REPORT 7: EQUIVALENT CONGRUENCE TRANSFORMS

by

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Report 7: Equivalent congruence transforms

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1 Introduction

In Report 6, we show that when a congruence transformation is applied to the MNA and output equations of an RLC network, the transformed network will be passive. However, in the case when the admittance or hybrid matrix of the network is of interest, if the MNA equations is written in the form of Eq(11) in Report 6 with $H = sA + b$, then the MNA equations are in the following form:

$$(sA + B)x = bu \tag{1}$$

where x contains all node voltages, inductor currents and source currents, and matrix A will have several zero rows and columns and be singular. This is not good for using the Anordi or Lanczos algorithm to find the congruence transformation, and we need to compact the equations to the form

$$(sA' + B')x' = b'u \tag{2}$$

so that x' contains only the unknown node voltages and inductor currents and there will be no zero rows and columns in A. For simplicity, we call the system described by Eq(1) and its associated output equations the original system, and the system described by Eq(2) and its associated output equations the compact system. We have shown that when a congruence transformation is applied to the original system, the reduced order system is passive. In this report, we will show that when a congruence transformation is applied to the compact system, the reduced order system is passive, too. In order to do that, we will show that when there is a congruence transform applied to the compact system, there is another corresponding congruence transform applied to the original matrix both of which result in the same admittance (or hybrid) matrix of the reduced order network. We call such a pair of congruence transforms equivalent.

In this report, we only consider that case when an admittance matrix is of interest. For the case when hybrid matrix is of concern, the derivation is similar and the conclusion is the same. In Sec.2, we will consider the typical case that all voltage sources are grounded at their negative terminals, and in Sec.3, we will talk about the case with floating voltage sources.

2 Grounded voltage sources

2.1 Compact form of MNA equations

When the admittance matrix of an RLC multiport is of interest, the network is excited by voltage sources and the response is the currents flowing out of the sources. By using the same symbols as in Report 6, the MNA equations are

$$Hx = bu \tag{3}$$

where

$$x = \begin{bmatrix} V_n \\ I_L \\ I_s \end{bmatrix} \tag{4}$$

$$H = \begin{bmatrix} G + sC & A_L & -A_v \\ -A_L^T & sL & \\ A_v^T & & \end{bmatrix} \tag{5}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \tag{6}$$

$$u = V_s \tag{7}$$

and the output equations are

$$y = b^T x \tag{8}$$

where

$$y = I_s \tag{9}$$

Now assume that the voltage sources applied to the network are grounded at their negative terminals. Then, the node voltage vector can be decomposed into

$$V_n = \begin{bmatrix} V_s \\ V_x \end{bmatrix} \tag{10}$$

where V_x is the vector of unknown node voltages, where all the nodes are internal nodes of the network. Correspondingly, matrices G, C, A_L and A_v can be decomposed as follows:

$$G = \begin{bmatrix} G_{ss} & G_{sx} \\ G_{xs} & G_{xx} \end{bmatrix} \tag{11}$$

with $G_{sx} = G_{xs}^T$, G_{ss} and G_{xx} symmetric and nonnegative-definite;

$$C = \begin{bmatrix} C_{ss} & C_{sx} \\ C_{xs} & C_{xx} \end{bmatrix} \tag{12}$$

with $C_{sx} = C_{xs}^T$, C_{ss} and C_{xx} symmetric and nonnegative-definite;

$$A_L = \begin{bmatrix} A_{Ls} \\ A_{Lx} \end{bmatrix} \tag{13}$$

and

$$A_v = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{14}$$

Let

$$x' = \begin{bmatrix} V_x \\ I_L \end{bmatrix} \tag{15}$$

Then the compact MNA equations are in the form of

$$H'x' = b'u \tag{16}$$

where

$$H' = \begin{bmatrix} G_{xx} + sC_{xx} & A_{Lx} \\ -A_{Lx}^T & sL \end{bmatrix} \tag{17}$$

$$b' = \begin{bmatrix} -(G_{xs} + sC_{xs}) \\ A_{Ls}^T \end{bmatrix} \tag{18}$$

Now the output vector I_s can be expressed as

$$I_s = (G_{ss} + sC_{ss})V_s + [(G_{sx} + sC_{sx}) \quad A_{Ls}]x' = c^T x' + d^T u \tag{19}$$

where

$$c = \begin{bmatrix} G_{xs} + sC_{xs} \\ A_{Ls}^T \end{bmatrix} \tag{20}$$

and

$$d = G_{ss} + sC_{ss} \tag{21}$$

Note that $c \neq b'$ in the general case.
From Eqs(3) and (8), the admittance matrix of the network can be expressed as

$$Y(s) = b^T H^{-1} b \tag{22}$$

and from Eqs(16) and (19), we have

$$Y(s) = d^T + c^T (H')^{-1} b'$$

$$= G_{ss} + sC_{ss} + [(G_{sx} + sC_{sx}) \quad A_{Ls}] \begin{bmatrix} G_{xx} + sC_{xx} & A_{Lx} \\ -A_{Lx}^T & sL \end{bmatrix}^{-1} \begin{bmatrix} -(G_{xs} + sC_{xs}) \\ A_{Ls}^T \end{bmatrix} \tag{23}$$

2.2 Congruence transformation

Let the dimension of vector V_s and I_s be m, and the dimension of vector x' be n. Now suppose that a congruence transform matrix $V \in R^{n \times p}$ has been found and applied to the compact form of the network representation, then we have the equations for the transformed network as follows:

$$x' = V\hat{x}' \tag{24}$$

$$\hat{H}'\hat{x}' = \hat{b}'u \tag{25}$$

where

$$\hat{H}' = V^T H' V = V^T \begin{bmatrix} G_{xx} + sC_{xx} & A_{Lx} \\ -A_{Lx}^T & sL \end{bmatrix} V \tag{26}$$

and

$$\hat{b}' = V^T b' = V^T \begin{bmatrix} -(G_{xs} + sC_{xs}) \\ A_{Ls}^T \end{bmatrix} \tag{27}$$

The output equations now become

$$y = \hat{c}'^T \hat{x}' + du \tag{28}$$

where

$$\hat{c}'^T = c^T V = [G_{sx} + C_{sx} \quad A_{Ls}]V \tag{29}$$

and the admittance matrix of the transformed system will be

$$\hat{Y}' = d + \hat{c}'^T \hat{H}'^{-1} \hat{b}'$$

$$= G_{ss} + sC_{ss} + [G_{sx} + sC_{sx} \quad A_{Ls}]V\hat{H}'^{-1} V^T \begin{bmatrix} -(G_{xs} + sC_{xs}) \\ A_{Ls}^T \end{bmatrix} \tag{30}$$

Now for the original system, we form a congruence transform W as follows:

$$W = \begin{bmatrix} I_{m \times m} & & \\ & V & \\ & & I_{m \times m} \end{bmatrix} \tag{31}$$

and we will show that W and V are a pair of equivalent congruence transforms. When W is applied to the original system, the transformed system becomes

$$\hat{H}\hat{x} = \hat{b}u \tag{32}$$

where

$$\hat{x} = \begin{bmatrix} V_s \\ \hat{x}' \\ I_s \end{bmatrix} \tag{33}$$

and

$$x = W\hat{x} = \begin{bmatrix} I_{m\times m} & & \\ & V & \\ & & I_{m\times m} \end{bmatrix} \begin{bmatrix} V_s \\ \hat{x}' \\ I_s \end{bmatrix} = \begin{bmatrix} V_s \\ x' \\ I_s \end{bmatrix} \tag{34}$$

$$\hat{b} = W^T b = b = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \tag{35}$$

Rewrite matrix H in the following form:

$$H = \begin{bmatrix} G_{ss} + sC_{ss} & \begin{bmatrix} G_{sx} + sC_{sx} & A_{Ls} \end{bmatrix} & -I \\ \begin{bmatrix} G_{xs} + sC_{xs} \\ -A_{Ls}^T \end{bmatrix} & \begin{bmatrix} G_{xx} + sC_{xx} & A_{Lx} \\ -A_{Lx}^T & sL \end{bmatrix} & \\ I & & \end{bmatrix} \tag{36}$$

Note that the central submatrix is H'.
Now,

$$\hat{H} = W^T H W = W^T \begin{bmatrix} G_{ss} + sC_{ss} & \begin{bmatrix} G_{sx} + sC_{sx} & A_{Ls} \end{bmatrix} V & -I \\ \begin{bmatrix} G_{xs} + sC_{xs} \\ -A_{Ls}^T \end{bmatrix} & H'V & \\ I & & \end{bmatrix}$$

$$= \begin{bmatrix} G_{ss} + sC_{ss} & \begin{bmatrix} G_{sx} + sC_{sx} & A_{Ls} \end{bmatrix} V & -I \\ V^T \begin{bmatrix} G_{xs} + sC_{xs} \\ -A_{Ls}^T \end{bmatrix} & V^T H'V & \\ I & & \end{bmatrix} \tag{37}$$

The output equations becomes

$$y = \hat{b}^T \hat{x} = I_s \tag{38}$$

From Eqs(33)-(35) and Eq(36) and note that $V^T H' V = \hat{H}'$, we have

$$\hat{x}' = \hat{H'}^{-1} V^T \begin{bmatrix} -(G_{xs} + sC_{xs}) \\ A_{Ls}^T \end{bmatrix} V_s \tag{39}$$

and

$$I_s = (G_{ss} + sC_{ss})V_s + [G_{sx} + sC_{sx} \quad A_{Ls}]V\hat{x}' \tag{40}$$

Then, the admittance matrix $\hat{Y}$ of the transformed system is

$$\hat{Y} = G_{ss} + sC_{ss} + [G_{sx} + sC_{sx} \quad A_{Ls}]V\hat{H'}^{-1} V^T \begin{bmatrix} -(G_{xs} + sC_{xs}) \\ A_{Ls}^T \end{bmatrix} \tag{41}$$

Compared with Eq(30), it can be seen that $\hat{Y} = \hat{Y}'$, so W and V are a pair of equivalent transforms. As $\hat{Y}$ is positive real, so is $\hat{Y}'$.

3 Floating voltage sources

3.1 Compact form of MNA equations

In this section, we will talk about the case when an RLC network is driven by floating voltage sources without common terminals, and the case when floating voltage sources with common terminals will be discussed in Sec.3.3. Under this assumption, the ungrounded node set can be divided into three subsects: $N = \{N_1, N_2, N_3\}$, where N_1 contains all positive terminals of the sources, N_2 contains all negative terminals of the sources, and N_3 contains other nodes. Let n_{1i} and n_{2i} be a pair of terminals of the i-th voltage source. Then, V_n, A_v, A_L, G and C can be decomposed as follows:

$$V_n = \begin{bmatrix} V_{n1} \\ V_{n2} \\ V_{n3} \end{bmatrix} \tag{42}$$

$$A_v = \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} \tag{43}$$

where I is an $m \times m$ identity matrix.

$$A_L = \begin{bmatrix} A_{L1} \\ A_{L2} \\ A_{L3} \end{bmatrix} \tag{44}$$

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \tag{45}$$

and

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{46}$$

Note that in this case, $V_{n1} - V_{n2} = V_s$. We do variable transformation as follows. Let $V_{na} = V_{n1} - V_{n2}$ and$V_{nb} = V_{n1} + V_{n2}$, so that $V_{n1} = \frac{1}{2}[V_{na} + V_{nb}]$ and $V_{n2} = \frac{1}{2}[-V_{na} + V_{nb}]$. Let

$$z = \begin{bmatrix} V_{na} \\ V_{nb} \\ V_{n3} \\ I_L \\ I_s \end{bmatrix} \tag{47}$$

Then the vector x in Eq(1) can be expressed as

$$x = Kz \tag{48}$$

where

$$K = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I & & & \\ -\frac{1}{2}I & \frac{1}{2}I & & & \\ & & I & & \\ & & & I & \\ & & & & I \end{bmatrix} \tag{49}$$

Now, we use K as a congruence transformation on Eq(3) and obtain

$$K^T H K z = K^T b u \tag{50}$$

or

$$H_z z = b_z u \tag{51}$$

where

$$b_z = b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} \tag{52}$$

$$H_z = K^T H K \tag{53}$$

Now,

$$HK = \begin{bmatrix} G_{11}+sC_{11} & G_{12}+sC_{12} & G_{13}+sC_{13} & A_{L1} & -I \\ G_{21}+sC_{21} & G_{22}+sC_{22} & G_{23}+sC_{23} & A_{L2} & I \\ G_{31}+sC_{31} & G_{32}+sC_{32} & G_{33}+sC_{33} & A_{L3} & \\ -A_{L1}^T & -A_{L2}^T & -A_{L3}^T & sL & \\ I & -I & & & \end{bmatrix} \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I & & & \\ -\frac{1}{2}I & \frac{1}{2}I & & & \\ & & I & & \\ & & & I & \\ & & & & I \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(G_{11}-G_{12}+s(C_{11}-C_{12})) & \frac{1}{2}(G_{11}+G_{12}+s(C_{11}+C_{12})) & G_{13}+sC_{13} & A_{L1} & -I \\ \frac{1}{2}(G_{21}-G_{22}+s(C_{21}-C_{22})) & \frac{1}{2}(G_{21}+G_{22}+s(C_{21}+C_{22})) & G_{23}+sC_{23} & A_{L2} & I \\ \frac{1}{2}(G_{31}-G_{32}+s(C_{31}-C_{32})) & \frac{1}{2}(G_{31}+G_{32}+s(C_{31}+C_{32})) & G_{33}+sC_{33} & A_{L3} & \\ \frac{1}{2}(-A_{L1}^T+A_{L2}^T) & -\frac{1}{2}(A_{L1}^T+A_{L2}^T) & -A_{L3}^T & sL & \\ I & & & & \end{bmatrix}$$

and

$$H_z = K^T H K = \begin{bmatrix} \frac{1}{2}I & -\frac{1}{2}I & & & \\ \frac{1}{2}I & \frac{1}{2}I & & & \\ & & I & & \\ & & & I & \\ & & & & I \end{bmatrix} HK$$

$$
= \begin{bmatrix}
\frac{1}{4}(G_{11}+G_{22}-G_{12}-G_{21}+s(C_{11}+C_{22}-C_{12}-C_{21})) & \frac{1}{4}(G_{11}+G_{12}-G_{21}-G_{22}+s(C_{11}+C_{12}-C_{21}-C_{22})) & \frac{1}{2}(G_{13}-G_{23}+s(C_{13}-C_{23})) & \frac{1}{2}(A_{L1}-A_{L2}) & -I \\
\frac{1}{4}(G_{11}-G_{12}+G_{21}-G_{22}+s(C_{11}-C_{12}+C_{21}-C_{22})) & \frac{1}{4}(G_{11}+G_{12}+G_{21}+G_{22}+s(C_{11}+C_{12}+C_{21}+C_{22})) & \frac{1}{2}(G_{13}+G_{23}+s(C_{13}+C_{23})) & \frac{1}{2}(A_{L1}+A_{L2}) & \\
\frac{1}{2}(G_{31}-G_{32}+s(C_{31}-C_{32})) & \frac{1}{2}(G_{31}+G_{32}+s(C_{31}+C_{32})) & G_{33}+sC_{33} & A_{L3} & \\
\frac{1}{2}(-A_{L1}^T+A_{L2}^T) & -\frac{1}{2}(A_{L1}^T+A_{L2}^T) & -A_{L3}^T & sL & \\
I & & & &
\end{bmatrix} \tag{54}
$$

Let

$$
V_x = \begin{bmatrix} V_b \\ V_3 \end{bmatrix} \tag{55}
$$

and

$$
x' = \begin{bmatrix} V_x \\ I_L \end{bmatrix} \tag{56}
$$

Then, the compact MNA equations still take the form of Eq(16) with H' and b' expressed in Eq(17) and (18), respectively, where $u = V_s$,

$$
G_{xx} + sC_{xx} = \begin{bmatrix} \frac{1}{4}(G_{11}+G_{12}+G_{21}+G_{22}+s(C_{11}+C_{12}+C_{21}+C_{22})) & \frac{1}{2}(G_{13}+G_{23}+s(C_{13}+C_{23})) \\ \frac{1}{2}(G_{31}+G_{32}+s(C_{31}+C_{32})) & G_{33}+sC_{33} \end{bmatrix} \tag{57}
$$

$$
A_{Lx} = \begin{bmatrix} \frac{1}{2}(A_{L1}+A_{L2}) \\ A_{L3} \end{bmatrix} \tag{58}
$$

$$
G_{xs} + sC_{xs} = \begin{bmatrix} \frac{1}{4}(G_{11}-G_{12}+G_{21}-G_{22}+s(C_{11}-C_{12}+C_{21}-C_{22})) \\ \frac{1}{2}(G_{31}-G_{32}+s(C_{31}-C_{32})) \end{bmatrix} \tag{59}
$$

and

$$
A_{Ls} = \frac{1}{2}(A_{L1}-A_{L2}) \tag{60}
$$

The output equations still take the form of Eq(19), where

$$c^T = [\frac{1}{4}(G_{11} + G_{12} - G_{21} - G_{22} + s(C_{11} + C_{12} - C_{21} - C_{22}))$$
$$\frac{1}{2}(G_{13} - G_{23} + s(C_{13} - C_{23})) \;\; \frac{1}{2}(A_{L1} - A_{L2})] \tag{61}$$

and

$$d^T = \frac{1}{4}(G_{11} + G_{22} - G_{12} - G_{21} + s(C_{11} + C_{22} - C_{12} - C_{21})) \tag{62}$$

and Eq(23) is valid for the admittance matrix.

3.2 Congruence transformation

Note that the compact form of the MNA equations and the output equations are the same as in Sec2.1, so when a congruence matrix V is applied to matrix H', Eqs(24) - (29) are still valid. Now, for the original system, the equivalent congruence matrix takes the following form

$$P = KW \tag{63}$$

where W has the same form of Eq(31).
In order to show that P is equivalent to V, Let

$$z = W\hat{x} \tag{64}$$

where

$$\hat{x} = \begin{bmatrix} V_s \\ \hat{x}' \\ I_s \end{bmatrix} \tag{65}$$

From Eq(50), we have

$$W^T K^T H K W \hat{x} = W^T K^T bu \tag{66}$$

or

$$P^T H P \hat{x} = P^T bu \tag{67}$$

From the same derivation as in Sec2.2, it can be proved that the admittance matrix resulted from Eq(67) is the same from the transformed system from the compact system.

3.3 Floating sources with common terminals

In this case, in some rows of matrix A_v, there are several nonzero elements, and correspondingly, there are several source currents in each of the node equations.

We use a decoupling transformation to let each of the node equations have only one source current. For example, suppose

$$A_v = \begin{bmatrix} A_{vs} \\ 0 \end{bmatrix}$$

where

$$A_{vs} = \begin{bmatrix} 1 & \\ -1 & 1 \\ & -1 \end{bmatrix}$$

When A_{vs} is premultiplied by the decoupling transformation matrix

$$D_v^T = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix}$$

then

$$A'_{vs} = D_v^T A_{vs} = \begin{bmatrix} 1 & \\ & 1 \\ & -1 \end{bmatrix}$$

and

$$(A'_{vs})^T = A_{vs}^T D_v = \begin{bmatrix} 1 & & \\ & 1 & -1 \end{bmatrix}$$

Note that the first column in A'_{vs} has only one nonzero element, which corresponds to a grounded source, and the second column has a pair of +1 and -1 elements, which corresponds to a floating source. In the general case, we let $N = \{N_s, N_x\}$, where N_s contains all s terminal nodes of the voltage sources, and N_x contains the other nodes. Correspondingly,

$$V_n = \begin{bmatrix} V_s \\ V_x \end{bmatrix} \tag{68}$$

and

$$A_v = \begin{bmatrix} A_{vs} \\ 0 \end{bmatrix} \tag{69}$$

Now we form a congruence transform

$$K = \begin{bmatrix} D_v & \\ & I \end{bmatrix} \tag{70}$$

where $D_v \in R_{s \times s}$ and can be formed by using the following algorithm.

Algorithm 1: Formulation of D_v^T
{

Initially set $D_v^T = I$.
for each row i in A_{vs} with two or more nonzero elements do
{If $\exists$ j such that $A_{vs}(i,j)$ is a column singleton
then let $jj = j$;
else
select any jj s.t. $A_{vs}(i,jj) \neq 0$;
for each $k \neq jj$ with $A_{vs}(i,k) \neq 0$ and $A_{vs}(j,k) \neq 0$ do
{set $D_v^T(i,j) = 1$;
set $A_{vs}(i,k) = 0$;}
}
}

Proposition 2
When there is no voltage source loop in the circuit, Algorithm 1 will terminate in finite steps with A_{vs} having one nonzero element (+1 or -1) in each row.
Proof.
The graph of the voltage source branches forms a forest in the general case, and row i in Algorithm 1 corresponds a node in a tree of the forest which connects two or more branches. Then, $A_{vs}(i,k) \neq 0$ and $A_{vs}(j,k) \neq 0$ corresponds to the k-th branch connecting nodes i and j, and letting $A_{vs}(i,k) = 0$ corresponds to setting the branch terminal at node i to the ground node. The algorithm would not result in an A_v having only one nonzero element in each row if during its implementation there were two column singletons in one row. But this means that the original graph had loops, which violates the assumption. □
After the decoupling transformation, A_v will have m_1 columns having only one nonzero elements, and m_2 columns with a pair of nonzero elements. The first set corresponds to m_1 grounded sources and the second set to m_2 floating sources without common terminals, and the techniques described in the previous sections can be used to form the compact system.

4 Summary

In this report, we provide the compact MNA equations of an RLC network for the generation of the admittance or hybrid matrix of its reduced order model. By introduction of the concept of equivalent congruence transform, we show that the matrix generated via congruence transform on the compact system is positive real, and the passivity of the reduced order model is preserved.