

Copyright © 1995, by the author(s).
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

**CONNECTING INFINITY: COMPLEX ENCIRCLEMENT
FOR FINDING ALL CIRCUIT SOLUTIONS**

by

Denise M. Wolf and Seth R. Sanders

Memorandum No. UCB/ERL M95/87

1 September 1995

COVER PAGE

**CONNECTING INFINITY: COMPLEX ENCIRCLEMENT
FOR FINDING ALL CIRCUIT SOLUTIONS**

by

Denise M. Wolf and Seth R. Sanders

Memorandum No. UCB/ERL M95/87

1 September 1995

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Connecting Infinity: Complex Encirclement for Finding All Circuit Solutions *

Denise M. Wolf and Seth R. Sanders

Department of Electrical Engineering and Computer Sciences
University of California, Berkeley CA 94720
e-mail: dwolf@eecs.berkeley.edu and sanders@eecs.berkeley.edu

Abstract

This memo contains a working version of a chapter of Denise Wolf's thesis entitled "Connecting Infinity: Complex Encirclement". The abstract, introduction and table of contents of this thesis, entitled *Multi-Parameter Homotopy and Complex Encirclement: Finding DC Operating Points and Periodic Orbits of Nonlinear Circuits*, are also included.

*This work was supported by SRC contract 95-DC-324, and grants from Tandem Computers and the UC Micro Program.

Abstract

Multi-Parameter Homotopy and Complex Encirclement: Finding DC Operating Points and Periodic Orbits of Nonlinear Circuits

by

Denise Miriam Wolf

Doctor of Philosophy in Electrical Engineering and Computer Science

University of California at Berkeley

Professor Seth R. Sanders, Chair

This thesis develops algorithms and underlying theory for calculating dc operating points and periodic orbits of nonlinear circuits. The focus is on homotopy continuation methods, a class of numerical techniques for solving systems of nonlinear algebraic equations based on higher-dimensional function *embedding* and solution *tracing*.

In Part I, the focus is on finding dc operating points. Part I introduces real and complex multi-parameter homotopy methods for finding dc solutions of nonlinear circuits, and shows that they can avoid folds and bifurcation points along solution paths, and can find multiple solutions. Results from algebraic geometry indicate that given an irreducible, analytic homotopy function H with a single complex parameter λ , regular paths through the complex parameter plane connect any solution of $H(x, \lambda_f) = 0$ to any other solution of $H(x, \lambda_f) = 0$.

This part also develops *complex encirclement*, a novel way of computing all solutions of nonlinear systems of equations, based on the connectivity properties of complex solution space. For circuit equations with a finite number of complex solutions, the idea is to *design a homotopy function that forces all circuit solutions to be locally connected around a single complex branch point*.

The connectivity of *real* homotopy functions is also investigated. Part I concludes with a preliminary analysis of the relationship between circuit dimensionality, real level set characteristics, and the number/placement of real parameters guaranteeing real operating point connectivity for a class of homotopy functions that uses additive terms in the circuit

equations as homotopy parameters

Part II of this thesis focuses on the dynamics of power electronic circuits, their bifurcations and Poincaré map properties, and the application of homotopy methods to finding periodic orbits. Part II studies in detail cyclic fold bifurcations, period doubling bifurcations, and bifurcations due to Poincaré map discontinuities. Multi-parameter homotopy methods are shown able to determine periodic orbits of circuits with sufficiently smooth Poincaré maps. The application of multi-parameter methods to finding periodic orbits is compared and contrasted to the respective problems of finding dc operating points.

Professor Seth R. Sanders
Dissertation Committee Chair

Contents

| | |
|--|--------------|
| List of Figures | viii |
| List of Tables | xviii |
| 1 Introduction and Outline | 1 |
| 1.1 Scope of Thesis | 2 |
| 1.2 Contents/Contributions | 5 |
| I DC Operating Points | 8 |
| 2 Background | 9 |
| 2.1 The DC Operating Point Problem | 9 |
| 2.1.1 Number of Solutions | 11 |
| 2.2 Homotopy Methods | 14 |
| 2.2.1 Homotopy Methods Defined | 15 |
| 2.2.2 Applicable Problems | 18 |
| 2.2.3 Previous Work - Finding One DC Operating Point | 19 |
| 2.2.4 Previous Work - Finding Multiple Operating Points | 22 |
| 2.3 Summary | 26 |
| 3 Multi-Parameter Homotopy for Finding DC Operating Points | 27 |
| 3.1 Chapter Contents | 28 |
| 3.2 Multi-Parameter Maps and Methods | 29 |
| 3.3 Avoiding Folds | 30 |
| 3.4 Avoiding Real Fork Bifurcations | 40 |
| 3.5 Finding Multiple Solutions: Global Connectivity in Complex Space | 47 |
| 3.6 Conclusion/Summary | 55 |
| 4 Connecting Infinity: Complex Encirclement | 57 |
| 4.1 Introduction | 58 |
| 4.1.1 Basic Idea and Research Goals: In a Nutshell | 62 |
| 4.2 Definitions and Mathematical Preliminaries | 66 |
| 4.2.1 Local Representation of an Analytic Function as a Polynomial | 69 |

| | | |
|-----------|---|------------|
| 4.2.2 | Fundamental Theorem on Connectivity | 70 |
| 4.3 | One Equation, One Unknown: Polynomials | 71 |
| 4.4 | Two Equations, Two Unknowns: Polynomials | 83 |
| 4.4.1 | 2-D Polynomial Example Series | 93 |
| 4.5 | Polynomials: n-D | 112 |
| 4.6 | Extension to Non-Polynomial-Bounded Analytic Functions, Transistor Circuit Models | 115 |
| 4.6.1 | Parameterized Scalar Equations With Exponential Terms | 116 |
| 4.7 | Conclusion/Future Work | 127 |
| 5 | A Perspective on Real Space Connectivity | 130 |
| 5.1 | Introduction | 130 |
| 5.2 | A Simple Class of Homotopy Functions | 132 |
| 5.3 | Connectivity: Real functions of 1 variable | 133 |
| 5.4 | Connectivity: Real functions of 2 variables | 139 |
| 5.4.1 | 1 Real Parameter: Solution Curves | 141 |
| 5.4.2 | 2 Real Parameters: Solution Surface | 152 |
| 5.5 | Connectivity: Functions of n variables | 154 |
| 5.5.1 | Comparison Between Real and Complex Connectivity in n-D | 156 |
| 5.6 | Application: 1-Port Circuits, Connectivity | 156 |
| 5.6.1 | Driving Point Characteristic Connectivity and Source Placement | 157 |
| 5.7 | Summary and Future Work | 163 |
| II | Periodic Orbits | 165 |
| 6 | Bifurcation of Periodic Orbits | 167 |
| 6.1 | Introduction | 168 |
| 6.2 | Mathematical Preliminaries | 170 |
| 6.3 | Cyclic Fold Bifurcations | 171 |
| 6.3.1 | Cyclic Folds - Differentiable Poincaré Maps | 173 |
| 6.4 | Period Doubling Bifurcations | 179 |
| 6.4.1 | An Introduction to Period Doubling | 179 |
| 6.4.2 | A Single Period Doubling | 180 |
| 6.4.3 | Period Doubling Cascade to Chaos | 186 |
| 6.4.4 | Period Doubling Genericity Depends upon Orbital Asymmetry | 192 |
| 6.5 | Bifurcations of Discontinuous Poincare Maps | 194 |
| 6.5.1 | Introduction | 194 |
| 6.5.2 | Circuit Example (Thyristor Controlled Reactor) | 197 |
| 6.5.3 | Generalization: Switched Circuits with Discontinuous Maps | 214 |
| 6.6 | Summary | 216 |

| | | |
|----------|---|------------|
| 7 | Multi-Parameter Homotopy for Finding Periodic Orbits | 217 |
| 7.1 | Introduction | 218 |
| 7.2 | Multi-Parameter Homotopy | 220 |
| 7.3 | Avoiding Cyclic Fold Bifurcations | 221 |
| 7.4 | Avoiding period-doubling bifurcations | 225 |
| 7.5 | Summary | 232 |
| 8 | Summary and Future Work | 234 |
| 8.1 | Future Work | 236 |
| A | Lyapunov-Schmidt Reduction | 239 |
| B | Global Connectivity Property of Analytic Varieties | 241 |
| C | Cyclic Folds, Continuous, Non-Differentiable Poincare Maps | 243 |
| | Bibliography | 248 |

Chapter 1

Introduction and Outline

The goal of this thesis is to develop algorithms and underlying theory for calculating all dc operating points of circuits with nonlinear elements like transistors and junction diodes, and all periodic solutions of periodically forced circuits, such as those found in power electronics. Because these circuits are known to possess multiple solutions, and some even depend upon their existence for operation, the ability to find multiple solutions and classify their stability is a potentially valuable feature of a circuit simulation program. Since Newton-type methods are not always convergent for nonlinear circuits, and are not well suited to finding multiple solutions, we focus on homotopy continuation approaches.

Homotopy continuation methods have been applied to the task of finding one or multiple dc solutions of nonlinear circuits, and more recently, to finding periodic solutions of circuits as well. Homotopy methods are numerical techniques for solving systems of nonlinear algebraic equations ($F(x) = 0, F : S \rightarrow S$) based on higher-dimensional function *embedding* and solution *tracing*. A continuously differentiable homotopy mapping $H : S \times U \rightarrow S$ satisfying the properties

1. $H(x, \lambda_0) = 0$ is relatively easy to solve and
2. $H(x, \lambda_f) = F(x)$

is obtained, and one or more solutions to $F(x) = 0$ are then traced by following the solution(s) of $H(x, \lambda) = 0$ from $\lambda = \lambda_0$ to $\lambda = \lambda_f$. From a circuit perspective, the idea behind a continuation method for finding a circuit solution is to embed a parameter in the circuit's nonlinear equations. By setting the parameter to zero, the problem is reduced to a simple problem that can be solved easily, or whose solution is known. The solution to the

simple problem is the starting point of a continuation path. The set of equations is then continuously deformed into the originally posed difficult problem. The main advantage of homotopy methods over Newton-type methods is that the region of convergence can be large or even global, and homotopy methods are well suited to finding multiple solutions.

Though better than locally convergent methods, existing homotopy and continuation methods may still encounter problems. In the absence of constraints, solution paths can bifurcate, fold sharply, diverge to infinity, or not lead to a solution. Solution curves may be closely spaced, and solution sets can be disconnected, which is a problem for so-called Lambda Threading type algorithms, which follow a real single solution path past $\lambda = \lambda_f$ in the hope that the curve will reverse direction and pass through $\lambda = \lambda_f$ multiple times. Efficiency is always an issue, and the problem of finding all solutions remains largely unsolved.

This thesis investigates novel approaches to developing homotopy methods for finding dc operating points and periodic orbits of nonlinear circuits. To this end, it synthesizes the subjects of algebraic geometry, homotopy-based computation, and circuit theory. The remainder of this chapter details our approach and summarizes the contents of this work.

1.1 Scope of Thesis

This thesis integrates methodologies from three disciplines: *Algebraic geometry*, *Computation*, and *Circuit Simulation/Analysis*. The following reasoning connects these three subjects into an interdisciplinary whole.

Designing a homotopy method can be thought of as involving the two steps:

1. Design a homotopy function H . Ideally, it should be one that guarantees certain algorithmic properties like existence of solutions, number of accessible solutions, and efficiency.
2. Design a solution curve (or surface) tracing scheme compatible with the chosen homotopy function - one that possesses desirable algorithmic properties.

A homotopy function H is a system of nonlinear equations with more variables than equations. Such a function has a solution set consisting of curves or surfaces. In order to develop 'good' homotopy methods, one would like to design homotopy functions

with qualities that lead to positive algorithmic properties like efficiency and accessibility of solutions. To do this, one must understand the nature of solution spaces, and what makes a homotopy solution curve or surface relatively good or bad, computationally speaking. This leads to the topic of algebraic geometry, the study of the nature of real and complex solution sets of analytic systems of equations.

This thesis seeks to understand fundamental properties of solution sets in real and complex space, and to see (1) what they have to do with homotopy method computation, and (2) whether this understanding can be used to come up with new and improved ways of computing solutions to nonlinear equations.

Questions also arise about the kinds of equations associated with calculating dc operating points, given the existence of a known set of circuit components and a range of circuit topologies. Ideally, circuit properties used in conjunction with insight into algebraic geometry should aid in the design of homotopy functions and associated methods.

The periodic orbit finding problem adds the complication of working with a two-point boundary value problem rather than directly with a system of nonlinear equations. The boundary value problem must be formulated as a system of nonlinear equations, and we seek to understand (1) what is the effect of having the natural domain of the problem be non-algebraic, and (2) for a particular class of circuits, say power electronic circuits, what properties do the associated algebraic formulations have, and how do these properties relate to the appropriateness of applying homotopy methods to finding periodic orbits.

Questions Addressed:

Some questions this thesis addresses are listed below, as are some fundamental properties of solution spaces that we explore in an effort to answer them.

- Can adding extra real or complex parameters to a typical single-parameter homotopy function, effectively increasing the dimension of the solution space available for maneuvering, lead to improved homotopy methods?
- If so, what is the minimum number of real and/or complex homotopy parameters that must be added to avoid folds, forks and other bifurcations along solution paths? How can this avoidance be accomplished, and will all locally linked solution branches be accessible?

- What is the minimum number of complex homotopy parameters that must be added to guarantee a solution set that connects all circuit solutions? What is the nature of these solution sets, and how might one exploit knowledge of the solution space topologies in order to develop algorithms for finding all solutions?
- Is it possible to convert a *global* connectivity property of complex space to a *local* connectivity property? If so, how? In particular, is it possible to connect all solutions of a circuit in a single algebraic element around infinity?
- Do the answers to the above questions differ for finding dc operating points and periodic orbits?
- Because the periodic-orbit finding problem is naturally posed as a two-point boundary value problem rather than a system of algebraic equations, a question arises regarding parameter placement. When finding periodic orbits of circuits, does it make a difference whether homotopy parameters are added directly to the differential circuit equations, or to the algebraic formulation of the two-point boundary value problem? If so, how and why?
- What are the properties of power electronic circuits and the periodic orbits associated with them? How about the relationship between the Poincare map class of such circuits, their periodic-orbit bifurcations, and the potential for applying the homotopy concepts developed for finding dc operating points to finding periodic orbits of power electronic circuits?

Fundamental Properties:

A preview of some fundamental properties of real and complex solution spaces that we use to provide answers to these questions are as follows. Given a homotopy function $H(x, \lambda): S \times U \rightarrow S$,

1. *Bifurcation sets* of $H(x, \lambda) = 0$ are *codimension* one in real space, and codimension two in complex space.
2. If the equations $H(x, \lambda) = 0$ are *irreducible*, then the complex solution space of $H(x, \lambda) = 0$ is *globally connected*.

3. There are two kinds of *branch points*, *singular* and *nonsingular*. There are two kinds of singular branch points, *locally reducible* and *locally irreducible*.
4. A branch point of multiplicity r serves to connect all r converging solution branches of $H(x, \lambda) = 0$ in a single *algebraic element*, if the function is locally irreducible at the branch point.
5. A system of equations may be parameterized in such a way that it has the same number of complex solutions for almost all parameter values. We call this property *conservation of solution number*.
6. The concept of *bifurcation set inheritance*. Given a homotopy function derived from an algebraic formulation of a two-point boundary value problem, the homotopy method will inherit the bifurcation set either of a generic algebraic system, or a generic dynamical system, depending on how the homotopy parameters are added.

Italicized terms are defined in Section 4.2, Chapter 2, and Chapter 7. All of these properties are woven into the coming chapters. Properties (1) and (2) help to answer questions on the potential of multi-parameter and complex-parameter homotopy in Chapter 3. Properties (4) and (5) provide a basis for Chapter 4, in which homotopy functions that connect all roots around a single branch point are designed. Concept (6) bedrocks the results in Chapter 7.

This thesis is divided into two parts. The first part deals with finding dc operating points, and the second part focuses on finding periodic orbits of power electronic circuits.

1.2 Contents/Contributions

- **Part I** focuses on applying homotopy methods to the dc operating point problem.
- *Chapter 2* presents background information on homotopy methods in general, and material related to part I of the thesis. It describes the dc operating point problem and homotopy methods. Previous work applying homotopy methods to finding one or multiple dc operating points is summarized, and the research presented in Part I of the thesis is placed in context.

- *Chapter 3* introduces multi-parameter homotopy methods for finding dc operating points. The question of whether adding extra real or complex parameters to a single-parameter homotopy function can lead to improved solution paths is investigated. It is shown that no number of added real parameters can lead to fold avoidance, but that generic folds may be efficiently avoided by complexifying the homotopy parameter and tracing a closed curve in complex parameter space around the critical fold value. A combination of real 2-parameter homotopy and complex parameter homotopy is shown to be sufficient for avoiding real fork bifurcations and enumerating all real, locally connected branches. Also, the potential of complex parameter homotopy methods for finding all circuit solutions is explored. Applying results from algebraic geometry indicates that given an irreducible, analytic homotopy function H with a single complex parameter λ , there exist regular paths through the complex parameter plane connecting any solution of $H(x, \lambda') = 0$ to any other solution of $H(x, \lambda') = 0$. So, in principle at least, complex parameter homotopy can be used to find all circuit solutions. *Appendices A and B* are associated with this chapter.
- *Chapter 4* is the heart of the thesis. The idea this chapter explores is that of transforming the property of distributed global connectivity of complex solution surfaces discussed in Chapter 2 to a *localized* connectivity property, through a well-reasoned choice of homotopy function. In particular, the idea is to *design homotopy functions that force all circuit solutions to be locally connected around a single complex branch point at infinity*. Given such a homotopy function, repeatedly encircling this branch point results in all solutions being calculated, with a built in stopping criterion inherent in the cyclic nature of the local solution structure. This thesis will refer to the notion of (1) designing a homotopy function to connect all solutions algebraically around a single point, and (2) the accompanying algorithm, which involves repeatedly encircling this branch point to find all solutions, *complex encirclement*.
- *Chapter 5* considers the subject of real homotopy functions and their associated solution sets, a topic of importance when considering homotopy methods that rely on real solution set connectivity to find all circuit solutions. This chapter contains a preliminary exploration of the real solution set connectivity of arbitrary, parameterized nonlinear circuits. The relationship between circuit dimensionality, real level set characteristics, and the number/placement of real parameters guaranteeing real operating

point connectivity is examined for a class of homotopy functions that uses constant terms in the circuit equations as homotopy parameters. A full set of results in 1- and 2-d are included, along with weaker results in higher dimensions.

- **Part II** focuses on applying homotopy methods to finding periodic orbits of periodically forced nonlinear circuits in general, and power electronic circuits in particular.
- *Chapter 6* studies various bifurcations of periodic orbits in power electronic circuits: cyclic fold bifurcations, period doubling bifurcations, and bifurcations due to Poincaré map discontinuities. An interesting feature of power electronic circuits is that they may have Poincaré maps that are continuous but not everywhere differentiable, or that are even discontinuous. Chapter 6 gives a comprehensive overview of period-doubling phenomena in closed-loop DC-DC conversion circuits. Also, bifurcation behavior in a thyristor controlled VAR compensator, understood in terms of Poincaré map discontinuities, is studied in detail. *Appendix C* is associated with this chapter. This chapter is a prelude to Chapter 7.
- *Chapter 7* applies material developed in Chapter 3 to the problem of finding periodic orbits of nonlinear circuits, such as those found in power electronic applications with differentiable Poincaré maps. This chapter focus on differences between applying multi-parameter homotopy to finding dc operating points and to finding periodic orbits. These differences flow from the special structure of the algebraic equations describing two point boundary value problems. The chapter shows that multi-parameter homotopy methods can avoid period-doubling and cyclic fold bifurcations along solution paths, and find all stable and unstable periodic solutions along folding or period-doubling paths. A distinction is made between *circuit-direct* and *formulation-indirect* homotopy (embedding parameters directly in the differential equations vs. embedding them in the algebraic formulation of the 2-point boundary value problem), and it is shown that the latter (with two real parameters) can avoid period-doubling bifurcations, while the former cannot. A formulation-indirect homotopy function may be interpreted dynamically as the addition of a time-varying, parameterized source to the circuit.
- *Chapter 8* summarizes the work presented in this thesis, and outlines some ideas for related future work.

Chapter 4

Connecting Infinity: Complex Encirclement

In principle at least, complex parameter homotopy has the potential for finding all dc operating points of nonlinear circuits with analytic circuit element models. This potential exists because complex solution manifolds of irreducible analytic homotopy functions are completely connected over the complex parameter plane, with an overall structure determined by the location of the branch points.

This chapter explores one approach to utilizing this global complex connectivity property, that of *deliberately designing a homotopy function that is constructed so that all roots are locally connected in the neighborhood of infinity*.

For analytic circuit equations with a finite number of complex roots (either polynomial, or polynomial-bounded), the idea is to design a complex homotopy function with all roots connected in a single algebraic element of finite order. Repeatedly encircling this branch point, located at infinity, results in the numerical calculation of all solutions, and there is a built in stopping criterion inherent in the cyclic nature of the local solution structure. We call the notion of (1) designing a homotopy function to connect all solutions algebraically around a branch point, and (2) the accompanying algorithm, which involves repeatedly encircling this branch point to find all solutions, *complex encirclement*.

The main geometric property on which this approach is based relates *local irreducibility* of a branch point to the existence of a single algebraic element connecting all coalescing solution branches. In short, a branch point of multiplicity r serves to connect all

r converging solution branches of $H(x, \lambda) = 0$ in a single algebraic element, if the function is locally irreducible at the branch point. Consequently, the goal is to design homotopy functions with a locally irreducible branch point of maximal multiplicity, in order to locally connect all circuit solutions.

This chapter also deals with analytic circuit equations with exponentials in them, which are fundamentally different from polynomial systems of equations in that they generally have an *infinite* number of complex roots. The notion of complex encirclement, initially developed on polynomial-type equations, can be generalized to systems with an infinite number of solutions. In this case the goal is to design a homotopy function that has all roots going to infinity along with a parameter, and that can be used to find all complex roots of the circuit equations *within a compact space* $|x| \leq r$, via complex encirclement. As for polynomial circuit equations, the notion of local irreducibility in the neighborhood of infinity is key to the success of this approach.

The chapter is organized by function dimension and type, and our development proceeds from scalar polynomials, to higher dimensional polynomials, to analytic systems with exponentials in them. Ideas are illustrated on circuit and function examples, and with line drawings.

4.1 Introduction

In the previous chapter, real and complex multi-parameter homotopy methods for solving nonlinear circuits were introduced, and their potential for avoiding folds and bifurcations along solution paths, and for finding all solutions, were explored. We found that, in principle at least, complex parameter homotopy has the potential for finding all solutions. This potential exists because complex solution manifolds are connected over the complex parameter plane. Given an irreducible, analytic homotopy function H with a complex parameter λ , there exist regular paths through the complex parameter plane connecting any solution of $H(x, \lambda') = 0$ to any other solution of $H(x, \lambda') = 0$. As explained in Section 3.5, this connectivity may be understood in terms of the location of branch points and branch cuts, which are generally not known a priori.

Given an arbitrary irreducible homotopy function, this *global connectivity property* is a composite effect of the local connectivity at each branch point. That is, each *branch point* generally connects only a small number of the n solution surfaces (perhaps two), as

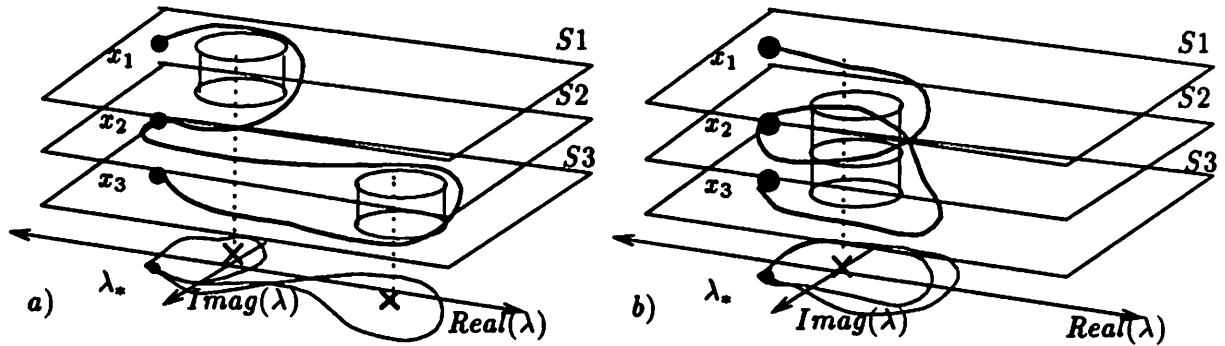


Figure 4.1: a) Complex homotopy function with distributed global connectivity. More than one branch point must be encircled to find all three solutions. b) Complex homotopy function with localized global connectivity. Only one branch point must be encircled (twice) to find all solutions.

illustrated in Figure 4.1a for the case of a function of degree three with two branch points corresponding to repeated roots of multiplicity two. In general, *no one branch point locally connects all n solution surfaces*. Thus, global connectivity is a *collective* property of the solution surface connections at all branch points. This is an important consideration in approaching the next logical step of this research - the development of a homotopy-type algorithm that uses the global complex connectivity property to its advantage.

Designing a homotopy method can be thought of as involving the two following steps.

1. Design a homotopy function. Ideally, it should be one that guarantees certain algorithmic properties like existence of solutions, number of accessible solutions, and efficiency.
2. Design a solution curve (or surface) tracing scheme compatible with the chosen homotopy function - one that possesses desirable algorithmic properties.

We want to develop a deterministic complex-space algorithm provably guaranteed to find all dc operating points, and to focus on step (1) of the homotopy method design process in an attempt to reach this objective.

The idea explored in this chapter is that of transforming the property of distributed global connectivity of complex solution surfaces discussed in Section 3.5 to a *localized* con-

nectivity property, through a well-reasoned choice of homotopy function. We *deliberately design a homotopy function that is constructed so that all roots are locally connected in the neighborhood of infinity.*

For analytic circuit equations with a finite number of complex roots (either polynomial, or polynomial-bounded), the idea is to *design homotopy functions that force all circuit solutions to be locally connected around a single complex branch point in a single algebraic element.* Given such a homotopy function, repeatedly encircling this branch point would result in all solutions being calculated, with a built in stopping criterion inherent in the cyclic nature of the local solution structure.

The idea is illustrated in Figure 4.1. In this figure we assume that the problem to be solved is to find all three isolated roots of a function $F(x) = 0$, and that two homotopy functions embedding F have been formed, H_1 and H_2 , the first being irreducible, with the overall connectivity distributed over a region (the typical scenario), and the second being *specifically designed to connect all three sheets at a single point.* Assume that $H_2(x, \lambda_*) = F(x)$, and that one solution x_1 of $H_2(x, \lambda_*) = F(x) = 0$ is given. Because all three sheets are connected in a single algebraic element, the other two solutions of $F(x) = 0$, x_2 and x_3 , can be found by encircling the branch point twice, as shown in Figure 4.1b. A third encirclement leads right back to x_1 , so we are automatically notified that all roots have been calculated and there is no need to look further. Thus, there is a built in stopping criterion in the cyclic nature of the algebraic element that connects all three solutions of the function.

The simple maneuver of locally encircling a single branch point in an attempt to calculate all three roots of $F(x) = 0$ would not be successful for our typical homotopy function H_1 with its distributed connectivity, as illustrated in Figure 4.1a. This is because the connectivity of this function is distributed across a pair of branch points, so encircling any one branch point would lead to a calculation of fewer than three roots.

We call the notion of (1) designing a homotopy function to connect all solutions algebraically around a single point, and (2) the accompanying algorithm, which involves repeatedly encircling this branch point to find all solutions, *complex encirclement.*

This chapter also deals with analytic circuit equations with exponentials in them, which are fundamentally different from polynomial systems of equations in that they generally have an *infinite* number of complex roots. For engineers, this is an important class of systems, because many circuit element models, such as the Ebers-Moll bipolar transistor

model, include exponential terms. The notion of complex encirclement, initially developed on polynomial-type equations, can be generalized to systems with an infinite number of solutions. In this case the goal is to design a homotopy function that has all roots going to infinity along with a parameter, and that can be used to find all complex roots of the circuit equations *within a compact space* $|x| \leq r$, via complex encirclement.

The remainder of the chapter focuses on developing the idea of complex encirclement around infinity. We begin with one polynomial equation in one unknown, where some of the answers can be found by inspection, or in classical root locus theory [49], or through other simple analyses. Then we move on to an analysis of higher dimensional polynomial equations, and then to non-polynomial equations and transistor circuits.

The chapter is organized as follows. In Section 4.2 we define terms, and present some mathematical preliminaries. Section 4.3 deals with one dimensional, parameterized polynomials. We design a homotopy function with an infinite branch point, and show that all roots of the equation are connected in a single algebraic element around infinity. We also address the question of exactly what it means, in practical terms, to encircle infinity, and provide insight on how to visualize complex solution space. We end the section with some examples, and a brief discussion of the kinds of complications one may encounter if the parameter appears non-linearly in the equation.

In Section 4.4, polynomial homotopy functions with two equations, two unknowns, and a complex parameter are investigated. A general homotopy function form is given, with the parameter appearing linearly, and various potential root constellations are discussed. Then a homotopy function design is presented, which is guaranteed to have all roots going to infinity with the parameter, along with accompanying sufficient conditions on the original problem of interest. Following that, some necessary and sufficient conditions ensuring that the homotopy function will have all roots connected in a single algebraic element around infinity are proved. The subsection ends with a series of examples illustrating the geometric and algebraic facets of these results. Concepts important to this section include winding numbers, complex solution curves seen as paths along tori, and local irreducibility around branch points.

Section 4.5 generalizes basic results of the previous sections to higher dimensional polynomial systems, and outlines a complete complex encirclement algorithm. Also, the issue of conservation of solution number and its importance to complex encirclement is discussed.

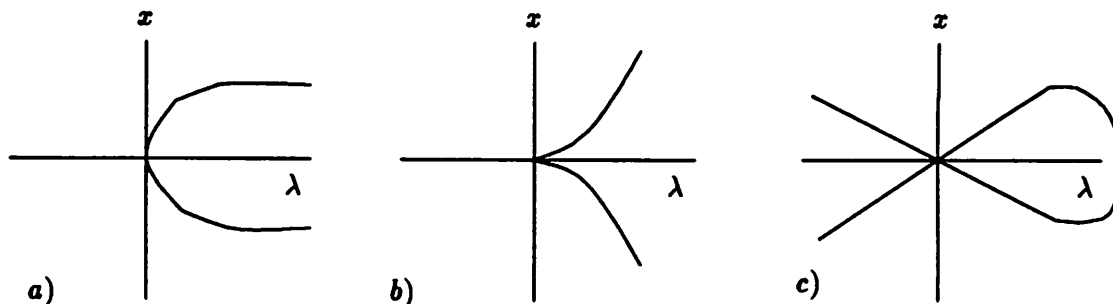


Figure 4.2: a) Nonsingular branch point. b) Singular, locally irreducible branch point. c) Singular, locally reducible branch point.

The chapter ends with Section 4.6, which considers analytic systems of nonlinear equations that include exponential as well as polynomial terms, systems that are fundamentally different from polynomials because they can have an infinite number of complex roots. The observations made in this section point to a more general notion of complex encirclement for systems with an infinite number of complex solutions and branch points. We conjecture that for certain types of functions, including those with exponential terms that are linear in the argument (some diode and transistor models), a homotopy function that has all roots going to infinity along with a parameter, and that is locally irreducible at infinity, can be used to find any number of roots of $h(x, \lambda_f) = 0$. Specifically, numerical experiments suggest that *complex encirclement* can be used to find all roots of $H(x, \lambda_f) = 0$ within the compact space $|x| \leq r$, along regular paths.

Also, ideas for related future work are presented in a conclusion. But first, we include a subsection containing a short self-contained summary of some of the main ideas and objectives of this chapter.

4.1.1 Basic Idea and Research Goals: In a Nutshell

The purpose of this subsection is to present a concise account of the main ideas and goals of the portion of this chapter that focuses on finding solutions of circuit equations with a *finite* number of complex roots. We define repeated root and branch point, and classify branch points as being either singular or nonsingular, locally reducible or locally irreducible. We state the main geometric property on which this portion of the chapter is based, that being the relationship between local irreducibility of a branch point and the

existence of a single algebraic element connecting all coalescing solution branches. Then an infinite branch point is defined, as is the encirclement of infinity. We end with a brief presentation of the goals of this research, primary among them that of designing homotopy functions with an infinite branch point connecting all circuit solutions in a single algebraic element. Accompanying algorithms are outlined. For a discussion of the issues associated with finding solutions of circuit equations with an infinite number of complex roots, see Section 4.6.

Repeated root, branch point:

A *branch point* is a parameter value $\lambda \in C$ at which the homotopy function $H(x, \lambda) = 0, H : C^n \times C \rightarrow C^n$ has a *repeated root* of multiplicity $r > 1$.

Two kinds of branch points, singular and nonsingular:

There are two kinds of branch points, singular and nonsingular. At a nonsingular branch point, the extended Jacobian matrix $[\partial H/\partial x, \partial H/\partial \lambda]$ has full rank. At a singular branch point, the extended Jacobian matrix drops rank. Figure 4.2a shows a nonsingular branch point $\lambda = 0$ with a repeated root $x = 0$. For example, the function $h(x, \lambda) = x^2 + \lambda = 0$ has a nonsingular branch point at $\lambda = 0$.

Two kinds of singular branch points, locally reducible and locally irreducible:

There are two kinds of singular branch points, those at which the function is locally reducible, and those at which the function is locally irreducible. Figure 4.2b shows a locally irreducible singular branch point at $\lambda = 0$ ($h(x, \lambda) = \lambda^3 + x^2 = 0$), while 4.2c shows a locally reducible branch point at $\lambda = 0$ ($h(x, \lambda) = x^2 + \lambda^2 + \lambda^3 = 0$). A function of two variables $f(x, \lambda) = 0$ that is locally reducible (and analytic) at a branch point may be locally represented as the product of two non-trivial power series, $f = gh$. A locally irreducible function is not locally reducible.

Locally irreducible implies connected manifold (Fundamental Property):

A branch point of multiplicity r serves to connect all r converging solution branches

of $H(x, \lambda) = 0$ in a single algebraic element, if the function is locally irreducible at the branch point. (Consequence of proposition A5 in Section 4.2)

Nonsingular branch points are irreducible, connected:

If a branch point is nonsingular, it is also irreducible. Therefore all branches coalescing in a nonsingular branch point are connected in a single algebraic element. This curve has a winding number of one, meaning that encircling the branch point r times in the parameter space λ results in the calculation of a complete solution cycle of $H = 0$, and the associated solution curve $x(\lambda)$ winds around the repeated root value x^* exactly once.

Some singular branch points are irreducible, connected:

Singular branch points that are irreducible have all coalescing branches connected in a single algebraic element. This curve will have a winding number greater than one. The functions $x(x + \lambda) = 0$ and $x^3 + \lambda^2 = 0$ are examples of reducible and locally irreducible functions, with branch points at $\lambda = 0$. The latter has a single algebraic element connecting all three roots with a winding number of two.

A branch point at infinity:

The function $H(x, \lambda) = 0$ has a branch point at infinity if more than one root x of H goes to infinity as the parameter λ goes to infinity. Meaning that $\lambda \rightarrow \infty$ implies that $(x_1, \dots, x_n) \rightarrow (\infty, \dots, \infty)$. $H(x, \lambda) = 0$ is said to have an infinite branch point at $\lambda = \infty$ of multiplicity r , where r is the number of coalescing complex branches.

Encircling infinity:

As explained in Section 4.3, tracing a circle through the complex parameter plane $\lambda = |\lambda_*|e^{j\theta}$, $\theta = 0 : 2\pi$, with the radius of the circle λ_* chosen to be large enough so that the complex circle contains *all finite branch points* of the homotopy function, is *equivalent to encircling infinity*. Since the location of complex branch points is not known apriori, locating the connecting branch point at infinity is useful. Any large enough circle traced through the complex parameter plane will connect all solutions of an algebraic element located at infinity (Figure 4.6).

Goals of this research:

1: Complex Encirclement: find all roots, real and complex.

- a) Design homotopy function H that forces all branches to coalesce in a single repeated root at a single branch point.
- b) Find additional conditions on the homotopy function H that guarantee that this branch point is either nonsingular, or singular but locally irreducible, so that all coalescing branches will be locally connected in a single algebraic element.

Associated algorithm: 1) Find all n roots of the designed homotopy function $H(x, \lambda_*) = 0$ via $n/2$ encirclements of infinity, with $\lambda = \lambda_* e^{j\theta}$, $\theta = 0 : n\pi$. Only $n/2$ are necessary because if $H(x, \lambda_*)$ is real, then the roots will appear in complex conjugate pairs. 2) Continue the n paths of $H(x, \lambda) = 0$ from $\lambda = \lambda_*$ to $\lambda = 0$, at which point $H(x, 0) = F(x)$, the circuit equations of interest.

Anticipated advantages: This algorithm finds all real and complex roots of a system, without apriori knowledge of the number of roots. Also, computation time is not wasted on homotopy curves that go to infinity rather than terminating on a solution, as is the case for typical multi starting point homotopy methods. This is useful for polynomial-bounded analytic equations and deficient polynomial systems (fewer roots than the Bezout upper degree bound).

The real circuit solutions of interest are then extracted from the complete set of real and complex solutions calculated by the algorithm.

2: A Complex Encirclement and real Lambda Threading hybrid: find all real roots.

- a) Design a homotopy function H that forces all *real disconnected* branches to have *at least one* root going to infinity.
- b) Find additional conditions on the homotopy function that guarantee that this branch point is either nonsingular, or singular but locally irreducible, so that the col-

lection of coalescing branches *including at least one representative of each disconnected real branch* will be locally connected in a single algebraic element.

Associated algorithm: 1) Use a real curve tracing algorithm to find all real solutions of $H(x, \lambda) = 0$ along a single real branch. 2) Trace out a root going to infinity from the real branch to a large $\lambda = \lambda_*$. 3) Apply infinite encirclement algorithm to find all branches connected at infinity corresponding to real disconnected branches. 4) Trace back roots to $\lambda = 0$ and check to see if any are on a different real branch from that in (1). If so, repeat (1) for all new branches.

Anticipated advantages: This algorithm is designed to find all real disconnected solution branches of a homotopy function, and thus all real roots of the circuit. Apriori knowledge of the number of real roots is not necessary, and not much computation time is wasted on complex roots.

This chapter mainly focuses on goal (1), with some observations along the lines of (2).

4.2 Definitions and Mathematical Preliminaries

Definition 4.2.1 (Analytic Function) A function $h(x, \lambda)$, $h : C^n \times C \rightarrow C$, is analytic if it has a local power series expansion in all variables at each $(x, \lambda) \in \{C^n \times C\}$. Examples of analytic functions are polynomials and exponentials.

Definition 4.2.2 (Branch Point) A branch point is a parameter value $\lambda \in C$ at which a complex-parameter homotopy function $H(x, \lambda) = 0$, $H : C^n \times C \rightarrow C^n$, has repeated roots. As such, the branch points correspond to parameter values where two or more solution surfaces contact each other.

Definition 4.2.3 (Regular Point) Given an analytic function $H(x, \lambda) = 0$, $H : C^n \times C \rightarrow C^n$, a parameter value $\lambda \in C$ is a regular point of H if it is not a branch point of H . Regular points correspond to parameter values at which the associated solution surfaces remain distinct.

Definition 4.2.4 (Irreducible) An analytic equation $H(x, \lambda) = 0$ is reducible if it may be written in product form $H(x, \lambda) = P(x, \lambda)Q(x, \lambda) = 0$, so that the roots of H are the union of

the roots of P and Q , both analytic functions in x and λ passing through zero. The equation $H(x, \lambda) = 0$ is irreducible if it is not reducible, and a system of equations $H(x, \lambda) = 0$, $H : C^n \times C \rightarrow C^n$, is irreducible if each h_i is irreducible, where $H = [h_1, h_2, \dots, h_n]'$, and if the equations h_i intersect generically. For example, $x_1^2 - x_2^2 \lambda = 0$ is irreducible, as is $e^{-x_1}(a_0 + a_1 x_1 + a_2 x_2 + \dots + a_n x_n) = 0$. The former function does not factor into the product of two analytic functions, while the latter function only factors into the product of a polynomial and an exponential, an analytic function which does not pass through zero called a unit. However, the solution set of $x_1^4 - x_2^2 \lambda^2 = 0$ reduces to the product of the solution sets of $x_1^2 - x_2 \lambda = 0$ and $x_1^2 + x_2 \lambda = 0$. Hence this function is reducible.

Definition 4.2.5 (Locally Irreducible at a Point) *An equation is considered locally reducible at a point if it may be locally represented by a product of non-trivial power series. The equation is locally irreducible at a point if it is not locally reducible at the same point. An equation may be both globally irreducible yet locally reducible at a point. For example, $h(x, \lambda) = x^2 - \lambda^2 - \lambda^3$ is irreducible, because it cannot be factored into the product of two polynomials, but is locally reducible at $\lambda = 0$. We factor $h(x, \lambda) = (x - \lambda\sqrt{1 + \lambda})(x + \lambda\sqrt{1 + \lambda})$, the product of two functions that, while not globally analytic, are analytic in the neighborhood of $\lambda = 0$, and thus are locally representable by power series. For contrast, look to $x^2 - \lambda^3 - \lambda^4 = 0$, which factors to $(x - \lambda\sqrt{\lambda}\sqrt{1 + \lambda})(x + \lambda\sqrt{\lambda}\sqrt{1 + \lambda})$, functions that are not analytic at $\lambda = 0$ because of the presence of the term $\sqrt{\lambda}$. This equation is both globally irreducible, and locally irreducible at $\lambda = 0$.*

Definition 4.2.6 (Degree of an Analytic Variety) *Let $f : S \rightarrow S'$ be an analytic map between compact Riemann surfaces S and S' . The degree of a variety V , the solution space of $f(x) = 0$, is the number of points of intersection of V with a generic linear subspace of complementary dimension ([25], p.171).*

The degree of a function $H(x, \lambda) = 0$, $H : C^n \times C \rightarrow C^n$, is the number of distinct solution points x satisfying $H(x, \lambda) = 0$ at a regular parameter value $\lambda \in C$. The degree of a function can be finite, as it is for polynomial-bounded functions, or infinite.

Definition 4.2.7 (Dimension of an Irreducible Analytic Variety) *The dimension of an irreducible analytic variety V is the dimension of the complex manifold V^* ($V^* = V - V_s$, the set V minus the set V_s , where V_s is the set the singular set of V). The singular set V_s is composed of all repeated roots in V .*

Definition 4.2.8 (Algebraic Element) Let $H : C^n \times C \rightarrow C^n$ be an analytic map. The solution set (x, λ) of $H(x, \lambda) = 0$ in the neighborhood of a branch point $\lambda_b \in C$ at the solution point $x \in C^n$ is called an algebraic element of order $d - 1$ if the function can be locally represented, via an isomorphic coordinate change, by a system of polynomial equations of the form

$$\begin{aligned} x_1^d &= \lambda^{v_1} \\ x_2^d &= \lambda^{v_2} \\ &\vdots \\ x_m^d &= \lambda^{v_m} \end{aligned} \tag{4.1}$$

where the integers (d, v_1, \dots, v_m) are coprime. A set of integers are coprime if they have no common divisor other than one (e.g. 2 and 3 are coprime, but 2 and 4 are not, as they have a common divisor of two).

Definition 4.2.9 (Algebraic Element - Scalar Case) Let $f : C \times C \rightarrow C$ be an analytic map. The solution set (x, λ) of $f(x, \lambda) = 0$ in the neighborhood of a branch point $\lambda_b \in C$ at the solution point $x \in C$ is called an algebraic element of order $d - 1$ if the function can be locally represented, via an isomorphic coordinate change, by a polynomial of the form $\lambda^v = x^d$, where d and v are coprime integers.

A branch point of order $d - 1$ must be circled d times in complex space in order to return to the original solution manifold. Each revolution has the effect of moving the solution point from one solution manifold to another (there are d), until the n th revolution returns the solution point to the original manifold. We refer to this local solution structure of d helicoid connected sheets as an algebraic element of order $d - 1$.

Definition 4.2.10 (Winding Numbers-Vector Function) The winding numbers of an algebraic element of order $d - 1$ are an n -tuple of integers $(w_1, w_2, \dots, w_i, \dots, w_n)$ signifying the number of times each variable winds around the repeated root value as a complex circle is traced around the associated branch point d times. Given an algebraic element of a vector function, which can be locally represented by Equations (4.1), each variable x_i of the solution vector $(x_1, x_2, \dots, x_i, \dots, x_n)$ of (4.1) has its own winding number $w_i = v_i$, which measures the number of times that the variable path $x_i(\lambda) = |\epsilon|^{v_i/d} e^{j\theta v_i/d}$ winds around $x_i = 0$ as the parameter $\lambda = \epsilon e^{j\theta}$, $\theta = 0 : d2\pi$ is traced around an infinitesimal circle of radius ϵ centered

Weierstrass Preparation Theorem ([25],p.8) : *If the function $f : C^n \rightarrow C$ is analytic around the origin in C^n and is not identically zero on the w -axis, then in some neighborhood of the origin f can be written uniquely as*

$$f(z, w) = g(z, w)h(z, w) \quad (4.3)$$

where g is a Weierstrass polynomial of degree d in w and $h(0, 0) \neq 0$.

4.2.2 Fundamental Theorem on Connectivity

From page 117 of [27], we take proposition (A.5), written below.

(A.5) Proposition: *A germ of analytic space $X \subset U \subset C^n$ at 0 is topologically unibranch if and only if it is irreducible (at 0).*

For a proof, see pages 117-118 of [27].

Interpretation:

Roughly speaking, a ‘germ’ of an analytic space is the solution set of a locally analytic system of equations in a sufficiently small neighborhood of a branch point located at 0. In [27], X is called topologically unibranch at 0 if there is a fundamental system of neighborhoods U_n of 0 such that $U_n \cap X - U_n \cap \text{Sing}(X)$ are connected, where $\text{Sing}(X)$ refers to the singular points of X . One can think of the neighborhoods U_n as successively smaller balls around 0. The term $U_n \cap X$ refers the intersection of the set X and the ball, and the term $U_n \cap \text{Sing}(X)$ refers to the set of all singular points of X in this ball, meaning the set of all repeated roots. Thus, the expression $U_n \cap X - U_n \cap \text{Sing}(X)$ refers to the set of regular points in a small neighborhood of the branch point. Proposition (A.5) states that roughly speaking, the set of regular points in a small neighborhood of a branch point is connected if the function is locally irreducible at that branch point.

Consequence:

A branch point of multiplicity r serves to connect all r converging solution branches of $H(x, \lambda) = 0$ in a single algebraic element, if the function is locally irreducible at the branch point.

Many of the results of this chapter can be understood as applications of Proposition (A.5).

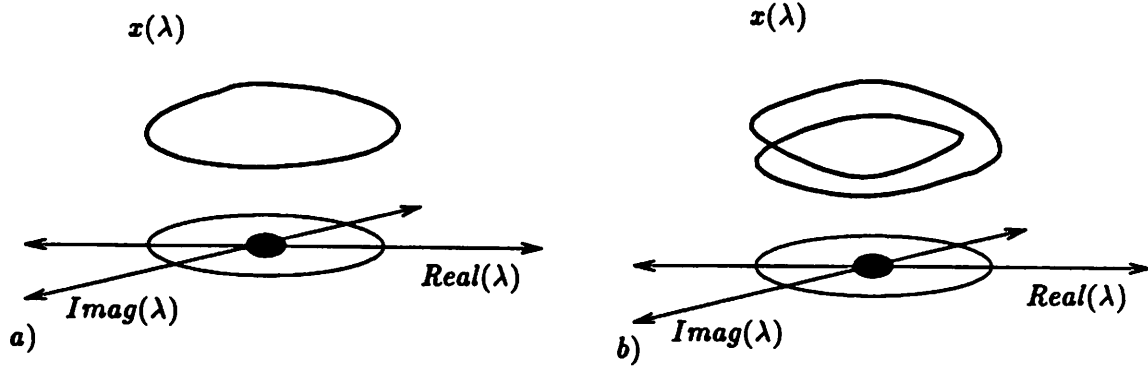


Figure 4.3: a) Algebraic element with winding number $w = 1$. b) Algebraic element with winding number $w = 2$.

around $\lambda = 0$ d times. See Figures 4.12 and 4.13 for examples of algebraic elements with different winding number vectors.

Definition 4.2.11 (Winding Number-Scalar Function) An algebraic element $\lambda^v = x^d$ has a winding number $w = v$. The winding number measures the number of times the solution path $x(\lambda) = |\epsilon|^{v/d} e^{j\theta v/d}$ winds around $x = 0$ as the parameter $\lambda = \epsilon e^{j\theta}$, $\theta = 0 : d2\pi$ is traced around an infinitesimal circle of radius ϵ centered around $\lambda = 0$ d times. It is the number of times the image of the complex parameter circle, a closed curve in solution space, winds around before closing in on itself, as shown in Figure 4.3.

Definition 4.2.12 (Generic [25]) When dealing with a family of objects parametrized locally by a complex manifold or an analytic subvariety of a complex manifold, the statement that “a generic member of the family has a certain property” means that the set of objects in the family that do not have that property is contained in a subvariety of strictly smaller dimension.

4.2.1 Local Representation of an Analytic Function as a Polynomial

Definition 4.2.13 (Weierstrass Polynomial ([25], p.7)) A Weierstrass polynomial in w is a polynomial of the form

$$w^d + a_1(z)w^{d-1} + \dots + a_d(z) \quad (4.2)$$

where $a_i(0) = 0$ and z is a complex vector.

4.3 One Equation, One Unknown: Polynomials

This section deals with one dimensional, parameterized polynomials. We design a homotopy function with an infinite branch point, and show that all roots of the equation are connected in a single algebraic element around infinity. We also address the question of exactly what it means, in practical terms, to encircle infinity, and provide insight on how to visualize complex solution space. We end the section with some examples, and a brief discussion of the kinds of complications one may encounter if the parameter appears nonlinearly in the equation.

First we look at homotopy functions with λ appearing linearly, as follows. For a polynomial equation $h(x, \lambda) = 0$, where

$$h(x, \lambda) = p(x) + \lambda r(x) \quad (4.4)$$

and $p(x)$ and $r(x)$ are polynomials, root locus theory [49] tells us that at $\lambda = 0$, any finite roots of $h(x, \lambda) = 0$ must satisfy $p(x) = 0$. At $\lambda = \infty$, any finite solutions of $h(x, \lambda) = 0$ must satisfy the equation $r(x) = 0$.

Homotopy Function Design:

As a consequence, if the function $r : C \rightarrow C$ is chosen to be a *constant*, $r(x) \equiv k, k \in C$, then the function $h(x, \lambda) = 0$ will have *no finite* solutions at infinite λ . Choosing a homotopy function of the form

$$h(x, \lambda) = p(x) + \lambda k \quad (4.5)$$

ensures that as the parameter $\lambda \rightarrow \infty$, all the solutions of $h(x, \lambda) = 0$ will also approach infinity ($|x| \rightarrow \infty$ as $|\lambda| \rightarrow \infty$). This implies that $h(x, \lambda) = p(x) + \lambda k = 0$ has an infinite root of *multiplicity* n at $\lambda = \infty$, where n is the degree of the polynomial $p(x)$.

In the proposition below (Proposition 4.3.1.1-2), we prove that a homotopy function of the form in Equation (4.5) has a repeated root of multiplicity n at $\lambda = \infty$. The next question is that of local connectivity in the neighborhood of $\lambda = \infty$. That is, how many algebraic elements are there at $\lambda = \infty$? What determines the answer to this question? If there are two or more algebraic elements, how are they grouped together? Proposition 4.3.1.3 provides the answer, which is that there is a single algebraic element at infinity that locally connects all roots of the polynomial. More formally stated,

Proposition 4.3.1:

1. *The polynomial homotopy function $h(x, \lambda) = p(x) + \lambda k = 0, k \in C$, has no finite roots as $\lambda \rightarrow \infty$.*
2. *At $\lambda = \infty$, $h(x, \lambda) = 0$ has an infinite root of multiplicity n , where n is the degree of the polynomial $p(x)$.*
3. *At $\lambda = \infty$, the polynomial homotopy function $h(x, \lambda) = p(x) + \lambda k = 0, k \in C$, has a single algebraic element of order $n - 1$, where n is the degree of the polynomial.*

Proof:

1. assertion: $\lambda \rightarrow \infty \Rightarrow x \rightarrow \infty$.

Assume not. Then $\exists M \in R$ s.t. $|x| < M, \forall \lambda \in C$. Let $h(x, \lambda) = p(x) + \lambda$, where p is a non-constant polynomial. Then $p(x) = -\lambda$, and $|p(x)| = |\lambda|$. $|x| < M \Rightarrow \exists N \in R$ s.t. $|p(x)| < N$. Thus $|x| < M \Rightarrow |p(x)| < N \Rightarrow |\lambda| < N$. But we let $\lambda \rightarrow \infty$, so proof by contradiction. ♣

2. (and 3.) Let $h(x, \lambda) = p(x) - \lambda = 0$. Without loss of generality, let the polynomial $p(x) = x^n + r(x)$, where n is the degree of p and r is a polynomial of degree less than n . As $\lambda \rightarrow \infty$, $x \rightarrow \infty$ and the highest degree term of $p(x)$ dominates and $h(x, \lambda) \approx x^n - \lambda$. Asymptotically as $\lambda \rightarrow \infty$, $x^n \approx \lambda = |\lambda|e^{j(\theta+k2\pi)}, k \in Z$, and $x \approx |\lambda|^{(1/n)}e^{j(\theta+k2\pi)/n}$, n roots equally spaced around a complex circle of radius $|\lambda|^{(1/n)}$, as shown in Figure 4.4. The entire solution structure of h in the neighborhood of $\lambda = \infty$ is thus a single algebraic element of order $n - 1$, and the branch point at $\lambda = \infty$ is an infinite repeated root of multiplicity n . ♣

Alternate Proof (Proposition 4.3.1.3):

- Let $h(x, \lambda) = p(x) - \lambda = 0$. From Proposition 4.3.1.1-2 we know that h has an infinite root of multiplicity n at $(x, \lambda) = (\infty, \infty)$, where n is the degree of p . Now, from [25] we know that if branch point is nonsingular, then a single algebraic element connects all coalescing roots of this function around the branch point.

To check for non-singularity, we first perform a change of variables $y = 1/x$ and $\lambda = 1/\mu$ and substitute into h in order to move the infinite branch point to $(y, \mu) =$

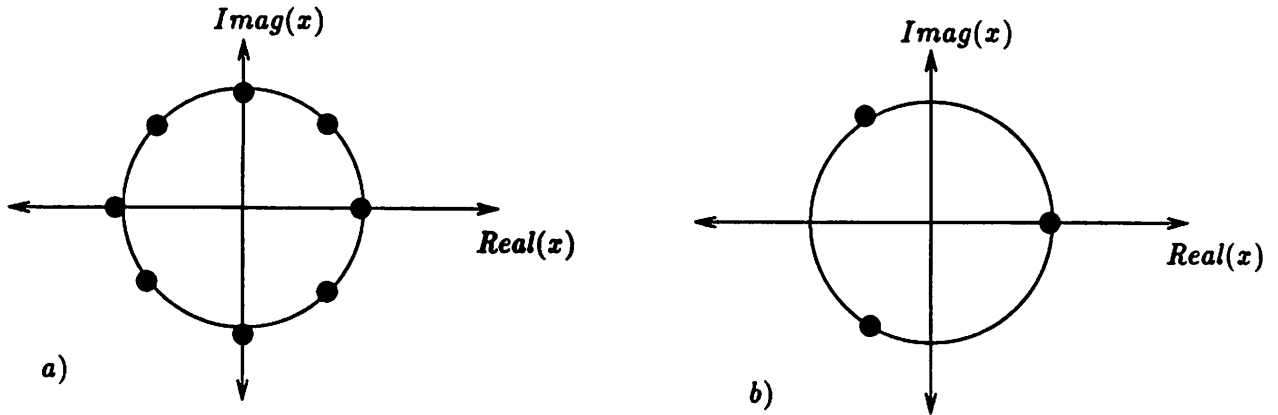


Figure 4.4: Algebraic elements of order $n - 1$, $hx^n = \lambda$. Solutions - n roots equally spaced around a circle of radius $|\lambda|^{(1/n)}$. a) $n = 8$. b) $n = 3$.

$(0,0)$. The new analytic function $h'(y, \mu) = \mu y^n h(1/y, 1/\mu) = \mu y^n p(1/y) - y^n$ is obtained by changing variables and multiplying out the denominator. Next we form the extended Jacobian matrix of h' , $[dh'/dx, dh'/d\lambda] = [r(y, \mu), 1]$, and check its rank at $(y, \mu) = (0,0)$ (r is a polynomial in y and μ). This matrix has a rank of one at $(y, \mu) = (0,0)$, which implies that the solution curve of h' passing through the repeated root $y = 0$ at $\mu = 0$ is non-singular. This in turn implies that the solution curve of h passing through the repeated infinite root at $\lambda = \infty$ is also nonsingular, and that a single algebraic element connects all n roots of h around infinity. ♣

Encircling Infinity:

In this subsection we address the question of exactly what it means, in practical terms, to encircle infinity. To understand this, first imagine the complex parameter plane, $\lambda \in \mathbb{C}$ extending out to infinity in all directions. Now curl up the far, infinite edges of this plane, the entire horizon, like a big drawstring bag with the string all around infinity. Then pull the edges upward and together until the infinite plane is deformed into a sphere with all of infinity contracted to a single point at the north pole of this sphere, as in Figure 4.5. This way of representing complex space as a sphere with a single infinite point is referred to as a *Riemann sphere* [25]. When thinking of the complex parameter plane λ , or sphere,

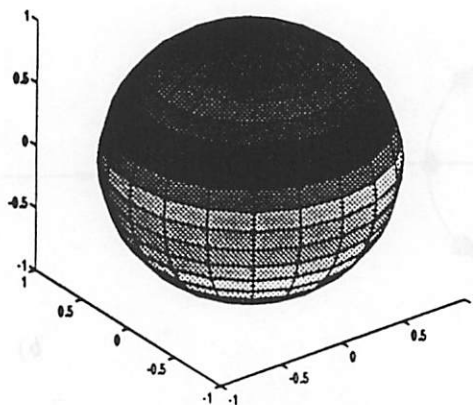


Figure 4.5: The complex parameter plane $\lambda \in \mathbb{C}$ represented as a Riemann Sphere, with the north pole acting as infinity.

as it relates to the solutions of the homotopy function $H(x, \lambda) = 0, H : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$, recall from Section 3.5 that there are at least two important types of points in the complex plane/sphere: regular points and branch points. At branch points the homotopy function has repeated roots, and at regular points the homotopy function has n distinct roots, where n is the degree of the solution space, or variety, of H . In the neighborhood of a branch point, solution surfaces are connected in one or more helicoid structures called algebraic elements, while small open sets of regular points have solution images that look like a set of n non-intersecting sheets.

Because of this, the important features on the Riemann parameter sphere associated with a homotopy function are the *branch points*, as they mark points at which the solution manifolds touch each other. A homotopy function will in general have some limited number of finite branch points, and an infinite branch point, marked as the north pole in Figure 4.6. If one is interested in encircling infinity, one may think of tracing a tiny circle around the north pole, as shown in Figure 4.6a. However, since all closed curves containing just one branch point (in this case the one at infinity) are homotopically equivalent, we might just as well pick a circle with a finite radius that is just large enough to contain all finite branch points, as shown in Figures 4.6b and 4.6c. This excursion is equivalent to a circle with a radius approaching infinity. Thus, in practical terms, one may encircle infinity with the complex parameter plane circle $\lambda = |\lambda_*|e^{j\theta}, \theta = 0 : 2\pi$, with a radius λ_* chosen to be just large enough to encircle all finite branch points. In the example that follows, a radius

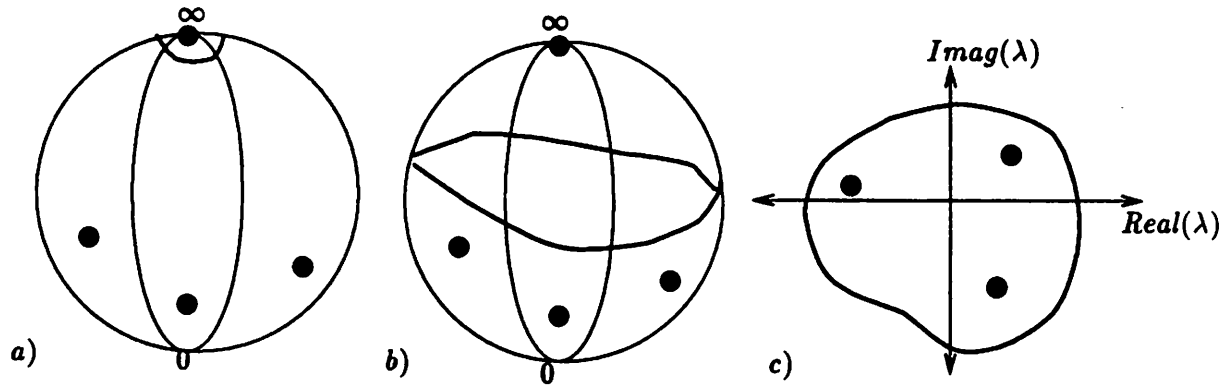


Figure 4.6: (a,b) The complex parameter plane $\lambda \in C$ represented as a Riemann Sphere, with the north pole acting as infinity. a,b,c) All paths encircle infinity.

of about $\lambda_* = 4.5$ is required to encircle infinity. In the next section on two dimensional functions, we will see examples where a radius of around $\lambda_* = 10^{12}$ is required to encircle infinity.

Implications of Proposition 4.3.1:

Given a single root $x_{1\lambda_*}$ of the equation $h(x, \lambda_*) = p(x) + \lambda_* k = 0, k \in C$ at a value of $\lambda_* \in C$ chosen such that $|\lambda_*| > r$, where r is the finite radius of a circle in the complex plane containing *all finite branch points of h* , then tracing a full circle in the complex parameter plane $\lambda = \lambda_* e^{j\theta}, \theta = 0 : 2\pi$, from the solution $x = x_{1\lambda_*}$ will lead to a second solution $x = x_{2\lambda_*}$ of $h(x, \lambda_*) = 0$. Tracing a second full circle in the complex parameter plane $\lambda = \lambda_* e^{j\theta}, \theta = 0 : 2\pi$, from the solution $x = x_{2\lambda_*}$ will then lead to a third solution of $h(x, \lambda_*) = 0, x = x_{3\lambda_*}$. Then, if one continues to trace circles in the complex parameter plane, moving from solution to solution, *all* solutions of $h(x, \lambda_*) = 0$ will be found, and eventually, on the n th revolution, the algebraic element will have been fully cycled through and the original root $x_{1\lambda_*}$ will re-appear. Thus, continuing to traverse a circle in the complex parameter plane $\lambda = \lambda_* e^{j\theta}, \theta = 0 : k2\pi$ will lead to the numerical calculation of a string of roots $(x_{1\lambda_*}, x_{2\lambda_*}, \dots, x_{n\lambda_*}, x_{1\lambda_*}, x_{2\lambda_*}, \dots)$ consisting of the repetition of the n roots of the function $h(x, \lambda_*) = 0$. Once all n roots of $h(x, \lambda_*) = 0$ are found, all the roots of $h(x, \lambda) = 0$ at any arbitrary value of λ may be found, including that of the original problem of interest to be solved.

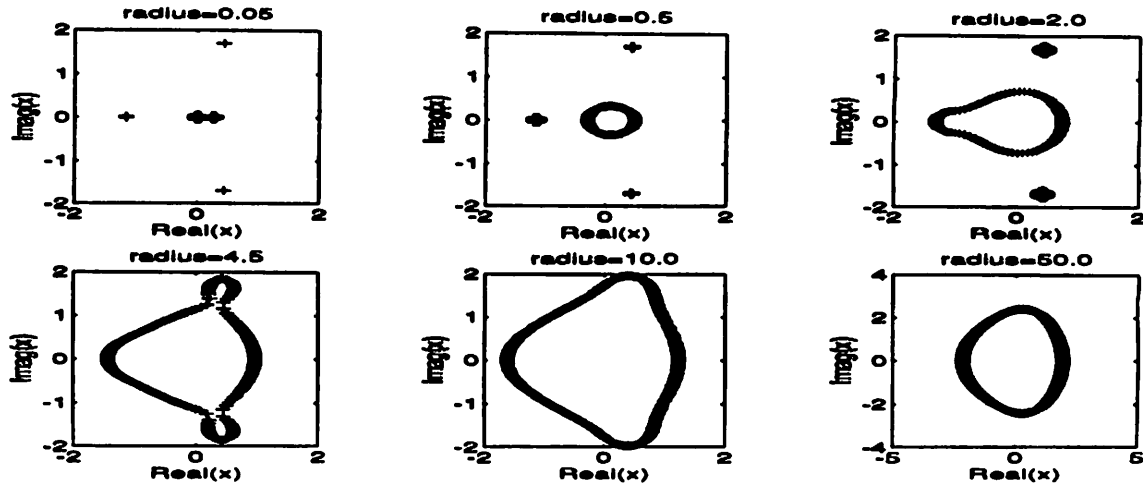


Figure 4.7: Root loci of $h(x, \lambda) = x^5 + 2x^3 + 3x^2 - x + \lambda = 0$, $\lambda = \lambda_* e^{j\theta}$, $\theta = 0 : 2\pi$, for various radii λ_* . a) Loci of h at $\lambda_* = 0.05$. b) Loci of h at $\lambda_* = 0.5$. c) Loci of h at $\lambda_* = 2.0$. d) Loci of h at $\lambda_* = 4.5$. e) Loci of h at $\lambda_* = 10$. f) Loci of h at $\lambda_* = 50$.

Example 1:

The first example we will go through in detail is the function

$$h(x, \lambda) = x^5 + 2x^3 + 3x^2 - x - \lambda \quad (4.6)$$

Root Loci:

To help build up intuition on the connectivity of solution spaces of such functions, look to the root loci plots of Figures 4.7a-f. These MATLAB plots were obtained by plotting all roots of $h(x, \lambda) = x^5 + 2x^3 + 3x^2 - x - \lambda = 0$ at complex parameter values sampled around a circle $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$. Each plot shown in Figures 4.7a-f is a locus corresponding to a different fixed value of the complex parameter circle radius λ_* .

Figure 4.7a shows a root locus plot of $h(x, \lambda) = 0$ at samples of λ around the circle $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, $\lambda_* = 0.05$. Tracing out this tiny circle in the parameter plane leads to five tiny circles being traced in the complex solution plane, three centered on the real axis, and two away from it. This means that the complex circle of radius $\lambda_* = 0.05$ does not encircle any finite branch points of h , and if one were to trace out a solution trajectory starting from any of the five roots of $h(x, \lambda_*) = 0$, such a revolution in the complex parameter plane would not lead to another root of $h(x, \lambda_*) = 0$.

Figures 4.7b,c show root loci plots of $h(x, \lambda) = 0$ at samples of λ around the circles $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, $\lambda_* = 0.5$ and $\lambda_* = 2.0$, respectively. Clearly, these larger complex circles do encircle one or more critical points of $h(x, \lambda_*) = 0$, since each locus contains fewer than five closed curves in complex solution space. Figure 4.7b shows four closed curves, meaning that two of the five roots are connected in a structure equivalent to an algebraic element of order one, while the other three roots are in disconnected zero-order elements. Figure 4.7c shows three closed curves, meaning that three of the five roots are connected in a structure akin to an algebraic element of order two, while the other two roots are in disconnected zero-order elements. If one were to start the trajectory $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, $\lambda_* = 2.0$, at any of the three algebraically connected roots of $h(x, 2.0) = 0$, $x = x1_{\lambda_*}$, a revolution in the complex parameter plane would lead to a second root of $h(x, 2.0) = 0$, $x = x2_{\lambda_*}$, while a third revolution in the complex parameter plane would lead to a third root of $h(x, 2.0) = 0$, $x = x3_{\lambda_*}$. A fourth revolution in the complex parameter plane would lead back to the original root of $h(x, 2.0) = 0$, $x = x1_{\lambda_*}$.

Figures 4.7d-f show root loci plots of $h(x, \lambda) = 0$ at samples of λ around the circles $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, with radii $\lambda_* = 4.5$, $\lambda_* = 10.0$, and $\lambda_* = 50.0$, respectively. These even larger complex circles encircle *all* finite critical values, as is evidenced by the fact that each locus contains exactly *one* closed curve in complex solution space. This means that tracing a complex circle in the complex parameter plane of large enough radius is *equivalent to encircling infinity*. If one were to start the trajectory $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, $\lambda_* \geq 4.5$, at any of the five algebraically connected roots of $h(x, \lambda_*) = 0$, tracing a revolution in the complex parameter plane would lead to a second root of $h(x, \lambda_*) = 0$. Tracing a second, third, fourth, and then fifth revolution in the complex parameter plane would lead to a third, fourth, and then fifth root of $h(x, \lambda_*) = 0$, and then to a return to the original root from which the path started. Also notice that as λ_* increases, the locus smooths out and approaches the shape of a circle with five equally spaced roots around it, the root structure in the neighborhood of infinity as shown below in the *analysis* section of the example.

Numerical Calculation:

Figure 4.8f shows the complex parameter circle $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, $\lambda_* = 10$, a circle of large enough radius to be equivalent to encircling infinity. In Figure 4.8a we start at $(x, \lambda) = (1.25, -10)$, the only real solution of $h(x, -10) = 0$, and trace the complex

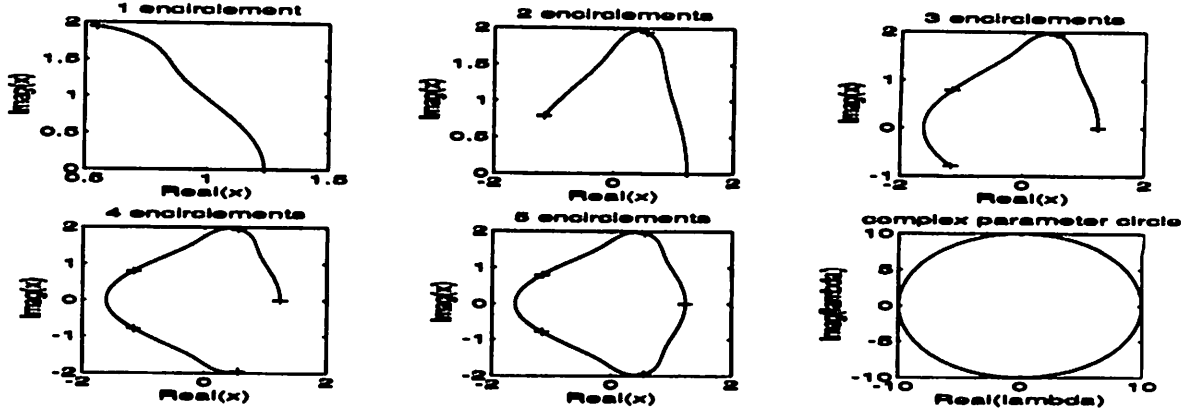


Figure 4.8: The complex solution trajectory obtained by traversing the circle in (f) r times, starting from the real solution of $h(x, -10) = 0$. a) $r = 1$. b) $r = 2$. c) $r = 3$. d) $r = 4$. e) $r = 5$. f) Complex parameter circle, radius equal 10.

parameter circle a single time to reach a second solution of $h(x, -10) = 0$, along the path shown in Figure 4.8a. Tracing a second, third, fourth, and then fifth revolution around the complex parameter circle leads to a third, fourth, and then fifth root of $h(x, -10) = 0$, as shown in Figures 4.8b-d, and then a return to the original root from which the path started. After five complex parameter revolutions, the closed curve shown in Figure 4.8e is traced out, with all five roots of $h(x, -10) = 0$ appearing at approximately equal intervals around the closed solution curve.

Analysis around ∞ :

Proposition 4.3.1 applies to the homotopy function $h(x, \lambda) = x^5 + 2x^3 + 3x^2 - x - \lambda = 0$, so we know that as $\lambda \rightarrow \infty$, all roots of h also go to infinity, and are connected in a single algebraic element around the branch point at $\lambda = \infty$.

As $\lambda \rightarrow \infty$, the highest order term in x of h dominates, and asymptotically $h(x, \lambda) \approx x^5 - \lambda = 0$. Therefore, for large enough $|\lambda|$, the root structure of h resembles $x = |\lambda|^{(1/5)} e^{j(\theta + k2\pi)/5}$, $k \in \mathbb{Z}$, a constellation of five roots equally distributed around a complex circle of radius $|\lambda|^{(1/5)}$.

Recall that as long as the radius of the circle traversed in complex parameter space, λ_* , is large enough so that the circle $\lambda = |\lambda_*| e^{j(\theta)}$, $\theta = 0 : 2\pi$ contains all finite branch points of $h(x, \lambda) = 0$, then the root structure will be homotopic to that around $\lambda = \infty$, the single algebraic element connecting all solutions of $h(x, \lambda) = 0$.

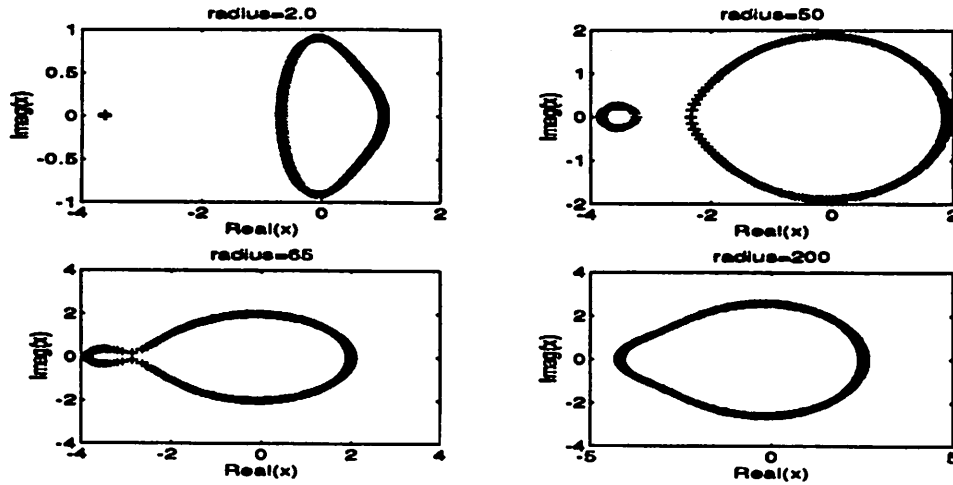


Figure 4.9: Root loci of $h(x, \lambda) = x^5 + 3x^4 - 2x^3 + .5x^2 - x + \lambda = 0$, $\lambda = \lambda_* e^{j\theta}$, $\theta = 0 : 2\pi$, for various radii λ_* . a) Loci of h at $\lambda_* = 2.0$. b) Loci of h at $\lambda_* = 50$. c) Loci of h at $\lambda_* = 65$. d) Loci of h at $\lambda_* = 200$.

For this example, a relatively small radius of $\lambda_* > 4.5$ suffices to connect all solutions, as is evident in Figures 4.7a-f, though in general the required radius λ_* may be much larger. For example, the equation $h(x, \lambda) = x^5 + 3x^4 - 2x^3 + .5x^2 - x + \lambda = 0$ requires a radius of $\lambda_* > 65$ to connect all roots, as is evident in the root locus plot shown in Figure 4.9. \square

A Note on Visualization:

One of the more difficult things about working with complex space, as compared to real space, is that of *visualizing* it. Complex functions take us one dimension out of our 3-d realm, since the very simplest scalar homotopy function in one variable and one parameter, $h(x, \lambda) = 0$, has a solution set that (generically) is a locally two dimensional object in 4-d.

A way of visualizing complex solution space that we have found to be intuitively rich is as follows. First, take the complex parameter plane and imagine it as a *continuum of concentric circles* of radii λ_* varying from zero to infinity, as shown in Figure 4.10a. Because of the nature of complex solution space, the image of a circle in the complex parameter plane is a *collection of closed curves* in the complex solution plane (as long as the

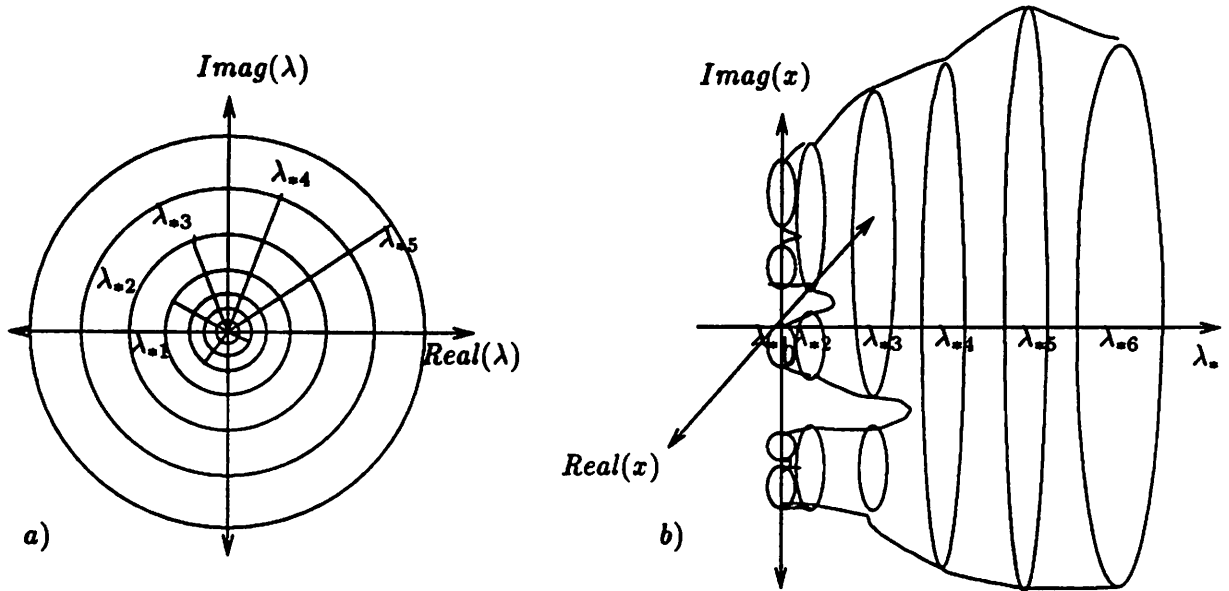


Figure 4.10: Visualizing complex solution space. a) Concentric circles of radii $0 < \lambda_* < \infty$ compose the complex parameter plane $\lambda \in C$. b) Visualize the solution space of $h(x, \lambda) = 0$ as a locally two dimensional surface in three-D, with the axes being the $Imag(x)$, $Real(x)$, vs. the parameter plane radius λ_* .

homotopy function has finite degree). For example, if a homotopy function $h(x, \lambda) = 0$ has five solutions at a fixed value of $\lambda = \lambda_*$, then the image of the circle $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$ is a collection of between one and five closed curves, depending on the chosen radius λ_* .

We represent the complex solution set of a homotopy function by plotting the *root locus* ($x/h(x, \lambda) = 0$) of the homotopy function along the parameter plane circle $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, as a function of circle radius λ_* . At a fixed circle radius λ_* , the locus consists of one or more closed complex solution curves. As the radius varies, the complex parameter circle will intersect branch points. When this happens, closed solution curves will merge or divide, in a sort of topological bifurcation. Viewed along the entire λ_* axis, solution surfaces of homotopy functions designed to connect around infinity, represented in this way, will resemble a hand, with the fingertips lined up against the $\lambda_* = 0$ plane, or tree roots merging into a trunk.

The above described 3-d representation is illustrated in Figure 4.10b for a homotopy function of degree five, with a completely connected solution structure in the neigh-

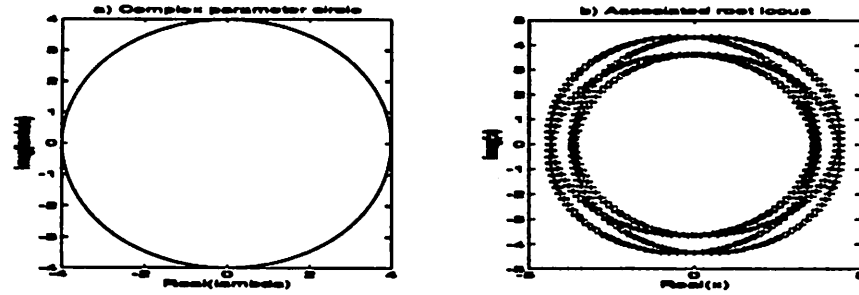


Figure 4.11: a) Complex parameter circle, $\lambda = \lambda_* e^{j(\theta)}$, $\theta = 0 : 2\pi$. b) Associated root locus of $h(x, \lambda) = x^4 - 2x^2\lambda + \lambda^4 = 0$

neighborhood of infinity. Observe that this picture is of the same general shape as the root loci plots of Examples 1 and 2, shown in Figures 4.7 and 4.9.

Remark on Real vs. Complex Connectivity in 1-D:

In Section 5.3, we explain that a homotopy function of the form $h(x, \lambda) = f(x) + \lambda = 0$ has a real solution curve $(x, \lambda \in \mathbb{R}, h(x, \lambda) = 0)$ that is *connected*. In this section we have shown that such a homotopy function also has a single algebraic element around the branch point $\lambda = \infty$ that serves to connect all real and complex roots of $h(x, \lambda_*) = 0$ for large enough $|\lambda_*|$. One is tempted to wonder what the relationship is between real solution curve connectivity and complex connectivity around infinity, and to guess that one might imply the other. As we shall see in the next section, in higher dimensions this proves not to be the case. In general, a connection around complex infinity does not imply that the real solution space is connected.

Other Homotopy Functions, Potential Complications:

The purpose of this subsection is to point out that the existence of a repeated root of multiplicity n of a function $h(x, \lambda) = 0$ at some value of the parameter λ *does not* in general imply the existence of a single algebraic element connecting all n coalescing branches in the neighborhood of the parameter value. For instance, if the parameter λ appears nonlinearly in the function, other scenarios may arise. The general concept is that a branch point may be either singular or nonsingular. Nonsingular branch points automatically connect all coalescing roots in a single algebraic element. At a singular branch point, the associated function must be *locally irreducible* for this branch point to connect

all coalescing roots in a single algebraic element. The following examples are of singular branch points, meaning that the extended Jacobians of the functions, $[\partial h/\partial x, \partial h/\partial \lambda]$ have rank zero at a branch point. Also, all of the following examples are locally reducible at zero.

The following equation provides an example.

$$h(x, \lambda) = x^2 - \lambda^2 = 0 \quad (4.7)$$

Notice that at $\lambda = 0$, we get $h(x, \lambda) = x^2 = 0$, so there is a root $x = 0$ of *multiplicity two* at the parameter value $\lambda = 0$. One might think that the existence of a double root at $\lambda = 0$ implies that there is an algebraic element joining the two coalescing solution branches, so that a revolution in the complex parameter plane $\lambda = e^{j(\theta)}$, $\theta = 0 : 2\pi$, would result in the permutation of the two branches. However, h is reducible to $h(x, \lambda) = (x - \lambda)(x + \lambda) = 0$, and there are *two intersecting regular solution branches* at $\lambda = 0$ rather than a single algebraic element of order 1. Tracing a full revolution in complex parameter space around the branch point from either real branch, $x = \epsilon$ or $x = -\epsilon$, results in a return to the same branch. For contrast, consider the function $h(x, \lambda) = x^2 - \lambda = 0$, which also has a double root $x = 0$ at $\lambda = 0$, but which has a single algebraic element about $\lambda = 0$. For this function, the two branches in the neighborhood of small λ are $x = |\epsilon|^{1/2} e^{j(\theta/2)}$ and $x = |\epsilon|^{1/2} e^{j(\theta+\pi)/2}$, which permute upon a full revolution $\lambda = e^{j(\theta)}$, $\theta = 0 : 2\pi$ around the branch point $\lambda = 0$.

Other, somewhat more complicated examples may be found in the following two equations.

$$h(x, \lambda) = x^2 - \lambda^2 - \lambda^3 = 0 \quad (4.8)$$

$$h(x, \lambda) = x^4 - 2x^2\lambda + \lambda^4 = 0 \quad (4.9)$$

These two equations are not reducible, as was Equation (4.7), but they are *locally reducible* in the neighborhood of zero, the branch point. First, we look at Equation (4.8), and notice that at $\lambda = 0$, $h(x, \lambda) = x^2 - \lambda^2 - \lambda^3 = 0$ has a double root $x = 0$, and at $\lambda = \infty$, $h(x, \lambda) = 0$ has an infinite double root. To check for local reducibility at $\lambda = 0$, we check to see whether $h(x, \lambda)$ can be locally represented as a *product of power series* in the neighborhood of zero. To test for local reducibility, we factor Equation (4.8) into $(x - \lambda\sqrt{1 + \lambda})(x + \lambda\sqrt{1 + \lambda})$. Then we check to see whether the factors are analytic at $\lambda = 0$. They are, because neither factor includes a term like $\sqrt{\lambda}$, so they may be represented locally

by power series. Thus, Equation (4.8) is locally reducible at the branch point $\lambda = 0$, and has a local solution structure that is not an algebraic element of order 1, but rather the superposition of two regular branches, like in Equation (4.7). In the neighborhood of $\lambda = \infty$, however, h is locally irreducible. In this neighborhood, $h(x, \lambda) \approx x^2 - \lambda^3$, an irreducible function. Thus, h cannot be written as a product of power series in the neighborhood of infinity; it is locally irreducible with a solution structure that is an algebraic element of order one, with a winding number of three (from the approximation $x^2 - \lambda^3 \approx 0$).

Next, consider Equation (4.9) in the neighborhood of $\lambda = 0$, at which the function has a root $x = 0$ of multiplicity four. Near $\lambda = 0$, we factor Equation (4.9) into the product $(x^2 - \lambda - \lambda\sqrt{1 - \lambda^2})(x^2 - \lambda + \lambda\sqrt{1 - \lambda^2})$. The next step is to check and see whether the factors are analytic at $\lambda = 0$. They are, so $h(x, \lambda)$ can be locally represented around zero as a product of power series, and thus is locally reducible at zero. Rather than finding a single algebraic element of order three connecting all branches at $\lambda = 0$, there are two algebraic elements of order two at $\lambda = 0$, each of which locally connects two of the four branches.

In the neighborhood of $\lambda = \infty$, the higher order terms dominate and $h(x, \lambda) \approx x^4 - \lambda^4$, resulting in four regular branches in the neighborhood of infinity, as shown in the root locus of Figure 4.11.

These examples were factored and tested for local reducibility near zero in an ad hoc manner. Systematic algorithms for testing local reducibility, implementable by a computer program, are based on applying the Weierstrass preparation theorem, given in Section 4.2, to the function of interest, and then using Hensels Lemma to develop an inductive procedure. This lemma can be found in [28].

4.4 Two Equations, Two Unknowns: Polynomials

In this subsection, we investigate polynomial homotopy functions with two equations, two unknowns, and a complex parameter. First, a general homotopy function form is given, with the parameter appearing linearly, and various potential root constellations are discussed. Then we present a homotopy function design, along with conditions on the original problem of interest that guarantees that all roots go to infinity with the parameter. Following that, we present conditions for ensuring that the homotopy function will have all roots connected in a single algebraic element around infinity. The subsection ends with a series of examples illustrating the geometric and algebraic facets of these results. Concepts

important to this portion of the chapter include winding numbers, complex solution curves seen as paths along tori, and local irreducibility around branch points.

Polynomial homotopy function: A general linear form:

Consider the following homotopy function form, where the parameter λ appears linearly.

$$\begin{aligned} h_1(x_1, x_2, \lambda) &= p_1(x_1, x_2) + \lambda r_1(x_1, x_2) \\ h_2(x_1, x_2, \lambda) &= p_2(x_1, x_2) + \lambda r_2(x_1, x_2) \end{aligned} \quad (4.10)$$

The functions r_1 and r_2 are as yet unspecified polynomials, and the problem of (eventual) interest is considered to be that of finding all roots of $p_1(x_1, x_2) = p_2(x_1, x_2) = 0$, or equivalently $H(x, \lambda) = (h_1(x, \lambda), h_2(x, \lambda))' = 0$ at $\lambda = 0$, where $x = (x_1, x_2)$.

The goal of this section is to design a homotopy function with a branch point at infinity that connects all roots of H in a single algebraic element, and we begin by looking at the possible root configurations of $H = 0$ at $\lambda = \infty$ for various functions r_1 and r_2 . But first, solution vectors at infinity must be classified.

For a single function in a single variable, infinity is a fairly simple concept. A solution x of $f(x, \lambda) = 0$, $f : C \times C \rightarrow C$, approaches infinity as λ approaches infinity if $x \rightarrow \infty$ as $\lambda \rightarrow \infty$. Infinity is a point on the Riemann sphere. For the case of two equations in two unknowns, infinity is a somewhat more difficult concept. The solution is now a vector, (x_1, x_2) , and infinity is no longer a single point, as it was in the scalar case, but is now the union of (∞, k_2) , (k_1, ∞) , and (∞, ∞) , $k_i \neq \infty$. That is, it is composed of the point (∞, ∞) , and the two sets $C \times \infty$ and $\infty \times C$. At $\lambda = \infty$ we classify a solution x of $H(x, \lambda) = 0$, $H : C^2 \times C \rightarrow C^2$, as being either finite or infinite, where an infinite solution vector $x = (x_1, x_2)$ has either a single component at infinity, $(x_1, x_2) = (\infty, k_2)$ or $(x_1, x_2) = (k_1, \infty)$, or both components at infinity, as in $(x_1, x_2) = (\infty, \infty)$. In general, Equations 4.10 can have finite roots, infinite roots of the form (∞, k_2) or (k_1, ∞) , or infinite roots of the form (∞, ∞) at $\lambda = \infty$.

Homotopy Function Design: No Finite Roots.

We show that by choosing either r_1 or r_2 to be a non-zero constant, Equations (4.10) are prevented from having any finite roots at $\lambda = \infty$.

Proposition 4.4.1 (No Finite Roots) *If either $r_1(x_1, x_2)$ is a non-zero constant or $r_2(x_1, x_2)$ is a non-zero constant, then the homotopy function given in Equations (4.10) can have no finite roots at $\lambda = \infty$.*

Proof: Assume not. Let $r_1(x_1, x_2) = k, k \in C, k \neq 0$. For H to have a finite solution $(x_1, x_2) = (k_1, k_2)$, equations

$$r_1(k_1, k_2) = 0 \quad (4.11)$$

$$r_2(k_1, k_2) = 0$$

must be satisfied. To see this, let $\mu = 1/\lambda$ and substitute into Equations 4.10. Then multiply out the denominators and take the limit as $\mu \rightarrow 0$. Equation (4.11) leads to the inconsistency $r_1(x_1, x_2) = k = 0$. ♣

As a consequence, the following homotopy function has no finite roots at an infinite parameter value, $\lambda = \infty$.

$$h_1(x_1, x_2, \lambda) = p_1(x_1, x_2) + \lambda \quad (4.12)$$

$$h_2(x_1, x_2, \lambda) = p_2(x_1, x_2) + \lambda r_2(x_1, x_2)$$

Homotopy Function Design: *All roots to (∞, ∞) .*

Next, we seek conditions on h_2 that force all roots of Equations (4.12) to go to (∞, ∞) as $\lambda \rightarrow \infty$. Specifically, we let r_1 be a non-zero constant, set $r_2 \equiv 0$, and derive conditions on p_2 .

The idea, more formally stated in the following proposition, is that if the polynomial p_2 consists of a sum of monomials such that the highest degreed terms of each variable, $x_1^{n_2}$ and $x_2^{m_2}$, appear alone - without crossterms - or at least not in form $x_1^{n_2} x_2^{m_2}$ where $\deg_{x_1}(p_2) = n_2$ and $\deg_{x_2}(p_2) = m_2$, then the homotopy function is structured so that all roots are forced to infinity along with the parameter.

Proposition 4.4.2 (Infinite Roots) *Assume that p_1 is a function of both variables, x_1 and x_2 , and that r_1 is a non-zero constant. If $r_2 \equiv 0$ and p_2 is a function of both x_1 and x_2 without a cross term of the form $x_1^{n_2} x_2^{m_2}$, where $\deg_{x_1}(p_2) = n_2$ and $\deg_{x_2}(p_2) = m_2$, then the homotopy function given by Equations 4.12 can have no roots of the form $(x_1, x_2) = (\infty, k_2)$ or $(x_1, x_2) = (k_1, \infty)$ at $\lambda = \infty$. All roots go to infinity, the point $(x_1, x_2) = (\infty, \infty)$, along with the parameter.*

Proof:

Assertion: No roots of the form $(x_1, x_2) = (\infty, k_2)$ or $(x_1, x_2) = (k_1, \infty)$.

Assume not. Substitute $x_1 = 1/y_1$, $x_2 = 1/y_2$ and $\mu = 1/\lambda$ into Equations (4.12), multiply out the denominators, and then take the limit as $\mu \rightarrow 0$ to get

$$y_1^{n_1} y_2^{m_1} = 0 \quad (4.13)$$

$$y_1^{n_2} y_2^{m_2} p_2(1/y_1, 1/y_2) = 0 \quad (4.14)$$

By Equation (4.13), either $y_1 = 0$ or $y_2 = 0$, or both. Without loss of generality, we assume the polynomial p_2 is of the form $p_2(x_1, x_2) = x_1^{n_2} x_2^{s_2} + x_2^{m_2} x_1^{s_1} + l(x_1, x_2)$, where $s_2 < m_2$, $s_1 < n_2$, and the polynomial l is such that $\deg_{x_1}(l) < n_2$ and $\deg_{x_2}(l) < m_2$. The term $y_1^{n_2} y_2^{m_2} p_2(1/y_1, 1/y_2)$ then equals $y_1^{n_2-s_1} + y_2^{m_2-s_2} + y_1^{n_2} y_2^{m_2} l(1/y_1, 1/y_2)$. Next, we assume that there exists a solution of the form $(x_1, x_2) = (\infty, k_2)$ at $\lambda = \infty$, where k_2 is a finite, non-zero number ($y_1 = 0$, $y_2 \neq 0$). Then setting $y_1 = 0$ in Equation (4.14) leads to $y_2^{m_2-s_2} = 0$, and by translation $x_2 = \infty$. However, we assumed $x_2 = k_2$, so proof by contradiction. The same reasoning excludes the possibility of a root of the form $(x_1, x_2) = (k_1, \infty)$. ♣

Connectivity at Infinity:

We have now seen that a homotopy function satisfying Proposition 4.4.2 of the form (4.12) with r_1 constant and $r_2 \equiv 0$ has an infinite repeated root of multiplicity n at $\lambda = \infty$. The next question is that of local connectivity in the neighborhood of $\lambda = \infty$. That is, how many algebraic elements are there at $\lambda = \infty$? What determines the answer to this question? If there are two or more algebraic elements, how are they grouped together?

In the following Proposition, the expression $\gcd(n, m)$ is defined below, and *winding numbers* w_i are defined in Section 4.2.

Definition 4.4.1 (Greatest Common Divisor) *The term $\gcd(n, m)$ refers to the greatest common divisor of integers n and m . For example, $\gcd(4, 8) = 4$, and $\gcd(5, 8) = 1$.*

Lemma 4.4: *A polynomial function of the form $h(x_1, x_2, \lambda) = p_1(x_1, x_2) + \lambda$ is irreducible. It is also locally irreducible in the neighborhood of the infinite point, $(x_1, x_2, \lambda) = (\infty, \infty, \infty)$.*

Proof: (1) If h is reducible, then it can be written as a product of polynomials $h = f(x_1, x_2, \lambda)g(x_1, x_2, \lambda)$. The polynomials f and g are composed of finite sums of monomials, $f(x, \lambda) = \sum_{k \geq 0} a_k \lambda^k + \sum_{k \geq 0} b_k x^k + \sum_{k, j > 0} c_{j,k} x^k \lambda^j$, and $g(x, \lambda) = \sum_{k \geq 0} d_k \lambda^k + \sum_{k \geq 0} r_k x^k + \sum_{k, j > 0} f_{j,k} x^k \lambda^j$. Multiplying out f and g , term by term, shows that it is impossible to form a polynomial where λ appears only as a scalar, additive term, as it does in $h(x_1, x_2, \lambda) = p_1(x_1, x_2) + \lambda$. Therefore, h is irreducible. (2) To prove that h is locally irreducible at infinity, we first move the infinite point $(x_1, x_2, \lambda) = (\infty, \infty, \infty)$ to zero by transforming coordinates $x_1 = 1/y_1$, $x_2 = 1/y_2$, and $\lambda = 1/\mu$, and multiplying out the denominator to get the polynomial $\mu(y_1^n y_2^m p_1(1/y_1, 1/y_2) + y_1^n y_2^m = 0$.¹ The term $y_1^n y_2^m p_1(1/y_1, 1/y_2)$ is a polynomial in y_1 and y_2 , which we will call q , so we write $\mu(y_1^n y_2^m p_1(1/y_1, 1/y_2) + y_1^n y_2^m$ as $\mu q(y_1, y_2) + y_1^n y_2^m$. If h is locally reducible at infinity, then $\mu q(y_1, y_2) + y_1^n y_2^m$ is locally reducible at zero, and may be locally represented as a product of power series around zero. Writing out two power series, and then multiplying them out term by term, reveals that it is impossible to form a power series where μ appears as it does in $\mu q(y_1, y_2) + y_1^n y_2^m = 0$, linearly but not reducibly so. Thus, $\mu q(y_1, y_2) + y_1^n y_2^m = 0$ is locally irreducible around zero, and $h(x_1, x_2, \lambda) = p_1(x_1, x_2) + \lambda$ is locally irreducible around infinity. ♣

Proposition 4.4.3 (Connecting Infinity): *Given the polynomial homotopy function $H = (h_1, h_2)'$*

$$\begin{aligned} h_1(x_1, x_2, \lambda) &= p_1(x_1, x_2) + \lambda \\ h_2(x_1, x_2) &= p_2(x_1, x_2) \end{aligned} \tag{4.15}$$

with p_1 and p_2 such that as $\lambda \rightarrow \infty$ all roots (x_1, x_2) of $H = 0$ approach $(x_1, x_2) = (\infty, \infty)$,

¹Though the coordinate transformation $x = 1/y$ is not in general isomorphic, it can be applied to move a point from infinity to zero if the function has only a single point at infinity. This is the case for the homotopy function specified in Propositions 4.4.2 and 4.4.3, by design. These functions have only a single point at infinity, $(x_1, x_2, \lambda) = (\infty, \infty, \infty)$ (and no points of the form $(x_1, x_2, \lambda) = (\infty, k, \infty)$.)

1. If p_2 is locally irreducible around $(x_1, x_2) = (\infty, \infty)$, then all solution branches of $H = 0$ are connected in a single algebraic element around $\lambda = \infty$.
2. If p_2 is of the form $p_2 = x_1^{n_2} + x_2^{m_2} + r(x_1, x_2)$, where the total degree of the polynomial r is less than or equal to $\min(n_2, m_2)$, and n_2 and m_2 are coprime, then p_2 is locally irreducible around $(x_1, x_2) = (\infty, \infty)$.

Proof:

1. Let h_2 be locally irreducible in the neighborhood of the infinite branch point. By Proposition A5 in Section 4.2, along with Bertini's Theorem² on generic intersections, if h_1 and h_2 are locally irreducible around the branch point $\lambda = \infty$, then all d branches coalescing to $(x_1, x_2) = (\infty, \infty)$ at $\lambda = \infty$ are joined in a single algebraic element around $\lambda = \infty$. It remains to be proved that h_1 is also irreducible in this neighborhood. By Lemma 4.4, any function of the form $h(x, \lambda) = p(x) + \lambda$ is locally irreducible around infinity. Locally, the algebraic element may be represented by equations of the form

$$\begin{aligned} x_1^d &= \lambda^{w_1} \\ x_2^d &= \lambda^{w_2} \end{aligned}$$

where $d-1$ is the order of the element and the integers (d, w_1, w_2) are relatively prime. Because the homotopy function is constructed so that all roots coalesce at infinity, d is also the degree of the solution space (variety) of $H(x, \lambda) = 0$.

2. Let $p_2 = x_1^{n_2} + x_2^{m_2} + r(x_1, x_2)$, where the total degree of the polynomial r is less than or equal to $\min(n_2, m_2)$. Move the branch point to zero by changing coordinates to $y_1 = 1/x_1$ and $y_2 = 1/x_2$, substituting the new coordinates into p_2 , and multiplying out the denominator to get the polynomial $p'_2(y_1, y_2) = y_1^{n_2} y_2^{m_2} p_2(1/y_1, 1/y_2) = y_1^{n_2} + y_2^{m_2} + y_1^{n_2} y_2^{m_2} r(1/y_1, 1/y_2)$. We investigate p'_2 in the neighborhood of $(y_1, y_2) = (0, 0)$. If p'_2 is locally reducible around zero, it may be locally represented as the product of power series $p'_2(y_1, y_2) = f(y_1, y_2)g(y_1, y_2)$. Let $m_2 \leq n_2$. By the degree assumption on r , $p'_2(y_1, y_2) = y_1^{n_2} + y_2^{m_2} + \sum_{i,j: i+j > m_2} a_{i,j} y_1^i y_2^j$. Near zero, $p'_2(y_1, y_2) \approx y_1^{n_2} + y_2^{m_2}$. It is known that if n_2 and m_2 are coprime, a polynomial of the form $y_1^{n_2} + y_2^{m_2}$ cannot

²We assume that the equations h_1 and h_2 intersect generically, as in Bertini's Theorem on page 8 of [29]. An example of what we do not expect, two irreducible equations that intersect in a non-generic way, is $x_1 - x_2\lambda = 0$ and $x_1 = 0$. Though each equation is irreducible, the intersection of the two equations is $x_2\lambda = 0$, a reducible equation.

be factored into a product of polynomials. Therefore, $p'_2(y_1, y_2)$ is locally irreducible around zero, and $p_2(x_1, x_2)$ is locally irreducible around $(x_1, x_2) = (\infty, \infty)$. ♣

Remark (Winding Numbers): If p_2 is of the form $p_2 = x_1^{n_2} + x_2^{m_2} + r(x_1, x_2)$, where the total degree of the polynomial r is less than or equal to $\min(n_2, m_2)$, and n_2 and m_2 are coprime, then the *winding numbers* (w_1, w_2) of the algebraic element connecting all solutions of Equations (4.15) around infinity, as defined in Section 4.2, can be *read directly off the equations*. The winding numbers $(w_1, w_2) = (m_2, n_2)$, meaning that as the parameter λ encircles infinity d times, where d is the number of solutions of the homotopy function at a regular value, the solution component x_1 winds around infinity a total of m_2 times, while that of x_2 winds around infinity a total of n_2 times.

Corollary 4.4.3: *Given the polynomial homotopy function $H = (h_1, h_2)'$*

$$\begin{aligned} h_1(x_1, x_2, \lambda) &= p_1(x_1, x_2) + \lambda \\ h_2(x_1, x_2) &= p_2(x_1, x_2) \end{aligned}$$

with p_1 and p_2 such that as $\lambda \rightarrow \infty$ all roots (x_1, x_2) of $H = 0$ approach $(x_1, x_2) = (\infty, \infty)$, and p_2 of the form $p_2 = x_1^{n_2} + x_2^{m_2} + r(x_1, x_2)$, with $\deg(r) \leq \min(n_2, m_2)$, then the following four statements are equivalent.

1. $\gcd(n_2, m_2) = 1$, where $\deg_{x_1}(p_2) = n_2$ and $\deg_{x_2}(p_2) = m_2$.
2. $x_1^{n_2} + x_2^{m_2}$ is irreducible. Equivalently, h_2 is locally irreducible at infinity.
3. \exists a single algebraic element connecting all solution branches of $H = 0$ around $\lambda = \infty$.
4. $\gcd(w_1, w_2) = 1$, where w_1 and w_2 are the winding numbers of x_1 and x_2 .

Proof: Follows directly from Proposition 4.4.3 and the above remark on winding numbers.

♣

Interpretation (Proposition 4.4.3, Corollary 4.4.3):

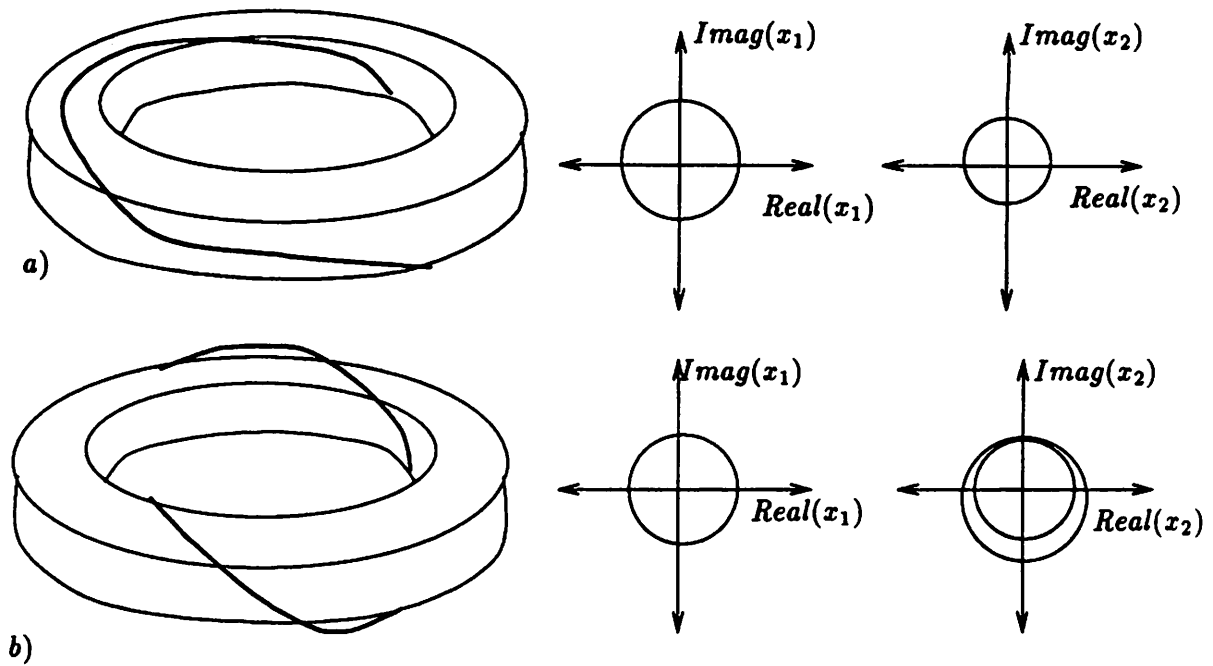


Figure 4.12: Visualize trajectories as tori with various windings (w_1, w_2) of (x_1, x_2) . a) Winding numbers $(w_1, w_2) = (1, 1)$. b) Winding numbers $(w_1, w_2) = (1, 2)$.

Geometric Interpretation: Tori windings

Because of the nature of complex solution space, the image of a circle in the complex parameter plane is a *collection of closed curves* in the complex solution plane (as long as the homotopy function has finite degree). In this section we deal with solution vectors (x_1, x_2) rather than scalars. A closed solution curve may be thought of as a pair of projected components, each of which forms a closed curve. This allows us to visualize a closed complex curve (x_1, x_2) , an object in 4-d, as a path along a *torus*, as shown in Figures 4.12 and 4.13. To do this, let x_1 revolve along one axis of rotation, and x_2 along the other. In this geometric context, the concept of winding number of an algebraic element is clear. A closed curve on a torus can be described by a set of winding numbers, the number of times the curve wraps around each axis. Figures 4.12a-b show closed complex solution trajectories with winding numbers $(w_1, w_2) = (1, 1)$ and $(w_1, w_2) = (1, 2)$. Solution trajectories with winding numbers $(w_1, w_2) = (1, 3)$ and $(w_1, w_2) = (2, 2)$ are illustrated in Figures 4.13a-b.

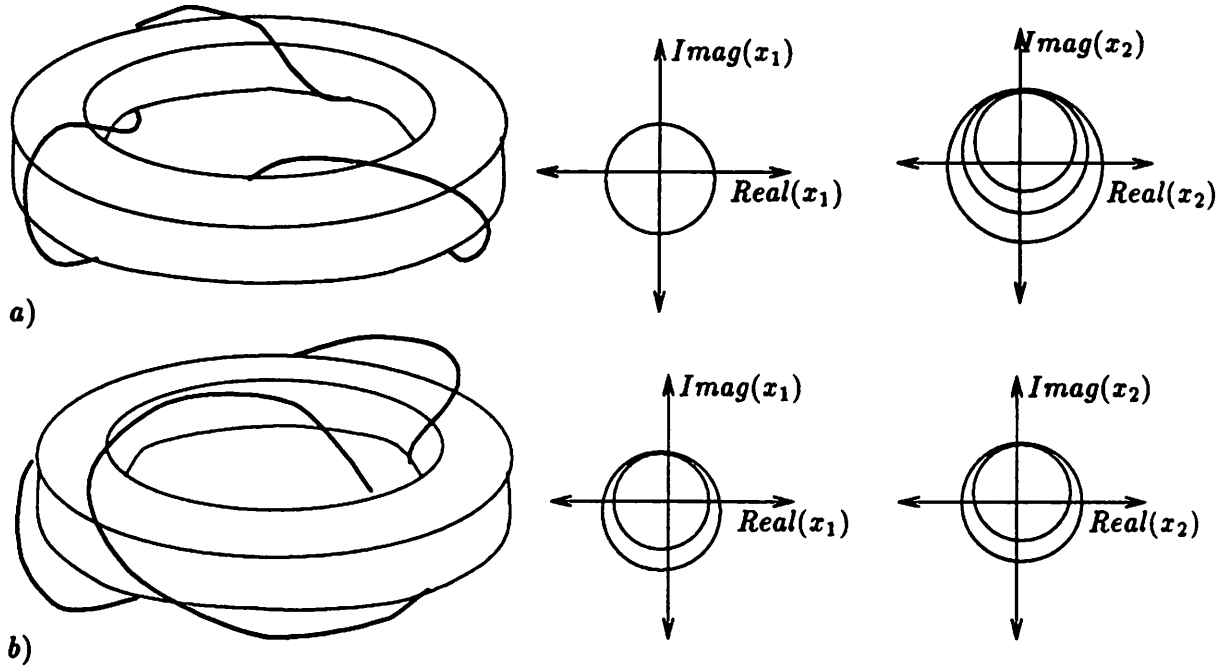


Figure 4.13: Visualize trajectories as tori with various windings (w_1, w_2) of (x_1, x_2) . a) Winding numbers $(w_1, w_2) = (1, 3)$. b) Winding numbers $(w_1, w_2) = (2, 2)$.

Numerical Consequences: Calculating Solutions

The implications of Proposition 4.4.3 are analogous to those of Propositions 4.6.1 and 4.3.1, but for two-dimensional polynomial functions. Assume that a single root $x1_{\lambda_*}$ of the homotopy function consisting of Equations (4.15) at a value of $\lambda_* \in \mathbb{C}$ is chosen such that $|\lambda_*| > r$, where r is the finite radius of a complex parameter circle containing *all finite branch points of H* , is known. Tracing a full circle in the complex parameter plane $\lambda = \lambda_* e^{j\theta}, \theta = 0 : 2\pi$, starting from the solution $x = x1_{\lambda_*}$ at $\lambda = \lambda_*$, will lead to a second solution $x = x2_{\lambda_*}$ of $H(x, \lambda_*) = 0$. Tracing a second full circle in the complex parameter plane $\lambda = \lambda_* e^{j\theta}, \theta = 0 : 2\pi$, from the solution $x = x2_{\lambda_*}$ will lead to a third solution of $H(x, \lambda_*) = 0$, $x = x3_{\lambda_*}$. If one continues to trace circles in the complex parameter plane, moving from solution surface to solution surface, then *all* solutions of $H(x, \lambda_*) = 0$ will be found, and eventually, on the n th revolution, the algebraic element will have been fully cycled through and the original root $x1_{\lambda_*}$ will re-appear. Thus, continuing to traverse a circle in the complex parameter plane $\lambda = \lambda_* e^{j\theta}, \theta = 0 : k2\pi$ will lead to the numerical

calculation a string of roots $(x_{1\lambda_*}, x_{2\lambda_*}, \dots, x_{n\lambda_*}, x_{1\lambda_*}, x_{2\lambda_*}, \dots)$ consisting of the repetition of the n roots of the function $H(x, \lambda_*) = 0$. Once all n roots of $H(x, \lambda_*) = 0$ are found, all the roots of $H(x, \lambda) = 0$ at any arbitrary value of λ may be found, including that of the original problem of interest to be solved, as discussed in Sections 4.1.1 and 4.5.

Disconnected Elements at Infinity:

Next we address the question of the number of algebraic elements at infinity should the conditions of Proposition 4.4.3 *not* be satisfied.

Proposition 4.4.4 (Disconnected Elements at ∞): Given the polynomial homotopy function $H = (h_1, h_2)'$

$$h_1(x_1, x_2) = p_1(x_1, x_2) + \lambda \quad (4.16)$$

$$h_2(x_1, x_2, \lambda) = p_2(x_1, x_2) \quad (4.17)$$

with p_1 and p_2 such that as $\lambda \rightarrow \infty$ all roots (x_1, x_2) of $H = 0$ approach $(x_1, x_2) = (\infty, \infty)$,

1. The number of algebraic elements at infinity, r , is equal to the maximum number of non-unit power series factors of p_2 at infinity.
2. If p_2 is of the form $p_2 = x_1^{n_2} + x_2^{m_2} + r(x_1, x_2)$, with $\deg(r) \leq \min(n_2, m_2)$, then the number of algebraic elements at infinity, r , is equal to the maximum number of non-trivial polynomial factors of $x_1^{n_2} + x_2^{m_2}$.

Proof: 1. Assume that at infinity, the power series expansion of p_2 factors maximally into r non-unit products $g_1 g_2 \dots g_r$. This means that locally, the solution set of $H = 0$ consists of the union of the solution sets of $(h_1 = 0 \cap g_i = 0)$ for all $i \leq r$. Each one of these r solution sets defines an algebraic element, whose only contact with the remaining algebraic elements is at the branch point (because of local reducibility). Per usual, we assume that h_1 and h_2 intersect generically, as discussed in Proposition 4.4.3. 2. Since $\deg(r) \leq \min(n_2, m_2)$, the number of power series factors of p_2 at infinity equals the number of polynomial factors of $x_1^{n_2} + x_2^{m_2}$. To see this, change coordinates $x_1 = 1/y_1$ and $x_2 = 1/y_2$ in p_2 and multiply out the denominator to move the infinite branch point to zero. See the proof of Proposition 4.4.3

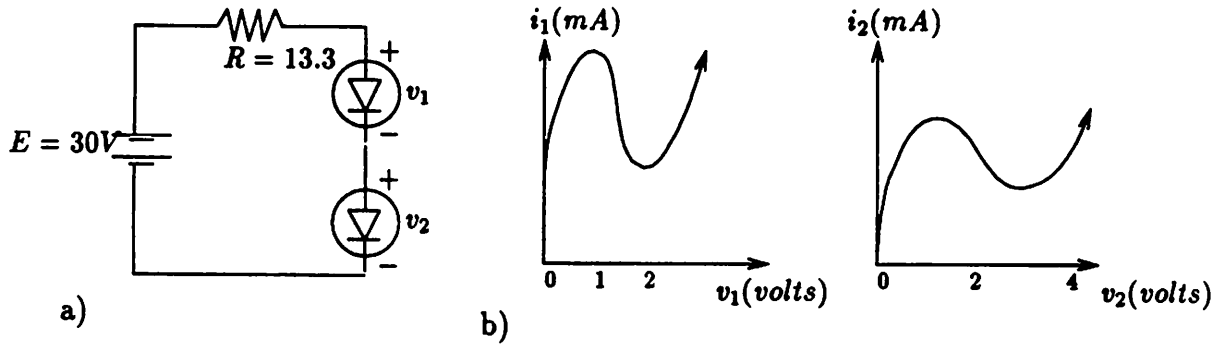


Figure 4.14: a) Tunnel diode circuit b) Diode characteristics.

to see how the higher order terms drop off and the number of factors of the power series equals the number of products of $x_1^{n_2} + x_2^{m_2}$. ♣

Examples:

See Examples 2 and 5 in Subsection 4.4.1.

4.4.1 2-D Polynomial Example Series

What follows is a series of examples that illustrate Proposition 4.4.3, Corollary 4.4.3, and Proposition 4.4.4. We begin with the tunnel diode example of Figure 3.7, repeated here in Figure 4.14 for convenience. In Example 1 of Section 5.4.1 we explore the real connectivity of this example for various real homotopy functions, in terms of function level sets. In this section we complexify the homotopy functions and examine their solution set structures around complex infinity, aiming for connectivity.

The nine operating points of the circuit in Figure 4.14 are determined by the loop equation $f_1(v) = E - Rg_1(v_1) - (v_1 + v_2) = 0$ and the node equation $f_2(v) = g_1(v_1) - g_2(v_2) = 0$, with $v = (v_1, v_2)$. The tunnel diode currents are given by $i_1 = g_1(v) = 2.5v^3 - 10.5v^2 + 11.8v$ and $i_2 = g_2(v) = 0.43v^3 - 2.69v^2 + 4.56v$. So the circuit equations are $F = (f_1, f_2)' = (0, 0)'$, with f_1 and f_2 as follows

$$f_1(x_1, x_2) = E + \alpha_1 x_1^3 + \beta_1 x_1^2 + \gamma_1 x_1 - (x_1 + x_2) \quad (4.18)$$

$$f_2(x_1, x_2) = \alpha'_1 x_1^3 + \beta'_1 x_1^2 + \gamma'_1 x_1 - (\alpha_2 x_2^3 + \beta_2 x_2^2 + \gamma_2 x_2) \quad (4.19)$$

and where the two variables are now written as $x = (x_1, x_2)$ rather than $v = (v_1, v_2)$. Let

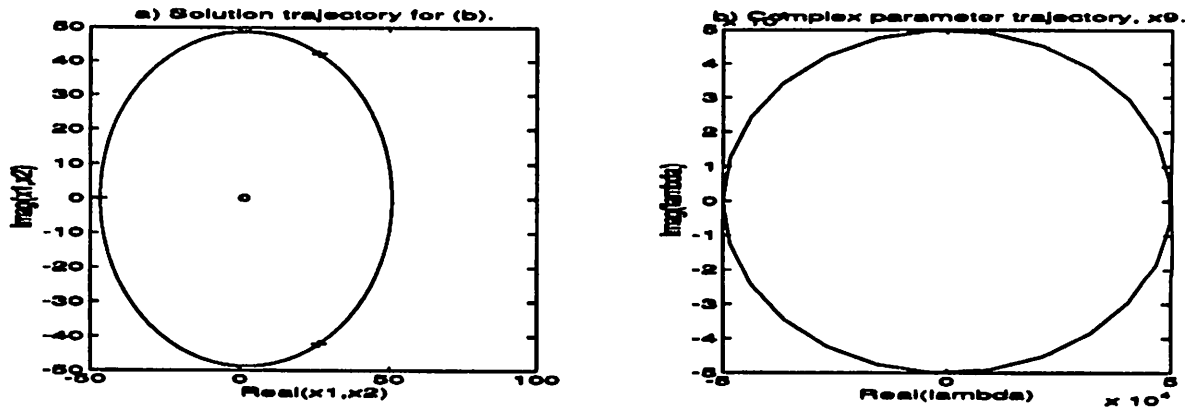


Figure 4.15: MATLAB plots for Example 1. a) Solution trajectory corresponding to encircling the parameter circle in (b) nine times. The solution trajectory passes through all nine solutions. Notice that both components of the solution vector, x_1 and x_2 , appear in the plot. The large closed curve, of radius approximately sixty, is x_2 , while the small curve in the center is x_1 . See the next figure for a close-up of the trajectory x_1 as λ encircles infinity repeatedly. b) The complex parameter circle.

the constants $E = 30, \alpha_1 = -33.25, \beta_1 = 139.6, \gamma_1 = -156.9, \alpha_2 = 0.43, \beta_2 = -2.69, \gamma_2 = 4.56, \alpha'_1 = 2.5, \beta'_1 = -10.5$, and $\gamma'_1 = 11.8$, to match the circuit of Figure 4.14.

Example 1 (parameter in second equation):

In this example we let the homotopy function be of the form

$$f_1(x_1, x_2) = 0 \quad (4.20)$$

$$f_2(x_1, x_2) - \lambda = 0 \quad (4.21)$$

where f_1 and f_2 are as defined in Equations (4.18) and (4.19). At $\lambda = 0$ the above function has the same solutions as those of Equations (4.18) and (4.19). The goal is to find all roots of Equations 4.20 and 4.21 at some regular value of the parameter λ . In the next sections we deal with the problem of how to get from a complete set of roots at an arbitrary regular parameter value to all the roots of the problem of interest, at $\lambda = 0$. For now, we divide the example into two parts, numerical computation and analysis.

Numerical Calculation:

Figures 4.15 and 4.16 show MATLAB plots of a simple, complex encirclement algorithm applied to the homotopy function in Equations (4.20) and (4.21). The algorithm

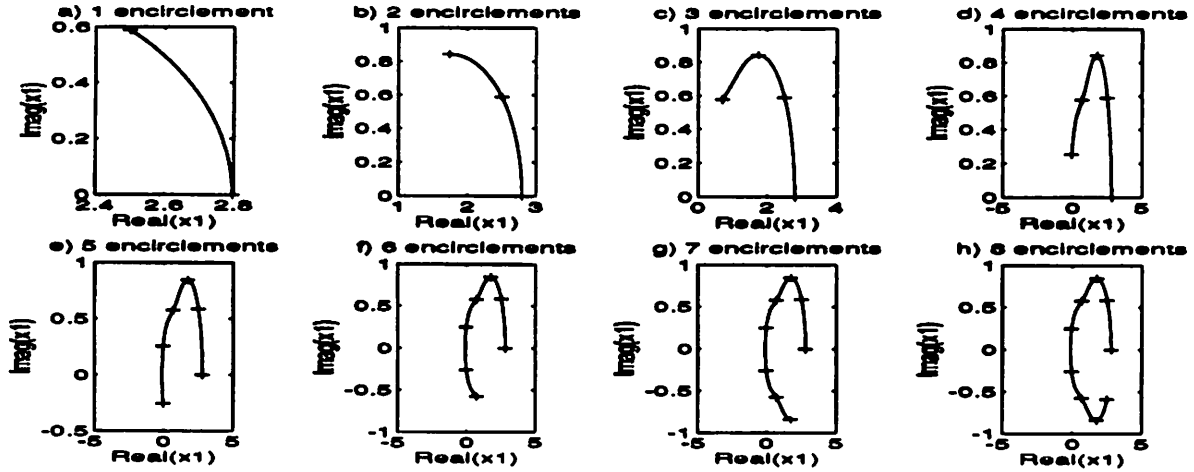


Figure 4.16: More MATLAB plots for Example 1. A close up of solution component x_1 as the complex parameter circle in the previous figure is traced 8 times. a) One encirclement. b) Two encirclements. c) Three encirclements. d) Four encirclements. d) Five encirclements. f) Six encirclements. g) Sevel encirclements. h) Eight encirclements.

involves tracing a root of $H(x, \lambda) = 0$ out to a large parameter value $\lambda = -\lambda_*$, and then stepping around a complex circle of radius λ_* repeatedly until all other available roots of $H(x, -\lambda_*) = 0$ are found. At each new step λ_i around the circle, a Newton-Raphson corrector is applied to get the next point along the trajectory.

Figure 4.15b shows the complex parameter circle $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, $\lambda_* = 5 \times 10^4$, a circle of large enough radius to be equivalent to encircling infinity for this example. Figure 4.15a shows the MATLAB solution trajectory corresponding to nine encirclements of the circle in Figure 4.15b. This trajectory passes through all nine solutions of $H(x, -\lambda_*) = 0$. Notice that the solution vector (x_1, x_2) has a winding number vector of $(w_1, w_2) = (1, 3)$, meaning that as the parameter λ encircles infinity nine times, x_1 winds around infinity only once, while x_2 winds around infinity three times. Figure 4.16 shows a close up of the path of solution component x_1 as the complex parameter circle is traversed repeatedly.

Analysis around ∞ :

By Proposition 4.4.2, the homotopy function of this example has all roots approaching the double infinity point $(x_1, x_2) = (\infty, \infty)$ as $\lambda \rightarrow \infty$. But to show this more concretely, perform the following change of variables $x_1 = 1/y_1$, $x_2 = 1/y_2$, and $\lambda = 1/\mu$, substitute the new variables into Equations (4.20) and (4.21), and then multiply out the

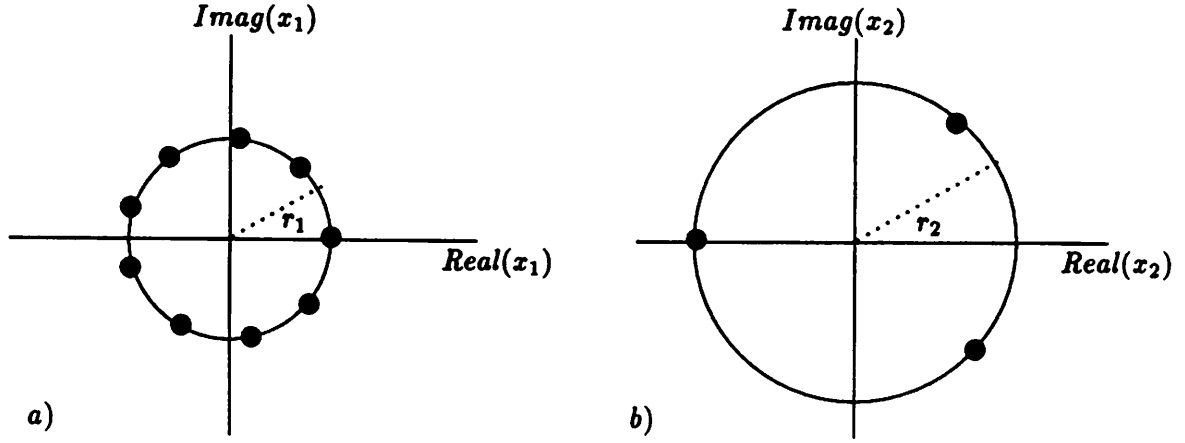


Figure 4.17: Calculated root structure for Example 1. a) The root components x_1 take the form of nine roots equally spaced around a complex circle of radius r_1 . b) The root components x_2 form an equally spaced triad around a complex circle of radius r_2 .

denominators to get

$$y_1^3 y_2 E + \alpha_1 y_2 + \beta_1 y_1 y_2 + \gamma_1 y_1^2 y_2 - (y_1^2 y_2 + y_1^3) = 0 \quad (4.22)$$

$$\mu g(y_1, y_2) + y_1^3 y_2^3 = 0 \quad (4.23)$$

where $g(y_1, y_2)$ is a polynomial in y_1 and y_2 . Since we are interested in what happens as $\lambda \rightarrow \infty$ we look for $\mu \rightarrow 0$ in the new coordinate system. As $\mu \rightarrow 0$, Equation (4.23) becomes $y_1^3 y_2^3 = 0$, so at $\mu = 0$ either $y_1 = 0$ or $y_2 = 0$ or both variables are zero. First, we check out the possibility that $y_1 = 0$ and $y_2 \neq 0$. With $y_1 = 0$, Equation (4.22) reduces to $\alpha_1 y_2 = 0$, so the hypothesis $y_2 \neq 0$ is shown to be false. Next, we check out the possibility that $y_2 = 0$ and $y_1 \neq 0$. With $y_2 = 0$, Equation (4.22) reduces to $y_1^3 = 0$, so the hypothesis $y_1 \neq 0$ is also ruled out. Thus the only possibility is that as $\mu \rightarrow 0$ both $y_1 \rightarrow 0$ and $y_2 \rightarrow 0$, which is equivalent to saying that all roots of Equations (4.20) and (4.21) approach $(x_1, x_2) = (\infty, \infty)$ as $\lambda \rightarrow \infty$.

The next step in our analysis is to investigate the structure of the solutions of Equations (4.20) and (4.21) in the neighborhood of infinity. To do this it is convenient to return to our original coordinate system (x_1, x_2, λ) and to proceed from there. As $\lambda \rightarrow \infty$, the variables x_1 and x_2 also go to infinity, and Equations (4.20) and (4.21) become dominated by the monomials of highest degree in each equation. Near infinity, Equations (4.20)

and (4.21) approach the following equations.

$$\alpha_1 x_1^3 - x_2 = 0 \quad (4.24)$$

$$\alpha_1' x_1^3 - \alpha_2 x_2^3 - \lambda = 0 \quad (4.25)$$

Then, substituting $x_1^3 = x_2/\alpha_1$ from Equation (4.24) into Equation (4.25) leads to the equation

$$(\alpha_1'/\alpha_1)x_2 - \alpha_2 x_2^3 - \lambda = 0 \quad (4.26)$$

which, in the neighborhood of infinity, is dominated by the higher degreed term and may be approximated by the equation

$$-\alpha_2 x_2^3 - \lambda = 0. \quad (4.27)$$

Setting the parameter $\lambda = |\lambda|e^{j\theta} = |\lambda|e^{j\theta+k2\pi}, k \in Z$, and solving Equation (4.27) for x_2 gives us $x_2 \approx |\lambda/\alpha_2|^{1/3}e^{j(\theta+k2\pi+\pi)/3}$, which is a triad of roots equally spaced around a complex circle of radius $r_2 = |\lambda/\alpha_2|^{1/3}$ as shown in Figure 4.17a. Since from Equation (4.24), $x_1^3 \approx x_2/\alpha_1$, then $x_1 \approx (r_2/\alpha_1)^{1/3}e^{j(\theta+k2\pi+\pi)/9}$, a set of nine roots equally distributed around a complex circle of radius $r_1 = (r_2/\alpha_1)^{1/3}$, as shown in Figure 4.17b.

Finally, we check the derived root structure of Figures 4.17a-b against the numerical computations shown in Figures 4.15a-b and 4.16, and see that the radii and root configurations shown match. At $|\lambda| = 50,000$, $r_2 = 48.8$ and $r_1 = 1.13$, which match Figures 4.15a-b closely. Also notice that the *winding numbers* of the two variables, $w_1 = 1$ for the variable x_1 and $w_2 = 3$ for the variable x_2 are consistent in the numerically obtained plots, the derived plots, and can be read off Equation (4.24) by looking at the degrees of each of the variables, as stated in the remark associated with Proposition 4.4.3. Additionally, observe that Equation (4.24) is irreducible and consists of monomials with degrees whose greatest common divisor is one, also consistent with Corollary 4.4.3. \square

Example 2 (parameter in first equation):

In this next example we let the homotopy function be of the form

$$f_1(x_1, x_2) - \lambda = 0 \quad (4.28)$$

$$f_2(x_1, x_2) = 0 \quad (4.29)$$

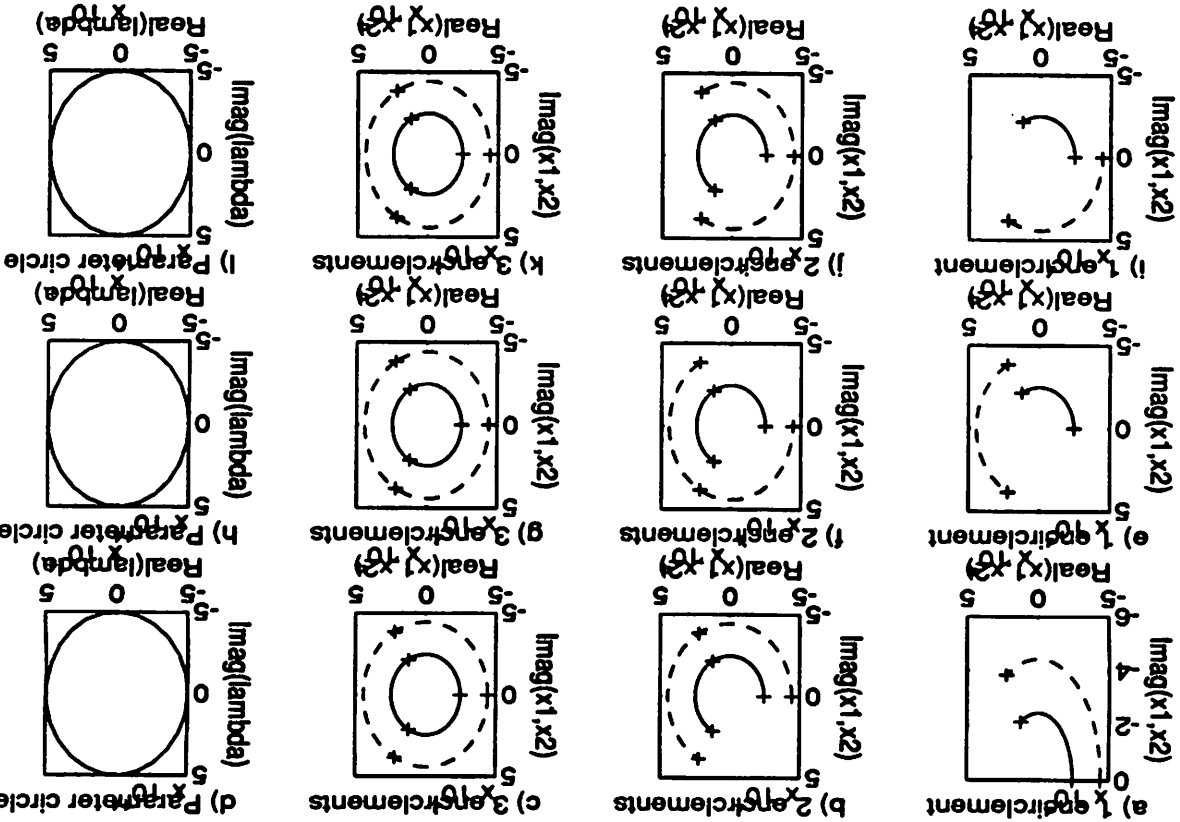


Figure 4.18: MATLAB plot for Example 2. There are three algebraic elements, each connecting three of the nine solutions. a-c) The solution path for one, two and then three excursions around the complex parameter circle in (d), starting from $(x_1, x_2) = (A, D)$ in the previous figure. e-g) Ditto, starting from $(x_1, x_2) = (A, F)$. i-k) Ditto, starting from $(x_1, x_2) = (A, E)$.

where f_1 and f_2 are as defined in Equations (4.18) and (4.19). This example differs from the previous one in that the homotopy function was derived by adding a parameter to the first equation rather than the second one. As in Example 1, the above function has the same solutions as those of Equations (4.18) and (4.19) at $\lambda = 0$. We numerically and analytically examine the connectivity of the root structure of the homotopy function consisting of Equations (4.28) and (4.29) in the neighborhood of infinity.

Numerical Calculation:

Figure 4.18 shows MATLAB plots of complex encirclement applied to the homotopy function in Equations (4.28) and (4.29). Figure 4.18d shows the complex parameter

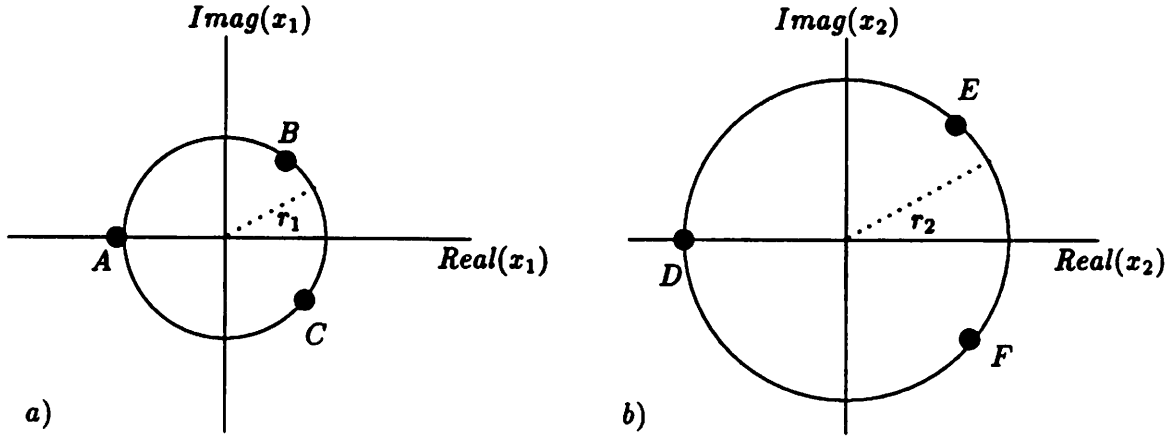


Figure 4.19: Calculated root structure for Example 2. a) The root components x_1 form an equidistant triad around a complex circle of radius r_1 . b) The root components x_2 form an equidistant triad around a complex circle of radius r_2 .

circle $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, $\lambda_* = 5 \times 10^4$, a circle of large enough radius to be equivalent to encircling infinity for this example. Figure 4.18c shows the solution trajectory corresponding to three encirclements of 4.18d. This trajectory passes through just three of the nine solutions of $H(x, -\lambda_*) = 0$. As the parameter λ encircles infinity three times, both x_1 and x_2 encircle infinity once. Figures 4.18(a,b) shows a close up of the solution path (x_1, x_2) as the complex parameter circle is traversed repeatedly. The two other algebraic elements at infinity, each of which connects three of the nine total solutions of $H(x, -\lambda_*) = 0$, are shown in Figures 4.18(e-g) and (i-k).

Analysis around ∞ :

By Proposition 4.4.2, the homotopy function of this example has all roots approaching the double infinity point $(x_1, x_2) = (\infty, \infty)$ as $\lambda \rightarrow \infty$. Since we went through a concrete derivation of this result in the previous example, we will not repeat the full exercise here, but the reader can check as follows. As in the last example, perform the change of variables $x_1 = 1/y_1$, $x_2 = 1/y_2$, and $\lambda = 1/\mu$ and then substitute the new variables into Equations (4.28) and (4.29) and multiply out the denominators. Setting $\mu = 0$ leads to the

following equations,

$$y_1^3 y_2 = 0 \quad (4.30)$$

$$\alpha'_1 y_2^3 + \beta'_1 y_2^3 y_1 + \gamma'_1 y_2^3 y_1^2 - (\alpha_2 y_1^3 + \beta_2 y_1^3 y_2 + \gamma_2 y_1^3 y_2^2) = 0 \quad (4.31)$$

From Equation (4.30) either $y_1 = 0$ or $y_2 = 0$ or both variables are zero. With $y_1 = 0$, Equation (4.31) reduces to $\alpha'_1 y_2^3 = 0$, so the hypothesis ($y_1 = 0, y_2 \neq 0$) is shown to be false. Setting $y_2 = 0$, reduces Equation (4.31) to $\alpha_2 y_1^3 = 0$, so the hypothesis ($y_1 \neq 0, y_2 = 0$) is also ruled out. The only remaining possibility is that as $\mu \rightarrow 0$ both $y_1 \rightarrow 0$ and $y_2 \rightarrow 0$, which is equivalent to saying that all roots of Equations (4.28) and (4.29) approach $(x_1, x_2) = (\infty, \infty)$ as $\lambda \rightarrow \infty$.

The next step in our analysis is to investigate the structure of the solutions of Equations (4.28) and (4.29) in the neighborhood of infinity. As $\lambda \rightarrow \infty$, the variables x_1 and x_2 also go to infinity, and Equations (4.28) and (4.29) become dominated by the monomials of highest degree in each equation. Near infinity, Equations (4.28) and (4.29) approach the following equations.

$$\alpha_1 x_1^3 - x_2 - \lambda = 0 \quad (4.32)$$

$$\alpha'_1 x_1^3 - \alpha_2 x_2^3 = 0 \quad (4.33)$$

Substituting $x_1^3 = (\alpha_2/\alpha'_1)x_2^3$ from Equation (4.33) into Equation (4.32) leads to the equation

$$(\alpha_1 \alpha_2 / \alpha'_1) x_2^3 - x_2 - \lambda = 0 \quad (4.34)$$

which, in the neighborhood of infinity, is dominated by the higher degreed term and may be approximated by the equation

$$(\alpha_1 \alpha_2 / \alpha'_1) x_2^3 - \lambda = 0. \quad (4.35)$$

In preparation for encircling infinity we represent the complex parameter in polar form, $\lambda = |\lambda|e^{j\theta}$, and note that $\lambda = |\lambda|e^{j\theta} = |\lambda|e^{j(\theta+k2\pi)}$, $k \in Z$. Then we solve Equation (4.35) for x_2 to get $x_2 \approx |(\lambda \alpha'_1)/(\alpha_2 \alpha_1)|^{1/3} e^{j(\theta+k2\pi)/3}$, three roots equally spaced around a complex circle of radius $r_2 = |(\lambda \alpha'_1)/(\alpha_2 \alpha_1)|^{1/3}$. From Equation (4.33), we get $x_1 \approx (r_2 \alpha_2 / \alpha'_1)^{1/3} e^{j(\theta+k2\pi)/3} e^{j(l2\pi)/3}$, $l, k \in Z$, a set of three roots equally spaced around a complex circle of radius $r_1 = |r_2 \alpha_2 / \alpha'_1|^{1/3}$. See Figures 4.19a,b for the two circles in complex solution space, with points

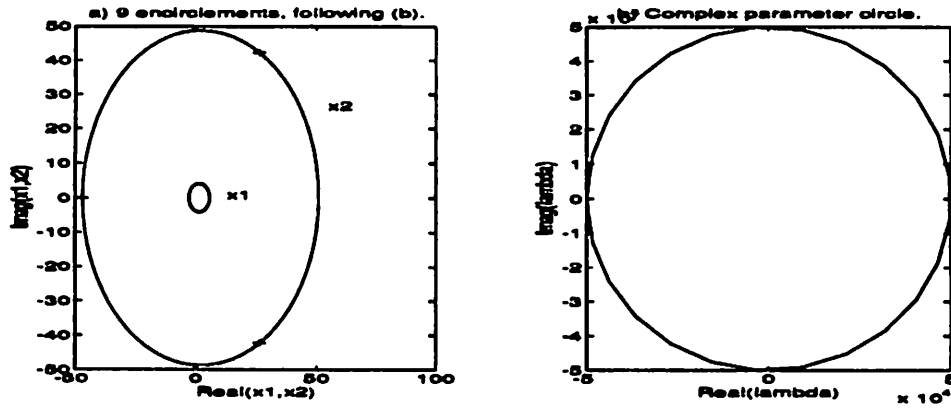


Figure 4.20: A MATLAB plot of Example 3. a) The solution path traced as the complex parameter circle in (b) is traversed nine times. A single algebraic element connects all nine solutions. b) The complex parameter circle.

marked (a, b, c) on the x_1 circle and (d, e, f) on the x_2 circle. The nine solutions of Equations (4.28) and (4.29) at large λ are the cross products of the sets (A, B, C) and (D, E, F) , $(x_1, x_2) = (A, D)$, $(x_1, x_2) = (B, E)$, $(x_1, x_2) = (C, F)$, $(x_1, x_2) = (A, E)$, $(x_1, x_2) = (B, F)$, $(x_1, x_2) = (C, D)$, $(x_1, x_2) = (A, F)$, $(x_1, x_2) = (B, D)$, and $(x_1, x_2) = (C, E)$. These nine solutions do not lie on a single closed curve in complex solution space associated with a large complex parameter circle. Rather, in this example, the inverse image of a circle in complex parameter space encircling infinity is a set of three non-intersecting closed curves in solution space, each of which connects three of the nine solutions of $H(x, -\lambda_\infty) = 0$. Each of these closed curves is an algebraic element of order two.

Next, we check the derived root structure of Figures 4.19a-b against the numerical computations shown in Figure 4.18a-b, and see that the radii and root configurations shown match. At $|\lambda| = 5 \times 10^{14}$, $r_1 = 2.4683 \times 10^4$ and $r_2 = 4.4383 \times 10^4$, which match Figures 4.19a-b closely. Also notice that the numerical calculations verify that all the solutions of Equations (4.28) and (4.29) are, as predicted by the analysis and by Propositions 4.4.3 and 4.4.4, *not connected* by a single algebraic element at infinity, but rather are distributed among *three disconnected algebraic elements*. Consistent with Propositions 4.4.3 and 4.4.4, Equation (4.33) is *reducible*, with monomials in x_1 and x_2 each of degree three, (a greatest common divisor of *three* instead than one). \square

Example 3 (change degree of a lead monomial in f_1):

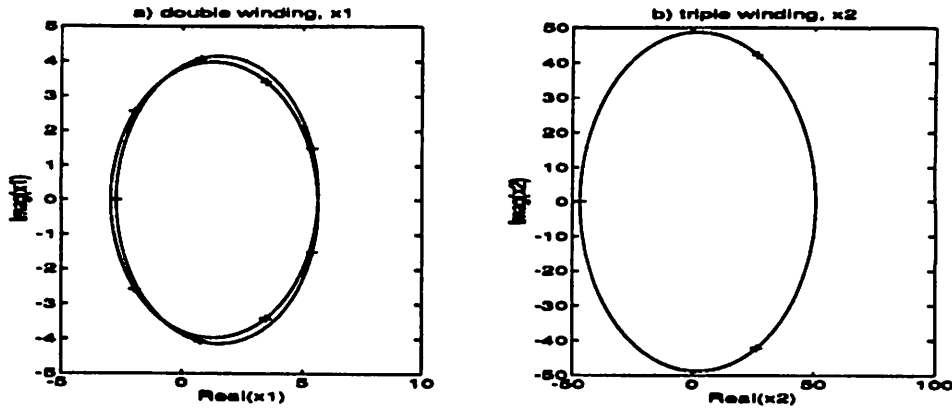


Figure 4.21: Another MATLAB plot of Example 3. a) A close-up of the solution component x_1 as the complex parameter circle shown in the previous figure, part (b), is traversed nine times. Notice the winding number $w_1 = 2$. b) A close-up of the solution component x_2 as the complex parameter circle shown in the previous figure, part (b), is traversed nine times. Notice the winding number $w_2 = 3$.

In this example we let the homotopy function be of the form

$$f_1(x_1, x_2^2) = 0 \quad (4.36)$$

$$f_2(x_1, x_2) - \lambda = 0 \quad (4.37)$$

where f_1 and f_2 are as written below.

$$f_1(x_1, x_2^2) = E + \alpha_1 x_1^3 + \beta_1 x_1^2 + \gamma_1 x_1 - (x_1 + x_2^2)$$

$$f_2(x_1, x_2) = \alpha'_1 x_1^3 + \beta'_1 x_1^2 + \gamma'_1 x_1 - (\alpha_2 x_2^3 + \beta_2 x_2^2 + \gamma_2 x_2)$$

This example has a parameter in the second equation, as does Example 1, but the variable x_2 appears *quadratically* rather than linearly. The purpose of this modification is to numerically investigate the effect of a monomial degree change on the root structure in the neighborhood of infinity. The degree change was deliberately chosen to be one that *does not result in a change of the total degree of the system* from that of Example 1 (it is still nine). In a later example, Example 7, the degree of a monomial is increased to four, thereby increasing the total degree of the system to twelve. Recall that the total degree of a system of polynomial equations is the product of the degrees of each of the equations.

Numerical Calculation:

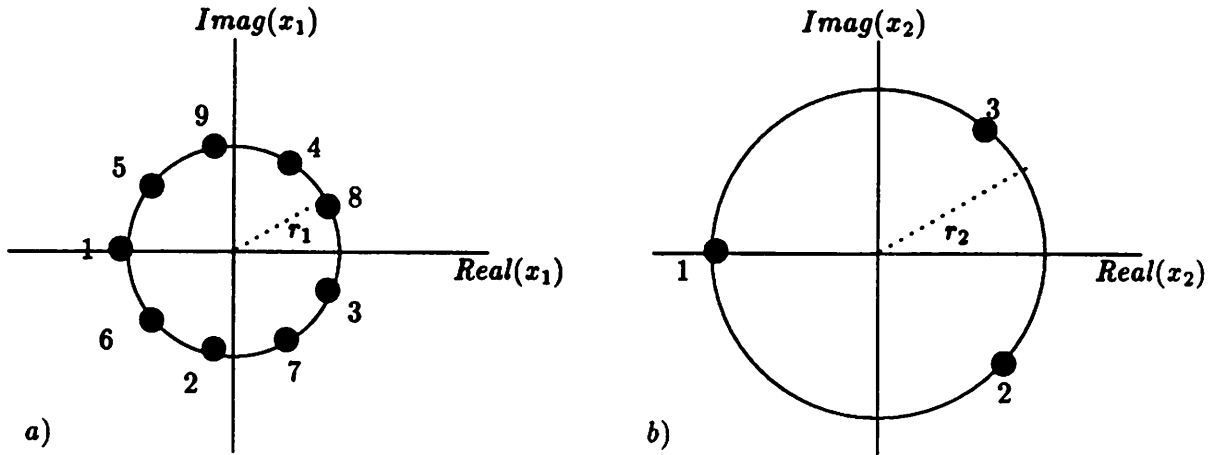


Figure 4.22: Calculated root structure for Example 3. a) The root components x_1 take the form of nine roots that wind twice around a complex circle of radius r_1 . b) The root components x_2 form an equally spaced triad around a complex circle of radius r_2 .

Figures 4.20 and 4.21 show MATLAB plots of complex encirclement applied to the homotopy function in Equations (4.36) and (4.37). Figure 4.20b shows the complex parameter circle $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, $\lambda_* = 5 \times 10^4$, a circle of large enough radius to be equivalent to encircling infinity for this example. Figure 4.20a shows the solution trajectory corresponding to nine encirclements of 4.15b. This trajectory passes through all nine solutions of $H(x, -\lambda_*) = 0$. Notice that the solution vector (x_1, x_2) has a winding number vector of $(w_1, w_2) = (2, 3)$, meaning that as the parameter λ encircles infinity nine times, x_1 winds around infinity twice, while x_2 winds around infinity three times. Figure 4.21 shows a close-up of the path of solution components x_1 and x_2 as the complex parameter circle is traversed repeatedly.

Analysis around ∞ :

By Proposition 4.4.2, the homotopy function of this example has all roots approaching the double infinity point $(x_1, x_2) = (\infty, \infty)$ as $\lambda \rightarrow \infty$. Since we went through derivations of this result for Examples 1 and 2, we will not go through another derivation, but it is analogous to the others.

Next, we investigate the structure of the solutions of Equations (4.36) and (4.37)

in the neighborhood of infinity. As $\lambda \rightarrow \infty$, the variables x_1 and x_2 also go to infinity, and Equations (4.36) and (4.37) become dominated by the monomials of highest degree in each equation. Near infinity, Equations (4.36) and (4.37) approach the following equations.

$$\alpha_1 x_1^3 - x_2^2 = 0 \quad (4.38)$$

$$\alpha'_1 x_1^3 - \alpha_2 x_2^3 - \lambda = 0 \quad (4.39)$$

Substituting $x_1^3 = (1/\alpha_1)x_2^2$ from Equation (4.38) into Equation (4.39) leads to the equation

$$(\alpha'_1/\alpha_1)x_2^2 - \alpha_2 x_2^3 - \lambda = 0 \quad (4.40)$$

which, in the neighborhood of infinity, is dominated by the higher degree term and may be approximated by the equation

$$-\alpha_2 x_2^3 - \lambda = 0 \quad (4.41)$$

Then we set the parameter $\lambda = |\lambda|e^{j\theta+k2\pi}$, $k \in \mathbb{Z}$, and solve Equation (4.41) for x_2 to get $x_2 \approx |\lambda/\alpha_2|^{1/3}e^{j(\theta+k2\pi)/3}$, three equally spaced roots (for θ held constant) around a complex circle of radius $r_2 = |\lambda/\alpha_2|^{1/3}$ as shown in Figure 4.22a. From Equation (4.38), $x_1^3 \approx x_2^2/\alpha_1$, so $x_1 \approx (r_2^{2/3}/\alpha_1^{1/3})e^{j(\theta+k2\pi)/2}$, a set of nine roots that *wind twice* around a complex circle of radius $r_1 = (r_2^{2/3}/\alpha_1^{1/3})$, as shown in Figure 4.22b.

Next, we compare the derived root structure of Figures 4.22a-b against the numerical computations shown in Figures 4.20a-b and 4.21, and see that the radii and root configurations shown are close. At $|\lambda| = 50,000$, $r_2 = 48.8$ and $r_1 = 4.15$, which are good approximations to the radii of the trajectories shown in Figures 4.22a-b. Also notice that the *winding numbers* of the two variables, $w_1 = 2$ for the variable x_1 and $w_2 = 3$ for the variable x_2 are consistent in the numerically obtained plots and the derived plots, and can be read off Equation (4.38) by looking at the degrees of each of the variables, as stated in Corollary 4.4.3. Additionally, observe that Equation (4.38) is irreducible and consists of monomials with degrees whose greatest common divisor is one, also consistent with Corollary 4.4.3. \square

Example 4 (add cross term to f_1):

In this example the homotopy function is as follows

$$f_1(x_1, x_2^2 + x_1 x_2) = 0 \quad (4.42)$$

$$f_2(x_1, x_2) - \lambda = 0 \quad (4.43)$$

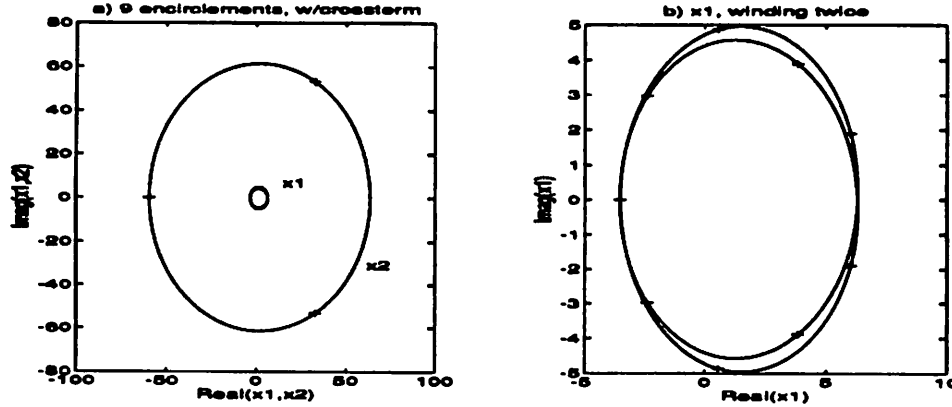


Figure 4.23: A MATLAB plot for Example 4. Like in Example 3, there is a single algebraic element around infinity connecting all nine solutions, and x_1 winds around zero twice while x_2 winds around zero three times. a) The solution trajectory as infinity is encircled nine times. b) A close up of the double winding variable, x_1 .

where f_1 and f_2 are as shown below.

$$\begin{aligned} f_1(x_1, x_2^2 + x_1 x_2) &= E + \alpha_1 x_1^3 + \beta_1 x_1^2 + \gamma_1 x_1 - (x_1 + x_2^2 + x_1 x_2) \\ f_2(x_1, x_2) &= \alpha'_1 x_1^3 + \beta'_1 x_1^2 + \gamma'_1 x_1 - (\alpha_2 x_2^3 + \beta_2 x_2^2 + \gamma_2 x_2) \end{aligned}$$

This example has a parameter in the second equation and the variable x_2 appears *quadratically* in the first equation, as it does in Example 3. In addition, a *crossterm* $x_1 x_2$ has been added to f_1 . The purpose of this added term is to numerically investigate the effect of the presence of crossterms on the root structure in the neighborhood of infinity. The crossterm was deliberately chosen to be of low enough degree so that Proposition 4.4.3-2 applies and the homotopy function is guaranteed to have all roots going to infinity with the parameter. This example illustrates another aspect of Proposition 4.4.3 - that the presence of crossterms of less than maximal degree does not effect the root structure of the homotopy function in the neighborhood of infinity.

Numerical Calculation:

Figure 4.23 shows MATLAB plots of complex encirclement applied to the homotopy function in Equations (4.42) and (4.43). Adding a cross term does not result in any significant topological difference from the solution structure around infinity of Example 3. To conclude, this example illustrates that crossterms of relatively low degree, as specified in Proposition 4.4.3, do not affect the solution set connectivity or winding topology around

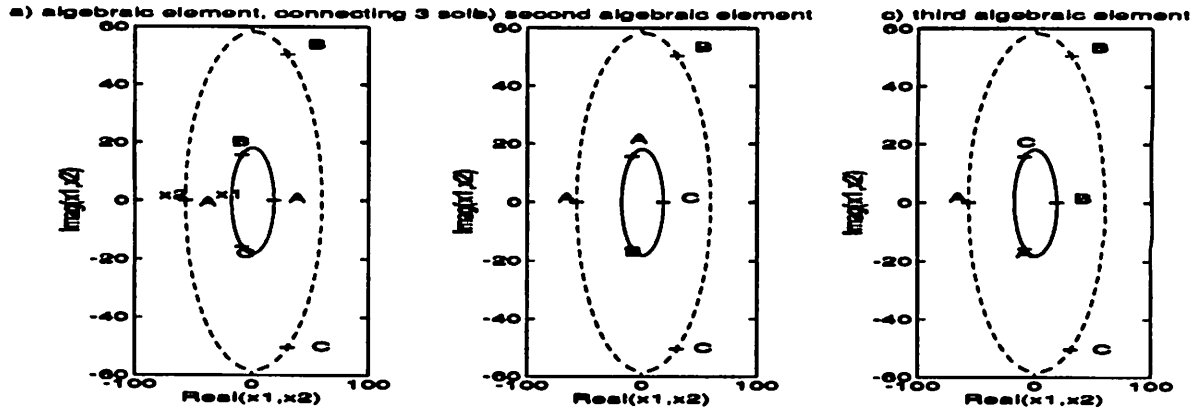


Figure 4.24: A MATLAB plot for Example 5. There are three algebraic elements around the infinite branch point, each of which connects three of the nine solutions. a-c) Algebraic elements 1-3.

infinity. \square

Example 5 (make f_1 locally reducible at ∞):

In this example the homotopy function is

$$f_1(x_1, x_2^3) = 0 \quad (4.44)$$

$$f_2(x_1, x_2) - \lambda = 0 \quad (4.45)$$

with f_1 and f_2 defined below.

$$f_1(x_1, x_2^3) = E + \alpha_1 x_1^3 + \beta_1 x_1^2 + \gamma_1 x_1 - (x_1 + x_2^3)$$

$$f_2(x_1, x_2) = \alpha'_1 x_1^3 + \beta'_1 x_1^2 + \gamma'_1 x_1 - (\alpha_2 x_2^3 + \beta_2 x_2^2 + \gamma_2 x_2)$$

Like Example 1, the parameter λ is embedded in the second equation, but the variable x_2 appears *cubically* in f_1 rather than linearly. The purpose of this example was to try to understand the effect of having both lead monomials of the function f_1 of the same degree on the root structure of the homotopy function in the neighborhood of infinity. As we shall see, though $f_1(x_1, x_2^3)$ is not globally reducible, it is *locally reducible in the neighborhood of infinity*, so the homotopy function given by Equations (4.44) and (4.45) has a disconnected set of algebraic elements around infinity. In fact, in the neighborhood of infinity, this example has a root structure that is nearly identical to that of Example 2, even though the homotopy functions are quite different. This example illustrates, through a counterexample,

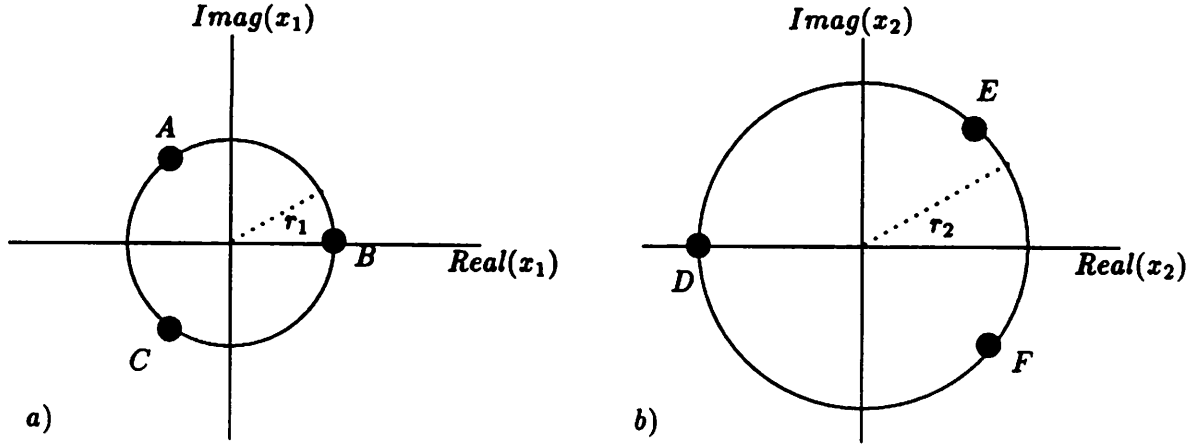


Figure 4.25: Calculated root structure for Example 5. a) The root components x_1 take the form of three roots equally spaced around a complex circle of radius r_1 . b) The root components x_2 take the form of three roots equally spaced around a complex circle of radius r_2 .

why Statement 4 of Corollary 4.4.3 is necessary for connectivity around the infinite branch point. It also illustrates, along with Example 2, the results of Proposition 4.4.4.

Analysis around ∞ :

By Proposition 4.4.4, the homotopy function of this example has all roots approaching the double infinity point $(x_1, x_2) = (\infty, \infty)$ as $\lambda \rightarrow \infty$. Because the variables x_1 and x_2 go to infinity as the parameter λ goes to infinity, Equations (4.44) and (4.45) become dominated by the monomials of highest degree in each equation. Near infinity, Equations (4.44) and (4.45) approach the equations

$$\alpha_1 x_1^3 - x_2^3 = 0 \quad (4.46)$$

$$\alpha'_1 x_1^3 - \alpha_2 x_2^3 - \lambda = 0 \quad (4.47)$$

Substituting $x_1^3 = (1/\alpha_1)x_2^3$ from Equation (4.46) into Equation (4.47) leads to

$$(\alpha'_1/\alpha_1)x_2^3 - \alpha_2 x_2^3 - \lambda = 0 \quad (4.48)$$

Setting the parameter $\lambda = |\lambda|e^{j\theta+k2\pi}$, $k \in \mathbb{Z}$, and solving Equation (4.48) for x_2 gives us $x_2 \approx |\lambda/((\alpha'_1/\alpha_1) - \alpha_2)|^{1/3}e^{j(\theta+k2\pi)/3}$, three roots around a circle of radius $r_2 =$

$|\lambda/((\alpha'_1/\alpha_1) - \alpha_2)|^{1/3}$ like the one shown in Figure 4.19b. From Equation (4.46), $x_1^3 \approx x_2^3/\alpha_1$, so $x_1 \approx (r_2)/\alpha_1^{1/3} e^{j(\theta+k2\pi+\pi)/3} e^{j(l2\pi)/3}$, $l, k \in Z$, a set of three roots equally spaced around a complex circle of radius $r_1 = |(r_2)/\alpha_1^{1/3}|$. From here on out, this discussion and analysis mimics that of Example 2, aside from a phase shift of $\pi/3$ of the roots of x_1 (compare Figures 4.25 and 4.18). See Example 2 for a more detailed discussion of the *three different, disconnected algebraic elements of order two* at infinity, each of which connects three of the nine solutions. An approximate calculation of the radii r_1 and r_2 at $\lambda = 100,000$, $r_1 = 20.38$ and $r_2 = 65.56$ show the MATLAB plots of Figure 4.24a-b to be a good match for the above analysis. \square

Example 6 (change degree of lead monomial in f_2):

For this example we let the homotopy function be of the form

$$f_1(x_1, x_2) = 0 \quad (4.49)$$

$$f_2(x_1, x_2) - \alpha_1 x_1^3 - \lambda = 0 \quad (4.50)$$

where f_1 and f_2 are as defined below

$$\begin{aligned} f_1(x_1, x_2) &= E + \alpha_1 x_1^3 + \beta_1 x_1^2 + \gamma_1 x_1 - (x_1 + x_2) \\ f_2(x_1, x_2) - \alpha_1 x_1^3 &= \beta'_1 x_1^2 + \gamma'_1 x_1 - (\alpha_2 x_2^3 + \beta_2 x_2^2 + \gamma_2 x_2) \end{aligned}$$

investigated in Example 1, except that the lead monomial in x_1 has been deleted from Equation (4.21), making the polynomial cubic in x_2 but only quadratic in x_1 .

Numerical Calculation/Analysis:

The purpose of this example is to see what effect changing the degree of x_1 in the second equation without changing the total degree of this equation has on the root structure of H at infinity. Figure 4.26 shows MATLAB plots of a simple, complex encirclement algorithm applied to the homotopy function in Equations (4.49) and (4.50).

The complex parameter trajectory traced is $\lambda = |\lambda_*| e^{j(\theta)}$, $\theta = 0 : 2\pi$, $\lambda_* = 5 \times 10^4$, a circle of large enough radius to be equivalent to encircling infinity for this example. Figure 4.26a shows the MATLAB solution trajectory corresponding to nine encirclements of the circle in Figure 4.15b. As in Example 1, this trajectory passes through all nine

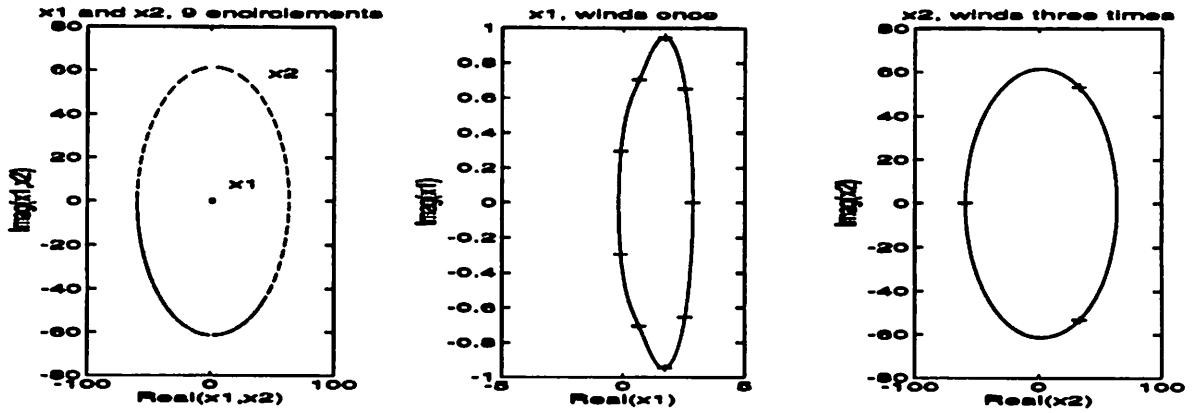


Figure 4.26: A MATLAB plot of Example 6. A single algebraic element connects all nine solutions. a) Solution trajectory as infinity is encircled nine times. b) A close up of x_1 , which winds around zero once. c) A close up of x_2 , which winds around zero three times.

solutions of $H(x, -\lambda_*) = 0$, and the solution vector (x_1, x_2) has a winding number vector of $(w_1, w_2) = (1, 3)$. Figure 4.26b-c shows a close up of the path of solution components x_1 and x_2 . Consistent with Proposition 4.4.3 and Corollary 4.4.3, reducing the monomial degree in this way has no effect on the connectivity or winding structure of the function at infinity.

□

Example 7 (change total degree from 9 to 12):

For the final example we choose the following homotopy function,

$$f_1(x_1, x_2^4) = 0 \quad (4.51)$$

$$f_2(x_1, x_2) - \lambda = 0 \quad (4.52)$$

with f_1 and f_2 defined as below.

$$f_1(x_1, x_2^4) = E + \alpha_1 x_1^3 + \beta_1 x_1^2 + \gamma_1 x_1 - (x_1 + x_2^4)$$

$$f_2(x_1, x_2) = \alpha'_1 x_1^3 + \beta'_1 x_1^2 + \gamma'_1 x_1 - (\alpha_2 x_2^3 + \beta_2 x_2^2 + \gamma_2 x_2)$$

This example has a parameter in the second equation, as does Example 3, but the variable x_2 appears *quartically* rather than cubically. This degree change was deliberately chosen to be one that results in a *change of total degree* of the system, from nine to twelve. This example serves to illustrate Proposition 4.4.3 on a function with twelve roots. As will

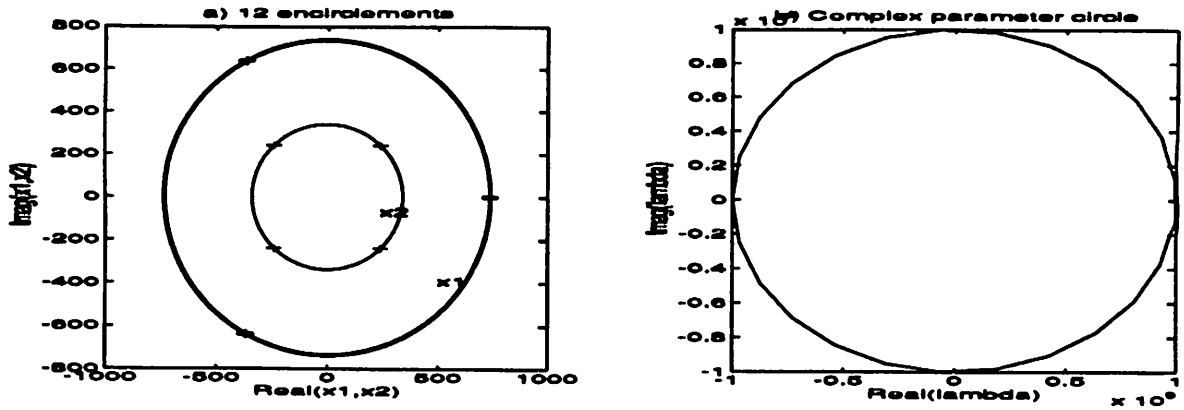


Figure 4.27: A MATLAB plot of Example 7. A single algebraic element connects all twelve solutions. The variables (x_1, x_2) have winding numbers $(w_1, w_2) = (4, 3)$ a) Solution trajectory as infinity is encircled twelve times, as in (b). b) The complex parameter circle.

be shown, in the neighborhood of infinity there is a single algebraic element connecting all twelve solutions, with winding numbers $w_1 = 4$ and $w_2 = 3$, the maximal monomial degrees of f_1 .

Numerical Calculation:

Figure 4.27 shows MATLAB plots of complex encirclement applied to the homotopy function in Equations (4.51) and (4.52). Figure 4.27b shows the complex parameter circle $\lambda = |\lambda_*|e^{j(\theta)}$, $\theta = 0 : 2\pi$, $\lambda_* = 10^9$, a circle of large enough radius to be equivalent to encircling infinity for this example. Figure 4.27a shows the solution trajectory corresponding to twelve encirclements of 4.27b. This trajectory passes through all twelve solutions of $H(x, -\lambda_*) = 0$. Notice that the solution vector (x_1, x_2) has a winding number vector of $(w_1, w_2) = (4, 3)$, meaning that as the parameter λ encircles infinity twelve times, x_1 winds around infinity four times, while x_2 winds around infinity three times.

Analysis around ∞ :

By Proposition 4.4.2, the homotopy function of this example has all roots approaching the double infinity point $(x_1, x_2) = (\infty, \infty)$ as $\lambda \rightarrow \infty$. As $\lambda \rightarrow \infty$, the variables x_1 and x_2 also go to infinity, and Equations (4.51) and (4.52) become dominated by the monomials of highest degree in each equation. Near infinity, Equations (4.51) and (4.52)

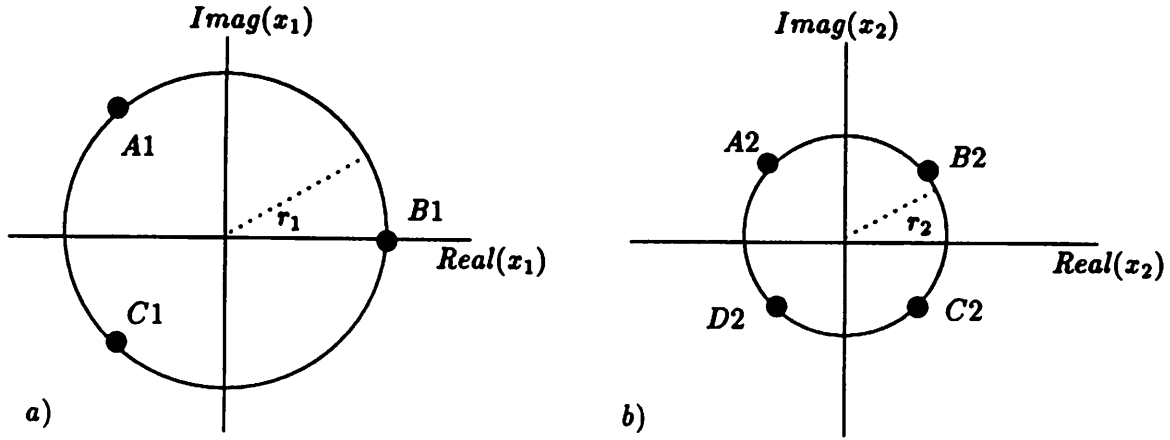


Figure 4.28: Calculated root structure for Example 7. a) The root components x_1 take the form of three roots equally spaced around a complex circle of radius r_1 . b) The root components x_2 take the form of four roots equally spaced around a complex circle of radius r_2 .

approach

$$\alpha_1 x_1^3 - x_2^4 = 0 \quad (4.53)$$

$$\alpha'_1 x_1^3 - \alpha_2 x_2^3 - \lambda = 0 \quad (4.54)$$

Substituting $x_1^3 = (1/\alpha_1)x_2^4$ from Equation (4.53) into Equation (4.54) leads to

$$(\alpha'_1/\alpha_1)x_2^4 - \alpha_2 x_2^3 - \lambda = 0 \quad (4.55)$$

which, as λ gets large, is asymptotically dominated by the higher degree term and may be approximated by

$$(\alpha'_1/\alpha_1)x_2^4 - \lambda = 0 \quad (4.56)$$

Then we set the parameter $\lambda = |\lambda|e^{j\theta+k2\pi}$, $k \in \mathbb{Z}$, and solve Equation (4.56) for x_2 to get $x_2 \approx |\lambda\alpha'_1/\alpha_1|^{1/4}e^{j(\theta+k2\pi)/4}$, four equally spaced roots around a complex circle of radius $r_2 = |\lambda\alpha'_1/\alpha_1|^{1/4}$ as shown in Figure 4.28b. From Equation (4.53), $x_1^3 \approx x_2^4/\alpha_1$, so $x_1 \approx (r_2^{4/3}/\alpha_1^{1/3})e^{j(\theta+k2\pi)/3}$, a set of three roots on a complex circle of radius $r_1 = (r_2^{4/3}/\alpha_1^{1/3})$ as shown in Figure 4.28a. Next, we compare the derived root structure of Figures 4.28a-b against the numerical computations shown in Figures 4.27a-b, and see that the topologies are identical. The winding numbers are $w_1 = 3$ for the variable x_1 and $w_2 = 4$ for

the variable x_2 , meaning that as the parameter space circle is traversed twelve times, the x_1 root circle of Figure 4.28a is traversed three times, while the x_2 root circle of Figure 4.28b is traversed four times. The twelve roots emerge from the winding considerations to be (A1,A2), (B1,B2), (C1,C2), (A1,D2), (B1,A2), (C1,B2), (A1,C2), (B1,D2), (C1,A2), (A1,B2), (B1,C2), and (C1,D2). This checks out with the MATLAB plots in Figures 4.27a-b at a parameter plane circle radius of $|\lambda| = 10^9$. Finally, notice that, consistent with Corollary 4.4.3, Equation (4.53) is irreducible and consists of monomials with degrees whose greatest common divisor is one. \square

Summary of Polynomial Example Series 1-7:

These examples serve to illustrate Proposition 4.4.2, Proposition 4.4.3, Corollary 4.4.3, and Proposition 4.4.4. The concepts of local irreducibility around infinity, winding numbers, and connectivity around infinity are discussed in the analyses. The example series is presented in a hypothesis testing style, in that each one is meant to answer a question about the effect of parameter placement, or adding or deleting certain monomials, on the solution set topology in the neighborhood of infinity.

4.5 Polynomials: n-D

This section generalizes basic results of the previous sections to higher dimensions and outlines a complete complex encirclement algorithm. Also, the issue of conservation of solution number and its importance to complex encirclement is discussed. We present a simple local analog to the results of Section 3.5, where it was pointed out that homotopy function irreducibility leads to global connectivity of the complex solution surface of $H(x, \lambda) = 0$ over the complex parameter plane $\lambda \in C$. Stated in Proposition 4.5.1 is a simple unifying principle of local connectivity around a branch point that extends across the dimensions.

Proposition 4.5.1 (Connectivity around infinity, n-d polynomial systems):

Given: a polynomial homotopy function

$$H(x, \lambda) = 0, H : C^n \times C \rightarrow C^n \quad (4.57)$$

designed such that as $\lambda \rightarrow \infty$ all solutions $x_\lambda = (x_1, x_2, \dots, x_n)$ of $H = 0$ approach $(x_1, x_2, \dots, x_n) =$

$(\infty, \infty, \dots \infty)$,

• If H is *locally irreducible* at the branch point at $\lambda = \infty$, then

1. *all solutions of $H(x, \lambda) = 0$ are connected in a single algebraic element around infinity.*
2. \exists a number $\lambda_{min} \in \Re$ s.t. tracing a circle $\lambda = |\lambda_*|e^{j\theta}, \theta = 0 : 2\pi, \lambda_* > \lambda_{min}, l$ successive times, where l is at least the number of solutions of $H = 0$, leads to the numerical calculation of a string of roots $(x_{\lambda_*}^1, x_{\lambda_*}^2, \dots, x_{\lambda_*}^m, x_{\lambda_*}^1, x_{\lambda_*}^2, \dots)$ consisting of the repetition of the m roots of the function $H(x, \lambda_*) = 0$.

Proof: Follows directly from Proposition (A5) in Section 4.2.

The above proposition summarizes the basic principle of solution set connectivity around infinity. It says that if the function H is designed so that all its roots go to infinity along with the parameter, and if H is locally irreducible at infinity, then all roots of $H(x, \lambda_*) = 0, \lambda_* > \lambda_{min}$, can be found by repeatedly traversing the parameter circle $\lambda = |\lambda_*|e^{j\theta}, \theta = 0 : 2\pi$. This parameter circle must be large enough to be equivalent to encircling infinity, meaning that it must contain all the finite branch points of H .

However, the goal is to find all roots of H at a particular value of the parameter, the one at which the homotopy function is identical to the original problem of interest ($H(x, \lambda_f) = F(x) = 0$). We must fill in a missing link in the argument. The missing link is the relationship between the number of solutions of $H(x, \lambda_*) = 0$ and $F(x) = 0$, assuming that $|\lambda_f| < |\lambda_*|$. If we are to continue the solutions of $H(x, \lambda_*) = 0$ to all solutions of $F(x) = 0$, the two functions must have the same number of solutions.

In fact, most natural parameterizations, including the homotopy functions designed in this chapter, have this property. We borrow the following theorem from [11], which states that all systems of polynomials with the same structure have the same number of roots (counting multiplicity).

Conservation of Solution-Number: Natural Maps

Let

$$p_1(c_1, \dots, c_r, x_1, \dots, x_n) = 0$$

$$\begin{array}{c} \dots \\ p_n(c_1, \dots, c_r, x_1, \dots, x_n) = 0 \end{array}$$

be a system of polynomial equations in the variables c_1, \dots, c_r and x_1, \dots, x_n . For each choice of $c = (c_1, \dots, c_r)$ in C^r , this is a system of polynomial equations in the variables x_1, \dots, x_n . Let d be the total degree of the system for a generic choice of c .

Theorem 4.5.1 [11]: Let c belong to C^r . There exists an open, dense, full-measure subset U of C^{n+r} such that for $(b_1^*, \dots, b_n^*, c_1^*, \dots, c_r^*) \in U$, the following holds:

a) The set X^* of solutions $x = (x_1, \dots, x_n)$ of

$$q_1(x_1, \dots, x_n) = p_1(c_1^*, \dots, c_r^*, x_1, \dots, x_n) + b_1^* = 0$$

..

$$q_n(x_1, \dots, x_n) = p_n(c_1^*, \dots, c_r^*, x_1, \dots, x_n) + b_n^* = 0$$

consists of d_0 isolated points, for some $d_0 \leq d$.

b) The smoothness and accessibility properties hold for the homotopy

$$H(x, \lambda) = P(\lambda c_1 + (1 - \lambda)c_1^*, \dots, \lambda c_r + (1 - \lambda)c_r^*, x_1, \dots, x_n) + (1 - \lambda)b^* \quad (4.58)$$

where $b^* = (b_1^*, \dots, b_n^*)$. It follows that every solution of $P(x) = 0$ is reached by a path beginning at a point of X^* . ♣

For a proof, see pages 1245 – 1249 of [11].

The above theorem implies that as long as a homotopy function H is derived from the problem of interest $F(x) = 0$ by adding scalar parameters and/or linearly embedding parameters in the polynomial coefficients, the system $H(x, \lambda) = 0$ will have the same number of solutions, at random, regular λ , as $F(x) = 0$ does. This is true whether or not $F(x) = 0$ is deficient (fewer solutions than the Bezout degree bound, as discussed in Chapter 2).

Complex Encirclement for Finding all Roots:

Now that the link between the problem this chapter concentrates on, that of solution set connectivity around an infinite branch point, and the original problem to be solved has been established, we outline an algorithm. All roots of a polynomial system of equations $P(x) = 0, P : C^n \rightarrow C^n$, may be found as follows:

1. *Design an analytic homotopy function $H : C^n \times C \rightarrow C^n$ such that..*
 - (a) *there exists a parameter value λ_f such that $H(x, \lambda_f) = P(x)$.*
 - (b) *all roots of $H(x, \lambda) = 0$ are connected in a single algebraic element around infinity for $\lambda > \lambda_*$.*
2. *Use the connectivity property to find all roots of $H(x, \lambda_*) = 0$ by encircling infinity m times, where m is the total number of roots.*
3. *Finally, use the homotopy function $H'(x, \lambda) = \lambda P(x) + (1 - \lambda)H(x, \lambda_*)$ to trace exactly m paths to all m roots of $P(x) = 0$. (Or use the homotopy function $H(x, \lambda_*)$ to trace m paths from $\lambda = \lambda_*$ to $\lambda = \lambda_f$.)*

This procedure takes the place of the first step of the *Cheater's Homotopy* discussed in [11], and is beneficial in that it makes the algorithm efficient and useful even if the equations need to be solved only once with a single set of coefficients, rather than repetitively with varying coefficients, or if the equations are not polynomial. Where such homotopy functions are available, the procedure is especially useful for the highly deficient polynomial systems found in practice, for example the indirect position problem for revolute-joint kinematic manipulators, which result in a system of equations with total degree of 256, but only 32 solutions [11].

4.6 Extension to Non-Polynomial-Bounded Analytic Functions, Transistor Circuit Models

In this section we consider analytic systems of nonlinear equations $F(x) = 0$, $F : C^n \rightarrow C^n$, that include exponential as well as polynomial terms. Such systems are fundamentally different from purely polynomial systems in that they can possess an *infinite* number of complex roots, rather than a finite number of complex roots as is the case for polynomials and functions that are bounded by polynomials. This is an important class of systems, because many circuit element models include exponential terms. For example, circuits that include bipolar transistors modeled by Ebers-Moll equations have circuit equations with exponential terms of the form $e^{x_1/k}$ and $e^{(x_1-x_2)/k}$, where x_1 and x_2 are device

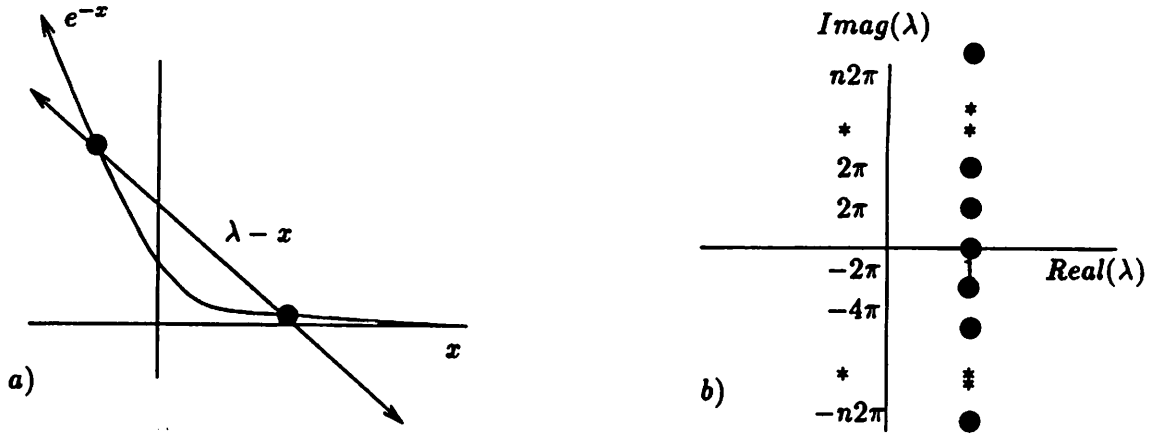


Figure 4.29: a) A drawing showing the two real roots of $h(x, \lambda) = e^{-x} + x - \lambda = 0$. The line $\lambda - x$ is drawn along with e^{-x} , and for a fixed value of λ , the intersection of these two curves gives the real roots of $h(x, \lambda) = 0$ (two or none). b) The non-polynomial function $h(x, \lambda)$ has an infinite number of complex branch points at $\lambda = 1 + (2n\pi)j$, $\forall n \in \mathbb{Z}$.

port voltages [47]. Typically, these circuit equations will consist of linear terms, exponential terms, and constants, corresponding to linear resistors, transistors, diodes, independent sources, and dependent sources.

In the following subsections, we *generalize the complex encirclement* ideas developed previously in the chapter, to systems with exponential terms. The observations made in this section, summarized in Conjecture 4.6.3, point to a more general notion of complex encirclement for systems with an infinite number of complex solutions and branch points. The idea is that for certain types of functions, including those with exponential terms that are linear in the argument (some diode and transistor models), a homotopy function that has all roots going to infinity along with a parameter, and is locally irreducible at infinity, can be used to find any number of roots of $h(x, \lambda_f) = 0$. Specifically, we conjecture that *complex encirclement* can be used to find all roots of $H(x, \lambda_f) = 0$ within the compact space $|x| \leq r$, along (generically) regular paths.

We start our development with the scalar case.

4.6.1 Parameterized Scalar Equations With Exponential Terms

This section examines homotopy functions of the form $h(x, \lambda) = p(x) + ce^{kx} - \lambda = 0$, where c and k are non-zero real constants, p is a non-constant polynomial, and λ is a

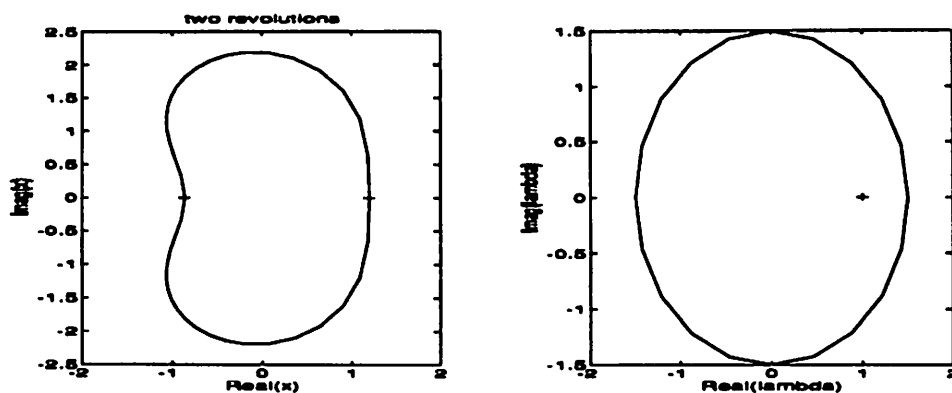


Figure 4.30: Complex encirclement of the homotopy function $h(x, \lambda) = e^{-x} + x - \lambda = 0$. a) The complex solution path of $h(x, \lambda) = 0$ corresponding to encircling the parameter plane circle in (b) twice. The two real roots of $h(x, 1.3) = 0$ are joined via an algebraic element of order one. b) The complex parameter circle contains a single branch point at $\lambda = 1$.

parameter. In general, at fixed real λ this function has a finite number of real roots and an infinite number of complex roots. Also, the equation $h(x, \lambda) = 0$ generally has an infinite number of complex branch points λ , and the property that $\lambda \rightarrow \infty \Rightarrow x \rightarrow \infty$. We illustrate some properties of this class of analytic functions through a detailed exploration of the equation $h(x, \lambda) = e^{-x} + x - \lambda = 0$.

As shown in Figure 4.29a, the equation $h(x, \lambda) = e^{-x} + x - \lambda = 0$ has two distinct real solutions at real $\lambda > 1$, no real solutions at real $\lambda < 1$, and a real double root at the branch point $\lambda = 1$. The line $\lambda - x$ is drawn along with e^{-x} , and for a fixed real value of λ the intersection of these two curves gives the real roots of $h(x, \lambda) = 0$. Also, the equation $h(x, \lambda) = 0$ has an infinite number of complex roots at most fixed λ , and an infinite number of branch points (parameter values λ at which $h = 0$ has repeated roots).

Figure 4.29b illustrates the location of the infinite number of complex branch points of $h(x, \lambda)$, calculated by differentiating h with respect to x to get $1 - e^{-x} = 0$, and then solving for $x = 2n\pi j, \forall n \in \mathbb{Z}$. Substituting $x = 2n\pi j$ into $e^{-x} + x - \lambda = 0$ gives the branch points $\lambda_n = 1 + (2n\pi)j, \forall n \in \mathbb{Z}$, as shown in the figure. Each finite branch point λ_n corresponds to a *double* repeated root of $h(x, \lambda) = 0$. This can be verified by noticing that differentiating h twice with respect to x leads to an equation with no finite solutions.

Next, we go through a series of complex encirclement experiments on this function using a MATLAB program, to illustrate the nature of the solution set of h and its connectivity in complex space.

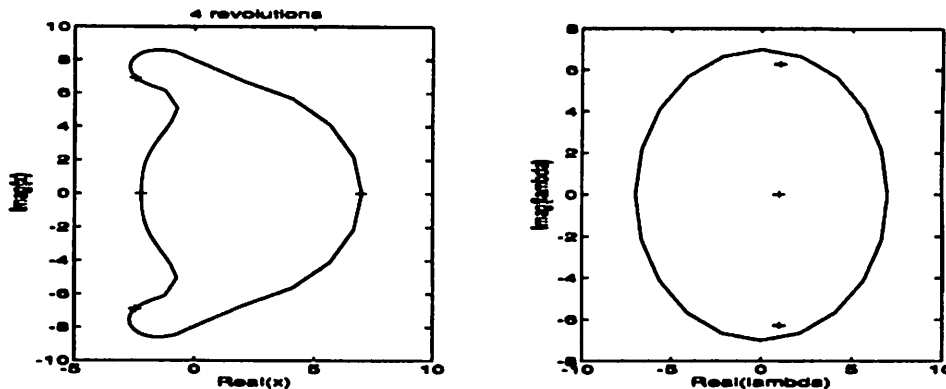


Figure 4.31: Complex encirclement of the homotopy function $h(x, \lambda) = e^{-x} + x - \lambda = 0$. a) The complex solution path of $h(x, \lambda) = 0$ corresponding to encircling the parameter plane circle in (b) four times. The two real roots of $h(x, 1.3) = 0$ and two complex roots of $h(x, 1.3) = 0$ are joined in closed solution structure akin to an algebraic element of order three. b) The complex parameter circle contains three branch points at $\lambda = 1$, $\lambda = 1 + 2\pi j$, and $\lambda = 1 - 2\pi j$.

Figure 4.30a shows a MATLAB plot of the complex solution path $x(\lambda)$ of $h(x, \lambda) = 0$ corresponding to encircling the parameter plane circle in Figure 4.30b twice, starting from one of the two real roots of $h(x, 1.3) = 0$. The parameter path traced is $\lambda = |\lambda_*|e^{j\theta}$, $\theta = 0 : 4\pi$, with $\lambda_* = 1.3$. Notice that the solution path shown in Figure 4.30a is a closed curve joining the two real roots of $h(x, 1.3) = 0$. The complex parameter circle in Figure 4.30b contains exactly *one* branch point, at $\lambda = 1$, which serves to connect the two real roots of $h(x, 1.3) = 0$ in an algebraic element of order one.

Figure 4.31b shows a larger complex parameter circle, with a radius of seven. This complex parameter circle contains not one, but *three* complex branch points of h , at $\lambda = 1$ and $\lambda = 1 \pm 2\pi j$. Figure 4.31a shows a MATLAB plot of the the complex solution path $x(\lambda)$ of $h(x, \lambda) = 0$ corresponding to encircling the parameter plane circle in Figure 4.31b four times, starting from one of the two real roots of $h(x, 7) = 0$. The parameter path traced is $\lambda = |\lambda_*|e^{j\theta}$, $\theta = 0 : 8\pi$, with $\lambda_* = 7$. Once again, the solution path shown in Figure 4.31a is a closed curve joining the two real roots of $h(x, \lambda_*) = 0$, but this larger circle also joins two *complex* roots of $h(x, \lambda_*) = 0$ as part of the solution cycle. Encircling two more branch points than were encircled in Figure 4.30b ($\lambda = 1 \pm 2\pi j$, along with the branch point $\lambda = 1$) results in the addition of two complex roots to the root cycle.

This trend continues indefinitely. As the complex parameter circle gets larger and larger, including more and more branch point pairs $\lambda = 1 \pm 2\pi j$, the associated solution

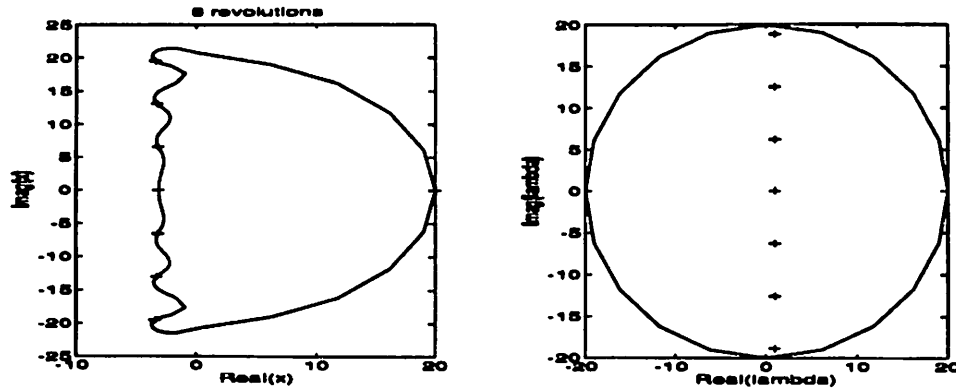


Figure 4.32: Complex encirclement of the homotopy function $h(x, \lambda) = e^{-x} + x - \lambda = 0$. a) The complex solution path of $h(x, \lambda) = 0$ corresponding to encircling the parameter plane circle in (b) eight times. The two real roots of $h(x, 1.3) = 0$ and six complex roots of $h(x, 1.3) = 0$ are joined in closed solution structure akin to an algebraic element of order seven. b) The complex parameter circle contains seven branch points at $\lambda = 1$ and $\lambda = 1 + r2\pi j, r = \pm 1, \pm 2, \pm 3$.

cycle includes more and more complex solutions of $H(x, \lambda_*) = 0$, two for each pair of encircled branch points. For example, Figure 4.32b shows a complex parameter circle of radius $\lambda_* = 20$ containing seven branch points. Starting from a real solution $x(\lambda_*)$ of $H(x, \lambda_*) = 0$, encircling the parameter circle eight times ($\lambda = |\lambda_*|e^{j\theta}, \theta = 0 : 16\pi$) leads to the calculation of a total of eight solutions of $h(x, 20) = 0$, which lie on the closed curve shown in Figure 4.32a. Notice that as the radius of the complex parameter circle gets larger and larger, the shape of the associated closed solution curve approaches that of a *half-disk*.

Another observation is that the finite solution cycles associated with parameter plane circles of radius r_i seem to have an *inclusion property* for increasing radii. In order to define an inclusion property, the notion of a root identification of $h(x, \lambda) = 0$ at different parameter values λ is necessary.

If one were to label a finite number (m) of solutions x of $h(x, \lambda_{*1}) = 0$ for real, regular, λ_{*1} , then it is possible to make an *identification* between these m solutions and m solutions of $h(x, \lambda) = 0$ at another regular parameter value, say real $\lambda = \lambda_{*2}$. We make this identification, which is in general not unique, by continuing m solution paths of $h(x, \lambda) = 0$ along the regular path $P(\lambda_{*1}, \lambda_{*2})$ in the parameter plane from $\lambda = \lambda_{*1}$ to $\lambda = \lambda_{*2}$, starting at the m chosen solutions at $\lambda = \lambda_{*1}$, and ending at m solutions of $h = 0$ at $\lambda = \lambda_{*2}$. We define the path $P(\lambda_{*1}, \lambda_{*2})$ to be a line segment connecting λ_{*1} and λ_{*2} , if there are no branch points along this segment. If this line segment does include a branch

point, we define the path $P(\lambda_{*1}, \lambda_{*2})$ from λ_{*1} to λ_{*2} to be the slightly perturbed curve $\lambda = \lambda_1 + (\lambda_{*2} - \lambda_1) \sin \theta/2 + i\epsilon \sin \theta, \theta = 0 : \pi$, with small ϵ chosen to ensure that the path traced is regular.

Once such an identification is made, the notion of a solution set inclusion property can be defined. Consider the following sequence of events. First, choose a large, real, regular parameter value λ_{*1} . Then, calculate a finite root cycle of $h(x, \lambda_{*1}) = 0$ associated with a radius of λ_{*1} , by repeatedly encircling a complex parameter circle with radius $r = \lambda_{*1}$ and tracing out the associated closed solution path $x(\lambda)$, as shown in Figures 4.30a and 4.31a. Once the m roots in this root cycle have been calculated, choose a new regular parameter value λ_{*2} , $|\lambda_{*2}| > |\lambda_{*1}|$, and continue the m calculated roots along the regular path $P(\lambda_{*1}, \lambda_{*2})$ to $\lambda = \lambda_{*2}$. Then, take one of the m continued roots of h at $\lambda = \lambda_{*2}$, and calculate the finite root cycle of $h(x, \lambda_{*2}) = 0$ associated with a radius of λ_{*2} . To do this, repeatedly encircle a complex parameter circle with radius $r = \lambda_{*2}$, starting from the continued root, until the solution path $x(\lambda)$ closes on itself. Though this root cycle is likely to contain more roots than does the root cycle of h at $\lambda = \lambda_{*1}$, if h satisfies the *solution inclusion property*, then the new root cycle at $\lambda = \lambda_{*2}$ will *include continuations of all m roots in the calculated root cycle of h at $\lambda = \lambda_{*1}$* . For a large enough parameter circle radius λ_1 , each root cycle corresponding to a parameter circle of larger radius λ_2 , $|\lambda_2| > |\lambda_1|$, includes the continuation of all the roots of an associated cycle at radius λ_1 . This is what is meant by a *solution set inclusion property*, which we illustrate on the example equation $e^{-x} + x - \lambda = 0$, which appears to have this property.

For example, the two real roots shown in Figure 4.30a, which form the root cycle associated with a parameter plane circle of radius 1.3 at $\lambda_* = 1.3$, can be continued to the two real roots in the root cycles shown in Figure 4.31a, and Figure 4.32a, by increasing λ_* from $\lambda_* = 1.3$, to $\lambda_* = 7$, and to $\lambda_* = 20$, respectively. Likewise, all four roots shown in Figure 4.31a, which form the root cycle associated with a parameter plane circle of radius 7 at $\lambda_* = 7$, can be continued to four of the eight roots in the root cycle shown in Figure 4.32a by increasing λ_* from $\lambda_* = 7$ to $\lambda_* = 20$.

It is the consideration of this inclusion property, along with the conjecture that all the roots of $h(x, \lambda_*) = 0$ within a compact region $|x| \leq r$ can be calculated by (1) choosing a large enough parameter circle radius r , and (2) encircling the complex parameter circle with radius r repeatedly to find an associated root cycle, and then (3) continuing these roots back to the parameter value of interest $\lambda = \lambda_*$, that forms the core of a generalized

complex encirclement approach for systems with an infinite number of solutions.

To formalize and generalize the above discussion, and our observations, we prove some properties and then state a conjecture. First, we prove the property that $\lambda \rightarrow \infty \Rightarrow x \rightarrow \infty$ for analytic functions of the form $f(x) + \lambda = 0$, in general.

Proposition 4.6.1: *Let $f(x)$ be an analytic function $f : C \rightarrow C$. Define a homotopy function $h(x, \lambda) = f(x) + \lambda k, k \in C$.*

- *The analytic homotopy function equation $h(x, \lambda) = f(x) + \lambda k = 0, k \in C$, has no finite roots as $\lambda \rightarrow \infty$.*

Proof:

- assertion: $\lambda \rightarrow \infty \Rightarrow x \rightarrow \infty$.

Assume not. Then $\exists x_* \in C, |x_*| < M$, such that $\lim_{x \rightarrow x_*} |f(x)| = \infty$ (since $f(x) = -k\lambda$). This implies that $f(x)$ has a singularity at $x = x_*$. However, f is assumed analytic, so proof by contradiction. ♣

This proposition is important because for complex encirclement to be successful at calculating all roots of interest of a function, the homotopy function must be designed so that all of its roots approach infinity as the parameter approaches infinity.

The next proposition sets the stage for Conjecture 4.6.3, by showing that the inverse-image of a simple, regular, closed curve of finite length in the complex parameter plane λ associated with $h(x, \lambda) = 0$ is a closed curve in the complex solution plane x . To see the importance of this property, consider the function $h(x, \lambda) = e^x + \lambda$, which violates the assumptions of Proposition 4.6.2 (because the polynomial p is a constant), and which does not have this property. For the equation $h(x, \lambda) = e^x + \lambda = 0$, the inverse-image of a closed curve in the parameter plane λ is *not* a closed a curve in the x plane, but rather a sort of helix that winds around zero infinitely many times. If one wanted apply complex encirclement to calculating all the roots of $h(x, \lambda) = e^x + \lambda = 0$ within some region of solution space $|x| \leq r$, it would not be possible to do so by tracing out an associated closed curve in solution space.

Proposition 4.6.2: *Let $f(x), f : C \rightarrow C$ be an analytic function of the form $f(x) = p(x) + ce^{kx}$, where p is a non-constant polynomial, and c and k are non-zero constants. Define a homotopy function $h(x, \lambda) = f(x) - \lambda$.*

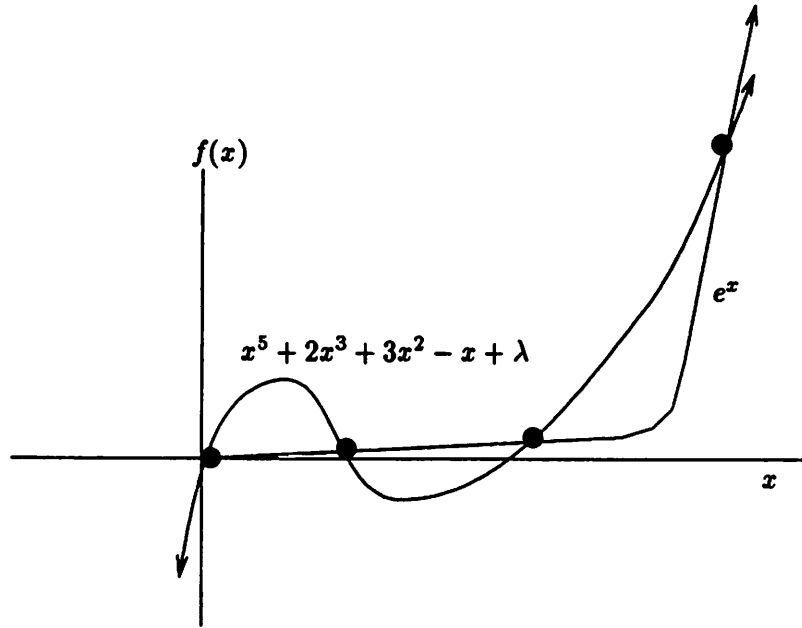


Figure 4.33: A drawing showing the four real roots of $h(x, \lambda) = x^5 + 2x^3 + 3x^2 - x - e^x + \lambda = 0$. The polynomial $x^5 + 2x^3 + 3x^2 - x + \lambda$ is drawn along with e^x , and for a fixed value of λ , the intersection of these two curves gives the roots of $h(x, \lambda) = 0$. This function has from zero to four real roots, depending on λ .

1. The analytic homotopy function equation $h(x, \lambda) = f(x) - \lambda = p(x) + ce^{kx} - \lambda = 0$ has no finite branch point λ corresponding to a finite repeated root x of infinite multiplicity. (i.e. each finite branch point of h corresponds to an algebraic element of finite order).
2. The analytic homotopy function equation $h(x, \lambda) = f(x) - \lambda = p(x) + ce^{kx} - \lambda = 0$ has a finite number of branch points in a compact region of the complex parameter plane, $|\lambda| \leq t$, for finite t .
3. The inverse-image of a simple, regular, closed curve of finite length in the complex parameter plane λ associated with $h(x, \lambda) = 0$ is a set of closed curves in the complex solution plane x .

Proof:

1. A finite branch point λ associated with a root x of multiplicity m satisfies the following

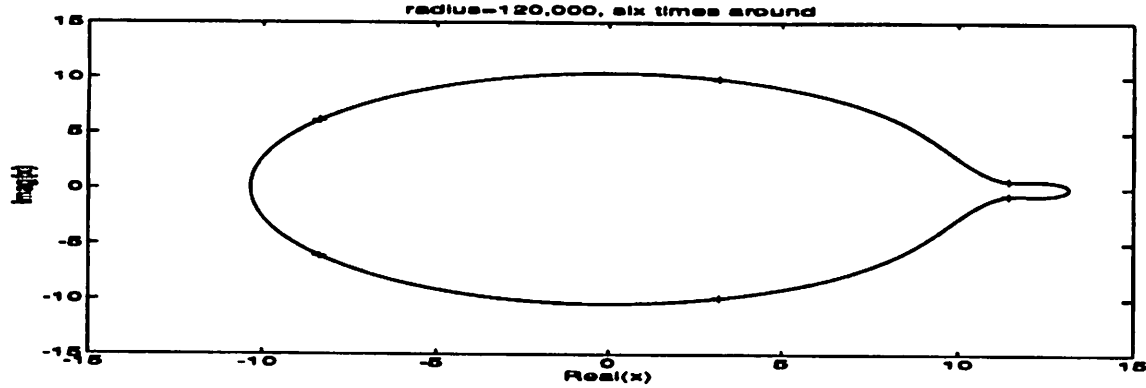


Figure 4.34: Complex encirclement of the homotopy function $h(x, \lambda) = x^5 + 2x^3 + 3x^2 - x - e^x + \lambda = 0$. The complex solution path of $h(x, \lambda) = 0$ corresponding to the parameter trajectory $\lambda = -\lambda_* e^{j\theta}$, $\theta = 0 : k2\pi$, with $k = 6$, starting at a continuation of one of the real roots of $h = 0$. In this case the radius $\lambda_* = 120,000$, and two extra complex roots are picked up in the cycle.

set of equations.

$$\begin{aligned} p(x) + ce^{kx} - \lambda &= 0 \\ \partial p(x)/\partial x + cke^{kx} &= 0 \\ &\vdots \\ \partial^{m-1} p(x)/\partial x^{m-1} + ck^{m-1}e^{kx} &= 0 \end{aligned}$$

However, if the polynomial p has degree d , then $\partial^i p(x)/\partial x^i = 0$ for $i \geq d$. This implies that differentiating h with respect to x d times leads to the equation $ck^d e^{kx} = 0$, which has no finite roots. Therefore, no finite branch point λ can correspond to a finite repeated root x of multiplicity greater than $d + 1$.

2. Branch points of h satisfy the equations $p(x) + ce^{kx} - \lambda = 0$ and $\partial p(x)/\partial x + cke^{kx} = 0$. By substitution, we get $kp(x) - \partial p(x)/\partial x = k\lambda$. Confining λ to a compact region $|\lambda| \leq t$ implies that at any branch point λ in this region, x is bounded. This then implies that in the compact region $|\lambda| \leq t$, the exponential term ce^{kx} can be represented to an arbitrary level of accuracy by a polynomial in x . Thus, the equations $p(x) + ce^{kx} - \lambda = 0$ and $\partial p(x)/\partial x + cke^{kx} = 0$ become bounded by polynomial equations within the region $|\lambda| \leq t$, and since polynomial equations only admit a finite number of roots, there are only a finite number of branch points in the region $|\lambda| \leq t$.

3. Items (1) and (2) of Proposition 4.6.2 imply item (3). ♣

Next, we formalize our observations about the connectivity and accessibility of solutions of $h(x, \lambda) = 0$, and apply the above propositions. The goal is to be able to find all roots x of $h(x, \lambda_f) = 0$ in any compact region $|x| \leq r$.

Conjecture 4.6.3: *Let $f(x), f : C \rightarrow C$ be an analytic function of the form $f(x) = p(x) + ce^{kx}$, where p is a non-constant polynomial, and c and k are non-zero constants. Define a homotopy function $h(x, \lambda) = f(x) - \lambda$.*

- For any finite number $r \in \mathbb{R}$, there exists a finite number $t \in \mathbb{R}$ such that:
 1. repeatedly encircling a regular complex parameter circle $\lambda = |\lambda_*|e^{j\theta}, \theta = 0 : 2n\pi$ of radius $|\lambda_*| > t$, leads to the calculation of a finite number (n) of roots of $h(x, \lambda_*) = 0$. This set of roots forms a closed cycle akin to an algebraic element of order $n - 1$.
 2. These n roots can then be continued with probability-1 along non-intersecting paths to n solutions of the homotopy function at any parameter value of interest, $\lambda = \lambda_f$, by applying multi-starter homotopy to the function $h_2(x, \lambda) = \lambda h(x, \lambda_f) + (1 - \lambda)h(x, \lambda_*) = 0$ (or directly to $h(x, \lambda) = 0$ from $\lambda = \lambda_*$ to $\lambda = \lambda_f$). The set of n calculated roots of $h(x, \lambda_f) = 0$ includes all roots x of $h(x, \lambda_f) = 0$ in the compact space $|x| \leq r$.

Supporting ideas, towards a proof :

The equation $h(x, \lambda) = f(x) - \lambda = 0$ has several properties that support the truth of Conjecture 4.6.3. Propositions 4.6.1-3 indicate that the function h has all roots going to infinity along with the parameter, and that the inverse-image of a closed curve in the complex parameter plane is a set of closed curves in the complex solution plane. Also, the description of a topologically unibranch set X , found in association with Proposition A.5 in Section 4.2, is general, and can be applied to functions with an infinite number of branch points in the neighborhood of infinity in order to define irreducibility of h in a neighborhood of infinity, $(x, \lambda) = (\infty, \infty)$. It must then be shown that h is in fact locally irreducible at infinity. Finally, one needs to know that 1) the homotopy function h_2 has the same number of roots at all regular parameter values, and 2) these roots can be regularly continued from one regular parameter value to another. Also, the idea that the branch points are not dense

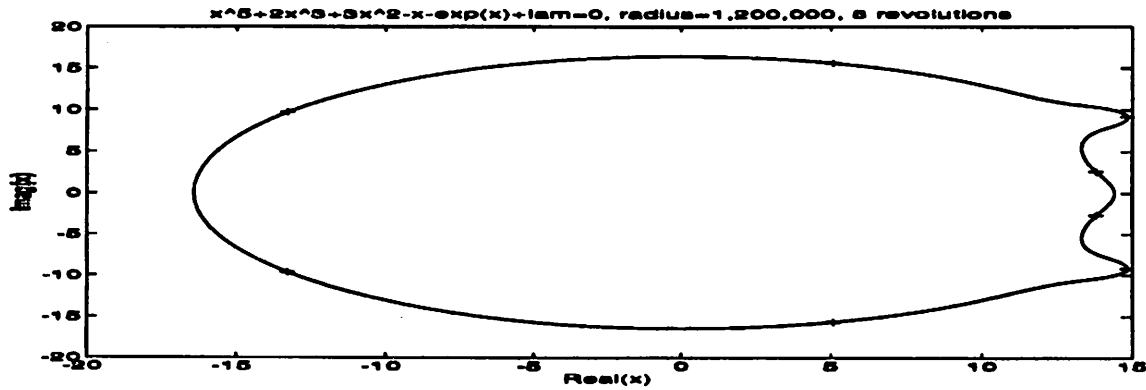


Figure 4.35: Complex encirclement of the homotopy function $h(x, \lambda) = x^5 + 2x^3 + 3x^2 - x - e^x + \lambda = 0$. The complex solution path of $h(x, \lambda) = 0$ corresponding to the parameter trajectory $\lambda = -\lambda_* e^{j\theta}$, $\theta = 0 : k2\pi$, with $k = 8$, starting at a continuation of one of the real roots of $h = 0$. In this case the radius $\lambda_* = 1,200,00$, and four extra complex roots are picked up in the cycle.

in λ is important to the notion of a *parameter circle* as an appropriate shape of closed parameter plane curve. Altogether, these ideas, when formalized, should contribute to a proof of this conjecture. ♣

Implications of Conjecture 4.6.3:

The observations made in this subsection, summarized in Conjecture 4.6.3, point to a more general notion of complex encirclement for systems with an infinite number of complex solutions and branch points. The generalization is that for certain types of functions, including those with exponential terms that are linear in the argument (some diode models), a homotopy function that has all roots go to infinity along with a parameter, and is locally irreducible at infinity, can be used to find any number of roots of $h(x, \lambda_f) = 0$. Specifically, *complex encirclement* can be used to find all roots of $h(x, \lambda_f) = 0$ within the compact space $|x| \leq r$, along (generically) regular paths.

Example 1:

An example function is $h(x, \lambda) = x^5 + 2x^3 + 3x^2 - x - e^x + \lambda = 0$, which is the same as Example 1 in Section 4.3 except for the addition of an exponential term.

The equation $h(x, \lambda) = 0$ has an infinite number of complex solutions, with a maximum of four of them being real at a given value of λ , and a minimum of none. Figure 4.33

illustrates the four real roots of $h(x, \lambda) = 0$. The polynomial $x^5 + 2x^3 + 3x^2 - x + \lambda$ is drawn along with e^x , and the intersection of these two curves gives the real roots of $h(x, \lambda) = 0$.

Notice that the polynomial in Example 1 of Section 4.3, $x^5 + 2x^3 + 3x^2 - x + \lambda = 0$, has a maximum of three real roots, but the addition of the exponential term has the effect of adding another real root whose position is much less sensitive to the parameter λ than are the other real roots of $h(x, \lambda) = 0$. Since the complex encirclement algorithm requires finding a radius λ_* that is large enough so that the complex parameter circle contains all branch points associated with the real roots of the function, having a real root that is relatively insensitive to the parameter implies that the radius required will be large, as it is in this example. In this example, $h(x, \lambda) = 0$ does not have the minimum number of real roots (none) until $\lambda \approx -120000$. This is why such a large radius λ_* is needed to connect all real solutions of h for this particular homotopy function. Figure 4.34 shows the closed solution curve of $h(x, \lambda) = 0$ associated with a circle $\lambda = \lambda_* e^{j\theta}$, $\theta = 0 : k2\pi$ in the complex parameter plane with radius $\lambda_* = 120,000$. This MATLAB plot was obtained by traversing the complex parameter circle six times, starting from one of the complex roots of $h(x, -\lambda_*) = 0$, a continuation of one of the four real roots of $h(x, 0) = 0$. After the first revolution, 1/6th of the closed curve shown in Figure 4.34 is traversed, and a second root of $h(x, -\lambda_*) = 0$ is found. A second, third, fourth, and then fifth parameter plane revolution leads to a third, fourth, fifth, and then a sixth solution of $h(x, -\lambda_*) = 0$. A final, sixth, revolution leads to the original solution from which the trajectory started. This six-root cycle includes continuations of all four real roots of $h(x, \lambda) = 0$ at real parameter values at which they exist, implying that the six roots can be traced back to the real solutions of interest. Also observe that since the function $h(x, -\lambda_*) = 0$ is real, its roots must occur in complex conjugate pairs, so *only three encirclements*, rather than six, are necessary to find all the roots of $h(x, -\lambda_*) = 0$ in the six-root cycle.

In keeping with Proposition 4.6.2, and the previous discussion in this subsection, repeatedly traversing a larger circle in the complex parameter plane results in tracing out a larger root cycle, with more complex roots drawn in the closed half-disk. Figure 4.35 shows the solution trajectory obtained by encircling a complex parameter circle with radius $\lambda_* = 1,200,000$ eight times, starting from a continuation of one of the four real roots of $h = 0$. As expected, the larger parameter circle picked up extra complex roots of h (two of them, in this case), though the new root cycle includes a continuation of the entire solution cycle at the smaller radius $\lambda_* = 120,000$ shown in Figure 4.34. Naturally, extensions of all

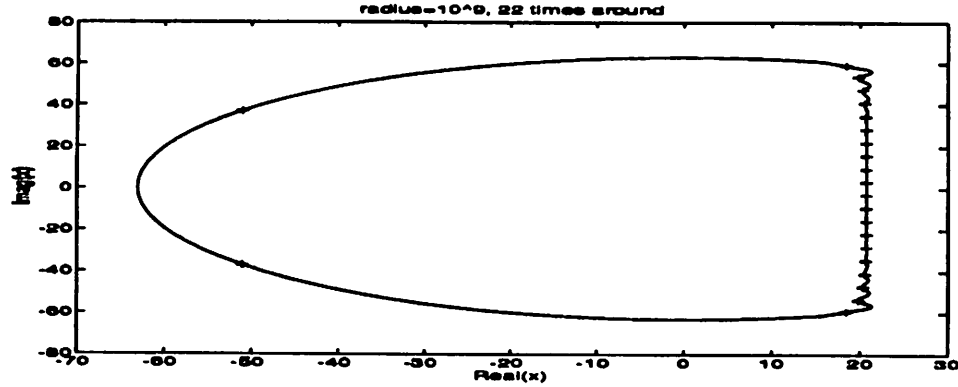


Figure 4.36: Complex encirclement of the homotopy function $h(x, \lambda) = x^5 + 2x^3 + 3x^2 - x - e^x + \lambda = 0$. The complex solution path of $h(x, \lambda) = 0$ corresponding to the parameter trajectory $\lambda = -\lambda_* e^{j\theta}$, $\theta = 0 : k2\pi$, with $k = 22$, starting at a continuation of one of the real roots of $h = 0$. In this case the radius $\lambda_* = 10^9$, and eighteen extra complex roots are picked up in the cycle.

four real roots of h are present in each, successively larger cycle. See Figure 4.36 to get a better sense of the half-disk shape the closed solution paths take on in the limit, as the complex parameter radius gets very large. The complex parameter circle associated with Figure 4.36 has a radius of $\lambda_* = 10^9$, and the solution cycle includes 22 complex roots of $h(x, -\lambda_*) = 0$. \square

4.7 Conclusion/Future Work

This chapter lays out the groundwork for a new way of computing all solutions of nonlinear systems of equations $F(x) = 0$, such as those corresponding to dc operating points of nonlinear circuits. For systems of equations with a finite number of complex solutions, the general idea is to *design homotopy functions that force all circuit solutions to be locally connected around a single complex branch point in a single algebraic element*. Such a homotopy function $H(x, \lambda) = 0$ must 1) force all solutions of a system of equations to go to a single branch point, say infinity, and 2) be constructed to be locally irreducible at this maximal branch point. All solutions of $H(x, \lambda_*) = 0$ (large parameter value λ_*) are then found by encircling the branch point (infinity, in this chapter, though other choices are possible) repeatedly, until the solutions begin to cycle, or until the first complex conjugate solution is calculated. Once all solutions of $H(x, \lambda_*) = 0$ are found, they can be used as starting points of multi-starter homotopy method to find all solutions of $H(x, \lambda_f) = F(x) =$

0, the problem of interest. In this chapter the notion of (1) designing a homotopy function to connect all solutions algebraically around a single point, and (2) the accompanying algorithm, which involves repeatedly encircling this branch point to find all solutions, is referred to as *complex encirclement*. The main advantage of complex encirclement over other homotopy methods is that complex encirclement finds all solutions without having to know, apriori, how many solutions a function has. This is especially useful for non-polynomial and deficient systems of equations, because the number of solutions may not be known.

This chapter also deals with analytic circuit equations with exponentials in them, which are fundamentally different from polynomial systems of equations in that they generally have an *infinite* number of complex roots. The notion of complex encirclement, initially developed on polynomial-type equations, can be generalized to systems with an infinite number of solutions. In this case the goal is to design a homotopy function that has all roots going to infinity along with a parameter, and that can be used to find all complex roots of the circuit equations *within a compact space* $|x| \leq r$, via complex encirclement.

In addition to developing the theoretical and algorithmic foundation of complex encirclement, this chapter goes through a detailed design and analysis of homotopy functions in one and two dimensions, for polynomials and certain analytic functions.

Section 4.3 deals with one dimensional, parameterized polynomials. A homotopy function with an infinite branch point is designed, and we prove that all coalescing roots of the equation are connected in a single algebraic element around infinity. The question of exactly what it means, in practical terms, to encircle infinity, is addressed, and insight is provided on how to visualize complex solution space. The section ends with some examples, and a brief discussion of the kinds of complications one may encounter if the parameter appears non-linearly in the equation.

In Section 4.4, polynomial homotopy functions with two equations, two unknowns, and a complex parameter are investigated. A general homotopy function form is given, with the parameter appearing linearly, and various potential root constellations are discussed. Then a homotopy function design is presented, which is guaranteed to have all roots going to infinity with the parameter, along with accompanying sufficient conditions on the original problem of interest. Following that, some necessary and sufficient conditions ensuring that the homotopy function will have all roots connected in a single algebraic element around infinity are proved. The section ends with a series of examples illustrating the geometric

and algebraic facets of these results. Concepts important to this section include winding numbers, complex solution curves seen as paths along tori, and local irreducibility around branch points.

Section 4.5 gives general connectivity results in n -d for polynomials. A complete *complex encirclement* algorithm is outlined, and the issue of solution-number conservation is discussed.

The chapter ends with Section 4.6, which considers analytic systems of nonlinear equations that include exponential as well as polynomial terms, systems that are fundamentally different from polynomials because they can have an infinite number of complex roots. The observations made in this section point to a more general notion of complex encirclement for systems with an infinite number of complex solutions and branch points. We conjecture that for certain types of functions, including those with exponential terms that are linear in the argument (some diode and transistor models), a homotopy function that has all roots going to infinity along with a parameter, and that is locally irreducible at infinity, can be used to find any number of roots of $h(x, \lambda_f) = 0$. Specifically, numerical experiments suggest that *complex encirclement* can be used to find all roots of $H(x, \lambda_f) = 0$ within the compact space $|x| \leq r$, along regular paths.

Our future work plans focus on applying the general principles developed in this chapter to designing homotopy functions that connect all solutions around infinity for special classes of equations in high dimensions. The goal is to duplicate the development found in this chapter to transistor circuit equations, and other special types of systems found in various electrical engineering applications.

Bibliography

- [1] A. Morgan. *Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems*. Prentice-Hall, Englewood Cliffs, NJ, 1987.
- [2] E. Allgower, K Georg. *Numerical continuation methods: an introduction*. Springer-Verlag, 1990.
- [3] J. Rinzel, R. Miller. "Numerical Calculation of Stable and Unstable Periodic Solution to the Hodgkin-Huxley Equations". *Mathematical Biosciences*, N 49, p.27-59, 1980.
- [4] Lj. Trajkoric and A.N. Wilson, Jr. " Theory of DC Operating Points of Transistor Networks", *Archiv fur Elektronik und Uebertragungstechnik*, July 1992, vol.46,(no.4):228-41.
- [5] Lj. Trajkoric, R. C. Melville, and S.C.Fang. "Finding DC operating points of transistor circuits using homotopy methods", *IEEE Int. Symp on Circuits and Systems*, Singapore, June,1991, pp758-761.
- [6] Lj. Trajkoric, R. C. Melville, and S.C.Fang. "Passivity and no-gain properties establish global convergence of a homotopy method for DC operating points." *IEEE Int. Symp on Circuits and Systems*,New Orleans, LA, May 1990,pp914-17.
- [7] L. O. Chua and A. Ushida. "A switching-parameter algorithm for finding multiple solutions of nonlinear resistive circuits", *Int. j. cir. theor. appl.*, 4, 215-239, 1976.
- [8] S. Chow, J. Mallet-paret, J.A Yorke, "Finding Zeroes of Maps that are Constructive with Probability One," *Mathematics of Computation*, vol. 32, no. 143, pp. 887-899, July 1978.

- [9] C.B. Garcia and W.I. Zangwill. "Determining all Solutions to Certain Systems of Nonlinear Equations," *Mathematics of Operations Research*, vol 4, No 1, February 1979.
- [10] R.C. Melville, L. Trajkovic, S-C Fang and L.T. Watson. "Artificial Parameter Homotopy Methods for the DC Operating Point Problem", *IEEE Trans on Computer-Aided Design of Integrated Circuits and Systems*, June, 1993, V12 N6:861-877.
- [11] T.Y. Li, T. Saueri, and A. Yorke. "The Cheater's Homotopy: An Efficient Procedure For Solving Systems of Polynomial Equations", *SIAM J. Numer. Anal.* Vol 26, No. 5, pp. 1241-1251, October 1989.
- [12] L.T. Watson. "Globally convergent homotopy methods: a tutorial", *Appl. Math. and Comp.*, vol. 31, pp. 369-396, May 1989.
- [13] D.M. Wolf and S.R. Sanders. "Multi-Parameter Methods for Finding DC Operating Points of Nonlinear Circuits", *IEEE Int. Symp on Circuits and Systems*, Chicago, IL, May, 1993. pp 2478-2481.
- [14] D.M. Wolf and S.R. Sanders. " Multi-Parameter Methods for Finding DC Operating Points of Nonlinear Circuits", To appear in *IEEE Trans. Cir. Sys.* Preprints available on request.
- [15] D.M. Wolf, M. Varghese, and S.R. Sanders. "Bifurcation of Power Electronic Circuits", to appear in the *Journal of the Franklin Institute*, 1995. Preprints available on request.
- [16] D.M. Wolf and S.R. Sanders. "Multi-Parameter Methods for Finding Periodic Solutions of Nonlinear Circuits", *IEEE Int. Symp on Circuits and Systems*, London, UK, June 1994
- [17] D.M. Wolf and S.R. Sanders. "Multi-Parameter Homotopy for Power Electronic Circuit Simulation", *Proceedings of 1994 IEEE Workshop on Computers in Power Electronics*, Trois-Rivieres, Que., Canada, Aug. 1994. pp. 300-6.
- [18] R. Kalaba and L. Tesfatsion. " Solving nonlinear equations by adaptive homotopy continuation", *Appl. Math. and Comp.*, vol. 41, pp. 99-115, Jan 1991.
- [19] M.M. Green and A.N. Willson Jr. "How to identify unstable DC operating points," *IEEE Trans. Cir. and Sys.*, vol.39, (no.10):820-32, Oct. 1992.

- [34] R.L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Benjamin/Cummings, 1986.
- [35] A. Ushida and L.O. Chua, "Tracing solution curves of nonlinear equations with sharp turning points", *Int. J. of Circuit Theory and Applications*, Vol 12, pp.1-21, 1984.
- [36] M. Golubitsky, D. Schaeffer. [1979a]. A theory for imperfect bifurcation via singularity theory. *Comm. Pure Appl. Math.*, 32, 21-98
- [37] M. Golubitsky, D. Schaeffer. [1979b]. Imperfect bifurcation in the presence of symmetry. *Comm. Math. Phys.*, 67, 205-232.
- [38] R.J. Duffin, "Impossible Behavior of Nonlinear Networks," 1953 *Proc. Symp. on Non-linear Circuit Analysis*, Polytechnic Inst. of Brooklyn, N.Y., pp.124-138.
- [39] D.H. Wolaver, *Fundamental Study of DC to DC Conversion Systems*, PhD Thesis, EECS Dept., MIT, 1969.
- [40] Middlebrook, R.D., "Modeling current-programmed buck and boost regulators," *IEEE Transactions on Power Electronics*, Jan. 1989, vol.4, (no.1):36-52.
- [41] Verghese, G.C.; Bruzos, C.A.; Mahabir, K.N., "Averaged and sampled-data models for current mode control: a re-examination," *PESC '89 Record. 20th Annual IEEE Power Electronics Specialists Conference*, pp. 484-91 vol.1.
- [42] S.R. Sanders, "On Limit Cycles and the Describing Function Method in Periodically Switched Circuits," *IEEE Trans. Circ. and Syst.*, vol. 40, no. 9, Sept. 1993, pp. 564-572.
- [43] K. Kundert, J. White, A. Sangiovanni-Vincentelli, *Steady-State Methods for Simulating Analog and Microwave Circuits*, 1990.
- [44] Chua, L.O.; Hasler, M; Neirynck, J; Verburgh, P, "Dynamics of a Piecewise-Linear Resonant Circuit," *IEEE Transactions on Circuits and Systems*, Vol. CAS-29, No. 8, August 1982, pp 535-546.
- [45] Deane, J. B.; Hamill, D. C., "Analysis, simulation and experimental study of chaos in the buck converter," *PESC '90 Record. 21st IEEE Power Electronics Specialists Conference*, pp. 491-98.

- [20] M.M. Green and A.N. Willson Jr. "An algorithm for identifying unstable operating points using SPICE," *IEEE Trans on Computer-Aided Design of Integrated Circuits and Systems*, March, 1993, V14 N3:360-370.
- [21] M.M. Green and R.C. Melville. "Sufficient Conditions for Finding Multiple Operating Points of DC Circuits Using Continuation Methods", To appear in *IEEE Int. Symp on Circuits and Systems*, 1995.
- [22] D. Manocha and J.F. Canny. "Multipolynomial resultant algorithms", *Journal of Symbolic Computation*, Feb. 1993, vol.15, (no.2):99-122.
- [23] E. Hille, *Analytic function theory*. Vols. I,II. Boston, Ginn, [1959-1962].
- [24] K. Knopp, *Theory of functions*. Vol II. New York, Dover publications, [1945-47].
- [25] P. Griffiths, J. Harris, *Principles of algebraic geometry*, New York, Wiley, 1978.
- [26] B. Mendelson, *Introduction to Topology*, Dover Publications Inc., New York, 1990. Third Edition.
- [27] D. Mumford, *Algebraic geometry I : Complex Projective Varieties*, New York, Springer-Verlag, 1976.
- [28] E. Artin, *Algebraic numbers and algebraic functions*, New York, Gordon and Breach, 1967. Series: Notes on mathematics and its applications.
- [29] R. Hartshorne, *Algebraic Geometry*, Springer Verlag
- [30] T.M. Rassias and H.M. Srivastana, "Some Recent Advances In The Theory of the Zeros and Critical Points of a Polynomial", in *Topics In Pol ynomials Of One and Several Variables and Their Applications*, 1993 World Scientific Publ. Co, pp. 463-481
- [31] S. Sastry, C. Desoer, "Jump Behavior of Circuits and Systems", *IEEE Trans on Circuits and Systems*, Vol. CAS-28, No. 12, December, 1981, pp1109-1124.
- [32] I. Stewart "Elementary catastrophe theory", *IEEE trans. Circuits and Systems*, August, 1983.
- [33] V.I. Arnold, *Catastrophe theory*, Springer-Verlag, 1984, 1986.

- [46] Deane, J. B.; Hamill, D. C., "Instability, Subharmonics, and Chaos in Power Electronic Systems," *IEEE Transactions on Power Electronics*, July 1990, vol.5, (no.3):260-8.
- [47] M. Hasler and J. Neirynck, *Nonlinear Circuits*, Artech House, inc, 1986.
- [48] L. O. Chua, C.A. Desoer, and E.S. Kuh, *Linear and Nonlinear Circuits*, New York : McGraw-Hill, c1987.
- [49] B. C. Kuo, *Automatic Control Systems*, Prentice-Hall, inc, 1982. Fourth edition.
- [50] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer Verlag, 1983.
- [51] Devaney, R. L., *An Introduction to Chaotic Dynamical Systems*, Menlo Park, Calif. : Benjamin/Cummings, c1986.
- [52] Jalali, S.G.; Dobson, I.; Lasseter, R.H., "Instabilities due to bifurcation of switching times in a thyristor controlled reactor", *PESC '92 Record. 23rd IEEE Power Electronics Specialists Conference*, ppp. 546-552.
- [53] Tse, C.K., "Chaos from a buck switching regulator operating in discontinuous mode," *International Journal of Circuit Theory and Applications*, July-Aug. 1994, vol.22, (no.4):263-78.
- [54] Tse, C.K., "Flip bifurcation and chaos in three-state boost switching regulators," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, Jan. 1994, vol.41, (no.1):16-23.
- [55] Swift, J.W.; Wiesenfeld, K., "Suppression of period doubling in symmetric systems," *Physical Review Letters*, 27 Feb. 1984, vol.52, (no.9):705-8.
- [56] Verghese, G.C., Elbuk, M.E., Kassakian, J.G., "A general approach to sampled-data modeling for power electronic circuits," *IEEE Transactions on Power Electronics*, April 1986, vol. PE-1, (no.2):76-89.
- [57] Sharkovsky, A.N., and Chua, L.O., "Chaos in Some 1-D Discontinuous Maps that Appear in the Analysis of Electrical Circuits," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, Oct. 1993, vol. 40, (no.10):722-732.

- [58] D.G. Bedrosian, J. Vlach. "An Accelerated Steady-State Method for Networks With Internally Controlled Switches". *IEEE Trans, Circuits Syst.*, v39, N7:520-530, July, 1992.
- [59] R.C. Wong, "Accelerated Convergence to the Steady-State Solution of Closed-Loop Regulated Switching-Mode Systems as Obtained Through Simulation", *IEEE Power Electron. Specialists Conf.*, pp 682-692, June 1987.
- [60] D.C. Hamill, "Improved Time-Domain Simulation of Resonant and Other DC-DC Converters". *IEEE Workshop on Computers in Power Electronics*, pp 93-109, 1990.
- [61] Y.Kuroe, H. Haneda, and T. Maruhashi, "A General Steady-State Analysis Program ANASP for Thyristor Circuits Based on the Adjoint-Network Approach", *IEEE Power Electron. Specialists Conf.*, pp 180-189, June 1980.
- [62] Y.Kuroe, "Homotopy Applied to Finding Steady-State of Power Electronic Circuits", Presented at the *IEEE Workshop on Computers in Power Electronics*, 1990.
- [63] J.H. Deane, D.C. Hamill, "Analysis, Simulation and Experimental Study of Chaos in the Buck Converter", *IEEE Power Electron. Specialists Conf.*, vol 2, pp 491-498, June 1990.
- [64] V.I. Gulyayev, A.L. Zubritskaya and V.L. Koshkin, "A Universal Sequence of Period-Doubling Bifurcations of the Forced Oscillations of a Pendulum", *PMM U.S.S.R.*, Vol. 53, No. 5, pp.561-565, 1989.