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Memorandum No. UCB/ERL M98/12

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Optimal Sequences and Sum Capacity of Synchronous CDMA Systems *

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1 Introduction

An important multiple access technique in wireless networks and other common channel communication systems is Code-Division Multiple Access (CDMA). Each user shares the entire bandwidth with all the other users and is distinguished from the others by its *signature sequence* or *code*. Each user spreads its information on the common channel through modulation using its signature sequence. Then, the receiver demodulates the transmitted messages upon observing the sum of the transmitted signals embedded in noise. We focus on symbol synchronous CDMA (S-CDMA) systems where in each symbol interval the receiver observes the sum of the transmitted signals in that symbol interval alone embedded in additive white Gaussian noise (AWGN). Although complete synchronization is too idealistic, the study of such system provides fundamental limits on the best that can be achieved by practical systems.

Of fundamental interest in such a system is the capacity region defined as the set of information rates at which users can transmit while still having reliable transmission. This problem was addressed in [5] and the capacity region was characterized as a function of the signature sequences and average input-energy constraints of the users. However, the choice of the signature sequences of the users is left open to the designer of the CDMA system and it was suggested in [5] that the signature sequences could be optimized given the constraints of the problem. We address this issue and focus on finding the "sum capacity" (maximum sum of the rates of all users; maximum over all choices of signature sequences).

This problem has been attempted in part in [4] where the authors derive an upper bound on the sum capacity. This upper bound is just the capacity of the system with "no spreading",

*Research supported by the National Science Foundation under grants NCR 94-22513 and IRI 95-05561.

i.e., of the system with processing gain 1 and an appropriate power constraint on its input. It turns out that this upper bound on the sum capacity is not achievable for all values of the average input-energy constraints of the users. In this report, we completely characterize the sum capacity. Furthermore, we identify the sequences (as a function of the average input-energy constraints of the users) that achieve sum capacity and indicate a method to construct them. Let L be the processing gain of the S-CDMA system and let there be M users. We show the following:

1. $M \geq L$ is a necessary condition for sum capacity to equal that of the system with no spreading.
2. When $M \geq L$, the sum of input-energy constraint of the users being greater than the product of L and input-energy constraint of every user is a necessary and sufficient condition for sum capacity to equal that of the system with no spreading.
3. Suppose the condition mentioned above holds. Then we identify a method of constructing sequences for the users that achieve the sum capacity.

This report is organized as follows: We discuss the model of the S-CDMA system briefly and recall the characterization of the capacity region for fixed choice of signature sequences in Section 2. Our main result, the characterization of sum capacity is in Section 3.

2 S-CDMA Model and Capacity Region

We consider the discrete-time, baseband S-CDMA channel model. There are M users in the system and the processing gain is L. Both M and L will be fixed throughout this report. Since we have assumed a synchronous model we can restrict our attention to one symbol interval. As is traditional, we model the information transmitted (symbol) by each user as independent random variables $X_1, \ldots, X_M$. We assume that there is an average input-energy constraint on the transmit symbols given by

$$E\left[X_i^2\right] \leq p_i \quad \forall i = 1 \ldots M$$

Let $D = \operatorname{diag}\{p_1, \ldots, p_M\}$ and the maximum total average input-energy be $p_{tot} = \sum_{i=1}^{M} p_i$. Let the signature sequence of user i be represented by s_i, a vector in $\mathcal{R}^L$. Each signature sequence has energy equal to L i.e., for each user i, we have $s_i^t s_i = L$. We assume an ambient white Gaussian noise, denoted by $W \sim \mathcal{N}(0, N_0 I)$ independent of the transmitted symbols. Then the received signal, represented by Y, can be written as:

$$Y = \sum_{i=1}^{M} s_i X_i + W$$

Let us represent the $L \times M$ matrix $[s_1 \ldots s_M]$ by S. The S-CDMA channel above is a special case of the M user Gaussian multiple access channel and the capacity region (set of rates at which reliable communication is possible) is well known (see Section 7 of [1]) as the closure of the convex hull of the union over all product probability densities $p_{\bar{X}}$ on the inputs $X_1, \ldots, X_M$ of the rate regions

$$\mathcal{C}(S, p_{\bar{X}}) = \bigcap_{J \subseteq \{1,\ldots,M\}} \left\{ (R_1, \ldots, R_M) : 0 \leq \sum_{i \in J} R_i \leq I\left(Y; X_i, i \in J | X_i, i \in J^c\right) \right\} \quad (1)$$

Continuing as in [4], the union region over the product distributions, as a function of S, can be simply written as

$$\mathcal{C}(S) = \bigcap_{J \subseteq \{1,\ldots,M\}} \left\{ (R_1, \ldots, R_M) : 0 \leq \sum_{i \in J} R_i \leq \frac{1}{L} \log \left[\det \left(I + \frac{1}{N_0} S_J D_J S_J^t \right) \right] \right\} \quad (2)$$

where J is a non-null set and R_i is the rate in bits/chip of user i. Also, S_J is $L \times |J|$ matrix $[s_i : i \in J]$ and D_J is the $|J| \times |J|$ matrix diag $\{p_i : i \in J\}$. Observe that the choice of product distribution for each user i, X_i is distributed as $\mathcal{N}(0, p_i)$ makes the region in (1) equal to that in (2). The sum capacity represents the maximum sum of rates of all users at which users can transmit reliably. Following the notation in [4], the sum capacity is defined formally as

$$\mathcal{C}_{\text{sum}} = \max_{S \in \mathcal{S}} \max_{R \in \mathcal{C}(S)} \sum_{i=1}^{M} R_i \quad (3)$$

where $\mathcal{S}$ is the set of all $L \times M$ real matrices with all columns having l_2 norm equal to $\sqrt{L}$.

3 Sum Capacity

In this section, we shall characterize $\mathcal{C}_{\text{sum}}$. In [4] an upper bound was derived for $\mathcal{C}_{\text{sum}}$, this upper bound being the sum capacity of the "unrestricted" S-CDMA channel, i.e., the situation of no spreading when L = 1 with the appropriate power constraint on its input. The latter channel is just the K-user Gaussian multiple-access channel and its sum capacity is $\log\left(1 + \frac{p_{tot}}{N_0}\right)$ (see Chapter 7 in [1]) in bits/chip. When the input-energy constraints are equal, it was shown in [4] that WBE signature sequences maximize the sum capacity, and the sum capacity then equals the upper bound mentioned above. WBE sequences are so called because they meet the Welch-Bound-Equality (see [7]).

However, we will show that this bound is not tight for arbitrary choices of the average input-energy constraints of the users and indeed there is a loss in sum capacity when the energy constraints are "far apart"; we shall make this notion precise. We also indicate a method to construct these signature sequences.

When $M \leq L$, it is easy to verify that the sum capacity is maximized when the signature sequences are chosen orthogonal to each other and we have $\mathcal{C}_{\text{sum}} = \frac{1}{L} \sum_{i=1}^{M} \log\left(1 + \frac{Lp_i}{N_0}\right)$. Note that when $M = L$ and all the energy constraints p_i are the same, $\mathcal{C}_{\text{sum}} = \log\left(1 + \frac{p_{tot}}{N_0}\right)$

the same as the sum capacity of the system with no spreading; this is the well known fact that for equal-power users, orthogonal multiple access incurs no loss in capacity relative to unconstrained multiple access. When $M < L$ the claim is that $\mathcal{C}_{\text{sum}} < \log\left(1 + \frac{p_{tot}}{N_0}\right)$. To see this, observe that $(p_1, \ldots, p_M)$ majorizes the vector $\left(\frac{p_{tot}}{M}, \ldots, \frac{p_{tot}}{M}\right)$ (see Appendix A for the notation) It is verified that the map $(p_1, \ldots, p_M) \mapsto \frac{1}{L}\sum_{i=1}^{M} \log\left(1 + \frac{Lp_i}{N_0}\right)$ is Schur-concave (see Appendix A for the notation). Hence,

$$\begin{aligned}
\mathcal{C}_{\text{sum}} &= \frac{1}{L}\sum_{i=1}^{M}\log\left(1+\frac{Lp_i}{N_0}\right) \\
&\leq \frac{M}{L}\log\left(1+\frac{Lp_{tot}}{MN_0}\right) \\
&< \log\left(1+\frac{p_{tot}}{N_0}\right)
\end{aligned}$$

where in the last step we used the inequality $(1+x)^a < 1 + ax$ for $x > 0$ and $a \in (0,1)$.

For the case $M > L$, $\mathcal{C}_{\text{sum}}$ is characterized below. This is our main result:

Theorem 3.1 *Let $M > L$. Let (without loss of generality) $p_1 \geq \ldots \geq p_M$. Then,*

$$\mathcal{C}_{sum} = \frac{1}{L}\sum_{i=1}^{L}\log\left(1+\frac{L\lambda_i^*}{N_0}\right) \tag{4}$$

where $\lambda^* = (\lambda_1^*, \ldots, \lambda_L^*) \in \mathcal{R}_+^L$ is given by

$$\lambda^* = \begin{cases} \left(\frac{p_{tot}}{L}, \ldots, \frac{p_{tot}}{L}\right) & \text{if } p_{tot} \geq Lp_1 \\ \left(p_1, \ldots, p_k, \frac{p_{tot}-\sum_{j=1}^{k} p_j}{L-k}, \ldots, \frac{p_{tot}-\sum_{j=1}^{k} p_j}{L-k}\right) & \forall k = 1 \ldots L-1 \\ \text{if } \sum_{j=1}^{k-1} p_j + (L-k+1)\, p_k > p_{tot} \geq \sum_{j=1}^{k} p_j + (L-k)\, p_{k+1} & \end{cases} \tag{5}$$

Proof First, observe that λ^* is well defined by the expression in (5). Now,

$$\begin{aligned}
\mathcal{C}_{\text{sum}} &= \max_{S\in\mathcal{S}} \max_{R\in\mathcal{C}(S)} \sum_{i=1}^{M} R_i \\
&= \max_{S\in\mathcal{S}} \frac{1}{L}\log\left[\det\left(I + \frac{1}{N_0}S_J D_J S_J^t\right)\right] \quad \text{from (2), also see [4]} \\
&= \max_{S\in\mathcal{S}} \frac{1}{L}\sum_{i=1}^{L}\log\left(1+\frac{L}{N_0}\lambda_i(S)\right) \qquad (6)
\end{aligned}$$

where $\lambda(S) = (L\lambda_1(S), \ldots, L\lambda_L(S)) \in \mathcal{R}_+^L$ denotes the vector of eigenvalues of the matrix SDS^t arranged in descending order. Define the convex set $\mathcal{L}$ in the positive orthant of $\mathcal{R}^L$ by $\mathcal{L} = \left\{(\lambda_1, \ldots, \lambda_L) \in \mathcal{R}_+^L : (\lambda_1, \ldots, \lambda_L, 0, \ldots, 0) \text{ majorizes } (p_1, \ldots, p_M)\right\}$. We first identify the region of eigenvalues of the matrix $\frac{1}{L}SDS^t$ as S varies in $\mathcal{S}$ to be exactly $\mathcal{L}$. Formally, we claim that

$$\left\{\frac{1}{L}\lambda(S) : S \in \mathcal{S}\right\} = \mathcal{L} \tag{7}$$

First consider $S \in \mathcal{S}$. Let $\tilde{\lambda}(S) \in \mathcal{R}_+^M$ be the vector of eigenvalues of the matrix $D^{\frac{1}{2}}S^tSD^{\frac{1}{2}}$ and observe that $\tilde{\lambda}(S)$ is just the vector $\lambda(S)$ with $M-L$ appended zeros. The observation that the diagonal elements of $\frac{1}{L}D^{\frac{1}{2}}S^tSD^{\frac{1}{2}}$ are $p_1,\ldots,p_M$ coupled with an appeal to Lemma A.1, allows us to conclude that $\frac{1}{L}\lambda(S) \in \mathcal{L}$. To see the other direction, consider $\lambda = (\lambda_1,\ldots,\lambda_L) \in \mathcal{L}$. Then, by definition, the vector $(\lambda_1,\ldots,\lambda_L,0,\ldots,0)$ majorizes the vector $(p_1,\ldots,p_M)$. Appealing to Lemma A.2, there exists a matrix H with eigenvalues $\lambda_1,\ldots,\lambda_L,0,\ldots,0$ and diagonal elements $p_1,\ldots,p_M$. Let $v_1,\ldots,v_L \in \mathcal{R}^M$ be the normalized eigenvectors of H corresponding to the eigenvalues $\lambda_1,\ldots,\lambda_L$. Let $V^t = [v_1v_2\ldots v_L]$. If we let Λ to be the diagonal matrix with entries $\lambda_1,\ldots,\lambda_L$, then $H = V^t\Lambda V$. Now define $S = \sqrt{L}\Lambda^{\frac{1}{2}}VD^{-\frac{1}{2}}$. Then, since the square of the l_2 norms of the columns of S are the diagonal elements of S^tS, we verify that $S^tS = LD^{-\frac{1}{2}}HD^{-\frac{1}{2}}$ has diagonal entries equal to L concluding that $S \in \mathcal{S}$. This completes the proof of the claim in (7).

Then the sum capacity can be rewritten as, from (6),

$$\mathcal{C}_{\text{sum}} = \max_{\lambda\in\mathcal{L}}\sum_{i=1}^{L}\log\left\{1+\frac{L}{N_0}\lambda_i\right\} \tag{8}$$

The following lemma identifies a "minimal" element in L:

Lemma 3.1 *Let $\mathcal{L} = \left\{(\lambda_1,\ldots,\lambda_L) \in \mathcal{R}_+^L : (\lambda_1,\ldots,\lambda_L,0,\ldots,0) \text{ majorizes } (p_1,\ldots,p_M)\right\}$. Then λ^* defined in (5) is a Schur-minimal element of $\mathcal{L}$, i.e., if $\lambda \in \mathcal{L}$, then λ majorizes λ^*.*

Suppose this is true. As observed earlier, the map $(\lambda_1,\ldots,\lambda_L) \mapsto \frac{1}{L}\sum_{i=1}^{L}\log\left(1+\frac{L\lambda_i}{N_0}\right)$ is Schur-concave. Then the proof is complete by an appeal to the lemma above. We only need to prove the lemma.

Proof of Lemma 3.1:
It is straightforward from the definition of λ^* in (5) that $\lambda_1^* \geq \lambda_2^* \geq \ldots \geq \lambda_L^*$ and hence that $\lambda^* \in \mathcal{L}$. Let $\lambda = (\lambda_1,\ldots,\lambda_L) \in \mathcal{L}$ and let $\lambda_1 \geq \ldots \geq \lambda_L$. By the definition of λ^* in (5), it can be verified that the following relation is true among the elements of λ^*:

$$\begin{aligned}\lambda_1^* &= \max\left\{\frac{p_{tot}}{L}, p_1\right\} \\ \lambda_{k+1}^* &= \max\left\{\frac{p_{tot}-\sum_{i=1}^{k}\lambda_i^*}{L-k}, p_{k+1}+\sum_{i=1}^{k}(p_i-\lambda_i^*)\right\} \; \forall k = 1\ldots L-1\end{aligned} \tag{9}$$

Hence $\forall k = 1\ldots L-1$ we can write

$$\sum_{i=1}^{k+1}\lambda_i^* = \max\left\{\sum_{i=1}^{k+1}p_i, \frac{p_{tot}}{L-k}+\frac{L-k-1}{L-k}\sum_{i=1}^{k}\lambda_i^*\right\} \tag{10}$$

Now, since $\lambda \in \mathcal{L}$ we have $\sum_{i=1}^{L} \lambda_i = p_{tot}$ and hence $\lambda_1 \geq \frac{p_{tot}}{L}$. Furthermore, $\lambda_1 \geq p_1$. Hence,

$$\lambda_1 \geq \max\left\{\frac{p_{tot}}{L}, p_1\right\} = \lambda_1^*$$

We shall complete the proof of the claim that λ majorizes λ^* by induction. Suppose $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \lambda_i^*$ for some $1 \leq k < L$. Since $\sum_{i=1}^{L-k} \lambda_{k+i} = p_{tot} - \sum_{i=1}^{k} \lambda_i$ and $\lambda_{k+1} \geq \lambda_{k+2} \geq \ldots \geq \lambda_L$, we have $\lambda_{k+1} \geq \frac{p_{tot} - \sum_{i=1}^{k} \lambda_i}{L-k}$. Hence

$$\begin{aligned}
\sum_{i=1}^{k+1} \lambda_i &\geq \frac{p_{tot}}{L-k} + \left(\frac{L-k-1}{L-k}\right) \sum_{i=1}^{k} \lambda_i \\
&\geq \frac{p_{tot}}{L-k} + \left(\frac{L-k-1}{L-k}\right) \sum_{i=1}^{k} \lambda_i^* \text{ by induction hypothesis} \qquad (11)
\end{aligned}$$

Since $\sum_{i=1}^{k+1} (\lambda_i - p_i) \geq 0$, from (11), we have

$$\begin{aligned}
\sum_{i=1}^{k+1} \lambda_i &\geq \max\left\{\sum_{i=1}^{k+1} p_i, \frac{p_{tot}}{L-k} + \frac{L-k-1}{L-k} \sum_{i=1}^{k} \lambda_i^*\right\} \\
&= \sum_{i=1}^{k+1} \lambda_i^* \quad \text{from (10)}
\end{aligned}$$

This is true for all $k = 1 \ldots L-1$. Hence λ majorizes λ^* and λ^* is a Schur-minimal element of $\mathcal{L}$. This completes the proof of the lemma and hence that of the theorem. ∎

As mentioned earlier, a trivial upper bound for $\mathcal{C}_{\text{sum}}$ is $\log\left(1 + \frac{p_{tot}}{N_0}\right)$ the sum capacity of the channel with "no spreading". We saw that when $M = L$ and all the energy constraints are equal, then $\mathcal{C}_{\text{sum}}$ actually equaled this upper bound and was strictly less than this bound when $M < L$. For the case $M > L$, we shall now identify, as a corollary of Theorem 4.1, necessary and sufficient conditions on the input-energy constraints such that $\mathcal{C}_{\text{sum}}$ will equal this upper bound. We shall also identify the signature sequences for the users to achieve this sum capacity.

Corollary 3.1 *Let $M > L$. Then $\mathcal{C}_{sum} = \log\left(1 + \frac{p_{tot}}{N_0}\right)$ if and only if $p_{tot} \geq Lp_i$ for all $i = 1 \ldots M$.*

The proof is obvious from Theorem 3.1.

We now consider identification and construction of signature sequences of the users to achieve sum capacity. The general scheme to identify the sequences that achieve the sum capacity was outlined in the proof of Theorem 3.1, and we shall repeat that here: Suppose $p_{tot} \geq Lp_i$ for all $i = 1 \ldots M$. Then, $(p_{tot}, \ldots, p_{tot}, 0, \ldots, 0)$ majorizes the vector $(Lp_1, \ldots, Lp_M)$. Appealing to Lemma A.2, there exists a $M \times M$ symmetric matrix H such that its diagonal entries are $Lp_1, Lp_2, \ldots, Lp_M$ and it has L eigenvalues (of multiplicity both algebraic and geometric) equal to p_{tot} and $M-L$ null eigenvalues. Let $v_1, \ldots v_L$ be the normalized eigenvectors of H corresponding to the eigenvalue p_{tot} and denote $V^t = [v_1 \ldots v_L]$. Then we can write $H = p_{tot} V^t V$. As before, let $D = \text{diag}\{p_1, \ldots, p_M\}$. Define the $L \times M$ matrix $S = \sqrt{p_{tot}} V D^{-\frac{1}{2}}$. Since the diagonal entries of $S^t S$ are all equal to L, we have $S \in \mathcal{S}$. Furthermore L eigenvalues of $D^{\frac{1}{2}} S^t S D^{\frac{1}{2}} = H$ are p_{tot} and $M-L$ eigenvalues are null (notice that, by construction, $SDS^t = p_{tot} I$). Hence for this choice of signature sequences S, we have, from (8), that $\mathcal{C}_{\text{sum}} = \log\left(1 + \frac{p_{tot}}{N_0}\right)$. The proof of Lemma A.2 indicates a method to construct the matrix H from which S can be obtained.

In general, suppose $\sum_{j=1}^{k-1} p_j + (L-k+1) p_k > p_{tot} \geq \sum_{j=1}^{k} p_j + (L-k) p_{k+1}$ for some $k \in \{1, 2, \ldots, L-1\}$ and that $p_1 \geq p_2 \geq \ldots \geq p_M$. Then, for $i = 1, \ldots, k$ we let s_i be orthogonal sequences. The remaining signature sequences are now constructed as above with M replaced by $M-k$ and L replaced by $L-k$ and the $M-k$ users have power constraints $p_{k+1}, \ldots, p_M$. Now these $M-k$ users have signature sequences chosen in the subspace $\left(span\,(s_1, \ldots, s_k)^{\perp}\right)$ which has dimension $L-k$.

In the special case when all the energy-constraints are equal (to say p), then $p_{tot} = Mp \geq Lp$ and hence $\mathcal{C}_{\text{sum}} = \log\left(1 + \frac{Mp}{N_0}\right)$. Furthermore, the sequence matrix S constructed above from H now satisfies the relation $SS^t = MI$. This result was observed in [4] and the sequences that meet this constraint were denoted WBE sequences (such sequences were also identified in [3]). However, WBE sequences had been constructed only for special values of M and L. Recently, in [6], the authors prove the existence of WBE sequences and provide a simple method to construct them for arbitrary choices of $M \geq L$.

The corollary above says that even if the input-energy constraints are not equal, so long as they are not too "far apart" there is no loss in sum capacity. However, in the asymmetric energy-constraints situation, the sequences that achieve the sum capacity are no longer WBE sequences. When the sequences are far apart, then the optimal choice of sequences is to choose signature sequences orthogonal for those users with the "most relaxed" input power constraint as made precise above.

A Definitions and Relevant Results from Theory of Majorization

In this appendix we collect together relevant definitions and results from the theory of majorization. All of these results can be found in the comprehensive reference on majorization [2]. Majorization makes precise the vague notion that the components of a vector x are "less spread out" or "more nearly equal" than are the components of a vector y by the statement x is majorized by y.

For any $x = (x_1, \ldots, x_n) \in \mathcal{R}^n$, let

$$x_{[1]} \geq \cdots \geq x_{[n]}$$

denote the components of x in decreasing order. For $x, y \in \mathcal{R}^n$, define

$$x \prec y \quad \text{if} \quad \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, & k = 1 \ldots n-1 \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \end{cases}$$

When $x \prec y$ say that x is majorized by y (or y majorizes x). An important though trivial example of majorization is

$$\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) \prec (a_1, \ldots, a_n) \tag{12}$$

for every $a \in \mathcal{R}^n$ such that $\sum_{i=1}^{n} a_i = 1$. An important characterization of majorization is the result that $x \prec y$ if and only if there exists a doubly stochastic matrix P such that $x = yP$.

A real valued function $\phi : \mathcal{R}^n \to \mathcal{R}$ is said to be Schur-concave if for all $x, y \in \mathcal{R}^n$ such that $x \prec y$ we have $\phi(x) \geq \phi(y)$. ϕ is said to be Schur-convex if $-\phi$ is Schur-concave. Using the observation in (12), for any Schur-concave function ϕ and for any vector $x \in \mathcal{R}^n$

$$\phi(\bar{x}) \geq \phi(x)$$

where $\bar{x} = \left(\frac{\sum_{i=1}^{n} x_i}{n}, \frac{\sum_{i=1}^{n} x_i}{n}, \ldots, \frac{\sum_{i=1}^{n} x_i}{n}\right)$. A well known structure of Schur-convex functions is the following result (Theorem 3.C.1 in [2]): If $g : \mathcal{R} \to \mathcal{R}$ is convex then the symmetric convex function $\phi(x) = \sum_{i=1}^{n} g(x_i)$ is Schur-convex. It is obvious that if $g : \mathcal{R} \to \mathcal{R}$ is concave then the symmetric concave function $\phi(x) = \sum_{i=1}^{n} g(x_i)$ is Schur-concave.

It is well known that the sum of diagonal elements of a matrix is equal to the sum of its eigenvalues. When the matrix is symmetric the *precise* relationship between the diagonal elements and the eigenvalues is that of majorization: (Theorem 9.B.1, [2]).

Lemma A.1 *Let H be a $n \times n$ symmetric matrix with diagonal elements $h_1, \ldots, h_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $h \prec \lambda$*

That h and λ cannot be compared by an ordering stronger than majorization is the consequence of the following converse (Theorem 9.B.2, [2]):

Lemma A.2 *If $h_1 \geq \cdots \geq h_n$ and $\lambda_1 \geq \cdots \lambda_n$ are $2n$ numbers satisfying $h \prec \lambda$ in $\mathcal{R}^n$, then there exists a real symmetric matrix H with diagonal elements $h_1, \ldots, h_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$.*

The following interlacing lemma captures the interconnection between the eigenvalues of a symmetric matrix and those of its principal submatrix. (p 219, [2]).

Lemma A.3 *Let H be a $n \times n$ symmetric matrix and $\tilde{H}$ be a $n-1 \times n-1$ principal submatrix of H. Let the eigenvalues of H be $\lambda_1 \geq \ldots \geq \lambda_n$ and those of $\tilde{H}$ be $\tilde{\lambda}_1 \geq \ldots \geq \tilde{\lambda}_{n-1}$. Then, $\lambda_1 \geq \tilde{\lambda}_1 \geq \ldots \geq \lambda_{n-1} \geq \tilde{\lambda}_{n-1} \geq \lambda_n$.*

That the converse of this is true in some sense is made precise by the following (Lemma 9.B.3 in [2]):

Lemma A.4 *Given real numbers $c_1, \ldots, c_{n-1}$ and $\lambda_1, \ldots, \lambda_n$ satisfying the interlacing property*

$$\lambda_1 \geq c_1 \geq \lambda_2 \geq c_2 \geq \cdots \geq c_{n-1} \geq \lambda_n,$$

there exists a real symmetric $n \times n$ matrix of the form

$$W = \begin{bmatrix} D_c & v^t \\ v & v_n \end{bmatrix}$$

with eigenvalues $\lambda_1, \ldots, \lambda_n$ where D_c is a diagonal matrix with diagonal elements $c_1, \ldots, c_{n-1}$.

The proofs of Lemmas A.2 and A.4 are constructive in nature and they provide a scheme of constructing the symmetric matrix with the claimed properties.

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