derivatives. Notice that equation (3.9) can be rearranged (after a division by a cosine) to give the congruence:

$$dy^i \equiv \tan\theta^i dx^i \mod \alpha^i \tag{3.11}$$

This division by a cosine introduces a singularity; the resulting coordinate transformation will not be valid at points where $\theta^n = \pm\pi/2$. See Remark 1 for a brief discussion of singularities.

All of the (x^i, y^i) are related by the hitch relationships. The exterior derivatives of these relationships can be taken,

$$\begin{aligned} x^{i-1} &= x^i + L_i \cos\theta^i \\ y^{i-1} &= y^i + L_i \sin\theta^i \end{aligned} \quad \Longrightarrow \quad \begin{aligned} dx^{i-1} &= dx^i - L_i \sin\theta^i d\theta^i \\ dy^{i-1} &= dy^i + L_i \cos\theta^i d\theta^i \end{aligned}$$

and these expressions can then be substituted into the formula for α^{i-1} from (3.9), allowing the constraint for the $(i-1)^{st}$ axle to be rewritten as:

$$\begin{aligned}
\alpha^{i-1} &= \sin\theta^{i-1}dx^{i-1} - \cos\theta^{i-1}dy^{i-1} \\
&= \sin\theta^{i-1}dx^{i} - \cos\theta^{i-1}dy^{i} - L_i\cos(\theta^{i-1}-\theta^{i})d\theta^{i} \\
&\equiv (\sin\theta^{i-1} - \tan\theta^{i}\cos\theta^{i-1})dx^{i} - L_i\cos(\theta^{i-1}-\theta^{i})d\theta^{i} \mod \alpha^{i} \\
&\equiv \sec\theta^{i}\sin(\theta^{i-1}-\theta^{i})dx^{i} - L_i\cos(\theta^{i-1}-\theta^{i})d\theta^{i} \qquad \mod \alpha^{i}
\end{aligned} \tag{3.12}$$

after an application of the congruence (3.11). A rearrangement of terms and a division by cosine in equation (3.12) will give the congruence

$$\begin{aligned}
d\theta^{i} &\equiv \frac{1}{L_i}\sec\theta^{i}\tan(\theta^{i-1}-\theta^{i})dx^{i} \mod \alpha^{i},\alpha^{i-1} \\
&\equiv f_{\theta^{i}}dx^{i} \qquad \mod \alpha^{i},\alpha^{i-1}
\end{aligned} \tag{3.13}$$

The exact form of the function f_{θ^i} is unimportant; what will be needed is the relationship between $d\theta^i$ and dx^i.

The first lemma relates the exterior derivatives of the x coordinates,

Lemma 9 *The exterior derivatives of any of the x variables are congruent modulo the Pfaffian system, that is: $dx^i \equiv f_{x^{i,j}}dx^j \mod I$.*

Proof. For two adjacent axles, the relationship between the x coordinates is given by the hitching,

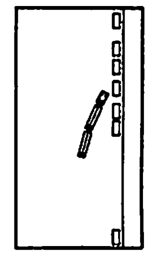
$$\begin{aligned}
x^{i-1} &= x^{i} + L_i\cos\theta^{i} \\
dx^{i-1} &= dx^{i} - L_i\sin\theta^{i}d\theta^{i} \\
&\equiv (1 - L_i\sin\theta^{i}f_{\theta^{i}})dx^{i} \mod \alpha^{i-1},\alpha^{i} \\
&\equiv f_{x^{i-1}}dx^{i} \qquad \mod \alpha^{i-1},\alpha^{i}
\end{aligned}$$

The congruence (3.13) was applied. □

A complement to the Pfaffian system $I = \{\alpha^0,\ldots,\alpha^n\}$ is given by

$$\{d\theta^0, dx^i\}$$

for any x^i, since by Lemma 9 their exterior derivatives are congruent modulo the system, and the complement is only defined modulo the system. These two one-forms, together with the codistribution I, form a basis for the space of all one-forms on the configuration manifold, or $\Omega^1(M)$.



Now consider the exterior derivative of the constraint corresponding to the i^{th} axle,

$$\begin{aligned} \alpha^i &= \sin\theta^i dx^i - \cos\theta^i dy^i \\ d\alpha^i &= d\theta^i \wedge (\cos\theta^i dx^i + \sin\theta^i dy^i) \\ &\equiv d\theta^i \wedge dx^i(\cos\theta^i + \sin\theta^i \tan\theta^i) \quad \text{mod } \alpha^i \\ &\equiv d\theta^i \wedge dx^i(\sec\theta^i) \quad \text{mod } \alpha^i \\ &\equiv 0 \quad \text{mod } \alpha^i, \alpha^{i-1} \end{aligned} \tag{3.14}$$

using (3.13). Thus, the exterior derivative of the constraint corresponding to the i^{th} axle is congruent to zero modulo itself and the constraint corresponding to the axle directly in front of it. The congruences (3.11) and (3.13) were useful in deriving this result.

This is all the information that is needed to find the derived flag for the system.

Theorem 10 (Derived flag for the N-trailer Pfaffian system) *Consider the Pfaffian system of the N-trailer system* (3.10) *with the one forms α^i defined by equations* (3.9). *The one-forms α^i are adapted to the derived flag in the following sense:*

$$\begin{aligned} I^{(0)} &= \{\alpha^0, \alpha^1, \ldots, \alpha^n\} \\ I^{(1)} &= \{\alpha^1, \ldots, \alpha^n\} \\ &\vdots \\ I^{(n)} &= \{\alpha^n\} \\ I^{(n+1)} &= \{0\}. \end{aligned} \tag{3.15}$$

Proof. The proof is merely a repeated application of equation (3.14). Noting that the exterior derivative of the i^{th} constraint is equal to zero modulo itself and the constraint corresponding to the axle directly in front of it, it is simple to check that the derived flag has the form given in equation (3.15). □

Note that $I^{(n+1)} = \{0\}$ implies that the N-trailer system is completely controllable (by Chow's theorem).

3.2.2 Conversion to Goursat normal form

In the preceding subsection, it was shown that basis $\{\alpha^0, \ldots, \alpha^n\}$ defined in equation (3.9) is adapted to its derived flag in the sense of (3.15). It remains to be checked whether the α^i satisfy the Goursat congruences and if they do, to find a transformation that puts them into the Goursat canonical form. The following theorem guarantees the existence of such a transformation.

Theorem 11 (Goursat congruences for the N-trailer system) *Consider the Pfaffian system $I = \{\alpha^0, \ldots, \alpha^n\}$ associated with the N-trailer system* (3.10) *with the one-forms α^i defined by equation* (3.9). *There exists a change of basis of the one forms α^i to $\bar{\alpha}^i$ which preserves the adapted structure, and a one-form π which satisfies the Goursat congruences for this new basis:*

$$\begin{aligned} d\bar{\alpha}^i &\equiv -\bar{\alpha}^{i-1} \wedge \pi \mod \bar{\alpha}^i, \ldots, \bar{\alpha}^n \qquad i = 1, \ldots, n \\ d\bar{\alpha}^0 &\not\equiv 0 \qquad\qquad\quad \mod I. \end{aligned}$$

The one-form which satisfies these congruences is given by

$$\pi = dx^n.$$

Proof. First of all, consider the original basis of constraints. The expression for α^i can be written in the configuration space coordinates from equation (3.9) together with the connection relations (3.7) and some bookkeeping as:

$$\alpha^i = \sin\theta^i dx^n - \cos\theta^i dy^n - \sum_{k=i+1}^{n} L_k \cos(\theta^i - \theta^k) d\theta^k \tag{3.16}$$

Before beginning the main part of the proof, it will be helpful to define a new basis of constraints $\tilde{\alpha}^i$, which are also adapted to the derived flag, but are somewhat simpler to work with. Each $\tilde{\alpha}^i$ will have only two terms.

Although the last constraint already has only two terms, it will be scaled by a factor,

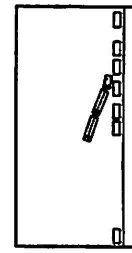
$$\tilde{\alpha}^n = \sec\theta^n \alpha^n = \tan\theta^n dx^n - dy^n$$

Note that a rearrangement of terms will give the congruence

$$dy^n \equiv \tan\theta^n dx^n \mod \tilde{\alpha}^n$$

Now consider the next to last constraint, α^{n-1}, and apply the preceding congruence:

$$\begin{aligned} \alpha^{n-1} &= \sin\theta^{n-1} dx^n - \cos\theta^{n-1} dy^n - L_n \cos(\theta^n - \theta^{n-1}) d\theta^n \\ &\equiv \sec\theta^n \sin(\theta^{n-1} - \theta^n) dx^n - L_n \cos(\theta^{n-1} - \theta^n) d\theta^n \mod \tilde{\alpha}^n \end{aligned}$$



Dividing once again by a cosine, the new basis element $\tilde{\alpha}^{n-1}$ is defined as

$$\tilde{\alpha}^{n-1} = \sec\theta^n \tan(\theta^{n-1} - \theta^n)dx^n - L_n d\theta^n \tag{3.17}$$

Thus, $\tilde{\alpha}^{n-1} \equiv f_{\alpha^{n-1}}\alpha^{n-1} \mod \alpha^n$. Also, the exterior derivative $d\theta^n$ is related to dx^n by the congruence

$$d\theta^n \equiv \frac{1}{L_n}\sec\theta^n \tan(\theta^{n-1} - \theta^n)dx^n \mod \tilde{\alpha}^{n-1}$$

This procedure of eliminating the terms $dy^n, d\theta^n, \ldots, d\theta^i$ from α^{i+1} can be continued.

Lemma 12 *A new basis of constraints $\tilde{\alpha}^i$ of the form*

$$\begin{aligned}
\tilde{\alpha}^n &= \tan\theta^n dx^n - dy^n \\
\tilde{\alpha}^i &= \sec\theta^n \sec(\theta^{n-1} - \theta^n)\ldots\sec(\theta^{i+1} - \theta^{i+2})\tan(\theta^i - \theta^{i+1})dx^n - L_{i+1}d\theta^{i+1} \\
& \quad i = 0, \ldots, n-1
\end{aligned} \tag{3.18}$$

is related to the original basis of constraints α^i through the following congruences:

$$\tilde{\alpha}^i \equiv f_{\alpha^i}\alpha^i \mod \alpha^{i+1}, \ldots, \alpha^n$$

and thus the basis $\tilde{\alpha}^i$ is also adapted to the derived flag.

Note that by the definition of $\tilde{\alpha}^i$, the exterior derivative $d\theta^{i+1}$ is related to dx^n by the congruence

$$d\theta^{i+1} \equiv \frac{1}{L_{i+1}}\sec\theta^n \sec(\theta^{n-1} - \theta^n)\ldots\sec(\theta^{i+1} - \theta^{i+2})\tan(\theta^i - \theta^{i+1})dx^n \mod \tilde{\alpha}^i$$

The lemma is proved by induction. It has already been shown that $\tilde{\alpha}^n = f_{\alpha^n}\alpha^n$ and $\tilde{\alpha}^{n-1} \equiv f_{\alpha^{n-1}}\alpha^{n-1} \mod \tilde{\alpha}^n$. Assume that $\tilde{\alpha}^i \equiv f_{\alpha^i}\alpha^i \mod \alpha^{i+1}, \ldots, \alpha^n$ for $i = j+1, \ldots, n$. Consider $\tilde{\alpha}^j$ as defined by equation (3.18),

$$\tilde{\alpha}^j = \sec\theta^n \sec(\theta^{n-1} - \theta^n)\ldots\sec(\theta^{j+1} - \theta^{j+2})\tan(\theta^j - \theta^{j+1})dx^n - L_{j+1}d\theta^{j+1}$$

Recall from equation (3.16) that α^j has the form

$$\alpha^j = \sin\theta^j dx^n - \cos\theta^j dy^n - \sum_{k=j+1}^{n} L_k \cos(\theta^j - \theta^k)d\theta^k$$

Now, applying the congruences

$$\begin{aligned}
dy^n &\equiv \tan\theta^n dx^n & \mod \tilde{\alpha}^n \\
d\theta^i &\equiv \frac{1}{L_i}\sec\theta^n \sec(\theta^{n-1} - \theta^n)\ldots\sec(\theta^i - \theta^{i+1})\tan(\theta^{i-1} - \theta^i)dx^n & \mod \tilde{\alpha}^{i-1}
\end{aligned}$$

to the expression for α^j, and expanding the summation, yields

$$\begin{aligned}\alpha^j \equiv\ & \sin\theta^j\,dx^n - \cos\theta^j\tan\theta^n dx^n - L_{j+1}\cos(\theta^j - \theta^{j+1})d\theta^{j+1} - \\ & - \cos(\theta^j - \theta^{j+2})\sec\theta^n\sec(\theta^{n-1} - \theta^n)\ldots\sec(\theta^{j+2} - \theta^{j+3})\tan(\theta^{j+1} - \theta^{j+2})dx^n \\ & - \cdots \\ & - \cos(\theta^j - \theta^{n-1})\sec\theta^n\sec(\theta^{n-1} - \theta^n)\tan(\theta^{n-2} - \theta^{n-1})dx^n \\ & - \cos(\theta^j - \theta^n)\sec\theta^n\tan(\theta^{n-1} - \theta^n)dx^n \\ & \quad \bmod \tilde{\alpha}^{j+1}, \ldots, \tilde{\alpha}^{n-2}, \tilde{\alpha}^{n-1}, \tilde{\alpha}^n\end{aligned}$$

To simplify the above expression, the trigonometric identity

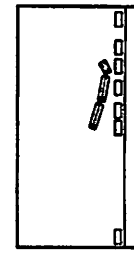
$$\sin a - \cos a\tan b = \sec b\sin(a - b)$$

is repeatedly applied and terms are collected. Start first with the first two terms in the first line,

$$\begin{aligned}\alpha^j \equiv\ & \sec\theta^n\sin(\theta^j - \theta^n)dx^n - L_{j+1}\cos(\theta^j - \theta^{j+1})d\theta^{j+1} - \\ & - \cos(\theta^j - \theta^{j+2})\sec\theta^n\sec(\theta^{n-1} - \theta^n)\ldots\sec(\theta^{j+2} - \theta^{j+3})\tan(\theta^{j+1} - \theta^{j+2})dx^n \\ & - \cdots \\ & - \cos(\theta^j - \theta^{n-1})\sec\theta^n\sec(\theta^{n-1} - \theta^n)\tan(\theta^{n-2} - \theta^{n-1})dx^n \\ & - \cos(\theta^j - \theta^n)\sec\theta^n\tan(\theta^{n-1} - \theta^n)dx^n \\ & \quad \bmod \tilde{\alpha}^{j+1}, \ldots, \tilde{\alpha}^{n-2}, \tilde{\alpha}^{n-1}, \tilde{\alpha}^n\end{aligned}$$

and now the first term and last term can be combined to yield

$$\begin{aligned}\alpha^j \equiv\ & -L_{j+1}\cos(\theta^j - \theta^{j+1})d\theta^{j+1} - \\ & - \cos(\theta^j - \theta^{j+2})\sec\theta^n\sec(\theta^{n-1} - \theta^n)\ldots\sec(\theta^{j+2} - \theta^{j+3})\tan(\theta^{j+1} - \theta^{j+2})dx^n \\ & - \cdots \\ & - \cos(\theta^j - \theta^{n-1})\sec\theta^n\sec(\theta^{n-1} - \theta^n)\tan(\theta^{n-2} - \theta^{n-1})dx^n \\ & + \sin(\theta^j - \theta^{n-1})\sec\theta^n\sec(\theta^{n-1} - \theta^n)dx^n \\ & \quad \bmod \tilde{\alpha}^{j+1}, \ldots, \tilde{\alpha}^{n-2}, \tilde{\alpha}^{n-1}, \tilde{\alpha}^n\end{aligned}$$



Collect the last two terms using the identity,

$$\begin{aligned}\alpha^j \equiv & -L_{j+1}\cos(\theta^j - \theta^{j+1})d\theta^{j+1} - \\ & - \cos(\theta^j - \theta^{j+2})\sec\theta^n \sec(\theta^{n-1} - \theta^n)\ldots\sec(\theta^{j+2} - \theta^{j+3})\tan(\theta^{j+1} - \theta^{j+2})dx^n \\ & - \ldots \\ & + \sin(\theta^j - \theta^{n-2})\sec\theta^n \sec(\theta^{n-1} - \theta^n)\sec(\theta^{n-2} - \theta^{n-1})dx^n \\ & \quad \mod \tilde{\alpha}^{j+1}, \ldots, \tilde{\alpha}^{n-2}, \tilde{\alpha}^{n-1}, \tilde{\alpha}^n\end{aligned}$$

and so forth. After all the terms are collected, it can be seen that the equation will read:

$$\begin{aligned}\alpha^j & \equiv \sin(\theta^j - \theta^{j+1})\sec\theta^n \sec(\theta^{n-1} - \theta^n)\ldots\sec(\theta^{j+1} - \theta^{j+2})dx^n - L_{j+1}\cos(\theta^j - \theta^{j+1})d\theta^{j+1} \\ & \qquad \mod \tilde{\alpha}^{j+1}, \ldots, \tilde{\alpha}^{n-2}, \tilde{\alpha}^{n-1}, \tilde{\alpha}^n \\ & \equiv \cos(\theta^j - \theta^{j+1})\tilde{\alpha}^j \mod \tilde{\alpha}^{j+1}, \ldots, \tilde{\alpha}^{n-2}, \tilde{\alpha}^{n-1}, \tilde{\alpha}^n\end{aligned}$$

and the lemma is proved.

The basis $\tilde{\alpha}^i$ will now be scaled to find the basis $\bar{\alpha}^i$ which will satisfy the congruences (3.6). Once again, the procedure will start with the last congruence, $\tilde{\alpha}^n$. The exterior derivative of $\tilde{\alpha}^n$ is given by

$$d\tilde{\alpha}^n = \sec^2\theta^n d\theta^n \wedge dx^n \tag{3.19}$$

Looking at the expression for $\tilde{\alpha}^{n-1}$ given in equation (3.17), it can be seen that π should be chosen to be some multiple of dx^n or $d\theta^n$. In fact, either $\pi = dx^n$ or $\pi = d\theta^n$ will work, although the computations are different for each case. The calculations here are for choosing $\pi = dx^n$. Choosing the new basis element $\bar{\alpha}^{n-1}$ as

$$\bar{\alpha}^{n-1} = \frac{1}{L_n}\sec^2\theta^n \tilde{\alpha}^{n-1}$$

will result in the desired congruence,

$$d\bar{\alpha}^n \equiv -\bar{\alpha}^{n-1} \wedge \pi \mod \alpha^n$$

Now consider the exterior derivative of $\bar{\alpha}^{n-1}$,

$$\begin{aligned}d\bar{\alpha}^{n-1} & = d(\frac{1}{L_n}\sec^3\theta^n \tan(\theta^{n-1} - \theta^n)dx^n - L_n \sec^2\theta^n d\theta^n) \\ & \equiv \frac{1}{L_n}\sec^3\theta^n \sec^2(\theta^{n-1} - \theta^n)d\theta^{n-1} \wedge dx^n \mod \bar{\alpha}^{n-1}\end{aligned}$$

since any terms $d\theta^n \wedge dx^n$ are congruent to 0 mod $\tilde{\alpha}^{n-1}$. Thus, in order to achieve the next Goursat congruence $d\tilde{\alpha}^{n-1} \equiv \tilde{\alpha}^{n-2} \wedge \pi$, the new basis element $\tilde{\alpha}^{n-2}$ should be chosen as

$$\bar{\alpha}^{n-2} = \frac{1}{L_n L_{n-1}} \sec^3 \theta^n \sec^2(\theta^{n-1} - \theta^n)\tilde{\alpha}^{n-2}$$

In general, the new basis is defined by

$$\bar{\alpha}^i = \frac{1}{L_n L_{n-1} \dots L_{i+1}} \sec^{n-i+1} \theta^n \sec^{n-i}(\theta^{n-1} - \theta^n) \dots \sec^3(\theta^{i+2} - \theta^{i+3}) \sec^2(\theta^{i+1} - \theta^{i+1})\tilde{\alpha}^i$$

It has already been shown that the congruences hold for $i = n$ and $i = n - 1$. Assume that the congruences

$$d\bar{\alpha}^i = -\bar{\alpha}^{i-1} \wedge \pi \mod \alpha^i, \dots, \alpha^n.$$

hold for $i = j + 1, \dots, n$. Consider the exterior derivative of $\bar{\alpha}^j$,

$$d\bar{\alpha}^j = d[(\frac{1}{L_n L_{n-1} \dots L_{j+1}} \sec^{n-j+1} \theta^n \sec^{n-j}(\theta^{n-1} - \theta^n) \dots \sec^3(\theta^{j+2} - \theta^{j+3}) \sec^2(\theta^{j+1} - \theta^{j+2}))$$
$$(\sec \theta^n \sec(\theta^{n-1} - \theta^n) \dots \sec(\theta^{j+1} - \theta^{j+2}) \tan(\theta^j - \theta^{j+1}) dx^n - L_{j+1} d\theta^{j+1})]$$

Before calculating all of the terms, recall that the following congruences hold:

$$d\theta^i \wedge dx^n \equiv 0 \mod \tilde{\alpha}^{i-1}$$
$$d\theta^i \wedge d\theta^k \equiv 0 \mod \tilde{\alpha}^{i-1}, \tilde{\alpha}^{k-1}$$

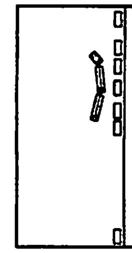
and thus the only term in $d\bar{\alpha}^j$ mod $\tilde{\alpha}^j, \dots, \tilde{\alpha}^n$ will be a multiple of $d\theta^j \wedge dx^n$,

$$\begin{aligned} d\bar{\alpha}^j \equiv {} & \frac{1}{L_n L_{n-1} \dots L_{j+1}} \sec^{n-j+2} \theta^n \sec^{n-j+1}(\theta^{n-1} - \theta^n) \dots \\ & \sec^4(\theta^{j+2} - \theta^{j+3}) \sec^3(\theta^{j+1} - \theta^{j+2}) \sec^2(\theta^j - \theta^{j+1}) \\ & \mod \tilde{\alpha}^j, \dots, \tilde{\alpha}^n \\ \equiv {} & \bar{\alpha}^{j-1} \wedge \pi \mod \tilde{\alpha}^j, \dots, \tilde{\alpha}^n \end{aligned}$$

This completes the proof that the Goursat congruences are satisfied. □

Since the one-forms $\bar{\alpha}^i$ do satisfy the Goursat congruences, Algorithm 1 can be used to find the coordinate transformation that will result in Goursat normal form. According to the Algorithm 1, there exist *possibly non-unique* functions f_1, f_2 which satisfy the Pfaff equations (3.1):

$$\begin{aligned} d\alpha^n \wedge \alpha^n \wedge df_1 &= 0 \\ \alpha^n \wedge df_1 \wedge df_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} \alpha^n \wedge df_1 &\neq 0 \\ df_1 \wedge df_2 &\neq 0. \end{aligned} \qquad \text{(Pfaff)}$$



The constraint corresponding to the last axle is once again given by[2]

$$\alpha^n = \sin\theta^n dx^n - \cos\theta^n dy^n$$

and its exterior derivative has the form

$$d\alpha^n = -\cos\theta^n dx^n \wedge d\theta^n - \sin\theta^n dy^n \wedge d\theta^n,$$

It follows that the exterior product of these two quantities is given by

$$d\alpha^n \wedge \alpha^n = -dx^n \wedge dy^n \wedge d\theta^n.$$

By the first equation of (3.1), f_1 may be chosen to be *any* function of x^n, y^n, θ^n *exclusively.*

Two different solutions of the equations (3.1) are explained here.

Transformation 1. Coordinates of the N^{th} trailer. Motivated by Sørdalen [47], f_1 can be chosen to be x^n. The second equation of (3.1) then becomes

$$\sin\theta^n dx^n \wedge dy^n \wedge df_2 = 0$$

with the proviso that $df_1 \wedge df_2 \neq 0$. A *non-unique* choice of f_2 is

$$f_2 = y^n.$$

The change of coordinates is defined by:

$$\begin{aligned} z_1 &= f_1(x) = x^n \\ z_{n+3} &= f_2(x) = y^n. \end{aligned} \tag{3.20}$$

The one form α^n may be written by dividing through by $\sin\theta^n$ as

$$\begin{aligned} \alpha^n &= dy^n - \tan\theta^n dx^n \\ &= dz_{n+3} - z_{n+2}dz_1, \end{aligned} \tag{3.21}$$

giving $z_{n+1} = \tan\theta^n$. The remaining coordinates are found by solving the equations

$$\alpha^i \equiv dz_{i+3} - z_{i+2}dz_1 \mod \alpha^{i+1}, \ldots, \alpha^n$$

[2]The basis that satisfies the Goursat congruences was a scaled version of the original basis, $\bar{\alpha}^n = f_{\alpha^n}\alpha^n$. However, it can be checked that

$$\begin{aligned} d\bar{\alpha}^n \wedge \bar{\alpha}^n &= (df_{\alpha^n} \wedge \alpha^n + f_{\alpha^n} d\alpha^n) \wedge f_{\alpha^n}\alpha^n \\ &= (f_{\alpha^n})^2 d\alpha^n \wedge \alpha^n \end{aligned}$$

and thus a function f_1 will satisfy $d\alpha^n \wedge \alpha^n \wedge df_1 = 0$ if and only if $d\bar{\alpha}^n \wedge \bar{\alpha}^n \wedge df_1 = 0$.

for $i = n-1, \ldots, 1$. In fact, because $dz_1 = \pi$ as chosen in the proof of Theorem 11, the one-forms $\bar{\alpha}^i$ already satisfy these equations,

$$\bar{\alpha}^i = \frac{1}{L_n L_{n-1} \ldots L_{i+1}} \sec^{n-i+1}\theta^n \sec^{n-i}(\theta^{n-1} - \theta^n) \ldots \sec^3(\theta^{i+2} - \theta^{i+3}) \sec^2(\theta^{i+1} - \theta^{i+1})$$
$$(\sec\theta^n \sec(\theta^{n-1} - \theta^n) \ldots \sec(\theta^{i+1} - \theta^{i+2}) \tan(\theta^i - \theta^{i+1}) dx^n - L_{i+1} d\theta^{i+1})$$

and so the coordinates z_i are given by the coefficients of dx^n in the expression for the $\bar{\alpha}^i$.

Transformation 2. Coordinates of the origin seen from the last trailer. Yet another choice for f_1 corresponds to writing the coordinates of the origin as seen from the last trailer. This is reminiscent of a transformation used by Samson [44] in a different context, and is given by

$$z_1 := f_1(x) = x^n \cos\theta^n + y^n \sin\theta^n.$$

This has the physical interpretation of being the origin of the reference frame when viewed from a coordinate frame attached to the n^{th} trailer. It satisfies the first of the equations of (3.1) simply by virtue of the fact that it is only a function of x^n, y^n, θ^n. It may be verified that a choice of f_2 given by

$$z_{n+3} := f_2 = x^n \sin\theta^n - y^n \cos\theta^n - \theta^n z_1$$

satisfies the Pfaff equation,

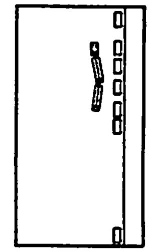
$$\alpha^1 \wedge df_1 \wedge df_2 = 0.$$

The remaining coordinates $z_2, \ldots, z_{n+2}$ corresponding to this transformation may be obtained by solving the equations

$$\alpha^i \equiv dz_{i+3} - z_{i+2} dz_1 \mod \alpha^{i+1}, \ldots, \alpha^n$$

for $i = n-1, \ldots, 1$. The details are tedious and are omitted.

Remark 1 (Singularities) There are two types of singularities associated with the transformation into chained form. At $\theta^n = \pi/2$, for example, the transformation will be singular, but this singularity can be avoided by choosing another coordinate chart at the singular point (such as by interchanging x and y, using the $SE(2)$ symmetry of the system). A singularity also occurs when the angle between two adjacent axles is equal to $\pi/2$; at this point, some of the codistributions in the derived flag will lose rank. The derived flag is not defined at these points; nor is the transformation. The methods described herein will not work for controlling the n-trailer system when the trailers must go through such a jack-knifed configuration.



There are no singularities of the second type in the unicycle ($n = 0$) or in the front-wheel drive car ($n = 1$).

3.2.3 The control system associated with the N-trailer system

Consider an exterior differential system on $\mathbb{R}^n$ of codimension 2, given by

$$I = \{\alpha^1, \dots, \alpha^{n-2}\},$$

and choose a basis g_1, g_2 for the 2-dimensional distribution Δ which is annihilated by the one-forms α^i, that is:

$$\alpha^i \cdot g_j = 0$$

for $i = 1, \dots, n-2$ and $j = 1, 2$. Any integral curve $\gamma(t)$ of I has its tangent vector $\dot{\gamma}(t)$ in the span of these two vector fields at every point, that is

$$\dot{\gamma}(t) = g_1(\gamma(t))\, u_1(t) + g_2(\gamma(t))\, u_2(t)$$

for some functions u_1 and u_2. All integral curves to I are defined by an initial condition and these two functions u_1 and u_2, which are called the inputs to the corresponding nonholonomic control system,

$$\Sigma: \qquad \dot{\mathbf{x}} = g_1 u_1 + g_2 u_2, \tag{3.22}$$

Note that this type of control system is drift-free, that is, if the inputs are set to zero, the system does not evolve. All configurations x are thus equilibria.

The nonholonomic control system associated with the N-trailer system is defined in the following proposition.

Proposition 13 *Consider an N-trailer system with $n+1$ rolling constraints α^i on $\mathbb{R}^{n+3}$,*

$$\alpha^i = \sin\theta^i dx^i - \cos\theta^i dy^i \qquad i = 0, \dots, n$$

with the x^i, y^i *related by* (3.7). *A basis for the distribution* Δ *which is annihilated by these one-forms* $\{\alpha^0, \ldots, \alpha^n\}$ *is given by*

$$g_1 = \begin{bmatrix} \cos\theta^n \\ \sin\theta^n \\ \frac{1}{L_n}\tan(\theta^{n-1} - \theta^n) \\ \vdots \\ \frac{1}{L_1}\prod_{i=2}^{n}\sec(\theta^{i-1} - \theta^i)\tan(\theta^0 - \theta^1) \\ 0 \end{bmatrix} \qquad g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Proof. From the connection equations (3.7), it can be shown by induction that the i^{th} constraint has an expression in coordinates given by

$$\alpha^i = \sin\theta^i dx^n - \sin\theta^i dy^n - \sum_{k=i+1}^{n} L_k\cos(\theta^i - \theta^k)d\theta^k. \tag{3.23}$$

(the induction proof of this is omitted here).

A basis for the distribution $\Delta = I^\perp$ is given by two linearly independent vector fields g_1, g_2 which satisfy:

$$\alpha^i \cdot g_j = 0 \qquad \text{for all } i = 0, \ldots, n,\ j = 1, 2.$$

Since none of the α^i have a term $d\theta^0$, one of the vector fields in Δ can be chosen to be

$$g_2 = \frac{\partial}{\partial\theta^0}.$$

Choosing the other vector field

$$g_1 = \cos\theta^n\frac{\partial}{\partial x^n} + \sin\theta^n\frac{\partial}{\partial y^n} + \sum_{k=1}^{n}\frac{1}{L_k}\tan(\theta^{k-1} - \theta^k)\prod_{i=k+1}^{n}\sec(\theta^{i-1} - \theta^i)\frac{\partial}{\partial\theta^k}$$

will result in $\alpha^i \cdot g_1 = 0\ \forall i$. In a more familiar notation, these two vector fields are written as

$$g_1 = \begin{bmatrix} \cos\theta^n \\ \sin\theta^n \\ \frac{1}{L_n}\tan(\theta^{n-1} - \theta^n) \\ \vdots \\ \frac{1}{L_1}\prod_{i=2}^{n}\sec(\theta^{i-1} - \theta^i)\tan(\theta^0 - \theta^1) \\ 0 \end{bmatrix} \qquad g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where the coordinates are written in the order $\mathbf{x} = (x^n, y^n, \theta^n, \ldots, \theta^0)$. □

Although there are many different choices of g_1, g_2 which will span the distribution $\Delta = I^{\perp}$, these two are natural in the sense that when the nonholonomic control system is written as:

$$\dot{\mathbf{x}} = g_1 u_1 + g_2 u_2$$

the input functions u_1 and u_2 have the physical interpretation of the linear velocity of the n^{th} trailer ($u_1 = v_n$), and the rotational velocity of the lead car ($u_2 = \omega$). From a practical point of view, the velocity v_0 of the lead car (not that of the last trailer) is the control input. This velocity is a function of v_n and the state of the system, given by the input transformation

$$v_0 = \sec(\theta^0 - \theta^1)\sec(\theta^1 - \theta^2)\cdots\sec(\theta^{n-1} - \theta^n)v_n.$$

3.2.4 Converting the N-trailer system into chained form

For two-input nonholonomic systems, a normal form called "chained form" for which point-to-point trajectories could easily be found was defined in [37]. It was called chained form because of the way the derivative of each state depended upon the one above it in a chained fashion,

$$\begin{aligned}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 \\
\dot{z}_3 &= z_2 u_1 \\
\dot{z}_4 &= z_3 u_1 \\
&\vdots \\
\dot{z}_{n+3} &= z_{n+2} u_1
\end{aligned} \tag{3.24}$$

This can also be written in a more compact form as:

$$\dot{\mathbf{z}} = g_1 u_1 + g_2 u_2$$

where the two input vector fields are:

$$\begin{aligned}
g_1 &= \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_3} + \cdots + z_{n+2} \frac{\partial}{\partial z_{n+3}} \\
g_2 &= \frac{\partial}{\partial z_2}.
\end{aligned} \tag{3.25}$$

These two vector fields g_1 and g_2 are annihilated by the one-forms in the Goursat normal form (3.4), and thus the chained form is the dual of the Goursat normal form.

The procedure for transforming a nonholonomic control system such as (3.22) into chained form requires both a coordinate transformation and state feedback. Although for the most general case, a state feedback is given by

$$\bar{u} = a(\mathbf{x}) + b(\mathbf{x})u,$$

for drift-free nonholonomic systems it is desirable to have $a(\mathbf{x}) = 0$. (If this were not the case, the state feedback would add a drift term to a drift-free system and could not result in a chained form, which is drift-free.) The purpose of the state feedback $\bar{u} = b(\mathbf{x})u$ is therefore to transform the basis of the distribution Δ into chained form in the new coordinate system.

Because chained form is the dual of Goursat form, the two transformations discussed in the previous section for converting the N-trailer system into Goursat form will also convert the associated nonholonomic control system into chained form. These transformations, along with the input transformations which they define, will be discussed in this section. It will also be verified that the resulting transformations are diffeomorphisms.

From looking at the chained form equations (3.24), it can be seen that the functions $z_1(t)$ and $z_{n+3}(t)$ completely define all the state variables and the inputs through the equations:

$$\begin{aligned} u_1 &= \dot{z}_1 \\ z_i &= \dot{z}_{i+1}/u_1 \qquad i = n+2, \ldots, 2 \\ u_2 &= \dot{z}_2. \end{aligned} \tag{3.26}$$

Consequently, a coordinate transformation into chained form is completely defined by the first and last coordinates of the chain, z_1 and z_{n+3}, as functions of the original coordinates x along with equation (3.26). (The fact that such a transform exists follows from the fact that the Goursat congruences have been verified.) One possible choice for these two coordinates z_1 and z_{n+3} is functions f_1 and f_2 from the solution to Pfaff's problem. If the Goursat congruences have not been verified, then it should be checked that the transformation which results from equation (3.26) is a diffeomorphism.

Remark 2 (Differential Flatness) Fliess and his coworkers [16, 41] define a control system to be differentially flat if there exists a set of functions $(h_1, \ldots, h_m)$ such that

1. The h_i's are differentially independent (not related by any differential equations).
2. The h_i's are functions of the system variables (states, inputs) and finitely many of their derivatives.
3. Any system variable is a differential function of the h_i's, that is, a function of the h_i's and a finite number of their derivatives.

For any differentially flat system, there exist many different choices of flat outputs. There is no constructive method for finding such outputs.

It has been pointed out by Martin [34] and Murray [40] that a two-input drift-free system is differentially flat if and only if it can be converted into Goursat or chained form. One possible set of flat outputs for such a system is $h_1 = z_1$ and $h_2 = z_{n+3}$. The possibilities for systems with more than two inputs are more complicated, and will be explored in the following chapters.

For a system which satisfies the Goursat conditions, there are many transformations into chained form; two are presented here for the N-trailer system. These are the same as those discussed in the previous subsection in the context of the Goursat normal form, the main different between the two treatments being the definition of the input transformation when converting into chained form.

Transformation 1. Coordinates of the N^{th} trailer. This transformation is defined as follows:

$$\begin{aligned} z_1 &= x^n \\ z_{n+3} &= y^n. \end{aligned}$$

The corresponding input transformation is given by:

$$\bar{u}_1 = \dot{z}_1 = \cos\theta^n \; v_n = \cos(\theta^0 - \theta^1)\cos(\theta^1 - \theta^2)\cdots\cos(\theta^{n-1} - \theta^n)v_0.$$

The other input $\bar{u}_2 = \dot{z}_2$ is a quite complicated function of x, v_0, ω for the general case with n trailers. However, it is easily verified that

$$\frac{\partial \bar{u}_2}{\partial \omega} \neq 0,$$

implying that the input transformation $\bar{u} = b(x)u$ is nonsingular. The remaining coordinates $z = f(x)$ are defined using equation (3.26).

The Jacobian of this coordinate transformation has the following form,

$$[\frac{\partial \mathbf{z}}{\partial \mathbf{x}}] = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & * & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & * & & * \end{array}\right]$$

where the coordinates are written in the order:

$$\begin{aligned} \mathbf{x} &= (x^n, y^n, \theta^n, \theta^{n-1}, \dots, \theta^0) \\ \mathbf{z} &= (z_1, z_{n+3}, z_{n+2}, \dots, z_2) \end{aligned} \qquad (3.27)$$

and $*$ represents any nonzero function. The ordering of the z coordinates was chosen to put the Jacobian matrix in a lower-triangular form, thereby highlighting its nonsingularity. That the Jacobian is nonsingular implies that the map $f : \mathbf{x} \to \mathbf{z}$ is a local diffeomorphism and thus a valid coordinate transformation.

It should be noted that this coordinate transformation is only defined locally. Since its definition requires a division by u_1, if any of the factors in u_1 are zero, the transformation is undefined for that particular configuration. For example, if $\theta^n = \pi/2$, corresponding to the last trailer being at right-angles with the coordinate frame, this coordinate transformation is no longer valid. In addition, if the i^{th} trailer is *jack-knifed*, that is to say, for some $1 \le i \le n$, $\theta^i = \theta^{i-1} \pm \pi/2$, the coordinate transformation is also singular. The nonsingular set of the coordinate transformation is large enough so that many practical path-planning problems can be solved using this transformation, as will be shown in Section 3.5

Transformation 2. Coordinates of the origin as seen from the last trailer. The other coordinate transformation discussed in this dissertation also has some singularities but will allow the trailer to be at any orientation with respect to the coordinate frame.

$$\begin{aligned} z_1 &= x^n \cos\theta^n + y^n \sin\theta^n \\ z_{n+3} &= x^n \sin\theta^n - y^n \cos\theta^n - \theta^n z_1. \end{aligned}$$

The input transformation and the rest of the coordinates follow from Equation (3.26). It can be verified that the input transformation has the form:

$$\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = \begin{bmatrix} b_{1,1}(x) & 0 \\ b_{2,1}(x) & b_{2,2}(x) \end{bmatrix} \begin{pmatrix} v_0 \\ \omega \end{pmatrix}$$

with $b_{1,1}$ and $b_{2,2}$ nonzero functions of x. This implies that the input transformation is nonsingular.

This transformation has a Jacobian given by the following matrix:

$$[\frac{\partial \mathbf{z}}{\partial \mathbf{x}}] = \left[\begin{array}{ccc|cc} \cos\theta^n & \sin\theta^n & * & 0 & 0 \\ \sin\theta^n & -\cos\theta^n & * & & \ddots \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & * & * & 0 \\ \vdots & \vdots & \vdots & & \ddots \\ 0 & 0 & * & * & * \end{array}\right]$$

where the coordinates are written in the order (3.27) and $*$ represents any nonzero function. Again, since the Jacobian is nonsingular, the map $f : \mathbf{x} \to \mathbf{z}$ is a local diffeomorphism. Singularities in this transformation also occur when division by u_1 is undefined. This happens when the expression

$$L_n + (y^n \cos\theta^n - x^n \sin\theta^n)\tan(\theta^n - \theta^{n-1}) = 0,$$

and also when any of the trailers is jack-knifed.

3.2.5 Other N-trailer configurations

Thus far, only the example of the Hilare mobile robot pulling a chain of trailers has been considered. In this section it is shown that this model can also be used not only for the more familiar system of a front-wheel drive car pulling trailers, but also for the luggage trains commonly found in airports.

The model of the front-wheel drive car is shown in Figure 3.2.5. In comparison with the Hilare model, another axle has been added to the front body of the chain, and a new variable ϕ represents the angle of the front wheels with respect to the car. The length of the wheelbase of the lead car is defined to be L_0.

The equivalence between the two models is most easily seen by looking at the form constraints. Each constraint corresponds to one axle rolling without slipping. Hilare with n trailers has $n+1$ axles; the car with n trailers has $n+2$ axles, and its Pfaffian system is therefore equivalent to that of Hilare pulling $n+1$ trailers.

Of course, the states and inputs for the car system are slightly different. By convention, the angle of the front axle is defined relative to the car instead of relative to the coordinate

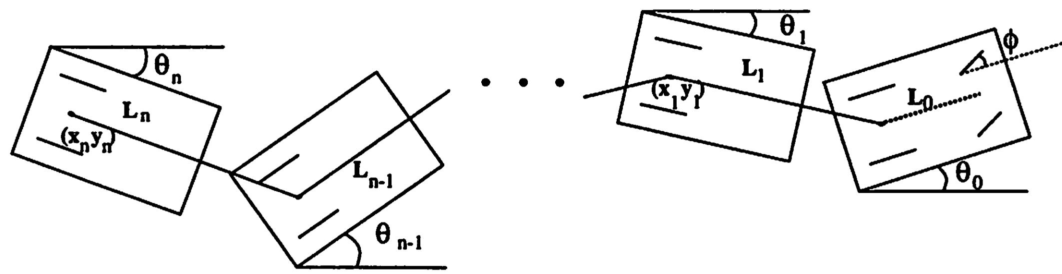


Figure 3.2: The front-wheel drive car with n trailers. This model is similar to that of Hilare with trailers (see Figure 3.1) with an extra axle added to the first body in the chain.

frame. This angle ϕ is merely $\theta^0 - \theta^1$ on the Hilare system. The velocity input is the same, assumed to be the linear velocity of the first body (it can be defined at either the front or rear axle depending on whether the car is front-wheel drive or rear-wheel drive), but the rotational input is usually taken as $\omega' = \dot{\phi}$ the steering wheel velocity. Since in the Hilare case, the velocity of the first body $\omega = \dot{\theta}^0$ is controlled, state feedback can be used to reconcile these differences. As mentioned in the proof of Proposition 13, there are many choices of vector fields orthogonal to a given Pfaffian system with each choice having a different (possibly physical) meaning.

The luggage carts used at most airports are also equivalent to the Hilare model. Each trailer on the luggage cart train has two sets of wheels; the front axle can spin freely about its center but the back axle is constrained to be aligned with the trailer (see Figure 3.2.5). Here the angles of the front wheels are defined with respect to each trailer, but looking at the form constraints it is easily seen that the cab with n luggage trailers is equivalent to a front-wheel drive car with $2n$ one-axle trailers. Again, a coordinate transformation is needed, since in the model of the luggage carts the angle of the front wheels of the trailers is defined relative to the trailer instead of relative to the coordinate frame.

3.3 Kingpin hitches

In all of the previous discussion, it was assumed that the hitch between adjacent axles was at the axle, as is commonly the case for tractor semi-trailer combinations. Most passenger cars, however, have a trailer hitch which is some distance behind the rear axle. Also, there is a mining vehicle in use in Québec which has an offset or "kingpin" hitch [19, 21]; this vehicle is sketched in Figure 3.4.

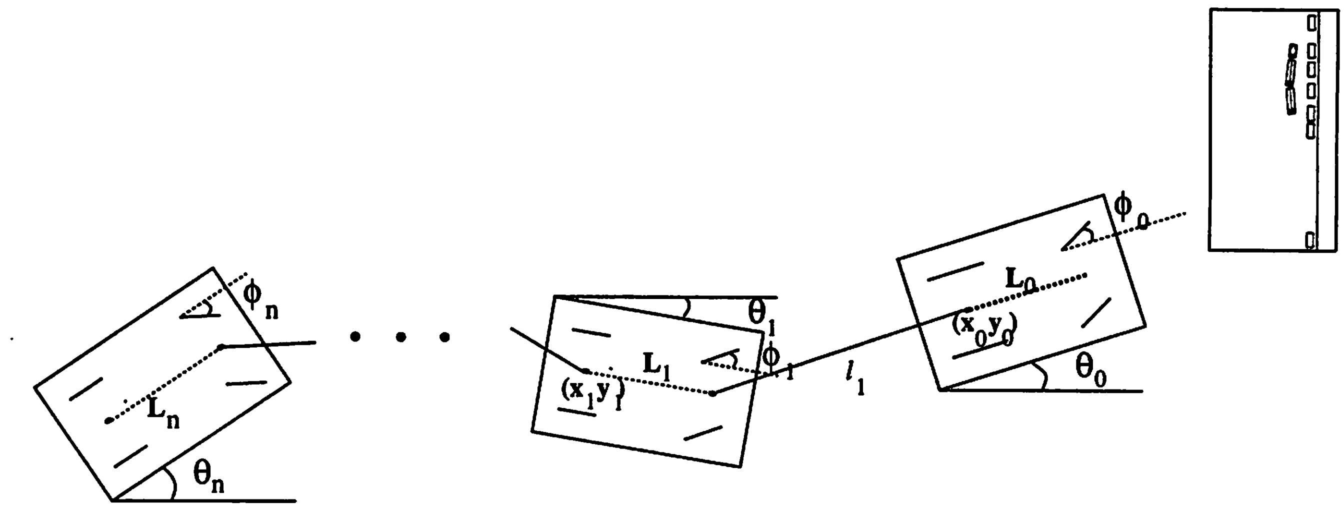


Figure 3.3: A car pulling n luggage carts. Each trailer has two axles; the front axle is free to spin about its midpoint but the rear axle is constrained to be aligned with the body of the trailer.

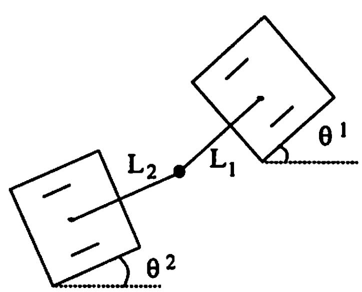


Figure 3.4: A two-axle system with kingpin hitch.

3.3.1 The 2-axle mining vehicle

There are two velocity constraints on the system, representing that the two axles must roll without slipping. These constraints can be written in coordinates as

$$\omega^i = \sin\theta^i dx^i - \cos\theta^i dy^i \qquad i = 1,2 \tag{3.28}$$

where (x^i, y^i, θ^i) represents the Cartesian position and angle of the i^{th} axle for $i = 1,2$. The variables (x, y) will be used to represent the position of the kingpin hitch. The relationship between the Cartesian positions of the two axles and the position of the hitch is given by the connection relations,

$$\begin{aligned} x^1 &= x + L_1\cos\theta^1 \qquad & x^2 &= x - L_2\cos\theta^2 \\ y^1 &= y + L_1\sin\theta^1 \qquad & y^2 &= y - L_2\sin\theta^2 \end{aligned}$$

Neither ω^1 nor ω^2 satisfies the condition that $d\omega^i \equiv 0 \mod \omega^1, \omega^2$. However, there does exist a linear combination of the two constraints,

$$\omega = L_1\omega^1 + L_2\omega^2$$

which has the property that

$$d\omega \equiv 0 \mod \omega^1, \omega^2$$

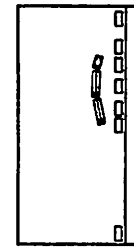
Also, ω is not integrable, that is $d\omega \not\equiv 0 \mod \omega$, and thus the system is controllable by Chow's theorem. The derived flag has the form

$$\begin{aligned} I^{(0)} &= \{\omega^1, \omega^2\} \\ I^{(1)} &= \{\omega\} \\ I^{(2)} &= \{0\} \end{aligned}$$

Note that the original basis of constraints which describes the system is not adapted to the derived flag.

Without finding a one-form π for this problem which will satisfy the conditions of Theorem 8, it can be noted that this system satisfies the conditions of Engel's problem (Theorem 7), and thus can be converted into Goursat form. The proof of Engel's theorem can be used to find the coordinates for conversion. The one-form ω which is in $I^{(1)}$ satisfies the conditions of Pfaff's problem by dimension count. That is, there exists a function f_1 which satisfies:

$$d\omega \wedge \omega \wedge df_1 = 0$$



In coordinates, ω has the form

$$\omega = (L_1 \sin\theta^1 + L_2 \sin\theta^2)dx - (L_1\cos\theta^1 + L_2\cos\theta^2)dy - {L_1}^2 d\theta^1 + {L_2}^2 d\theta^2$$

The calculations for the case when the lengths of the two hitches are equal will be done first. By choosing units appropriately, assume that $L_1 = L_2 = 1$. Then, ω has the form

$$\begin{aligned}\omega &= (\sin\theta^1 + \sin\theta^2)dx - (\cos\theta^1 + \cos\theta^2)dy - d\theta^1 + d\theta^2 \\ &= 2\sin(\frac{\theta^1+\theta^2}{2})\cos(\frac{\theta^1-\theta^2}{2})dx - 2\cos(\frac{\theta^1+\theta^2}{2})\cos(\frac{\theta^1-\theta^2}{2})dy - d\theta^1 + d\theta^2\end{aligned} \tag{3.29}$$

After a change of coordinates given by:

$$\alpha = \frac{\theta^1+\theta^2}{2} \qquad \beta = \frac{\theta^1-\theta^2}{2}$$

ω has the expression

$$\omega = 2\cos\beta(\sin\alpha\; dx - \cos\alpha\; dy - \sec\beta\; d\beta)$$

It can be checked that $d\omega\wedge\omega\wedge d\alpha = 0$, or equivalently $f_1 = \alpha$ will satisfy the Pfaff equation (3.1).

The other coordinates can be found from solving the second of the two Pfaff equations, or from the expression for ω in the new coordinates. It is helpful to perform another coordinate change given by

$$\begin{aligned}X &= x\cos\alpha + y\sin\alpha \\ Y &= x\sin\alpha - y\cos\alpha\end{aligned}$$

so that the one-form ω can be written as:

$$\omega = 2\cos\beta(dY - X\,d\alpha - \sec\beta d\beta)$$

Since ω is only defined up to a scale factor, the function $2\cos\beta$ can be ignored (that is, $\tilde{\omega} = \omega/(2\cos\beta)$ is also a basis for $I^{(1)}$). From the expression

$$\begin{aligned}\tilde{\omega} &= dY - \sec\beta d\beta - X\,d\alpha \\ &= dz^4 - z^3 dz^1\end{aligned}$$

the Goursat coordinates can be read off simply as

$$\begin{aligned}z^1 &= \alpha \\ z^4 &= Y - \int \sec\beta d\beta = Y - \log|\sec\beta + \tan\beta| \\ z^3 &= X\end{aligned}$$

The final coordinate z^2 is most easily found by differentiating and dividing according to equation (3.26),

$$z^2 = \dot{z}^3/\dot{z}^1 = \dot{X}/\dot{\alpha}$$

and the entire coordinate transformation is now defined.

Consider next the case where $L_1 \neq L_2$. The expression for ω does not simplify as was shown in (3.29); it is given by the complete expression

$$\omega = (L_1 \sin\theta^1 + L_2 \sin\theta^2)dx - (L_1 \cos\theta^1 + L_2 \cos\theta^2)dy - {L_1}^2 d\theta^1 + {L_2}^2 d\theta^2$$

Pfaff's problem still applies, and it can be checked that the function γ defined by

$$\tan\gamma = \frac{L_1 \sin\theta^1 + L_2 \sin\theta^2}{L_1 \cos\theta^1 + L_2 \cos\theta^2}$$

will satisfy the first Pfaff equation of (3.1), that is,

$$d\omega \wedge \omega \wedge d\gamma = 0$$

Note that if $L_1 = L_2$, then $\gamma = \alpha$ as defined above. Another function φ can be defined as:

$$\varphi = \theta^1 - \theta^2$$

It can be checked that φ is independent of γ (in fact, $d\gamma \wedge d\varphi = d\theta^1 \wedge d\theta^2$). Following the procedure detailed above, a coordinate transformation

$$X = x\cos\gamma + y\sin\gamma$$
$$Y = x\sin\gamma - y\cos\gamma$$

is performed, and in these coordinates ω has the expression

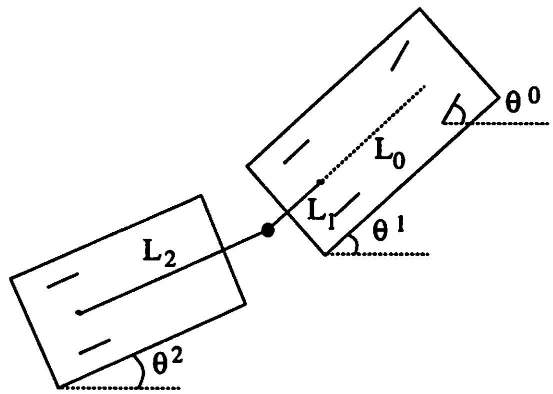
$$\omega = \ell dY - \ell X d\gamma - ({L_1}^2 - {L_2}^2)d\gamma - \frac{1}{\ell^2}(2{L_1}^2{L_2}^2 + {L_1}^3 L_2 \cos\varphi + L_1 {L_2}^3 \cos\varphi)d\varphi$$

with the function ℓ defined by

$$\begin{aligned}\ell^2 &= (L_1 \sin\theta^1 + L_2 \sin\theta^2)^2 + (L_1 \cos\theta^1 + L_2 \cos\theta^2)^2 \\ &= {L_1}^2 + {L_2}^2 + 2L_1 L_2 \cos\varphi\end{aligned}$$

After scaling by ℓ, the one-form ω has the expression

$$\begin{aligned}\frac{1}{\ell}\ \omega &= \left(dY - L_1 L_2 \frac{2L_1L_2 + ({L_1}^2 + {L_2}^2)\cos\varphi}{({L_1}^2 + {L_2}^2 + 2L_1L_2\cos\varphi)^{3/2}} d\varphi\right) - (X + \frac{{L_1}^2 - {L_2}^2}{\ell})d\gamma \\ &= dz_4 - z_3 dz_1\end{aligned}$$



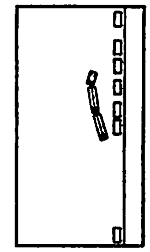


Figure 3.5: A car pulling a trailer.

The expression within the large parentheses is integrable, and so the Goursat coordinates can be read off as:

$$
\begin{aligned}
z^1 &= \gamma \\
z^4 &= Y - \frac{2L_1L_2\sin\varphi}{\sqrt{{L_1}^2+{L_2}^2+2L_1L_2\cos\varphi}} + L_1L_2\int\frac{\cos\varphi}{\sqrt{{L_1}^2+{L_2}^2+2L_1L_2\cos\varphi}}\,d\varphi \\
z^3 &= X + \frac{{L_1}^2-{L_2}^2}{\ell}
\end{aligned}
\qquad (3.30)
$$

Note the elliptic integral that has appeared. The final coordinate z^2 is most easily found using the definition $z^2 = \dot{z}^3/\dot{z}^1$.

3.3.2 A car pulling one offset trailer

Now consider the car and trailer example of Figure 3.5. Although this system has three axles, it bears many similarities to the two-axle mining system. The third constraint (corresponding to the front axle) can be expressed in coordinates as

$$\omega^0 = \sin\theta^0 dx^0 - \cos\theta^0 dy^0$$

where the (x, y) position of the front axle is given by

$$x^0 = x^1 + L_0\cos\theta^1 \qquad y^0 = y^1 + L_0\sin\theta^1$$

It can be checked that the derived flag has the form:

$$\begin{aligned} I &= \{\omega^0, \omega^1, \omega^2\} \\ I^{(1)} &= \{\omega^1, \omega^2\} \\ I^{(2)} &= \{\omega\} \\ I^{(3)} &= \{0\} \end{aligned}$$

for the same $\omega = L_1\omega^1 + L_2\omega^2$ as before. Thus, the Pfaff equations that need to be solved for this system are the same as those for the two-axle mining system. The Goursat coordinates are given as (for the case $L_1 = L_2 = 1$):

$$\begin{aligned} z^1 &= \frac{\theta^1 + \theta^2}{2} \\ z^5 &= x \sin(\frac{\theta^1 + \theta^2}{2}) - y \cos(\frac{\theta^1 + \theta^2}{2}) - \log|\sec(\frac{\theta^1 - \theta^2}{2}) + \tan(\frac{\theta^1 - \theta^2}{2})| \\ z^4 &= x \cos(\frac{\theta^1 + \theta^2}{2}) + y \sin(\frac{\theta^1 + \theta^2}{2}) \end{aligned}$$

and similarly to (3.30) for the general case. The other two coordinate are most easily found by differentiating and dividing according to equation (3.26),

$$\begin{aligned} z^3 &= \dot{z}^4 / \dot{z}^1 \\ z^2 &= \dot{z}^3 / \dot{z}^1 \end{aligned}$$

after which the entire coordinate transformation is defined.

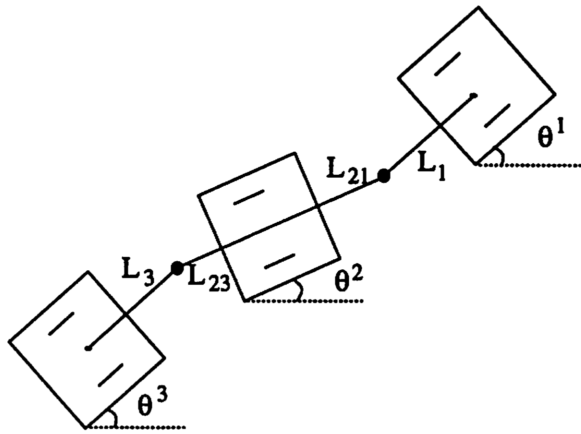
Remark 3 It has been pointed out by Rouchon et al. [42] that the system of Figure 3.5 is differentially flat. They give as a set of flat outputs the two functions:[3]

$$\begin{aligned} out_1 &= x + L_1 \cos\theta^1 - L_2 \cos\theta^2 + f(\theta^1 - \theta^2) \frac{L_1 \sin\theta^1 + L_2 \sin\theta^2}{\sqrt{L_1{}^2 + L_2{}^2 + 2L_1L_2\cos(\theta^1 - \theta^2)}} \\ out_2 &= y + L_1 \sin\theta^1 - L_2 \sin\theta^2 + f(\theta^1 - \theta^2) \frac{L_1 \cos\theta^1 + L_2 \cos\theta^2}{\sqrt{L_1{}^2 + L_2{}^2 + 2L_1L_2\cos(\theta^1 - \theta^2)}} \end{aligned}$$

with the function f defined as

$$f(\theta^1 - \theta^2) = L_1L_2 \int_{\pi}^{\pi + \theta^1 - \theta^2} \frac{\cos\sigma}{\sqrt{L_1{}^2 + L_2{}^2 + 2L_1L_2\cos\sigma}} d\sigma$$

[3]It is perhaps worth noting here that in the angle β which they define as the angle of the trailer with respect to the horizontal axis is equal to $\theta^2 + \pi$ according to the notation used in this dissertation.



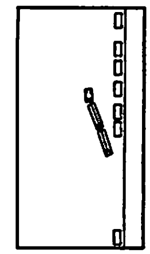


Figure 3.6: A three-axle system with two kingpin hitches.

The calculations require a good deal of organization, but it can be checked analytically that both out_1 and out_2 satisfy the Pfaff equations 3.1, that is

$$d\omega \wedge \omega \wedge d(out_i) = 0$$

and thus will give another possible set of Goursat coordinates for the kingpin hitched system,

$$\begin{aligned} z_1 &= out_1 \\ z_5 &= out_2 \\ z_i &= \dot{z}_{i+1}/\dot{z}_1 \quad i = 4,3,2 \end{aligned}$$

3.3.3 Three axles with two offset hitches

If there are three axles connected by two kingpin hitches as shown in Figure 3.6, the situation is substantially different. The constraints that the three axles roll without slipping can be written in coordinates as

$$\omega^i = \sin\theta^i dx^i - \cos\theta^i dy^i$$

for $i = 1,2,3$. The relationship between the (x,y) coordinates is

$$\begin{aligned} x^1 &= x^2 - L_{21}\cos\theta^2 - L_1\cos\theta^1 \qquad & y^1 &= y^2 - L_{21}\sin\theta^2 - L_1\sin\theta^1 \\ x^3 &= x^2 + L_{23}\cos\theta^2 + L_3\cos\theta^3 \qquad & y^3 &= y^2 + L_{23}\sin\theta^2 + L_3\sin\theta^3 \end{aligned}$$

The derived flag can be shown to have the form

$$I^{(0)} = \{\omega^1, \omega^2, \omega^3\}$$
$$I^{(1)} = \{\alpha^1, \alpha^3\}$$
$$I^{(2)} = \{0\}$$

where the constraints in $I^{(1)}$ have the expression:

$$\alpha^1 = \omega^1 + \frac{L_{21}}{L_1}\omega^2 \qquad \alpha^3 = \omega^3 + \frac{L_{23}}{L_1}\omega^2$$

and not only is the basis of constraints not adapted to the derived flag, but the dimension count for converting into Goursat normal form is no longer satisfied. In fact, this is an example of Cartan's famous five-variable problem, the problem of three constraints in five variables [9]. The path planning problem for such a system is unsolved.

3.4 Steering two-input chained form systems

The problem that is addressed in this section is: given a 2-input, n-state system in chained form, with an initial state z^0 and a goal state z^f, find some control inputs $u_1(t), u_2(t)$ which will steer the system from z^0 to z^f after some time T. Three methods will be presented for steering chained form systems, using sinusoidal, piecewise constant, and polynomial inputs.

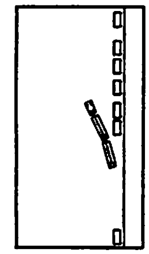
3.4.1 Sinusoidal inputs

The first steering method considered in this dissertation uses sinusoidal inputs. Steering chained form systems with sinusoids was originally proposed in [37]. The method described here is different from the original algorithm in that it steers all the states in one step, instead of one state at a time.

Given an n-state chained form system, it is easily seen that the first two states, z_1 and z_2, can be steered from their initial to their final positions using constant inputs over any time period T. Of course, the states $z_3, \ldots, z_n$ will drift as a consequence of this.

By direct integration, it may be verified that a combination of out of phase sinusoids applied to the inputs,

$$u_1(t) = \alpha \sin \omega t \qquad u_2(t) = \beta \cos \omega t$$



over one period $T = 2\pi/\omega$, will cause a motion in the z_3 variable as follows:

$$\begin{aligned} z_1(T) &= z_1(0) \\ z_2(T) &= z_2(0) \\ z_3(T) &= z_3(0) + \frac{\alpha\beta}{2\omega}. \end{aligned}$$

The states $z_4, \ldots, z_n$ will drift in some fashion. Further, using inputs with u_2 having k times the frequency of u_1, namely:

$$u_1(t) = \alpha \sin \omega t \qquad u_2(t) = \beta \cos k\omega t$$

applied over one period $T = 2\pi/\omega$, will result (as may be verified directly by integration) in the following expressions for $z(T)$:

$$\begin{aligned} z_1(T) &= z_1(0) \\ &\vdots \\ z_{k+1}(T) &= z_{k+1}(0) \\ z_{k+2}(T) &= z_{k+2}(0) + \frac{\alpha^k \beta}{k!(2\omega)^k}. \end{aligned}$$

The intuition behind this steering scheme lies in the different levels of Lie brackets. The Lie bracket between the two input vector fields g_1, g_2 of a chained form system has the expression $[g_1, g_2] = [0\ 0\ 1\ 0 \cdots 0]^T$, and corresponds to the z_3 direction. Motion in this first level Lie bracket is generated by cycling between the two input vector fields in a continuous manner described by the out of phase sinusoids. To get motion in the second level Lie bracket, $[g_1, [g_1, g_2]] = [0\ 0\ 0\ 1\ 0 \cdots 0]^T$ or equivalently the z_4 direction, the input u_2 completes two cycles for one cycle on u_1. More generally, motion in the $\mathrm{ad}_{g_1}^k g_2 = [0 \cdots 1 \cdots 0]^T$ or the z_{k+2} direction is achieved by using k times the frequency of u_1 on u_2.

The steering algorithm of [37] is step-by-step: It first steers z_1, z_2 to their final position using constant inputs, disregarding the other states. Then it steers z_3 to its desired final position using sinusoids, z_1, z_2 will return to their final values. Now z_4 can be steered, and similarly on down the chain, until all states are at their final positions. This is a simple algorithm that is easy to implement, but can be time-consuming when there are many states to be steered.

To alleviate the tedium of steering the states one at a time, an "all-at-once" sinusoids method has been proposed in [55], combining all the frequencies on u_2 together in one step,

$$\begin{aligned} u_1 &= a_0 + a_1 \sin \omega t \\ u_2 &= b_0 + b_1 \cos \omega t + b_2 \cos 2\omega t + \cdots + b_{n-2} \cos(n-2)\omega t. \end{aligned} \tag{3.31}$$

It is no longer as simple to choose appropriate values for the parameters $(a_0, a_1, b_0, \ldots, b_{n-2})$ because of the drift that could be ignored when each state was considered individually. However, it is still possible to integrate the chained form equations sequentially, finding $z_1(t), z_2(t), z_3(t), \ldots, z_n(t)$ which result from the inputs (3.31) above. The state $z(t)$ is a function of the initial condition z^0 as well as the input parameters $a_0, a_1, b_0, \ldots, b_{n-2}$. If the state $z(t)$ is evaluated at a time T corresponding to one period on the first input, $T = 2\pi/\omega$, all the sinusoidal functions will be either 0 or 1. A set of n polynomial equations in the $(n+1)$ input parameters $(a_0, a_1, b_0, \ldots, b_{n-2})$ are obtained from setting $z(T) = z^f$. The following proposition guarantees the existence of solutions to these equations at least locally around z^0.

Proposition 14 *Consider an n-state chained form system with initial and final states z^0, z^f. If $|z^0 - z^f| < \delta$ small, then there exist input parameters $(a_0,\ a_1,\ b_0,\ \ldots,\ b_{n-2})$ such that the inputs*

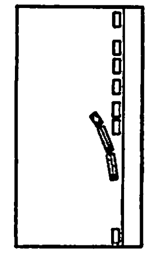
$$\begin{aligned} u_1 &= a_0 + a_1 \sin \omega t \\ u_2 &= b_0 + b_1 \cos \omega t + b_2 \cos 2\omega t + \cdots + b_{n-2} \cos(n-2)\omega t \end{aligned}$$

will steer the system from z^0 to z^f in time $T = 2\pi/\omega$.

Proof. Consider the map

$$\phi_{z^0} : \mathbb{R}^n \to \mathbb{R}^n$$

which takes values in the parameter space $(a_0, b_0, \ldots, b_{n-2})$ and maps them to values in the state space $(z_1^f, \ldots, z_n^f)$. Define $\phi_{z^0}(a_0, b_0, \ldots, b_{n-2})$ to be the value of $z(T)$ when the chained form system (3.24) is integrated starting at the initial condition z^0 and applying the inputs (3.31) over the time period $[0, T]$. Choose $a_1 \neq 0$. It can be shown that ϕ_{z^0} is a local diffeomorphism (about the origin) by demonstrating that the Jacobian of ϕ_{z^0} is nonsingular.



Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$ and let ϵ be small. Set $a_1 \neq 0$. Now consider the input parameterized by ϵe_1,

$$u_1 = \epsilon + a_1 \sin \omega t \qquad u_2 = 0.$$

Integrating the chained form equations and evaluating it at time T will give

$$\phi_{z^0}(\epsilon e_1) = z^0 + [\epsilon T \;\; 0 \;\; O(\epsilon) \cdots O(\epsilon)]^T$$

where $O(\epsilon)$ represents terms that are of linear and higher order in ϵ.

Now consider the input parameterized by ϵe_2

$$u_1 = a_1 \sin \omega t \qquad u_2 = \epsilon.$$

Integrate and evaluate at T as before,

$$\phi_{z^0}(\epsilon e_2) = z^0 + [0 \;\; \epsilon T \;\; O(\epsilon) \cdots O(\epsilon)]^T.$$

In this case it may be verified that $O(\epsilon)$ terms are linear in ϵ. In general, for an input parameterized by ϵe_k,

$$u_1 = a_1 \sin \omega t \qquad u_2 = \epsilon \cos(k-2)\omega t,$$

the directional derivative of ϕ in the e_k direction is given by the limit of its flow divided by ϵ as ϵ goes to zero. The flow is of the form:

$$\phi_{z^0}(\epsilon e_k) = z^0 + [0 \cdots 0 \;\; p(\epsilon) \;\; O(\epsilon) \cdots O(\epsilon)]^T,$$

where the term $p(\epsilon)$ in the k^{th} position is given by

$$p(\epsilon) = \frac{a_1^{k-2}\epsilon}{(k-2)!(2\omega)^{k-2}}.$$

The n directional derivatives are linearly independent; implying that the Jacobian of ϕ_{z^0} is nonsingular (indeed triangular) at the origin, and thus ϕ_{z^0} is a local diffeomorphism about this point. □

Remark 4 The overparameterization of the input ($n+1$ parameters: $a_0, a_1, b_0, \ldots, b_{n-2}$ and n states) has been dealt with in this case by initially choosing a value for a_1 and then solving the n equations for the remaining n input parameters.

By choosing a fixed value for a_1, the input u_1 must go through one period. Since u_1 roughly corresponds to the driving input in a mobile robot system, paths planned using the

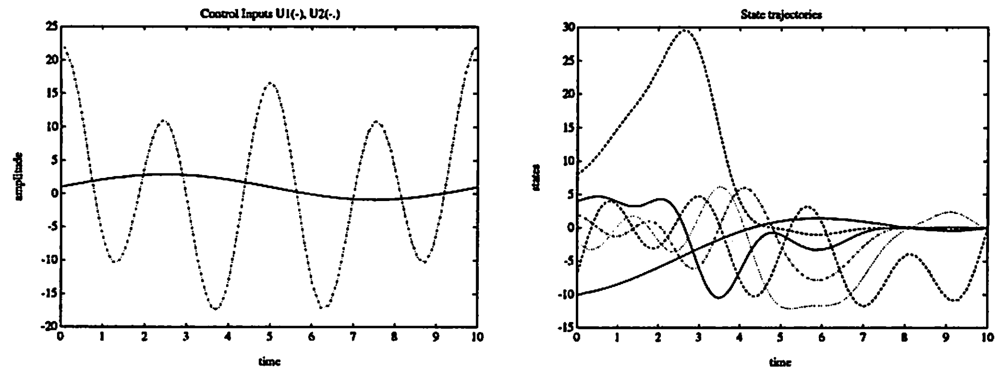


Figure 3.7: The inputs and state trajectories for a six-state, chained form system, steering from $(-10, -7, -2, 2, 4, 8)$ to the origin. The input u_1 is sinusoidal of one period; u_2 is a sum of sinusoids, of which the highest frequency is 4ω.

sinusoidal method generally have one back-up or speed reversal, corresponding to the zero-crossing of u_1. Parallel-parking type maneuvers seem particularly well-suited to sinusoidal trajectories.

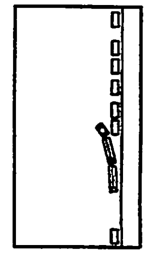
A sample of the input functions and state trajectories for a sinusoidal steering problem is shown in Figure 3.7. There are six states, in chained form, steering from an initial position of $(z_1, z_2, z_3, z_4, z_5, z_6) = (-10, -7, -2, 2, 4, 8)$ to the origin. The parameters were chosen to be $T = 10$ seconds and $a_1 = \frac{6\pi}{T}$.

3.4.2 Piecewise constant inputs

The second method described in this dissertation for steering chained form systems uses piecewise constant inputs. This method was originally proposed by Monaco and Normand-Cyrot [32], and was inspired by multirate digital control. It is most easily understood in the context of nonholonomic motion planning simply as piecewise constant inputs.

Consider holding the inputs u_1 and u_2 constant over some small time period $[0, \bar{\delta})$,

$$\begin{aligned} u_1(\tau) &= u_{1,1} \\ u_2(\tau) &= u_{2,1}. \end{aligned} \qquad \tau \in [0, \bar{\delta})$$



The chained form state equations can then be integrated, and evaluated at time $\bar{\delta}$ to yield

$$\begin{aligned}
z_1(\bar{\delta}) &= z_1(0) + u_{1,1}\bar{\delta} \\
z_2(\bar{\delta}) &= z_2(0) + u_{2,1}\bar{\delta} \\
z_3(\bar{\delta}) &= z_3(0) + z_2(0)u_{1,1}\bar{\delta} + u_{1,1}u_{2,1}\frac{\bar{\delta}^2}{2} \\
&\vdots \\
z_n(\bar{\delta}) &= z_n(0) + z_{n-1}(0)u_{1,1}\bar{\delta} + \cdots + u_{2,1}u_{1,1}^{n-2}\frac{\bar{\delta}^{n-1}}{(n-1)!}.
\end{aligned} \tag{3.32}$$

Now consider another pair of constant inputs on the time interval $[\bar{\delta}, 2\bar{\delta})$,

$$\begin{aligned}
u_1(\tau) &= u_{1,2} \\
u_2(\tau) &= u_{2,2}.
\end{aligned} \qquad \tau \in [\bar{\delta}, 2\bar{\delta})$$

Integration of the state equations gives $z(2\bar{\delta})$ as a function of $u_{1,2}, u_{2,2}, z(\bar{\delta})$. Using $z(\bar{\delta})$ from equation (3.32), an expression for $z(2\bar{\delta})$ in terms of $u_{1,1}$, $u_{1,2}$, $u_{2,1}$, $u_{2,2}$, $z(0)$ is obtained. This procedure of piecewise integration and substitution can be repeated as many times as necessary.

For path planning, u_1 is kept at a constant value over the entire trajectory. The equations (3.32) must then be integrated $n-1$ times so as to have exactly n parameters for which to solve: $u_1, u_{2,1}, \ldots, u_{2,n-1}$. The total time needed for steering is $\delta = (n-1)\bar{\delta}$. Although δ can be chosen arbitrarily, a smaller time δ will result in larger inputs u to achieve the same path.

The n equations which result from setting $z(0) = z^0$ and $z(\delta) = z^f$ are polynomial (of order $n-2$) in u_1 but are linear in $u_{2,1}, \ldots, u_{2,n-1}$. Since u_1 is easily determined from

$$u_1 = \frac{z_1^f - z_1^0}{\delta}$$

the remaining $n-1$ linear equations can be solved for u_2 quite easily. This is one of the reasons that u_1 is kept constant over the entire trajectory; if u_1 varied, high-order polynomial equations in the $u_{1,k}$ parameters would need to be solved to find the path.

It should be noted that if $z_1^f = z_1^0$, i.e. the initial and final states agree in the first coordinate, this method as stated so far will fail to yield a solution. From looking at the chained form equations, it is obvious that if $u_1 = 0$, only the second state z_2 can move; all other states must remain stationary. In practice, this case is dealt with by planning two

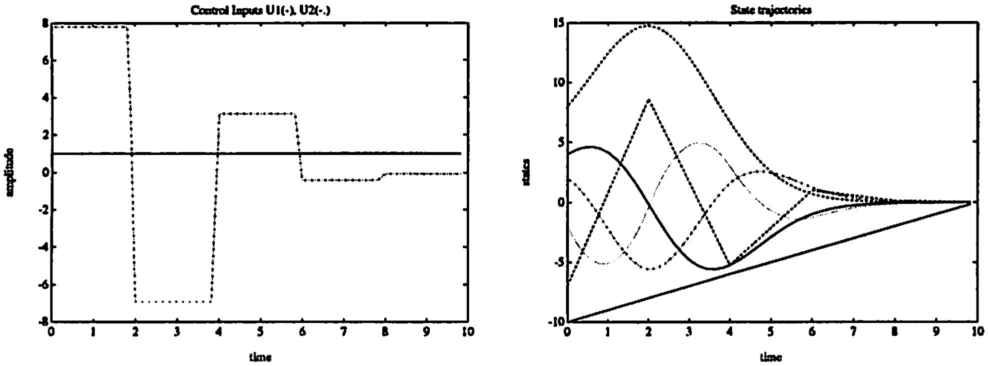


Figure 3.8: Sample inputs and state trajectories for steering a six-state chained form system with piecewise constant inputs. The initial position is $(-10, -7, -2, 2, 4, 8)$ and the goal point is the origin.

paths, the first of which takes the initial condition to an intermediate state, the second of which joins the intermediate state with the goal position. The concatenation of these two paths is a valid trajectory between the start and goal. The intermediate point z^m is chosen to be halfway between the initial and final points in all coordinates except the first, which is chosen to be offset from the starting position by a constant amount,

$$z_k^m = (z^f - z_k^0)/2, \quad k = 2, \ldots, n$$
$$z_1^m = z_1^0 + \text{const.}$$

The magnitude of the constant offset can be adjusted to fit the situation.

The procedure detailed in the previous paragraph is used when a parallel-parking trajectory is desired for the mobile robot with trailers, since the z^1 direction in chained form corresponds to "sideways" in the original coordinates. It is practical to choose the constant offset at approximately twice the length of the entire robot and trailer system. A smaller offset will result in tighter turns and more lateral motion. If there are obstacles in the field, this constant offset gives a parameter that can be adjusted in an effort to avoid collisions.

Another reason for choosing u_1 to be constant over the entire trajectory is that in the mobile robot and trailer system, this input is roughly equivalent to the driving velocity. Because of the coordinate transformation that maps u_1 to the actual velocity v_0, the actual velocity of the robot will not be constant, but in most cases it will not cross zero and change sign. This means that the robot will not have to execute backing-up maneuvers to achieve its final goal position.

The main drawback of the piecewise constant inputs is the discontinuity of u_2. The models used in this dissertation are purely kinematic using as inputs the driving and steering velocities. In a real robot system, the inputs are not velocities but accelerations, or torques. When a path satisfying the velocity constraints is found, the input velocities need to be differentiated to find the corresponding accelerations. Of their very nature, the piecewise constant trajectories are not differentiable at the switching points.

From looking at the chained form equations, it can be noted that if the input u_1 is constant and the input u_2 is discontinuous, then the state z_2 will be C^0, the state z_3 will be C^1, and each successive state in the chain will have one more degree of continuity. A modified version of this algorithm could be considered which plans a path using piecewise constant inputs for a chained form system of a higher dimension than what is required; the number of extra states would be determined by the degree of continuity on the input that is desired. Once a suitable path was found, the states which are not continuous enough could be discarded, and the new input taken as the derivative of the highest state which remains in the chain.

3.4.3 Polynomial inputs

Yet another possibility for steering systems in chained form is to use inputs which are polynomial functions of time:

$$\begin{aligned} u_1 &= 1 \\ u_2 &= c_0 + c_1 t + \cdots + c_{n-2} t^{n-2}. \end{aligned}$$

This approach has the advantage of a constant input on u_1 with the added advantage of the differentiability of u_2.

The time needed to steer the system from z^0 to z^f is determined by the change desired in the first coordinate,

$$T = z_1^f - z_1^0.$$

Once T has been found, the state equations (3.24) can be integrated using the initial condition $z(0) = z^0$,

$$\begin{aligned}
z_1(t) &= z_1(0) + t \\
z_2(t) &= z_2(0) + c_0 t + \frac{c_1 t^2}{2} + \cdots + \frac{c_{n-2} t^{n-1}}{n-1} \\
&\vdots \\
z_i(t) &= z_i(0) + \sum_{k=0}^{n-2} \frac{k! c_k t^{i+k-1}}{(i+k-1)!} + \sum_{k=2}^{i-1} \frac{t^{i-k}}{(i-k)!} z_k(0) \\
&\vdots
\end{aligned}$$

Evaluating the foregoing at time T and setting $z(T) = z^f$ yields a total of $n-1$ equations affine in the $n-1$ parameters $c_0, \ldots, c_{n-2}$,

$$M(T) \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \end{bmatrix} + f(z(0), T) = \begin{bmatrix} z_2^f \\ z_3^f \\ \vdots \\ z_n^f \end{bmatrix}$$

The matrix entries $M_{i,j}(T)$ have the form:

$$M_{i,j} = \frac{(j-1)!\, T^{i+j-1}}{(i+j-1)!}.$$

It may be shown that this matrix is nonsingular for $T \neq 0$.

Note that if $z_1^f - z_1^0 < 0$, the solution specifies a negative time period. This situation is easily remedied by choosing $u_1 = -1$.

As in the case of steering with piecewise constant inputs, this method will yield no solution when $z_1^f - z_1^0 = 0$. The same procedure outlined in Section 3.4.2 can be applied to deal with this scenario.

3.4.4 Other choices

Because of the simple form of the chained form system, many different classes of input functions other than the three described above could be used to steer systems in this form. The chief requirement is that there should be at least as many parameters in the input functions as there are states. For multi-trailer systems, a desirable characteristic of the input functions is that u_1 have few or no zero-crossings since these will correspond to fewer

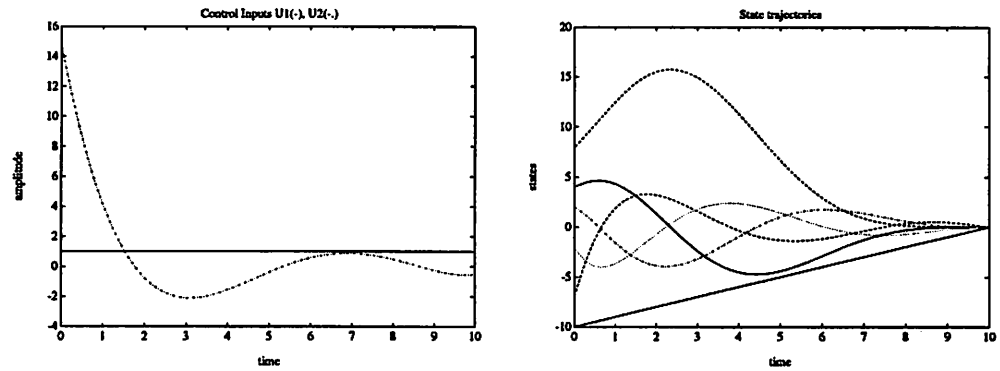


Figure 3.9: Sample trajectories and input traces for steering with polynomial inputs. The initial position is $(-10, -7, -2, 2, 4, 8)$ and the goal point is the origin.

backups. In fact, the number of backups needed to complete a maneuver may be taken as a measure of complexity of an input class.

3.5 Sample paths for a two-trailer system

An extensive toolbox now exists for steering an N-trailer system. With two different coordinate transformations which bring the system into chained form, and at least three different methods for steering the system once it is in chained form, an effort can be made to choose the best combination of coordinate transformation and input type for each start and goal point. There is as yet no formal way to define when one path is "better" than another, but as was mentioned earlier, desirable paths can be generally described as those that have few backups and do not stray too far from the vicinity of the start and goal points.

One of the things that must be considered is coordinate singularities. Although all three methods proposed here will find a path between any start and goal points in the chained form coordinates, there is no guarantee that this path, when transformed back into the actual coordinates, will avoid the transformation singularities. This must be checked for each desired path. If a singularity does result, another steering method might yield a valid path, or perhaps an intermediate point will need to be chosen, and the path planned in two or more steps.

In Figures 3.10 and 3.11, two different paths are shown for a front-wheel drive car with two trailers. The wheelbase of the car has been chosen to be $L_1 = 0.5$ units, and each trailer was given a length of $L_2 = L_3 = 2$ units. Each path was generated using techniques described in this dissertation: first, transforming the start and goal points into the chained form coordinates; second, steering the chained form system using one of the methods from Section 3.4; and finally, transforming the trajectory back into the original coordinates.

The trajectory shown in Figure 3.10 represents the truck backing into a loading dock. The initial condition is $(x^3, y^3, \theta^3, \theta^2, \theta^1, \theta^0) = (10, 10, 0, 0, 0, 0)$ and the final position is $(0, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$. Coordinate transformation 2 is used, since the first coordinate transformation is singular at the goal position. In the figure, the trajectory of the front of the car (x^0, y^0) is presented instead of the back of the second trailer (x^3, y^3) to amplify the difference between the two steering methods; the trajectories of the second trailer are virtually identical.

In Figure 3.11, the path taken by the front car is once again shown. Here two different coordinate transformations are used with the same steering method. The trajectories in the chained form coordinates are identical; however, a difference can be seen in the physical coordinates. Once again, the trajectory traced by the rear of the second trailer is very similar in both cases. Some scenes from a movie animation of this trajectory are shown in Figure 3.12 and also in the right-hand margins of this chapter; in the movie the coordinates derived from transformation 1 were used.

With the sinusoidal steering method, there is one parameter that can be adjusted independently of the start and goal positions; this is the magnitude of the sinusoid on the first input, or a_1 in the terminology of Section 3.4.1. When constructing this movie, several different values of a_1 were considered; a larger value of a_1 will correspond to the car driving out farther before it starts backing into the space. For this particular configuration, it was possible to choose a value for this parameter so that the car and trailer system did not hit any of the obstacles along its path.

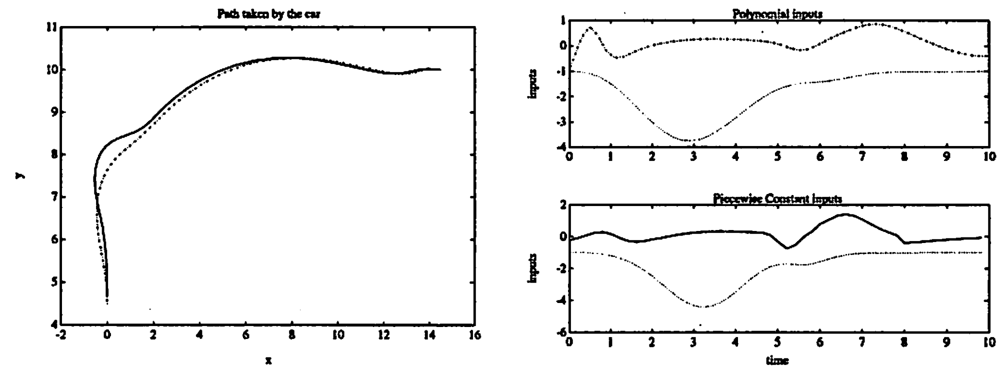


Figure 3.10: Backing a car with two trailers into a loading dock. The trajectories shown here were found by two different steering methods for the same initial and final conditions. The solid line corresponds to the piecewise constant inputs and the dashed line to the polynomial inputs. The x, y trace of the front of the car is shown, since the trajectory of the rear trailer is virtually identical in the two cases. Both trajectories use the second coordinate transformation. The input v_0 is the dotted line in both graphs. Clips from a movie simulation of this trajectory can be seen in Figure 3.13.

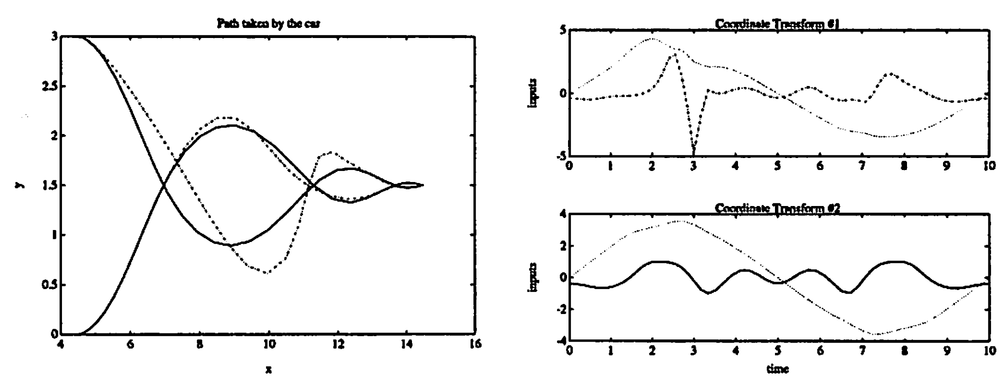


Figure 3.11: Parallel-parking a car with two trailers using sinusoids. the trace of the front car is shown for two different choices of coordinates: Transformations 1 (solid line) and 2 (dashed line). The steering input differs on with the two transformations, although for this path, the driving input v_0 (dotted line) is similar in both cases.

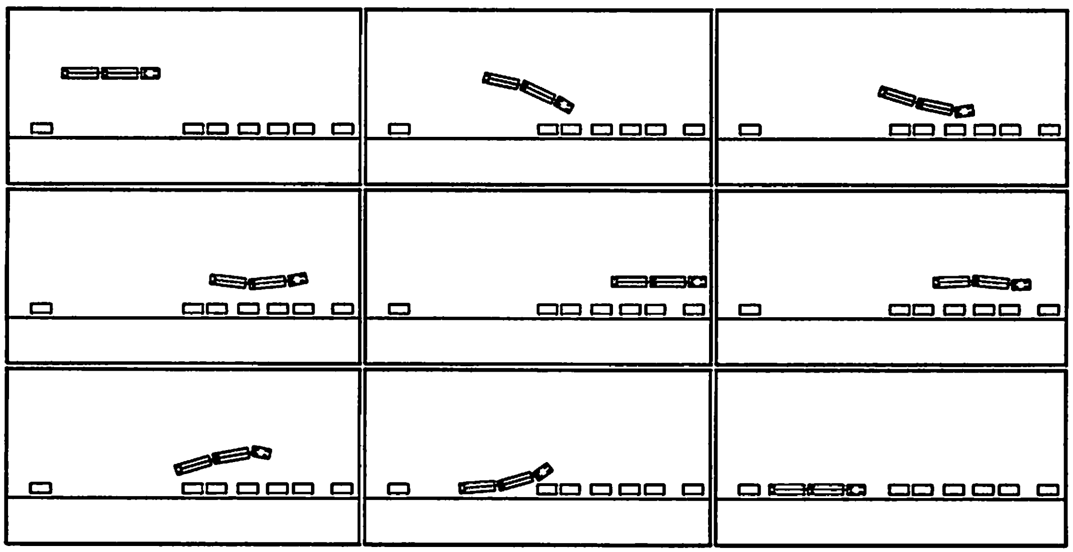
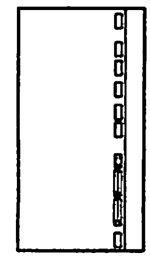


Figure 3.12: Scenes from a movie animation, showing the front-wheel drive car with two trailers (a six-state system) parallel-parking in the presence of obstacles. Sinusoidal inputs were used for steering, and the magnitude of the periodic part of the driving input (a_1 in the terminology of Section 3.4) was adjusted so that the obstacles were avoided. The first coordinate transformation was used.



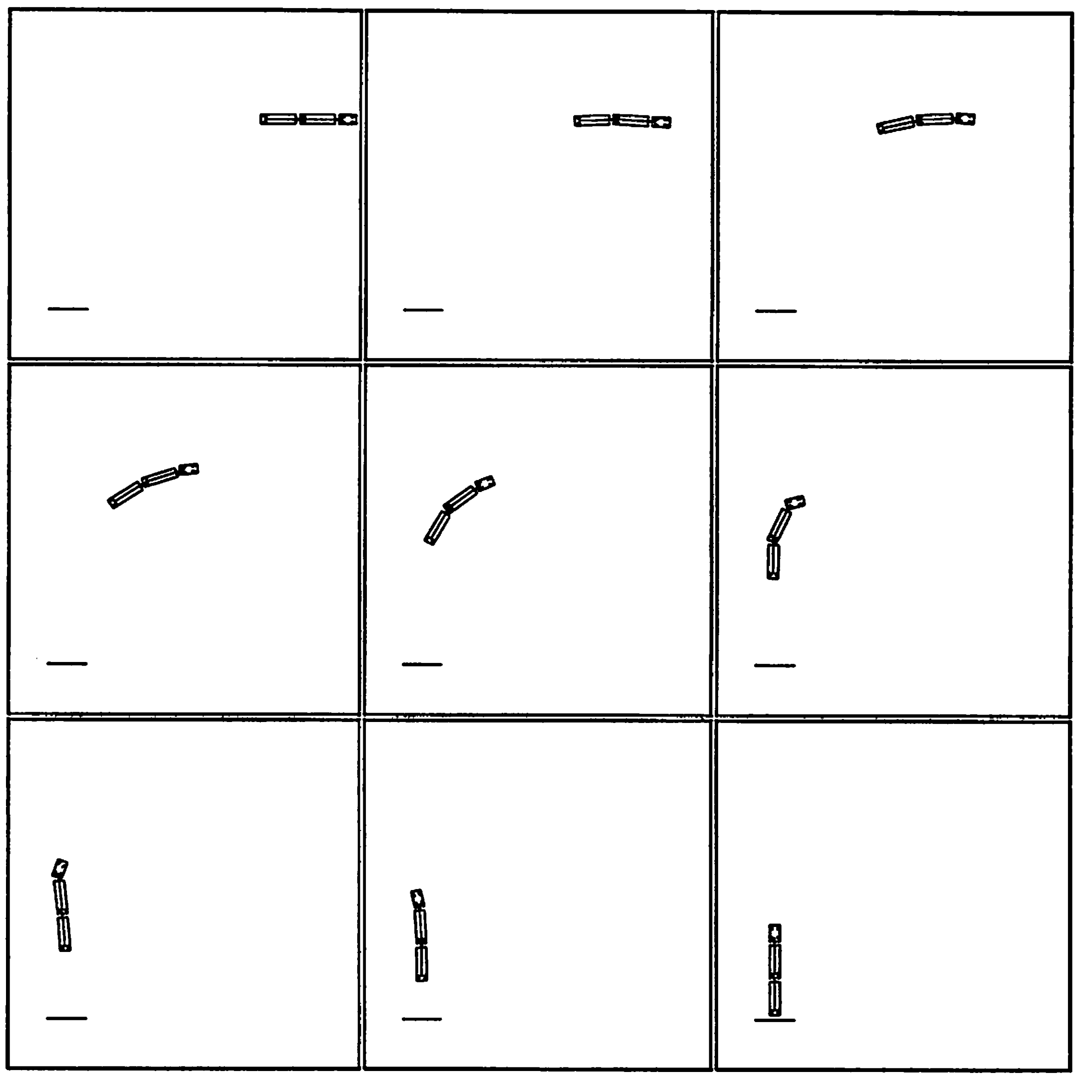


Figure 3.13: These are scenes from a movie animation, showing the front-wheel drive car with two trailers backing into a loading dock. Piecewise constant inputs were used to steer the chained form system.

Chapter 4

Extended Goursat Normal Forms and the Multi-steering Trailer System

The previous chapter analyzed the system of a car-like mobile robot towing n trailers. A similar system consisting of a chain of wheeled trailers, several of which are steerable, will be examined in this chapter. This system was originally proposed in [58], where it was shown how the kinematic equations could be converted into a multi-input chained form using dynamic feedback (interpreted as adding virtual trailers to the system). In [57], the same system was analyzed using the framework of exterior differential systems, and it was shown in exactly which cases such a dynamic feedback was necessary. The steering methods described in the last section, which are generalizations of those discussed in the previous chapter, were first presented in [58].

The appropriate normal form for the Pfaffian system associated with the multi-steering trailer system system is the extended Goursat normal form, which is a generalization of the Goursat normal form for systems of codimension greater than two. Two theorems will be stated which give necessary and sufficient conditions for converting Pfaffian systems into extended Goursat normal form, and in addition, sufficient conditions will be given for converting systems into extended Goursat normal form after a *prolongation* of the Pfaffian system. If the prolonged system can be converted into extended Goursat normal form, paths can be found for this higher-dimensional system using the methods described in Section 4.5, and the projection of these paths onto the original system will give integral curves of the original Pfaffian system.

It will be shown that, allowing for prolongations, the multi-steering trailer system can always be transformed into extended Goursat normal form, and thus the path-planning problem for this system can be solved. There are many arrangements possible for the order of the steerable and passive axles in a multi-steering trailer system. Those arrangements which can be converted into extended Goursat normal form without using prolongations will be identified

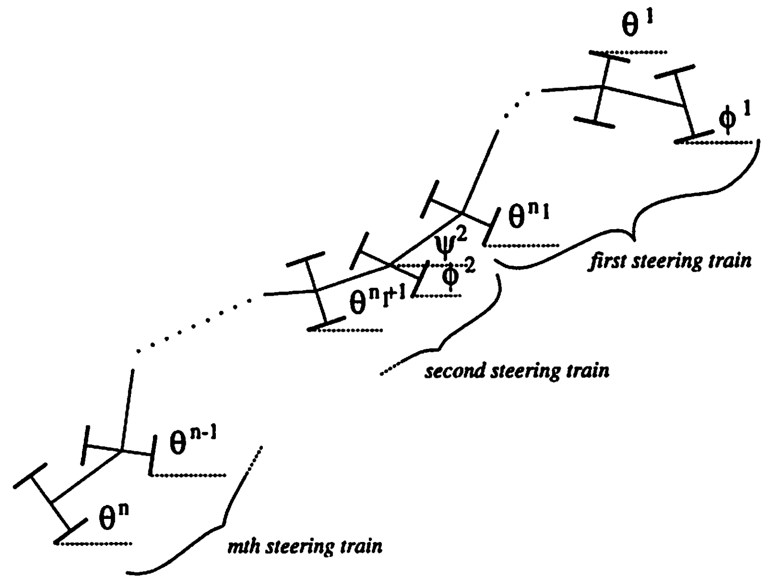
More specifically, the organization of this chapter is as follows. First, the configuration space for the system is introduced, and the notation is explained. The nonholonomic constraints (that each axle of wheels must roll without slipping) are then defined. Much of the treatment will parallel that of the n-trailer system of Chapter 3. After the extended Goursat normal form theorems and the definition of prolongations, a general theorem is given, stating that the system as defined can be converted into Goursat normal form after prolongation. Conditions are also given for an exact transformation (without prolongation) to exist. Finally, some examples of systems which do not satisfy these conditions are investigated, and it is shown explicitly how prolongations can be used to convert these systems into extended Goursat form. A parallel-parking maneuver for one of these example systems is illustrated through the movie animation in the upper right-hand corner of the pages in this chapter.

4.1 A multi-steering trailer system

First, consider a system of n (passive) trailers and m (steerable) cars linked together by rigid bars, as sketched in Figure 4.1. It is assumed that each body (trailer or car) has only one axle, since, as was described in Section 3.2.5, a two-axle car is equivalent (under coordinate transformation and state feedback) to a one-axle car towing one trailer.

4.1.1 Configuration space

The active or steering axles are numbered from front to back, starting with 1 and going up to m, and the passive axles are numbered similarly from 1 to n. There are a total of $n+m$ axles in the system. The angle of each passive axle with respect to the horizontal will



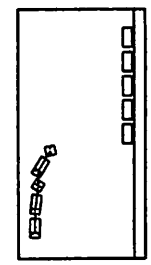


Figure 4.1: A multi-trailer system with n (passive) trailers and m (active) steering wheels.

be represented by θ^i where $i \in \{1, \ldots, n\}$ is the axle number. Each steerable axle together with the passive axles directly behind it will be called a *steering train*.

The steerable axles may be interspersed among the passive axles in any fashion. The indices of the passive axles which are directly in front of the steerable axles will be denoted by $n_1, \ldots, n_{m-1}$. The first axle is always assumed to be steerable, and thus $n_0 = 0$. The angle of the first axle with respect to the horizontal is denoted by ϕ^1. If there are n_1 passive trailers in the first steering train, their angles are denoted $\theta^1, \ldots, \theta^{n_1}$. The axle directly behind the first steering train is steerable, and its angle with respect to the horizontal will be ϕ^2. The (passive) axles behind the second steering wheel are thus $\theta^{n_1+1}, \ldots, \theta^{n_2}$; the angle of the third steering wheel will be ϕ^3, and so forth. For convenience of notation, let $n_m = n$, although in general the last axle will not be steerable. If the last axle is steerable, then $n_{m-1} = n_m$.

Remark 5 (Control Inputs) It is perhaps natural to think of the linear velocity of the lead car as well as the steering velocities $\dot{\phi}^1, \ldots, \dot{\phi}^m$ as the kinematic control inputs to this type of system. However, for the analysis that is performed in this dissertation on finding feasible paths for such a system, the exact controller structure is unimportant. It is possible to imagine such a system which has a Hilare-type robot at the front of the chain, controlling the driving and steering velocities of the first axle, and motors on each of the "steerable" axles to command the linear velocities of those axles. If both the linear and angular velocities of these axles were to be controlled, slipping would necessarily occur, unless some extra variable (the link length?) were allowed to vary.

This system as defined is a very general system, and includes the following as special cases:

1. the standard n trailer system, examined in Chapter 3 and [25, 37, 41, 47, 55] corresponds to $m = 1$.
2. the fire truck of [7, 53] corresponds to $m = 2$, $n_2 = n_1 = 1$.

Let ψ^j denote the absolute angle (with respect to the horizontal) of the bar connecting the j^{th} steered axle to the last axle of the $(j-1)^{st}$ steering train (which may be either steered or passive). This can be considered to be the angle of the bar connecting the j^{th} steering train to the $(j-1)^{st}$ steering train. The Cartesian position (x, y) of *any one* of the axles, along with all of the angles described above, will determine the state of the system. The choice of which (x, y) will be deferred for the time being, but it is noted that only one pair is needed.

The configuration of a trailer system consisting of n trailers and m steerable cars is thus completely given by

$$\xi = [\theta^1, \dots, \theta^n, \phi^1, \dots, \phi^m, \psi^2, \dots, \psi^m, x, y]^T \in (S^1)^{n+2m-1} \times \mathbb{R}^2 .$$

4.1.2 Pfaffian system

The nonholonomic constraints on the velocities, representing the fact that each axle of wheels rolls without slipping, form a codistribution of one-forms in the cotangent bundle to the configuration manifold and thus generate a Pfaffian system.

If the variables (x^i, y^i) are used to represent the Cartesian position of the i^{th} passive axle, then the constraint that the i^{th} passive axle roll without slipping can be written in these coordinates as:

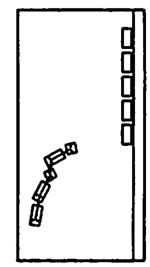
$$\omega^i = \sin\theta^i dx^i - \cos\theta^i dy^i \tag{4.1}$$

Similarly, let (x_s^j, y_s^j) represent the Cartesian position of the j^{th} steerable axle (where the subscript s stands for steerable). The constraint that the j^{th} steerable axle roll without slipping may be written as:

$$\alpha^j = \sin\phi^j dx_s^j - \cos\phi^j dy_s^j \tag{4.2}$$

Of course, as noted before, only one pair of (x, y), along with all of the angles, is needed to specify the state of the system.

The Pfaffian system generated by this mobile robot system is the collection of all the nonholonomic (rolling without slipping) constraints:



$$I = \{\omega^1, \ldots, \omega^n, \alpha^1, \ldots, \alpha^m\}$$

Thus I has dimension $n + m$ in a space of dimension $n + 2m + 1$; the codimension of I is $m + 1$, or one more than the number of steering angles.

Notice that from equations (4.1) and (4.2) it can be seen that:

$$dy^i \equiv \tan\theta^i dx^i \mod \omega^i \tag{4.3}$$

$$dy_s^j \equiv \tan\phi^j dx_s^j \mod \alpha^j \tag{4.4}$$

All of the (x^i, y^i)'s and (x_s^i, y_s^i)'s are related by the hitch relationships. The exterior derivatives of these relationships can be taken, yielding

$$\begin{matrix} x^{i-1} = x^i + L_i\cos\theta^i \\ y^{i-1} = y^i + L_i\sin\theta^i \end{matrix} \implies \begin{matrix} dx^{i-1} = dx^i - L_i\sin\theta^i d\theta^i \\ dy^{i-1} = dy^i + L_i\cos\theta^i d\theta^i \end{matrix}$$

and substituting these quantities into the expression for ω^{i-1} from (4.1), the constraint for the $(i-1)^{st}$ passive axle can be rewritten as:

$$\begin{aligned} \omega^{i-1} &= \sin\theta^{i-1}dx^{i-1} - \cos\theta^{i-1}dy^{i-1} \\ &= \sin\theta^{i-1}dx^i - \cos\theta^{i-1}dy^i - L_i\cos(\theta^i - \theta^{i-1})d\theta^i \\ &\equiv (\sin\theta^{i-1} - \tan\theta^i\cos\theta^{i-1})dx^i - L_i\cos(\theta^i - \theta^{i-1})d\theta^i \mod \omega^i \\ &\equiv \sec\theta^i\sin(\theta^{i-1} - \theta^i)dx^i - L_i\cos(\theta^i - \theta^{i-1})d\theta^i \qquad \mod \omega^i \end{aligned} \tag{4.5}$$

where the congruence (4.3) has been used. Once again, a rearrangement of terms and a division by cosine in (4.5) will give the congruence

$$\begin{aligned} d\theta^i &\equiv \frac{1}{L_i}\sec\theta^i\tan(\theta^{i-1} - \theta^i)dx^i \mod \omega^i, \omega^{i-1} \\ d\theta^i &\equiv f_{\theta^i}dx^i \qquad \mod \omega^i, \omega^{i-1} \end{aligned} \tag{4.6}$$

The exact form of the function f_{θ^i} is unimportant; what will be needed is the relationship between $d\theta^i$ and dx^i.

The first lemma can now be proved,

Lemma 15 *The exterior derivatives of any of the x variables are congruent modulo the Pfaffian system, that is:* $dx^i \equiv f_{x^{i,j}} dx^j \equiv f_{x_s^{i,k}} dx_s^k \mod I$

Proof. For two passive axles, the relationship between the x coordinates is given by the hitching relationship,

$$\begin{aligned} x^{i-1} &= x^i + L_i \cos\theta^i \\ dx^{i-1} &= dx^i - L_i \sin\theta^i d\theta^i \\ &\equiv (1 - L_i \sin\theta^i f_{\theta^i}) dx^i \mod \omega^{i-1}, \omega^i \\ &\equiv f_{x^{i-1}} dx^i \mod \omega^{i-1}, \omega^i \end{aligned} \tag{4.7}$$

where the congruence (4.6) was used.

The computations are similar when there is a steerable axle involved instead of two passive axles. If the i^{th} passive axle is located in front of the j^{th} steerable axle, then the hitch relationship and its exterior derivative are given by:

$$\begin{aligned} x^i &= x_s^j + l_j \cos\psi^j \\ dx^i &= dx_s^j - l_j \sin\psi^j d\psi^j \end{aligned} \tag{4.8}$$

In this case, the constraint corresponding to the i^{th} passive axle has the form

$$\begin{aligned} \omega^i &= \sin\theta^i dx^i - \cos\theta^i dy^i \\ &= \sin\theta^i dx_s^j - \cos\theta^i dy_s^j - l_j \cos(\theta^i - \psi^j) d\psi^j \\ &\equiv (\sin\theta^i - \cos\theta^i \tan\phi^j) dx_s^j - l_j \cos(\theta^i - \psi^j) d\psi^j \mod \alpha^j \\ &\equiv \sec\phi^j \sin(\theta^i - \phi^j) dx_s^j - l_j \cos(\theta^i - \psi^j) d\psi^j \mod \alpha^j \end{aligned} \tag{4.9}$$

Again, the standard trick of dividing through by a cosine and rearranging terms will result in the congruence

$$\begin{aligned} d\psi^j &\equiv \frac{1}{l_j} \sec\phi^j \sin(\theta^i - \phi^j) \sec(\theta^i - \psi^j) dx_s^j \mod \alpha^j, \omega^i \\ d\psi^j &\equiv f_{\psi^j} dx_s^j \mod \alpha^j, \omega^i \end{aligned} \tag{4.10}$$

Now, combining (4.10) with (4.8), it can be seen that

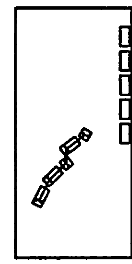
$$dx^i \equiv f_{x_s^j} dx_s^j \mod \alpha^j, \omega^i$$

The case where there are two adjacent steerable axles is done exactly the same way, with different notation, and will not be written out in detail here. □

A complement to the Pfaffian system $I = \{\omega^1, \ldots, \omega^n, \alpha^1, \ldots, \alpha^m\}$ is given by

$$\{d\phi^1, \ldots, d\phi^m, dx\}$$

for any $x \in \{x^1, \ldots, x^n, x_s^1, \ldots, x_s^m\}$, since by Lemma 15 their exterior derivatives are congruent modulo the system, and the complement is only defined modulo the system. Since the derivatives $d\phi^j$ do not appear in any of the constraints, they are in the complement to I.



From the exterior derivative of the constraint corresponding to the i^{th} passive axle, it can be seen that

$$\begin{aligned} \omega^i &= \sin\theta^i dx^i - \cos\theta^i dy^i \\ d\omega^i &= d\theta^i \wedge (\cos\theta^i dx^i + \sin\theta^i dy^i) \\ &\equiv d\theta^i \wedge (\cos\theta^i + \sin\theta^i \tan\theta^i) dx^i \quad \text{mod } \omega^i \\ &\equiv \sec\theta^i d\theta^i \wedge dx^i \quad \text{mod } \omega^i \\ &\equiv 0 \quad \text{mod } \omega^i, \omega^{i-1} \end{aligned} \tag{4.11}$$

where the congruences (4.3) and (4.6) have been used. That is, the exterior derivative of the constraint corresponding to the i^{th} passive axle is equal to zero modulo itself and the constraint which corresponds to the axle most directly in front. Without redoing the calculations, which are identical except for the notation, it can be seen that if the i^{th} passive axle is behind a steerable axle with angle ϕ^k instead of a passive axle with angle θ^{i-1}, that is, $i = n_{k-1} + 1$, then the following congruence will result:

$$d\omega^i \equiv 0 \mod \omega^i, \alpha^k \tag{4.12}$$

Proceeding similarly, the exterior derivatives of the constraints associated with the steerable axles can be found,

$$\begin{aligned} \alpha^j &= \sin\phi^j dx_s^j - \cos\phi^j dy_s^j \\ d\alpha^j &= d\phi^j \wedge (\cos\phi^j dx_s^j + \sin\phi^j dy_s^j) \\ &\equiv d\phi^j \wedge (\cos\phi^j + \sin\phi^j \tan\phi^j) dx_s^j \quad \text{mod } \alpha^j \\ &\equiv \sec\phi^j d\phi^j \wedge dx_s^j \quad \text{mod } \alpha^j \\ &\not\equiv 0 \quad \text{mod } I \end{aligned} \tag{4.13}$$

and it can be seen that their exterior derivatives are nonzero modulo the Pfaffian system I.

Recalling the definition of the derived flag from Chapter 2, it is now easy to see that all of the constraints corresponding to the passive axles are in the first derived system, and none of those corresponding to the steerable axles are. That is, the first derived system is given by:

$$I^{(1)} = \{\omega^1, \dots, \omega^n\}$$

In fact, the entire derived flag can be found just from the three congruences, (4.11), (4.12), and (4.13),

Lemma 16 (Derived Flag) *The derived flag associated with the m-steering, n-trailer system has the form:*

$$I^{(k)} = \{\omega^i : n_{j-1} + k \leq i \leq n_j, j = 1, \dots, m\}$$

for $k = 1, \dots, n$. In addition,

$$I^{(n+1)} = \{0\}.$$

Proof. The proof is just a one-time application of (4.13), to show that none of the constraints α^j corresponding to the steering axles are in $I^{(1)}$, and then a repeated application of (4.11) to show at which level each constraint falls out of the derived flag. □

If n_1 is the greatest of the indices n_i, the derived flag has the structure:

$$\begin{array}{rlllll}
I & = \{ & \alpha^1, \omega^1, \omega^2, \dots, \omega^{n_1}, & \alpha^2, \omega^{n_1+1}, \omega^{n_1+2}, \dots, & \alpha^m, \omega^{n_{m-1}+1}, \dots, \omega^n \} \\
I^{(1)} & = \{ & \omega^1, \omega^2, \dots, \omega^{n_1}, & \omega^{n_1+1}, \omega^{n_1+2}, \dots, & \omega^{n_{m-1}+1}, \dots, \omega^n \} \\
I^{(2)} & = \{ & \omega^2, \dots, \omega^{n_1}, & \omega^{n_1+2}, \dots, & \dots, \omega^n \} \\
\vdots & & \vdots & & \\
I^{n_1} & = \{ & \omega^{n_1} \} & & \\
I^{n_1+1} & = \{ & 0 \} & &
\end{array}$$

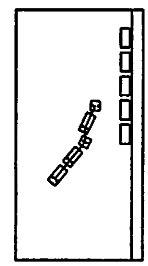
In the general case, the Pfaffian system I consists of the constraints corresponding to all the axles, the first derived system lacks the steerable axles, the second derived system lacks those passive axles that are directly behind steerable axles, and at every subsequent level, the constraint which is most toward the front of each steering train will drop off. Since the longest possible chain of contiguous passive axles is equal to n, the total number of passive axles that are in the chain, the $(n+1)^{st}$ derived system must be equal to $\{0\}$.

4.2 Extended Goursat normal form

Consider the following definition of the extended Goursat normal form,

Definition 4 (Extended Goursat Normal Form) *A Pfaffian system I on $\mathbb{R}^{n+m+1}$ of codimension $m+1$ is in* extended Goursat normal form *if it is generated by n constraints of the form:*

$$I = \{dz_i^j - z_{i+1}^j dz^0 : i = 1, \dots, s_j; j = 1, \dots, m\}, \tag{4.14}$$



This is a direct extension of the Goursat normal form, and all integral curves of (4.14) are determined by the $m+1$ functions $z^0(t), z_1^1(t), \dots, z_1^m(t)$ and their derivatives with respect to the parameter t. The notation has been changed slightly; the canonical constraints are now $dz_i^j - z_{i+1}^j dz^0$ whereas before they were $dz_i - z_{i-1} dz_1$. For the Goursat form, the constraint in the last nontrivial derived system was $dz^n - z^{n-1} dz^1$; in the extended Goursat normal form, it will be $dz_1^j - z_2^j dz^0$ (if indeed there is one tower which is longest).

The distribution annihilated by the constraints which define the extended Goursat normal form is spanned by the $m+1$ vector fields:

$$g_0 = \frac{\partial}{\partial z^0} + z_2^1 \frac{\partial}{\partial z_1^1} + \cdots + z_{s_1+1}^1 \frac{\partial}{\partial z_{s_1}^1} + \cdots + z_2^m \frac{\partial}{\partial z_1^m} + \cdots + z_{s_m+1}^m \frac{\partial}{\partial z_{s_m}^m}$$
$$g_i = \frac{\partial}{\partial z_{s_i+1}^i} \qquad i = 1, \dots, m$$

which is the same as the multi-input, single-generator chained form defined in [7, 37].

Remark 6 (Differential Flatness) It is clear that any system which admits a transformation into extended Goursat normal form is differentially flat. The flat outputs (defined in Remark 2) can be chosen as $z^0, z_1^1, \dots, z_1^m$.

There are conditions due to Murray [39] for converting a Pfaffian system to extended Goursat normal form. This theorem is restated and proved here with the additional condition (correction) that π needs to be integrable:

Theorem 17 (Extended Goursat Normal Form) *Let I be a Pfaffian system of codimension $m+1$. If (and only if) there exists a set of generators $\{\alpha_i^j : i = 1, \dots, s_j; j = 1, \dots, m\}$ for I and an* integrable *one-form π such that for all j,*

$$\begin{aligned} d\alpha_i^j &\equiv -\alpha_{i+1}^j \wedge \pi \mod I^{(s_j - i)} \quad i = 1, \dots, s_j - 1 \\ d\alpha_{s_j}^j &\not\equiv 0 \qquad\qquad \mod I \end{aligned} \tag{4.15}$$

then there exists a set of coordinates z such that I is in extended Goursat normal form,

$$I = \{dz_i^j - z_{i+1}^j dz^0 : i = 1, \dots, s_j; j = 1, \dots, m\}.$$

Proof. If the Pfaffian system is already in extended Goursat normal form, the congruences are satisfied with $\pi = dz^0$ (which is integrable) and the basis of constraints $\alpha_i^j = dz_i^j - z_{i+1}^j dz^0$.

Now assume that a basis of constraints for I has been found which satisfies the congruences (4.15). It is easily checked that this basis is adapted to the derived flag, that is:

$$I^{(k)} = \{\alpha_i^j : \quad i = 1, \dots, s_j - k; j = 1, \dots, m\}$$

The coordinates z which comprise the Goursat normal form can now be constructed.

Since π is integrable, any first integral of π can be used for the coordinate z^0. If necessary, the constraints α_i^j can be scaled so that the congruences (4.15) are satisfied with dz^0:

$$d\alpha_i^j \equiv -\alpha_{i+1}^j \wedge dz^0 \mod I^{(s_j - i)} \quad i = 1, \dots, s_j - 1$$

and the constraints can be renumbered so that $s_1 \geq s_2 \geq \cdots \geq s_m$.

Now consider the last nontrivial derived system, $I^{(s_1-1)}$. The one-forms $\{\alpha_1^1, \dots, \alpha_1^{r_1}\}$ form a basis for this codistribution, where $s_1 = s_2 = \cdots = s_{r_1}$. From the fact that

$$d\alpha_1^j \equiv -\alpha_2^j \wedge dz^0 \mod I^{(s_1-1)},$$

it follows that the one-forms $\alpha_1^1, \dots, \alpha_1^{r_1}$ satisfy the Frobenius condition:

$$d\alpha_1^j \wedge \alpha_1^1 \wedge \cdots \wedge \alpha_1^{r_1} \wedge dz^0 = 0$$

and thus, by the Frobenius theorem, coordinates $z_1^1, \dots, z_1^{r_1}$ can be found such that

$$\begin{bmatrix} \alpha_1^1 \\ \vdots \\ \alpha_1^{r_1} \end{bmatrix} = A \begin{bmatrix} dz_1^1 \\ \vdots \\ dz_1^{r_1} \end{bmatrix} + B dz^0$$

The matrix A must be nonsingular, since the α_1^j's are a basis for $I^{(s_1-1)}$ and they are independent of dz^0. Therefore, a new basis $\bar{\alpha}_1^j$ can be defined as:

$$\begin{bmatrix} \bar{\alpha}_1^1 \\ \vdots \\ \bar{\alpha}_1^{r_1} \end{bmatrix} := A^{-1} \begin{bmatrix} \alpha_1^1 \\ \vdots \\ \alpha_1^{r_1} \end{bmatrix} = \begin{bmatrix} dz_1^1 \\ \vdots \\ dz_1^{r_1} \end{bmatrix} + (A^{-1}B) dz^0$$

and the coordinates $z_2^j := -(A^{-1}B)_j$ are defined so that the one-forms $\bar{\alpha}_1^j$ have the form

$$\bar{\alpha}_1^j = dz_1^j - z_2^j dz^0$$

for $j = 1, \ldots, r_1$. In these coordinates, the exterior derivative of $\bar{\alpha}_1^j$ is equal to

$$d\bar{\alpha}_1^j = -dz_2^j \wedge dz^0$$

If there were some coordinate z_2^k which could be expressed as a function of the other z_2^j's and z_1^j's, then there would be some linear combination of the $\bar{\alpha}_1^j$'s whose exterior derivative would be zero modulo $I^{(s_1-1)}$, which is a contradiction. Thus, this is a valid choice of coordinates.

By the proof of the standard Goursat theorem, all of the coordinates in the j^{th} tower can be found from z_1^j and z^0. By the above procedure, all the coordinates in the first r_1 towers can be found.

To find the coordinates for the other towers, the lowest derived systems in which they appear must be considered . The coordinates for the longest towers were found first, next those for the next-longest tower(s) will be found.

Consider the smallest integer k such that dim $I^{(s_1-k)} > k\, r_1$; more towers will appear at this level. A basis for $I^{(s_1-k)}$ is

$$\{\bar{\alpha}_1^1, \ldots, \bar{\alpha}_k^1, \ldots, \bar{\alpha}_1^{r_1}, \ldots, \bar{\alpha}_k^{r_1}, \alpha_1^{r_1+1}, \ldots, \alpha_1^{r_1+r_2}\}$$

where $\bar{\alpha}_i^j = dz_i^j - z_{i+1}^j dz^0$ for $j = 1, \ldots, r_1$, as found in the first step, and α_1^j for $j = r_1 + 1, \ldots, r_2$ are the one-forms which satisfy the congruences (4.15) and are adapted to the derived flag. The lengths of these towers are $s_{r_1+1} = \cdots s_{r_1+r_2} = s_1 - k + 1$. For notational convenience, define $z_{(k)}^j := (z_1^j, \ldots, z_k^j)$ for $j = 1, \ldots, r_1$.

By the Goursat congruences, $d\alpha_1^j \equiv -\alpha_2^j \wedge dz^0 \mod I^{(s_1-k)}$ for $j = r_1 + 1, \ldots, r_1 + r_2$, thus the Frobenius condition

$$d\alpha_1^j \wedge \alpha_1^{r_1+1} \wedge \cdots \wedge \alpha_1^{r_1+r_2} \wedge dz_1^1 \wedge \cdots \wedge dz_k^1 \wedge \cdots \wedge dz_1^{r_1} \wedge \cdots \wedge dz_k^{r_1} \wedge dz^0 = 0$$

is satisfied for $j = r_1 + 1, \ldots, r_1 + r_2$. Using the Frobenius theorem, new coordinates $z_1^{r_1+1}, \ldots, z_1^{r_1+r_2}$ can be found such that

$$\begin{bmatrix} \alpha_1^{r_1+1} \\ \vdots \\ \alpha_1^{r_1+r_2} \end{bmatrix} = A \begin{bmatrix} dz_1^{r_1+1} \\ \vdots \\ dz_1^{r_1+r_2} \end{bmatrix} + B dz^0 + C \begin{bmatrix} dz_{(k)}^1 \\ \vdots \\ dz_{(k)}^{r_1} \end{bmatrix}$$

Since the congruences are only defined up to mod $I^{(s_1-k)}$, the last group of terms (those multiplied by the matrix C) can be eliminated by adding in the appropriate multiples of $\tilde{\alpha}_i^j = dz_i^j - z_{i+1}^j dz^0$ for $j = 1, \ldots, r_1$ and $i = 1, \ldots, k$. This will change the B matrix, leaving the equation

$$\begin{bmatrix} \tilde{\alpha}_1^{r_1+1} \\ \vdots \\ \tilde{\alpha}_1^{r_1+r_2} \end{bmatrix} = A \begin{bmatrix} dz_1^{r_1+1} \\ \vdots \\ dz_1^{r_1+r_2} \end{bmatrix} + \tilde{B} dz^0$$

Again, note that A must be nonsingular because the α_1^j's are linearly independent mod $I^{(s_1-k)}$ and also independent of dz^0. Define

$$\begin{bmatrix} \bar{\alpha}_1^{r_1+1} \\ \vdots \\ \bar{\alpha}_1^{r_1+r_2} \end{bmatrix} := A^{-1} \begin{bmatrix} \tilde{\alpha}_1^{r_1+1} \\ \vdots \\ \tilde{\alpha}_1^{r_1+r_2} \end{bmatrix} = \begin{bmatrix} dz_1^{r_1+1} \\ \vdots \\ dz_1^{r_1+r_2} \end{bmatrix} + (A^{-1}\tilde{B}) dz^0$$

and then define the coordinates $z_2^j := -(A^{-1}\tilde{B})_j$ for $j = r_1 + 1, \ldots, r_1 + r_2$ so that $\bar{\alpha}_1^j = dz_1^j - z_2^j dz^0$. Again, by the standard Goursat theorem, all of the coordinates in the towers $r_1 + 1, \ldots, r_1 + r_2$ are now defined.

The coordinates for the rest of the towers are defined in a manner exactly analogous to that of the second-longest tower. □

If the one-form π which satisfies the congruences (4.15) is not integrable, then the Frobenius theorem cannot be used to find the coordinates. In the special case where $s_1 > s_2$, that is, there is one tower which is strictly longer than the others, it can be shown that if there exists *any* π which satisfies the congruences, then there also exists an *integrable* π' which also satisfies the congruences (with a rescaling of the basis forms), see [6, 39]. However, if $s_1 = s_2$, or there are at least two towers which are longest, this is no longer true. Thus, the assumption that π is integrable is necessary for the general case.

If I can be converted to extended Goursat normal form, then the derived flag of I has the structure:

$$
\begin{array}{rllllllll}
I = & \{\alpha_1^1, & \cdots, & \cdots, & \alpha_{s_1-1}^1, \ \alpha_{s_1}^1, & \cdots, & \alpha_1^m, & \cdots, & \alpha_{s_m-1}^m, \ \alpha_{s_m}^m\} \\
I^{(1)} = & \{\alpha_1^1, & \cdots, & \cdots, & \alpha_{s_1-1}^1, & \cdots, & \alpha_1^m, & \cdots, & \alpha_{s_m-1}^m\} \\
 & \vdots & & \iddots & & & \vdots & \iddots & \\
I^{(s_m-1)} = & \{\alpha_1^1, & \cdots, & \alpha_{s_1-s_m+1}^1, & & \cdots, & \alpha_1^m\} & & \\
 & \vdots & \iddots & & & & & & \\
I^{(s_1-2)} = & \{\alpha_1^1, & \alpha_2^1\} & & & & & & \\
I^{(s_1-1)} = & \{\alpha_1^1\} & & & & & & & \\
I^{(s_1)} = & \{0\} & & & & & & &
\end{array}
$$

where the forms in each level have been arranged to show the different "towers" which result. The superscripts j indicate the tower to which each form belongs, and the subscripts i index the position of the form within the j^{th} tower. There are s_j forms in the j^{th} tower.

Another version of the extended Goursat normal form theorem is given here, which is easier to check, since it does not require finding a basis which satisfies the congruences but only one which is adapted to the derived flag. One special case of this theorem is proven in [49].

Theorem 18 (Extended Goursat Normal Form) *A Pfaffian system I of codimension $m+1$ on $\mathbb{R}^{n+m+1}$ can be converted to extended Goursat normal form if and only if $I^{(N)} = \{0\}$ for some N and there exists a one-form π such that $\{I^{(k)}, \pi\}$ is integrable for $k = 0, \ldots, N-1$.*

Proof. The only if part is easily shown by taking $\pi = dz^0$ and noting that

$$
\begin{aligned}
I^{(k)} &= \{dz_i^j - z_{i+1}^j dz^0 : i = 1, \ldots, s_j - k; j = 1, \ldots, m\} \\
\{I^{(k)}, \pi\} &\equiv \{dz_i^j, dz^0 : \qquad i = k+1, \ldots, s_j; j = 1, \ldots, m\}
\end{aligned}
$$

which is integrable for every k.

Now assume that such a π exists. After the derived flag of the system, $I =: I^{(0)} \supset I^{(1)} \supset \cdots \supset I^{(s_1)} = \{0\}$, has been found, a basis which is adapted to the derived flag and which satisfies the Goursat congruences (4.15) can be iteratively constructed.

The lengths of each tower are determined from the dimensions of the derived flag. Indeed, the longest tower of forms has length s_1. If the dimension of $I^{(s_1-1)}$ is r_1, then there are r_1 towers which each have length s_1; and we have $s_1 = s_2 = \cdots = s_{r_1}$. Now, if the dimension of $I^{(s_1-2)}$ is $2r_1 + r_2$, then there are r_2 towers with length $s_1 - 1$, and $s_{r_1+1} = \cdots = s_{r_1+r_2} = s_1 - 1$. Each s_j is found similarly.

A π which satisfies the conditions must be in the complement of I, for if π were in I, then $\{I, \pi\}$ integrable means that I is integrable, and this contradicts the assumption that $I^{(N)} = \{0\}$ for some N.

Consider the last nontrivial derived system, $I^{(s_1-1)}$. Let $\{\alpha_1^1, \ldots, \alpha_1^{r_1}\}$ be a basis for $I^{(s_1-1)}$. The definition of the derived flag, specifically $I^{(s_1)} = \{0\}$, implies that

$$d\alpha_1^j \not\equiv 0 \mod I^{(s_1-1)} \quad j = 1, \ldots, r_1 \tag{4.16}$$

Also, the assumption that $\{I^{(k)}, \pi\}$ is integrable gives the congruence

$$d\alpha_1^j \equiv 0 \mod \{I^{(s_1-1)}, \pi\} \quad j = 1, \ldots, r_1 \tag{4.17}$$

combining equations (4.16) and (4.17), the congruence

$$d\alpha_1^j \equiv \pi \wedge \beta^j \mod I^{(s_1-1)} \quad j = 1, \ldots, r_1 \tag{4.18}$$

must be satisfied for some $\beta^j \not\equiv 0 \mod I^{(s_1-1)}$.

Now, from the definition of the derived flag,

$$d\alpha_1^j \equiv 0 \mod I^{(s_1-2)} \quad j = 1, \ldots, r_1$$

which combined with (4.18) implies that β^j is in $I^{(s_1-2)}$.

Claim. $\beta^1, \ldots, \beta^{r_1}$ are linearly independent mod $I^{(s_1-1)}$.

Proof of Claim. The proof is by contradiction. Suppose there exists some combination of the β^j's, say

$$\beta = b_1\beta^1 + \cdots + b_{r_1}\beta^{r_1} \equiv 0 \mod I^{(s_1-1)}$$

with not all of the b_j's equal to zero. Consider $\alpha = b_1\alpha_1^1 + \cdots + b_{r_1}\alpha_1^{r_1}$. This one-form $\alpha \neq 0$ because the α_1^j are a basis for $I^{(s_1-1)}$. The exterior derivative of α can be found by the product rule,

$$\begin{aligned} d\alpha &= \sum_{j=1}^{r_1} b_j d\alpha_1^j + \sum_{j=1}^{r_1} db_j \wedge \alpha_1^j \\ &\equiv \sum_{j=1}^{r_1} b_j(\pi \wedge \beta^j) \mod I^{(s_1-1)} \\ &\equiv \pi \wedge (\sum_{j=1}^{r_1} b_j\beta^j) \mod I^{(s_1-1)} \\ &\equiv 0 \mod I^{(s_1-1)} \end{aligned}$$

which implies that α is in $I^{(s_1)}$. However, this contradicts the assumption that $I^{(s_1)} = \{0\}$. Thus the β^j's must be linearly independent mod $I^{(s_1-1)}$.

Define $\alpha_2^j := \beta^j$ for $j = 1, \ldots, r_1$. Note that these basis elements satisfy the first level of Goursat congruences, that is:

$$d\alpha_1^j \equiv -\alpha_2^j \wedge \pi \mod I^{(s_1-1)} \quad j = 1, \ldots, r_1$$

If the dimension of $I^{(s_1-2)}$ is greater than $2r_1$, then choose one-forms $\alpha_1^{r_1+1}, \ldots, \alpha_1^{r_1+r_2}$ such that that

$$\{\alpha_1^1, \ldots, \alpha_1^{r_1}, \alpha_2^1, \ldots, \alpha_2^{r_1}, \alpha_1^{r_1+1}, \ldots, \alpha_1^{r_1+r_2}\}$$

is a basis for $I^{(s_1-2)}$.

For the induction step, assume that a basis for $I^{(i)}$ has been found,

$$\{\alpha_1^1, \ldots, \alpha_{k_1}^1, \alpha_1^2, \ldots, \alpha_{k_2}^2, \ldots, \alpha_1^c, \ldots, \alpha_{k_c}^c\}$$

which satisfies the Goursat congruences up to this level:

$$d\alpha_k^j = -\alpha_{k+1}^j \wedge \pi \mod I^{(s_j-k)} \quad k = 1, \ldots, k_j - 1; \quad j = 1, \ldots, c$$

Note c towers of forms have appeared in $I^{(i)}$. Consider only the last form in each tower that appears in $I^{(i)}$, that is $\alpha_{k_j}^j, j = 1, \ldots, c$. By the construction of this basis (or from the Goursat congruences), $\alpha_{k_j}^j$ is in $I^{(i)}$ but is not in $I^{(i+1)}$, thus

$$d\alpha_{k_j}^j \not\equiv 0 \mod I^{(i)} \quad j = 1, \ldots, c$$

The assumption that $\{I^{(i)}, \pi\}$ is integrable assures

$$d\alpha_{k_j}^j \equiv 0 \mod \{I^{(i)}, \pi\} \quad j = 1, \ldots, c$$

thus $d\alpha_{k_j}^j$ must be a multiple of π mod $I^{(i)}$,

$$d\alpha_{k_j}^j \equiv \pi \wedge \beta^j \mod I^{(i)} \quad j = 1, \ldots, c$$

for some $\beta^j \not\equiv 0 \mod I^{(i)}$. From the fact that $\alpha_{k_j}^k$ is in $I^{(i)}$ and the definition of the derived flag,

$$d\alpha_{k_j}^j \equiv 0 \mod I^{(i-1)} \quad j = 1, \ldots, c$$

which implies that $\beta^j \in I^{(i-1)}$. By a similar argument to the claim above, it can be shown that the β^j's are independent mod $I^{(i)}$. Define $\alpha_{k_j+1}^j := \beta^j$, and thus

$$\{\alpha_1^1, \ldots, \alpha_{k_1+1}^1, \alpha_1^2, \ldots, \alpha_{k_2+1}^2, \ldots, \alpha_1^c, \ldots, \alpha_{k_c+1}^c\}$$

forms part of a basis of $I^{(i-1)}$. If the dimension of $I^{(i-1)}$ is greater than $k_1+k_2+\cdots+k_c+c$, then complete the basis of $I^{(i-1)}$ with any linearly independent one-forms $\alpha_1^{c+1},\ldots,\alpha_1^{c+r_c}$ such that

$$\{\alpha_1^1,\ldots,\alpha_{k_1+1}^1,\alpha_1^2,\ldots,\alpha_{k_2+1}^2,\ldots,\alpha_1^c,\ldots,\alpha_{k_c+1}^c,\alpha_1^{c+1},\ldots,\alpha_1^{c+r_c}\}$$

is a basis for $I^{(i-1)}$.

Repeated application of this procedure will construct a basis for I which is not only adapted to the derived flag, but also satisfies the Goursat congruences.

By assumption, π is integrable mod the last nontrivial derived system, $I^{(s_1-1)}$. Looking at the congruences (4.15), any integrable one-form π' which is congruent to π up to a scaling factor f,

$$\pi' = dz^0 \equiv f\pi \mod I^{(s_1-1)}$$

will satisfy the same set of congruences up to a rescaling of the constraint basis by multiples of this factor f. □

4.3 Prolongations

Consider a Pfaffian system I in extended Goursat normal form:

$$I = \{dz_i^j - z_{i+1}^j dz^0 : i = 1,\ldots,s_j; j = 1,\ldots,m\},$$

with independence condition dz^0. Let the Pfaffian system J be defined by:

$$J = \{dz_i^j - z_{i+1}^j dz^0 : i = 1,\ldots,s_1+1 \text{ and } i = 1,\ldots,s_j; j = 2,\ldots,m\},$$

The coordinate $z_{s_1+2}^1$ has been added, but the new system is also in extended Goursat normal form. It is clear that there is a one-to-one correspondence of integral curves between I and J although they are defined on manifolds of different dimensions. J is said to be a *prolongation by differentiation* (of order one) of I with respect to the independence condition dz^0.

Prolongations by differentiation can also be defined for systems which are not *a priori* in extended Goursat normal form. Let I be a Pfaffian system on a manifold M with independence condition dt, and let $d\eta$ be a one-form in the complement of I. Define the system J on $M \times \mathbb{R}$ given by

$$J = \{I, d\eta - y\,dt\}$$

to be a prolongation by differentiation of I, where the new coordinate y is the fiber coordinate on $\mathbb{R}$. In effect, this adds the derivative of η (with respect to the independence condition) as a state variable.

In general, many of these partial prolongations by differentiation may be taken.

Definition 5 (Prolongation by differentiation) *Let I be a Pfaffian system of codimension $m+1$ on $\mathbb{R}^{n+m+1}$ with coordinates (z, v, t) for which dt is an independence condition and $\{dv_1, \ldots, dv_m, dt\}$ forms a complement. Let $b_1, \ldots, b_m$ be nonnegative integers and let b denote their sum. The system I augmented by the b one-forms*

$$\begin{array}{llll} dv_1 - v_1^1 dt, & \ldots, & dv_1^{b_1-1} - v_1^{b_1} dt, & \\ dv_2 - v_2^1 dt, & \ldots, & & dv_2^{b_2-1} - v_2^{b_2} dt \\ \vdots & & & \ddots \\ dv_m - v_m^1 dt, & \ldots, & \ldots, & dv_m^{b_m-1} - v_m^{b_m} dt, \end{array}$$

is a prolongation by differentiation *of I. The augmented system is defined on $\mathbb{R}^{n+m+b+1}$.*

If a Pfaffian system I does not satisfy the necessary and sufficient conditions of Theorems 17 and 18, then I cannot be converted into extended Goursat normal form. It is possible, however, that there exists a prolongation by differentiation J of I which does satisfy the extended Goursat conditions. In this case, the prolonged system J can be put into Goursat normal form, paths can be found for the transformed system using the methods described in Section 4.5, and these paths can be projected down onto the original Pfaffian system I to give integral curves.

Although the general problem of determining which Pfaffian systems can be converted into extended Goursat normal form after prolongation is still an open one, the following theorem gives some sufficient conditions under which such a transformation exists.

Theorem 19 (Conversion to Goursat form with prolongation by differentiation) *Consider a Pfaffian system $I = \{\alpha^1, \ldots, \alpha^n\}$ on $\mathbb{R}^{n+m+1}$ with independence condition dz^0 and complement $\{dv_1, \ldots, dv_m, dz^0\}$. If there exists a list of integers $b_1, \ldots, b_m$ such that the prolonged system*

$$\begin{aligned} J = \{ \; & \alpha^1, \ldots, \alpha^n, \\ & dv_1 - v_1^1 dz^0, \ldots, dv_1^{b_1-1} - v_1^{b_1} dz^0, \\ & \vdots \\ & dv_m - v_m^1 dz^0, \ldots, dv_m^{b_m-1} - v_m^{b_m} dz^0 \; \} \end{aligned}$$

satisfies the condition that $\{J^{(k)}, dz^0\}$ is integrable for all k, then I can be transformed to extended Goursat normal form using a prolongation by differentiation.

Proof. The proof is by application of Theorem 18 to the prolonged system J. □

4.4 Converting the multi-steering trailer system to extended Goursat normal form

In this section, it will be shown how the general multi-steering trailer system can be converted into extended Goursat normal form after prolongation. The configurations of this system which satisfy the conditions for conversion without prolongation will also be detailed. The first lemma gives a candidate choice for π, since for a system in extended Goursat normal form it must always be true that $\{I, \pi\}$ is integrable.

Lemma 20 *$\{I, dx\}$ is integrable for any $x \in \{x^1, \dots, x^n, x_s^1, \dots, x_s^m\}$.*

Proof. Each constraint in I satisfies the congruence

$$\begin{aligned} d\omega^i &\equiv d\theta^i \wedge dx^i \mod \omega^i \\ &\equiv 0 \mod \{I, dx^i\} \\ d\alpha^j &\equiv d\phi^j \wedge dx_s^j \mod \alpha^j \\ &\equiv 0 \mod \{I, dx_s^j\} \end{aligned}$$

(by equations (4.11) and (4.13)). Also, all of the dx^i, dx_s^j are congruent by Lemma 15. Thus, the exterior derivative of any constraint in $\{I, dx\}$ is congruent to zero mod $\{I, dx_s^j\}$, which is the condition for integrability. □

It can be shown that for the general case, there does not exist a dx (or any other one-form) which will satisfy the condition that $\{I^{(i)}, dx\}$ is integrable for every i. However, the general multi-steering system can be transformed into Goursat normal form after prolongation.

The concept of "virtual trailers" was first introduced in [58] as a type of dynamic state feedback for the multi-steering trailer system. A chain of these virtual trailers, each analogous to a physical trailer, was added in front of each actual steering wheel, and a virtual steering wheel was added at the front of each virtual chain. The sketch of this augmented system in Figure 4.2 helps make the concept more clear.

Each virtual trailer adds one state to the system, as well as one constraint. Thus the codimension of the extended system is the same as that of the original system, $m+1$.

Consider the following theorem.

Theorem 21 (Converting the multi-steering system to Goursat form) *The multi-steering system with n trailers and m steering wheels can be put into extended Goursat normal form, for any n, m and for any configuration of steerable cars and passive trailers, using a prolongation of degree less than or equal to $n_1 + \cdots + n_{m-1}$.*

Proof. Consider the n-trailer, m-steering system with virtual extension as shown in Figure 4.2. That is, in front of each steerable axle, imagine that there are n_{j-1} virtual axles, and that only the front axle in each virtual chain is steerable. Note that with this virtual axle formulation, the actual steerable axles within the multi-trailer chain are no longer assumed to be directly steerable, but rather are controlled through the virtual steering axles and the the chains of virtual trailers.

Let ϕ_v^j represent the angle of the j^{th} virtual steering axle, where the subscript v stands for virtual. The angles of the passive axles that are added are denoted by θ_j^i, where the subscript j stands for the index of the virtual chain that they are in, and the superscript i indexes their position from the front of the virtual train.

A total of $n_1 + \cdots + n_{m-1}$ states have been added to the system, corresponding to the angles of the virtual axles. The same number of constraints have also been added. The first axle is always assumed to be steerable, and no virtual axles are added in front of the front steering wheel.

Because the constraints that were added have the same form as those in the system already, it is easy to see that they can be written in coordinates as

$$\nu_j^i = \sin\theta_j^i dx_j^i - \cos\theta_j^i dy_j^i$$

for the passive virtual axles and

$$\alpha_v^j = \sin\phi_v^j dx_v^j - \cos\theta_v^j dy_v^j$$

for the steerable virtual axles at the front of each chain. Although these constraints do not immediately appear to be of the same form as a prolongation by differentiation, it can be shown that

$$\nu_j^i \equiv d\theta_j^{i+1} - \tan\theta_j^i dx^n \mod J^{(i)}$$

for $i < n_{j-1} - 2$ and

$$\nu_j^{n_{j-1}-1} \equiv d\phi^j - \tan\theta_j^{n_{j-1}-1} dx^n \mod J^{(n_{j-1}-1)}$$

where x^n is the x-position of the last passive axle. This particular form of a prolongation by differentiation was chosen so that the constraints which were added to the system would have the same expression (in coordinates) as the physical constraints; the computations are somewhat simplified by this choice. Because of the equivalence, a standard prolongation by differentiation could have been used; it would be difficult to interpret the meaning of the added states.

The prolonged Pfaffian system is given by the collection of actual and virtual constraints,

$$J = \{\alpha^1, \dots, \alpha^m, \alpha_v^2, \dots, \alpha_v^m, \omega^1, \dots, \omega^n, \nu_j^i : j = 2, \dots, m; i = 1, \dots, n_{j-1} - 1\}$$

The derived flag corresponding to the extended Pfaffian system can now be found. First, performing a similar calculation to that in equation (4.11), it can be seen that

$$d\nu_j^i \equiv 0 \mod \nu_j^i, \nu_j^{i-1}$$

Then, similar to equation (4.13),

$$d\alpha_v^j \not\equiv 0 \mod J$$

It is also not difficult to show that

$$d\alpha^j \equiv 0 \mod \alpha^j, \nu_j^{n_{j-1}-1}$$

From these three congruences, the structure of the derived flag is seen to be

$$\begin{aligned}
J^{(1)} &= \{\alpha^2, \dots, \alpha^m, \omega^1, \dots, \omega^n, \nu_j^i : j = 2, \dots, m; i = 1, \dots, n_{j-1} - 1\} \\
J^{(2)} &= \{\alpha^2, \dots, \alpha^m, \omega^2, \dots, \omega^n, \nu_j^i : j = 2, \dots, m; i = 2, \dots, n_{j-1} - 1\} \\
&\vdots \\
J^{(k+j_k-1)} &= \{\alpha^{j_k+1}, \dots, \alpha^m, \omega^k, \dots, \omega^n, \nu_j^i : j = 2, \dots, m; i = k, \dots, n_{j-1} - 1\} \\
&\vdots \\
J^{(n+m-1)} &= \{\omega^n\} \text{ or } \{\alpha^m\} \\
J^{(n+m)} &= \{0\}
\end{aligned}$$

where j_k is defined to be the number of steerable axles that are in front of the k^{th} passive axle in the actual chain of trailers, and $J^{(n+m-1)} = \{\omega^n\}$ if the last axle in the chain is passive, and $J^{(n+m-1)} = \{\alpha^m\}$ if the last axle in the chain is steerable. In words, the (extended) Pfaffian system J consists of all the constraints corresponding to both the actual and the virtual axles. The first derived system consists of all constraints except the ones at the front of each (virtual) chain. At the second level, the constraints corresponding to the axles directly behind each virtual steering wheel fall off, and at the k^{th} level, the constraints corresponding to the axles which are k behind each virtual steering wheel fall off, until at the last level, there is only the constraint corresponding to the last axle in the chain (ω^n if it is passive, α^m if it is steerable). The $(n+m)^{th}$ derived system is trivial, which implies that the augmented system is controllable.

At each level of the derived flag, exactly one of the constraints which falls out of the flag corresponds to a real axle, and all the rest which fall out correspond to virtual axles.

The one-form π which satisfies the Goursat conditions of Theorem 18 is equal to the exterior derivative of the x coordinate of the last body in the actual multi-steering chain; dx^n (if the last axle in the chain is passive) or dx_s^m (if the last axle in the chain is steerable). The rest of the details are straightforward, although the notation is cumbersome. □

Now that it has been shown that the system with virtual trailers can always be converted into extended Goursat normal form, some special cases of the multi-steering trailer system which can be converted into extended Goursat normal form without any prolongation will be examined.

Theorem 22 *If there is only one steering train which has passive axles in it, that is, all the passive axles are contiguous, then the system can be converted into extended Goursat normal form without prolongation.*

Proof. The Pfaffian system has the form,

$$I = \{\alpha^1, \dots, \alpha^k, \omega^1, \dots, \omega^n, \alpha^{k+1}, \dots, \alpha^m\}$$

where the constraints have been arranged in the order in which the axles appear in the chain. Choose $\pi = dx^n$, and note that by Lemma 20, $\{I, dx^n\}$ is integrable.

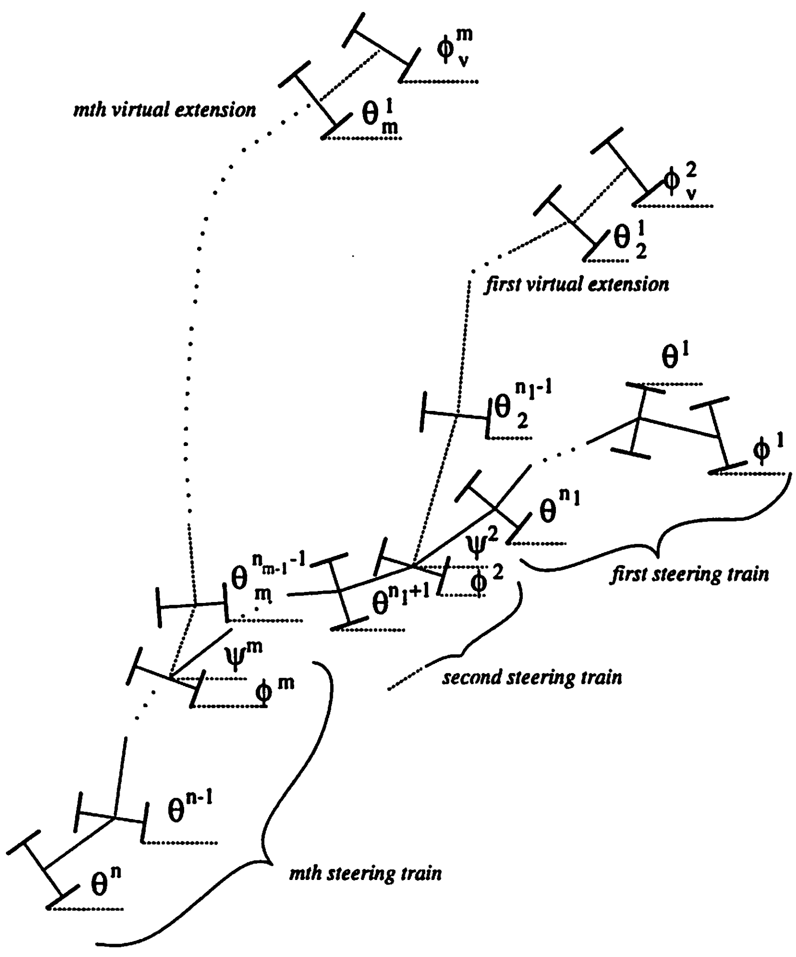
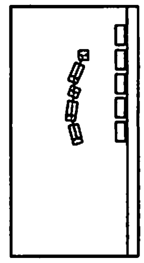


Figure 4.2: A multi-trailer system with n (passive) trailers and m (active) steering wheels, with a virtual extension of n_{j-1} virtual trailers in front of each steering wheel.

The derived flag associated to this case is simply found using either Lemma 16 or equation (4.11). It has the form:

$$\begin{aligned} I^{(1)} &= \{\omega^1, \omega^2, \ldots, \omega^n\} \\ I^{(2)} &= \{\omega^2, \ldots, \omega^n\} \\ &\vdots \\ I^{(n)} &= \{\omega^n\} \\ I^{(n+1)} &= \{0\} \end{aligned}$$



which is reminiscent of the N-trailers case from Chapter 3.

Equation (4.11),

$$d\omega^i \equiv d\theta^i \wedge \sec\theta^i dx^i \quad \mathrm{mod}\ \omega^i$$

combined with equation (4.7),

$$dx^{i-1} \equiv f_{x^{i-1}} dx^i \quad \mathrm{mod}\ \omega^{i-1}, \omega^i$$

gives the congruence

$$d\omega^i \equiv f_{\omega^i} d\theta^i \wedge dx^n \quad \mathrm{mod}\ \omega^i, \omega^{i+1}, \ldots, \omega^n$$

which implies that $\{I^{(i)}, dx^n\}$ is integrable for $i = 1, \ldots, n+1$, and thus by Theorem 18, the system can be converted into extended Goursat normal form. □

The Goursat coordinates are defined by (x^n, y^n), the Cartesian position of the last passive axle, along with $\phi^1, \ldots, \phi^{m-1}$, the angles of the hitches.

Corollary 22.1 (Special cases) *As special cases of the general case described in Theorem 22, the following systems can be converted into Goursat form without prolongation:*

- *There is only one steering wheel, $m = 1$, which by convention is located at the front of the chain. This is the n-trailer problem of Chapter 3.*
- *There is one steering wheel at the front of the chain and another at the end of the chain, as in the firetruck example [7, 53].*
- *All the steering wheels are at the front, that is $n_1 = n_2 = \cdots = n_{m-1} = 0$.*

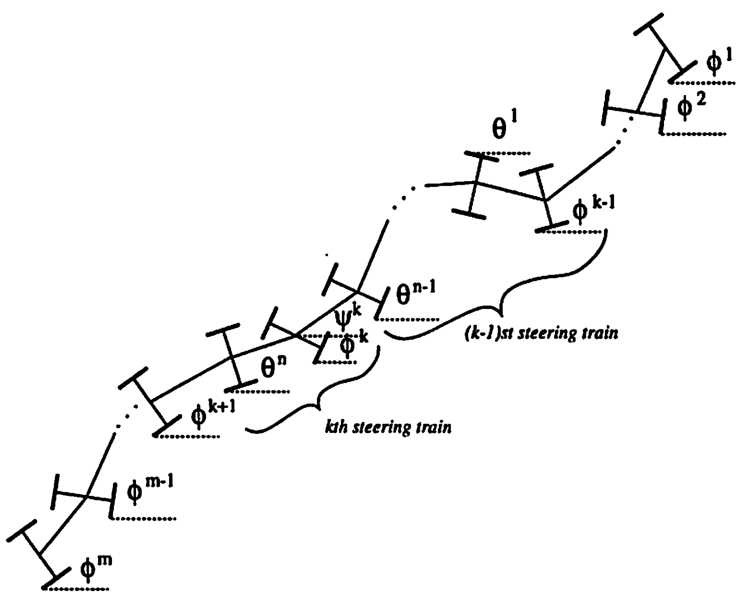


Figure 4.3: A multi-trailer system with n (passive) trailers and m (active) steering wheels, with $n-1$ of the passive trailers in one steering train.

- *All the steering wheels are either at the front or the back of the chain, in a generalized firetruck situation.*
- *All the axles are steerable,* $n = 0$.
- *There is only one passive trailer,* $n = 1$.

The other special case which does not require prolongation to achieve Goursat normal form is slightly more complicated.

Proposition 23 *If there are two sets of passive axles, separated by only one steerable axle, and the set towards the back has only one axle, then the system can be converted to extended Goursat normal form without prolongation.*

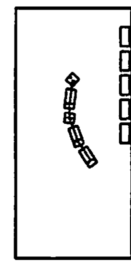
Proof. A sketch of this case is given in Figure 4.3. The Pfaffian system associated with this particular multi-steering system is:

$$I = \{\alpha^1, \alpha^2, \ldots, \alpha^{k-1}, \omega^1, \omega^2, \ldots, \omega^{n-1}, \alpha^k, \omega^n, \alpha^{k+1}, \ldots, \alpha^m\}$$

Note that since the last passive axle is separated from the passive axle in front of it by only one steerable axle, the coordinates x^n and y^n can be written in terms of x^{n-1} and y^{n-1} as

$$x^n = x^{n-1} - l_k \cos\psi^k - L_n \cos\theta^n$$
$$y^n = y^{n-1} - l_k \sin\psi^k - L_n \sin\theta^n$$

Taking the exterior derivative, it can be seen that



$$dx^n = dx^{n-1} + l_k \sin\psi^k d\psi^k + L_n \sin\theta^n d\theta^n$$
$$dy^n = dy^{n-1} - l_k \cos\psi^k d\psi^k - L_n \cos\theta^n d\theta^n$$

Now, rewrite ω^n in terms of dx^{n-1} and dy^{n-1}:

$$\begin{aligned}\omega^n &= \sin\theta^n dx^n - \cos\theta^n dy^n \\ &= \sin\theta^n dx^{n-1} - \cos\theta^n dy^{n-1} + l_k \cos(\psi^k - \theta^n) d\psi^k + L_n d\theta^n\end{aligned}$$

which implies that

$$d\psi^k \equiv \frac{-1}{l_k} \sec(\psi^k - \theta^n)(\sin\theta^n dx^{n-1} - \cos\theta^n dy^{n-1} + L_n d\theta^n) \quad \text{mod } \omega^n$$

and thus, the exterior derivative of ω^n can be written as:

$$\begin{aligned}d\omega^n &= d\theta^n \wedge (\cos\theta^n dx^{n-1} + \sin\theta^n dy^{n-1} + l_k \sin(\psi^k - \theta^n) d\psi^k) \\ &\equiv d\theta^n \wedge ((\cos\theta^n - \sin\theta^n \tan(\psi^k - \theta^n)) dx^{n-1} + (\sin\theta^n + \cos\theta^n \tan(\psi^k - \theta^n)) dy^{n-1} \\ &\qquad -L_n \tan(\psi^k - \theta^n) d\theta^n) \\ &\quad \text{mod } \omega^n \\ &\equiv f_{\omega^n} d\theta^n \wedge dx^{n-1} \\ &\quad \text{mod } \omega^n, \omega^{n-1}\end{aligned}$$

for some function f_{ω^n}. Thus, for $\pi = dx^{n-1}$, $\{I^{(i)}, \pi\}$ is integrable for every i, and the system can be converted into extended Goursat normal form. □

For the configuration of Proposition 23, the Goursat coordinates can be found as follows. Since $\pi = dx^{n-1}$, one of the coordinates will be x^{n-1}. The last nontrivial derived system is $I^{(n-1)} = \{\omega^{n-1}\}$. From the results on the n-trailer system of Chapter 3, it can be shown that y^{n-1} will be another of the coordinates. If there are more than two steering wheels, the steering wheel angles $\psi^1, \ldots, \psi^{k-1}, \psi^{k+1}, \ldots, \psi^{m-1}$ will also be coordinates. However, the final coordinate which defines the Goursat form will not be ψ^k. This can be seen from looking at the one-form ω^n,

$$\begin{aligned}\omega^n &= \sin\theta^n dx^n - \cos\theta^n dy^n \\ &= \sin\theta^n dx^{n-1} - \cos\theta^n dy^{n-1} + l_k\cos(\psi^k - \theta^n)d\psi^k + L_n d\theta^n \\ &= \sin(\theta^n - \theta^{n-1})\sec\theta^{n-1}dx^{n-1} + [l_k\cos(\psi^k - \theta^n)d\psi^k + L_n d\theta^n] \mod \omega^{n-1}\end{aligned}$$

The expression inside the square brackets is integrable, but is obviously not a multiple of $d\psi^k$. After an integral for this expression has been found, that is, some functions f and g such that the expression is equal to fdg, the one-form ω^n can be rewritten as:

$$\omega^n = \sin(\theta^n - \theta^{n-1})\sec\theta^{n-1}dx^{n-1} - fdg$$

The final coordinate which defines the Goursat form will be the function g.

To find such an integral, the method of separation of variables is used. Consider for the moment only the expression under consideration,

$$fdg = l_k\cos(\psi^k - \theta^n)d\psi^k + L_n d\theta^n$$

Perform a change of variables given by

$$\alpha = \frac{\psi^k - \theta^n}{2} \qquad \beta = \frac{\psi^k + \theta^n}{2}$$

and note that in these coordinates, the expression takes the form

$$\begin{aligned}fdg &= l_k\cos 2\alpha\,(d\alpha + d\beta) + L_n\,(d\beta - d\alpha) \\ &= l_k(\cos 2\alpha - L_n/l_k)\,d\alpha + l_k(\cos 2\alpha + L_n/l_k)\,d\beta \\ &= l_k(\cos 2\alpha + L_n/l_k)\left(\frac{\cos 2\alpha - L_n/l_k}{\cos 2\alpha + L_n/l_k}\,d\alpha + d\beta\right)\end{aligned}$$

Consider first the simplest case, where the lengths of the two hitches are equal, or $L_n/l_k = 1$. Using the identity,

$$\frac{\cos 2\alpha - 1}{\cos 2\alpha + 1} = -\tan^2\alpha,$$

the expression above can be rewritten as

$$fdg = l_k(\cos 2\alpha + L_n/l_k)[-\tan^2\alpha\,d\alpha + d\beta]$$

Now the quantity inside the square brackets is exact, and can be integrated to give the function g

$$g = -\tan\alpha + \alpha + \beta$$

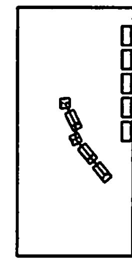
or, in the original coordinates,

$$g = \psi^k - \tan\frac{(\psi^k - \theta^n)}{2}$$

and

$$f = l_k(\cos(\psi^k - \theta^n) + 1)$$

Now consider the case when $L_n \neq l_k$, and let $\ell = L_n/l_k$. The expression fdg has the form



$$fdg = l_k(\cos 2\alpha + \ell)\left[\frac{\cos 2\alpha - \ell}{\cos 2\alpha + \ell}\, d\alpha + d\beta\right]$$

Consider only the dg part of the expression,

$$\begin{aligned} dg &= \frac{\cos 2\alpha - \ell}{\cos 2\alpha + \ell}\, d\alpha + d\beta \\ &= \frac{-2\ell}{\cos 2\alpha + \ell}\, d\alpha + d\alpha + d\beta \end{aligned}$$

which can be integrated by consulting a table of integrals. Note that the result depends on whether the ratio of the two hitches involved L_n/l_k is greater than or less than one,

$$g = \psi^k + \begin{cases} \frac{2}{\sqrt{\ell^2-1}} \arctan \frac{\sqrt{\ell^2-1}\tan\alpha}{1+\ell} & \text{if } \ell > 1 \\ \frac{1}{\sqrt{1-\ell^2}} \log\left(\frac{\sqrt{1-\ell^2}\tan\alpha+1+\ell}{\sqrt{1-\ell^2}\tan\alpha-1-\ell}\right) & \text{if } \ell < 1 \end{cases}$$

As before, f has the form:

$$f = l_k(\cos(\psi^k - \theta^n) + \ell)$$

All configurations which do not satisfy either Theorem 22 or Proposition 23 require prolongation to be converted into extended Goursat normal form. The minimum dimension of the prolongation can be computed as follows. Recall that there are a total of n passive trailers and m steerable axles, and let k equal the index of the first steerable axle which has no passive trailers behind it. That is, $n_k = n_{k+1} = \cdots = n_m = n$ and $n_{k-1} < n$. There are two possible cases:

1. If $n_{k-1} = n - 1$, then the minimum dimension of prolongation is $n_1 + \cdots + n_{k-2}$.

2. Otherwise, a prolongation of dimension $n_1 + \cdots + n_{k-1}$ is needed to convert the system into extended Goursat normal form.

Now some specific multi-steering mobile robot systems will be considered and it will be shown how their associated Pfaffian systems satisfy the extended Goursat conditions.

Example 1 (Two, Three, or Four Axles) It is a simple exercise in combinatorics to check that all of the possible configurations with two or three axles and one, two or three steering wheels satisfy the conditions of Theorem 22. Note particularly that the firetruck example [7], sketched in Figure 4.4, satisfies these conditions with $n = 1$.

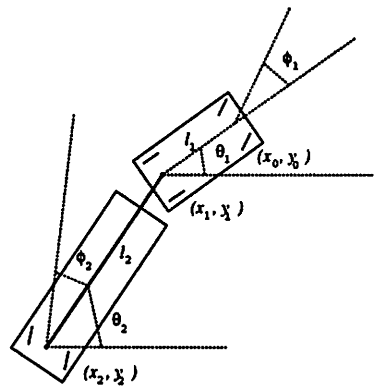


Figure 4.4: A sketch of the firetruck, with steering wheels on the front and back axles.

In addition, it can be shown that all except one configuration of a system with four axles will satisfy the conditions of Theorem 22. The exception is $m = 2$, two steerable axles, two passive axles, alternating. That is, the first and third axles are steerable, and the second and fourth axles are passive. This situation would arise if a car were towing another car and both of the cars had drivers at the steering wheels. This example satisfies Proposition 23, and thus can be converted into Goursat form without prolongation.

The 5-axle system with two steering wheels is the lowest-dimensional case where interesting things begin to happen.

Example 2 (5-axle, 1-4 steering) First consider the 5-axle system with the first and fourth axles steerable, as sketched in Figure 4.5.

The constraints are that each axle rolls without slipping:

$$\omega^i = \sin\theta^i dx^i - \cos\theta^i dy^i \quad i = 1,2,3 \qquad \alpha^j = \sin\phi^j dx_s^j - \cos\phi^j dy_s^j \quad j = 1,2$$

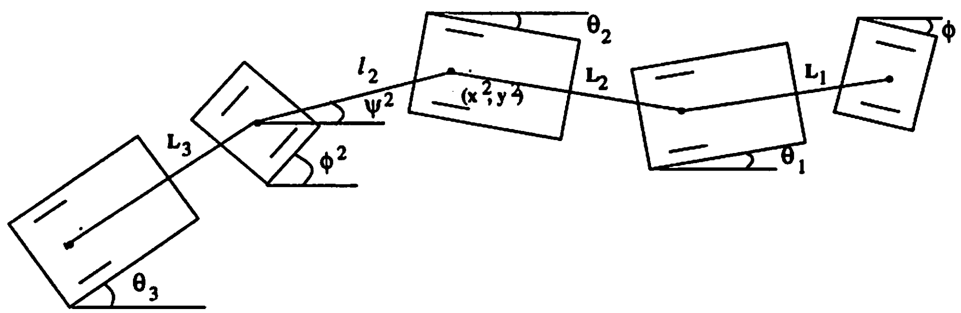
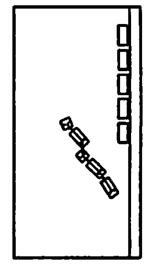


Figure 4.5: A 5-axle trailer system with the first and fourth axles steerable.



The Pfaffian system is thus $I = \{\alpha^1, \omega^1, \omega^2, \alpha^2, \omega^3\}$ and a complement to this system is $\{d\phi^1, d\phi^2, dx^2\}$. This basis is adapted to the the derived flag,

$$\begin{array}{rl} I = & \{\alpha^1,\ \omega^1,\ \omega^2,\ \ \alpha^2,\ \omega^3\} \\ I^{(1)} = & \quad\ \{\omega^1,\ \omega^2,\qquad\ \ \omega^3\} \\ I^{(2)} = & \qquad\quad \{\omega^2\} \\ I^{(3)} = & \qquad\quad\ \{0\} \end{array}$$

and it can be checked that each $\{I^{(i)}, dx^2\}$ is integrable. The coordinates which put the system into Goursat form are the following:

$$\begin{aligned} z^0 &= x^2 \\ z_1^1 &= y^2 \\ z_1^2 &= \begin{cases} \psi^2 - \tan\frac{(\psi^2-\theta^3)}{2} & L_3 = l_2 \\ \psi^2 + \frac{2L_3/l_2}{\sqrt{(\frac{L_3}{l_2})^2-1}} \arctan\left(\frac{\sqrt{(\frac{L_3}{l_2})^2-1}\tan(\frac{\psi^2-\theta^3}{2})}{1+L_3/l_2}\right) & L_3 > l_2 \\ \psi^2 + \frac{L_3/l_2}{\sqrt{1-(\frac{L_3}{l_2})^2}} \log\left(\frac{\sqrt{1-(\frac{L_3}{l_2})^2}\tan(\frac{\psi^2-\theta^3}{2})+1+L_3/l_2}{\sqrt{1-(\frac{L_3}{l_2})^2}\tan(\frac{\psi^2-\theta^3}{2})-1-L_3/l_2}\right) & L_3 < l_2 \end{cases} \end{aligned}$$

Note the dependence on the relative lengths of the hitches in the system. The remaining coordinates are defined by the relationships

$$\begin{aligned} z_k^1 &= \dot{z}_{k-1}^1/\dot{z}^0 \quad k = 2, \ldots, 4 \\ z_k^2 &= \dot{z}_{k-1}^2/\dot{z}^0 \quad k = 2, 3 \end{aligned}$$

Of course, by Theorem 19, this system can also be converted into an extended Goursat normal form using a prolongation of dimension two, and the coordinates in this case are given by:

$$\zeta^0 = x^3 \qquad \zeta_1^1 = y^3 \qquad \zeta_1^2 = \psi^2$$

together with the relations

$$\begin{aligned} \zeta_k^1 &= \dot{\zeta}_{k-1}^1/\dot{\zeta}^0 \quad k = 2,\dots,5 \\ \zeta_k^2 &= \dot{\zeta}_{k-1}^2/\dot{\zeta}^0 \quad k = 2,\dots,4 \end{aligned}$$

The two sets of coordinates (z^0, z_1^1, z_1^2) and $(\zeta^0, \zeta_1^1, \zeta_1^2)$ defined above are valid flat outputs for this system, in the sense that all the states and inputs to the system can be found by taking derivatives of these quantities. More differentiations will be required for the ζ coordinates.

Both coordinate transformations have two types of singularities. Because of the division by the derivative of z^0 (or ζ^0), whenever this coordinate is constant (corresponding to $\cos\theta^2$ or $\cos\theta^3$ respectively being zero), the transformations will be undefined. This type of singularity can be avoided by choosing a different coordinate chart at the singular point (interchanging x and y for example). A singularity also occurs when the angle between two adjacent axles is equal to $\pi/2$; at this point, some of the codistributions in the derived flag will lose rank. The derived flag is not defined at these points; nor is the transformation. The methods described herein will not work for controlling the multi-steering trailer system when the trailers must go through such a jack-knifed configuration.

Example 3 (5-axle, 1-3 steering) The only instance of the 5-axle trailer system with two steering wheels which satisfies neither Theorem 22 or Proposition 23 has the first and third axles steerable, as shown in Figure 4.6.

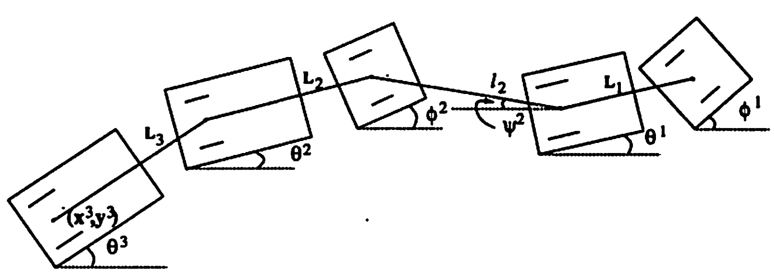


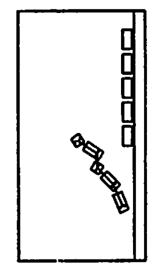
Figure 4.6: A 5-axle trailer system with the first and third axles steerable. This is the only configuration of the 5-axle system with two steering wheels which does not satisfy the conditions for converting to extended Goursat normal form without prolongation.

The constraints are that each axle roll without slipping:

$$\omega^i = \sin\theta^i dx^i - \cos\theta^i dy^i \quad i = 1,2,3 \qquad \alpha^j = \sin\phi^j dx^j_s - \cos\phi^j dy^j_s \quad j = 1,2$$

The Pfaffian system is $I = \{\alpha^1, \omega^1, \alpha^2, \omega^2, \omega^3\}$, and a complement to the system is given by $\{d\phi^1, d\phi^2, dx^3\}$.

By Lemma 16, the derived flag has the form



$$\begin{aligned} I &= \{\alpha^1,\ \omega^1,\ \alpha^2,\ \omega^2,\ \omega^3\} \\ I^{(1)} &= \{\omega^1,\ \omega^2,\ \omega^3\} \\ I^{(2)} &= \{\omega^3\} \\ I^{(3)} &= \{0\} \end{aligned}$$

In order to have $\{I^{(2)}, \pi\}$ integrable, π must be dx^3 (mod ω^3). This will also give $\{I^0, dx^3\}$ integrable by Lemma 20, but a simple check will show that $\{I^{(1)}, dx^3\}$ is *not* integrable. Thus, as predicted by the theorems, this system does not satisfy the conditions for conversion to extended Goursat normal form without prolongation.

The system I can be prolonged by differentiation, adding the additional form $\nu = d\phi^2 - v\, dx^3$. The new coordinate v can be thought of as the tangent of the angle of the virtual axle that is added to the system in Theorem 19. The derived flag of the augmented system is:

$$\begin{aligned} J &= \{\alpha^1,\ \omega^1,\ \nu,\ \alpha^2,\ \omega^2,\ \omega^3\} \\ J^{(1)} &= \{\omega^1,\ \alpha^2,\ \omega^2,\ \omega^3\} \\ J^{(2)} &= \{\omega^2,\ \omega^3\} \\ J^{(3)} &= \{\omega^3\} \\ J^{(4)} &= \{0\} \end{aligned}$$

and the systems $\{J^{(k)}, dx^3\}$ are integrable for all k. Thus, the prolonged system J can be converted into extended Goursat normal form.

For the case of a 5-axle system with three steering wheels (two passive trailers), if the two passive trailers are connected we know from Theorem 22 that the system can be converted into extended Goursat normal form without prolongation. If the two passive trailers are separated by only one steerable axle, then we apply Proposition 23. The only configuration which does not satisfy one of these two conditions has the passive axles in the second and fifth positions, and this configuration will again require prolongation to convert it to extended Goursat normal form.

4.5 Steering multi-input chained form systems

As was stated before, extended Goursat normal form is the dual of multi-input single-generator chained form. This section is devoted to discussing several different methods for finding paths for multi-input chained form systems. These methods are the direct extension of the methods considered in the previous chapter for two-input chained form systems.

In this section no particular system of trailers or nonlinear equations will be considered, only the multi-chained form equation:

$$
\begin{array}{llll}
\dot{z}_0^0 = u^0 & \dot{z}_0^1 = u^1 & \dot{z}_0^2 = u^2 \quad \cdots & \dot{z}_0^m = u^m \\
 & \dot{z}_1^1 = z_0^1 u^0 & \dot{z}_1^2 = z_0^2 u^0 & \dot{z}_1^m = z_0^m u^0 \\
 & \vdots & \vdots & \vdots \\
 & \dot{z}_{n_1+1}^1 = z_{n_1}^1 u^0 & \vdots & \vdots \\
 & & \dot{z}_{n_2+1}^2 = z_{n_2}^2 u^0 \quad \ddots & \vdots \\
 & & & \dot{z}_{n_m+1}^m = z_{n_m}^m u^0 .
\end{array}
\tag{4.19}
$$

The indices are slightly changed (actually inverted) from the definition of extended Goursat normal form (4.14), but it is clear that the equations are the same.

The problem that is considered in this section is: given a system of equations in the above form, and a desired initial and final state, find inputs $\{u^i(t) \; : \; t \in [0,T), \; i = 0,\ldots,m\}$ which will steer the system from the initial state to the final state.

4.5.1 Steering with polynomial inputs

One approach to the point-to-point steering problem is to hold the first input u^0 constant and identically equal to one over the entire trajectory. The time needed to steer is then determined from the change in the z_0^0 coordinate,

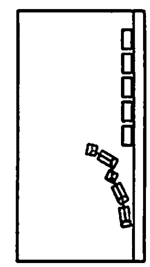
$$T = (z_0^0)^f - (z_0^0)^i. \tag{4.20}$$

The parameters for the remaining inputs are coefficients of a Taylor polynomial,

$$
\begin{aligned}
u^1 &= a_0 + a_1 t + \ldots + a_{n_1+1} t^{n_1+1} \\
u^2 &= b_0 + b_1 t + \ldots + b_{n_2+1} t^{n_2+1} \\
&\vdots \\
u^m &= \nu_0 + \nu_1 t + \ldots + \nu_{n_m+1} t^{n_m+1}
\end{aligned}
\tag{4.21}
$$

with the number of parameters on each input chosen to be equal to the number of states in its chain. The chained form equations can be integrated symbolically and the input parameters $a_j, b_j, \ldots, \nu_j$ can be found in terms of the initial and final states. This is a fairly simple procedure since all of the equations that need to be solved are linear. A symbolic manipulation program can be used quite readily to do this.

Of course, if $T = 0$ from equation (4.20), then this method will not work. This case corresponds in the physical system to the "parallel-parking" direction, or no change in the x coordinate. The easiest way to remedy this situation is to first choose an intermediate point and then plan the path in two pieces, as was described earlier in Section 3.4.2



4.5.2 Steering with piecewise constant inputs

This steering method was originally inspired by multirate digital control [32], but is most easily understood in terms of motion planning simply as piecewise constant inputs. The first input u^0 is chosen to be constant over the entire trajectory. This choice will ensure the linearity of the equations that need to be solved for the other input parameters, as well as generate "nice" trajectories (since this input is related to the driving input of the multi-trailer system, a constant u^0 will usually transform to a uni-directional velocity, or equivalently no backups).

The other inputs are chosen to be piecewise constant, and to ensure that the resulting equations have a solution, each input should have at least as many switches as there are states in its chain. There will need to be the largest number of switches on the m^{th} input since it will always have the longest chain.

The time for the trajectory can be chosen arbitrarily as T. As stated before, the first input is chosen to be constant over the entire trajectory,

$$u^0(t) = u_D^0 \quad \text{for } t \in [0, T)$$

where the magnitude of the first input is chosen such that the first chained form state will go from its initial to its final position over the time period,

$$u_D^0 = \left[(z_0^0)^f - (z_0^0)^i\right]/T \,. \tag{4.22}$$

The other inputs are chosen to be piecewise constant. Let the switching times be chosen as

$$0 = t_0^j < t_1^j < \ldots < t_{n_j+2}^j = T \,,$$

where $n_j + 2$ switching times are needed for each input since there are $n_j + 2$ states in the j^{th} chain. There are many different methods available for choosing these times. They are most commonly chosen so that for the m^{th} input, which has the most switching times, the holding times will be equal. The switching times for the other inputs are then chosen to be some subset of the switching times for the m^{th} input. The j^{th} input will be of the form:

$$u^j(t) = u_D^{j,k} \quad \text{for } t \in [t_k, t_{k+1}) .$$

When the chained form equations are integrated using these input values, the final state can be expressed in terms of the inputs and the initial state as

$$\begin{bmatrix} z_0^j \\ z_1^j \\ \vdots \\ z_{n_j+1}^j \end{bmatrix}(T) = \Delta^j(u_D^0, z^j(0)) \begin{bmatrix} u_D^{j,0} \\ u_D^{j,1} \\ \vdots \\ u_D^{j,n_j+1} \end{bmatrix}$$

where the matrices Δ^j are assured to be nonsingular whenever the first input u_D^0 is nonzero [32].

Similarly to the previous section, if the first input does come out to be zero from equation (4.22), then a slight modification of this method is necessary. A multirate input can also be added on u^0, using at least two time periods, or an intermediate point can be chosen and the path can be planned in two steps. This case corresponds in the physical system to the parallel-parking direction.

The inputs which are chosen to be piecewise constant are not the velocities of the steering wheels, but the chained form inputs, which are nonlinear functions of the states and the *virtual* steering inputs.

4.5.3 Steering with sinusoidal inputs

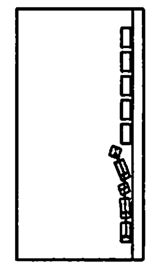
A method for steering multi-chained systems with sinusoids was proposed in [7]. This method is step-by-step and uses one step to steer each level of the chain (although the states of all chains at the same level can be steered simultaneously). Since the longest chain has length $n_m + 2$, this is the number of steps that will be needed.

The algorithm is sketched as follows:

Step 0 Steer the top-level coordinates, $\{z_0^j,\ j = 0, \ldots, m\}$ by choosing constant values for $u^0, u^1, \ldots, u^m$ on the time interval $[0, T)$.

Step 1 Steer the coordinates at the first level down by choosing a sinusoid on u^0 and out-of-phase sinusoids on u^j,

$$\begin{aligned} u^0 &= \alpha \sin \omega t \\ u^1 &= \beta \cos \omega t \\ u^2 &= \gamma \cos \omega t \\ &\vdots \\ u^m &= \nu \cos \omega t \end{aligned}$$



over a time period $[T, 2T)$, with appropriate choice of $\alpha, \beta, \ldots, \nu$ so that at time $2T$, the states $\{z_1^j,\ j = 1, \ldots, m\}$, have achieved their desired final values.

Step k ($k = 2, \ldots, n_m + 1$). Steer the coordinates at the k^{th} level from the top. If $n_i < k \leq n_{i+1}$, then only chains $i+1, \ldots, m$ will be affected. Again, choose a single frequency sinusoid on the first input, but now choose multiple frequency sinusoids on the other inputs:

$$\begin{aligned} u^0 &= \alpha \sin \omega t \\ u^1 &= 0 \\ &\vdots \\ u^i &= 0 \\ u^{i+1} &= \zeta \cos k\omega t \\ &\vdots \\ u^m &= \nu \cos k\omega t \end{aligned}$$

over a time period $[kT, (k+1)T)$, with appropriate choice of $\zeta, \ldots, \nu$ so that at time kT, the states $\{z_k^j,\ j = i+1, \ldots, m\}$, are at their desired final values.

After each step k, the states closer to the top of the chain than level k will have returned to their values after the previous step $(k-1)$. The states lower in the chain than level k will move as a result of the inputs at step k by some amount; these are disregarded because those states are steered to their desired final values in subsequent iterations.

Although this method works well, and the magnitudes of the sinusoids are simple to solve for, the algorithm can be tedious in practice because of the many steps that are needed.

The trajectories that are generated consist of many segments and do not always follow a direct path between the start and goal.

Therefore, an "all-at-once" sinusoids method, which is an extension of that detailed in Section 3.4.1 for the two input single chain case, was proposed in [58]. Only one step is needed, all of the necessary frequencies are put into the inputs.

$$\begin{aligned} u^0 &= \alpha_0 + \alpha \sin \omega t \\ u^1 &= \beta_0 + \beta_1 \cos \omega t + \cdots + \beta_{n_1+1} \cos(n_1+1)\omega t \\ &\vdots \\ u^m &= \nu_0 + \nu_1 \cos \omega t + \cdots + \nu_{n_m+1} \cos(n_m+1)\omega t \; . \end{aligned}$$

The input parameters are found in the same manner as in the other methods; that it, the chained form equations are integrated symbolically, evaluated at time T, and the parameters are solved for as a function of the initial and final states.

The main drawback to this approach is that there will be some interference between the levels (although not between chains) and solving for the input parameters will require solving nonlinear algebraic equations. In low-dimensional cases, this does not appear to be a problem for a symbolic manipulator. It can be shown (in a manner similar to the proof of Proposition 14) that solutions are guaranteed to exist at least locally.

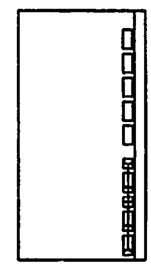
When this method is implemented on a multi-steering trailer system, the first input, which always goes through one period, will transform back to the driving input, which will usually change direction (at least one backup). This seems to work well when parallel-parking type maneuvers are desired. The free parameter α can be adjusted to change the distance that the trailer system drives forward before it backs up.

4.6 Path planning for the multi-steering trailer system

Once the kinematic equations are in multi-input chained form, the system can be steered by one of the algorithms discussed in Section 4.5. As an illustration, consider the parallel parking maneuver shown in Figure 4.7 for the five-axle, two-steering system described in Example 3. The system parameters were chosen to be $n = 3$ (three passive axles), $m = 2$ (two steering wheels), and the lengths of the hitches as $L_1^1 = L_2^2 = L_3^2 = 5$, and $L_2^1 = 3$.

Polynomial inputs in the chained form equations are used to plan a trajectory from an initial point of $(x, y) = (0, 20)$ to a final point of $(x, y) = (0, 0)$, where (x, y) are the coordinates of the midpoint of the last axle, and all of the body angles aligned with the horizontal axis in both the initial and final configurations.

As noted in Section 4.5.1, polynomial inputs are not immediately suited to this type of trajectory since the time needed to steer the system, computed from equation (4.20), would come out to be zero and the algorithm would fail. Therefore the trajectory was planned in two steps, choosing an intermediate point $(x, y) = (30, 10)$. The virtual angles were chosen equal to zero in both the initial and final states, and the virtual hitch length were chosen as $L_1^2 = 1$. The procedure is first to transform the initial and final states into the chained form coordinates. Using the polynomial inputs methods discussed in Section 4.5.1, the chained form inputs needed to steer the system are found. These inputs can then be transformed back to the original coordinates to find the virtual inputs, and the real inputs can finally be calculated from this.



The simulation was performed on the system in the chained form coordinates, then the inverse coordinate transformation was used on the simulation data to obtain the trajectory in the original coordinates. A movie animation was made of this trajectory; scenes from this movie are shown in Figure 4.7 and the complete movie can be viewed in the margins of this chapter. The path taken by the virtual axle is not shown.

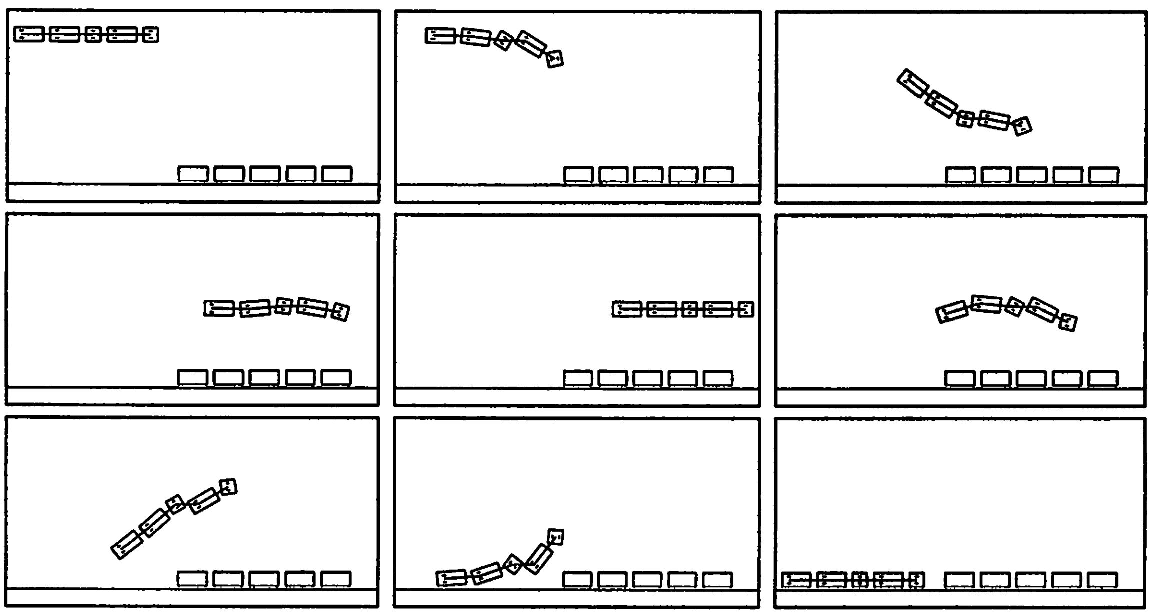


Figure 4.7: A parallel-parking trajectory for the five-axle, two-steering system. The planning algorithm as described in this paper does not account for obstacle avoidance; however, it does plan "nice" paths which may can be used in conjunction with an obstacle-avoidance algorithm to achieve a complete solution to the path-planning problem.

Chapter 5

Control Systems as Pfaffian Systems

This chapter gives some preliminary results on the use of exterior differential systems to analyze control systems. Some of the first efforts in this area can be found in the dissertation of Sluis [46] and a paper by Gardner and Shadwick [20]. Although this area of research is rather recent, the tools available in exterior differential systems are very appropriate to the study of control systems, particularly the problems of trajectory generation (or path planning) and linearization. Most of the results which appear in this chapter are taken from [56].

A control system is a system of underdetermined differential equations:

$$\dot{x} = f(x, u, t)$$

where the state $x \in \mathbb{R}^n$ and the derivative of the state is taken with respect to time $t \in \mathbb{R}$. The control inputs $u \in \mathbb{R}^m$ are assumed to be freely specifiable (the problems associated with bounded controls will not be considered here). The state trajectories $x(t)$ as functions of time are of interest. They can be studied as integral curves of an associated Pfaffian system, formulated as follows:

Definition 6 (Control System.) *A* control system $\dot{x} = f(x, u, t)$ *generates a Pfaffian system I on $\mathbb{R}^{n+m+1}$*

$$I = \{dx_i - f^i(x, u, t)\, dt : i = 1, \ldots, n\} \tag{5.1}$$

with independence condition dt and complement $\{du_1, \ldots, du_m, dt\}$.

Any Pfaffian system I of codimension $m+1$ on $\mathbb{R}^{n+m+1}$ with coordinates (x, u, t) can be called a control system if it has a set of generators of the form (5.1).

5.1 Exact linearization

Brunovsky [5] showed that any controllable linear system $\dot{x} = Ax + Bu$ with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ can be converted to a canonical form given by

$$\begin{array}{cccc} \dot{x}_1^1 = u_1 & \dot{x}_1^2 = u_2 & \cdots & \dot{x}_1^m = u_m \\ \dot{x}_2^1 = x_1^1 & \dot{x}_2^2 = x_1^2 & \cdots & \dot{x}_2^m = x_1^m \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \dot{x}_{k_m}^m = x_{k_m-1}^m \\ \vdots & \dot{x}_{k_2}^2 = x_{k_2-1}^2 & & \\ \dot{x}_{k_1}^1 = x_{k_1-1}^1 & & & \end{array} \tag{5.2}$$

with $n = k_1 + \cdots + k_m$. Thus, any control system $\dot{x} = f(x, u, t)$ which can be converted to a linear form $\dot{\xi} = A\xi + Bv$ by a coordinate transformation $\xi = \phi(x)$ and state feedback $v = \nu(u, x)$ can be converted into Brunovsky's normal form (5.2). Since Brunovsky linear form for a control system is a special case of extended Goursat normal form (4.14) with $dz^0 = dt$, the theorems for transforming to Goursat form can be specialized to give conditions for exact linearization of control systems.

Theorem 24 (Exact Linearization [20]) *If a control system I defined on $\mathbb{R}^{n+m+1}$ has a set of generators $\{\alpha_i^j : i = 1, \ldots, s_j; j = 1, \ldots, m\}$ such that for all j,*

$$\begin{array}{lll} d\alpha_i^j \equiv -\alpha_{i+1}^j \wedge dt & \bmod I^{(s_j-i)} & i = 1, \ldots, s_j - 1 \\ d\alpha_{s_j}^j \not\equiv 0 & \bmod I & \end{array} \tag{5.3}$$

then there exists a set of coordinates z such that I is in Brunovsky normal form,

$$I = \{dz_i^j - z_{i+1}^j dt : i = 1, \ldots, s_j; j = 1, \ldots, m\}.$$

An algorithm for converting systems to Brunovsky normal form is also given in [20], and it is shown that if the control system is time-invariant and affine in the inputs, then the resulting feedback transformation is also autonomous and input-affine.

The control systems version of Theorem 18 is given by

Theorem 25 (Exact Linearization [46]) *A control system I can be converted to linear form if and only if $\{I^{(k)}, dt\}$ is integrable for every k.*

Another version of this theorem is given in [1] with slightly different notation.

5.2 Linearization by time-scaling

By way of example, consider a control system which is not linearizable but can be converted to Goursat normal form. The transformation scales time by a function of one of the states.

Example 4 (Goursat normal form for a control system) Consider the single-input control system [12],

$$\begin{aligned}\dot{x}_1 &= x_2 + {x_3}^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u\end{aligned}$$

This control system generates a Pfaffian system,

$$I = \{dx_1 - (x_2 + {x_3}^2)dt, dx_2 - x_3 dt, dx_3 - u dt\} \tag{5.4}$$

I is of codimension two on $\mathbb{R}^5$ with coordinates (x, u, t). The derived flag of I is

$$\begin{aligned}I &= \{\alpha^1, \alpha^2, \alpha^3\} \\ I^{(1)} &= \{\alpha^1, \alpha^2\} \\ I^{(2)} &= \{\alpha^1\} \\ I^{(3)} &= \{0\}\end{aligned}$$

where the one-forms adapted to the derived flag are given by

$$\begin{aligned}\alpha^1 &= dx_1 - 2x_3 dx_2 + ({x_3}^2 - x_2)dt \\ \alpha^2 &= dx_2 - x_3 dt \\ \alpha^3 &= dx_3 - u dt\end{aligned}$$

Note that this is not the basis of (5.4) which generated I. Since $\{I^{(2)}, dt\}$ is not integrable, Theorem 25 implies that the system is not feedback linearizable.

The Goursat congruences (3.6), however, *are* satisfied, for $\pi = d\tau = dt - 2dx_3$:

$$\begin{aligned} d\alpha^1 &= \alpha^2 \wedge d\tau \\ d\alpha^2 &= c(u)\, \alpha^3 \wedge d\tau \\ d\alpha^3 &= c(u)\, du \wedge d\tau \mod \alpha^3 \end{aligned}$$

Thus, there does exist a transformation $\Phi(x,u,t) = (z,v,\tau)$ to Goursat normal form, which is given by

$$\begin{aligned} \tau &= t - 2x_3 \\ v &= \frac{u}{1-2u} \\ z_1 &= x_3 \\ z_2 &= x_2 - {x_3}^2 \\ z_3 &= x_1 - 2x_2x_3 + \frac{2}{3}{x_3}^3 \end{aligned}$$

and it is easily checked that

$$\begin{aligned} \frac{dz_1}{d\tau} &= v \\ \frac{dz_2}{d\tau} &= z_1 \\ \frac{dz_3}{d\tau} &= z_2 \end{aligned}$$

The difference between this form and the Brunovsky normal form of (5.2) is that time has been scaled by a function of one of the states. Not all trajectories of the scaled system correspond to feasible trajectories of the original control system; it must be carefully checked that the time coordinate evolves in an appropriate manner.

The problem of feedback linearization by time-scaling was also studied in [45]. The authors considered an autonomous, input-affine control system and gave a modified set of conditions for feedback linearization allowing for the fact that time could be scaled by a function of the states. The Goursat formulation gives a simple set of conditions to check if a time-scaled version of a system can be linearized.

Proposition 26 *A control system I of the form* (5.1) *can be feedback linearized by time-scaling if and only if*

1. *The system is controllable, that is $I^{(N)} = \{0\}$ for some N,*

2. *there exists a function τ such that $\{I^{(k)}, d\tau\}$ is integrable for $k = 0, \ldots, N-1$,*

3. The new coordinate τ which is the scaled version of time must not be independent of time, $\frac{\partial \tau}{\partial t} \neq 0$.

Just as in the case of conversion to Goursat form, the candidate one-form $d\tau$ will be determined by the last nontrivial derived system, and it is a matter of algebra to check whether the other derived systems, together with $d\tau$, are also integrable.

5.3 Dynamic extension

Linearizing control system using dynamic extensions is a problem that has been studied extensively. The exterior differential systems framework allows the problem to be viewed from a slightly different angle. A *dynamic extension* of a control system is an augmented system with integrators added to the inputs; for example, a simple first-order dynamic extension is given by:

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{u}_k &= v \end{aligned}$$

where an integrator is added to the k^{th} input channel. .

The prolongation by differentiation which was defined in Section 4.3 is exactly the dual of dynamic extension in the language of forms. Thus, the control systems version of Theorem 19 can be stated as:

Theorem 27 (Linearization by dynamic extension) *Consider a control system I on $\mathbb{R}^{n+m+1}$ with coordinates (x, u, t), independence condition dt, and complement*

$$\{du_1, \dots, du_m, dt\}$$

The system I is linearizable by dynamic extension if (and only if) there exists a prolongation by differentiation of dimension $b = b_1 + \cdots + b_m$ such that the augmented system

$$\begin{aligned} J = \{\alpha^i = dx_i - f^i(x, u)dt : \quad & i = 1, \dots, n; \\ \beta_j^k = du_j^{k-1} - u_j^k dt : \quad & j = 1, \dots, m; k = 0, \dots, b_i\} \end{aligned}$$

on $\mathbb{R}^{n+m+b+1}$ satisfies the condition $\{J^{(k)}, dt\}$ is integrable for every k.

Proof. Apply Theorem 25 to the extended system J. □

This theorem is similar to the one stated by Charlet, Lévine, and Marino [12] which gave sufficient conditions for linearizing systems by dynamic extension. Their conditions also relied on the existence of some integers b_i which are the number of integrators added to the i^{th} input channel. However, the existence of a dynamic extension of order $b = (b_1, \ldots, b_m)$ which is linearizable does not imply that the conditions of their theorem are satisfied for that b; whereas if there exists a dynamic extension of order $b = (b_1, \ldots, b_m)$ which can be linearized, the conditions of Theorem 27 will always be satisfied for that b. Unfortunately, no bounds are yet known on the number of extensions that must be checked to find out if the system is linearizable by this technique.

The problem of linearization by dynamic extension has also been studied by Aranda-Bricaire, Moog, and Pomet [1] using differential forms and differential algebra. They also give necessary and sufficient conditions, which can be fairly tedious (if not actually impossible) to check in practice.

A simple example will be presented to show how Theorem 27 can be applied to linearize control systems using dynamic extension.

Example 5 **([12])** Consider a 4-state, 2-input control system:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u_1 \\
\dot{x}_3 &= u_2 \\
\dot{x}_4 &= x_3 - x_3 u_1
\end{aligned}$$

The corresponding Pfaffian system on $\mathbb{R}^7$ is

$$I = \{dx_1 - x_2 dt, dx_2 - u_1 dt, dx_3 - u_2 dt, dx_4 - (x_3 - x_3 u_1)dt\} \tag{5.5}$$

with independence condition dt and complement $\{du_1, du_2, dt\}$. The derived flag has the form:

$$\begin{aligned}
I &= \{\alpha^1, \alpha^2, \alpha^3, \alpha^4\} \\
I^{(1)} &= \{\alpha^1, \alpha^4\} \\
I^{(2)} &= \{0\}
\end{aligned}$$

the one-forms α^i which are adapted to the derived flag are not the same as those of (5.5) which generated I,

$$\begin{aligned}
\alpha^1 &= dx_1 - x_2 dt \\
\alpha^2 &= dx_2 - u_1 dt \\
\alpha^3 &= dx_3 - u_2 dt \\
\alpha^4 &= dx_4 + x_3 dx_2 - x_3 dt
\end{aligned}$$

The structure equations are fairly simple to find,

$$\begin{aligned} d\alpha^1 &= -\alpha^2 \wedge dt \\ d\alpha^2 &= -du_1 \wedge dt \\ d\alpha^3 &= -du_2 \wedge dt \\ d\alpha^4 &= -\alpha^2 \wedge \alpha^3 + (u_1 - 1)\, \alpha^3 \wedge dt - u_2\, \alpha^2 \wedge dt \end{aligned}$$

note that $\{I^{(1)}, dt\}$ is not integrable, thus the system is not linearizable by static state feedback.[1]

Consider a prolongation by differentiation of I defined by

$$J = \{I, \omega = du_1 - v_1 dt\}$$

corresponding to adding an integrator to the first input channel. The derived flag is now

$$\begin{aligned} J &= \{\alpha^1,\ \alpha^2,\ \omega,\ \alpha^3,\ \tilde{\alpha}^4\} \\ J^{(1)} &= \{\alpha^1,\ \alpha^3,\ \tilde{\alpha}^4\} \\ J^{(2)} &= \{\alpha^1\} \\ J^{(3)} &= \{0\} \end{aligned}$$

where the basis adapted to the derived flag has changed slightly, with $\tilde{\alpha}^4 = dx_4 + (u_1 x_3 - x_3)dt$. It is now easy to see just from the expressions of the forms adapted to the derived flag that $\{I^{(i)}, dt\}$ is integrable for $i = 0, 1, 2$, and thus the extended system can be converted to Brunovsky form.

[1] The system is, however, linearizable by time-scaling. Let $\tau = t - x_2$, and note that $\{I^{(i)}, d\tau\}$ is integrable for $i = 1, 2$.

Chapter 6

Conclusions and Future Work

This dissertation focused on the problem of path planning for robotic systems with nonholonomic velocity constraints. Several systems were considered, including a car towing n trailers, a two-axle mining vehicle, and a multi-steering trailer system. The systems were studied from the point of view of their velocity constraints. The constraints were written as one-forms on the configuration manifold, and generated a special type of exterior differential system called a Pfaffian system. Integral curves to the Pfaffian system correspond to feasible paths for the robotic system.

Integral curves were found for the Pfaffian system by converting it into a normal form, finding a path for the system in normal form, and then converting this trajectory back into the original coordinates. The most important normal forms in this dissertation was the Goursat normal form for systems of codimension two, and the extended Goursat normal form for systems with codimension greater than two. The N-trailer system and the mining vehicle were converted to Goursat normal form, and the multi-steering and firetruck systems were converted into extended Goursat normal form.

The Goursat forms are equivalent to the chained forms studied in previous work, and many algorithms are known for finding paths for chained form systems. Several of these methods were described in this dissertation, and then applied to the mobile robot systems to demonstrate the types of paths which result.

Several avenues are still open for future work.

1. The problem of path planning for mobile robot systems with nonholonomic velocity constraints in the presence of obstacles remains unsolved. Some work has been done for

the system consisting of a single Hilare-like robot in [27, 30]. The approach given in the first reference can be extended to a system of a car towing trailers; some preliminary work in this direction is presented in [29]. Tilbury et al. [54] applied high-frequency inputs to a two-trailer system, generating feasible paths which were arbitrarily close to a given path through configuration space which avoided the obstacles but did not satisfy the velocity constraints. The resulting trajectories were highly oscillatory and would be difficult to implement in practice. Sahai, Secor, and Bushnell considered the obstacle avoidance problem for a multi-trailer system with kingpin hitching [50]. Other methods for obstacle avoidance which use optimization based approaches may be found in [13, 14].

2. Although the paths generated by the methods proposed in this dissertation are "nice" in some aesthetic sense, nothing can be said about their optimality. Reeds and Shepp [43] explicitly characterized the shortest-length paths for a car with bounded turning radius. Their proof relies heavily on the geometry of the problem and the generalization even to the case of a car towing one trailer is unknown. In particular, the path length taken by each axle is different, and it is not clear what would be the best way to define "shortest." The optimal path planning problem with minimum input effort is also unsolved; although it is thought that the driving and steering inputs should somehow be weighted differently.

3. The paths generated by the methods proposed in this dissertation are purely open-loop. For a practical implementation of these methods, a feedback loop can be closed around the path, and the system stabilized to the path, perhaps using a method such as that outlined in [59].

4. Not much activity has focused on the problem of planning paths for systems which must necessarily go through singular points. Although the transformations into Goursat normal form are only guaranteed to exist locally, the transformations considered in this dissertation are defined on almost the entire configuration space. Singular points do exist; both the derived flag and the chained form transformations are undefined whenever the angle between two adjacent axles is 90 degrees. The methods described herein cannot be applied to a path which must go through such a point. Practically, this may not be too much of a limitation, but theoretically, it has been shown [28, 48]

that the N-trailer system is controllable even at these points. The number of Lie brackets of the input vector fields which much be taken to span the tangent space at such singular points is exponential in the number of trailers (in fact, it is a Fibonacci number). The problem of constructing control inputs which steer the system through such singular points has not yet been considered.

5. Any completely nonholonomic Pfaffian system with two constraints in $\mathbb{R}^4$ can be converted into Goursat (Engel's) normal form. However, in $\mathbb{R}^5$, the generic codimension two Pfaffian system does *not* satisfy the conditions for conversion into Goursat form, and thus the path-planning techniques described in this dissertation cannot be used to find paths for such systems.

   There are some interesting path-planning problems to consider even in this relatively low-dimensional realm. As noted in Section 3.3.3, the three-axle trailer system with two kingpin hitches is a codimension two Pfaffian system in five dimensions. The system of two surfaces rolling against each other without slipping has similar dimensions; the configuration of the system can be represented locally by the position of the contact point in the coordinate chart on each of the two objects in addition to the relative angle between the two charts. The nonholonomic constraints arise from specifying that the two objects roll in contact without slipping or rotating about the point of contact; Montana's equations [33] can be used to write down these constraints in local coordinates; see also [31].

   A third type of system which has the same dimension count is the two-chained form of Murray and Sastry [37]:

$$\begin{aligned} \dot{x}_1 &= u_1 & \dot{y}_1 &= u_2 \\ \dot{x}_2 &= y_1 u_1 & (y_2 &:= x_2) \\ \dot{x}_3 &= x_2 u_1 & \dot{y}_3 &= y_2 u_2 \end{aligned}$$

   In general, the x and y chains can be arbitrarily long; only the five-state version is shown here. This system is of interest because a simple algorithm (similar to that of Section 3.4.1) can be used to steer this system using sinusoids [37].

The derived flag for all three of these systems has the form

$$
\begin{aligned}
I &= \{\alpha^1, \alpha^2, \alpha^3\} \\
I^{(1)} &= \{\alpha^1, \alpha^2\} \\
I^{(2)} &= \{0\}
\end{aligned}
$$

Cartan [9] characterized the possibilities for systems which have this derived flag.

Although general conditions are not known for converting nonholonomic systems into two-chained form, the method of equivalence described by Gardner [17] can be used to check whether there exists a transformation between any two particular exterior differential systems. This equivalence calculation was performed by Shadwick, Sluis and Grossman [51] for the very simple system of a sphere rolling on a plane and the two-chained form, and they concluded that these two systems were not equivalent. It would be interesting to know of other normal forms for nonholonomic systems for which integral curves could easily be found.

6. All of the analysis of nonholonomic systems in this dissertation has been completely kinematic. For the wheeled mobile robots considered in this dissertation, a kinematic analysis may be sufficient. In practice, the controls are not the velocities but the accelerations; however, if the trailer system is moving "slowly enough" the dynamics will play only a small part and a kinematic path can be followed closely. Nonholonomic constraints also arise when mechanical systems conserve angular momentum; satellites and space robots are two interesting examples. These systems have nontrivial dynamics which should be taken into account when the path planning problem is considered.

7. As mentioned earlier, the results on applying the theory of exterior differential systems to the particular problems found in control systems are still preliminary. In particular, easy to check, necessary and sufficient conditions for linearizing control systems using dynamic state feedback remain unknown.

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