PASSIVE MODEL ORDER REDUCTION ALGORITHM FOR MULTIPOINT MOMENT MATCHING OF MULTIPORT DISTRIBUTED INTERCONNECT NETWORKS

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Passive model order reduction algorithm for multipoint moment matching of multiport distributed interconnect networks

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Abstract

In this paper, we introduce a general method of interconnect simulation based on distributed circuits. The algorithm is very efficient and consists of two main steps. In the first step, ecah distributed line is modeled by a finite order system with passivity preservation and multipoint moment matching of its input admittance/impedance matrix. In the second step, an Arnoldi-based congruence transform is applied to the network to form its reduced order model. The main feature of the algorithm is in its first step, where a passive multipoint moment matching model of a distributed line can be generated without any discretization of the line. We provide an integrated-congruence transform which can be directly applied to the partial differential equations of a distributed line and generate a passive finite order system as its model. We also provide an algorithm based on the L^2 Hilbert space theory so that exact moment matching at multiple points can be obtained. We demonstrate the accuracy of our method with examples and show the advantage of ours over conventional ones based on lumped circuit models.

1 Introduction

With the rapid increase of signal frequency and decrease of feature sizes of high speed electronic circuits, interconnect has become a dominating factor in determining circuit performance and reliability in deep submicron designs. Not only interconnect delay has dominated gate delay, but also transmission line effects, such as reflection, distortion, dispersion and crosstalk, have severe impact on circuit performance. In recent years, interconnect modeling, simulation and design has become a hot topic in the research of advanced CAD techniques [1].

There are two kinds of interconnect models: lumped and distributed. The interconnects on chip are mostly modeled as RC circuits in today's technology, but it has been found [2] that long wires (data buses, control and clock wires, etc.) on chip need to be modeled as RLC circuits. The interconnects among chips and on the printed circuit board are modeled as RLGC circuits or transmission lines. As interconnects are distributed in nature, so an accurate model of an interconnect should be a distributed network, and a lumped model is only its approximation.

In the case of a lumped model of an interconnect, in order to approximate its distributed nature, the order of the model is usually very high, and the formation of a reduced order model is a key to the fast simulation and optimal design of large interconnect circuits. In the case of a distributed model, the circuit is described by a set of partial differential equations. Though a number of numerical methods exist to directly perform the time-domain analysis of transmission line networks [3], they are not as efficient as the frequency domain analysis methods when the network is large. As a distributed network is of infinite order in the frequency domain description, a reduced order model with a finite low order is needed so that efficient simulation and analysis can be done.

The reduced order model should meet with two basic requirements. First, as interconnects themselves are passive in nature, its model should be passive to guarantee the stability. Second, the model should maintain to some extent the characteristics of the original circuit. In the frequency domain, moment matching methods are widely used to form the model. Experiments have shown that if the model keeps moment matching at suitable points with suitable orders, the model will perform well in both frequency and time domains.

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In the past decade, a number of papers have been published. AWE [5] was first introduced to use *Padé* approximation to do moment matching for a transfer function of a linear lumped circuit. This method has been intensively studied and extended [6]- [10]. About the same time, the use of *Padé* approximation to deal with the modeling of a transmission line was introduced [11]. This method and the extension of moment matching method from lumped circuits to distributed circuits have also been extensively studied [12]-[20]. There are two problems related to these moment matching methods. First, *Padé* approximation cannot guarantee the stability of the model in the general case, and among the models developed in these papers, only very few of them are passive [17, 18]. Second, when the moment matching order is high, numerical ill-conditioning problems occur. To overcome the numerical ill-conditioning problem, *Padé Via Lanczos* (PVL) was introduced [21], which was later extended to $MPVL$ and $Sy - PVL$ [22, 23]. As PVL and $MPVL$ are *Padé* based algorithms, they also have the instability problem when dealing with general RLC circuits, but it has been shown that when $Sy - PVL$ is used for two-kind element passive circuits (RC, RL or LC), the reduced order model is passive. [24] was the first to use the congruence transform to guarantee the passivity of the reduced order model of RC circuits, and the Arnoldi-based congruence transform has been successfully extended to the passive model-order reduction algorithms for general RLC circuits [25]-[28].

Since the successful use of the Lanczos and/or Arnoldi algorithms to do model-order reduction for lumped RLC circuits, an attempt to extend the use of these algorithms to distributed circuits has been made. In [29], a transmission line is first discretized by using a spectral approximation, then the PVL algorithm is applied. The stability of the reduced order model is not guaranteed because of the discretization method and the *Padé* approximation. In [30], we provided another algorithm, where a transmission line is first discretized by a number of RLGC sections, then the Arnoldi algorithm is applied. This algorithm can guarantee the passivity of the reduced order model and provide multipoint moment matching. Note that in both algorithms, discretization for a transmission line is the first step. This is because that both Lanczos and Arnoldi algorithms can only be applied to finite order systems. But discretization is not ideal either in theory or in practice, and because of this step, the moment matching model generated cannot be exact. The development of an algorithm to do model order reduction for distributed networks without any discretization has interested many researchers. To the best of our knowledge, there has been no such an algorithm published in the literature.

In this paper, we provide a new algorithm to do model order reduction of distributed interconnect networks. The distributed network may consist of any number of RC lines and RLGC transmission lines together with some linear passive lumped elements (such as capacitors, etc.). Our algorithm consists of two main steps. In the first step, each RC or RLGC line is modeled by a finite order system with passivity preservation and multipoint moment matching of its input admittance/impedance matrix. In the second step, an Arnoldi-based congruence transform is applied to the network to form its reduced order model. Our algorithm can guarantee the passivity of the reduced order model and provide the multi-point moment matching as required. The main contribution of this paper is that we provide a passive reduced order model algorithm for RC and RLGC lines with multipoint moment matching without any discretization of the lines first. We developed an integrated-congruence transform, which can be directly applied to the partial differential equations of a line and generate a passive finite order system so that the discretization step can be eliminated. To meet with the moment matching requirement, we extend the Krylov subspace algorithm for finite dimensional space to an approach based on the L^2 Hilbert space theory [31] , which can provide an exact moment matching model. Our algorithm is new, and experiments show it works well in practice.

The rest of the paper is organized as follows. In Sec.2, we provide the integrated-congruence transform and prove the passivity of the reduced order model. In Sec.3, we provide an algorithm for multipoint moment matching which is based on the L^2 Hilbert space theory. These two sections are mainly devoted to RLGC transmission lines, and can be applied to RC lines, too. In Sec.4, we provide a more efficient algorithm specially dealing with RC lines. In Sec.5, we provide a passive model order reduction algorithm with multi-point moment matching for a general distributed passive network. We give examples and conclusions in Sec.6 and 7, respectively.

2 Integrated-congruence transform and passivity preservation

2.1 General form of distributed line equations

For ease of understanding, we begin with single uniform line, and extend the result to multi-conductor lines and nonuniform lines.

Suppose that R', L', G' and C' are the resistance, inductance, conductance and capacitance per unit length of a line, respectively, and d is its length. We first normalize the length to 1, and let $R = R'd$, $L = L'd$, $G = G'd$ and $C = C'd$ be the normalized resistance, inductance, conductance and capacitance per unit length, respectively, of the normalized line. Let z be the axis along the line, and $z = 0$ and $z = 1$ correspond to its near and far end, respectively. Let $i(z,t)$ and $v(z,t)$ be the current and voltage along the line. The partial differential equations of the line are as follows:

$$\frac{\partial v(z,t)}{\partial z} = -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t}$$
$$\frac{\partial i(z,t)}{\partial z} = -Gv(z,t) - C\frac{\partial v(z,t)}{\partial t} \tag{1}$$

When we are dealing with the input admittance of the line, it is assumed that two voltage sources $v_{s1}(t)$ and $v_{s2}(t)$ are applied to the two ends of the line, so the boundary conditions of the line equations are

$$\begin{bmatrix} v(0,t) \\ v(1,t) \end{bmatrix} = v_s(t) = \begin{bmatrix} v_{s1}(t) \\ v_{s2}(t) \end{bmatrix} \tag{2}$$

Apply the Laplace transform on the equations. Let $V(z,s) = \mathcal{L}[v(z,t)]$, $I(z,s) = \mathcal{L}[i(z,t)]$, $V_{s1}(s) = \mathcal{L}[v_{s1}(t)]$ and $V_{s2}(s) = \mathcal{L}[v_{s2}(t)]$, and assume that $v(z,0_-) = 0$ and $i(z,0_-) = 0$ for $z \in [0,1]$, we have the differential equations in the frequency domain as follows:

$$\frac{dV(z,s)}{dz} = -(R + sL)I(z,s)$$
$$\frac{dI(z,s)}{dz} = -(G + sC)V(z,s) \tag{3}$$

with the boundary conditions

$$\begin{bmatrix} V(0,s) \\ V(1,s) \end{bmatrix} = V_s(s) = \begin{bmatrix} V_{s1}(s) \\ V_{s2}(s) \end{bmatrix} \tag{4}$$

We rewrite Eq(3) in the following form:

$$(sM + N + T\frac{d}{dz})X(z,s) = 0 \tag{5}$$

where

$$X(z,s) = \begin{bmatrix} I(z,s) \\ V(z,s) \end{bmatrix} \tag{6}$$

$$M = \begin{bmatrix} L & \\ & C \end{bmatrix} \tag{7}$$

$$N = \begin{bmatrix} R & \\ & G \end{bmatrix} \tag{8}$$

and

$$T = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \tag{9}$$

Eq(5) is the general form of line equations. In the case of an RLGC coupled line system with m lines, $I(z,s)$ and $V(z,s)$ are m- dimensional vectors, R, L, G and C are $m \times m$ dimensional matrices, $X(z,s)$ is a $2m$ dimensional vector, and M, N and T are $2m \times 2m$ matrices, where the element 1 in Eq(9) is replaced by an $m \times m$ identity matrix. The boundary conditions remain in the form of Eq(4), where $V_{s1}(s)$ and $V_{s2}(s)$ are m- dimensional vectors. In the case of a nonuniform line, R, L, G, C, M and N are functions of variable z. For an RC line, G and L are null matrices. Note that for all practical lines, M and N are non-negative definite matrices.

In the later discussion, we assume that m nonuniform coupled lines are considered, which includes single lines and uniform lines as special cases.

2.2 Integrated-congruence transform

Suppose that $u(z)$ is a $2m \times n$ matrix, which can be expressed as

$$u(z) = \begin{bmatrix} u_1(z) & u_2(z) & \dots & u_n(z) \end{bmatrix} = \begin{bmatrix} u_i(z) \\ u_v(z) \end{bmatrix} = \begin{bmatrix} u_{i,1}(z) & u_{i,2}(z) & \dots & u_{i,n}(z) \\ u_{v,1}(z) & u_{v,2}(z) & \dots & u_{v,n}(z) \end{bmatrix} \tag{10}$$

where $u_j(z)$ is its j-th column vector, which consists of two subvectors

$$u_j(z) = \begin{bmatrix} u_{i,j}(z) \\ u_{v,j}(z) \end{bmatrix} \tag{11}$$

each of which has m components, i.e.,

$$u_{i,j}(z) = [u_{i,1j}(z), u_{i,2j}(z), \dots, u_{i,mj}(z)]^T \tag{12}$$

and

$$u_{v,j}(z) = [u_{v,1j}(z), u_{v,2j}(z), \dots, u_{v,mj}(z)]^T \tag{13}$$

$u_i(z)$ and $u_v(z)$ are two $m \times n$ submatrices of matrix u. Note that matrix u is a function of variable z but not a function of s.
Let

$$X(z,s) = u(z)\hat{x}(s) \tag{14}$$

where vector $\hat{x}(s) = [\hat{x}_1(s), \hat{x}_2(s), \dots, \hat{x}_n(s)]^T$ is of dimension n. Substitute Eq(14) to Eq(5), premultiply $u^T(z)$ on both sides of the equations and integrate them w.r.t. variable z from 0 to 1, we obtain

$$(s\hat{M} + \hat{N}_1 + \hat{T})\hat{x}(s) = 0 \tag{15}$$

where

$$\hat{M} = \int_0^1 u^T(z)M(z)u(z)dz \tag{16}$$

$$\hat{N}_1 = \int_0^1 u^T(z)N(z)u(z)dz \tag{17}$$

and

$$\hat{T} = \int_0^1 u^T(z)T\frac{du(z)}{dz}dz \tag{18}$$

The transform from Eq(5) to Eq(15) is called an integrated-congruence transform (w.r.t. the transformation matrix $u(z)$). Note that the order of the system described by Eq(15) is n, and $\hat{x}(s)$ can be regarded as the state vector of the finite order system.
We expand the expressions of matrices $\hat{M}$, $\hat{N}_1$ and $\hat{T}$ for computational purposes and combine Eq(15) with the boundary conditions to generate a typical form of the state equations of state vector $\hat{x}(s)$.
It is easy to show that

$$\hat{M} = \int_0^1 (u_i^T(z)L(z)u_i(z) + u_v^T(z)C(z)u_v(z))dz \tag{19}$$

$$\hat{N}_1 = \int_0^1 (u_i^T(z)R(z)u_i(z) + u_v^T(z)G(z)u_v(z))dz \tag{20}$$

and

$$\hat{T} = \int_0^1 (u_i^T(z)\frac{du_v(z)}{dz} + u_v^T(z)\frac{du_i(z)}{dz})dz \tag{21}$$

We divide $\hat{T}$ into two matrices: $\hat{T} = \hat{N}_2 + P$, where

$$P = \int_0^1 \frac{d(u_i^T(z)u_v(z))}{dz}dz = u_i^T(1)u_v(1) - u_i^T(0)u_v(0) \tag{22}$$

and

$$\hat{N}_2 = \int_0^1 (u_i^T(z)\frac{du_v(z)}{dz} + u_v^T(z)\frac{du_i(z)}{dz})dz - \int_0^1 \frac{d(u_i^T(z)u_v(z))}{dz}dz$$

$$= \int_0^1 (u_v^T(z)\frac{du_i(z)}{dz} - \frac{du_i^T(z)}{dz}u_v(z))dz \tag{23}$$

Note that

$$\hat{N}_2^T = -\hat{N}_2 \tag{24}$$

Let $\hat{N} = \hat{N}_1 + \hat{N}_2$, then Eq(15) becomes

$$(s\hat{M} + \hat{N})\hat{x}(s) = -P\hat{x}(s) \tag{25}$$

From Eq(22),

$$-P\hat{x}(s) = u_i^T(0)u_v(0)\hat{x}(s) - u_i^T(1)u_v(1)\hat{x}(s)$$

and from Eq(14),

$$u_v(0)\hat{x}(s) = V(0,s) = V_{s1}(s)$$

and

$$u_v(1)\hat{x}(s) = V(1,s) = V_{s2}(s)$$

so that

$$-P\hat{x}(s) = bV_s(s) \tag{26}$$

where

$$b = \begin{bmatrix} u_i(0) \\ -u_i(1) \end{bmatrix}^T \tag{27}$$

Substitute Eq(26) to Eq(25), we have the state equations for the state vector $\hat{x}$ as follows:

$$(s\hat{M} + \hat{N})\hat{x}(s) = \hat{b}V_s(s) \tag{28}$$

When we are interested in the input admittance of the line, the output vector $y(s)$ consists of $I(0,s)$ and $-I(1,s)$. From Eq(14), we have

$$y(s) = \begin{bmatrix} I(0,s) \\ -I(1,s) \end{bmatrix} = \begin{bmatrix} u_i(0) \\ -u_i(1) \end{bmatrix} \hat{x}(s) \tag{29}$$

Compared with Eq(27), it can be seen that

$$y(s) = \hat{b}^T \hat{x}(s) \tag{30}$$

and the input admittance matrix of the transformed system can be expressed as

$$\hat{Y}(s) = \hat{b}^T(s\hat{M} + \hat{N})^{-1}\hat{b} \tag{31}$$

2.3 Passivity of the reduced order system

Theorem 1

Consider a practical RLGC transmission line system, where matrices L and C are positive-definite. Suppose that the transformation matrix $u(z)$ is of full rank and each element of $u(z)$ in the region $z \in [0,1]$ is in C^1, then the reduced order model generated by using an integrated-congruence transform w.r.t. matrix u on the line system is passive.

The proof of Theorem 1 is given in Appendix A.

3 Moment matching

3.1 Block form of line equations

We will talk about moment matching of the input admittance matrix of an m-coupled line system in this section. In order to make the statements and the formulas more compact, we first introduce the block form of the line equations.

An m-coupled line system is a $2m$ port. To find its input admittance, we can apply a set of $2m$ input voltage vectors with the j-th source voltage vector being the unit vector e_j. Let

$$X_j(z,s) = \begin{bmatrix} I_j(z,s) \\ V_j(z,s) \end{bmatrix} \tag{32}$$

be the solution to Eq(5) when the source vector e_j is applied, and

$$Y_j(s) = \begin{bmatrix} I_j(0,s) \\ -I_j(1,s) \end{bmatrix} \tag{33}$$

Then,

$$Y(s) = [Y_0(s), Y_1(s), \ldots, Y_{2m}(s)] \tag{34}$$

Let

$$J(z,s) = [I_1(z,s), I_2(z,s), \ldots, I_{2m}(z,s)] \tag{35}$$

$$U(z,s) = [V_1(z,s), V_2(z,s), \ldots, V_{2m}(z,s)] \tag{36}$$

and

$$W(z,s) = \begin{bmatrix} J(z,s) \\ U(z,s) \end{bmatrix} \tag{37}$$

Then, from Eq(5), we have

$$(sM + N + T\frac{d}{dz})W(z,s) = 0 \tag{38}$$

This is the block form of the line equations, and its boundary conditions are

$$\begin{bmatrix} U(0,s) \\ U(1,s) \end{bmatrix} = I \tag{39}$$

where I is a $2m \times 2m$ identity matrix.

Similarly, Let $\hat{x}_j(s)$ be the solution to Eq(28) when $V_s = e_j$. Let

$$\hat{W}(s) = [\hat{x}_1(s), \hat{x}_2(s), \ldots, \hat{x}_{2m}(s)] \tag{40}$$

Then, we have

$$(s\hat{M} + \hat{N})\hat{W}(s) = \hat{b} \tag{41}$$

and

$$\hat{Y}(s) = \hat{b}^T \hat{W}(s) \tag{42}$$

The above two equations are the state and output equations of the reduced order system in block form.

3.2 Moment matrices and equations

Let matrix $W(z,s)$ be expanded into Tylor series at some point $s = s_0$, where s_0 may be zero, a real value or a complex one.

$$W(z,s) = \sum_{k=0}^{\infty} W^{(k)}(z,s_0)(s-s_0)^k \tag{43}$$

Then, $W^{(k)}(z, s_0)$ is called the k-th order moment of $W(z, s)$ at $s = s_0$.
The moment matrices $W^{(k)}(z, s_0)'s$ meet with the following equations.
Rewrite Eq(38) into the following form

$$((s - s_0)M + N(s_0) + T\frac{d}{dz})W(z, s) = 0 \tag{44}$$

where

$$N(s_0) = s_0 M + N \tag{45}$$

Substitute Eq(43) to Eq(44), and let the coefficient matrix of the same power of $s - s_0$ be equal, we have the following recursive equations. For $k = 0$,

$$(N(s_0) + T\frac{d}{dz})W^{(0)}(z, s_0) = 0 \tag{46}$$

Note the fact that $T^{-1} = T$, so the above equation can be rewritten in the following form:

$$(TN(s_0) + \frac{d}{dz})W^{(0)}(z, s_0) = 0 \tag{47}$$

where

$$TN(s_0) = \begin{bmatrix} & G + s_0 C \\ R + s_0 L & \end{bmatrix} \tag{48}$$

The boundary conditions of this equation are

$$\begin{bmatrix} U^{(0)}(0, s_0) \\ U^{(0)}(1, s_0) \end{bmatrix} = I \tag{49}$$

where I is an $2m \times 2m$ identity matrix.
For $k > 0$, we have

$$(N(s_0) + T\frac{d}{dz})W^{(k)}(z, s_0) = -MW^{(k-1)}(s_0, z) \tag{50}$$

or

$$(TN(s_0) + \frac{d}{dz})W^{(k)}(z, s_0) = -TMW^{(k-1)}(s_0, z) \tag{51}$$

where

$$TM = \begin{bmatrix} & C \\ L & \end{bmatrix} \tag{52}$$

and the boundary conditions of the equation are

$$\begin{bmatrix} U^{(0)}(0, s_0) \\ U^{(0)}(1, s_0) \end{bmatrix} = 0 \tag{53}$$

For the matrix of input admittance $Y(s)$, let it be expanded into Tylor series at $s = s_0$

$$Y(s) = \sum_{k=0}^{\infty} Y^{(k)}(s_0)(s - s_0)^k \tag{54}$$

Then, $Y^{(k)}(s_0)$ is called the k-th order moment of matrix $Y(s)$ at $s = s_0$.
From the definition of $W^{(k)}(z, s_0)$, we have

$$Y^{(k)}(s_0) = \begin{bmatrix} J^{(k)}(0, s_0) \\ -J^{(k)}(1, s_0) \end{bmatrix} \tag{55}$$

The definition for the moment of the input admittance matrix $\hat{Y}(s)$ at $s = s_0$ is similar. From Eq(31), we have

$$\hat{Y}^{(k)}(s_0) = \hat{b}^T(-\hat{N}(s_0)^{-1}\hat{M})^k \hat{N}(s_0)^{-1}\hat{b} \tag{56}$$

where

$$\hat{N}(s_0) = s_0\hat{M} + \hat{N} \tag{57}$$

3.3 Moment matching theorem

To state the moment matching theorem, we first refer to some definitions in the $L^2[0,1]$ Hilbert space [31].
Def.3.1. Inner product
The inner product of two m-dimensional real vectors $x(z)$ and $y(z)$ defined in $z \in [0,1]$ is denoted by $x(z)|y(z)$ and is defined as

$$x(z)|y(z) = \int_0^1 x^T(z)y(z)dz = \int_0^1 y^T(z)x(z)dz \tag{58}$$

Def.3.2 Orthonormal matrix
An $m \times n$ real matrix $u(z)$ defined in $z \in [0,1]$ is called orthonormal, if the inner product of any two of its column vectors

$$u_i(z)|u_j(z) = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \tag{59}$$

Def.3.3 Containment
Given an m-dimensional vector $v(z)$, which may be real or complex, and a set of m-dimensional real vectors $u(n,z) = \{u_1(z), u_2(z), \ldots, u_n(z)\}$ defined in $z \in [0,1]$, $v(z)$ is said to be contained in $span(u(n,z))$ denoted by $v(z) \in span(u(n,z))$ if there exists a set of coefficients $c = \{c_1, c_2, \ldots, c_n\}$ such that $v(z) = \sum_{i=1}^{n} c_i u_i(z)$. Note that the elements in c may be complex.
Let $u(z) = [u_1(z), u_2(z), \ldots, u_n(z)]$ be a matrix consisting of column vectors $u_j(z), j = 1, \ldots, n$, and $c = [c_1, c_2, \ldots, c_n]^T$ be a vector, then when $v(z) \in span(u(n,z))$, it can be expressed as

$$v(z) = u(z)c \tag{60}$$

When the above equation exists, we also say that $v(z)$ is contained in the column span of matrix $u(z)$, denoted by $v(z) \in colspan(u(z))$. Likewise, a set of vectors $v_1(z), v_2(z), \ldots, v_k(z)$ is said to be contained in span(u(n,z)) if each $v_j(z) \in span(u(n,z)), j = 1, \ldots, k$. In this case, let $v(z) = [v_1(z), v_2(z), \ldots, v_k(z)]$ be a matrix, then there exists an $n \times k$ matrix c such that Eq(60) exists.
Theorem 2
Let $W(k,z,s_0) = \{W^{(0)}(z,s_0), W^{(1)}(z,s_0), \ldots, W^{(k)}(z,s_0)\}$, i.e.,$W(k,z,s_0)$ is a set of moment matrices of $W(z,s)$ at $s = s_0$ from order 0 to k. Let $u(z) = [u_1(z), u_2(z), \ldots, u_n(z)]$ be the transformation matrix of the integrated-congruence transform. If matrix $u(z)$ is orthonormal and $W(k,z,s_0) \in colspan(u(z))$, then

$$\hat{Y}^{(j)}(s_0) = Y^{(j)}(s_0) \qquad 0 \leq j \leq k \tag{61}$$

The proof is given in Appendix B.

3.4 Multipoint moment matching algorithm

Based on Theorem 2, we give the multipoint moment matching algorithm as follows. For a matching point s_i and its matching order k_i, we define a matching pair $m_i = (s_i, k_i)$, and we define a matching set $MS = \{m_1, m_2, \ldots, m_p\}$. Given the matching set, the algorithm can be stated as follows.

Multipoint Moment Matching Algorithm 1
{ Input: Line number m, line parameters R, L, G and C and matching set MS.
  Output: Transformation matrix u.
  $u = \phi$; $n = 0$;
  for $i = 1$ to $|MS|$ do
    $\{(s_i, k_i) = m_i$;
     for $j = 0$ to k_i do
      {compute $[r_1(z), r_2(z), \ldots, r_{2m}(z)] = W^{(j)}(z, s_i)$;
       for $k = 1$ to $2m$ do
        {if(s_i is real)
          n=orthonormal($u, r_k(z), n$);
        else

```
        {ra = real(r_k(z));     rb = imag(r_k(z));
         n=orthonormal(u, ra, n);
         n=orthonormal(u, rb, n);
        }
      }
    }
  }
}
```

In the above algorithm, the function orthonormal(u, r, n) is based on the Gram-Schmidt process [33] and can be stated as follows.

```
function orthonormal(u,r,n)
 {for i=1 to n do
   {λ = r|u_i;
    r = r − λu_i;
   }
   r = r/ || r ||_2;
   n = n + 1;
   u_n = r;
   return(n);
 }
```

The reason that when s_i is complex, the function "orthonormal" should be implemented twice for the real and imaginary part of the vector r_k respectively is that we need a real transformation matrix u [34], and it is easy to see that $r_k \in colspan(u)$ is guaranteed.
From the algorithm it can be seen that the number of columns of matrix u

$$n = \sum_{i, real\ s_i} (k_i + 1) + \sum_{i, complex\ s_i} 2(k_i + 1) \tag{62}$$

The computation of moment matrices $W^{(j)}(z, s_i)$ can be implemented by solving the moment differential equations given in Sec.3.2. The details are given in Appendix C.
Theorem 3
Given R, L, G and C, the parameter matrices of a line and MS, the moment matching set, let u be the transformation matrix generated by using "Multipoint Moment Matching Algorithm 1", and $\hat{Y}(s)$ be the input admittance matrix of the reduced order system by the integrated-congruence transform w.r.t. the transformation matrix u, then $\hat{Y}(s)$ is positive-real and

$$\hat{Y}^{(j)}(s_i) = Y^{(j)}(s_i), \qquad \forall m_i = (s_i, k_i) \in MS, \;\; 0 \le j \le k_i \tag{63}$$

Proof
Matrix u is orthonormal, so is of full rank. Each column of u is a linear combination of some moment matrix functions, each of which is first order continuous differentiable. So the conditions of Theorem 1 exist and $\hat{Y}(s)$ is positive-real. From Theorem 2 Eq(63) exists. □

4 Moment matching model of uniform RC lines

In the RC line case, the theorems and algorithms stated in Sec.2 and 3 can be applied, but we can take advantage of the special case so that for a given moment matching set, the order of the reduced model would be smaller than that given by Eq(62).

4.1 Variable transform

In this section, we deal with the input impedance matrix rather than the input admittance matrix of a uniform RC line system, which consists of m coupled lines in the general case. The line equations are as follows:

$$
\begin{aligned}
\frac{dV(z,s)}{dz} &= -RI(z,s) \\
\frac{dI(z,s)}{dz} &= -CV(z,s)
\end{aligned}
\tag{64}
$$

where $V(z,s)$ and $I(z,s)$ are m-dimensional vectors. The boundary conditions are

$$
\begin{bmatrix} I(0,s) \\ -I(1,s) \end{bmatrix} = I_s(s) = \begin{bmatrix} I_{s1}(s) \\ I_{s2}(s) \end{bmatrix} \tag{65}
$$

where $I_{s1}(s)$ and $I_{s2}(s)$ are m dimensional vectors.
We first use a variable transform, whose benefit will be seen later. Let

$$
\bar{I}(z,s) = R^{\frac{1}{2}} I(z,s) \tag{66}
$$

and

$$
\bar{V}(z,s) = R^{-\frac{1}{2}} V(z,s) \tag{67}
$$

Then, we have

$$
\frac{d\bar{V}(z,s)}{dz} = -\bar{I}(z,s) \tag{68}
$$

and

$$
\frac{d\bar{I}(z,s)}{dz} = -sM\bar{V}(z,s) \tag{69}
$$

where

$$
M = R^{\frac{1}{2}} C R^{\frac{1}{2}} \tag{70}
$$

is positive definite for a practical line system. Let

$$
\begin{bmatrix} V(0,s) \\ V(1,s) \end{bmatrix} = Z(s) \begin{bmatrix} I(0,s) \\ -I(1,s) \end{bmatrix} \tag{71}
$$

where $Z(s)$ is the input impedance matrix of the line, and

$$
\begin{bmatrix} \bar{V}(0,s) \\ \bar{V}(1,s) \end{bmatrix} = \bar{Z}(s) \begin{bmatrix} \bar{I}(0,s) \\ -\bar{I}(1,s) \end{bmatrix} \tag{72}
$$

Then,

$$
Z(s) = R^{\frac{1}{2}} \bar{Z}(s) R^{\frac{1}{2}} \tag{73}
$$

From the above equations, it is known that if $\bar{Z}(s)$ is approximated by $\tilde{Z}(s)$ with moment matching at $s = s_i$ from order 0 to k_i, then $\hat{Z}(s) = R^{\frac{1}{2}} \tilde{Z}(s) R^{\frac{1}{2}}$ will be an approximation of $Z(s)$ with moment matching at $s = s_i$ from order 0 to k_i, too. Also, if $\tilde{Z}(s)$ is positive real, then $\hat{Z}(s)$ is positive real, too. Therefore, we will use Eqs(68) and (69) instead of Eqs(64). Also, the symbols $\bar{V}(z,s)$, $\bar{I}(z,s)$ and $\bar{Z}(s)$ will be replaced by $V(z,s)$, $I(z,s)$ and $Z(s)$, respectively, in the later use for simplicity.

4.2 Model order reduction and passivity preservation by integrated-congruence transform

From Eqs(68) and (69), we have

$$
\frac{d^2V(z,s)}{dz^2} = sMV(z,s) \tag{74}
$$

Substitute $V(z,s) = u(z)\hat{x}(s)$ into the above equation, premultiply both sides by $u^T(z)$ and integrate both side w.r.t. variable z from 0 to 1, we have

$$\int_0^1 u^T(z)\frac{d^2u(z)}{dz^2}dz\hat{x}(s) = s\int_0^1 u^T(z)Mu(z)dz\hat{x}(s) \tag{75}$$

Let

$$\hat{M} = \int_0^1 u^T(z)Mu(z)dz \tag{76}$$

where $\hat{M}$ is positive definite provided that $u(z)$ is of full rank.
By using integral by parts, the l.h.s. of Eq(75) becomes:

$$\int_0^1 u^T(z)\frac{d^2u(z)}{dz^2}dz\hat{x}(s) = u^T(z)\frac{du(z)}{dz}\mid_0^1 \hat{x}(s) - \int_0^1 \frac{du^T(z)}{dz}\frac{du(z)}{dz}dz\hat{x}(s) \tag{77}$$

Let

$$\hat{N} = \int_0^1 \frac{du^T(z)}{dz}\frac{du(z)}{dz}dz \tag{78}$$

It can be seen that $\hat{N}$ is nonnegative definite, and if there exits a nonzero length region $[z_1, z_2] \in [0,1]$ such that matrix $\frac{du(z)}{dz}$ is of full rank, than $\hat{N}$ is positive-definite, which is the most practical case. Note that matrix $\hat{N}$ is symmetric, which is different from the case of an RLGC transmission line, and it is the symmetry of the matrices $\hat{M}$ and $\hat{N}$ that we get the benefit in moment matching.
We combine the first term on the r.h.s. of Eq(77) with the boundary conditions. Note that from Eq(68),

$$I(z,s) = -\frac{du(z)}{dz}\hat{x}(s)$$

so

$$-\frac{du(z)}{dz}\mid_{z=0} \hat{x}(s) = I(0,s) = I_{s1}(s)$$

and

$$\frac{du(z)}{dz}\mid_{z=1} \hat{x}(s) = -I(1,s) = I_{s2}(s)$$

Therefore,

$$\begin{aligned} u^T(z)\frac{du(z)}{dz}\mid_0^1 \hat{x}(s) &= u^T(0)[-\frac{du(z)}{dz}\mid_{z=0} \hat{x}(s)] + u^T(1)[\frac{du(z)}{dz}\mid_{z=1} \hat{x}(s)] \\ &= u^T(0)I_{s1}(s) + u^T(1)I_{s2}(s) = \begin{bmatrix} u(0) \\ u(1) \end{bmatrix}^T I_s(s) \end{aligned}$$

Let

$$\hat{b} = \begin{bmatrix} u(0) \\ u(1) \end{bmatrix}^T \tag{79}$$

Then, the state equations of the reduced order model can be written in the following form

$$(s\hat{M} + \hat{N})\hat{x}(s) = \hat{b}I_s(s) \tag{80}$$

The output equations of the model are

$$V_s(s) = \begin{bmatrix} V(0,s) \\ V(1,s) \end{bmatrix} = \begin{bmatrix} u(0) \\ u(1) \end{bmatrix}\hat{x}(s) = \hat{b}^T\hat{x}(s) \tag{81}$$

and the input impedance matrix of the model is

$$\hat{Z}(s) = \hat{b}^T(s\hat{M} + \hat{N})^{-1}\hat{b} \tag{82}$$

Compared Eq(82) with Eq(31), it can be seen that Theorem 1 exits in this case, i.e., if matrix $u(z)$ is of full rank and if each element of $u(z)$ is first order continuous differentiable, the input impedance $\hat{Z}(s)$ of the reduced order model is positive-real, and the model is passive.

4.3 Moment matching

Similar to the case in Sec3.1, let $V_j(z,s)$ and $I_j(z,s)$ be the solution to Eqs(68) and (69) when $I_s = e_j$, $U(z,s) = [V_1(z,s), V_2(z,s), \ldots, V_{2m}(z,s)]$, and $J(z,s) = [I_1(z,s), I_2(z,s), \ldots, I_{2m}(z,s)]$, then we have the block forms of Eqs (68), (69), and (74) as follows:

$$\frac{dU(z,s)}{dz} = -J(z,s) \tag{83}$$

$$\frac{dJ(z,s)}{dz} = -sMU(z,s) \tag{84}$$

and

$$\frac{d^2U(z,s)}{dz^2} = sMU(z,s) \tag{85}$$

with the boundary conditions being

$$\begin{bmatrix} -\frac{dU(z,s)}{dz}\big|_{z=0} \\ \frac{dU(z,s)}{dz}\big|_{z=1} \end{bmatrix} = I \tag{86}$$

where I is a $2m \times 2m$ identity matrix. From the above definitions, the input impedance matrix can be expressed as

$$Z(s) = \begin{bmatrix} U(0,s) \\ U(1,s) \end{bmatrix} \tag{87}$$

Now we have the moment matching theorem as follows:
Theorem 4
Let $U(k,z,s_0) = \{U^{(0)}(z,s_0), U^{(1)}(z,s_0), \ldots, U^{(k)}(z,s_0)\}$, i.e.,$U(k,z,s_0)$ is a set of moment matrices of $U(z,s)$ at $s = s_0$ from order 0 to k. Let $u(z) = [u_1(z), u_2(z), \ldots, u_n(z)]$ be the transformation matrix of the integrated-congruence transform. If matrix $u(z)$ is orthonormal and $U(k,z,s_0) \in colspan(u(z))$, then

$$\hat{Z}^{(j)}(s_0) = Z^{(j)}(s_0) \qquad 0 \le j \le 2k+1 \tag{88}$$

Note the difference between Theorem 4 and Theorem 2. When the transformation matrix $u(z)$ is formed based on the moment matching requirement, for the same starting moment matching pair, the final moment matching order in the RC line case is doubled. On the other hand, given a moment matching set MS for the whole network, for each RC line, for each $m_i = (s_i, k_i) \in MS$, the matching order can be reduced to $\lceil \frac{k_i - 1}{2} \rceil$.
The proof of Theorem 4 is given in Appendix D.

5 Model order reduction of distributed network

5.1 MNA equations

Suppose that we have a p-port distributed network, which consists of n RLGC transmission lines, m RC lines, some linear resistors, capacitors and inductors and is driven by p voltage sources and its input admittance matrix is of interest. We first apply the model order reduction algorithm to all the lines. Suppose that for the i-th transmission line, its state equations and output equations of the reduced order model are as follows:

$$F_{ai}(s)x_{ai} = b_{ai}V_{ai} \tag{89}$$

and

$$I_{ai} = b_{ai}^T x_{ai} \tag{90}$$

Where V_{ai} and I_{ai} are the port voltage and current vector of the line, respectively. Let V_n be the node voltage vector of the entire network, and A_{ai} be the node-branch incidence matrix of the port branches of the line. Then, $V_{ai} = A_{ai}^T V_n$, and Eq(89) can be rewritten in the following form:

$$F_{ai}(s)x_{ai} = c_{ai}V_n \tag{91}$$

where

$$c_{ai} = b_{ai} A_{ai}^T \tag{92}$$

and the contribution of I_{ai} to the KCL equations of the nodes can be expressed as

$$A_{ai} I_{ai} = c_{ai}^T x_{ai} \tag{93}$$

Let $x_a = [x_{a1}^T, x_{a2}^T, \ldots, x_{an}^T]^T$, then we have

$$F_a(s) x_a = c_a V_n \tag{94}$$

where

$$F_a(s) = block\_diag(F_{a1}(s), F_{a2}(s), \ldots, F_{an}(s)) \tag{95}$$

and

$$c_a = [c_{a1}^T, c_{a2}^T, \ldots, c_{an}^T]^T \tag{96}$$

and the contribution of $I_{ai},\ 1 \leq i \leq n$ to the KCL equations can be expressed as $c_a^T x_a$.
Similarly, suppose that I_{ci} and V_{ci} are the input current and voltage vector of the i-th RC line, respectively, and the state and the output equations of its reduced order model are in the following form:

$$F_{ci}(s) x_{ci} = b_{ci} I_{ci} \tag{97}$$

and

$$V_{ci} = b_{ci}^T x_{ci} \tag{98}$$

Let A_{ci} be the node-input branch incidence matrix of the line, then the above equation can be written in the form of

$$A_{ci}^T V_n = b_{ci}^T x_{ci} \tag{99}$$

Let

$$x_c = [x_{c1}^T, x_{c2}^T, \ldots, x_{cm}^T]^T \tag{100}$$

$$I_c = [I_{c1}^T, I_{c2}^T, \ldots, I_{cm}^T]^T \tag{101}$$

$$F_c(s) = block\_diag(F_{c1}(s), F_{c2}(s), \ldots, F_{cm}(s)) \tag{102}$$

$$b_c = block\_diag(b_{c1}, b_{c2}, \ldots, b_{cm}) \tag{103}$$

and

$$A_c = [A_{c1}, A_{c2}, \ldots, A_{cm}] \tag{104}$$

Then, from Eqs(97) and (99), we have

$$F_c(s) x_c = b_c I_c \tag{105}$$

and

$$A_c^T V_n = b_c^T x_c \tag{106}$$

Suppose that there are lumped resistors, capacitors and inductors in the network. Let the nodal conductance and capacitance matrix of single resistors and capacitors be G and C, respectively. Let R and L be the branch resistance and inductance matrix of the serially connected resistor-inductor branches, and I_L and A_L be their branch current vector I_L and node- branch incidence matrix, respectively. Let the port voltage and current vector be V_s and I_s, respectively, and their node-branch incidence matrix be A_s. Note that the current of each voltage source flows out of its positive terminal. Then, the unknown vector of the MNA equations of the network $x = [x_a^T, x_c^T, V_n^T, I_L^T, I_c^T, I_s^T]^T$, and the MNA equations can be written as follows:

$$H(s) x = b V_s \tag{107}$$

where

$$H(s) = \begin{bmatrix} A & -Q \\ Q^T & B \end{bmatrix} \tag{108}$$

with

$$A = block\_diag(F_a(s), F_c(s)) \tag{109}$$

$$Q = \begin{bmatrix} c_a & 0 & 0 & 0 \\ 0 & 0 & c_b & 0 \end{bmatrix} \tag{110}$$

$$B = \begin{bmatrix} sC+G & A_L & A_c & -A_s \\ -A_L^T & SL+R & & \\ -A_c^T & & & \\ A_s^T & & & \end{bmatrix} \tag{111}$$

and

$$b = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad I]^T \tag{112}$$

The output equations of the network are

$$I_s = b^T x \tag{113}$$

and the input admittance matrix of the network is

$$Y(s) = b^T H(s)^{-1} b \tag{114}$$

Theorem 5
Matrix $H(s)$ in Eq(114) is positive-real.
Proof.
It is easy to check that Conditions 1 and 2 of a positive-real matrix are satisfied by $H(s)$. To check Condition 3, note that

$$H(s) + H^*(s)^T = \begin{bmatrix} A + A^T(s^*) & \\ & B + B^T(s^*) \end{bmatrix}$$

$$A + A^T(s^*) = block\_diag(F_a(s) + F_a^T(s^*), F_c(s) + F_c^T(s^*))$$

When $Re(s) > 0$, $F_a(s) + F_a^T(s^*) \geq 0$ and $F_c(s) + F_c^T(s^*) \geq 0$ so $A + A^T(s^*) \geq 0$. From Eq(111), let $\sigma = Re(s)$, then

$$B + B^T(s^*) = block\_diag(2\sigma C + 2G, 2\sigma L + 2R, 0, 0) \geq 0$$

when $\sigma > 0$. Therefore, Condition 3 is satisfied. □

5.2 Model order reduction

Suppose that $|x| = r$, and we apply a congruence transform with the transformation matrix $V \in R^{r \times q}$ with $q < r - |I_c| - |I_s|$ on Eq(107), then we have a q-th order system as follows:

$$\hat{H}(s)\hat{x} = \hat{b}V_s \tag{115}$$

where

$$x = V\hat{x} \tag{116}$$

$$\hat{H}(s) = V^T H(s) V \tag{117}$$

and

$$\hat{b} = V^T b \tag{118}$$

The output equations become

$$I_s = \hat{b}^T \hat{x} \tag{119}$$

and the input admittance matrix of the reduced order model is

$$\hat{Y}(s) = \hat{b}^T \hat{H}(s)^{-1} \hat{b} \tag{120}$$

Theorem 6
If the transform matrix V is of full rank, than $\hat{Y}(s)$ is positive- real and the reduced order model is passive.
The proof of Theorem 6 is based on Theorem 5 and can be done by the same arguments as in the proof of Theorem 1.

5.3 Moment matching

As in the case of modeling a line, we apply p sets of input voltages such that the j-th input voltage vector is the unit vector e_j, and let x_j be the solution to Eq(107) in this case. Let

$$X = [x_1, x_2, \ldots, x_p] \tag{121}$$

Then, the block form of the MNA equations will be

$$H(s)X = b \tag{122}$$

and the output equations become

$$Y(s) = b^T X \tag{123}$$

We have the moment matching theorem as follows.
Theorem 7
Let $H(s) = sM + N$, $N(s_0) = s_0 M + N$, $X^{(j)}(s_0) = (-N(s_0)^{-1}M)^j N(s_0)^{-1} b$ and $K(n, s_0) = \{X^{(0)}(z, s_0), X^{(1)}(z, s_0), \ldots, X^{(n)}(z, s_0)\}$. If the congruence transform matrix V is orthonormal and $K(n, s_0) \in colspan(V)$, then

$$\hat{Y}^{(j)}(s_0) = Y^{(j)}(s_0), \qquad 0 \le j \le n \tag{124}$$

The proof of Theoprem 7 in given in Appendix E.
Based on Theorem 7, given the moment matching set MS, the algorithm for the formation of congruence transform matrix V is as follows.

Multipoint Moment Matching Algorithm 2

```
{ Input: Port number p, matrix H(s) = sM + N, vector b and matching set MS.
  Output: Transformation matrix V.
  V = φ; n = 0;
  for i = 1 to |MS| do
    {(s_i, k_i) = m_i;
     for j = 0 to k_i do
       {compute [r_1, r_2, ..., r_p] = X^(j)(s_i);
        for k = 1 to p do
          {if(s_i is real)
            n=orthonormal1(V, r_k, n);
           else
            {ra = real(r_k); rb = imag(r_k);
             n=orthonormal1(V, ra, n);
             n=orthonormal1(V, rb, n);
            }
          }
       }
    }
}
```

In the above algorithm, the function *orthonormal*1 is as follows:

```
function orthonormal1(V, x, n)
{for i = 1 to n do
    x = x - x^T V_i;
  a = sqrt(x^T x);
  n = n + 1;
  V_n = x/a;
  return(n); }
```

6 Examples

We have successfully tested a number of examples. We show some of them here.
Example 1.
This is a simple example to show the advantage of our model over the discrete model of a transmission line. The circuit consists of a single line with parameters $R = 0.01\Omega/cm$, $L = 2.5nH/cm$, $C = 1pF/cm$, $d = 1cm$, a load resistor and a source resistor of 50Ω, which match the characteristic impedance of the line at high frequencies. The voltage source is a pulse. Fig.1 shows the output voltage waveform V_2 obtained by the SPICE simulation with the segmentation model of the line, where the solid and dashed lines correspond to the number of segments equal to 50 and 100, respectively. The ripple in the waveform is obvious, which should not exist when exact model is used, and when the number of segments increases, the magnitude of the ripple does not decrease much. Fig.2 shows the time domain simulation by using a three point moment matching model generated by our algorithm, where the ripple is missing, and the waveform is nearly exact. This example shows that discretization is not ideal in practice as we stated in "Introduction".
Example 2.
This is a clock net consisting of 73 lossy transmission lines, 2895 resistors and 2777 capacitors driven by a cascade of two inverters, as shown in Fig.3. The waveforms at PIN117 are shown in Fig.4, where the solid and dashed lines correspond to the result of SPICE simulation and the time domain simulation with our model, where for each line moment matching at 0 frequency with order 4 and at a high frequency with order 0 is used. These two waveforms are close.
Example 3.
The circuit is shown in Fig.5, where 4 coupled lines with neighbor coupling are presented. The waveforms of V_5 are shown in Fig.6, where the solid and dashed lines correspond to the result of SPICE simulation and the time domain simulation with our model, which is obtained by moment matching at zero frequency with order 4. These two waveforms are close.
Example 4.
This is an example borrowed from [29]. The circuit is shown in Fig.7, where two coupled line systems, each of which consists of three coupled lines, are presented. The frequency domain and time-domain response of V_{out} are shown in Fig.8 and 9, respectively. The solid line represents the exact solution where the coupled lines are modeled by their exact multiport characteristic model. The dashed line corresponds to our model, where a moment matching set $MS = \{(0Hz, 4), (1.5Ghz, 0), (3Ghz, 0), (4Ghz, 0), (5Ghz, 0)\}$ for each line system is used. The solid and dashed lines are indistinguishable. Compared with the model used in [29], not only that our model is guaranteed passive and theirs not, but also that our model order is much lower (a 40-th moment matching model at zero frequency is used for each coupled line system in [29]). This also shows the advantage of a multipoint moment matching model over a single point one.
Example 5.
It is a single RC line in $0.25\mu m$ technology with length 4cm, which is connected to a capacitor load and a source resistor, and driven by a pulse signal. The line model is formulated by matching with order 2 at a very small (near 0) and a large real value of s. The waveforms at the load are shown in Fig.10, where the solid and dashed lines correspond to the results from SPICE simulation and from our model, respectively. They match well.
Example 6.
It is a 3 RC couple line system in $0.45\mu m$ technology, loaded in capacitors and connected to voltage sources with inner resistors, as shown in Fig.11. The first and third lines are aggressors, where two synchronous pulse signals are added, and the second line in between is a victim with no input signal. The model of the line system is formulated by using the same matching set as in Example 5. The waveforms of the coupling noise at the victim load are shown in Fig.12, where the solid and dashed lines correspond to the results from SPICE simulation and from our model, respectively. They match well.

7 Conclusions

We have presented a new algorithm for passive model order reduction with multipoint moment matching for distributed interconnect networks. Given the moment matching requirement, for each distributed lines, moment matrix functions are computed, then the Gram-Schmidt process is implemented to form an orthonormal transformation matrix in the Hilbert $L^2[0,1]$ space. By using an integrated-congruence transform on the partial differential equations of the line, a finite order passive system satisfying the moment matching requirement is obtained. Then, the MNA equations of the whole network are formulated, and an orthonormal matrix based on the moment matching requirement is formed. By using a congruence transform with such a matrix, a passive reduced order model of the network with multipoint moment matching is obtained. Based on the theory presented, an exact moment matching model is obtained which eliminates the need of discretization of a distributed line. Experiments show that the model generated by the algorithm works well. This algorithm can also be useful to do model order reduction in elecro-mechanical systems.

A Appendix A: Proof of Theorem 1

To prove Theorem 1, we refer to the following definitions and conclusions [32].
Def.A.1.
An $n \times n$ matrix $F(s)$ is called positive real, if the following conditions are met:

1. Each element of $F(s)$ is analytic in $Re(s) > 0$;
2. $F(s^*) = F^*(s)$, where "*" stands for complex conjugate;
3. $[F^*(s)]^T + F(s)$ is nonnegative-definite for all $Re(s) > 0$.

Lemma A.1
An n-port network is passive iff its input immittance matrix is positive-real.
Lemma A.2
If $F(s)$ is positive real, and $det(F(s)) \neq 0$ at one point $Re(s) > 0$, then $F^{-1}(s)$ is positive-real.
Lemma 2.3
If $F(s)$ is positive-real and b is real, then $G(s) = b^T F(s) b$ is positive-real.
Proof.
It is easy to show that $G(s)$ meets with Conditions 1 and 2 for a positive-real matrix. For Condition 3, note that

$$G(s) + [G^*(s)]^T = b^T(F(s) + [F^*(s)]^T)b$$

As $F(s)$ is positive-real, $F(s) + [F^*(s)]^T \geq 0$ when $Re(s) > 0$, and $G(s) + [G^*(s)]^T \geq 0$ when $Re(s) > 0$. □
Proof of Theorem 1.
From Lemma A.1, it is known that the reduced order model is passive iff $\hat{Y}(s)$ is positive-real.
Let $\hat{H}(s) = s\hat{M} + \hat{N}$, we first show that $\hat{H}(s)$ is positive-real. From the condition that each element of the transformation matrix u is in C^1 when $z \in [0,1]$, $\hat{H}(s)$ is well defined. It is obvious that conditions 1 and 2 are satisfied. To check condition 3, from the fact that matrices M and N are nonnegative definite, it is easy to show that matrices $\hat{M}$ and $\hat{N}_1$ are nonnegative definite. Let $s = \sigma + j\omega$.

$$\hat{H}(s) + \hat{H}^*(s)^T = 2\sigma\hat{M} + \hat{N} + \hat{N}^T$$

From Eq(24), $\hat{N} + \hat{N}^T = 2\hat{N}_1$, so that $\hat{H}(s) + \hat{H}^*(s)^T = 2(\sigma\hat{M} + \hat{N}_1)$ is nonnegative definite when $Re(s) = \sigma > 0$. For a practical RLGC line, matrix L and C are positive definite, so matrix M is positive definite. As u is of full rank, $\hat{M}$ is positive definite, and there exists a point s with $Re(s) > 0$ such that $det(\hat{H}(s)) \neq 0$. From Lemma A.2, it is known that $\hat{H}(s)^{-1}$ is positive-real, and from Lemma A.3, it is clear that $\hat{Y}(s)$ is positive-real.□

B Appendix B: Proof of Theorem 2

To prove Theorem 2, we first prove several lemmas.
Lemma B.1
If $u(z)$ is orthonormal and $w(z) \in colspan(u(z))$, then

$$u(z)\int_0^1 u^T(x)w(x)dx = w(z) \tag{125}$$

Proof.
When $w(z) \in colspan(u(z))$, then there exists a matrix c such that $w(z) = u(z)c$. As $u(z)$ is orthonormal, so $\int_0^1 u^T(z)u(z)dz = I$ where I is an identity matrix. So

$$u(z)\int_0^1 u^T(x)w(x)dx = u(z)\int_0^1 u^T(x)u(x)cdx = u(z)c = w(z) \qquad \square$$

Lemma B.2
If $W^{(j)}(z, s_0) \in colspan(u(z))$ for $0 \le j \le k$. Then there exists $\hat{W}^{(j)}(s_0)$ with

$$W^{(j)}(z, s_0) = u(z)\hat{W}^{(j)}(s_0) \quad 0 \le j \le k \tag{126}$$

such that

$$\hat{N}(s_0)\hat{W}^{(0)}(s_0) = \hat{b} \tag{127}$$

and for $0 < j \le k$,

$$\hat{M}\hat{W}^{(j-1)}(s_0) + \hat{N}(s_0)\hat{W}^{(j)}(s_0) = 0 \tag{128}$$

Proof.
Since $W^j(z, s_0) \in colspan(u(z))$ for $0 \le j \le k$, there exists $\hat{W}^{(j)}(s_0)$ such that

$$W^{(j)}(z, s_0) = u(z)\hat{W}^{(j)}(s_0) \tag{129}$$

Substitute Eq(129) to Eq(46) for $j = 0$ and Eq(50) for $j > 0$, premultiply $u^T(z)$ on both side of the equation, integrate them from 0 to 1 w.r.t. variable z, and apply the same mathematical process as done in Sec.2.2, we obtain Eqs(127) and (128). $\square$
Lemma B.3
Under the conditions of Theorem 2,

$$(-\hat{N}(s_0)^{-1}\hat{M})^j\hat{N}(s_0)^{-1}\hat{b} = \int_0^1 u^T(z)W^{(j)}(z, s_0)dz, \qquad 0 \le j \le k \tag{130}$$

Proof.
We prove it by induction.
For $j = 0$, from Eq(126) in Lemma B.2,

$$\int_0^1 u^T(z)W^{(0)}(z, s_0)dz = (\int_0^1 u^T(z)u(z)dz)\hat{W}^{(0)}(s_0) = \hat{W}^{(0)}(s_0)$$

and from Eq(127), we have

$$\hat{W}^{(0)}(s_0) = \hat{N}(s_0)^{-1}\hat{b}$$

So Lemma B.3 is true for $j = 0$.
Now suppose that Lemma B.3 is true for $0 \le j < k$, we prove it is true for $j + 1$.
From the induction hypothesis,

$$(-\hat{N}(s_0)\hat{M})^{j+1}\hat{N}(s_0)^{-1}\hat{b} = -\hat{N}(s_0)^{-1}\hat{M}\int_0^1 u^T(z)W^{(j)}(z, s_0)dz$$

From Lemma B.1 and B.2,

$$\begin{aligned}\hat{M}\int_0^1 u^T(z)W^{(j)}(z, s_0)dz &= \int_0^1 u^T(w)M[u(w)\int_0^1 u^T(z)W^{(j)}(z, s_0)dz]dw \\ &= \int_0^1 u^T(w)MW^{(j)}(w, s_0)dw = (\int_0^1 u^T(w)Mu(w)dw)\hat{W}^{(j)}(s_0) \\ &= \hat{M}\hat{W}^{(j)}(s_0) = -\hat{N}(s_0)\hat{W}^{(j+1)}(s_0)\end{aligned}$$

$$= -\hat{N}(s_0)\int_0^1 u^T(z)W^{(j+1)}(z,s_0)dz$$

So,

$$(-\hat{N}(s_0)^{-1}\hat{M})\int_0^1 u^T(z)W^{(j)}(z,s_0)dz = \int_0^1 u^T(z)W^{(j+1)}(z,s_0)dz \qquad \square$$

Proof of Theorem 2.
From Eq(56) and Lemma B.3,

$$\hat{Y}^{(j)}(s_0) = \hat{b}^T(-\hat{N}(s_0)^{-1}\hat{M})^j\hat{N}(s_0)^{-1}\hat{b} = \hat{b}^T\int_0^1 u^T(z)W^{(j)}(z,s_0)dz$$

Now,

$$\hat{b}^T = \begin{bmatrix} u_i(0) \\ -u_i(1) \end{bmatrix} = \begin{bmatrix} [I \quad 0]u(0) \\ -[I \quad 0]u(1) \end{bmatrix}$$

from Lemma B.1,

$$u(z)\int_0^1 u^T(z)W^{(j)}(z,s_0)dz = W^{(j)}(z,s_0)$$

so

$$[I \quad 0]u(z)\int_0^1 u^T(z)W^{(j)}(z,s_0)dz = J^{(j)}(z,s_0)$$

and from Eq(55)

$$\hat{Y}^{(j)}(s_0) = \begin{bmatrix} J^{(j)}(0,s_0) \\ -J^{(j)}(1,s_0) \end{bmatrix} = Y^{(j)}(s_0) \quad 0 \le j \le k. \qquad \square$$

C Appendix C: Computation of moment matrices of RLGC lines

C.1 Moment matrices of uniform lines

In the case of a uniform line system consisting of m lines, for the two excitations $V_s = e_j$, $j \le m$, and $V_s = e_{j+m}$, we have $V_j(z,s) = V_{j+m}(1-z,s)$ and $I_j(z,s) = -I_{j+m}(1-z,s)$. In order to save computation, in this case, we redefine

$$U(z,s) = [V_1(z,s), V_2(z,s), \ldots, V_m(z,s)] \tag{131}$$

$$J(z,s) = [I_1(z,s), I_2(z,s), \ldots, I_m(z,s)] \tag{132}$$

i.e., we reduce the number of vectors in $U(z,s)$ and $J(z,s)$ from $2m$ to m. $W(z,s)$ is still defined as Eq(37) which satisfies Eq(38), but now the boundary conditions become

$$\begin{bmatrix} U(0,s) \\ U(1,s) \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{133}$$

C.1.1 RLC lines with $s_0 = 0$

In the case of RLC lines with the matching point $s_0 = 0$, the moment matrices meet with the following equations. For the 0-th order moment matrices,

$$\frac{dJ^{(0)}(z,0)}{dz} = 0 \tag{134}$$

$$\frac{dU^{(0)}(z,0)}{dz} = -RJ^{(0)}(z,0) \tag{135}$$

with the boundary conditions $U^{(0)}(0,0) = I$ and $U^{(0)}(1,0) = 0$.
It is easy to find that

$$J^{(0)}(z,0) = R^{-1} \tag{136}$$

and

$$U^{(0)}(z,0) = I(1-z) \tag{137}$$

where I is an $m \times m$ identity matrix.
For $j > 0$, we have

$$\frac{dJ^{(j)}(z,0)}{dz} = -CU^{(j-1)}(z,0) \tag{138}$$

and

$$\frac{dU^{(j)}(z,0)}{dz} = -RJ^{(j)}(z,0) - LJ^{(j-1)}(z,0) \tag{139}$$

with the boundary conditions $U^{(j)}(0,0) = U^{(j)}(1,0) = 0$.
The general solution to Eq(138) is

$$J^{(j)}(z,0) = J^{(j)}(0,0) - C\int_0^z U^{(j-1)}(x,0)dx \tag{140}$$

and the solution to Eq(139) is

$$U^{(j)}(z,0) = -RJ^{(j)}(0,0)z + RC\int_0^z\int_0^y U^{(j-1)}(x,0)dxdy - L\int_0^z J^{(j-1)}(x,0)dx \tag{141}$$

From $U^{(j)}(1,0) = 0$, we have

$$J^{(j)}(0,0) = C\int_0^1\int_0^y U^{(j-1)}(x,0)dxdy - R^{-1}L\int_0^1 J^{(j-1)}(x,0)dx \tag{142}$$

The above three equations are the recursive formulas for the computation of $J^{(j)}(z,0)$ and $U^{(j)}(z,0)$.
From the fact that $J^{(0)}(z,0)$ and $U^{(0)}(z,0)$ are polynomials of z with order 0 and 1, it can be deduced that $J^{(j)}(z,0)$ and $U^{(j)}(z,0)$ are polynomials of z with order $2j$ and $2j+1$, respectively; and the computation of the integrals in the expressions can be done exactly.

C.1.2 RLGC line with matching point $s = s_0$

In the general case of an RLGC line with matching point $s = s_0$, for the 0-th order moment, we have the following equations:

$$\frac{dW^{(0)}(z,s_0)}{dz} + AW^{(0)}(z,s_0) = 0 \tag{143}$$

where

$$A = \begin{bmatrix} & G+s_0C \\ R+s_0L & \end{bmatrix} \tag{144}$$

The general solution to Eq(143) is

$$W^{(0)}(z,s_0) = exp(-Az)W^{(0)}(0,s_0) \tag{145}$$

Note that A and $exp(-Az)$ are $2m \times 2m$ matrices. Let

$$exp(-A) = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{146}$$

and

$$W^{(0)}(0,s_0) = \begin{bmatrix} J^{(0)}(0,s_0) \\ I \end{bmatrix} \tag{147}$$

From the boundary condition $U^{(0)}(1,s_0) = 0$, we have

$$J^{(0)}(0,s_0) = -K_{21}^{-1}K_{22} \tag{148}$$

Eqs(145), (147) and (148) give the solution $W^{(0)}(0,s_0)$.
For the j-th order moment with $j > 0$, we have

$$\frac{dW^{(j)}(z,s_0)}{dz} + AW^{(j)}(z,s_0) = F^{(j)}(z,s_0) \tag{149}$$

where

$$F^{(j)}(z,s_0) = -\begin{bmatrix} & C \\ L & \end{bmatrix} W^{(j-1)}(z,s_0) \tag{150}$$

The general solution to Eq(149) can be written as follows:

$$W^{(j)}(z,s_0) = exp(-Az)[\int_0^z exp(Ax)F^{(j)}(x,s_0)dx + W^{(j)}(0,s_0)] \tag{151}$$

Let

$$exp(-A)\int_0^1 exp(Ax)F^{(j)}(x,s_0)dx = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{152}$$

where each submatrix is of dimension $m \times m$. Then, from the boundary conditions, we have

$$P_2 + K_{21}J^{(j)}(0,s_0) = 0 \tag{153}$$

and

$$J^{(j)}(0,s_0) = -K_{21}^{-1}P_2 \tag{154}$$

From Eqs(151), (152) and , (154) $W^{(j)}(z,s_0)$ can be computed.
It can be seen from the above equations that $W^{(j)}(z,s_0)$ is in the form of a sum of products of exponential and polynomial functions. There are several ways to compute the moment matrices.
1. Exact solutions.
In this case, we compute the eigenvalues and eigenvectors of matrix A, and functions $exp(Az)$ and $exp(-Az)$ can be expressed as a sum of exponential functions with matrix coefficients. Using such expressions and the formulas given before, we can compute the moment matrices exactly. When the matching order goes high, the expressions become more complicated, so does the data structure.
2. Polynomial approximations.
The second way is to use polynomials to approximate matrix exponentials. The data structure is simple in this case, and the order of polynomial is low when the magnitudes of the eigenvalues of matrix A are much less than 1, e.g., when $s_0 = 0$ for most practical lines. In the general case, the order of the polynomial depends on both the largest magnitude of the eigenvalues of the matrix and the maximum moment matching order. When $|s_0|$ is very large, we usually need high order polynomial to do approximation.
3. Numerical solution.
For a pair of polynomials with orders m and n, to compute their inner product, we need $(m+1)\times(n+1)$ multiplications. When m and n are large, the computation cost will be high. A fast approximation is to use numerical methods to solve the differential equations of moment matrices. In this case, the moment matrix functions are expressed by the data at sampling points. The difference between this method and the segmentation model is that in this case we can use high order numerical integration methods, but the segmentation model is a first order approximation.
The numerical solution can be obtained as follows. For solving $W^{(0)}(z,s_0)$, we first solve Eq(143) with the initial condition

$$W^{(0)}(0,s_0) = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{155}$$

and get a function

$$f_1(z,s_0) = \begin{bmatrix} f_1^J(z,s_0) \\ f_1^U(z,s_0) \end{bmatrix} \tag{156}$$

and second we solve Eq(143) with another initial condition

$$W^{(0)}(0,s_0) = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{157}$$

and get a function

$$f_2(z,s_0) = \begin{bmatrix} f_2^J(z,s_0) \\ f_2^U(z,s_0) \end{bmatrix} \tag{158}$$

Then,

$$W^{(0)}(z,s_0) = f_1(z,s_0) + f_2(z,s_0)J^{(0)}(0,s_0) \tag{159}$$

By using the boundary condition

$$U^{(0)}(1,s_0) = f_1^U(1,s_0) + f_2^U(1,s_0)J^{(0)}(0,s_0) = 0$$

$$J^{(0)}(0,s_0) = -f_2^U(1,s_0)^{-1}f_1^U(1,s_0) \tag{160}$$

is obtained, and the solution $W^{(0)}(z,s_0)$ is known.
For the case $j > 0$, we use similar method to solve Eq(149) with different initial conditions twice, where the initial condition for the first case is a null matrix.

C.2 Nonuniform line

In the case of a nonuniform line system, we have to use numerical solution to represent the moment matrix function by its data at sampling points. In that case, we divide the source voltage matrix into two parts: $V_{s1} = diag(I,0)$ and $V_{s2} = diag(0,I)$, where V_{s1} and V_{s2} correspond to the excitations at the near end and far end, respectively. Then we solve the differential equations Eq(143) and Eq(149) corresponding to V_{s1} and V_{s2}, and obtain the first m and the next m columns of the moment matrix, respectively.

D Appendix D: Proof of Theorem 4

To prove Theorem 4, other than Lemma B.1 to B.3, which still exit in this case, we give two more lemmas.
Lemma D.1

$$-\int_0^1 U^{(0)}(z,s_0)^T M U^{(j)}(z,s_0)dz = Z^{(j+1)}(s_0) \qquad j \geq 0 \tag{161}$$

Proof.
From Eqs(83) and (84), we have

$$\frac{dU^{(j)}(z,s_0)}{dz} = -J^{(j)}(z,s_0) \tag{162}$$

$$\frac{dJ^{(0)}(z,s_0)}{dz} = -s_0 M U^{(0)}(z,s_0) \tag{163}$$

and

$$\frac{dJ^{(j+1)}(z,s_0)}{dz} = -MU^{(j)}(z,s_0) - s_0 M U^{(j+1)}(z,s_0) \quad j \geq 0 \tag{164}$$

From Eq(164), we have

$$-\int_0^1 U^{(0)}(z,s_0)^T M U^{(j)}(z,s_0) = I_1 + I_2$$

where

$$I_1 = \int_0^1 U^{(0)}(z,s_0)^T \frac{dJ^{(j+1)}(z,s_0)}{dz}dz$$

and

$$I_2 = \int_0^1 U^{(0)}(z,s_0)^T s_0 M U^{(j+1)}(z,s_0)dz$$

By using integral by parts,

$$I_1 = U^{(0)}(z,s_0)^T J^{(j+1)}(z,s_0)\,|_0^1 - \int_0^1 \frac{dU^{(0)}(z,s_0)^T}{dz} J^{(j+1)}(z,s_0)dz$$

From the boundary conditions $J^{(j+1)}(z, s_0) = 0$ at $z = 0$ and $z = 1$ when $j \geq 0$, the first term in the r.h.s. of the above expression is zero. From Eq(162),

$$I_1 = -\int_0^1 \frac{dU^{(0)}(z, s_0)^T}{dz} J^{(j+1)}(z, s_0)dz = \int_0^1 J^{(0)}(z, s_0)^T J^{(j+1)}(z, s_0)dz$$

$$= -\int_0^1 J^{(0)}(z, s_0)^T \frac{dU^{(j+1)}(z, s_0)}{dz}dz = -J^{(0)}(z, s_0)^T U^{(j+1)}(z, s_0) \mid_0^1 + \int_0^1 \frac{dJ^{(0)}(z, s_0)^T}{dz} U^{(j+1)}(z, s_0)dz$$

From Eq(163) and the symmetry of matrix M, the second term of the above expression is equal to

$$-\int_0^1 U^{(0)}(z, s_0)^T s_0 M U^{(j+1)}(z, s_0)dz = -I_2$$

and

$$-\int_0^1 U^{(0)}(z, s_0)^T M U^{(j)}(z, s_0)dz = -J^{(0)}(z, s_0)^T U^{(j+1)}(z, s_0) \mid_0^1$$

As

$$J^{(0)}(0, s_0) = [I \quad 0]$$

and

$$J^{(0)}(1, s_0) = -[0 \quad I]$$

So,

$$-J^{(0)}(z, s_0)^T U^{(j+1)}(z, s_0) \mid_0^1 = \begin{bmatrix} U^{(j+1)}(0, s_0) \\ U^{(j+1)}(1, s_0) \end{bmatrix} = Z^{(j+1)}(s_0) \qquad \square$$

Lemma D.2

$$-\int_0^1 U^{(k)}(z, s_0)^T M U^{(j)}(z, s_0)dz = -\int_0^1 U^{(k-1)}(z, s_0)^T M U^{(j+1)}(z, s_0)dz \qquad k > 0, j \geq 0 \qquad (165)$$

Proof.
From Eq(164),

$$-\int_0^1 U^{(k)}(z, s_0)^T M U^{(j)}(z, s_0)dz = I_1 + I_2$$

where

$$I_1 = \int_0^1 U^{(k)}(z, s_0)^T \frac{dJ^{(j+1)}(z, s_0)}{dz}dz$$

and

$$I_2 = \int_0^1 U^{(k)}(z, s_0)^T s_0 M U^{(j+1)}(z, s_0)dz$$

By using integral by parts and $J^{j+1}(z, s_0) = 0$ at $z = 0$ and $z = 1$ when $j \geq 0$,

$$I_1 = -\int_0^1 \frac{dU^{(k)}(z, s_0)^T}{dz} J^{(j+1)}(z, s_0)dz$$

From Eq(162),

$$I_1 = \int_0^1 J^{(k)}(z, s_0)^T J^{(j+1)}(z, s_0)dz = -\int_0^1 J^{(k)}(z, s_0)^T \frac{dU^{j+1)}(z, s_0)}{dz}dz$$

$$= -J^{(k)}(z, s_0)^T U^{(j+1)}(z, s_0) \mid_0^1 + \int_0^1 \frac{dJ^{(k)}(z, s_0)^T}{dz} U^{(j+1)}(z, s_0)dz$$

When $k > 0$, the first term in the above expression is zero. From Eq(164), the second term can be expressed as

$$\int_0^1 \frac{dJ^{(k)}(z, s_0)^T}{dz} U^{(j+1)}(z, s_0)dz = -\int_0^1 U^{(k-1)}(z, s_0)^T M U^{(j+1)}(z, s_0)dz - I_2$$

and Lemma 4.2 is true. $\square$

Proof of Theorem 4.

Let $\hat{N}(s_0) = s_0\hat{M} + \hat{N}$, then

$$\hat{Z}^{(j)}(s_0) = \hat{b}^T(-\hat{N}(s_0)^{-1}\hat{M})^j \hat{N}(s_0)^{-1}\hat{b} \tag{166}$$

We first prove that Theorem 4 is true when $j = 0$. In this case, from Lemma B.2 and Lemma B.1,

$$\hat{Z}^{(0)}(s_0) = \hat{b}^T \hat{N}(s_0)^{-1}\hat{b} = \hat{b}^T \int_0^1 u^T(z)U^{(0)}(z, s_0)dz$$

$$= \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} \int_0^1 u^T(z)U^{(0)}(z, s_0)dz$$

$$= \begin{bmatrix} U^{(0)}(0, s_0) \\ U^{(0)}(1, s_0) \end{bmatrix} = Z^{(0)}(s_0)$$

For $1 \le j \le 2k+1$, let $j = j_1 + j_2$ with $1 \le j_1 \le k+1$ and $0 \le j_2 \le k$. Then,

$$\hat{Z}^j = \hat{b}^T(-\hat{N}(s_0)^{-1}\hat{M})^j \hat{N}(s_0)^{-1}\hat{b} = -A\hat{M}B = -\int_0^1 Au^T(z)Mu(z)Bdz$$

where

$$A = \hat{b}^T(-\hat{N}(s_0)^{-1}\hat{M})^{j_1-1}\hat{N}(s_0)^{-1}$$

and

$$B = (-\hat{N}(s_0)^{-1}\hat{M})^{j_2}\hat{N}(s_0)\hat{b}$$

From Lemma B.3 and B.1,

$$u(z)B = u(z)\int_0^1 u^T(z)U^{(j_2)}(z, s_0)dz = U^{(j_2)}(z, s_0)$$

As $\hat{M}$, $\hat{N}$ and $\hat{N}(s_0)$ are symmetric, so

$$A^T = (-\hat{N}(s_0)^{-1}\hat{M})^{j_1-1}\hat{N}(s_0)^{-1}\hat{b}$$

and

$$Au^T(z) = [u(z)A^T]^T = U^{(j_1-1)}(z, s_0)^T$$

So,

$$\hat{Z}^{(j)}(s_0) = -\int_0^1 U^{(j_1-1)}(z, s_0)^T M U^{(j_2)}(z, s_0)dz$$

If $j_1 = 1$, then $j = j_2 + 1$ and from Lemma D.1,

$$\hat{Z}^{(j)} = Z^{(j_2+1)} = Z^{(j)}$$

Otherwise, by repeatedly using Lemma D.2, we have

$$\hat{Z}^{(j)} = -\int_0^1 U^{(0)}(z, s_0)^T M U^{(j_2+j_1-1)}(z, s_0)dz = Z^{(j_2+j_1)} = Z^{(j)} \qquad \square$$

E Appendix E: Proof of Theorem 7

Let $N(s_0) = s_0M + N$ and $\hat{N}(s_0) = s_0\hat{M} + \hat{N}$, where $\hat{M} = V^TMV$ and $\hat{N} = V^TNV$, then

$$Y^{(j)}(s_0) = b^TX^{(j)}(s_0) \tag{167}$$

and

$$\hat{Y}^{(j)}(s_0) = \hat{b}^T\hat{X}^{(j)}(s_0) \tag{168}$$

We first give two lemmas.

Lemma E.1

If $X^{(j)}(s_0) \in colspan(V)$ and V is orthonormal, then

$$VV^T X^{(j)}(s_0) = X^{(j)}(s_0) \tag{169}$$

Proof.

Let $u = X^{(j)}(s_0)$ and u can be expressed as $u = Va$. Then

$$VV^T u = V(V^T V)a = Va = u \qquad \square$$

Lemma E.2

Under the conditions of Theorem 7, for each $X^{(j)}(s_0)$ with $0 \le j \le k$, there exists $\hat{X}^{(j)}(s_0)$ with $X^{(j)}(s_0) = V\hat{X}^{(j)}(s_0)$ and

$$X^{(j)}(s_0) = (-\hat{N}(s_0)^{-1}\hat{M})^j \hat{N}(s_0)^{-1}\hat{b} \tag{170}$$

Proof.

As $K(n, s_0) \in colspan(V)$, so for each $X^{(j)}(s_0)$ with $0 \le j \le k$, there exists $\hat{X}^{(j)}(s_0)$ with $X^{(j)}(s_0) = V\hat{X}^{(j)}(s_0)$.
Now we prove that Eq(170) exists.
From

$$N(s_0)X^{(0)}(s_0) = b$$

we have

$$V^T N(s_0)V\hat{X}^{(0)}(s_0) = V^T b = \hat{b}$$

$$\hat{N}(s_0)\hat{X}^{(0)}(s_0) = \hat{b}$$

So Eq(170) exists for $j = 0$.
Now suppose that Eq(170) exists for $0 \le j < k$, we prove it exists for $j+1$. From the fact

$$MX^{(j)}(s_0) + N(s_0)X^{(j+1)}(s_0) = 0$$

we have

$$\hat{V}MV\hat{X}^{(j)}(s_0) + \hat{V}N(s_0)V\hat{X}^{(j+1)}(s_0) = 0$$

so

$$\hat{X}^{(j+1)}(s_0) = -\hat{N}(s_0)^{-1}\hat{M}\hat{X}^{(j)}(s_0) = (-\hat{N}(s_0)^{-1}\hat{M})^{j+1}\hat{N}(s_0)^{-1}\hat{b} \qquad \square$$

Remark. When $X^{(j)}(s_0) = V\hat{X}^{(j)}(s_0)$, $V^T X^{(j)}(s_0) = V^T V\hat{X}^{(j)}(s_0) = \hat{X}^{(j)}(s_0)$.

Proof of Theorem 7.

Note that $Y^{(j)}(s_0) = b^T X^{(j)}(s_0)$. from Lemma E.2 and E.1,

$$\hat{Y}^{(j)}(s_0) = \hat{b}^T\hat{X}^{(j)}(s_0) = b^T V\hat{X}^{(j)}(s_0) = b^T VV^T X^{(j)}(s_0)$$

$$= b^T X^{(j)}(s_0) = Y^{(j)}(s_0), \qquad 0 \le j \le n \qquad \square$$

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Fig. 1

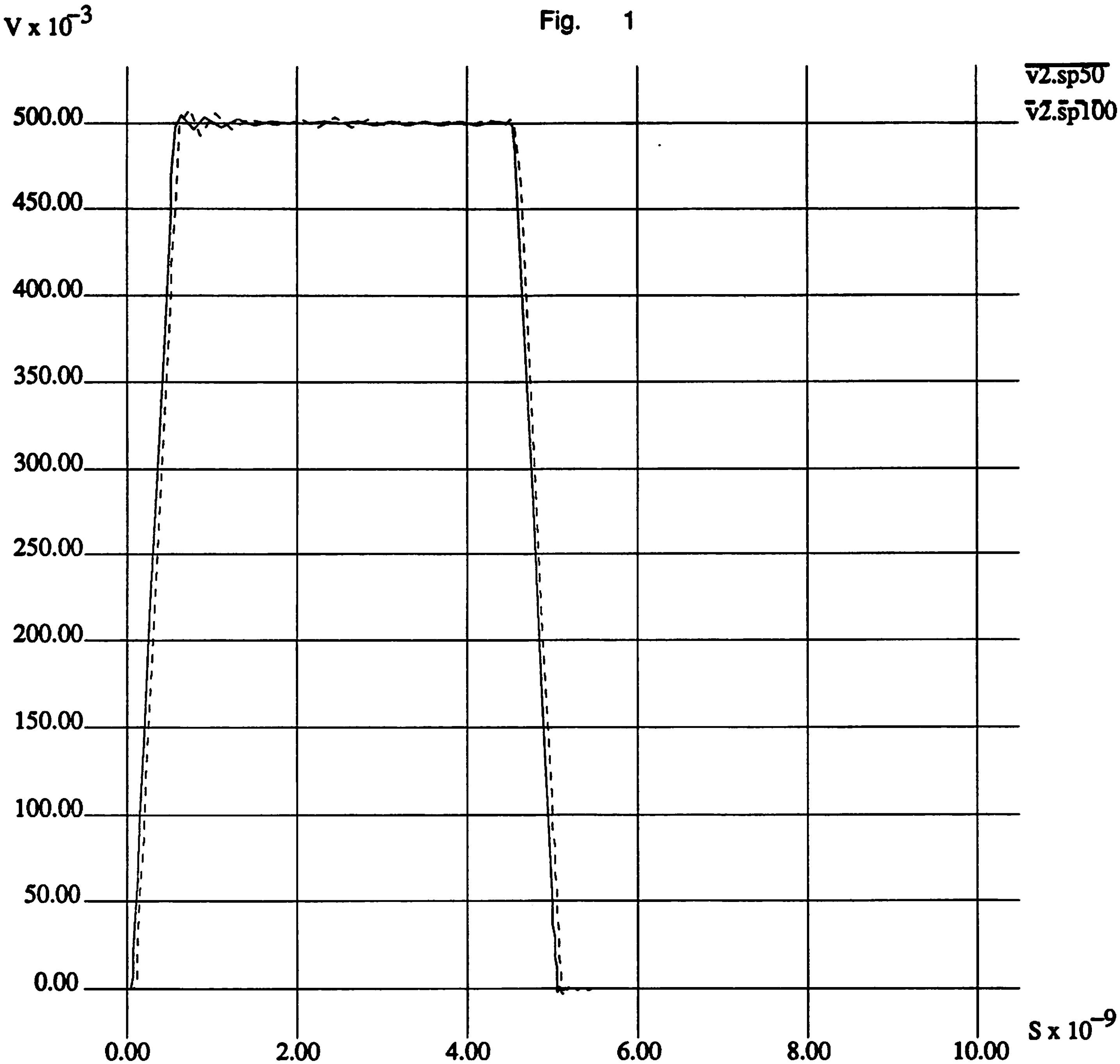

V x 10^-3
v2.sp50
v2.sp100
500.00
450.00
400.00
350.00
300.00
250.00
200.00
150.00
100.00
50.00
0.00
0.00
2.00
4.00
6.00
8.00
10.00
S x 10^-9

Fig. 2

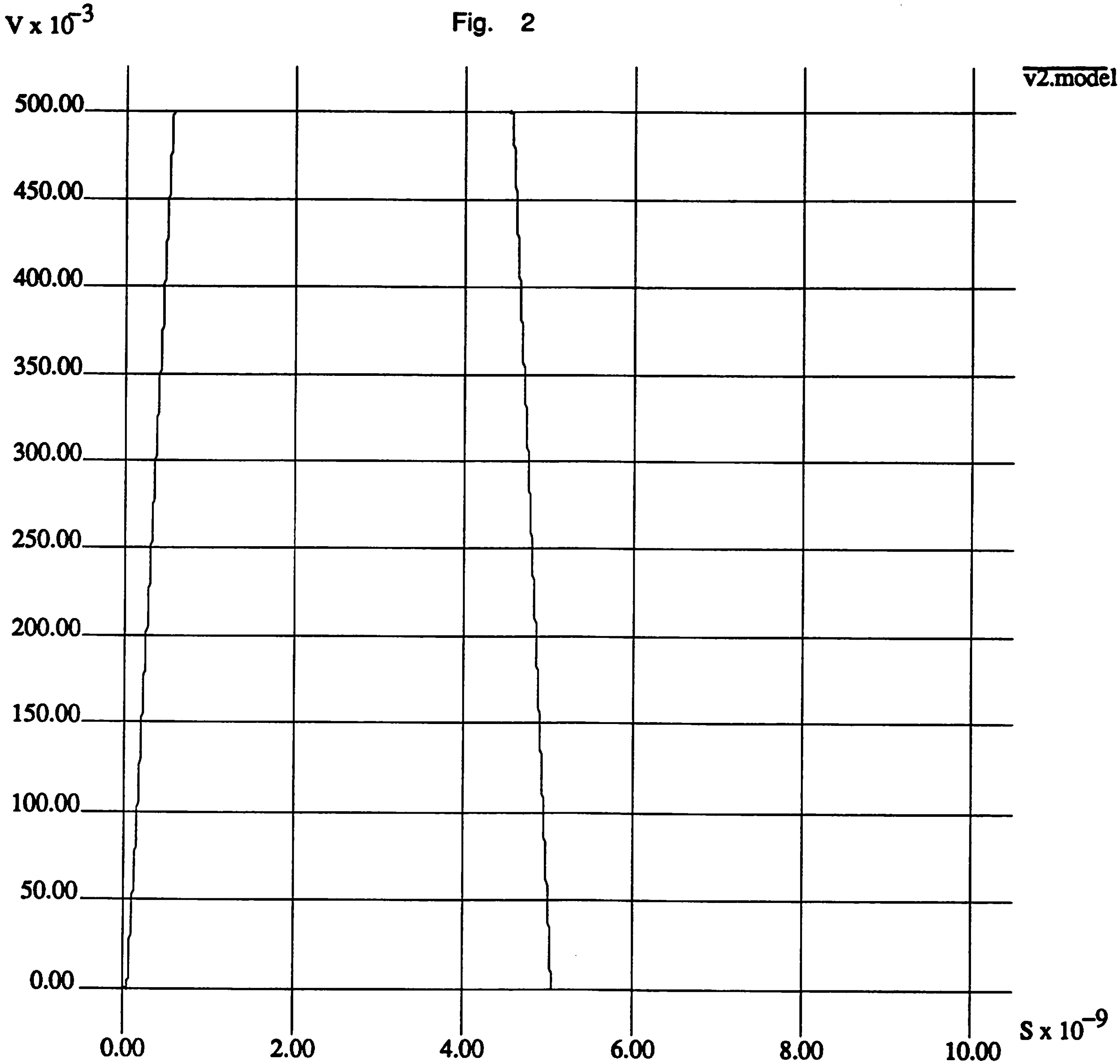

V x 10^{-3}
v2.model
500.00
450.00
400.00
350.00
300.00
250.00
200.00
150.00
100.00
50.00
0.00
0.00
2.00
4.00
6.00
8.00
10.00
S x 10^{-9}

Fig. 3

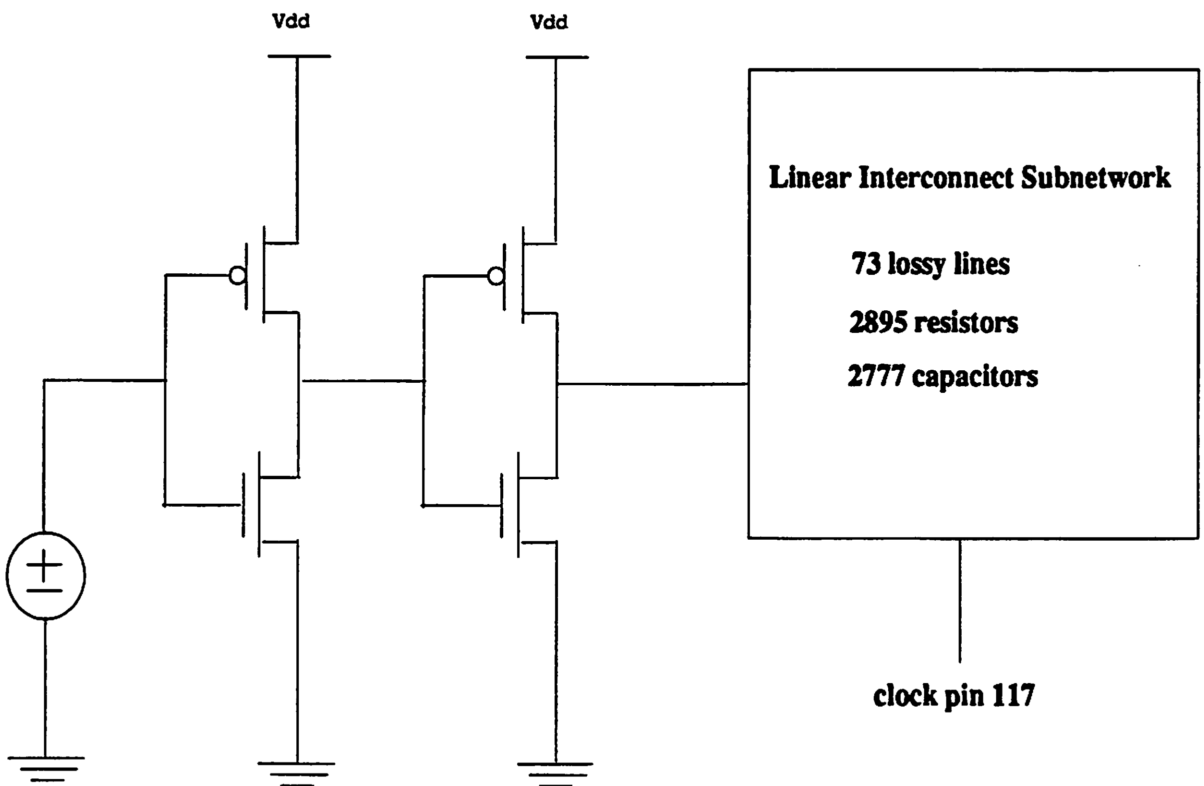

Vdd
Vdd
Linear Interconnect Subnetwork
73 lossy lines
2895 resistors
2777 capacitors
clock pin 117

Fig. 4

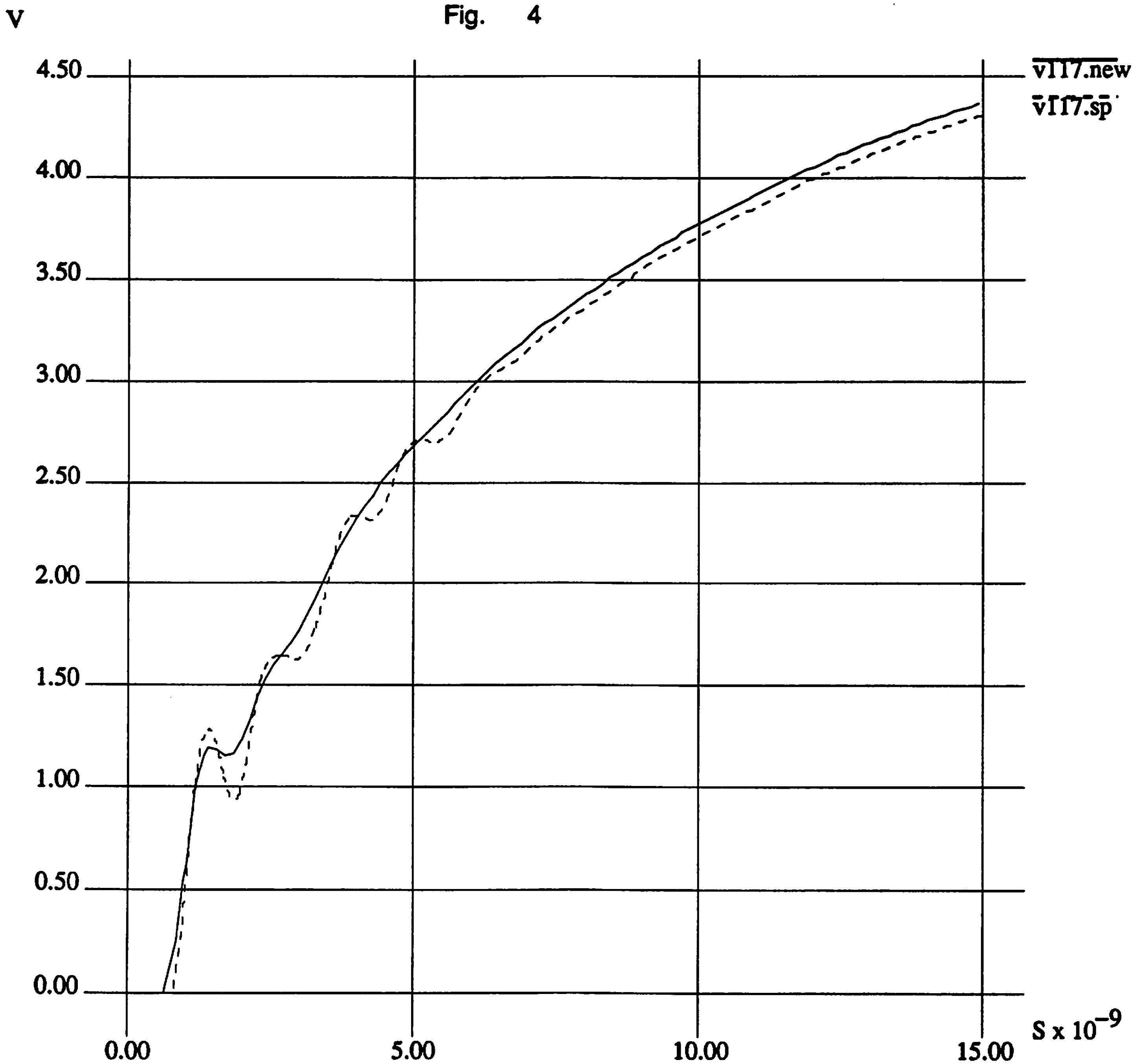
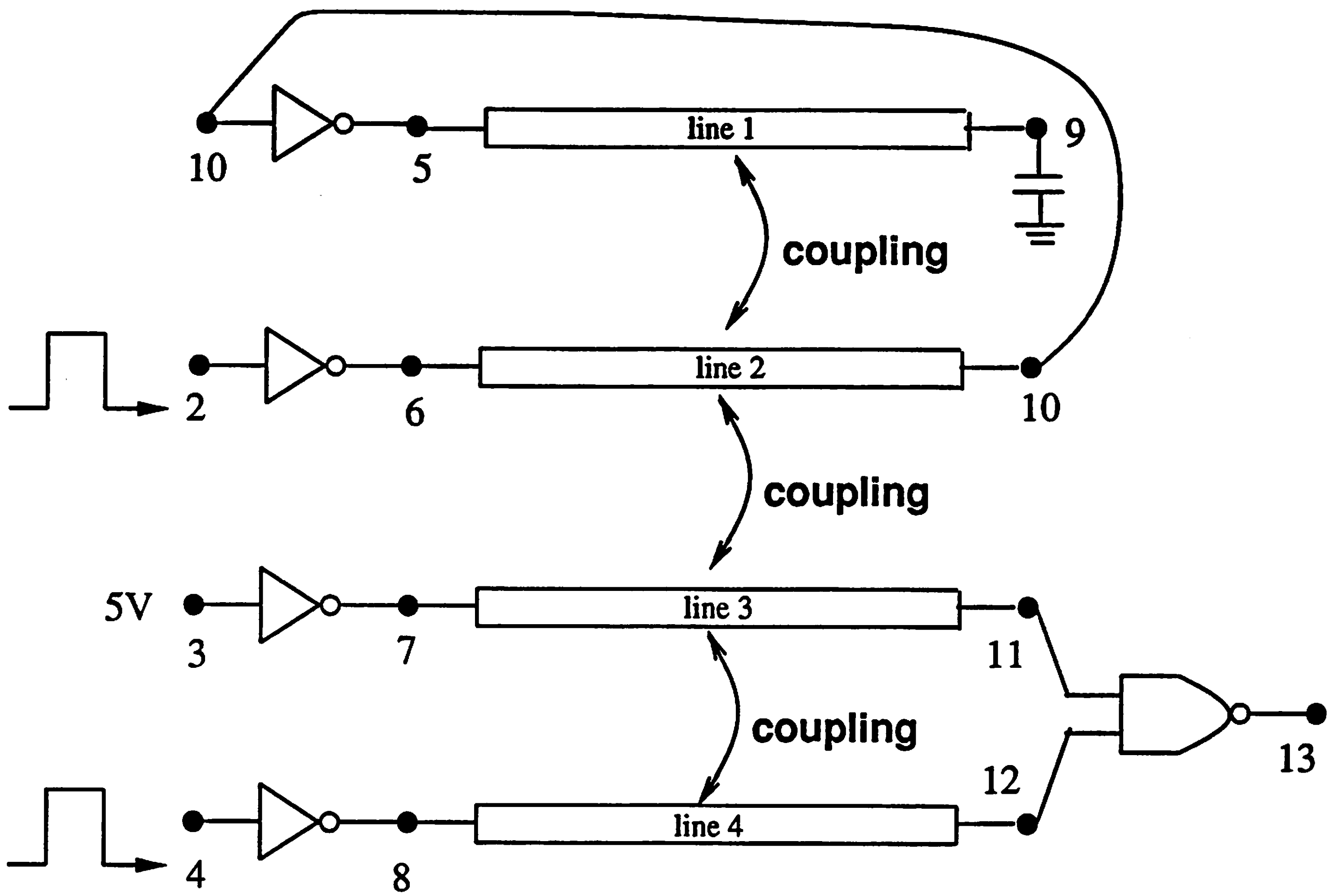

V
4.50
4.00
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
0.00
5.00
10.00
15.00
S x 10^{-9}
v117.new
v117.sp

FiG. 5



10
line 1
9
5
coupling
line 2
2
6
10
coupling
5V
line 3
3
7
11
coupling
13
12
line 4
4
8

Fig. 6

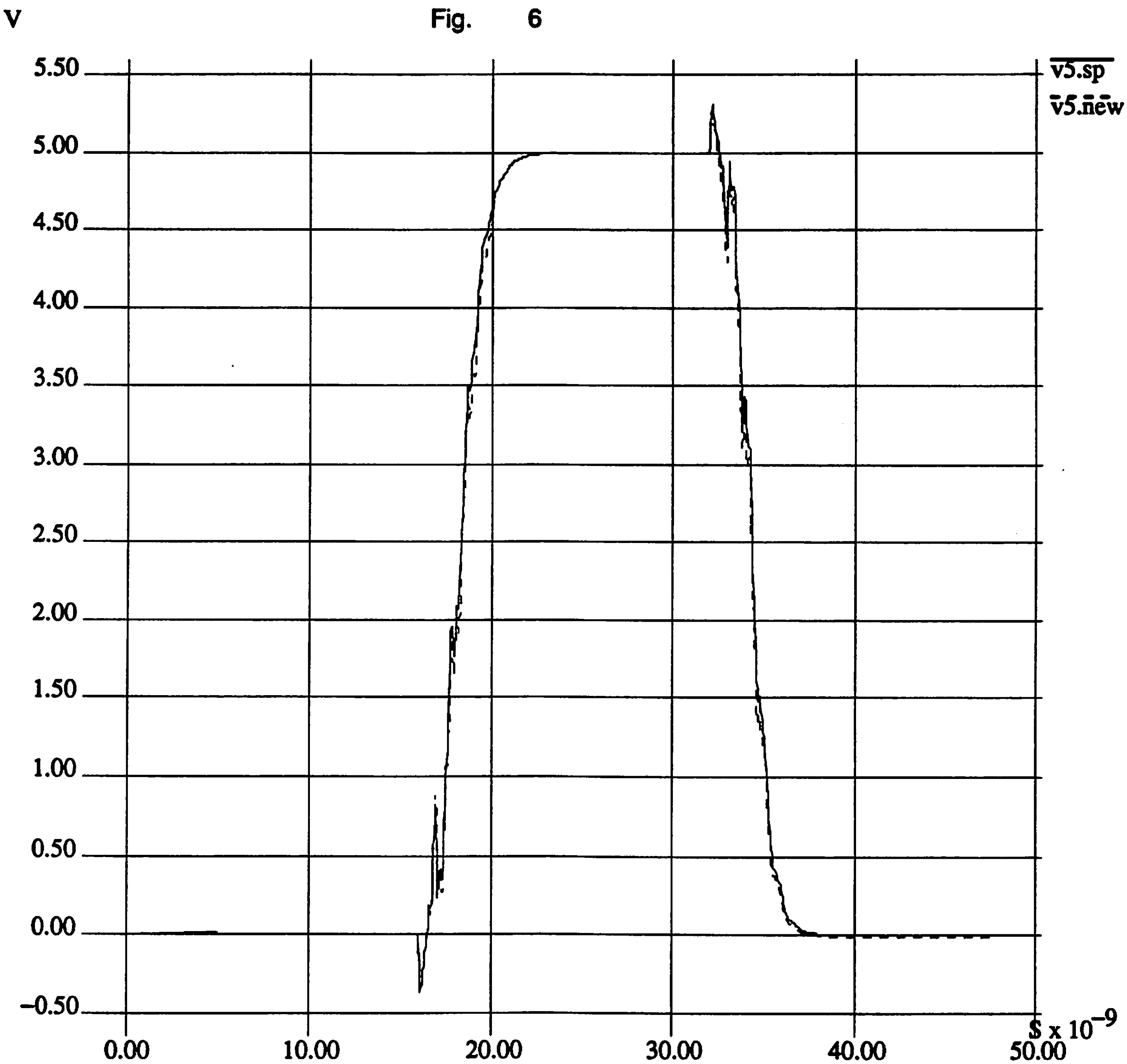

V
5.50
5.00
4.50
4.00
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
-0.50
0.00
10.00
20.00
30.00
40.00
50.00
S x 10^-9
v5.sp
v5.new

Fig. 7

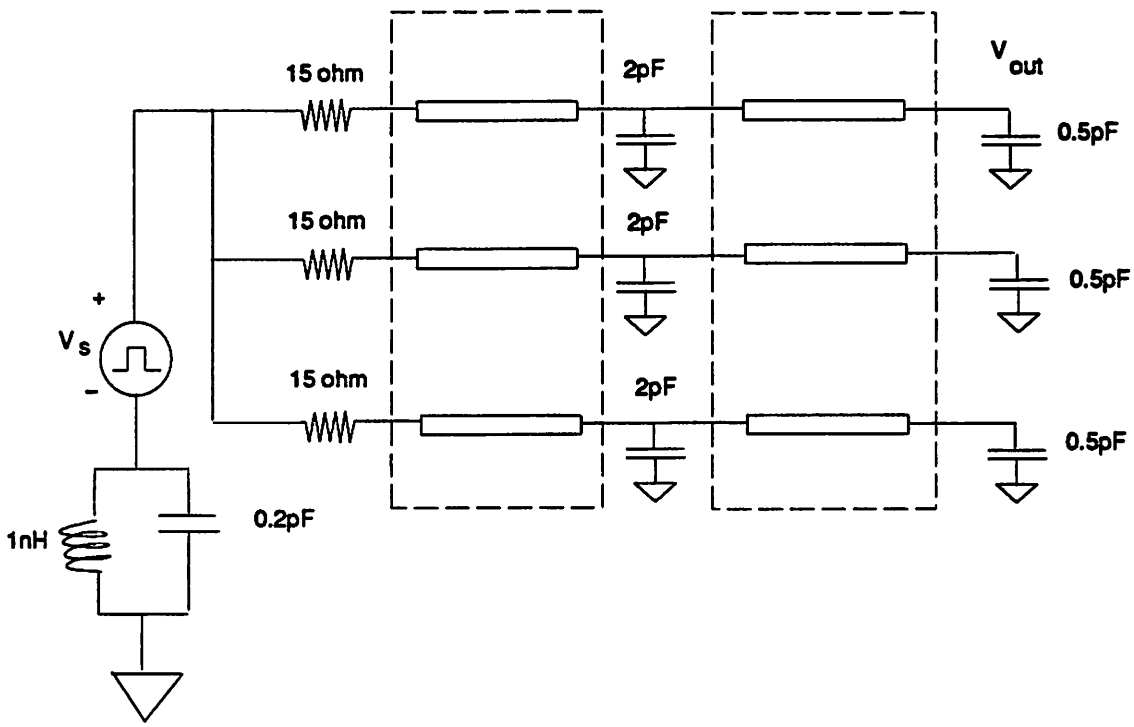


Fig. 8

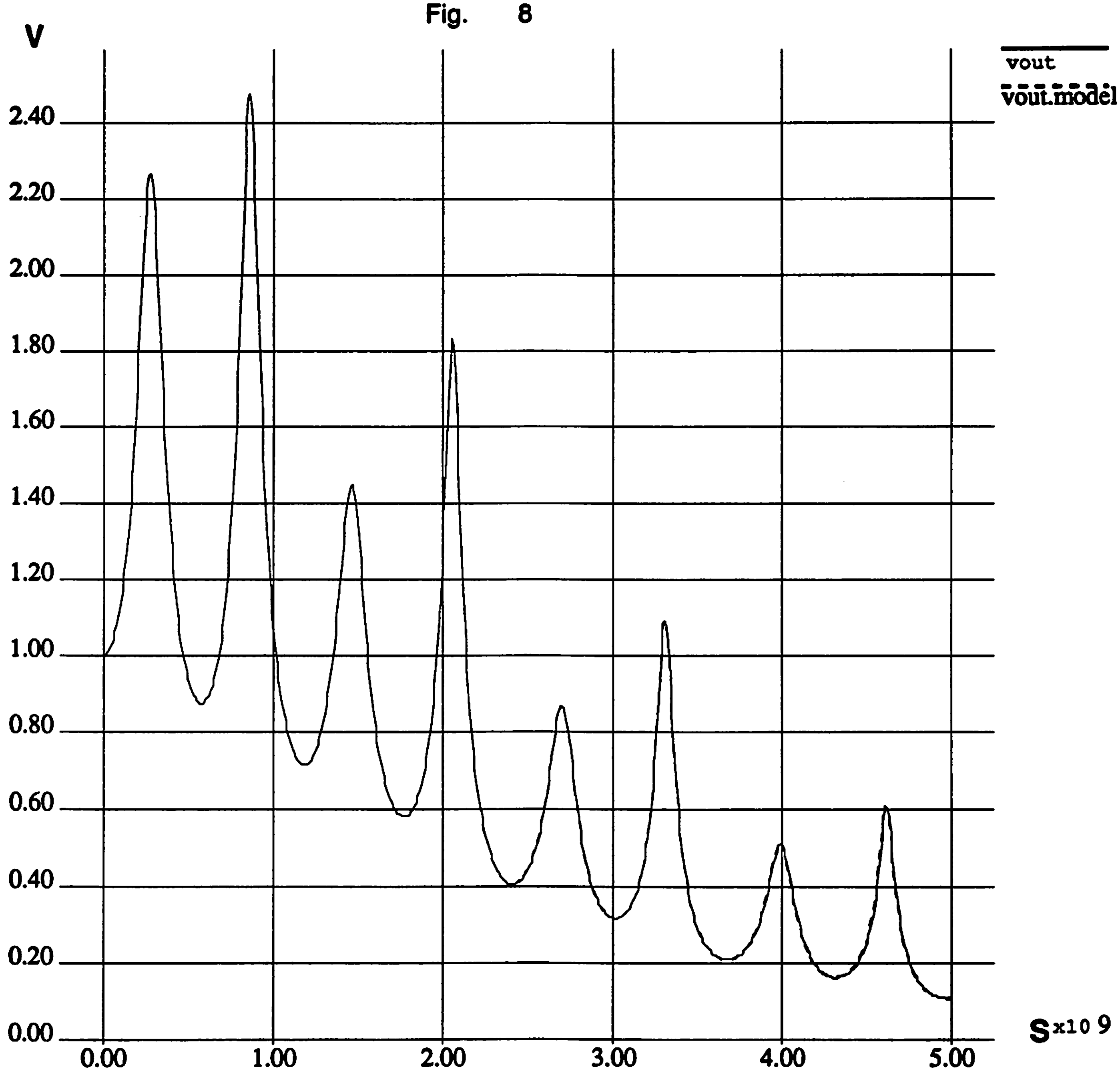
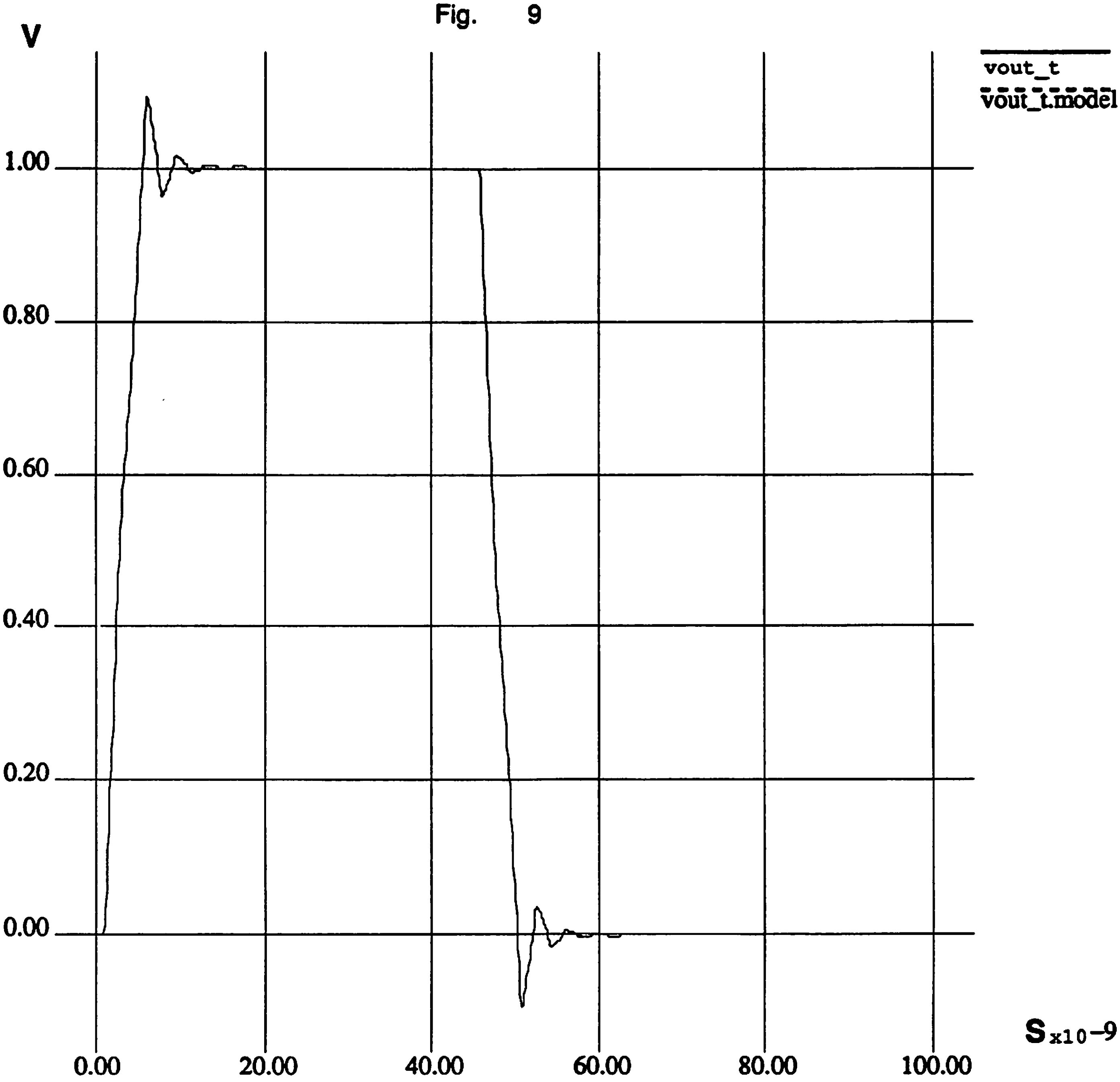

V
vout
vout.model
2.40
2.20
2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
0.00
1.00
2.00
3.00
4.00
5.00
S x10 9

Fig. 9



V
vout_t
vout_t.model
1.00
0.80
0.60
0.40
0.20
0.00
0.00
20.00
40.00
60.00
80.00
100.00
S x10-9

Fig. 10

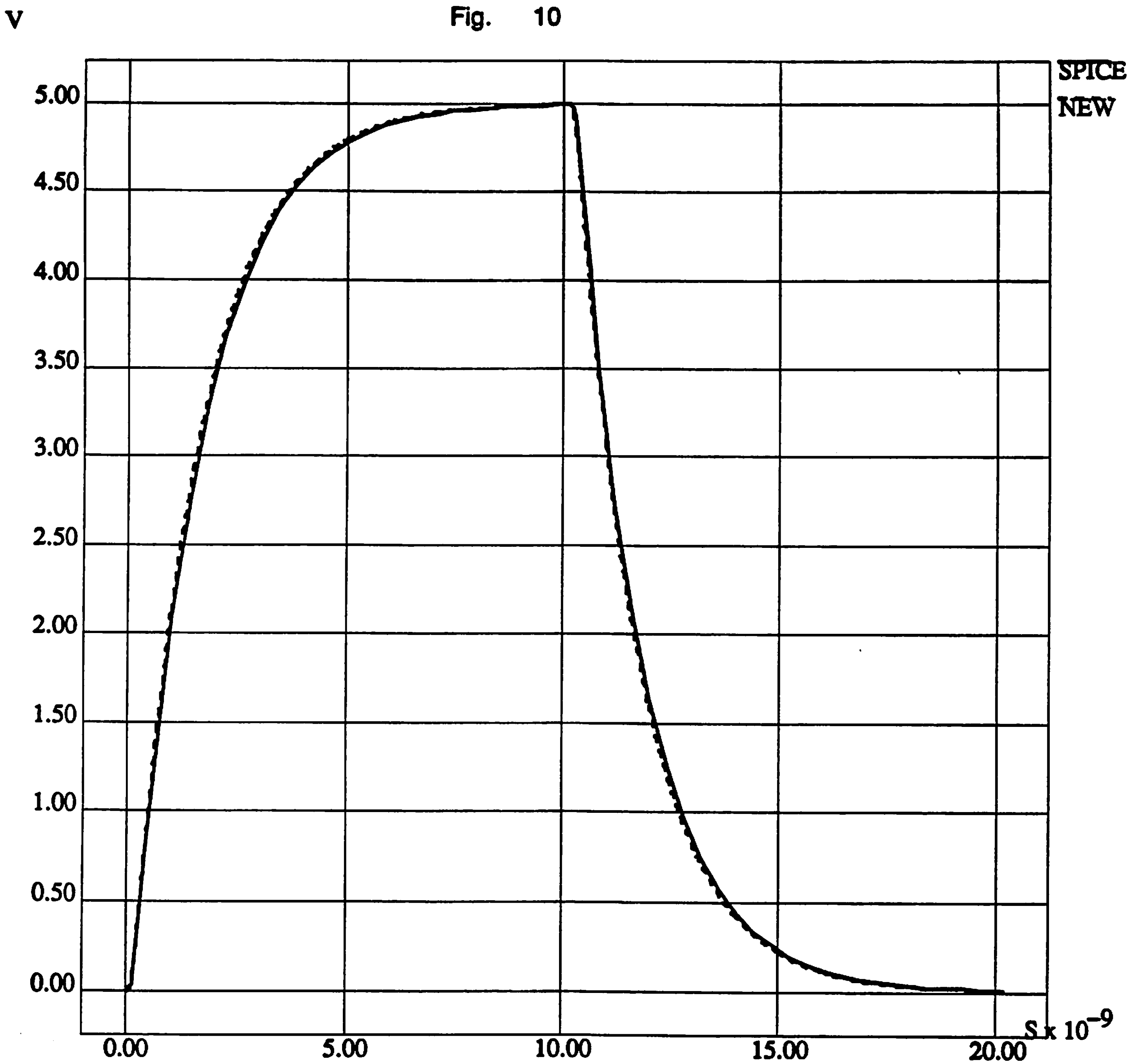

V
SPICE
NEW
5.00
4.50
4.00
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
0.00
5.00
10.00
15.00
20.00
S x 10−9

Fig. 11

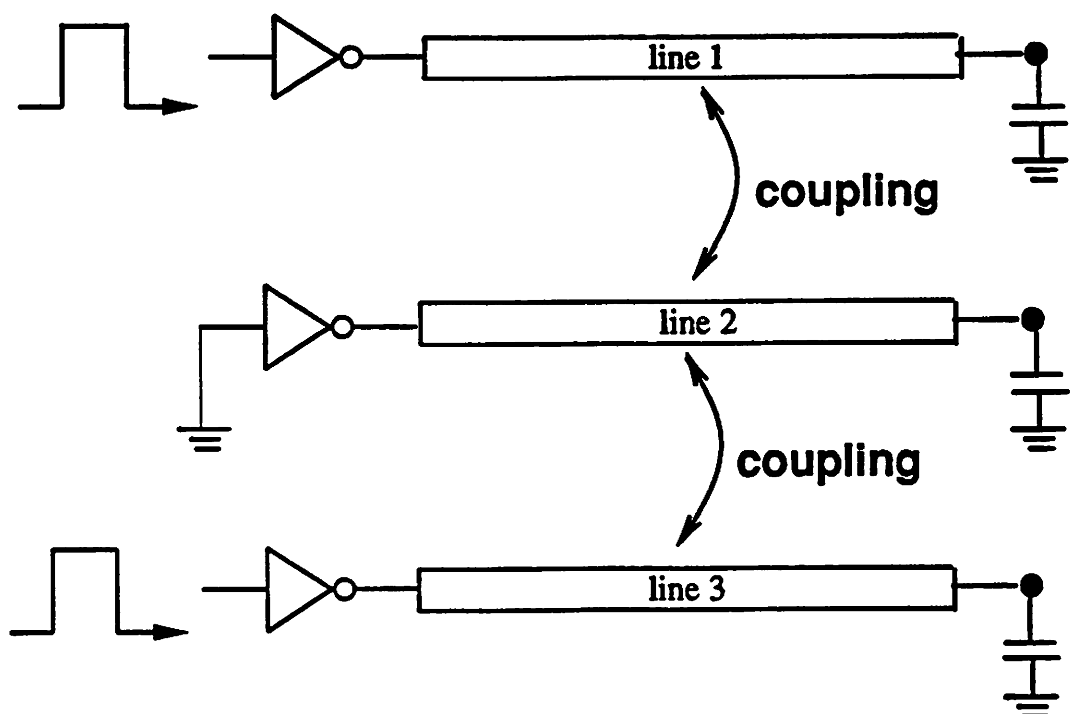
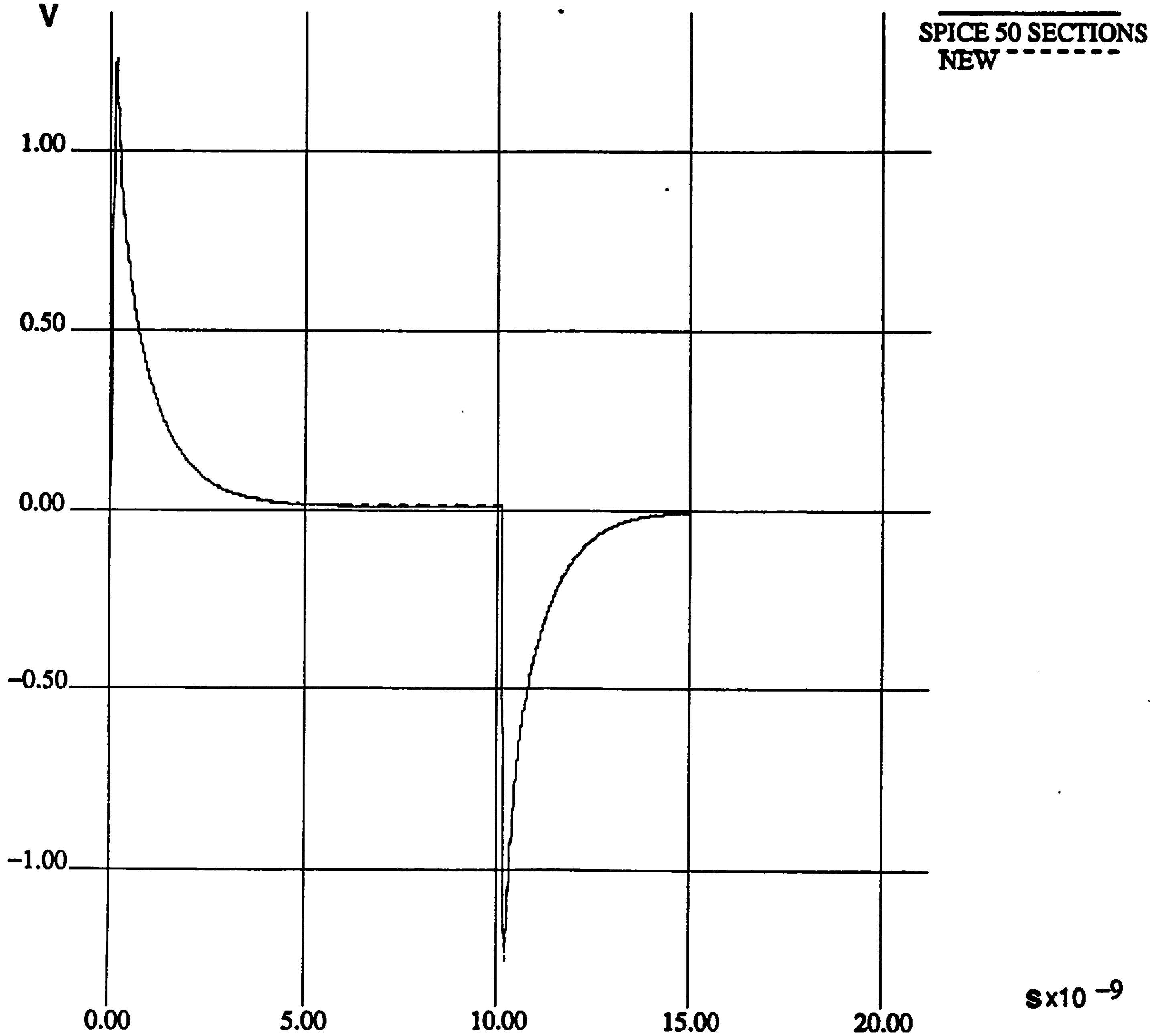


Fig. 12


V
SPICE 50 SECTIONS
NEW
1.00
0.50
0.00
-0.50
-1.00
0.00
5.00
10.00
15.00
20.00
s x10 $^{-9}$