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# On the Departure Process of a Leaky Bucket System with Long-Range Dependent Input Traffic: The Finite Cell Buffer Case [†]

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November 18, 1996

**Abstract**

Due to the strong experimental evidence that the traffic to be offered to future broadband networks will display long - range dependence, it is important to examine in more detail the possible implications that these new traffic sources may have on the design and performance of networks. In particular, one important question is whether the offered traffic preserves its long - range dependent nature after passing through the policing mechanism at the interface of the network. One of the proposed solutions for flow control in the context of the emerging ATM standard is the so-called leaky bucket scheme. In this note we consider a leaky bucket system with long - range dependent input traffic. We examine the departure process of the system and show that it, too, is long - range dependent for any token buffer size and any finite cell buffer size.

## 1 Introduction - Problem Formulation

Recent experimental studies of traffic to be carried by broadband networks have pointed out the possible importance of analyzing the performance of communication networks using traffic models with long - range dependence. Long - range dependence in network traffic has been reported, for instance, in [4], where statistical analysis of measurements of Ethernet traffic at Bellcore demonstrated its self - similar nature; in [2] long - range dependence has been established in variable bit rate video traffic generated by a number of different codecs;

[†]Research supported by NSF under NCR 9422513 and by an U. C. Berkeley Regents Fellowship.
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and in [6] the presence of long - range dependence in TELNET and other wide area network traffic was concluded.

Due to the strong experimental evidence that the traffic to be offered to future broadband networks will display long - range dependence, it is important to examine in more detail the possible implications that these new traffic models may have on the design and performance of networks. In particular, one important question is whether the offered traffic preserves its long - range dependent nature after passing through the policing mechanism at the interface of the network. One of the proposed solutions for flow control in the context of the emerging ATM standard is the so-called *leaky bucket* scheme which is shown schematically in Figure 1. Fixed - size cells arrive in a buffer of size $B < \infty$. The departure of cells from the buffer is controlled by *tokens* that are stored in a buffer of fixed size $C$. An arriving cell can be transmitted only if it finds a token in the token buffer, in which case it is transmitted instantaneously by consuming a token. If the token buffer is empty, the cell has to wait for the generation of a new token. Time is assumed to be discrete and exactly one token is generated at the beginning of each unit of time. A stored cell is transmitted immediately upon the generation of a new token. Moreover, the outgoing capacity of the link is assumed to be at

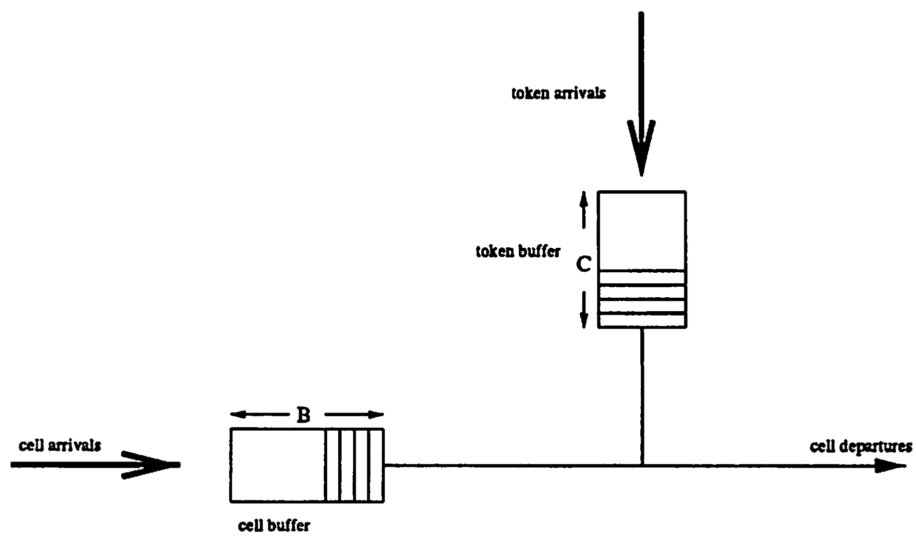


Figure 1: The leaky bucket scheme.

least $C$, so that it imposes no limitations on the number of cells that can be transmitted instantaneously.

We assume that the cell arrival process belongs to a class of discrete - time long - range dependent traffic models which includes as a special case the one proposed in [5]. In an arrival model of this class a number of sessions are initiated at the beginning of each unit of time, which is a Poisson random variable with parameter $\lambda$. Each of those sessions consists of a random number $\tau$ of cells which has finite mean, infinite variance and a regularly varying tail, i.e. $P(\tau > k) \sim k^{-\alpha}L(k)$, where $1 < \alpha < 2$ and $L(\cdot)$ is a slowly varying function. Once a session is initiated, it generates one cell at the beginning of each unit of time until its termination.

In [7] it was shown that the departure process of a leaky bucket system with long - range dependent input traffic is long - range dependent in the case of infinite cell buffer size $B$. However, since it is sometimes argued that in real communication networks the finite size of the buffers will drastically reduce the importance of the long - range dependent behaviour of the sources, it is important to investigate the case of finite cell buffer size as well. Thus, in this note we consider a leaky bucket system with finite cell and token buffer size $B$ and $C$ respectively, fed by an arrival process having a long - range dependent behaviour as discussed above. We examine the departure process of the system and show that it, too, is long - range dependent.

# 2 Statement of Results

We first examine the departure process in the case $C = 1$ for any $B \geq 0$. In this case the leaky bucket system is equivalent to a single - server queue that is served at a constant rate equal to one cell per unit time. We prove the following:

**Lemma 1.** *The departure process of the leaky bucket system in the case $C = 1$ is long - range dependent for any cell buffer size $B$.*

**Proof**

We recall that a necessary and sufficient condition for a stationary second-order stochastic process to be long - range dependent is that the sum of the absolute values of its covariances be infinite, i.e.

$$\sum_{m=1}^{\infty} \mid r(m) \mid = \infty, \tag{1}$$

where $r(m)$ the covariance function of the process.

If we denote the departure process by $\{d^{(1,B)}(k)\}$, then it is obvious that $d^{(1,B)}(k) \in \{0,1\}$, $\forall k$. Let $Prob(d^{(1,B)}(k) = 1) \triangleq p_{busy} < 1$ and $Prob(d^{(1,B)}(k) = 0) \triangleq p_{idle}$ in stationarity. It is obvious then that $E[d^{(1,B)}(k)] = p_{busy}$ in the stationary regime. The covariance of the departure process is:

$$\begin{aligned} r^{(1,B)}(m) &= E[(d^{(1,B)}(k) - p_{busy})(d^{(1,B)}(k+m) - p_{busy})] \\ &= p_{busy}^2 P(d^{(1,B)}(k+m) = 0, d^{(1,B)}(k) = 0) \\ &\quad -p_{busy}(1 - p_{busy})P(d^{(1,B)}(k+m) = 0, d^{(1,B)}(k) = 1) \\ &\quad -p_{busy}(1 - p_{busy})P(d^{(1,B)}(k+m) = 1, d^{(1,B)}(k) = 0) \\ &\quad +(1 - p_{busy})^2 P(d^{(1,B)}(k+m) = 1, d^{(1,B)}(k) = 1). \end{aligned}$$

If we define $P_{i|j}^m \triangleq P(d^{(1,B)}(k+m) = i \mid d^{(1,B)}(k) = j)$, we get from the above relation:

$$r^{(1,B)}(m) \;=\; p_{busy}^2(1 - p_{busy})P_{0|0}^m - p_{busy}^2(1 - p_{busy})P_{0|1}^m$$

$$\begin{aligned} & -p_{busy}(1-p_{busy})^2 P^m_{1|0} + p_{busy}(1-p_{busy})^2 P^m_{1|1} \\ = \; & p_{busy}(1-p_{busy})(P^m_{0|0} - P^m_{0|1}) \qquad (2) \\ = \; & p_{busy}(1-p_{busy})(P^m_{1|1} - P^m_{1|0}) \qquad (3) \end{aligned}$$

In the above we used the fact that $P^m_{0|0} + P^m_{1|0} = P^m_{0|1} + P^m_{1|1} = 1$. From equation (2) it is obvious that $r^{(1,B)}(m)$ is positive, since $0 < p_{busy} < 1$ and $P^m_{0|0} > P^m_{0|1}$ which can be easily seen from the definition of $P^m_{i|j}$.

We will henceforth refer to cells belonging to sessions that have been initiated at or before time $k$ as cells of type 1, while cells belonging to sessions that arrived after time $k$ will be referred to as cells of type 2. Assume that beginning at time $k+1$ priority is given to cells of type 2 over cells of type 1 in the case that $d^{(1,B)}(k) = 1$ (obviously if $d^{(1,B)}(k) = 0$, then there are no cells of type 1 after time $k$). If at some point there are no cells of type 2 in the system, then the service of any remaining cells of type 1 may be resumed. Since the arrival process after time $k+1$ is independent of the past, we may assume that it is pathwise the same for both the initially active system with $d^{(1,B)}(k) = 1$ and the initially idle system with $d^{(1,B)}(k) = 0$. Then it follows easily that we get the following relation:

$$P^m_{1|1} - P^m_{1|0} = P\{d^{(1,B)}(k+m) = 1 \textit{ and the cell departing at } k+m \textit{ is of type } 1 \mid d^{(1,B)}(k) = 1\} \qquad (4)$$

Thus, according to the above discussion in order to show that the departure process is long - range dependent, it suffices to show that

$$\sum_{m=1}^{\infty} P\{d^{(1,B)}(k+m) = 1 \textit{ and the cell departing at } k+m \textit{ is of type } 1 \mid d^{(1,B)}(k) = 1\} = \infty \qquad (5)$$

Let $S_k$ denote the set of sessions present in the system at time $k$ and let $X_i$, $i \in S_k$ denote the number of cells belonging to session $i$ that arrive in the system after time $k$. We define $X \triangleq \max_{i \in S_k}\{X_i\}$. Also let $A_{1,l}$ denote the event that the cell departing at time $l$ is of type 1. Then using the above definitions we may write:

$$\begin{aligned} & \sum_{m=1}^{\infty} P\{d^{(1,B)}(k+m) = 1 \textit{ and } A_{1,k+m} \mid d^{(1,B)}(k) = 1\} \\ & = \sum_{m=1}^{\infty}\sum_{l=0}^{\infty} P\{d^{(1,B)}(k+m) = 1 \textit{ and } A_{1,k+m} \mid X = l, d^{(1,B)}(k) = 1\} \cdot P\{X = l \mid d^{(1,B)}(k) = 1\} \\ & = \sum_{l=0}^{\infty} P\{X = l \mid d^{(1,B)}(k) = 1\} E[N_{1,k} \mid X = l, d^{(1,B)}(k) = 1] \qquad (6) \end{aligned}$$

where $N_{1,k}$ denotes the number of cells of type 1 that get released after time $k$. We now use the following coupling argument: Consider a second identical leaky bucket system with token buffer size $C = 1$ and cell buffer size $B$, that is driven by an arrival process that is independent of and identically distributed to the arrival process of the original system up to and including time $k$. Starting at $k+1$ the part of the arrival process that consists of sessions that are generated at or after time $k+1$ is assumed to be pathwise the same for both systems. Of course, since the arrival processes in the two systems were independent for times prior to $k+1$, it is obvious that the part of the arrival process that consists of cells belonging to sessions which were generated before time $k+1$ will in general be different. In what follows we will refer to the cells which belong to sessions that were generated before or at time $k$ in the second system as cells of type 3. Note that the cells of type 2 - as defined above - are common in both systems. It should also be pointed out that, by its definition, the second system is in stationarity at all times. Finally, we assume that after time $k$ we have the service discipline described above, i.e. cells of type 2 are given priority over cells of type 1 and 3 in the first (original) and second system respectively. Thus, at any time instant $k+j,\ j>0$ we may have only one of the following cases:

- There is no departure in any of the systems (case 0).
- A cell of type 1 departs in the original system and there is no departure in the second system (case 1).
- There is no departure in the original system and a cell of type 3 departs in the second system (case 2).
- A cell of type 1 departs in the original system and a cell of type 3 departs in the second system (case 3).
- A cell of type 2 departs in both systems (case 4).

We denote the events corresponding to the cases 0, 1, 2, 3 and 4 as $00(j)$, $10(j)$, $03(j)$, $13(j)$ and $22(j)$ respectively. Using those results we may write:

$$
\begin{aligned}
E[N_{1,k} \mid X = l, d^{(1,B)}(k) = 1] &= E[\sum_{j=1}^{\infty}(1(10(j)) + 1(13(j))) \mid X = l, d^{(1,B)}(k) = 1] \\
&\geq E[\sum_{j=1}^{l}(1(10(j)) + 1(13(j))) \mid X = l, d^{(1,B)}(k) = 1] \\
&\geq E[\sum_{j=1}^{l} 1(10(j)) \mid X = l, d^{(1,B)}(k) = 1] \\
&= E[l - \sum_{j=1}^{l}(1(03(j)) + 1(13(j)) + 1(22(j))) \mid X = l, d^{(1,B)}(k) = 1] \qquad (7)
\end{aligned}
$$

where in the above $1(A)$ is the indicator function of the event $A$. In the last equation we used the fact that given $X = l$, there is always a departure in the original system between $k$ and $k+l$. Since the second system is in stationarity despite the conditioning, we have that

$$
E[\sum_{j=1}^{l}(1(03(j)) + 1(13(j)) + 1(22(j))) \mid X = l, d^{(1,B)}(k) = 1] = p_{busy}l \qquad (8)
$$

where we used the fact that the union of the events in the summation is the event that at some time instant there is a departure in the second system. Thus we get:

$$E[N_{1,k} \mid X = l, d^{(1,B)}(k) = 1] \geq p_{idle} l \tag{9}$$

From (6) using this result we obtain:

$$\begin{aligned}
\sum_{m=1}^{\infty} P\{d^{(1,B)}(k+m) = 1 \; and \; A_{1,k+m} \mid d^{(1,B)}(k) = 1\} &\geq p_{idle} \sum_{l=0}^{\infty} l \cdot P\{X = l \mid d^{(1,B)}(k) = 1\} \\
&= p_{idle} \sum_{l=1}^{\infty} P\{X \geq l \mid d^{(1,B)}(k) = 1\} \\
&\geq p_{idle} \sum_{l=1}^{\infty} P\{X \geq l\} \qquad (10)
\end{aligned}$$

Let $\chi_k(m)$ denote the number of sessions starting at time $k$ that are active during slot $k+m$, $m \geq 1$. Since the number of sessions generated at any time instant is Poisson with parameter $\lambda$ and the durations of sessions are independent of each other, we get that $\chi_k(m)$ is a Poisson random variable with parameter $\lambda q_{m+1}$, with $q_j = Prob\{\tau \geq j\}$, where $\tau$ is the generic random variable that denotes the duration of a session. If we define $u_k(m)$ to be the total work brought in during slot $k+m$, $m \geq 1$ by sessions that started at or prior to $k$, we have that

$$u_k(m) = \sum_{n=-\infty}^{k} \chi_n(k+m-n) = \sum_{n=m}^{\infty} \chi_{k+m-n}(n) \tag{11}$$

Since the variables $\chi_k(m)$ are independent for different values of $k$, we get that $u_k(m)$ is a Poisson random variable with parameter $\lambda \sum_{n=m+1}^{\infty} q_n$. We now note that $P\{X \geq l\} = P\{u_k(l) \geq 1\}$ so that we get

$$P\{X \geq l\} = 1 - \exp(-\lambda \sum_{n=l+1}^{\infty} q_n) \tag{12}$$

We recall that $q_n \sim n^{-\alpha} L(n)$, where $1 < \alpha < 2$ and $L(\cdot)$ is a slowly varying function. Further, since $E\tau < \infty$, we have that $\sum_{n=l+1}^{\infty} q_n \to 0$ as $l \to \infty$, so that $\lambda \sum_{n=l+1}^{\infty} q_n \leq 1$ for all $l \geq L$, for some $L < \infty$. Note also that $1 - e^x \geq \frac{x}{2}$ for $0 \leq x \leq 1$. Hence:

$$\begin{aligned}
\sum_{l=1}^{\infty} P\{X \geq l\} &\geq \sum_{l=L}^{\infty} [1 - \exp(-\lambda \sum_{n=l+1}^{\infty} q_n)] \\
&\geq \frac{1}{2} \sum_{l=L}^{\infty} \sum_{n=l+1}^{\infty} q_n \\
&= \frac{1}{2} \sum_{n=l+1}^{\infty} (n-L) q_n \qquad (13)
\end{aligned}$$

Since $q_n \sim n^{-\alpha} L(n)$, we have that $(n-L) q_n \sim n^{-\alpha+1} L(n)$. From [3], Proposition 1.3.6, part (v), page 16, we have that for any $\epsilon > 0$, $n^{\epsilon} L(n) \to \infty$ as $n \to \infty$. Choosing $\epsilon > 0$ such that

$1-\alpha-\epsilon > -1$, we see that $(n-L)q_n \sim n^{-\alpha+1}L(n) = n^{-\alpha+1-\epsilon}n^{\epsilon}L(n) > n^{-1}$ for all large enough $n$, so that

$$\sum_{l=1}^{\infty} P\{X \geq l\} = \infty \tag{14}$$

from which we conclude that the departure process is long-range dependent. □

We will now consider the general case of $C > 1$. In accordance with the notation used above we denote by $\{d^{(C,B)}(k)\}$ the departure process of the leaky bucket system in the case that the token buffer size is $C$ and the cell buffer size is $B$. We will first prove the following:

**Lemma 2.** *For every sample path of the arrival process the following inequalities hold:*

$$\begin{aligned} & d^{(C_1,B_1)}(k) + d^{(C_1,B_1)}(k+1) + \ldots + d^{(C_1,B_1)}(k+m) \\ & \quad \leq d^{(C_2,B_2)}(k) + d^{(C_2,B_2)}(k+1) + \ldots + d^{(C_2,B_2)}(k+m) + (C_2 - C_1), \end{aligned} \tag{15}$$

$$\begin{aligned} & d^{(C_1,B_1)}(k) + d^{(C_1,B_1)}(k+1) + \ldots + d^{(C_1,B_1)}(k+m) \\ & \quad \geq d^{(C_2,B_2)}(k) + d^{(C_2,B_2)}(k+1) + \ldots + d^{(C_2,B_2)}(k+m) - (C_2 - C_1), \end{aligned} \tag{16}$$

*for $C_1 + B_1 = B_2 + C_2$ and $C_1 \leq C_2$ and for any selection of $k$ and $m \geq 0$.*

**Proof**

We will use the pathwise construction of the departure process of a leaky bucket system presented in [1]. Let $X^{(C,B)}(k)$, $(Y^{(C,B)}(k))$ denote the number of cells (tokens) in the cell (token) buffer immediately after time $k$ in the system with token buffer size $C$ and cell buffer size $B$. By convention we assume that cells at time $k$ arrive immediately after the arrival of a token at time $k$ and that $X^{(C,B)}(k)$ and $Y^{(C,B)}(k)$ are measured immediately after the arrival of the cells. Obviously, $0 \leq X^{(C,B)}(k) \leq B$ and $0 \leq Y^{(C,B)}(k) \leq C$. We may clearly assume that $X^{(C,B)}(k)Y^{(C,B)}(k) = 0$ for all $k$, so that the system can be described by the parameter $Z^{(C,B)}(k) = X^{(C,B)}(k) - Y^{(C,B)}(k)$. If $0 < Z^{(C,B)}(k) \leq B$ there are $Z^{(C,B)}(k)$ cells and no tokens, if $-C \leq Z^{(C,B)}(k) < 0$ there are $-Z^{(C,B)}(k)$ tokens and no cells, and if $Z^{(C,B)}(k) = 0$ then there are neither cells nor tokens. Let $a(k)$ denote the number of cells arriving into the leaky bucket at time $k$. Then it is easy to see that $Z^{(C,B)}(k)$ is given by the recursion

$$Z^{(C,B)}(k+1) = \min\{B, Z^{(C,B)}(k) + a(k+1) - I\{Z^{(C,B)}(k) \geq -C+1\}\} \tag{17}$$

If we now define the quantity $W^{(C,B)}(k) = Z^{(C,B)}(k) + C = X^{(C,B)}(k) - Y^{(C,B)}(k) + C$, we get the following recursion for $W^{(C,B)}(k)$ from (17):

$$\begin{aligned} W^{(C,B)}(k+1) &= \min\{C+B, W^{(C,B)}(k) + a(k+1) - I\{W^{(C,B)}(k) > 0\}\} \\ &= \min\{C+B, (W^{(C,B)}(k) - 1)^{+} + a(k+1)\} \end{aligned} \tag{18}$$

where in the above $(x)^+ = \max\{x, 0\}$. Note that $0 \leq W^{(C,B)}(k) \leq C + B$, $\forall k$. It is important also to point out that the recursion for $W^{(C,B)}$ depends on the quantities $C$ and $B$ only through the sum $C + B$. Hence, if for two systems $C_1 + B_1 = C_2 + B_2$, then $W^{(C_1,B_1)}$ and $W^{(C_2,B_2)}$ satisfy the same recursion. This means that if the two systems are assumed to be driven by the same sample path, then the corresponding $W$-processes are also pathwise the same.

It is easy to see that the departure process from the leaky bucket can be constructed from a sample path of the $W^{(C,B)}$ process as shown in Figure 2. In order to determine the

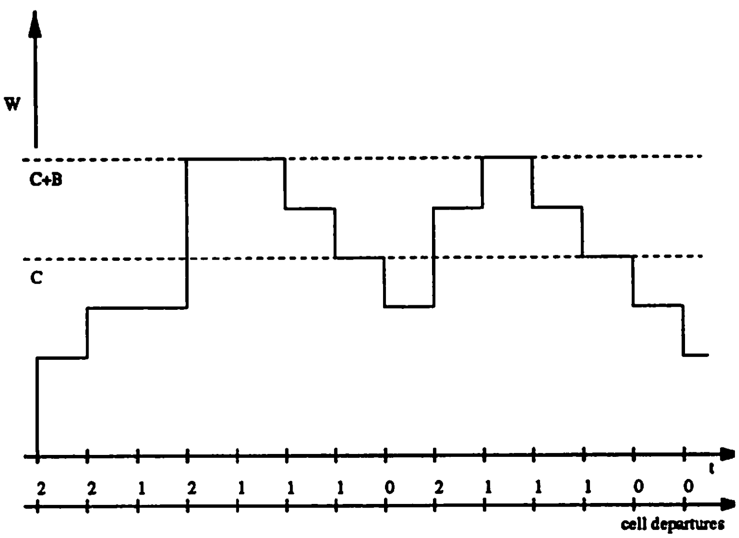


Figure 2: Construction of the departure process. The numbers on the axis labeled 'cell departures' give the number of cells leaving the system at each time instant for the specific realization of the process $W^{(C,B)}$.

number of cells $d^{(C,B)}(k)$ that leave the system at time $k$ we consider the following cases:
a) $W^{(C,B)}(k-1) = 0$: In that case the number of departing cells is $d^{(C,B)}(k) = \min\{W^{(C,B)}(k), C\}$.
b) $0 < W^{(C,B)}(k-1) \leq C$: In that case the number of departing cells is $d^{(C,B)}(k) = \min\{W^{(C,B)}(k), C\} - W^{(C,B)}(k-1) + 1$.
c) $C < W^{(C,B)}(k-1) \leq C + B$: In that case there is exactly one departure at time $k$ due to the arriving token, i.e. $d^{(C,B)}(k) = 1$.
From the above we deduce that the departure processes for a whole range of systems, corresponding to different values of $C$ and $B$, can be read off directly from the sample path of $W^{(C,B)}$.

Let $d_{k,m}^{(C,B)} = d^{(C,B)}(k) + \ldots + d^{(C,B)}(k+m)$ be the number of departing cells from the system with token buffer of size $C$ and cell buffer of size $B$ during the interval $[k, k+m]$, $m \geq 0$. Also let $\Delta d_{k,m} = d_{k,m}^{(C_1,B_1)} - d_{k,m}^{(C_2,B_2)}$ be the difference in the number of departing cells from two systems in the same interval, where $C_1 + B_1 = C_2 + B_2$. We are interested in upper and lower bounds of the quantity $\Delta d_{k,m}$. Without loss of generality we may assume that $C_1 \leq C_2$. Using the above results (a) - (c) we have the following cases, where we take into account that the $W$-processes of the two systems are pathwise the same, as discussed above, and therefore are represented simply by $W(\cdot)$:

- $W(k-1) \leq C_1$ and $W(k+m) \leq C_1$: Then $\Delta d_{k,m} = 0$.
- $W(k-1) \leq C_1$ and $C_1 + 1 \leq W(k+m) \leq C_2$: Then $\Delta d_{k,m} = C_1 - W(k+m)$.
- $W(k-1) \leq C_1$ and $C_2 + 1 \leq W(k+m) \leq C_1 + B_1$: Then $\Delta d_{k,m} = C_1 - C_2$.
- $C_1 + 1 \leq W(k-1) \leq C_2$ and $W(k+m) \leq C_1$: Then $\Delta d_{k,m} = W(k-1) - C_1$.
- $C_1+1 \leq W(k-1) \leq C_2$ and $C_1+1 \leq W(k+m) \leq C_2$: Then $\Delta d_{k,m} = W(k-1) - W(k+m)$.
- $C_1+1 \leq W(k-1) \leq C_2$ and $C_2+1 \leq W(k+m) \leq C_1+B_1$: Then $\Delta d_{k,m} = W(k-1) - C_2$.
- $C_2 + 1 \leq W(k-1) \leq C_1 + B_1$ and $W(k+m) \leq C_1$: Then $\Delta d_{k,m} = C_2 - C_1$.
- $C_2+1 \leq W(k-1) \leq C_1+B_1$ and $C_1+1 \leq W(k+m) \leq C_2$: Then $\Delta d_{k,m} = C_2 - W(k+m)$.
- $C_2 + 1 \leq W(k-1) \leq C_1 + B_1$ and $C_2 + 1 \leq W(k+m) \leq C_1 + B_1$: Then $\Delta d_{k,m} = 0$.

We see that the greatest value of $\Delta d_{k,m}$ is achieved for any interval $[k, k+m]$ with $C_2 \leq W(k-1) \leq C_1 + B_1$ and $W(k+m) \leq C_1$ and for those intervals we get $\Delta d_{k,m} = C_2 - C_1$. Similarly, the smallest value is achieved for any interval $[k, k+m]$ with $W(k-1) \leq C_1$ and $C_2 \leq W(k+m) \leq C_1 + B_1$ and for those intervals we get $\Delta d_{k,m} = -(C_2 - C_1)$. Hence, $-(C_2 - C_1) \leq d_{k,m}^{(C_1,B_1)} - d_{k,m}^{(C_2,B_2)} \leq C_2 - C_1$, $\forall m > 0, k$, if $C_1 + B_1 = C_2 + B_2$ with $C_1 \leq C_2$ and the proof is complete. □

It should be noted here that due to the above lemma and the fact that the departure process for any $C$ and $B$ is stationary and ergodic, the expected values of the departure processes when the conditions $C_1 + B_1 = C_2 + B_2$, $C_1 \leq C_2$ hold are the same, since

$$\lim_{m\to\infty} \frac{d_{k,m}^{(C_2,B_2)} - (C_2 - C_1)}{m} \leq E[d^{(C_1,B_1)}] = \lim_{m\to\infty} \frac{d_{k,m}^{(C_1,B_1)}}{m} \leq \lim_{m\to\infty} \frac{d_{k,m}^{(C_2,B_2)} + (C_2 - C_1)}{m}$$

and the first and third limits are the same and equal to $E[d^{(C_2,B_2)}] = \lim_{m\to\infty} \frac{d_{k,m}^{(C_2,B_2)}}{m}$.

Now we can prove the following:

**Theorem** *The departure process of the leaky bucket system is long - range dependent for any token buffer size $C$ and any cell buffer size $B$.*

**Proof**

We have shown the result for $C = 1$. To prove the result in the case $C > 1$ we proceed as follows: Both sides of (15) are nonnegative, so we may square both sides and the inequality will still hold. We also assume that $C_1 = 1$. If we also take expectations on both sides and subtract the quantity $E^2 = E[d^{(1,B_1)}(k)]^2 = E[d^{(C_2,B_2)}(k)]^2$ from each expectation term on both sides we get the following relation:

$$(m+1)r^{(1,B_1)}(0) + 2\sum_{i=1}^{m}(m-i+1)r^{(1,B_1)}(i)$$
$$\leq (m+1)r^{(C_2,B_2)}(0) + 2\sum_{i=1}^{m}(m-i+1)r^{(C_2,B_2)}(i) + 2(m+1)C_2E + C_2^2 \qquad (19)$$

where in the above we took into account that the output process is stationary. Let $\Sigma_0^{(C_2,B_2)}(l) \triangleq \sum_{i=0}^{l} \left| r^{(C_2,B_2)}(i) \right|$ and $\Sigma_1^{(1,B_1)}(l) \triangleq \sum_{i=1}^{l} r^{(1,B_1)}(i)$. Also let $d \triangleq C_2 E$ and $g \triangleq 2C_2E + C_2^2$. Since $r^{(1,B_1)}(0), r^{(C_2,B_2)}(0) \geq 0$ we get from (19) after a few simple algebraic manipulations.

$$\sum_{l=1}^{m} (\Sigma_1^{(1,B_1)}(l) - d) \leq \sum_{l=0}^{m} \Sigma_0^{(C_2,B_2)}(l) \ + g \tag{20}$$

In order to show that $\{d^{(C_2,B_2)}(k)\}$ is long - range dependent, it suffices to show that for every $M > 0$ there exists some integer $L$, such that

$$\Sigma_0^{(C_2,B_2)}(l) > M, \quad \forall l \geq L \tag{21}$$

Note that since $\Sigma_0^{(C_2,B_2)}(l)$ is monotone increasing in $l$, then if (21) holds for some index $L$, it will definitely hold for all indices $l \geq L$. To show (21) we may argue by contradiction. Suppose that for some $M_0$ there is no integer $L$, such that $\Sigma_0^{(C_2,B_2)}(L) > M_0$. But, since $\lim_{l\to\infty} \Sigma_1^{(1,B_1)}(l) = \infty$, this means that there exists some integer $K$, such that $\Sigma_1^{(1,B_1)}(l) - d > M_0 > \Sigma_0^{(C_2,B_2)}(l)$, $\forall l \geq K$. This obviously contradicts the inequality in (20) and therefore we must have that (21) holds or equivalently that $\{d^{(C,B)}(k)\}$ is long - range dependent for any values of $C$ and $B$. □

# 3 Concluding Remarks

We have studied the departure process of a leaky bucket system in an ATM network fed by a class of proposed models for long - range dependent input traffic. We established the fact that the departure process is long - range dependent for any cell and token buffer size.

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