ERROR ANALYSIS IN OVERSAMPLED A/D CONVERSION AND QUANTIZATION OF WEYL-HEISENBERG FRAME EXPANSIONS

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Error Analysis in Oversampled A/D Conversion and Quantization of Weyl-Heisenberg Frame Expansions

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Abstract

Modern techniques for A/D conversion exploit the trade-off between resolutions of discretization in time and discretization in amplitude for improvements of conversion accuracy. For implementation reasons, oversampling is used to compensate for the loss of information induced by quantization. Classical reconstruction reduces variance of quantization error, usually modeled as a white noise [1], by a factor which is equal to the oversampling ratio r. However, it does not fully exploit the information contained in the digital representation. It was demonstrated in [2-4] that periodic bandlimited signals can be reconstructed with error whose squared norm decays as $O(1/r^2)$ rather than $O(1/r)$. In this paper, the error in oversampled A/D conversion of bandlimited signals in $L^2(\mathbf{R})$ is studied using a deterministic approach. The deterministic analysis demonstrates that under certain reasonable assumptions, the error squared norm behaves as $O(1/r^2)$.

Another instance of the redundancy-robustness interplay has been observed with frame expansions in $L^2(\mathbf{R})$. For a given accuracy of reconstruction, frame expansions allow for a progressively coarser coefficient quantization as the redundancy is increased. Error analysis, based on the white noise model, indicates that the squared norm of the quantization error decreases inversely to the frame redundancy factor r [7, 8]. As a generalization of results on oversampled A/D conversion, this paper gives a deterministic analysis of quantization error in Weyl-Heisenberg frame expansions. It shows that in cases when either the frame window functions or input signals have a compact support in time or frequency, under certain assumptions, the accuracy of the reconstruction after coefficient quantization can be further improved, giving the error which decays as $O(1/r^2)$.

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1 Introduction

Analog to digital (A/D) conversion involves discretization of an analog signal in time followed by discretization in amplitude. A block diagram of the converter together with a classical reconstruction scheme is illustrated in Figure 1. The discretization in time, implemented as sampling with a time interval τ, amounts to signal expansion in terms of a family of *sinc* functions,

$$\{sinc\,(\pi\,(t-n\tau)\,/\tau)\}_{n\in\mathbf{Z}}\,. \tag{1}$$

According to the sampling theorem, this family is complete in any space of σ-bandlimited functions with $\sigma \leq \pi/\tau$. Hence, σ-bandlimited signals sampled at or above the Nyquist rate, $\tau_N = \pi/\sigma$, can be reconstructed perfectly after the time discretization. On the other hand, the discretization of amplitude, that is quantization, induces an irreversible loss of information.

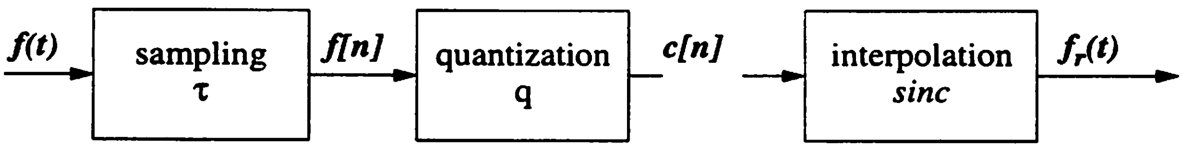

Figure 1: *Block diagram of simple A/D conversion followed by classical reconstruction. Input σ-bandlimited signal $f(t)$ is first sampled at a frequency $f_s = 1/\tau$, which is above the Nyquist frequency $f_N = \sigma/\pi$. Sequence of samples $f[n]$ is then discretized in amplitude with a quantization step q. Classical reconstruction gives a signal $f_r(t)$, which is obtained as a low-pass filtered version of some signal having the same digital version as the original $f(t)$.*

Accuracy of the conversion can be improved by reducing the quantization interval q. Statistical analysis [1], which models the quantization error as a zero mean white noise independent of the input, shows that the quantization error variance, σ_e^2, is proportional to q^2. However, circuit complexity and precision of analog components impose limits on the quantization step rather than on sampling interval refinement. Modern techniques for high resolution A/D conversion are therefore based on oversampling (in this paper, we concentrate on simple oversampled A/D conversion as opposed to $\Delta\Sigma$ modulation). Sampling a signal at a rate higher than the Nyquist rate, means expanding in terms of *sinc* functions (as given in (1)) which are an overcomplete set for the signal space. Redundancy of the expansion can be exploited for error reduction. Error analysis, based on the white noise model, indicates that this method used with classical linear reconstruction (see Figure 1) gives an error with the variance which decreases proportionally to the sampling interval, $\sigma_e^2 = O(\tau)$ [1, 2], or inversely to the oversampling ratio r, $\sigma_e^2 = O(1/r)$. This result is in accordance with measured data [1], under conditions such as a large number of quantization levels and small quantization step size compared to the input amplitude range, so that the quantization error can be satisfactorily approximated by a uniformly distributed white noise. However, it was shown in [2, 3, 4] that signals in the space of periodic bandlimited functions, i.e. trigonometric polynomials, can be reconstructed with error which behaves as $O(1/r^2)$ rather than $O(1/r)$, provided that the number of their quantization threshold crossings is greater or equal to the dimension of the space. The error measure considered in [2, 4, 3] was its squared norm, $\|e\|^2$, that is, energy in an interval equal to the signal period. An experimental evidence of the $O(1/r^2)$ result was also reported in [5, 6]. The reason for suboptimality of the classical reconstruction is the fact that in the presence of oversampling it does not fully exploit the information contained in the digital signal. Any consistent reconstruction scheme, that is a scheme which yields a signal which has the same digital version as the original, gives $\|e\|^2 = O(1/r^2)$ [2]. The scope of this result is limited to spaces of trigonometric polynomials, and the purpose of this paper is to find an upper bound of the quantization error in the case of bandlimited signals in $L^2(\mathbf{R})$.

In Section 2 we give an analysis of oversampled A/D conversion based on a study of the structure of reconstruction sets. That analysis provides an intuitive explanation of the idea that in terms of error reduction oversampling should not be inferior to quantization refinement. Section 3 gives a proof of the fact that bandlimited signals in $L^2(\mathbf{R})$ can be reconstructed after A/D conversion with an error which depends on the oversampling ratio as $\|e\|^2 = O(1/r^2)$. The proof uses in an essential manner the condition that samples taken at the points of quantization threshold crossings of a considered signal provide a complete and stable characterization of a given space of bandlimited signals. This is in analogy to the requirement for a sufficient number of quantization threshold crossings in the case of periodic bandlimited signals [3, 4].

A more general form of oversampled A/D conversion would be to use expansions other than those based on the *sinc* functions for the discretization in time. In the past decade, signal expansions in terms of overcomplete families of vectors, in particular wavelet and Weyl-Heisenberg frames, received considerable attention. Motivation for the study of the overcomplete expansions stems mainly from the fact that by relaxing the requirement for orthogonality or linear independence, a certain design freedom can be attained, but it has also been observed that increased redundancy can be exploited for gaining robustness. It was first pointed out by Morlet (as reported in [7, pp. 97-99]), that it was possible to reconstruct signals from their wavelet frame coefficients with a precision much higher than the precision with which coefficients were known.

An intuitive explanation of this effect has been given by Daubechies [7, pp. 97-99]. The range of a frame expansion, $\mathcal{V}_\phi$, is a subspace of some $\ell^2(\mathbf{J})$ space, and quantized coefficients, $\{c_j\}_{j\in\mathbf{J}}$, are usually not inside this subspace. Starting with the quantized family of coefficients, another set of coefficients which is closer to the originals is the orthogonal projection of $\{c_j\}_{j\in\mathbf{J}}$ onto $\mathcal{V}_\phi$. As the redundancy of the frame is increased, the subspace $\mathcal{V}_\phi$ becomes more and more constrained ("smaller" in some sense), so that on average, the projection approaches the original values, giving progressively better reconstruction results. Daubechies further showed, using a heuristic argument and the white noise model for the quantization error, that in this way expected value of error energy, $E(\|e\|^2)$, decreases in inverse proportion to the oversampling ratio, $E(\|e\|^2) = O(1/r)$, for signals which are "essentially localized" in a bounded region of the time-frequency plane. A rigorous treatment of the frame noise reduction property was given by Munch [8], for the case of tight Weyl-Heisenberg frames and integer oversampling ratios. Munch showed that by projecting noisy coefficients onto $\mathcal{V}_\phi$ the average noise power is reduced by the factor $1/r^2$. The effect on the error energy, which is considered here, is that it decays as $1/r$, on any given region of the phase space. Note that this is a result on a local error behavior, and that the white noise model generally gives a total error of infinite norm regardless of the oversampling ratio. Besides, reconstruction precision reported by Morlet was higher than it could be expected even from the $E(\|e\|^2) = O(1/r)$ result, indicating that the error in some cases decays faster.

Analogously to the case of oversampled A/D conversion, one may argue that the linear reconstruction, which amounts to projecting $\{c_j\}_{j\in\mathbf{J}}$ onto $\mathcal{V}_\phi$ and then finding the inverse of the frame transform, does not utilize all available information and is therefore suboptimal. More specifically, the reconstructed signal and the original need not have the same set of quantized coefficients. It can be expected that a consistent reconstruction, that is a reconstruction which always restores a signal which has the same set of quantized coefficients as the original, should give an error with a faster decay. Section 4 of this paper gives a deterministic analysis of the quantization error of Weyl-Heisenberg frame expansions in $L^2(\mathbf{R})$, as a generalization of results on simple oversampled A/D conversion. The cases which are considered are: 1) frames with bandlimited window functions, 2) timelimited signals, 3) frames with timelimited window functions, 4) bandlimited signals. The analysis shows that for instance in the first case, under certain conditions, signals can be reconstructed from the quantized coefficients with an error $\|e\|^2 = O(1/r^2)$, if the oversampling ratio is increased by decreasing the frame time step for a fixed frequency step. Analogous results hold in the other three cases.

Notations

The convolution of two signals $f(t)$ and $g(t)$ will be denoted as

$$f(t) * g(t) = \int_{\infty}^{\infty} f(s)g(t-s)ds.$$

The Fourier transform of a signal $f(t)$, $\mathcal{F}\{f(t)\}$, will be written as $\hat{f}(\omega)$. We say that a signal $f(t)$ is σ-bandlimited if

$$\hat{f}(\omega) = 0 \text{ for } |\omega| > \sigma, \text{ and } \|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty.$$

Similarly, a signal $f(t)$ is said to be T-timelimited if

$$f(t) = 0 \text{ for } |t| > T, \text{ and } \|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty.$$

A sequence of vectors $\{\varphi_1, \varphi_2, \varphi_3, \ldots\}$ in an infinite-dimensional Banach space Φ is said to be a *Schauder basis* for Φ if to each vector φ in the space, there corresponds a unique sequence of scalars $\{c_1, c_2, c_3, \ldots\}$ such that $\varphi = \sum_{i=1}^{n} c_i \varphi_i$. The term *basis* will here mean Schauder basis.

A basis for a Hilbert space is a *Riesz basis* if it is obtained from an orthonormal basis by means of a bounded invertible operator.

A sequence of real numbers $\{\lambda_n\}$ is *separated* if there is an $\epsilon > 0$ such that $|\lambda_n - \lambda_m| \geq \epsilon$ if $m \neq n$. It is said to have *uniform density d*, $d > 0$, if it is separated and there exists a constant L such that

$$|\lambda_n - \frac{n}{d}| \leq L, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{2}$$

2 Reconstruction Sets of Oversampled A/D Converters

The digital version of an analog signal is traditionally viewed as representation bearing information on its amplitude, with the uncertainty determined by the quantization step q, at the sequence of time instants $t = \ldots, -\tau, 0, \tau, 2\tau, \ldots$. Oversampled A/D conversion refers to the case when the signal is in a space of σ-bandlimited functions and is sampled at a rate higher than the Nyquist rate. The knowledge on the signal, contained in its digital version, defines a set of signals which can not be distinguished from the original based on the output of the A/D converter. This set is called a *reconstruction set* [2], Ω_r, and can be represented as the intersection of two sets (see Figure 2),

$$\Omega_r = \mathcal{V}_\sigma \cap \mathcal{C}(q), \tag{3}$$

one of which is the space of σ-bandlimited signals, $\mathcal{V}_\sigma$, while the other is a convex set, $\mathcal{C}(q)$, consisting of signals in $L^2(\mathbf{R})$ which have amplitudes within the same quantization intervals as the original at all sampling instants.

The size the reconstruction set can be considered as a measure of the uncertainty about the original analog signal. Hence, one way to obtain more accurate representation, or equivalently to reduce the size of the reconstruction set, is to refine the quantization by decreasing q. As an illustration, let us assume that the quantization step is decreased by an integral factor, m. As the result, the reconstruction set becomes smaller, since $\mathcal{C}(q/m)$ is a proper subset of

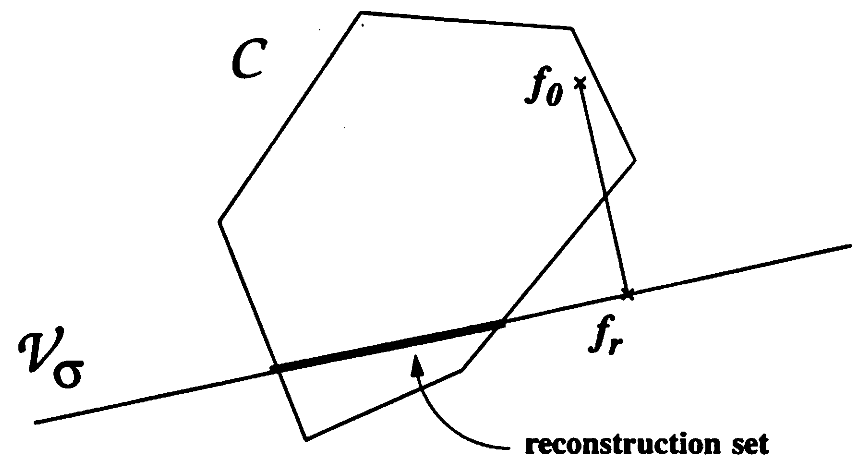

Figure 2: *Structure of the reconstruction set of simple oversampled A/D conversion and classical reconstruction scheme. The reconstruction set is the intersection of two sets. One of them is the set of signals which have the same digital representation as the original, and is denoted by C, while the other is the space of bandlimited signals $\mathcal{V}_\sigma$. The classical reconstruction scheme gives a signal f_r which is the orthogonal projection of some $f_0 \in C$ onto $\mathcal{V}_\sigma$. The figure illustrates a case when the reconstructed signal f_r is not in the reconstruction set, i.e. it is not a consistent estimate of the original.*

$C(q)$ (see Figure 3). In the limit, when $q \to 0$, there is no uncertainty about the amplitude values, and the signal can be perfectly reconstructed, which means that the reconstruction set reduces to a single point.

An alternative interpretation comes into play if the quantization threshold crossings are separated and sampling is sufficiently fine so that at most one quantization threshold crossing can occur in each of the sampling intervals. If this is satisfied, quantized samples of the signal are uniquely determined by the sampling intervals in which its quantization threshold crossings occur, and vice a versa. So, we can think of the digital representation as carrying information on the instants, with uncertainty τ in time, when the signal assumes values $..., -q, 0, q, 2q, ...$. Hence, the reconstruction set can also be viewed as lying in the intersection,

$$\Omega_r \subseteq \mathcal{V}_\sigma \cap \mathcal{D}(\tau), \tag{4}$$

where $\mathcal{D}(\tau)$ is the set of all signals in $L^2(\mathbf{R})$ which have the same quantization threshold crossing as the original, in all of the sampling intervals where the original goes through a quantization threshold. If in addition we require that signals in $\mathcal{D}(\tau)$ can not have more than one quantization threshold crossing per sampling interval, then in (4) equality holds.

This approach indicates that instead of refining the quantization step, a higher accuracy of the digital representation can be achieved by decreasing the sampling interval. For instance, if we reduce the sampling interval by an integral factor, m, the set of signals sharing the sampling intervals of quantization threshold crossings with the original, for this new sampling interval, is a proper subset of $\mathcal{D}(\tau)$, $\mathcal{D}(\tau/m) \subset \mathcal{D}(\tau)$. As a result, the size of the reconstruction set is also reduced, and the situation seems to be analogous to the one with quantization step refinement. In this perspective the result about $O(\tau)$ error variance behavior is disappointing, compared to improvements of accuracy achieved through quantization interval reduction, which gives $\sigma_e^2 = O(q^2)$. However, this lower gain of time-resolution refinement compared to amplitude-resolution refinement has no origin in fundamental phenomena, but is rather a consequence of inadequate reconstruction.

It has been observed that the classical reconstruction is not consistent in the presence of oversampling [2-6]. It does not necessarily give a signal in the reconstruction set. Figure

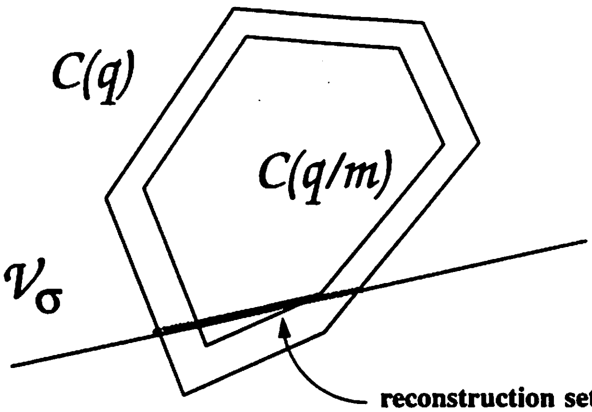

Figure 3: *The effect of the quantization refinement on the reconstruction set. The set of signals which share the same digital version with the original, $C(q)$, becomes smaller as the result of quantization refinement, $C(q/m) \subset C(q)$. Consequently, the reconstruction set is also reduced giving a higher accuracy of the representation*

2 depicts such a scenario. The reconstructed signal, $f_r(t)$, is obtained by low-pass filtering of an initial estimate, $f_0(t)$, which has the same digital representation as the original. This gives the orthogonal projection of $f_0(t)$ onto the space of bandlimited signals, $\mathcal{V}_\sigma$. This projection is closer to the original than $f_0(t)$ itself, but need not lie in $C(q)$, even if $f_0(t)$ does. By further alternating projections of the initial estimate onto $C(q)$ and $\mathcal{V}_\sigma$, the accuracy of the reconstruction is improved and the reconstruction set is asymptotically reached [3, 4, 5]. Any such reconstruction algorithm, which gives a signal in the reconstruction set, is called *consistent reconstruction*, while the signals in the reconstruction set are denoted as *consistent estimates* of the original [2]. It is shown in [3, 4] that consistent reconstruction of trigonometric polynomials gives an error which decreases as $\|e\|^2 = O(\tau^2)$. In the next section we consider the error of consistent reconstruction in the case of bandlimited signals in $L^2(\mathbf{R})$.

3 Deterministic Analysis of Error in Oversampled A/D Conversion

3.1 Asymptotic Behavior and Frames of Complex Exponentials in $L^2[-\sigma, \sigma]$

The representation of the reconstruction set of the oversampled A/D converter in (4) indicates that its size gets smaller with higher oversampling. Quantization refinement, which also improves the accuracy of the reconstruction, asymptotically gives the reconstruction set reduced to a single point, which is the original signal itself. Does this also happen in the former case, when the sampling interval goes to zero? In other words, is it possible to reconstruct perfectly an analog σ-bandlimited signal after simple A/D conversion in the case of infinitely high oversampling?

In the limit, when the sampling interval assumes an infinitely small value, the time instants in which the signal goes through the quantization thresholds are known with infinite precision. The information on the input analog signal, in this limiting case, is its values at a sequence of irregularly spaced points $\{\lambda_n\}$, which are its quantization threshold crossings. If $\{e^{j\lambda_n\omega}\}$ is complete in $L^2[-\sigma, \sigma]$, then the reconstruction set asymptotically does reduce to a single point, giving the perfectly restored input analog signal. Completeness of $\{e^{j\lambda_n\omega}\}$ in $L^2[-\sigma, \sigma]$ means that any σ-bandlimited signal $f(t)$ is determined by the sequence of samples $\{f(\lambda_n)\}$. However, this does not necessarily mean that $f(t)$ can be reconstructed in a numerically stable way from the samples $\{f(\lambda_n)\}$, unless another constraint on $\{\lambda_n\}$ is introduced which would ensure that any two σ-bandlimited signals which are close at points $\{\lambda_n\}$ are also close in L^2 norms. A precise formulation of the stability requirement is that there exists a constant $A > 0$ such that for any σ-bandlimited signal $f(t)$

$$A\|f\|^2 \leq \sum_n |f(\lambda_n)|^2. \tag{5}$$

In the subsequent analysis of the quantization error we shall also assume that the quantization threshold crossings satisfy

$$\sum_n |f(\lambda_n)|^2 \leq B\|f\|^2, \tag{6}$$

for any σ-bandlimited $f(t)$. Note, that (6) is satisfied for any reasonable sequence of quantization threshold crossings. For instance, it holds for any separated sequence $\{\lambda_n\}$ [9, pp. 162-166]. These two conditions together, (5) and (6), mean that $\{e^{j\lambda_n\omega}\}$ is a frame in $L^2[-\sigma,\sigma]$ [10] (see Appendix A). For such a sequence $\{\lambda_n\}$ we shall say that it is a *frame sequence* for the space of σ-bandlimited signals, with bounds A and B. Problems such as completeness of complex exponentials $\{e^{j\lambda_n\omega}\}$ in $L^2[-\sigma,\sigma]$, as well as conditions under which bounds in (5) and (6) are satisfied, are the core issues of nonharmonic Fourier analysis. An excellent introductory treatment of this subject can be found in [9]. Here, only a few illuminating results are discussed in order to provide some intuition about frames of complex exponentials.

An interesting point is to see what are "minimal" frames. The removal of a vector from a frame leaves either a frame or an incomplete set [10]. A frame which ceases to be a frame when any of its elements is removed is said to be an *exact frame*. A remarkable fact is that a sequence of vectors in a separable Hilbert space is an exact frame if and only if it is a Riesz basis [9, pp. 184-189]! Supplementing a Riesz basis $\{e^{j\lambda_n\omega}\}$ by another set of complex exponentials still gives a frame unless there are too many vectors in excess so that the upper bound in (6) can not be satisfied for any finite B. Are there any bases $\{e^{j\lambda_n\omega}\}$ which are not Riesz bases? This is still an open problem. Every example of a basis of complex exponentials for $L^2[-\sigma,\sigma]$ so far has been proven to be a Riesz basis [9, pp. 190-197]. For the end of this subsection we review two results on frame sufficient conditions.

If a sequence of real numbers $\{\lambda_n\}$ satisfies

$$|\lambda_n - n\frac{\pi}{\sigma}| \leq L < \frac{1}{4}\frac{\pi}{\sigma}, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{7}$$

then $\{e^{j\lambda_n\omega}\}$ is a Riesz basis for $L^2[-\sigma,\sigma]$. This condition is known as Kadec's 1/4 Theorem [11]. How realistic is this condition in the quantization context? For a quantization step small compared to a signal amplitude, a sufficiently dense sequence of quantization threshold crossings could be expected, thus satisfying (7), at least on a time interval before the signal magnitude eventually falls below the lowest nonzero quantization threshold. Suppose that the analog signal $f(t)$, at the input of an A/D converter satisfies: $\hat{f}(\omega)$ is continuous on $[-\sigma,\sigma]$ and $\hat{f}(\sigma) \neq 0$. Such a signal $f(t)$, in terms of its zero-crossings, asymptotically behaves as $sinc(\sigma t)$. So, if one of the quantization thresholds is set at zero, then for large n, quantization threshold crossings should be close to points $n\pi/\sigma$.

Another criterion was given by Duffin and Schaeffer [10]. It states that if $\{\lambda_n\}$ has a uniform density $d > \sigma/\pi$, then for any σ-bandlimited signal $f(t)$

$$A\|f\|^2 \leq \sum_n |f(\lambda_n)|^2 \leq B\|f\|^2, \tag{8}$$

for some positive constants $0 < A \leq B < \infty$. Feasibility of a σ-bandlimited signal with a sequence of quantization threshold crossings having uniform density greater than σ/π is unlikely for a quantization scheme with a fixed quantization step. Amplitude of a bandlimited signal $f(t)$ decays at least as fast as $1/t$, and outside of some finite time interval, $[-T, T]$, it is confined in the range between the two lowest nonzero quantization thresholds. Zero-crossings are the only possible quantization threshold crossings of $f(t)$ on $(-\infty, -T] \cup [T, \infty)$. However, it doesn't seem to be plausible that zero-crossings of a σ-bandlimited signal can have a uniform density greater than the density of zero-crossings of $sinc(\sigma t)$, which is σ/π. Therefore, in order to meet this criterion the quantization step has to change in time following the decay of the signal. One way to achieve this is a scheme with quantization steps which are fixed on given time segments but eventually decrease with segment order.

3.2 $O(1/r^2)$ Error Bound

Consider a σ-bandlimited signal, $f(t)$, at the input of an oversampled A/D converter with a sampling interval τ. Suppose that its sequence of quantization threshold crossings, $\{x_n\}$, is a frame sequence for the space of σ-bandlimited signals. If $g(t)$ is a consistent estimate of $f(t)$, there exists a sequence $\{y_n\}$ of quantization threshold crossings of $g(t)$, such that every x_n has a corresponding y_n in the same sampling interval. Hence. for each pair (x_n, y_n), $f(x_n) = g(y_n)$ and $|x_n - y_n| \leq \tau$.

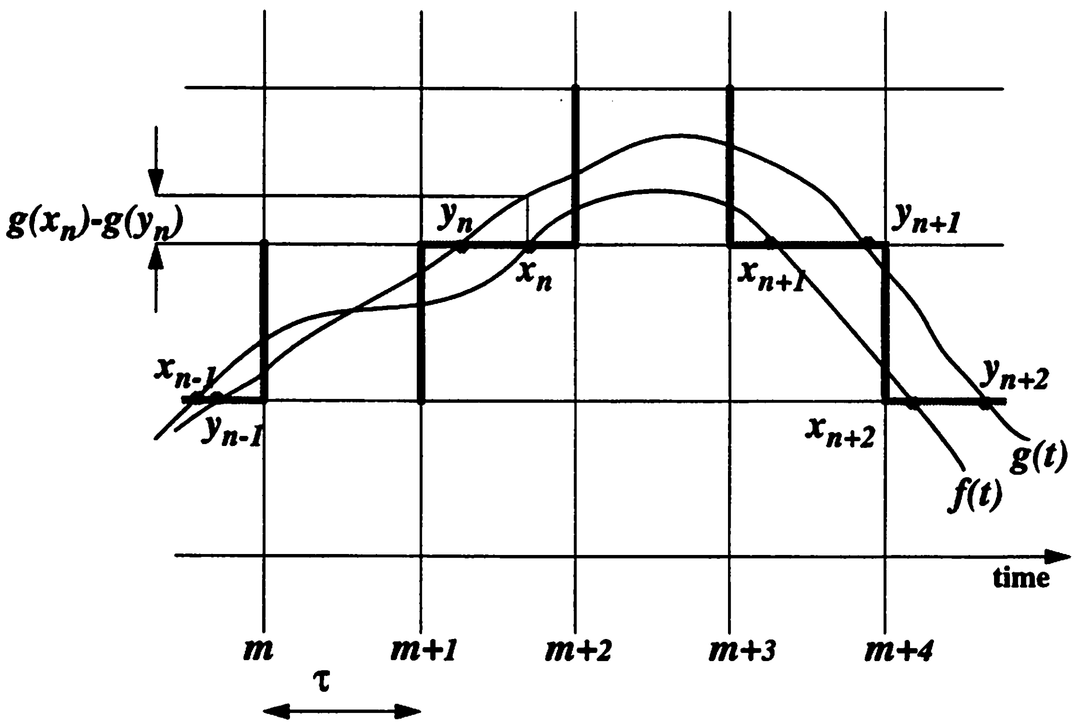

Figure 4: *Quantization threshold crossings of an analog signal $f(t)$ and its consistent estimate $g(t)$. The sequence of quantization threshold crossings, $\{x_n\}$, of $f(t)$, uniquely determines its digital version and vice versa, provided that all the crossings occur in distinct sampling intervals. If $f(t)$ goes through a certain quantization threshold at the point x_n, then $g(t)$ has to cross the same threshold at a point y_n which is in the same sampling interval with x_n. At a point x_n, the error amplitude is equal to $|g(x_n) - f(x_n)| = |g(y_n) - g(x_n)|$.*

At the points $\{x_n\}$, the error amplitude is bounded by the variation of $g(t)$ on the interval $[y_n, x_n]$ (without loss of generality we assume that $y_n < x_n$). as shown in Figure 4. Since $g(t)$ is bandlimited, which also means that it has finite energy, it can have only a limited variation on $[y_n, x_n]$. This variation is bounded by some value which is proportional to the sampling interval, so

$$|g(x_n) - g(y_n)| \leq c_n \cdot \tau. \tag{9}$$

The constant c_n in this relation can be the maximum slope of $g(t)$ on the interval $[y_n, x_n]$, $c_n = \sup_{t \in (y_n, x_n)} g'(t)$. The error signal, $e(t) = g(t) - f(t)$, is itself a σ-bandlimited signal. At the points $\{x_n\}$ its amplitude is bounded by values which are proportional to τ (9). If the sequence $\{c_n\}$ is square summable, it can be expected that the energy of the error signal is bounded as $\|e\|^2 \leq const \cdot \tau^2$, or in terms of the oversampling ratio r,

$$\|e\|^2 \leq \frac{const}{r^2}. \tag{10}$$

This result is the content of the following theorem.

Theorem 1 *Let $f(t)$ be a real σ-bandlimited signal at the input of an A/D converter with a sampling interval $\tau < \pi/\sigma$. If the sequence of quantization threshold crossings of $f(t)$, $\{x_n\}$, forms a frame sequence for the space of σ-bandlimited signals, then there exists a positive constant δ such that if $\tau < \delta$, for every consistent estimate of $f(t)$, $g(t) \in C^1$,*

$$\|f(t) - g(t)\|^2 \leq k\|f(t)\|^2 \tau^2, \tag{11}$$

where k is a constant which does not depend on τ.

Proof Let A and B be bounds of the frame $\{e^{jx_n\omega}\}$ in $L^2[-\sigma, \sigma]$, so that for any σ-bandlimited $s(t)$

$$A\|s(t)\|^2 \leq \sum_n |s(x_n)|^2 \leq B\|s(t)\|^2. \tag{12}$$

At the points x_n, the error amplitude is bounded by the variation of the reconstructed signal on corresponding sampling intervals, as described by the following relations:

$$\begin{aligned} |e(x_n)| &= |f(x_n) - g(x_n)| \\ &= |f(x_n) - g(y_n) + g(y_n) - g(x_n)| \\ &= |g(y_n) - g(x_n)| \\ &< \tau \cdot g'(\epsilon_n), \quad min(x_n, y_n) \leq \epsilon_n \leq max(x_n, y_n). \end{aligned} \tag{13}$$

Here, ϵ_n denotes a point on the interval $[y_n, x_n]$. The error norm then satisfies

$$\begin{aligned} \|e(t)\|^2 &\leq \frac{1}{A}\sum_n |e(x_n)|^2 \\ &< \frac{\tau^2}{A}\sum_n |g'(\epsilon_n)|^2. \end{aligned} \tag{14}$$

If τ is smaller than $\delta = \delta_{1/4}(\{x_n\}, \sigma)$ (see Appendix A, Lemma 3), then $|\epsilon_n - x_n| < \delta$ and consequently

$$\sum_n |g'(\epsilon_n)|^2 \leq \frac{9B}{4}\|g'(t)\|^2. \tag{15}$$

This gives

$$\begin{aligned}\|e(t)\|^2 &< \tau^2\frac{9B}{4A}\|g'(t)\|^2 \\ &\le \tau^2\sigma^2\frac{9B}{4A}\|g(t)\|^2\end{aligned} \tag{16}$$

so that energy of the error can be bounded as

$$\|e(t)\|^2 \le \frac{\tau^2\sigma^2 9B}{4A}\|g(t)\|^2. \tag{17}$$

It remains to find a bound for the norm of $g(t)$.

According to Lemma 3, since $|x_n - y_n| < \tau < \delta_{1/4}(\{x_n\}, \sigma)$, the following holds

$$\begin{aligned}\|g(t)\|^2 &\le \frac{4}{A}\sum_n g(y_n)^2 \\ &= \frac{4}{A}\sum_n f(x_n)^2 \\ &\le \frac{4B}{A}\|f(t)\|^2.\end{aligned} \tag{18}$$

As the consequence of the last inequality and the error bound in (17) we obtain

$$\|e(t)\|^2 \le 9\sigma^2\frac{B^2}{A^2}\|f(t)\|^2\tau^2. \tag{19}$$

QED

An extension of the result of Theorem 1 to complex bandlimited signals is straightforward, provided that the real and imaginary parts are quantized separately.

Corollary 1 *Let the real and imaginary parts of a complex σ-bandlimited signal $f(t)$, at the input of an A/D converter with the sampling interval $\tau < \pi/\sigma$, be quantized separately. If sequences of quantization threshold crossings of both $\mathrm{Re} f(t)$ and $\mathrm{Im} f(t)$ form frame sequences for the space of σ-bandlimited signals, then there exists a positive constant δ such that if $\tau < \delta$ then for every consistent estimate $g(t) \in C^1$*

$$\|f(t) - g(t)\|^2 \le k\|f(t)\|^2\tau^2, \tag{20}$$

for some constant k which does not depend on τ.

Proof Let $\{x_n^r\}$ and $\{x_n^i\}$ be the sequences of quantization threshold crossings of the real and imaginary parts of $f(t)$, respectively, and $0 < A_r \le B_r < \infty$, $0 < A_i \le B_i < \infty$ the corresponding frame bounds. Then the relation (20) holds for

$$\tau < \min\left(\delta_{1/4}(\{x_n^r\}, \sigma), \delta_{1/4}\left(\{x_n^i\}, \sigma\right)\right)$$

and

$$k = 9\sigma^2\mu, \quad \mu = \max\left(\frac{B_i^2}{A_i^2}, \frac{B_r^2}{A_r^2}\right). \tag{21}$$

QED

In Appendix B we give another, more intuitive error bound for the case when quantization threshold crossings form a sequence of uniform density greater than σ/π.

4 $O(1/r^2)$ Behavior of Error in Quantization of Weyl-Heisenberg Frame Expansions

An immediate generalization of the results on oversampled A/D conversion is in error analysis in the case of quantization of coefficients of Weyl-Heisenberg frame expansions (see Appendix C). The two cases which are considered first are: 1) frames derived from bandlimited window functions without restrictions on input signals, except that they are in $L^2(\mathbf{R})$, 2) timelimited input signals, and no restrictions on the window function in addition to the requirement that it is in $L^2(\mathbf{R})$.

CASE 1. σ-bandlimited window function

Let

$$\left\{\varphi_{m,n}(t) : \varphi_{m,n}(t) = \varphi(t - nt_0)e^{jm\omega_0 t}\right\} \tag{22}$$

be a Weyl-Heisenberg frame in $L^2(\mathbf{R})$, with the bounds

$$A\|f\|^2 \leq \sum_{m,n\in\mathbf{Z}^2} |\langle\varphi_{m,n}, f\rangle|^2 \leq B\|f\|^2. \tag{23}$$

Frame coefficients $\{c_{m,n} : c_{m,n} = \langle\varphi_{m,n}, f\rangle\}$ of a signal f can be expressed in the Fourier domain as

$$c_{m,n} = \int_{-\infty}^{\infty} \hat{f}(\omega - m\omega_0)\hat{\varphi}^*(\omega)e^{j\omega nt_0}d\omega. \tag{24}$$

The system which for an input signal gives these coefficients can be viewed as a multichannel system, containing a separate channel for each frequency shift $m\omega_0$, such that the m-th channel performs modulation of an input signal with $e^{jm\omega_0 t}$, then linear filtering with $\varphi(-t)$ and finally sampling at points $\{nt_0\}$. Such a system is shown in Figure 5. For a fixed m, coefficients $c_{m,n}$ are samples of the signal

$$f_m(t) = \left(f(t)e^{jm\omega_0 t}\right) * \varphi(-t), \tag{25}$$

which will be called *m-th subband component* of $f(t)$. In the sequel, the m-th subband component of a signal $s(t)$ will be denoted by $s_m(t)$ and the sequence obtained by sampling $s_m(t)$ with the interval t_0 will be denoted by $S_m[n]$, $S_m[n] = s_m(nt_0)$. Using this notation, frame coefficients of $f(t)$ are given by $c_{m,n} = F_m[n]$.

Such an interpretation of Weyl-Heisenberg frame coefficients of the signal $f(t)$ means that their quantization amounts to simple A/D conversion of the subband components of $f(t)$. Note that these coefficients are in general complex, and it is assumed here that real and imaginary parts are quantized separately. If the frame window, $\varphi(t)$, is a σ-bandlimited function, each of the subband components is also a σ-bandlimited signal. In this context, a signal $g(t)$ is said to be a consistent estimate of $f(t)$ if they have the same quantized values of the frame coefficients and each subband component of $g(t)$ is continuously differentiable, $g_m(t) \in C^1$ (note that the subband signals, being bandlimited, are continuously differentiable a.e.). In the light of the results of the previous section, this indicates that if the frame redundancy is increased by decreasing the time step t_0 for a fixed ω_0, the quantization error of consistent reconstruction should decay as $O(t_0^2)$. This result is established by the following corollary of Theorem 1, and can be expressed in terms of the oversampling ratio, $r = \frac{2\pi}{\omega_0 t_0}$, as $\|e\|^2 = O(1/r^2)$.

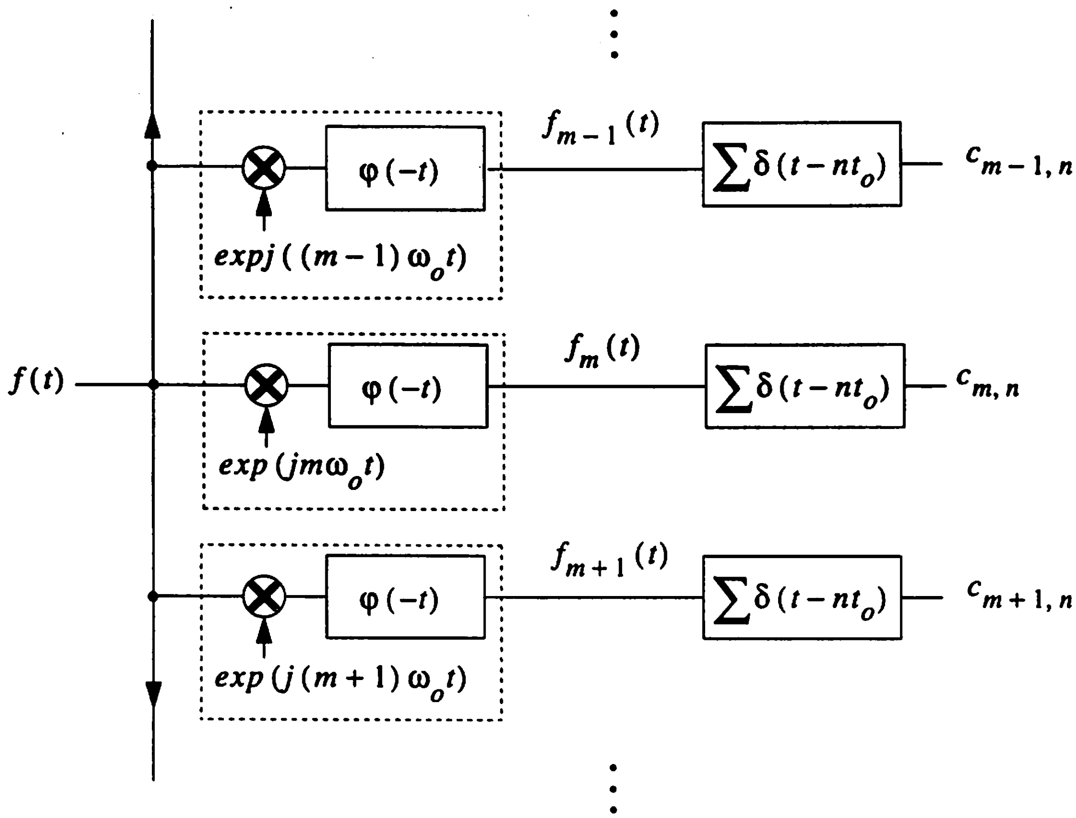

Figure 5: *Evaluation of Weyl-Heisenberg frame coefficients. For a fixed m, coefficients $c_{m,n}$ are obtained as samples, with t_0 sampling interval, of the signal $f_m(t)$, which is the result of modulating the input signal with $e^{jm\omega_0 t}$, followed by filtering with $\varphi(-t)$.*

Corollary 2 *Let $\{\varphi_{m,n}(t)\}$ be a Weyl-Heisenberg frame in $L^2(\mathbf{R})$, with a time step t_0 and a frequency step ω_0, derived from a σ-bandlimited window function $\varphi(t)$. Consider quantization of the frame coefficients of a signal $f(t) \in L^2(\mathbf{R})$ and suppose that for a certain ω_0 the following hold:*

i) *quantization threshold crossings of both real and imaginary parts of all the subband components $f_m(t) = (f(t)e^{jm\omega_0 t}) * \varphi(-t)$ form frame sequences for the space of σ-bandlimited signals, with frame bounds $0 < \alpha^r_m \leq \beta^r_m < \infty$ and $0 < \alpha^i_m \leq \beta^i_m < \infty$;*

ii)

$$\sup_{m\in\mathbf{Z}} \max\left(\frac{\beta^r_m}{\alpha^r_m}, \frac{\beta^i_m}{\alpha^i_m}\right) = M < \infty.$$

Then there exists a constant δ, such that if $t_0 < \delta$, for any consistent estimate $g(t)$ of $f(t)$, the reconstruction error satisfies

$$\|f(t) - g(t)\|^2 \leq k\|f(t)\|^2 t_0^2, \tag{26}$$

where k is a constant which does not depend on t_0.

Proof Let $f(t)$ be reconstructed from its quantized coefficients as $g(t)$. Suppose that $g(t)$ is a consistent estimate of $f(t)$, that is, frame coefficients of $g(t)$ are quantized to the same values as those of $f(t)$. Under the bandlimitedness condition on $\varphi(t)$, all subband signals $f_m(t)$, are also σ-bandlimited, and each $g_m(t)$ is a consistent estimate of the corresponding $f_m(t)$, in the sense discussed in Section 2.

Under assumption *i)*, and as a consequence of Corollary 1, for each m there is a δ_m such that the m-th subband error component, $f_m(t) - g_m(t)$, satisfies

$$\|f_m(t) - g_m(t)\|^2 \leq 9\sigma^2 \mu_m \|f_m(t)\|^2 t_0^2, \quad \mu_m = \max\left(\left(\frac{\beta_m^r}{\alpha_m^r}\right)^2, \left(\frac{\beta_m^i}{\alpha_m^i}\right)^2\right). \tag{27}$$

For a sampling interval $t_0 \leq \pi/\sigma$, and any $s \in L^2(\mathbf{R})$ we have that norms of subband signals s_m and their sampled versions satisfy:

$$\|s_m\|^2 = t_0 \|S_m\|^2. \tag{28}$$

The frame condition (23) then implies

$$\begin{aligned} \|s\|^2 &\leq \frac{1}{A}\sum_m \|S_m\|^2 \\ &= \frac{1}{At_0}\sum_m \|s_m\|^2. \end{aligned} \tag{29}$$

and

$$\begin{aligned} \|s\|^2 &\geq \frac{1}{B}\sum_m \|S_m\|^2 \\ &= \frac{1}{Bt_0}\sum_m \|s_m\|^2. \end{aligned} \tag{30}$$

From assumption *ii)* and Lemma 3 in Appendix A, it follows that $\delta = \inf_{m\in\mathbf{Z}} \delta_m$ is strictly greater than zero, $\delta \geq \epsilon > 0$. For $t_0 \leq \delta$, we have the following as a consequence of relations (27), (29) and (30):

$$\begin{aligned} \|f(t) - g(t)\|^2 &\leq \frac{1}{At_0}\sum_m \|f_m(t) - g_m(t)\|^2 \\ &\leq \frac{1}{At_0}\sum_m 9\sigma^2\mu_m \|f_m(t)\|^2 t_0^2 \\ &\leq 9\sigma^2 M^2 \frac{B}{A}\|f\|^2 t_0^2. \end{aligned} \tag{31}$$

This finally gives

$$\|f(t) - g(t)\|^2 \leq k\|f(t)\|^2 t_0^2, \tag{32}$$

which for $\omega_0 = const$ can be expressed as $\|f(t) - g(t)\|^2 = O(1/r^2)$.

QED

CASE 2. T-timelimited signals

Another way to interpret expansion coefficients of $f(t)$ with respect to the frame (22) is to consider them as samples of signals

$$\hat{\varphi}_n(\omega) = \left(\hat{\varphi}(\omega)e^{-jnt_0\omega}\right) * \hat{f}(\omega), \tag{33}$$

with ω_0 sampling interval. This is illustrated in Figure 6. Suppose that $f(t)$ is a T-timelimited

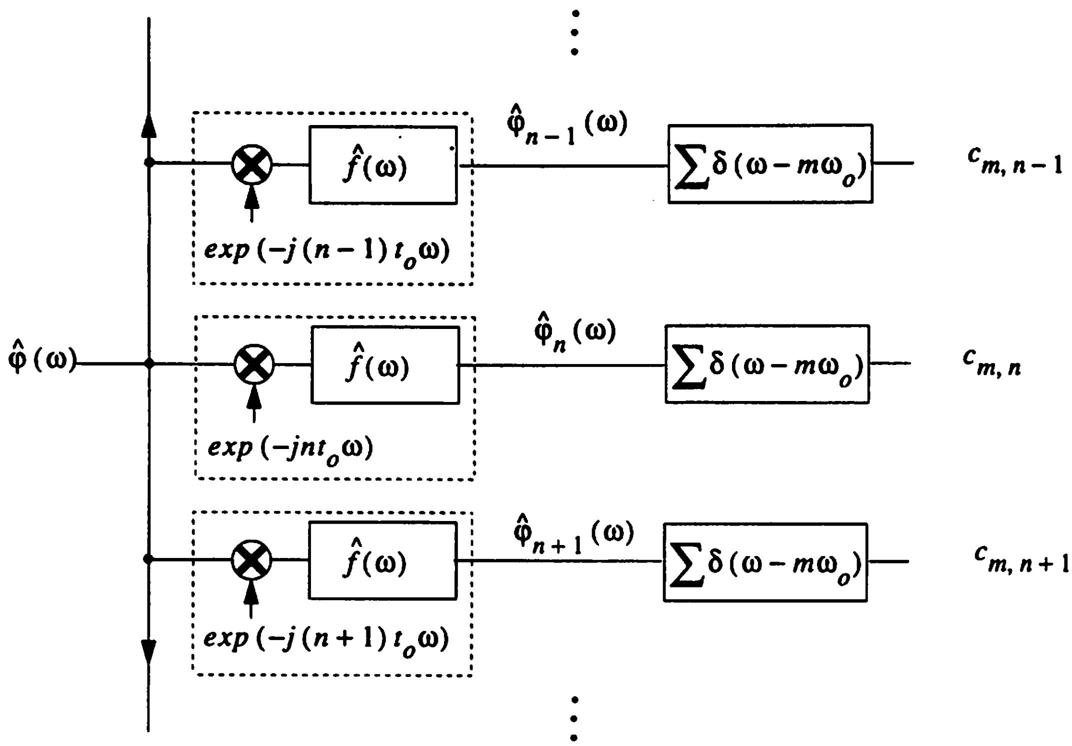

Figure 6: *Evaluation of Weyl-Heisenberg frame coefficients. For a fixed n, coefficients $c_{m,n}$ are obtained as samples, with ω_0 sampling interval, of the signal $\hat{\varphi}_n(\omega)$, which is itself result of modulating $\hat{\varphi}(\omega)$ by $e^{-jnt_0\omega}$, followed by filtering with $\hat{f}(\omega)$.*

signal. Then $\mathcal{F}\{\hat{\varphi}_n(\omega)\}$ is also T-timelimited for each n, which makes all subband signals $\hat{\varphi}_n(\omega)$ bandlimited. From an argument completely analogous to the one in the previous case, it can be concluded that if for some fixed t_0, sequences of quantization threshold crossings of subband signals $\hat{\varphi}(\omega)$ satisfy certain frame properties the quantization error of consistent reconstruction can be bounded as $\|e(t)\|^2 \leq k\|f(t)\|^2\omega_0^2$. Since we consider the case when $t_0 = const$ and $\omega_0 \to 0$, this can be also expressed as $\|e\|^2 = O(1/r^2)$. The precise formulation of this result is established as follows.

Corollary 3 *Let $\{\varphi_{m,n}(t)\}$ be a Weyl-Heisenberg frame in $L^2(\mathbf{R})$, with a time step t_0 and a frequency step ω_0, derived from a window function $\varphi(t)$. Consider quantization of the frame coefficients of a T-timelimited signal $f(t) \in L^2(\mathbf{R})$. Suppose that for a certain t_0 the following hold:*

i) quantization threshold crossings of both real and imaginary parts of all the subband components $\hat{\varphi}_n(\omega) = (\hat{\varphi}(\omega)e^{-jnt_0\omega}) * \hat{f}(\omega)$ form frames for the space of T-timelimited signals, with frame bounds $0 < \alpha_n^r \leq \beta_n^r < \infty$ and $0 < \alpha_n^i \leq \beta_n^i < \infty$;

ii)

$$\sup_{n\in\mathbf{Z}} \max\left(\frac{\beta_n^r}{\alpha_n^r}, \frac{\beta_n^i}{\alpha_n^i}\right) = M < \infty.$$

Then there exists a constant δ, such that if $\omega_0 < \delta$, for any consistent estimate $g(t)$ of $f(t)$, the reconstruction error satisfies

$$\|f(t) - g(t)\|^2 \leq k\|f(t)\|^2\omega_0^2, \tag{34}$$

where k is a constant which does not depend on ω_0.

If the Weyl-Heisenberg frame is redefined as

$$\left\{\varphi_{m,n}(t) : \varphi_{m,n}(t) = \varphi(t - nt_0)e^{jm\omega_0(t-nt_0)}\right\},$$

then analogous results hold for the cases when input signals are bandlimited or the window function has a compact support in time.

The assumptions on bounded support of frame window functions or input signals in either time or frequency, introduced in the above considerations, are natural assumptions of time-frequency localized signal analysis. A question which naturally arises is whether in the case when both the window function is bandlimited and the considered signals have finite support, we have that the error decays as $\|e\|^2 = O(\omega_0^2 t_0^2)$. Another interesting case is the one when none of these assumptions is introduced. Is then also possible to exploit frame redundancy for quantization error reduction so that the error norm tends to zero as the redundancy is increased, or even more $\|e\|^2 = O(1/r^2)$? These are still open problems.

5 Conclusion

The error of oversampled A/D conversion was studied here using deterministic analysis. The analysis showed that for signals with quantization threshold crossings which form frame sequences of the corresponding space of bandlimited signals, the information contained in the digital version allows for reconstruction with an error which decreases in inversely to the square of the oversampling ratio, $\|e\|^2 = O(1/r^2)$. This result was generalized giving a quantitative characterization of redundancy-robustness interplay in Weyl-Heisenberg frame expansions. The deterministic analysis showed again that in cases when either the frame window function or input signals are bandlimited or timelimited, under certain reasonable assumptions, the quantization error of consistent reconstruction is reduced as $\|e\|^2 = O(1/r^2)$ rather than $\|e\|^2 = O(1/r)$ which is expected with the linear reconstruction.

Acknowledgements

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Appendix A

The results of nonharmonic Fourier analysis, reviewed here, are adapted from [10].

Definition 1 *A set of functions $\{\exp(j\lambda_n\omega)\}$ is a frame in $L^2[-\gamma,\gamma]$ if there exist positive constants A and B which depend exclusively on γ and the set of functions $\{\exp(j\lambda_n\omega)\}$ such that*

$$A \leq \frac{\frac{1}{2\pi}\sum_n \left|\int_{-\gamma}^{\gamma} g(\omega)e^{j\lambda_n\omega}d\omega\right|^2}{\int_{-\gamma}^{\gamma}|g(\omega)|^2 d\omega} \leq B \tag{35}$$

for every function $g(\omega) \in L^2[-\gamma, \gamma]$. *Constants A and B are called frame bounds of the frame* $\{\exp(j\lambda_n\omega)\}$.

Lemma 1 *A system of complex exponentials* $\{\exp(j\lambda_n\omega)\}$ *is a frame in* $L^2[-\gamma, \gamma]$ *with frame bounds* $0 < A \le B < \infty$ *if and only if*

$$A\|f(x)\|^2 \le \sum_n |f(\lambda_n)|^2 \le B\|f(x)\|^2 \tag{36}$$

for every entire function $f(z)$ *of exponential type* γ *which is square integrable on the real axis,* $f(x) \in L^2(\mathbf{R})$.

Note that if $f(x)$ is a γ-bandlimited function, then $f(z)$ is an entire function of exponential type γ. Conversely, if $f(z)$ is an entire function of exponential type γ, which is square integrable on the real axis, then $f(x)$ is a γ-bandlimited function.

Estimates of bounds on the quantization error (Section 3) are derived based on the next two lemmas.

Lemma 2 *Let* $\{\exp(j\lambda_n\omega)\}$ *be a frame in* $L^2[-\gamma, \gamma]$. *If M is any constant and* $\{\mu_n\}$ *is a sequence satisfying* $|\mu_n - \lambda_n| \le M$, *then there is a number* $C = C(M, \gamma, \{\lambda_n\})$ *such that*

$$\frac{\sum_n |f(\mu_n)|^2}{\sum_n |f(\lambda_n)|^2} \le C \tag{37}$$

for every entire function $f(z)$ *of of exponential type* γ.

Lemma 3 *Let* $\{\exp(j\lambda_n\omega)\}$ *be a frame in* $L^2[-\gamma, \gamma]$, *with bounds* $0 < A \le B < \infty$ *and* δ *a given positive number. If a sequence* $\{\mu_n\}$ *satisfies* $|\lambda_n - \mu_n| \le \delta$ *for all* n, *than for every entire function* $f(z)$ *of exponential type* γ *which is square integrable on the real axis*

$$A(1 - \sqrt{C})^2\|f\|^2 \le \sum_n |f(\mu_n)|^2 \le B(1 + \sqrt{C})^2\|f\|^2, \tag{38}$$

where

$$C = \frac{B}{A}\left(e^{\gamma\delta} - 1\right)^2. \tag{39}$$

Remark

If δ in the statement of Lemma 3 is chosen small enough, so that C is less then 1, then $\{\exp(j\mu_n\omega)\}$ is also a frame in $L^2[-\gamma, \gamma]$. Moreover, there exists some $\delta_{1/4}(\{\lambda_n\}, \gamma)$, such that whenever $\delta < \delta_{1/4}(\{\lambda_n\}, \gamma)$, $\{\exp(j\mu_n\omega)\}$ is a frame with frame bounds $A/4$ and $9B/4$.

Appendix B

Recall that in order to have a sequence of quantization threshold crossings of a σ-bandlimited signal $f(t)$, having a uniform density $d > \sigma/\pi$, the quantization step q must change in time following the decay of the signal. Since $f(t)$ is square integrable, $q = q(t)$ has to decay at least as fast as $1/t$, although it can be fixed on a given set of time segments.

Let A and B denote again the lower and the upper bound, respectively, of the frame determined by the quantization threshold crossings, $\{x_n\}$, of $f(t)$, and let $g(t)$ be a consistent estimate of $f(t)$. Following the discussion in the proof of Theorem 1 (see (14)), the error norm is bounded as

$$\|e(t)\|^2 < \frac{\tau^2}{A} \sum_n |g'(\epsilon_n)|^2. \tag{40}$$

Since τ is smaller then the Nyquist sampling interval, $|x_n - \epsilon_n| < \pi/\sigma$ for all n. According to Lemma 2, there exists a constant $C = C(\{x_n\}, \sigma)$ such that for all sampling intervals $\tau < \pi/\sigma$

$$\sum_n |g'(\epsilon_n)|^2 \leq C \sum_n |g'(x_n)|^2. \tag{41}$$

This gives

$$\|e(t)\|^2 \leq \frac{\sigma^2 BC\tau^2}{A} \|g(t)\|^2. \tag{42}$$

Being a consistent estimate of $f(t)$, $g(t)$ can not differ from $f(t)$ by more than $q(n\tau)$, at time instants $\{n\tau\}$. The energy of $g(t)$ can be bounded, by considering its samples at these points, as

$$\begin{aligned} \|g(t)\|^2 &= \tau \sum_n |g(n\tau)|^2 && (43) \\ &\leq \tau \sum_n \left(|f(n\tau)|^2 + 2|q(n\tau)f(n\tau)| + |q(n\tau)|^2\right) && (44) \\ &= \|f(t)\|^2 + E_s(\tau). && (45) \end{aligned}$$

Note that

$$E_s(\tau) = \tau \sum_n \left(2|q(n\tau)g(n\tau)| + |q(n\tau)|^2\right)$$

converges to

$$\lim_{\tau \to 0} E_s(\tau) = \int_{-\infty}^{+\infty} \left(2|q(t)g(t)| + |q(t)|^2\right) dt, \tag{46}$$

which has to be finite since $q(t) = O(f(t))$ and $f(t)$ is square integrable. Therefore, $E_s(\tau)$ has to be bounded by some E which does not depend on τ. This gives the following error bound

$$\|e(t)\|^2 \leq \frac{\sigma^2 BC}{A} \left(\|f(t)\|^2 + E\right) \tau^2. \tag{47}$$

Appendix C

The notion of frames has been extended by Duffin and Schaeffer beyond the context of nonharmonic Fourier expansions [10]. It is given by the following definition.

Definition 2 *A family of functions* $\{\varphi_j\}_{j\in J}$ *in a Hilbert space* $\mathcal{H}$ *is called a frame, if there exist two constants* $0 < A \leq B < \infty$ *such that for all* f *in* $\mathcal{H}$

$$A\|f\|^2 \leq \sum_{j\in J} |\langle \varphi_j, f\rangle|^2 \leq B\|f\|^2. \tag{48}$$

Constants A and B are called frame bounds. If they are equal,

$$\sum_{j\in J} |\langle \varphi_j, f\rangle|^2 = A\|f\|^2, \tag{49}$$

and the frame is called *tight frame*. The frame condition (48) ensures that any f in $\mathcal{H}$ can be reconstructed in a numerically stable way from coefficients $\{c_j : c_j = \langle \varphi_j, f\rangle\}_{j\in J}$. We will refer to these coefficients as coefficients of frame expansion, or frame coefficients.

A frame $\{\varphi_{i,j}\}_{(i,j)\in \mathbf{Z}^2}$ in $L^2(\mathbf{R})$ is said to be a Weyl-Heisenberg frame if the frame vectors are obtained by translating a window function $\varphi(t)$ in time and frequency with steps t_0 and ω_0. The frame vectors then have the form:

$$\varphi_{m,n}(t) = \varphi(t - nt_0)e^{jm\omega_0 t}. \tag{50}$$

If a Weyl-Heisenberg frame is tight and its window function is normalized to unit energy, then the frame bound is equal to $2\pi/\omega_0 t_0$ [7, pp. 81-83]. A limitation of Weyl-Heisenberg frames is that no frames exist for $\omega_0 t_0 > 2\pi$ [7, pp. 81-83], which actually means that the frame vectors have to be distributed in the time-frequency plane with a sufficient density in order to span the whole space. The critical density is given by $\omega_0 t_0 = 2\pi$, which corresponds to bases. For $\omega_0 t_0 < 2\pi$ the frame is redundant, so that the ratio $r = 2\pi/\omega_0, t_0$ can be interpreted as its redundancy factor, or oversampling ratio.

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