OPTIMAL ROUTING CONTROL: GAME THEORETIC APPROACH

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Optimal Routing Control: Game Theoretic Approach*

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June 9, 1998

Abstract

Communication networks shared by selfish users are considered and modeled as noncooperative *repeated* games. Each user is interested only in optimizing its own performance by controlling the routing of its load. We investigate the existence of a Nash equilibrium point (NEP) that achieves the system-wide optimum cost. The existence of a subgame-perfect NEP that not only achieves the system-wide optimum cost but also yields a cost for each user no greater than its stage game NEP cost is shown for two-node multiple link networks. It is shown that more general networks where all users have the same source-destination pair have a subgame-perfect NEP that achieves the minimum total system cost, under a mild technical condition. It is shown that general networks with users having multiple source-destination pairs do not necessarily have such an NEP.

1 Introduction

Traditionally the network was designed and operated as a single entity with a single objective under the assumption that users were passive and would cooperate for the good of the entire network. In modern networking, however, this assumption is no longer valid since many networks, each of which belongs to a different administration and shares resources with others, are internetworked to form a coalition of networks. Furthermore, different service providers compete to provide services to users over the network.

Thus, an alternative approach is required that views the network as a resource shared by active players, where players may have different performance measures and demands, which may even be contradictory in some cases. One natural way of managing such a resource is letting the players compete with one another and allow themselves to settle to an equilibrium where each of them reaches its optimum working state. Obviously, in this kind of environment, players change their behavior according to those of others, trying to achieve the best performance, and this gives a rise to a dynamic system. The behaviour of players in such an environment can be addressed in the framework of game theory. Of key importance here is the notion of an equilibrium where no user finds it beneficial to change its behavior unilaterally. Such an equilibrium is called a Nash equilibrium point (NEP).

There has been some prior work on applying game theory in many different areas of networking. For instance, Douligeris and Mazumdar [2], Bovopoulos and Lazar [12] and Hsiao and Lazar [13]

*Research support by NSF grant NCR 9422513.

discuss flow control problems. Lee and Cohen [15] study the problem of customer allocation in a system of parallel M/M/c queues. Dziong and Mason [3] consider a call admission control problem. In [14], Lazar, Orda, and Pendarakis investigate the problem of assigning bandwidth to different virtual paths and show that the Nash equilibrium satisfies a certain fairness criterion. Shenker in [19] investigates an internetwork gateway problem, where users are assumed to be selfish, and discusses the issue of designing resource allocation mechanisms that produce efficient throughput and congestion allocations despite the selfish user behavior. Routing problem is also studied in game theoretic framework by Economides and Silvester [5] and Yamaoka and Sakai [20], [21].

Most of the past research on the use of game theory in networking problems, including all of the papers cited above, has restricted itself to the use of *static* games as models. In an attempt to understand the dynamics of modern networks, we address the routing problem from a game theoretic point of view, but using the concepts and techniques of *dynamic* game theory. Our starting point is a paper of Orda, Rom, and Shimkin [16], where a routing problem, formulated precisely in section 3, is considered using a static game theoretic model. The agents in this routing game are naturally thought of as being the network access providers. In this context the users would typically interact with each other several times before the nature of the game changes significantly, which might happen, for instance, because of the addition of new network access providers, a change in the topology of the network, or a significant change in the net load being handled by a network access provider. Thus, it is interesting to approach this routing problem as a repeated game rather than just a single shot game. While a network access provider might typically handle loads between several origin-destination pairs, we focus here on the situation where each of the competing users carries flow between a specific origin-destination pair. In a repeated game there is the possibility of strategies that result in NEPs which are more efficient than in the single shot game. In this paper we are interested in investigating the implications of the existence of such new NEPs in the dynamic routing game. In particular, we are interested in how efficient such Nash equilibria are relative to the system wide optimum cost that might be achievable if the network could force all the users to operate cooperatively to minimize the overall cost.

In the static game model considered in [16], the uniqueness of the NEP in the routing problem for a two node network with parallel links is proved, for a wide class of cost functions. It is shown that the overall system cost at this unique Nash equilibrium can be substantially larger than the minimum system cost that could be achieved if all the users cooperate. In more general networks, the situation is even more complicated, because uniqueness of NEP does not hold even for rather natural cost functions. An example is given in [16] where there is more than one NEP when the cost functions are of what they call type-A.

In this paper we restrict ourselves to cost functions that are of type-C in the terminology of [16]. Such cost functions are motivated by the delay formula in an exponential server queue. See section 3 for more details. We prove that, in the two node routing problem with parallel links, in a dynamic game theoretic framework, there are NEPs where the agents operate at the unique system-wide optimum point, while at the same time, each user's cost is no greater than it would be in the unique stage game NEP. Such strategies are supported by credible threats or rewards that the users might make or offer to one another. In the language of game theory, this says that any such strategy is not only an NEP but also a subgame-perfect NEP (SPNEP); in fact, it can be shown that no user will be able to reduce its own cost in any subgame by deviating from such an equilibrium. Section 2 contains the basic definitions and results from dynamic game theory that we need to develop our results. In more general networks, it is much harder to determine if strategies exist in the repeated game that yield a cost for each user that is smaller than or equal to its cost in every stage game NEP. Much of the difficulty lies in the analysis of the stage game and the characterization of its NEPs. Nevertheless, in networks where there is a fixed source node and a fixed destination node common to all the users, we show, under a mild technical condition, that there exists a SPNEP that drives users to operate at a system-wide optimum point. When different users may have different source-destination pairs, we show by means of an example that the existence of an NEP for the repeated game which achieves the systemwide optimum cost cannot be guaranteed in general. On the positive side, we show that there always exists an NEP in the repeated game which achieves a total system cost that is no more than the minimum total system cost over all the NEPs of the static game played between "class users", where a class user between a given source node and a given destination node is defined to be the coalition of all the actual users that have that particular source-destination pair. However, we show by means of an example that the systemwide minimum cost could sometimes be strictly smaller than this minimum, so the preceding result, while encouraging, is not strong enough.

We should mention that, since the appearance of [16], a number of other works have also appeared that discuss extensions of this model. Korilis, Lazar, and Orda [9] show that in a parallel link network, with selfish users that attempt to optimize their individual performance objective, the capacity allocation problem has a simple and intuitive optimum solution that coincides with that of the single user case. In another paper [10] they show that in a parallel link network environment, under certain conditions, the network manager, who is aware of the noncooperative behavior of the users, can use its own flow to drive the users to operate at an equilibrium point that is oftentimes more efficient than the Nash equilibrium point. The existence of maximally efficient strategies for the manager is investigated, and the necessary and sufficient conditions for the existence of a maximally efficient strategy are derived in that paper.

This paper is organized in the following way : we begin in section 2 with a brief summary on the language of game theory. Here we also state, with precise references, the results from the theory of dynamic games that we will be referring to. We discuss two node parallel link networks in section 3. In section 4 we discuss general networks, both in the case where all the users have a single source-destination pair and in the case when there are multiple source-destination pairs. Some summarizing remarks are made in the final section.

2 Game Theory

In this section we will briefly review the language of game theory. For more details, refer to [6], [7], and [17]. One can model a game in many different ways, depending on the properties and information available to the users. In *static* games, the interaction between users occurs only once, while in *dynamic* games the interaction occurs several times. An example of a dynamic game is a repeated game where the same static game is played many times.

Depending on whether each player knows the other players's payoff functions or not, a game can be formulated either as a *complete* or *incomplete* information game. If every player is aware of history of all the plays made, the game is said to have *perfect information.* If not, the game is of *imperfect information.*

A Nash equilibrium of a game is a choice of strategies by the players where each player's strategy is a best response to the other players' strategies. This implies that no player can increase its payoffs by unilaterally deviating from the equilibrium. One problem with Nash equilibria is that some Nash equilibria involve players choosing irrational plays. A simple refinement, called a *subgame-perfect* Nash equilibrium eliminates many such Nash equilibria involving irrational plays. A subgame is by itself a well defined game that starts from a decision node n that is not the beginning node of the game, and includes all the decision nodes and terminating nodes following n in the game tree. A Nash equilibrium is a subgame-perfect Nash equilibrium if the players' strategies constitute a Nash equilibrium in every subgame. Consider the example in Figure 1, where the pairs of numbers at the terminal nodes of the game tree give the payoffs of the two players, and the player who plays at any node of the game tree is indicated in the figure. This is a game of complete information.

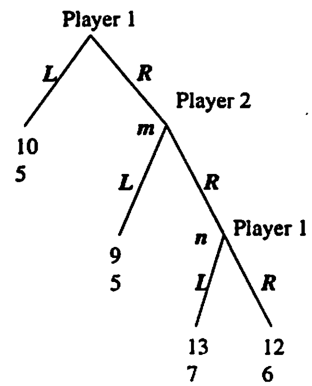

Figure 1: A Nash equilibrium that is not a subgame-perfect Nash equilibrium.

Playing (L, L, L) is a Nash equilibrium in this game since neither player can increase its own payoffs by deviating from the equilibrium. However, (L, L, L) is not a subgame-perfect Nash equilibrium because player 2 playing L at the decision node m does not lead to a Nash equilibrium in the subgame starting at m. If for some reason player 1 plays R at the beginning, then player 2 should play R whether player 1 plays L or R at the information node n. This implies that player 2 playing L at node m is not a credible threat and that player 1 should always play R at the beginning of the game.

We will now discuss a result due to Rosen [18], which is used to establish the existence of a Nash equilibrium of the stage games that will be discussed in the following sections. Consider an I-player game, where the strategy of the ith player is represented by the vector f^i in the subset F^i of the Euclidean space $R^{m_i}, i = 1,...,I$, and the vector $f \in R^m$ denotes the simultaneous strategies of all players, where $m = m_1 + \ldots + m_I$. Thus $f \in F = F^1 \times \ldots \times F^I$, which we assume is a convex, closed, and bounded set. The cost function for the ith player depends on the strategies of all the other players as well as on its own strategy, and is given by the function $J^i(f) = J^i(f^1, ..., f^I)$. We assume that for all $f \in F$, $J^i(f)$ is continuous in f and is convex in f^i for each fixed $(f^1, ..., f^{i-1}, f^{i+1}, ..., f^I)$. Then Theorem 1 in [18] states that there always exists a Nash equilibrium point for such a game.

Let us now define player i's reservation cost, a concept which is used later in stating the folk theorem for repeated games. Player i's reservation cost is denoted by $\underline{v}_i$ and defined as

$$\underline{v}_i = max_{f^{-i} \in F^{-i}}[min_{f^i \in F^i} J^i(f^i, f^{-i})],$$

where F^i is player i's strategy space, F^{-i} is the product space of the strategies of all players except for player i, and J^i is player i's cost function. The significance of player i's reservation cost is as follows : if the other players are colluding to punish player i, player i can guarantee itself a cost no greater than its reservation cost. In other words, player i's reservation cost is the highest cost player i's opponents can hold it to by any choice of f^{-i}.

We will now state two key theorems in game theory for repeated games, which will be used to prove the results of this paper. For more details on the theorems and their proofs, refer to [6], [7], and [17]. These theorems are about repeated games with discounting. Namely, the same static game, called the stage game, is played an infinite number of times. At the end of each stage n, each player is aware of all the actions of all the players at times 1 through $n-1$. The overall cost of player i is the discounted sum of its costs at each stage, for some discount factor $0 < \delta < 1$, normalized by $1-\delta$.

The first theorem of interest to us is the *folk theorem* for repeated games, which says that for every feasible cost vector v with $v_i < \underline{v}_i$ for all players i, where $\underline{v}_i$ is player i's reservation cost, there exists a $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$ there is a Nash equilibrium in the repeated game with cost vector v. Moreover, if rational feasible region, i.e. the portion of the feasible region that Pareto-dominates the reservation cost vector, i.e. $\{v : v_i \le \underline{v}_i \text{ for all } 1 \le i \le I\}$, satisfies the full dimensionality condition, i.e., it has nonempty interior or has same dimension as the number of players, then there is a subgame-perfect Nash equilibrium with cost vector v.

The second theorem of interest to us is called *Friedman's theorem*, and says that if a stage game Nash equilibrium α^* has the corresponding cost vector e, then for any feasible cost vector v with $v_i < e_i$ for all players i, there is a $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$ there is a subgame-perfect Nash equilibrium in the repeated game with cost vector v.

In the following sections these theorems will be used to establish the existence of Nash equilibria and/or subgame-perfect Nash equilibria in the dynamic routing game of interest in this paper.

3 A Network of Parallel Links

3.1 Model and Problem Formulation

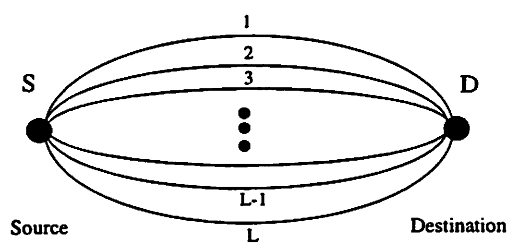

Figure 2: Parallel link network.

We are given a network with a set $I = \{1, 2, ..., \mathrm{I}\}$, $\mathrm{I} \ge 2$, of users that share a set of parallel communication links $L = \{1, 2, ..., \mathrm{L}\}$ interconnecting a common source to a common destination node. Without loss of generality, we assume that the links are ordered by decreasing capacity, i.e., $C_1 \ge C_2 \ge \cdots \ge C_{\mathrm{L}}$. Each user $i \in I$ is assumed to be selfish in the sense that it attempts to minimize its own cost regardless of what the other users are doing. User i has a demand, which is some ergodic process with an average rate of r^i. Without loss of any generality, we will assume that users are ordered in order of decreasing average rate, $r^1 \ge r^2 \ge \cdots \ge r^{\mathrm{I}}$.

We will first describe the stage game, which we assume is repeated many times. If each user is not sure when the game will end, we can model this as an infinitely repeated game with an appropriate discounting factor $0 < \delta < 1$. In the stage game, each user splits its demand over the communication links, i.e., user i decides how much of its demand, f_l^i, it will send on link l. We must have $f_l^i \ge 0$ (nonnegativity constraint) and $\sum_{l \in L} f_l^i = r^i$ (demand constraint). Let f_l denote the total flow on link l, i.e., $f_l = \sum_{i \in I} f_l^i$. The flow configuration of user i is denoted by f^i. and the system flow configuration by $f = (f^1, f^2, ..., f^{\mathrm{I}})$. A user flow configuration f^i is said to be feasible if it satisfies the nonnegativity and demand constraints. We denote the set of all feasible flow configurations for user i by F^i. Similarly, a system flow configuration f, is feasible if every user flow configuration is feasible, and $F = F^1 \times \ldots \times F^I$ denotes the set of all feasible system flow configurations.

In order to compare the performance of each user $i \in I$, we need to have a performance measure. This performance measure is given by a cost function $J^i(f)$ defined for each user i. The goal of each user is to minimize its cost by distributing its demand over the links. The cost of user i generally depends not only on its flow configuration but also on those of other users. Since we are assuming that every user is selfish, the problem can be modeled as a noncooperative game where each user is trying to minimize its cost [16]. A natural question that arises in this type of setting is whether there is a Nash equilibrium of the game or not. In other words, we are interested in finding a system flow configuration such that no user finds it beneficial to change its own flow configuration assuming that no other users do. Mathematically a system flow configuration $\tilde{f} = (\tilde{f}^1, \tilde{f}^2, ..., \tilde{f}^I)$ is a Nash Equilibrium Point (NEP) if, for all $i \in I$, the following holds:

$$\begin{aligned} J^i(\tilde{f}) &= J^i(\tilde{f}^1, ..., \tilde{f}^{i-1}, \tilde{f}^i, \tilde{f}^{i+1}, ..., \tilde{f}^I) \\ &= min_{f^i \in F^i} J^i(\tilde{f}^1, ..., \tilde{f}^{i-1}, f^i, \tilde{f}^{i+1}, ..., \tilde{f}^I) \end{aligned}$$

The importance of an NEP is that it is a point at which no user has an incentive to deviate. However, one problem with an NEP is that it is not necessarily very efficient. In fact, Korillis, Lazar, and Orda [10] give numerical examples with natural cost functions where the difference between the total cost at the system-wide optimum point and that at the NEP could be more than 20 percent.

In our analysis, we use what are called type-C cost functions in [16]. The cost function for user i, J^i, is the sum of link cost functions, J^i_l, i.e., $J^i(f) = \sum_{l \in L} J^i_l(f_l)$, and $J^i_l(f_l) = J^i_l(f^i_l, f_l) = f^i_l \cdot T_l(f_l)$, where

$$T_l(f_l) = \begin{cases} \frac{1}{C_l - f_l}, & f_l < C_l \\ \infty, & f_l \geq C_l \end{cases} \tag{1}$$

where C_l is the capacity of link l. Such a cost function is very natural in this context as it is directly motivated by the delay formula in an exponential server queue for a given throughput.

Throughout the paper, we will assume that the stability condition is satisfied, i.e., $\sum_{i \in I} r^i < \sum_{l \in L} C_l$. Then it is easy to see that at any NEP the costs of all users are finite. Otherwise, at least one user with infinite cost can change its own flow configuration to have finite cost.

It turns out that, under these assumptions, the routing game satisfies the conditions of the theorem of Rosen [18] described in section 2 and so the existence of an NEP is guaranteed. Also, Kuhn-Tucker conditions constitute the necessary and sufficient conditions for a feasible system configuration to be an NEP. These say that for every $i \in I$ there must exist a Lagrange multiplier, λ^i, such that, for every link $l \in L$,

$$\begin{aligned} f^i_l > 0 &\Rightarrow K^i_l(f_l) = \lambda^i \\ f^i_l = 0 &\Rightarrow K^i_l(f_l) \geq \lambda^i \end{aligned}$$

where K^i_l is the partial derivative of J^i_l with respect to f^i_l. We will call λ^i user i's marginal cost at the NEP.

Orda, Rom, and Shimkin prove in [16] that in the parallel link case with type-C cost functions there is a unique NEP. An important consequence of this is that given a set of demands for all the users, there is a unique system flow configuration, f', that achieves the smallest total system cost. This can also be seen directly by observing that this flow configuration is the solution to a convex optimization problem. Unfortunately, in most cases where multiple selfish users compete over the network, the resulting unique NEP does not result in the same link flows as the system-wide optimum point.

From the theory of dynamic games we know that it is possible to get NEPs in the repeated game that are different from merely repeating the NEP of the static game at every period. Some of these NEPs may be more efficient than repeating the stage game NEPs at every period. Thus, it would be interesting to see if the system-wide optimum point can be supported by an NEP in the repeated game. The folk theorem and Friedman's theorem, which were stated in section 2 give us considerable information about the cost vectors that can be supported by NEPs in the repeated game. We will use these theorems to analyze the issue of when it is possible to get NEPs in the repeated game that are efficient (achieve a total cost equal to the minimum cost achievable if all the users were to collaborate for social good).

3.2 Properties of the unique stage game NEP

In this subsection, we will briefly describe the properties of the unique stage game NEP. For more details, refer to [16]. Assume that users are ordered by decreasing demand and links are ordered by decreasing capacity. Let $\check{f}$ denote the unique stage game NEP and L_i denote the set of links used by user i at the NEP, i.e., $L_i = \{l \in L : \check{f}_l^i > 0\}$. Then, at the unique stage game NEP $\check{f}$, the following are true.

(1) $\check{f}_1 \geq \check{f}_2 \geq \cdots \geq \check{f}_{\mathrm{L}}$. For $1 \leq l \leq L$, if $C_l > C_{l+1}$ and $\check{f}_l > 0$, then $\check{f}_l > \check{f}_{l+1}$, while if $C_l = C_{l+1}$ then $\check{f}_l = \check{f}_{l+1}$.

(2) $\check{f}_1^i \geq \check{f}_2^i \geq \cdots \geq \check{f}_{\mathrm{L}}^i$, for all $i \in I$. For $1 \leq l \leq L$, if $C_l > C_{l+1}$ and $\check{f}_l^i > 0$, then $\check{f}_l^i > \check{f}_{l+1}^i$, while if $C_l = C_{l+1}$ then $\check{f}_l^i = \check{f}_{l+1}^i$.

(3) $C_1 - \check{f}_1 \geq C_2 - \check{f}_2 \geq \cdots \geq C_{\mathrm{L}} - \check{f}_{\mathrm{L}}$. For $1 \leq l \leq L$, if $C_l > C_{l+1}$ and $\check{f}_{l+1} > 0$, then $C_l - \check{f}_l > C_{l+1} - \check{f}_{l+1}$.

(4) $\check{f}_l^i \geq \check{f}_l^j$ for all $l \in L$ if $i < j$, i.e., $r^i \geq r^j$. Moreover, if $r^i > r^j$ and $\check{f}_l^j > 0$, then $\check{f}_l^i > \check{f}_l^j$.
(4′) $L_i \supseteq L_j$ if $i < j$.

3.3 Existence of SPNEP that yields the optimum system cost

We now assume that the stage game of section 3.1 is repeated infinitely often, with discounting. We will show the existence of a system flow configuration that is an NEP of the repeated game and achieves the minimum total system cost C'. Furthermore we will also show the existence of an SPNEP for the repeated game achieving the minimum total system cost, where the cost of each user is no more than its cost if play proceeds by repeating the unique NEP of the stage game.

We first need to define user i's reservation cost. This is defined to be

$$\underline{v}_i = max_{f^{-i}}[min_{f^i} J^i(f^i, f^{-i})] \tag{2}$$

where $f^{-i} = (f^1, ..., f^{i-1}, f^{i+1}, ..., f^I)$. In words, if other users collude to punish user i, it can guarantee that it incurs a cost no more than $\underline{v}_i$. Let $\underline{v} = (\underline{v}_1, \underline{v}_2, ..., \underline{v}_I)$. The folk theorem for repeated games, which was also stated in section 2, says that any cost vector that strictly Pareto-dominates $\underline{v}$ in each coordinate can be supported by an NEP in the repeated game, for any discount factors sufficiently close to 1. Our first result is the following :

Theorem 1 *If the discount factor in the repeated game is sufficiently close to* 1 *there is an NEP in the repeated game that achieves the minimum total system cost,* C'.

Let $\check{f}$ denote the unique stage game NEP and $\check{J}^i$ the cost of user i at the stage game NEP. To prove theorem 1 we need the following :

Lemma 1 $\frac{\check{J}^i}{\check{J}^j} \geq \frac{r^i}{r^j}$ *if* $i \leq j$.

Proof: We will first show

$$\frac{\check{f}_l^j}{\check{f}_l^i} \geq \frac{\check{f}_{l+1}^j}{\check{f}_{l+1}^i} \; if \; \check{f}_{l+1}^i \; > \; 0, \tag{3}$$

and show that equation (3) with property(3) of section 3.2 with $C_l - \check{f}_l \geq C_{l+1} - \check{f}_{l+1}; 1 \leq l \leq$ L-1, implies that

$$\check{J}^j = \sum_{l=1}^{\mathrm{L}} \frac{\check{f}_l^j}{C_l - \check{f}_l} \leq \sum_{l=1}^{\mathrm{L}} \frac{\check{f}_l^i}{C_l - \check{f}_l} \frac{r^j}{r^i} = \check{J}^i \cdot \frac{r^j}{r^i} \tag{4}$$

and this will prove the lemma.

Let us prove equation (3). There are two cases to consider :

(1) If $\check{f}_{l+1}^j = 0$, it's trivial.

(2) If $\check{f}_{l+1}^j > 0$, then from the K-T conditions and property (2) of section 3.2, we have

$$\begin{aligned} \lambda^j &= \frac{1}{C_l - \check{f}_l} + \frac{\check{f}_l^j}{(C_l - \check{f}_l)^2} \\ &= \frac{1}{C_{l+1} - \check{f}_{l+1}} + \frac{\check{f}_{l+1}^j}{(C_{l+1} - \check{f}_{l+1})^2}, \end{aligned} \tag{5}$$

while from the K-T conditions and properties (2) and (4) of section 3.2, we have

$$\begin{aligned} \lambda^i &= \frac{1}{C_l - \check{f}_l} + \frac{\check{f}_l^i}{(C_l - \check{f}_l)^2} \\ &= \frac{1}{C_{l+1} - \check{f}_{l+1}} + \frac{\check{f}_{l+1}^i}{(C_{l+1} - \check{f}_{l+1})^2}. \end{aligned} \tag{6}$$

Let $\alpha = \frac{1}{C_{l+1} - \check{f}_{l+1}} - \frac{1}{C_l - \check{f}_l}$. If we subtract $\frac{1}{C_l - \check{f}_l}$ from both sides of equations (6) and (5), we have

$$\frac{\check{f}_l^i}{(C_l - \check{f}_l)^2} = \alpha + \frac{\check{f}_{l+1}^i}{(C_{l+1} - \check{f}_{l+1})^2}, \tag{7}$$

and

$$\frac{\check{f}_l^j}{(C_l - \check{f}_l)^2} = \alpha + \frac{\check{f}_{l+1}^j}{(C_{l+1} - \check{f}_{l+1})^2}. \tag{8}$$

Divide equation (8) by equation (7) and simplify.

$$\frac{\check{f}_l^j}{\check{f}_l^i} = \frac{\alpha' + \check{f}_{l+1}^j}{\alpha' + \check{f}_{l+1}^i}, \tag{9}$$

where $\alpha' = \alpha \cdot (C_{l+1} - \check{f}_{l+1})^2$. Since, by property (3) of section 3.2, we have $\alpha' \geq 0$ and, by property (4) of section 3.2, we have $\check{f}_l^j \leq \check{f}_l^i$, we get

$$\frac{\check{f}_l^j}{\check{f}_l^i} = \frac{\alpha' + \check{f}_{l+1}^j}{\alpha' + \check{f}_{l+1}^i} \geq \frac{\check{f}_{l+1}^j}{\check{f}_{l+1}^i}, \tag{10}$$

which proves equation (3).

Now we will show that equation (3) implies that

$$\sum_{k=1}^{l} \check{f}_k^j \geq \frac{r^j}{r^i} \cdot \sum_{k=1}^{l} \check{f}_k^i \; \forall \, l \in L. \tag{11}$$

Assume the contrary, i.e., suppose that there exists $l \in L$ such that

$$\sum_{k=1}^{l} \check{f}_k^j < \frac{r^j}{r^i} \sum_{k=1}^{l} \check{f}_k^i \, . \tag{12}$$

If there is such an $l \in L$, there must also be one for which we have, in addition, that

$$\frac{r^j}{r^i} \cdot \check{f}_l^i > \check{f}_l^j \, , \tag{13}$$

so we assume that both equations (12) and (13) hold. First note that equation (12) together with property (2) of section 3.2 implies that $\check{f}_{l+1}^j > 0$. This is because we cannot have $\sum_{k=1}^{l} \check{f}_k^j = r^j$ without violating the demand constraint for user i. Then, from equations (3) and (13), we have

$$\frac{\check{f}_{l+1}^i}{\check{f}_{l+1}^j} \geq \frac{\check{f}_l^i}{\check{f}_l^j} > \frac{r^i}{r^j}, \tag{14}$$

Equation (14) implies

$$\frac{r^j}{r^i} \cdot \check{f}_{l+1}^i > \check{f}_{l+1}^j \tag{15}$$

and from equations (12) and (15)

$$\sum_{k=1}^{l+1} \check{f}_k^j < \frac{r^j}{r^i} \sum_{k=1}^{l+1} \check{f}_k^i \tag{16}$$

Since equations (15) and (16) are just the same as equations (13) and (12) respectively except that they now hold for link $l+1$ instead of link l, we may repeatedly argue in exactly the same way and conclude that

$$\sum_{k=1}^{L} \check{f}_k^j < \frac{r^j}{r^i} \sum_{k=1}^{L} \check{f}_k^i \, . \tag{17}$$

But this contradicts the demand constraints for users i and j. Therefore we have proved equation (11).

Now, for any $0 \le r \le r^i$, let

$$J^i(r) = \sum_{l=1}^{k} \frac{\check{f}_l^i}{C_l - \check{f}_l} + \frac{q}{C_{k+1} - \check{f}_{k+1}}, \tag{18}$$

where k and q are defined in terms of r by

$$0 \le r = \sum_{l=1}^{k} \check{f}_l^i + q \le r^i \text{ and } 0 < q \le \check{f}_{k+1}^i.$$

Similarly, for any $0 \le r' \le r^j$, let

$$J^j(r') = \sum_{l=1}^{k'} \frac{\check{f}_l^j}{C_l - \check{f}_l} + \frac{q'}{C_{k'+1} - \check{f}_{k'+1}}, \tag{19}$$

where k' and q' are defined in terms of r' by

$$0 \le r' = \sum_{l=1}^{k'} \check{f}_l^j + q' \le r^j \text{ and } 0 < q' \le \check{f}_{k'+1}^j.$$

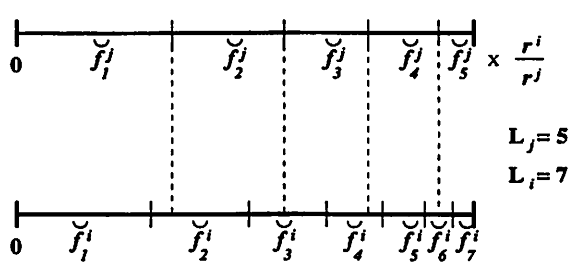

Figure 3: Comparison of cost per unit flow.

Let L_i, $i \in I$, be the largest $l \in L$ such that $\check{f}_l^i > 0$. Now, since $C_1 - \check{f}_1 \ge \cdots \ge C_L - \check{f}_L$, as illustrated in Fig 3, using equation (11), the cost per unit flow of each block l, where $l = 1,\ldots,L_j$, in the top graph is smaller than or equal to that of the corresponding block in the bottom graph. This means that

$$\frac{r^i}{r^j} \cdot J^j(\check{f}_1^j) \le J^i(\frac{r^i}{r^j} \cdot \check{f}_1^j) \tag{20}$$

and

$$\begin{aligned} &\frac{r^i}{r^j} \cdot (J^j(\sum_{l=1}^{\bar{l}} \check{f}_l^j) - J^j(\sum_{l=1}^{\bar{l}-1} \check{f}_l^j)) \\ &\le J^i(\frac{r^i}{r^j} \cdot \sum_{l=1}^{\bar{l}} \check{f}_l^j) - J^i(\frac{r^i}{r^j} \cdot \sum_{l=1}^{\bar{l}-1} \check{f}_l^j), \\ &\qquad 2 \le \bar{l} \le L_j. \end{aligned} \tag{21}$$

Sum equations (20) and (21) for all $1 \le l \le L_j$ and multiply by $\frac{r^j}{r^i}$. Then, we get

$$\check{J}^j = J^j(r^j) \le \frac{r^j}{r^i} J^i(\frac{r^i}{r^j} \cdot r^j) = \frac{r^j}{r^i} J^i(r^i) = \frac{r^j}{r^i} \check{J}^i. \tag{22}$$

Another way of writing equation (22) is

$$\frac{\check{J}^i}{\check{J}^j} \geq \frac{r^i}{r^j}. \tag{23}$$

This completes the proof of the lemma. ■

Let C' denote the minimum total system cost that can be achieved given the set of demands $(r^1, \ldots, r^I)$, and let $R = \sum_{i=1}^{I} r^i$. Also, let $f' = (f_1', \ldots, f_L')$ denote the unique system configuration that achieves C'. Then we have:

Lemma 2 *$\underline{v}_i \geq C' \cdot \frac{r^i}{R}$ for all $i \in I$. Moreover, if there exists $l \in L$ such that $f_l' > f_{l+1}' > 0$, then $\underline{v}_i > C' \cdot \frac{r^i}{R}$ for all $i \in I$.*

Proof: From Lemma 1, we know that at the unique stage game NEP with a finite number of no less than two users

$$\frac{\check{J}^i}{\check{J}^j} \geq \frac{r^i}{r^j} \; if \; r^i > r^j, \tag{24}$$

where $\check{J}^i$ is the cost of user i at the NEP. Suppose the other users want to punish user i. We can break up the other users to punish user i in the following way : split the other users into n identical users each with demand $r^p = \frac{\sum_{j \neq i} r^j}{n} < r^i$. Let $\check{J}^i(n)$ denote the cost of user i at the unique NEP of this new problem. Then, we know at the unique NEP of the new problem with the n identical users and user i

$$\frac{\check{J}^i(n)}{\check{J}^p(n)} \geq \frac{r^i}{r^p},$$

where user p is one of the identical users and $\check{J}^p(n)$ denotes the NEP cost of such a user in the new problem. Therefore,

$$\check{J}^i(n) \geq \frac{r^i}{R} \cdot C_{system}(n) \geq \frac{r^i}{R} \cdot C', \tag{25}$$

where $C_{system}(n)$ is the total system cost at the NEP of the problem with user i and the n identical users. Thus,

$$\underline{v}_i \geq \check{J}^i(n) \geq \frac{r^i}{R} \cdot C'. \tag{26}$$

Now suppose there exists $l \in L$ such that $f_l' > f_{l+1}' > 0$. Note that, by property (1) of section 3.2 applied to the "game" consisting of a single user, this implies that $C_l > C_{l+1}$. Pick the smallest such l. From the K-T conditions for the "game" consisting of a single user, and because $f_l' > f_{l+1}' > 0$, we have

$$\begin{aligned} & \frac{1}{C_l - f_l'} + \frac{f_l'}{(C_l - f_l')^2} \\ = & \frac{1}{C_{l+1} - f_{l+1}'} + \frac{f_{l+1}'}{(C_l - f_{l+1}')^2}. \end{aligned} \tag{27}$$

We will show that any stage game NEP with any finite set of no less than two users, $\check{f}$, results in a strictly higher total system cost than C' by showing that $\check{f} \neq f'$. More specifically, we will show that $\frac{\breve{f}_l}{\breve{f}_{l+1}} \neq \frac{f'_l}{f'_{l+1}}$. Let $I = \{1,...,\mathrm{I}\}$ be the set of users, where $\mathrm{I} \geq 2$. Let B be the set of users that use link $l+1$ at the stage game NEP. For all $i \in B$, from the K-T conditions and property (2) of section 3.2, we have

$$\begin{aligned} \breve{\lambda}^i &= \frac{1}{C_l - \breve{f}_l} + \frac{\breve{f}^i_l}{(C_l - \breve{f}_l)^2} \\ &= \frac{1}{C_{l+1} - \breve{f}_{l+1}} + \frac{\breve{f}^i_{l+1}}{(C_{l+1} - \breve{f}_{l+1})^2} \end{aligned} \tag{28}$$

Suppose $\breve{f}_l = f'_l$ and $\breve{f}_{l+1} = f'_{l+1}$. Let $\frac{1}{C_l - \breve{f}_l} = \frac{1}{C_l - f'_l} = \alpha$ and $\frac{1}{C_{l+1} - \breve{f}_{l+1}} = \frac{1}{C_{l+1} - f'_{l+1}} = \beta$. By property (3) of section 3.2 we have $C_l - f'_l > C_{l+1} - f'_{l+1}$, so that $\beta > \alpha$. Let $\beta = \alpha + \gamma$ with some $\gamma > 0$. Subtract α from both sides of equations (28) and (27). Because we assumed $\breve{f}_l = f'_l$ and $\breve{f}_{l+1} = f'_{l+1}$, we have

$$\frac{\breve{f}_l}{(C_l - \breve{f}_l)^2} = \gamma + \frac{\breve{f}_{l+1}}{(C_l - \breve{f}_{l+1})^2} \tag{29}$$

and

$$\frac{\breve{f}^i_l}{(C_l - \breve{f}_l)^2} = \gamma + \frac{\breve{f}^i_{l+1}}{(C_{l+1} - \breve{f}_{l+1})^2} \tag{30}$$

Divide equation (29) by equation (30) and simplify. Then we get

$$\frac{\breve{f}_l}{\breve{f}^i_l} = \frac{\gamma' + \breve{f}_{l+1}}{\gamma' + \breve{f}^i_{l+1}} \tag{31}$$

where $\gamma' = \gamma \cdot (C_{l+1} - \breve{f}_{l+1})^2$. Note that $\breve{f}_{l+1} > \breve{f}^i_{l+1}$, because if not, then equation (31) implies that $\breve{f}^i_l = \breve{f}_l$ which contradicts the choosing of l as the smallest index such that $f'_l > f'_{l+1}$, because for the smallest such l, by property (1) of section 3.2, $C_1 = \cdots = C_l$ and so, by property (2) of section 3.2, all I users use link l. Since $\gamma' > 0$ and $\breve{f}_{l+1} > \breve{f}^i_{l+1}$, we have

$$\frac{\gamma' + \breve{f}_{l+1}}{\gamma' + \breve{f}^i_{l+1}} < \frac{\breve{f}_{l+1}}{\breve{f}^i_{l+1}} \tag{32}$$

Therefore, we have

$$\frac{\breve{f}_l}{\breve{f}^i_l} < \frac{\breve{f}_{l+1}}{\breve{f}^i_{l+1}} \tag{33}$$

If we multiply equation (33) by $\frac{\breve{f}^i_{l+1}\breve{f}^i_l}{\breve{f}_{l+1}}$, we get

$$\frac{\breve{f}_l}{\breve{f}_{l+1}} \cdot \breve{f}^i_{l+1} < \breve{f}^i_l \tag{34}$$

Let A be the set of users that use link l at the stage game NEP. Then, for all i in $A \setminus B$, equation (34) trivially holds since $\check{f}^i_{l+1} = 0$. If we sum equation (34) over all $i \in A$, we have

$$\begin{aligned}\frac{\check{f}_l}{\check{f}_{l+1}} \cdot \sum_{i \in A} \check{f}^i_{l+1} &= \frac{\check{f}_l}{\check{f}_{l+1}} \cdot \check{f}_{l+1} \\ &= \check{f}_l < \sum_{i \in A} \check{f}^i_l = \check{f}_l,\end{aligned} \tag{35}$$

which is a contradiction. Therefore, $\check{f} \neq f'$. Since this is true with any finite set of more than one user, the second inequality in equation (25) becomes a strict inequality in this case because C_{system} is greater than C'. Hence, from equation (26), $\underline{v}_i > \frac{r^i}{R} \cdot C'$. ■

Proof: (*Theorem 1*) We will show that, if the discount factor is sufficiently close to 1, then playing $\tilde{f} = (\tilde{f}^1, ..., \tilde{f}^I)$ every period gives rise to an NEP in the repeated game, where

$$\tilde{f}^i = (f'_1 \cdot \tfrac{r^i}{R}, ..., f'_L \cdot \tfrac{r^i}{R}) \ .$$

We need to consider two cases. First, suppose there exists $l \in L$ such that $f'_l > f'_{l+1} > 0$. From Lemma 2, $\tilde{f}$ yields a cost to each user i that is strictly smaller than its reservation cost. By the folk theorem it follows that, if the discount factor is sufficiently close to 1, there is an NEP of the repeated game with cost vector $(\frac{r^1}{R}C', \ldots, \frac{r^I}{R}C')$, and this has total cost C', which is optimum. Such an NEP can be supported by the following strategy profile in the repeated game : initially, every user uses $\tilde{f}$ every period until exactly one user i deviates from $\tilde{f}$ at some period k. If exactly one user i deviates at period k, then starting at period $k+1$ other users collaborate and use a flow configuration that yields to user i, when it best responds, a cost equal to its reservation cost, which we saw is strictly greater than its cost under $\tilde{f}$. Then, one can show that, if the discount factor is sufficiently close to 1, the normalized total discounted cost of user i will be greater than $C'.\frac{r^i}{R}$ if it deviates at any period. Hence, no user i has the incentive to deviate from $\tilde{f}$.

Now, suppose that for every $l \in L$ we either have $f'_l = f'_{l+1} > 0$ or $f'_{l+1} = 0$. Let L^0 be the set of links such that $f'_l > 0$. Then, by property (1) of section 3.2, C_l is same for all $l \in L^0$. We can now show that $\tilde{f}$ is a stage game NEP. First, K^i_l, the partial derivative of J^i_l with respect to f^i_l, is same for all $l \in L^0$ because both C_l and $\tilde{f}^i_l$ are same for all $l \in L^0$. Hence, in order to show that $\tilde{f}$ is a stage game NEP, we only need to show that no users are tempted to use any other links not in L^0. Clearly, $K^i_l(\tilde{f}_l) = \frac{1}{C_l - f'_l} + \frac{f'_l \cdot \frac{r^i}{R}}{(C_l - f'_l)^2} < \frac{1}{C_l - f'_l} + \frac{f'_l}{(C_l - f'_l)^2} = K_l(f'_l) \leq \frac{1}{C_{l'}}$ for all $l \in L^0$ and $l' \notin L^0$. This proves that no users are tempted to use any $l' \notin L_0$. Thus, $\tilde{f}$ is a stage game NEP, and repeating a stage game NEP is an NEP in the repeated game. In fact, it is an SPNEP. ■

One thing to notice about $\tilde{f}$ is that the costs of some users may be greater than their cost at the stage game NEP. Friedman in [4] shows that any cost vector that strictly Pareto-dominates a stage game NEP cost vector in each coordinate can be supported by an SPNEP for discount factors sufficiently close to 1. Thus, the following theorem shows that there exists an SPNEP, $\hat{f}$, that achieves C' and yields a cost for each user that is smaller than or equal to its cost at the stage game NEP, i.e., $J^i(\hat{f}) \leq J^i(\check{f})$ for all $i \in I$.

Theorem 2 *There exists a system flow configuration, $\hat{f}$, that yields the optimum total system cost C' and a cost for each user that is smaller than or equal to its cost at the stage game NEP. Also, if $\check{C} = \sum_{i=1}^{I} J^i(\check{f}) > C'$, then the cost of each user at this configuration is strictly smaller than its NEP cost. A consequence is that, if the discount factor is sufficiently close to* 1*, there is an SPNEP* *for the repeated game in which every user has a cost that is at most equal to its cost in the unique stage game NEP and the overall cost is optimum.*

We need several lemmas to prove Theorem 2. Let $\check{f}$ denote the stage game NEP and f' the flow configuration that achieves the minimum total system cost.

Lemma 3 *For all $l \in L$, we have $\check{f}_l > 0 \Rightarrow f'_l > 0$.*

Proof: We will show this lemma by contradiction. Suppose $\exists\, l \in L$ such that

$$\check{f}_l > 0 \text{ and } f'_l = 0. \tag{36}$$

Let L_0 denote the set of links with $f'_l > 0$ and L_1 the set of links that user 1 uses at the stage game NEP. Then, by properties (1) and (2) of section 3.2, equation (36) implies that $L_0 \subset L_1$, which means that $\exists\, l' \in L_0$ such that

$$\check{f}_{l'} < f'_{l'}. \tag{37}$$

Suppose user 0 is the global user that attempts to minimize the total system cost and uses the flow configuration f'. Let λ^0 and λ^1 be the marginal cost for user 0 and user 1, respectively. From the K-T conditions, we have

$$\lambda^1 = \frac{C_l^1}{(C_l - \check{f}_l)^2} = \frac{C_{l'}^1}{(C_{l'} - \check{f}_{l'})^2}\,, \tag{38}$$

where $C_l^i = C_l - \sum_{j \neq i} \check{f}_l^j$, and from the K-T conditions for user 0, i.e. for the "game" consisting of a single user, we have

$$\lambda^0 = \frac{C_{l'}}{(C_{l'} - f'_{l'})^2} \leq \frac{1}{C_l}\,. \tag{39}$$

With equation (39), equation (36) implies

$$\frac{C_{l'}}{(C_{l'} - f'_{l'})^2} \leq \frac{1}{C_l} < \frac{C_l^1}{(C_l - \check{f}_l)^2} \tag{40}$$

But equation (37) implies

$$\frac{C_{l'}^1}{(C_{l'} - \check{f}_{l'})^2} < \frac{C_{l'}}{(C_{l'} - f'_{l'})^2}. \tag{41}$$

Now from equations (38), (40), and (41), we have

$$\begin{aligned} \frac{C_{l'}^1}{(C_{l'} - \check{f}_{l'})^2} &< \frac{C_{l'}}{(C_{l'} - f'_{l'})^2} \leq \frac{1}{C_l} \\ &< \frac{C_l^1}{(C_l - \check{f}_l)^2} = \frac{C_{l'}^1}{(C_{l'} - \check{f}_{l'})^2} \end{aligned}$$

which is a contradiction. This proves the lemma. ∎

Lemma 4 $\sum_{l=1}^{\bar{l}} \check{f}_l \geq \sum_{l=1}^{\bar{l}} f'_l$, $1 \leq \bar{l} \leq$ L.

Proof: We will draw a contradiction by using induction. Suppose that for some $l \in L$ we have

$$\sum_{k=1}^{l} \check{f}_k < \sum_{k=1}^{l} f'_k . \tag{42}$$

Then there must also be some $l \in L$ for which, in addition, we have

$$\check{f}_l < f'_l , \tag{43}$$

so we assume that both the equations (42 and (43) hold. We will show that this means that equations (43) and (42) hold for all $\bar{l} \geq l$. Let A be the set of users that use link l and B set of users that use link l+1 at the stage game NEP. By property (2) of section 3.2 we have $A \supseteq B$. Since $\sum_{k \in L} \check{f}_k = \sum_{k \in L} f'_k$, equation (42) implies that $l < L$. Further, equation (42), together with property (1) of section 3.2 implies that $\check{f}_{l+1} > 0$, which means $B \neq \emptyset$. Let $C = A \setminus B$. From the K-T conditions, we have for all $i \in A$

$$\lambda^i = \frac{C_l^i}{(C_l - \check{f}_l)^2} = \frac{C_l - \check{f}_l^{-i}}{(C_l - \check{f}_l)^2} , \tag{44}$$

where $\check{f}_l^{-i} = \sum_{j \neq i} \check{f}_l^j = \check{f}_l - \check{f}_l^i$, and for all $i \in B$ we have

$$\lambda^i = \frac{C_{l+1}^i}{(C_{l+1} - \check{f}_{l+1})^2} = \frac{C_{l+1} - \check{f}_{l+1}^{-i}}{(C_{l+1} - \check{f}_{l+1})^2} . \tag{45}$$

Lemma 3 tells us that $\check{f}_{l+1} > 0$ implies $f'_{l+1} > 0$. From the K-T conditions for user 0 we have

$$\lambda^0 = \frac{C_l}{(C_l - f'_l)^2} = \frac{C_{l+1}}{(C_{l+1} - f'_{l+1})^2} \tag{46}$$

Let B be the number of users in B. If we sum equations (44) and (45) over $i \in B$, we get

$$\begin{aligned}
&\frac{\mathrm{B}C_l - \sum_{i \in B}(\check{f}_l - \check{f}_l^i)}{(C_l - \check{f}_l)^2} \\
&= \frac{\mathrm{B}C_l - \mathrm{B}\check{f}_l + \sum_{i \in B} \check{f}_l^i}{(C_l - \check{f}_l)^2} \\
&= \frac{\mathrm{B}C_l - \mathrm{B}\check{f}_l + (\check{f}_l - \sum_{i \in C} \check{f}_l^i)}{(C_l - \check{f}_l)^2} \\
&= \frac{\mathrm{B}C_l - (\mathrm{B}-1)\check{f}_l - \sum_{i \in C} \check{f}_l^i}{(C_l - \check{f}_l)^2} \qquad (47) \\
&= \frac{\mathrm{B}C_{l+1} - \sum_{i \in B}(\check{f}_{l+1} - \check{f}_{l+1}^i)}{(C_{l+1} - \check{f}_{l+1})^2}
\end{aligned}$$

$$= \frac{\mathrm{B}C_{l+1} - \mathrm{B}\check{f}_{l+1} + \sum_{i \in B} \check{f}^i_{l+1}}{(C_{l+1} - \check{f}_{l+1})^2}$$

$$= \frac{\mathrm{B}C_{l+1} - (\mathrm{B}-1)\check{f}_{l+1}}{(C_{l+1} - \check{f}_{l+1})^2} \tag{48}$$

If we pull out (B-1) terms from equations (47) and (48), we get

$$\frac{(\mathrm{B}-1)(C_l - \check{f}_l)}{(C_l - \check{f}_l)^2} + \frac{C_l - \sum_{i \in C} \check{f}^i_l}{(C_l - \check{f}_l)^2}$$
$$= \frac{(\mathrm{B}-1)(C_{l+1} - \check{f}_{l+1})}{(C_{l+1} - \check{f}_{l+1})^2} + \frac{C_{l+1}}{(C_{l+1} - \check{f}_{l+1})^2}$$

$$\Rightarrow \frac{\mathrm{B}-1}{(C_l - \check{f}_l)} + \frac{C_l - \sum_{i \in C} \check{f}^i_l}{(C_l - \check{f}_l)^2}$$
$$= \frac{\mathrm{B}-1}{(C_{l+1} - \check{f}_{l+1})} + \frac{C_{l+1}}{(C_{l+1} - \check{f}_{l+1})^2} \tag{49}$$

Suppose $\check{f}_{l+1} \geq f'_{l+1}$. From the assumption in equation (43) we have $\check{f}_l < f'_l$. Hence, from equation (49) we get

$$\frac{\mathrm{B}-1}{(C_l - f'_l)} + \frac{C_l - \sum_{i \in C} \check{f}^i_l}{(C_l - f'_l)^2}$$
$$> \frac{\mathrm{B}-1}{(C_{l+1} - f'_{l+1})} + \frac{C_{l+1}}{(C_{l+1} - f'_{l+1})^2} \tag{50}$$

If we subtract equation (46) from equation (50), we get

$$\frac{\mathrm{B}-1}{(C_l - f'_l)} - \frac{\sum_{i \in C} \check{f}^i_l}{(C_l - f'_l)^2} > \frac{\mathrm{B}-1}{(C_{l+1} - f'_{l+1})}$$

If B > 1, then it implies

$$\frac{\mathrm{B}-1}{(C_l - f'_l)} > \frac{\mathrm{B}-1}{(C_{l+1} - f'_{l+1})}$$
$$\Rightarrow \frac{1}{(C_l - f'_l)} > \frac{1}{(C_{l+1} - f'_{l+1})}$$
$$\Rightarrow C_l - f'_l < C_{l+1} - f'_{l+1}$$

which contradicts that $C_l - f'_l \geq C_{l+1} - f'_{l+1}$. If B $= 1$, then we get the contradiction that

$$-\frac{\sum_{i \in C} \check{f}^i_l}{(C_l - f'_l)^2} > 0.$$

Therefore, $\check{f}_{l+1} < f'_{l+1}$ and $\sum_{k=1}^{l+1} \check{f}_k < \sum_{k=1}^{l+1} f'_k$. We can successively apply this argument for all $\bar{l} \geq l$ and show that for all $\bar{l} \geq l$ equations (43) and (42) hold. This leads to the equation

$$\sum_{l \in L} \check{f}_l < \sum_{l \in L} f'_l\ , \tag{51}$$

which contradicts the demand constraint. This contradiction proves the lemma. ∎

For $0 \leq r \leq R$, let

$$\check{C}(r) = \sum_{l=1}^{\check{k}} \frac{\check{f}_l}{C_l - \check{f}_l} + \frac{\check{q}}{C_{\check{k}+1} - \check{f}_{\check{k}+1}}, \tag{52}$$

where $\check{k}$ and $\check{q}$ are defined in terms of r by

$$\sum_{l=1}^{\check{k}} \check{f}_l + \check{q} = r,\ 0 < \check{q} \leq \check{f}_{\check{k}+1}. \tag{53}$$

Similarly, for $0 \leq r \leq R$, let

$$C'(r) = \sum_{l=1}^{k'} \frac{f'_l}{C_l - f'_l} + \frac{q'}{C_{k'+1} - f'_{k'+1}}, \tag{54}$$

where k' and q' are defined in terms of r by

$$\sum_{l=1}^{k'} f'_l + q' = r,\ 0 < \ q' \leq f'_{k'+1}. \tag{55}$$

Lemma 5 *$\check{C}(r) \geq C'(r)$ for all $0 \leq r \leq R$. Moreover, if $C' < \check{C}$, then $\check{C}(r) > C'(r)$ for all $0 < r \leq R$.*

Proof: We will prove the first part of lemma by contradiction. We know from Lemma 4 that

$$\sum_{l=1}^{\bar{l}} \check{f}_l \geq \sum_{l=1}^{\bar{l}} f'_l,\ 0 \leq \bar{l} \leq \mathrm{L}. \tag{56}$$

Suppose $\exists\ 0 < r' \leq R$ such that

$$\check{C}(r') < C'(r'). \tag{57}$$

From Lemma 4, we know $\check{k}+1 \leq k'+1$, where $\check{k}$ and k' satisfy

$$\sum_{l=1}^{\check{k}} \check{f}_l + \check{q} = r',\ 0 < \ \check{q} \leq \check{f}_{\check{k}+1}$$

and

$$\sum_{l=1}^{k'} f'_l + q' = r',\ 0 < \ q' \leq f'_{k'+1}.$$

We will show that the assumption in equation (57) implies that either

(a) $\check{C}(\sum_{l=1}^{\check{k}} \check{f}_l) < C'(\sum_{l=1}^{\check{k}} \check{f}_l)$

or

(b) $\check{C}(\sum_{l=1}^{\check{k}+1} \check{f}_l) < C'(\sum_{l=1}^{\check{k}+1} \check{f}_l)$.

Suppose $C_{\check{k}+1} - \check{f}_{\check{k}+1} \geq C_{k'+1} - f'_{k'+1}$, then from equation (57) and property (3) of section 3.2 we have

$$\check{C}(\sum_{l=1}^{\check{k}+1} \check{f}_l) < C'(\sum_{l=1}^{\check{k}+1} \check{f}_l) \tag{58}$$

because

$$\check{C}(\sum_{l=1}^{\check{k}+1} \check{f}_l) - \check{C}(r') = \frac{1}{(C_{\check{k}+1} - \check{f}_{\check{k}+1})} \cdot (\check{f}_{\check{k}+1} - \check{q}) \leq \frac{1}{C_{k'+1} - f'_{k'+1}} \cdot (\check{f}_{\check{k}+1} - \check{q}) \leq C'(\sum_{l=1}^{\check{k}+1} \check{f}_l) - C'(r') , \quad (59)$$

where the last inequality follows from $C_l - f'_l \leq C_{k'+1} - f'_{k'+1}\ \forall\ l \geq k'+1$.

Similarly, if $C_{\check{k}+1} - \check{f}_{\check{k}+1} < C_{k'+1} - f'_{k'+1}$, then from property (3) of section 3.2 we have

$$\check{C}(r') - \check{C}(\sum_{l=1}^{\check{k}} \check{f}_l) = \frac{1}{(C_{\check{k}+1} - \check{f}_{\check{k}+1})} \cdot \check{q} > \frac{1}{C_{k'+1} - f'_{k'+1}} \cdot \check{q} \geq C'(r') - C'(\sum_{l=1}^{\check{k}} \check{f}_l) , \quad (60)$$

where the last inequality follows from $C_l - f'_l \geq C_{k'+1} - f'_{k'+1}\ \forall\ l \leq k'+1$. Thus, from equation (57), we have

$$\check{C}(\sum_{l=1}^{\check{k}} \check{f}_l) < C'(\sum_{l=1}^{\check{k}} \check{f}_l) \quad (61)$$

We will now show that f' is not system-wide optimum, which shows that the assumption in equation (57) leads to a contradiction. If (a) is true, let $l' = \check{k}$. Otherwise (b) must be true, and we let $l' = \check{k}+1$. With this notation, we have

$$\check{C}(\sum_{l=1}^{l'} \check{f}_l) < C'(\sum_{l=1}^{l'} \check{f}_l) \quad (62)$$

First note that $l' \geq 1$ because $\check{C}(0) = C'(0) = 0$. Let $\bar{f}_1 = (\check{f}_1, ..., \check{f}_{l'}, 0, ..., 0)$ and $\bar{f}_2 = (0, ..., 0, g_{\bar{l}}, f'_{\bar{l}+1}, ..., f'_L)$, where $g_{\bar{l}} + \sum_{l=\bar{l}+1}^{L} f'_l = R - \sum_{l=1}^{l'} \check{f}_l = \sum_{l=l'+1}^{L} \check{f}_l$ and $0 < g_{\bar{l}} \leq f'_{\bar{l}}$. From Lemma 4 we have $\sum_{l=1}^{l'} \check{f}_l \geq \sum_{l=1}^{l'} f'_l$, which means that $\bar{l} \geq l'+1$. Let us consider the following configuration.

$$\bar{f} = \bar{f}_1 + \bar{f}_2 = (\check{f}_1, ..., \check{f}_{l'}, 0, ..., 0, g_{\bar{l}}, f'_{\bar{l}+1}, ..., f'_L)$$

Note that there are no zeros between $\check{f}_{l'}$ and $g_{\bar{l}}$ if $\bar{l} = l'+1$. The cost incurred by $\bar{f}$ is

$$\sum_{l=1}^{l'} \frac{\check{f}_l}{C_l - \check{f}_l} + \frac{g_{\bar{l}}}{C_{\bar{l}} - g_{\bar{l}}} + \sum_{l=\bar{l}+1}^{L} \frac{f'_l}{C_l - f'_l} \quad (63)$$

From equation (62), the first term in equation (63) is strictly smaller than $C'(\sum_{l=1}^{l'} \check{f}_l)$, and because $\bar{l} \geq l'+1$ and $g_{\bar{l}} \leq f_{\bar{l}}$, the sum of the second and third terms is no greater than $C'(R) - C'(\sum_{l=1}^{l'} \check{f}_l) = \frac{g_{\bar{l}}}{C_{\bar{l}} - f'_{\bar{l}}} + \sum_{l=\bar{l}+1}^{L} \frac{f'_l}{C_l - f'_l}$. Therefore, the total system cost incurred by $\bar{f}$ is smaller than that of f', which is a contradiction. This completes the proof of the first part of Lemma 5.

We now proceed to prove the second half of Lemma 5. Suppose that $C' < \check{C}$. We will first show that this implies that $\check{f}_1 > f'_1$. We already know from Lemma 4 that $\check{f}_1 \geq f'_1$. So suppose that $\check{f}_1 = f'_1$. We first argue that there is some l for which $\check{f}_l > 0$ and $C_l < C_1$. Suppose otherwise. Then we have $C_l = C_1$ for all $l \in L$ such that $\check{f}_l > 0$, and by property (1) of section 3.2 and $\check{f}_1 = f'_1$ we then get that $\check{f}_l = f'_l$ for all $l \in L$. This, however, contradicts the assumption that $\check{C} > C'$.

Consider now any l for which $\check{f}_l > 0$ and $C_l < C_1$, with the assumption that $\check{f}_1 = f'_1$ still in force. Let I_l denote $\{i | \check{f}^i_l > 0\}$. Then $I_l \neq \emptyset$. Also recall from Lemma 3 that $\check{f}_l > 0$ implies $f'_l > 0$. We now argue that we must have $\check{f}_l < f'_l$. Suppose not, i.e.,

$$\check{f}_l \geq f'_l. \quad (64)$$

Then, we may show that $f_l' \geq \check{f}_l^i$ for $i \in I_l$ as follows. First, from the K-T conditions for user $i \in I_l$ and because $\check{f}_l > 0$, we have

$$\begin{aligned}\lambda^i &= \frac{1}{C_l - \check{f}_l} + \frac{\check{f}_l^i}{(C_l - \check{f}_l)^2} \\ &= \frac{1}{C_1 - \check{f}_1} + \frac{\check{f}_1^i}{(C_1 - \check{f}_1)^2}\,, \end{aligned} \tag{65}$$

while, from the K-T conditions for user 0 and because Lemma 3 implies that $f_l' > 0$, we have

$$\begin{aligned}\lambda^0 &= \frac{1}{C_1 - f_1'} + \frac{f_1'}{(C_1 - f_1')^2} \\ &= \frac{1}{C_l - f_l'} + \frac{f_l'}{(C_l - f_l')^2}\,. \end{aligned} \tag{66}$$

Because $\check{f}_1^i < \check{f}_1 = f_1'$, and by the assumption that $\check{f}_1 = f_1'$, we have from equations (65) and (66) that

$$\lambda^0 = \frac{1}{C_1 - f_1'} + \frac{f_1'}{(C_1 - f_1')^2} > \frac{1}{C_l - \check{f}_l} + \frac{\check{f}_l^i}{(C_l - \check{f}_l)^2} = \lambda^i\,. \tag{67}$$

Also from the assumption in equation (64) that $\check{f}_l \geq f_l'$ and equation (64), we get

$$\begin{aligned}\lambda^i &= \frac{1}{C_l - \check{f}_l} + \frac{\check{f}_l^i}{(C_l - \check{f}_l)^2} \\ &\geq \frac{1}{C_l - f_l'} + \frac{\check{f}_l^i}{(C_l - f_l')^2}\,. \end{aligned} \tag{68}$$

Thus, from equations (66), (67) and (68), we have $f_l' > \check{f}_l^i$ for $i \in I_l$.

From the assumption in equation (64)

$$\frac{1}{C_l - f_l'} + \frac{f_l'}{(C_l - f_l')^2} \leq \frac{1}{C_l - \check{f}_l} + \frac{f_l'}{(C_l - \check{f}_l)^2} \tag{69}$$

Let $\gamma = \frac{1}{C_l - \check{f}_l} - \frac{1}{C_1 - \check{f}_1} > 0$. If we subtract $\frac{1}{C_1 - \check{f}_1}$ from equation (65), we get

$$\frac{\check{f}_1^i}{(C_1 - \check{f}_1)^2} = \gamma + \frac{\check{f}_l^i}{(C_l - \check{f}_l)^2}, \tag{70}$$

and, using equations (64) and (66), if we subtract $\frac{1}{C_1 - \check{f}_1}$ from (69), we get

$$\frac{f_1'}{(C_1 - f_1')^2} = \frac{\check{f}_1}{(C_1 - \check{f}_1)^2} \leq \gamma + \frac{f_l'}{(C_l - \check{f}_l)^2} \tag{71}$$

Divide equation (71) by equation (70) and simplify.

$$\frac{\check{f}_1}{\check{f}_1^i} \leq \frac{\gamma' + f_l'}{\gamma' + \check{f}_l^i} < \frac{f_l'}{\check{f}_l^i} \tag{72}$$

where $\gamma' = \gamma \cdot (C_l - \check{f}_l)^2$. The last inequality follows from that $\gamma' > 0$ and $f_l' > \check{f}_l^i$. From equation (72) we have

$$\frac{\check{f}_1}{f_l'} \cdot \check{f}_l^i < \check{f}_1^i \tag{73}$$

Equation (73) holds trivially for all users in $I \setminus I_l$ because $\check{f}_1^i > 0$ for all i. Thus, if we sum equation (73) over all i, we get

$$\frac{\check{f}_1}{f_l'} \cdot \check{f}_l < \check{f}_1 \tag{74}$$

Since $\frac{\check{f}_l}{f_l'} \geq 1$ from the assumption in equation (64), this is a contradiction. This proves that $\check{f}_l < f_l'$ for all $l > 1$ such that $\check{f}_l > 0$ and $C_l < C_1$. This proves that $\check{f}_1 > f_1'$ because we argued earlier that there is at least one link $l > 1$ such that $\check{f}_l > 0$ and $C_l < C_1$, and so the assumption that $\check{f}_1 = f_1'$ would lead to the contradiction that $\sum_{l \in L} \check{f}_l < \sum_{l \in L} f_l'$.

Continuing the proof of the second part of Lemma 5, now assume that there is some $\bar{r} > 0$ such that

$$\check{C}(\bar{r}) \leq C'(\bar{r}). \tag{75}$$

We will first show that $\bar{r} \geq f_1'$. This is easy to see from observing that $\frac{1}{C_1 - f_1'} < \frac{1}{C_1 - \check{f}_1}$ since $\check{f}_1 > f_1'$.

We will now show $\bar{r} > \check{f}_1$. Assume $f_1' < \bar{r} \leq \check{f}_1$. We will show that this implies that f' is not system-wide optimum. Since $C'(f_1') = \frac{f_1'}{C_1 - f_1'} < \frac{f_1'}{C_1 - \check{f}_1} = \check{C}(f_1')$, the assumption that $\check{C}(\bar{r}) \leq C'(\bar{r})$, together with the fact that $C_1 - f_1' \geq C_2 - f_2' \geq \cdots \geq C_L - f_L'$ would imply that $C'(\check{f}_1) \geq \check{C}(\check{f}_1)$. Now, let $\ddot{f} = (\check{f}_1, 0, ..., 0, g_{\bar{l}}, f_{\bar{l}+1}', ..., f_L')$, where $0 < g_{\bar{l}} \leq f_{\bar{l}}'$ and $\sum_{l=\bar{l}+1}^{L} f_l' + g_{\bar{l}} = \sum_{l=2}^{L} \check{f}_l$. Note that since $\check{f}_1 > f_1'$, the second element in $\ddot{f}$, $\ddot{f}_2$, is strictly smaller than f_2'. This implies that

$$\sum_{l=2}^{L} \frac{\ddot{f}_l}{C_l - \ddot{f}_l} < \sum_{l=2}^{L} \frac{f_l'}{C_l - f_l'}\,, \tag{76}$$

because $\ddot{f}_2 < f_2'$ and $\ddot{f}_l \leq f_l'$ for all $l > 2$. Since $C'(\check{f}_1) \geq \check{C}(\check{f}_1)$, the total cost incurred by $\ddot{f}$ is strictly smaller than C'. This, however, contradicts the assumption that f' is system-wide optimum. This proves that $\bar{r} > \check{f}_1$.

We will now complete the proof of the second half of Lemma 5 by showing that equation (75) cannot hold for any $\bar{r} > 0$. From the above result, $\bar{r} > \check{f}_1$. Define $\check{k}$ and $\check{q}$ by

$$\sum_{l=1}^{\check{k}} \check{f}_l + \check{q} = \bar{r},\ 0 < \check{q} \leq \check{f}_{\check{k}+1} \tag{77}$$

and define k' and q' by

$$\sum_{l=1}^{k'} f_l' + q' = \bar{r},\ 0 < q' \leq f_{k'+1}'. \tag{78}$$

Note that $\check{k} \geq 1$ since $\bar{r} > \check{f}_1$. As shown in the proof of the first part of the lemma, we can show that $\check{C}(\bar{r}) \leq C'(\bar{r})$ implies that either

(a) $\check{C}(\sum_{l=1}^{\check{k}} \check{f}_l) \leq C'(\sum_{l=1}^{\check{k}} \check{f}_l)$
or
(b) $\check{C}(\sum_{l=1}^{\check{k}+1} \check{f}_l) \leq C'(\sum_{l=1}^{\check{k}+1} \check{f}_l)$.

If (a) is true, let $l' = \check{k}$. Otherwise (b) must be true, and we let $l' = \check{k}+1$. Note that, since $\check{k} \geq 1$, we have $l' \geq 1$. With this notation, we have

$$\check{C}(\sum_{l=1}^{l'} \check{f}_l) \leq C'(\sum_{l=1}^{l'} \check{f}_l) \tag{79}$$

We can construct a system flow configuration $\bar{f} = (\check{f}_1, ..., \check{f}_{l'}, 0, ..., 0, g_{\bar{l}}, f'_{\bar{l}+1}, ..., f'_L)$ as in the first part of the proof. The total system cost under $\bar{f}$ is given by equation (63). Since the first term is no greater than $C'(\sum_{l=1}^{l'} \check{f}_l)$ and the sum of the second and third terms is no greater than $C'(R) - C'(\sum_{l=1}^{l'} \check{f}_l)$ as explained before, the total system cost under $\bar{f}$ is smaller than or equal to C'. Since C' is minimum possible achievable cost, the cost under $\bar{f}$ is equal to C'. However, this contradicts the uniqueness of system-wide optimum flow configuration because $\check{f}_1 > f'_1$ if $C' < \check{C}$ as shown before. This proves that $C'(r) < \check{C}(r)$, $0 < r \leq R$. ■

Proof: (*Theorem 2*) We consider three cases. The first case is when $f' = \check{f}$. In this case the theorem is trivial, since we may take $\hat{f} = f' = \check{f}$. Repeating the NEP of the stage game at every stage of the repeated game is an SPNEP for the repeated game in which every player incurs a cost equal to its cost at the stage game NEP.

In the remaining two cases we assume that $f' \neq \check{f}$. Think of flow on each link l at the stage game NEP as a flow vector $y_l = (0,...,0,\check{f}_l,0,...,0)$. This flow has an associated cost per unit flow, $\beta_l = \frac{1}{C_l - \check{f}_l}$. By property (3) of section 3.2, this cost per unit flow, β_l, is nondecreasing in l. We can create another set of flows based on the optimum flow configuration f' by filling up f'_l, $l \in L$, in increasing order of l. The l-th such flow is written as a vector $x^l = (a^l_1, ..., a^l_L)$, where $\sum_{m=1}^{L} a^l_m = \check{f}_l$, and for all $\bar{l} \in L$, $\sum_{l=1}^{\bar{l}} x^l = (f'_1, .., f'_{l'}, g_{l'+1}, 0, ..., 0)$ such that $\sum_{l=1}^{l'} f'_l + g_{l'+1} = \sum_{l=1}^{\bar{l}} \check{f}_l$ and $g_{l'+1} \leq f'_{l'+1}$. There is a unique set of x_l's that satisfies the above conditions.

Let α_l, $1 \leq l \leq L$, denote the cost per unit flow of the flow x^l. This is evaluated by taking the cost per unit flow on link l to be $\frac{1}{C_l - f'_l}$. The second case is when $\alpha_l \leq \beta_l$ for all $l \in L$. Suppose we allocate to user i the fraction $\frac{\check{f}^i_l}{\check{f}_l}$ of x_l, for all $l \in L$. Call the resulting flow configuration $\hat{f}$. The demand constraint of user i is met, because we have

$$\sum_{l \in L} \sum_{l' \in L} \frac{\check{f}^i_l}{\check{f}_l} x^l_{l'} = \sum_{l \in L} \frac{\check{f}^i_l}{\check{f}_l} \check{f}_l = \sum_{l \in L} \check{f}^i_l = r^i \; . \tag{80}$$

The overall cost under $\hat{f}$ is optimum, because the flow on each link under $\hat{f}$ is the same as that under f'. Also, the cost of user i under $\hat{f}$ is strictly smaller than its cost at the unique NEP of the stage game because, first of all

$$\sum_{l \in L} \alpha_l \cdot \check{f}^i_l \leq \sum_{l \in L} \beta_l \cdot \check{f}^i_l \; , \tag{81}$$

and further, since $f' \neq \check{f}$ by assumption, the proof of the second part of Lemma 5 showed that $C'(\check{f}_1) < \check{C}(\check{f}_1)$, and this implies that $\alpha_1 < \beta_1$, so that the inequality in equation (81) is strict for all i. Thus, the vector of costs under $\hat{f}$ is strictly Pareto dominated by the stage game NEP costs, strictly in each coordinate. By Friedman's theorem, which was stated in section 2, for discount factors sufficiently close to 1 there is an SPNEP of the repeated game with cost vector equal to the cost vector under $\hat{f}$. The overall cost at this SPNEP is optimum and the cost for each user is strictly less than its stage game NEP cost.

The final case is when there is at least one link $l \in L$ for which $\alpha_l > \beta_l$. In this case, let $A_0 = \{l : \alpha_l < \beta_l\}$, $B_0 = \{l : \alpha_l > \beta_l\}$, and $C_0 = \{l : \alpha_l = \beta_l\}$. Lemma 5 says that $\check{C}(R) \geq C'(R)$. Hence, if $B_0 \neq \emptyset$, then $A_0 \neq \emptyset$.

We will first argue that if $\bar{l} \in B_0$, then there exists another link $l' < \bar{l}$ in A_0. Suppose not. Then, this implies that $\alpha_l \geq \beta_l$ for all $l < \bar{l}$. However, this says that $\check{C}(\sum_{l=1}^{\bar{l}} \check{f}_l) = \sum_{l=1}^{\bar{l}} \beta_l \cdot \check{f}_l < \alpha_l \cdot \check{f}_l = C'(\sum_{l=1}^{\bar{l}} \check{f}_l)$, which contradicts Lemma 5. Thus, for each $\bar{l} \in B_0$, there exists another $l' \in A_0$ smaller than $\bar{l}$.

Let us describe a mixing procedure between $x^{\bar{l}}$ and $x^{l'}$, where $l' < \bar{l}$, so that $\alpha_{l'} < \alpha_{\bar{l}}$, $\beta_{l'} < \beta_{\bar{l}}$, and $\check{f}_{\bar{l}} \leq \check{f}_{l'}$. For $0 \leq u \leq 1$ consider the flows $z^{\bar{l}}(u)$ and $z^{l'}(u)$ defined by

$$z^{\bar{l}}(u) = (1-u) \cdot x^{\bar{l}} + u \cdot \frac{\check{f}_{\bar{l}}}{\check{f}_{l'}} x^{l'}, \tag{82}$$

$$z^{l'}(u) = (1 - u \cdot \frac{\check{f}_{\bar{l}}}{\check{f}_{l'}}) x^{l'} + u \cdot x^{\bar{l}} . \tag{83}$$

Note that the total amount of flow being sent in these flows does not depend on u, because

$$\sum_{l \in L} z_l^{\bar{l}}(u) = (1-u) \cdot \check{f}_{\bar{l}} + u \cdot \frac{\check{f}_{\bar{l}}}{\check{f}_{l'}} \check{f}_{l'} = \check{f}_{\bar{l}} , \tag{84}$$

$$\sum_{l \in L} z_l^{l'}(u) = (1 - u \cdot \frac{\check{f}_{\bar{l}}}{\check{f}_{l'}}) \check{f}_{l'} + u \cdot \check{f}_{\bar{l}} = \check{f}_{l'} . \tag{85}$$

Further, note that the sum of the flow configurations $z_l^{\bar{l}}(u)$ and $z_l^{l'}(u)$ equals the sum of the flow configurations $x^{\bar{l}}$ and $x^{l'}$, for all $0 \leq u \leq 1$.

Now, let us define $\mu_{\bar{l}}(u)$ to be the cost associated with $z^{\bar{l}}(u)$ and $\mu_{l'}(u)$ to be the cost associated with $z^{l'}(u)$. From equations (82) and (83) we have

$$\mu_{\bar{l}}(u) = \alpha_{\bar{l}} \cdot (1-u) \check{f}_{\bar{l}} + \alpha_{l'} \cdot u \check{f}_{\bar{l}} \tag{86}$$

and

$$\mu_{l'}(u) = \alpha_{l'} \cdot (\check{f}_{l'} - u \check{f}_{\bar{l}}) + \alpha_{\bar{l}} \cdot u \check{f}_{\bar{l}} . \tag{87}$$

Since $\alpha_{l'} < \beta_{l'} \leq \beta_{\bar{l}} < \alpha_{\bar{l}}$, we see that $\mu_{\bar{l}}(u)$ is decreasing in u and $\mu_{l'}(u)$ is increasing in u. Since $\check{f}_{l'} \geq \check{f}_{\bar{l}}$, we also observe that

$$\mu_{\bar{l}}(1) = \alpha_{l'} \cdot \check{f}_{\bar{l}} < \beta_{l'} \cdot \check{f}_{\bar{l}} \leq \beta_{\bar{l}} \cdot \check{f}_{\bar{l}} , \tag{88}$$

while

$$\mu_{\bar{l}}(0) = \alpha_{\bar{l}} \cdot \check{f}_{\bar{l}} > \beta_{\bar{l}} \cdot \check{f}_{\bar{l}} . \tag{89}$$

Further,

$$\mu_{l'}(0) = \alpha_{l'} \check{f}_{l'} < \beta_{l'} \check{f}_{l'} , \tag{90}$$

while

$$\mu_{l'}(1) = \alpha_{l'}(\check{f}_{l'} - \check{f}_{\bar{l}}) + \alpha_{\bar{l}}\check{f}_{\bar{l}} \ . \tag{91}$$

This raises the natural question of whether, as u increases from 0 to 1, we first have $\mu_{\bar{l}}(u)$ becoming equal to $\beta_{\bar{l}} \cdot \check{f}_{\bar{l}}$ for some u or first have $\mu_{l'}(u)$ becoming equal to $\beta_{l'}\check{f}_{l'}$. That one or the other of these possibilities (or both) must occur as u ranges from 0 to 1 can be seen from equations (88) -(91). Some algebra from equation (91) shows that if $\alpha_{\bar{l}} \cdot \check{f}_{\bar{l}} + \alpha_{l'} \cdot \check{f}_{l'} \leq \beta_{\bar{l}} \cdot \check{f}_{\bar{l}} + \beta_{l'} \cdot \check{f}_{l'}$ then the former eventuality occurs first, while if $\alpha_{\bar{l}} \cdot \check{f}_{\bar{l}} + \alpha_{l'} \cdot \check{f}_{l'} > \beta_{\bar{l}} \cdot \check{f}_{\bar{l}} + \beta_{l'} \cdot \check{f}_{l'}$ the latter occurs first, and it is not the case that both occur simultaneously.

For each $\bar{l} \in B_0$ in increasing order, we may now perform the following procedure. Note that this procedure is guaranteed to terminate after a finite number of steps with $\alpha_l \leq \beta_l$ for all $l \leq \bar{l}$.

(1) Pick l', the largest $l \in A_0$ smaller than $\bar{l}$. Such an l' is guaranteed to exist, as argued earlier.

(2) If $\alpha_{\bar{l}} \cdot \check{f}_{\bar{l}} + \alpha_{l'} \cdot \check{f}_{l'} \leq \beta_{\bar{l}} \cdot \check{f}_{\bar{l}} + \beta_{l'} \cdot \check{f}_{l'}$, let u' be such that $\mu_{\bar{l}}(u') = \beta_{\bar{l}} \cdot \check{f}_{\bar{l}}$ and $\mu_{l'}(u') \leq \beta_{l'} \cdot \check{f}_{l'}$. Replace the flow configuration $x^{\bar{l}}$ by $z^{\bar{l}}(u')$. As shown in equation (84), the total amount of flow in this flow configuration is still $\check{f}_{\bar{l}}$. The total cost associated to this flow configuration, with costs still being evaluated based on cost per unit flow $\frac{1}{C_l - \check{f}_l}$ on link l, is $\mu_{\bar{l}}(u') = \beta_{\bar{l}} \cdot \check{f}_{\bar{l}}$. Thus the cost per unit flow of this flow configuration is equal to $\beta_{\bar{l}}$. We may thus remove $\bar{l}$ from B_0 and put it in C_0. Likewise, we replace the flow configuration $x^{l'}$ by $z^{l'}(u')$. Either $\mu_{l'}(u') < \beta_{l'} \cdot \check{f}_{l'}$, in which case we keep l' in A_0, or $\mu_{l'}(u') = \beta_{l'} \cdot \check{f}_{l'}$, in which case we remove l' from A_0 and put it in C_0.

If $\alpha_{\bar{l}} \cdot \check{f}_{\bar{l}} + \alpha_{l'} \cdot \check{f}_{l'} > \beta_{\bar{l}} \cdot \check{f}_{\bar{l}} + \beta_{l'} \cdot \check{f}_{l'}$, let u' be such that $\mu_{\bar{l}}(u') > \beta_{\bar{l}} \cdot \check{f}_{\bar{l}}$ and $\mu_{l'}(u') = \beta_{l'} \cdot \check{f}_{l'}$ We replace the flow configuration $x^{l'}$ by the flow configuration $z^{l'}(u')$. Since the cost per unit flow of new flow configuration is $\beta_{l'}$, we may remove l' from A_0 and put it in C_0. Similarly, we replace the flow configuration $x^{\bar{l}}$ by the flow configuration $z^{\bar{l}}(u')$. Since the cost per unit flow of new $J_{\bar{l}}$ is still strictly higher than $\beta_{\bar{l}}$, go back to (1).

After the above procedure is completed for every $\bar{l} \in B_0$, every link $l \in L$ is either in A_0 or in C_0, and thus, we have a new set of flows, $J_l, l \in L$, whose cost per unit flow, α_l, is no greater than β_l. Hence, for the same reason as in the second case, allocating a fraction, $\frac{\check{f}_l^i}{\check{f}_l}$, of x_l, for all $l \in L$, to user i yields to user i a cost smaller than or equal to its stage game NEP. Again, let $\hat{f}$ be such flow configuration.

Suppose $C' < \check{C}$. We will show that $\hat{f}$ yields to each user a cost strictly smaller than its stage game NEP cost. In order to show this it suffices to show that after each stage of the above procedure we still have $\alpha_1 < \beta_1$, where α_1 is now defined for the flow configuration x^1 that results after the corresponding stage is completed. This may be argued as follows : at any stage the link $\bar{l}$ under consideration satisfies $\bar{l} \geq 2$, by the inductive hypothesis that $\alpha_1 < \beta_1$ at the end of the previous stage. If the stage is being carried out with a value of $l' \geq 2$ then α_1 does not change, so there is nothing to worry about. Thus we may assume that we have $l' = 1$ during this stage. Recall that we always choose l' to be the link in A_0 with the largest index strictly less than $\bar{l}$. Further we choose $\bar{l}$ to be the link in B_0 with the smallest index. Thus, each of the links $2, \ldots \bar{l} - 1$ must be in C_0. If, at the end of this stage we have $\alpha_1 = \beta_1$, then we must also have $C'(\sum_{l=1}^{\bar{l}} \check{f}_l) = \check{C}(\sum_{l=1}^{\bar{l}} \check{f}_l)$, and because $\bar{l} \geq 2$, this contradicts Lemma 5. Thus, after the procedure terminates, $\alpha_1 < \beta_1$, and this proves that the cost of each user under $\hat{f}$ is strictly smaller than its stage game NEP cost. We may now argue, using Friedman's theorem that the statements of the theorem are true in this case also, exactly as we argued in the second case.

The SPNEP of the repeated game guaranteed by Friedman's theorem in the second and third case can be supported by the following strategy profiles : initially every user uses $\hat{f}$ until exactly one user deviates from $\hat{f}$. If exactly one user deviates at period k, then staring at period $k+1$ other users use the stage game NEP flow configuration. Then one can see that, if the discount factor is sufficiently close to 1, any gain at one period will be outweighed by the loss in later periods. Also, since no user can gain anything by deviating from the second stage, where stage game NEP is played every period, this is an SPNEP in the repeated game.

∎

4 General Networks

4.1 Model

In this section we consider a network $G(V, L)$, where V is a finite set of nodes and $L \subseteq V \times V$ is a set of directed links. We will assume that there is at most one link between each pair of nodes in each direction. As before we have a finite number of selfish users $I = \{1,2,...,I\}$, $I \geq 2$, that share the network, and the demand for user i is denoted by r^i. Again, assume that users are ordered in order of decreasing demand.

A user sends its demand from its source node to its destination by splitting its flow on the different paths that connect its source and destination nodes. A user is able to decide how to split its flow as in the parallel links case. An important difference from the parallel link network is that, in the general network, many paths can come together at some node, share certain links, and split again at another node. Let L_p be the set of links on path p. The incoming link of path p at node u is defined to be the link $l = (v, u)$ such that $l \in L_p$, and the outgoing link of path p at node u is the link $l' = (u, v')$ such that $l' \in L_p$. We say that path p_1 and p_2 split at node v if their incoming links to node v are the same but their outgoing links at node v are different, and that path p_1 and p_2 meet at node v if they both go through node v but their incoming links at node v are different. The fact that many paths may share some links makes the analysis of the network much harder. Some of the properties that hold in the parallel link networks and were crucial in the proofs of the results in section 3 no longer hold for more general networks.

We will first discuss the single source-destination pair case, where each user has the same source node and the same destination node. We then discuss the multiple source-destination pair case in the next subsection.

4.2 Single Source-Destination Pair Case

4.2.1 Model and Problem Formulation

We are given a network modeled as a directed graph $G = (V, E)$ with a set $I = \{1,2,...,I\}$, $I \geq 2$, of users that share the network. Let r^i denote user i's demand. All users have the same source node and the same destination node. We will assume that the total demand of users, denoted by R, is strictly smaller than the minimal cut capacity from the source node to the destination node. If this were not to hold, the static game would be of no interest since every user would have infinite cost at every NEP.

A path is a sequence of links that leads from the source node to the destination node. Without loss of generality, we may restrict attention to paths all of whose nodes are distinct. We may do this because the problem of each user is one of minimizing it cost. If any user sends strictly positive flow along a path with a loop, it can reduce its cost by removing the flow in the loop. Note that

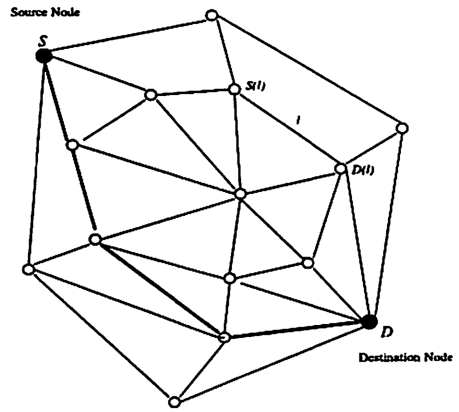

Figure 4: Single source-destination pair case.

the paths might not be disjoint, i.e., there might be two paths that share some links. Let $\mathcal{P}$ denote the set of paths available from the source node to the destination node, and L_p the set of links on path $p \in \mathcal{P}$. Let $\mathrm{P} = |\mathcal{P}|$. Since we only consider paths without loops, P is finite. We will assume that $\mathrm{P} \geq 2$.

The users share the P paths available from the source node to the destination node. Each user splits its own flow over the paths available. Let $\bar{f}_p^i$ denote the amount of flow user i sends on path p. We use the bar notation simply to avoid confusion between path flows and link flows. Then, we must have $\bar{f}_p^i \geq 0$ (nonnegativity constraint) and $\sum_{p \in \mathcal{P}} \bar{f}_p^i = r^i$ (demand constraint). Note that the parallel link case is a special case of the single source-destination pair case. As in the case of parallel link network, C_l denotes the capacity of link l, f_l^i denotes user i's flow on link l and f_l denotes the total flow on link l.

User i's performance measure is given by the same cost function used in the parallel link network case, i.e., $J^i(f) = \sum_{l \in L} J_l^i(f) = \sum_{l \in L} f_l^i \cdot T_l(f_l)$, where $T_l(f_l)$ is given by equation (1). Again since we are faced with selfish users, we have a noncooperative game played by users that attempt to minimize their own costs.

4.2.2 Existence of SPNEP that yields the optimum system cost

The first thing we investigate is the existence of an NEP in the stage game. This is guaranteed by the result in [18] that was described in section 2. Kuhn-Tucker conditions for an NEP can be written as follows : for every $i \in I$, there exists a set of Lagrange multipliers $\{\lambda_u^i\}_{u \in V}$ such that, for every link $(u, v) \in L$:

$$\begin{aligned} f_{uv}^i > 0 &\Rightarrow \lambda_u^i = K_{uv}^i(\mathrm{f}_{uv}) + \lambda_v^i \\ f_{uv}^i = 0 &\Rightarrow \lambda_u^i \leq K_{uv}^i(\mathrm{f}_{uv}) + \lambda_v^i \end{aligned} \tag{92}$$

where $K_{uv}^i(\mathrm{f}_{uv}) = \frac{\partial}{\partial f_{uv}^i} J_{uv}^i(\mathrm{f}_{uv})$. See the paper of Orda et al. [16] for more details, if necessary. Uniqueness of the NEP has been proven for only a few special cases. However, the uniqueness of the system flow configuration that achieves the minimum total system cost can be easily seen by observing that this is just the solution to a convex optimization problem. Therefore, the minimum total system cost, C', and the corresponding system flow configuration are well defined.

Suppose f' is the system flow configuration that achieves the minimum total system cost, C'. Throughout this section, we will assume that f' is such that there are two paths used under f'

with different cost per unit flow, $\sum_{l\in L_p}\frac{1}{C_l-f_l^i}$. This is the technical assumption alluded to in the abstract.

We will first show the existence of a system flow configuration that is an NEP in the repeated game with discount factor sufficiently close to 1, and achieves the total system cost, C'. Then, we will show that any such system flow configuration is an SPNEP.

Theorem 3 *If the discount factor in the repeated game is sufficiently close to* 1 *there is an NEP in the repeated game that achieves the minimum total system cost, C'.*

In order to give the proof of the theorem, we need the following.

Lemma 6 *Given a fixed total demand $R < C$, where C is the minimal cut capacity from the source node to the destination node, there exists a uniform bound on the total system cost at any stage game NEP regardless of distribution of demands among any finite number of users.*

Proof:

In order to show the existence of such a bound it suffices to show that there exists $\epsilon > 0$ such that, for any arbitrary set of users with an arbitrary distribution of demands among the users with total demand $R < C$, at any stage game NEP f of the routing game corresponding to these users with these demands, we have $C_l - f_l \geq \epsilon$ for all $l \in L$. We will show the existence of such ϵ by contradiction.

Let M be the maximum number of links on any path and recall that P denotes the number of paths available from the source node to the destination node. Order the paths in some way from 1 to P.

Suppose there exists no such $\epsilon > 0$. Then, for any $\epsilon > 0$, there exist a set of finitely many users and a distribution of demands among these users with total demand $R < C$ and some link $\bar{l} \in L$ such that, at some stage game NEP for the routing game corresponding to these users with these demands, we have $C_{\bar{l}} - f_{\bar{l}} < \epsilon$. Let user i be such that $f_{\bar{l}}^i > 0$. Such a user must exist if $\epsilon < \min_l C_l$. From the K-T conditions at this NEP, see equation (92), we have $\lambda^i > \frac{1}{\epsilon}$, where λ^i is $\lambda_S^i - \lambda_D^i$ in the notation of equation (92) and S and D are source and destination nodes, respectively. This is because $\lambda_S^i - \lambda_D^i \geq \frac{1}{C_{\bar{l}}-f_{\bar{l}}} + \frac{f_{\bar{l}}^i}{(C_{\bar{l}}-f_{\bar{l}})^2}$, which is easily seen from equation (92) and the existence of a path from S to D going through $\bar{l}$ along which user i sends strictly positive flow, which must be the case because $f_{\bar{l}}^i > 0$. From the K-T conditions, for any $p \in \mathcal{P}$,

$$
\begin{aligned}
&\sum_{l\in L_p}\frac{1}{C_l-f_l}+\frac{R}{(C_l-f_l)^2}\\
&>\sum_{l\in L_p}\frac{1}{C_l-f_l}+\frac{f_l^i}{(C_l-f_l)^2}\\
&\geq \lambda^i > \frac{1}{\epsilon}
\end{aligned}
\tag{93}
$$

This means that for all $p \in P$, there exists $l \in L_p$ such that

$$
\frac{1}{C_l-f_l}+\frac{R}{(C_l-f_l)^2} > \frac{1}{M\epsilon} \tag{94}
$$

Let δ^1 be the positive solution of the following quadratic equation.

$$
\frac{x^2}{M\epsilon} - x - R = 0
$$

It is easy to see that the above equation has only one positive solution. Equation (94), now, implies that

$$C_l - f_l < \delta^1 \ . \tag{95}$$

Note that $\delta^1 \downarrow 0$ as $\epsilon \downarrow 0$. The set of all links that satisfy equation (95) is a disconnecting set because there is no path available from the source node S to the destination D after removing all such links. Denote this disconnecting set by $\check{L}$.

Let δ^2 be the positive solution of

$$\frac{y^2}{M\delta^1} - y - R = 0$$

The importance of δ^2 is that if a path p with positive flow meets another path $\bar{p}$ at some node v and path p has a link l such that $C_l - f_l < \delta^1$ between S and v, then there is a link $\bar{l}$ on path $\bar{p}$ between S and v such that $C_{\bar{l}} - f_{\bar{l}} < \delta^2$. This follows similarly as in equation (93). Again, $\delta^2 \downarrow 0$ as $\delta^1 \downarrow 0$. Therefore, as $\epsilon \downarrow 0$, $\delta^2 \downarrow 0$ because $\delta^1 \downarrow 0$ as $\epsilon \downarrow 0$. Now, we can see that if δ^{m+1}, $1 \leq m < M\mathrm{P}$, is the positive solution of the equation

$$\frac{y^2}{M\delta^m} - y - R = 0, \tag{96}$$

then as $\epsilon \downarrow 0$, $\delta^m \downarrow 0$, $1 \leq m \leq M\mathrm{P}$. Note that $\epsilon < \delta^1 < \delta^2 < \cdots < \delta^{M\mathrm{P}}$.

We will now associate a link, which will be denoted by l_p, to each path $p \in P$. Initially, we will select l_p for each $p \in P$ to be the link in $\check{L}$ that is closest to the destination node along path p. Each l_p may be updated throughout the procedure described below. Let $L^1 = \{l_p : p \in P\}$ and $L_p^1 = L^1 \cap L_p$ for all $p \in P$.

Let $m = 2$ and execute the following procedure :

(*) Let $\bar{L}_p$ be the set of links in L_p that lie between the source node S and l_p along path p, not including l_p. If for every path p with strictly positive flow we have $\bar{L}_p \cap L^1 = \emptyset$, then the procedure terminates and we are done.

Otherwise, pick a path $p \in \mathcal{P}$ with strictly positive flow for which $\bar{L}_p \cap L^1 \neq \emptyset$. Let $P_p = \{p' \mid l_{p'} \in \bar{L}_p \cap L^1\}$. For each path $p' \in P_p$, find a link $\bar{l}_{p'}$ between $l_{p'}$ and the destination node D, such that $C_{\bar{l}_{p'}} - f_{\bar{l}_{p'}} < \delta^m$. The existence of such a link is guaranteed by that l_p lies between $l_{p'}$ and the destination node D along path p and that $\bar{f}_p > 0$. Let $L' = \{l_p \mid p \in P \setminus P_p\} \cup \{\bar{l}_p \mid p' \in P_p\}$. Let the new l_p for each $p \in P$ be the last link in L' along the path p. Let the new $L^1 = \{l_p \mid p \in P\}$. Increment m by 1 and return to (*).

Note that every time we perform procedure (*), $l_{\bar{p}}$ for some $\bar{p} \in P$ gets at least one link closer to the destination node D along path $\bar{p}$. Thus we can repeat (*) at most (M-1)P times before the procedure terminates, because there are at most MP links along all P paths. When the procedure terminates, we have L^1 such that, for each path $p \in P$ with $\bar{f}_p > 0$, $L^1 \cap \bar{L}_p = \emptyset$. Since for each $p \in P$, when we update l_p at the end of the procedure we pick the link in L^1 that is closest to the destination node D along path p (and this was also true at the beginning of the procedure), one can see that each path with positive flow goes through exactly one link in L^1, and the total flow on links in L^1 must therefore equal the total demand of users R. Moreover, since (1) cannot be repeated more than $(M-1)\mathrm{P}$, every link in L^1 satisfies $C_l - f_l < \delta^{(M-1)\mathrm{P}+1} \leq \delta^{M\mathrm{P}}$, because $\mathrm{P} \geq 2$.

From the beginning of the proof, we know that if ϵ is sufficiently small, then $\delta^{MP} < \frac{C-R}{\mathrm{P}}$. We will show that this leads to a contradiction. If $\delta^{MP} < \frac{C-R}{\mathrm{P}}$, then we get $\sum_{l \in L^1}(C_l - f_l) = \sum_{l \in L^1} C_l - R < \mathrm{P} \cdot \delta^{MP} < \mathrm{P} \cdot \frac{C-R}{\mathrm{P}} = C - R$. This is, however, a contradiction because C is the minimal cut capacity and no greater than $\sum_{l \in L^1} C_l$.

■

Proof: (*Theorem 3*) Let $\underline{v}_i$ denote the reservation cost of user i. In order to show the existence of such an NEP, we will first show that the reservation value of each user is greater than or equal to $\frac{r^i}{R} \cdot C'$. We call $\frac{r^i}{R} \cdot C'$ user i's cost of proportional sharing.

Let r^{-i} denote $R - r^i$. Since the reservation cost denotes the worst cost a user can guarantee itself when the others attempt to punish it, we consider a game where one user, denoted $-i$, with demand r^{-i} attempts to punish user i. Now suppose that user $-i$ splits itself into n identical users with demand $\frac{r^{-i}}{n}$. Take an NEP f of the game with user i and these n identical users. Since all user i does at the NEP is take away some capacity from each link it uses, we may reduce C_l by $f_l^i(n)$ for each $l \in L$ and consider the resulting network with n identical users. From Theorem 5 of [16] each of the identical users in this network uses the same set of paths. Let $\mathcal{P}(n)$ be the set of paths used by user i at the NEP and $\mathcal{P}_0(n)$ the set of paths used by each of the other n identical users. For any $p \in \mathcal{P}(n)$, let

$$\lambda^i(n) = \sum_{l \in L_p} \frac{1}{C_l - f_l(n)} + \frac{f_l^i(n)}{(C_l - f_l(n))^2} . \tag{97}$$

By the Kuhn-Tucker conditions for an NEP, this is the same for all $p \in \mathcal{P}(n)$. We also have, by the Kuhn-Tucker conditions, that for every $p \in \mathcal{P}$,

$$\sum_{l \in L_p} \frac{1}{C_l - f_l(n)} + \frac{f_l^i(n)}{(C_l - f_l(n))^2} \geq \lambda^i(n) . \tag{98}$$

For any $p \in \mathcal{P}_0(n)$ let

$$\lambda^0(n) = \sum_{l \in L_p} \frac{1}{C_l - f_l(n)} + \frac{f_l^0(n)}{(C_l - f_l(n))^2} . \tag{99}$$

By the Kuhn-Tucker conditions for an NEP, this is the same for all $p \in \mathcal{P}_0(n)$. We also have, by the Kuhn-Tucker conditions, that for every $p \in \mathcal{P}$,

$$\sum_{l \in L_p} \frac{1}{C_l - f_l(n)} + \frac{f_l^0(n)}{(C_l - f_l(n))^2} \geq \lambda^0(n) . \tag{100}$$

In each of equations (97) and (99) the sum of the first terms is the cost per unit flow along the corresponding path. From the previous lemma, because the total system cost is bounded as n goes to infinity, we can see that the sum of the second terms of the RHS in equation (99) or the LHS of (100) goes to zero, i.e., for every $p \in \mathcal{P}$,

$$\lim_{n \to \infty} \sum_{l \in L_p} \frac{f_l^0(n)}{(C_l - f_l(n))^2} = 0 . \tag{101}$$

Since the total system cost is bounded from the lemma, the sequence $(\lambda^0(n), n \geq 1)$ has a converging subsequence, i. e.

$$\lim_{k \to \infty} \lambda^0(n_k) = \lambda^* \tag{102}$$

for some subsequence $(n_k, k \geq 1)$. Combining equations (100), (101), and (102), we see that, for every $p \in \mathcal{P}$,

$$\liminf_{k\to\infty} \sum_{l\in L_p} \frac{1}{C_l - f_l(n_k)} \geq \lambda^* , \tag{103}$$

while, if we combine equations (99), (101), and (102), we see that for $p \in \mathcal{P}_0$ we have

$$\liminf_{k\to\infty} \sum_{l\in L_p} \frac{1}{C_l - f_l(n_k)} = \lambda^* , \tag{104}$$

The cost per unit flow along a path p is $\sum_{l\in L_p} \frac{1}{C_l - f_l}$. We have just seen that the average cost per unit flow of user i along any path that it uses is at least as large as the average cost per unit flow of the n_k identical users into which user $-i$ (the coalition of the users other than user i) splits itself, in the limit as $k \to \infty$. As the total system cost in the limit is no smaller than C', the cost of user i is greater than or equal to that of proportional sharing. This proves that the reservation cost of user i is greater than or equal to the cost of proportional sharing. This gives rise to three different possible scenarios.

(1) Every user's reservation cost is strictly greater than that of proportional sharing.

(2) Some users have reservation cost strictly greater than that of proportional sharing, and some users have reservation cost equal to that of proportional sharing.

(3) Every user's reservation cost is equal to that of proportional sharing.

In case (1), the existence of an NEP that achieves the minimum total system cost follows directly from the folk theorem, which was described in section 2, see [6]. Consider the following strategy profile in the repeated game. Initially each user i uses the flow configuration $\tilde{f}^i = \frac{r^i}{R} \cdot f'$ until exactly one user deviates at some period k. If exactly one user deviates at period k, then starting at period $k+1$, other users use a flow configuration that yields to user i, for its best response, a cost of $\underline{v}_i$. Then, it is easy to see that, if the discount factor is sufficiently close to 1, no user has an incentive to deviate from $\tilde{f} = (\tilde{f}^1, ..., \tilde{f}^I)$, which we call *proportional sharing.*

In case (3), since any NEP in the stage game Pareto-dominates the reservation cost vector and the total system cost at the stage game NEP cannot be smaller than C', the stage game NEP achieves the total system cost C'. Repeating a stage game NEP in each period is an NEP, in fact an SPNEP, in the repeated game.

Consider case (2). Let A be the set of users whose reservation cost is equal to that of proportional sharing, and let B be the complement, $I \setminus A$. Recall the technical assumption we made that, under f', two paths exist that are used and have different cost per unit flow. For any user k, let $\tilde{g}^k = (\tilde{g}^k_1, ..., \tilde{g}^k_{\mathrm{P}_0})$ be the flow configuration under proportional sharing, where $\tilde{g}^k_p$ denotes the amount of flow user k sends on path p, and P_0 is the number of paths used under f' by the global user 0 that attempts to minimize the total system cost. Without loss of generality assume that paths are ordered by increasing cost per unit flow, i.e., $\sum_{l\in L_1} \frac{1}{C_l - f_l} \leq \cdots \leq \sum_{l\in L_{\mathrm{P}_0}} \frac{1}{C_l - f_l}$. Suppose user i is in B and user j is in A. Now, consider the following flow configurations of user i and user j.

$$\bar{g}^i = (\tilde{g}^i_1 - \epsilon, ..., \tilde{g}^i_{\mathrm{P}_0} + \epsilon)$$

and

$$\bar{g}^j = (\tilde{g}^j_1 + \epsilon, ..., \tilde{g}^j_{\mathrm{P}_0} - \epsilon)$$

Since the cost per unit flow of path 1 is strictly smaller than that of path P_0 from the technical assumption, the cost yielded to user j by $\bar{g}^j$ is strictly smaller than its proportional sharing cost, which is equal to its reservation cost. Also, since the reservation cost of user i is strictly bigger than that of proportional sharing, if ϵ is sufficiently small, then the cost yielded by $\bar{g}^i$ will still be strictly smaller than its reservation cost. For every user in A this can be done with some user in B. Thus, we can find a system configuration that yields to every user a cost strictly smaller than its reservation cost, while the overall cost is optimum. By the folk theorem, this proves that if the discount factor is sufficiently close to 1 there is an NEP in the repeated game for which the overall cost is optimum. ■

Let us first define the rational feasible cost region to be the subset of the feasible cost region that Pareto-dominates the reservation cost vector. We will now define full dimensionality. Suppose V^* is the rational feasible cost region. Then, we say that V^* has full dimensionality if there exists $\underline{v} \in V^*$ and $\varepsilon > 0$ such that all $\underline{v}'$ for which $| v_i' - v_i | < \varepsilon$ for all $i \in I$ are in V^*. This means that the interior of V^* is nonempty in R^{I}.

We will now show that any NEP of the repeated game is also an SPNEP of the game by showing that full dimensionality holds.

Theorem 4 *If the discount factor is sufficiently close to* 1, *there exists an SPNEP in the repeated game that achieves the minimum total system cost, C'.*

Proof: We need to show that the rational feasible cost region has nonempty interior. We need to look at two cases.

(1) Every user's reservation cost is equal to that of proportional sharing.

(2) Some user's reservation cost is strictly greater than that of proportional sharing.

In case (1), the stage game NEP achieves the minimum total system cost C', and repeating a stage game NEP is obviously an SPNEP in the repeated game.

In case (2), we know that there is a system flow configuration that yields every user a cost strictly smaller than its reservation cost, from the proof of Theorem 3. Let one of such system flow configurations be $\bar{f}$ and the associated cost vector $\bar{v}$. Further, from examining the proof of Theorem 3, we see that there is an open neighborhood of cost vectors whose intersection with the hyperplane of constant total cost equal to the optimum cost consists entirely of feasible cost vectors that Pareto dominate the reservation cost vector (i.e. rational feasible cost vectors). Since the rational feasible region is convex, we see that there is an open neighbourhood in the rational feasible region. This means that the full dimensionality condition holds, and so, by the folk theorem, if the discount factor is sufficiently close to 1 there will be an SPNEP for the repeated game that achieves optimum total cost.

We now describe the nature of the strategies that result in such an SPNEP. Since the cost vector corresponding to $\bar{f}$ Pareto dominates the reservation cost vector, we may in fact find an open neighbourhood in the rational feasible region centered around a vector v that satisfies for each user i, $\underline{v}_i > v_i > \bar{v}_i$. Also, for each i let $f^*(i)$ denote the system flow configuration that yields cost vector $v^*(i) = (v_1 - \varepsilon, ..., v_{i-1} - \varepsilon, v_i, v_{i+1} - \varepsilon, ..., v_{\mathrm{I}} - \varepsilon)$ such that each $v^{*j}(i)$, user j's cost in $v^*(i)$, is strictly greater than $\bar{v}_j$. Also, let $\underline{f}(i)$ denote the system flow configuration that yields to user i its reservation cost $\underline{v}_i$ when it best responds. Then, consider the following strategy profile in the repeated game.

(X) Initially users use $\bar{f}$ until exactly one user deviates from $\bar{f}^i$ at some period k. If exactly one user i deviates at period k, then go to (X_i).

(X_i) Users play $\underline{f}(i)$ for N periods. If every user cooperates for N periods, go to (Y_i). If exactly one user j deviates at some period k', then go to (X_j).

(Y_i) Users play $f^*(i)$ in each period. If exactly one user j deviates, then go to (X_i).

The punishment period N above is chosen such that, for all i, $min_{f^i} J^i(f^{*-i}(i), f^i) + N \cdot \underline{v}_i > (N+1) \cdot v_i$, where $f^{*-i}(i)$ denotes the flow configurations of all users other than user i under $f^*(i)$.

Then, for a discount factor sufficiently close to 1, one can show that if user i deviates from $\bar{f}$, i.e., (X), its average cost will be bigger than $C' \cdot \frac{r^i}{R}$. Also, any user who deviates from the punishment period, i.e., (X_i), gets a bigger average cost greater than the average cost it would receive if it cooperated in the punishment stage. The intuition behind this is that any gain by some user i at any period will be outweighed by a penalty imposed on it during the N punishment periods plus the penalty imposed on it forever after the punishment periods. The details are part of the proof of the folk theorem, see [6], pp. 158 -160. Thus, playing $\bar{f}$ every period leads to an SPNEP in the repeated game. ∎

4.3 Multiple Source-Destination Pairs Case

4.3.1 Model and Problem Formulation

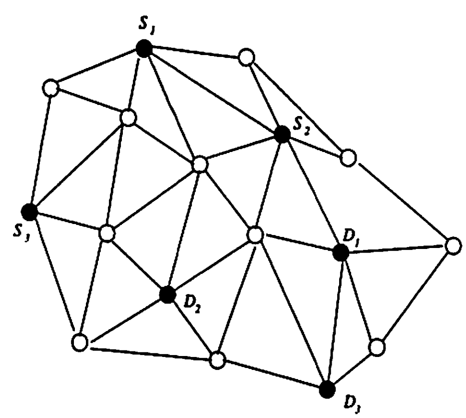

Figure 5: Multiple source-destination pairs case.

In this subsection, as in the preceding subsection, we consider a network with a set of users, $I = \{1,2,...,I\}$, $I \geq 2$, but now different users may have different source-destination pairs. User i's demand is again denoted by r^i and the capacity of link l is denoted by C_l. Each user splits its demand among the available paths from its source node to its destination node. The performance measure of user i is given by the same cost function $J^i(f) = \sum_{l \in L} J_l^i(f) = \sum_{l \in L} f_l^i \cdot T_l(f_l)$, where $T_l(f_l)$ is given by equation (1). User i's flow configuration can be written similarly as in the single source-destination pair case. However, one has to be careful with notation since there are different paths available for different source-destination pairs. Again, since each user attempts to minimize its own cost, the problem is modeled as a noncooperative game.

The existence of an NEP for the stage game is guaranteed by the result due to Rosen [18] which was described in section 2. However, the uniqueness of the system flow configuration that achieves minimum total system cost hasn't been proven yet. Rather than attempting to prove or disprove such uniqueness, we will show that even when there exists a unique system flow configuration that achieves the minimum total system cost, C', and the full dimensionality condition is satisfied, there may not exist an NEP of the repeated game that achieves C', however close the discount factor is to 1.

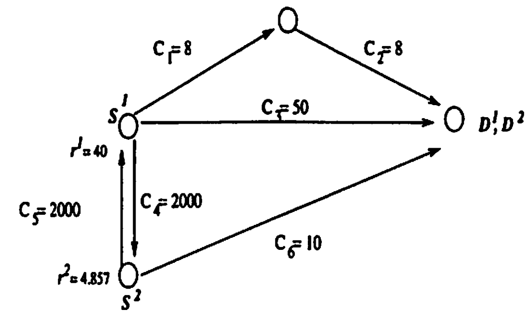

Figure 6: Multiple source-destination pairs case.

Consider the network with two users as shown in Figure 6. The reservation costs of user 1 and user 2 are 4.326 and 0.6416, respectively. Also, if we consider a global user who attempts to minimize the overall system cost, the unique system flow configuration that minimizes the total cost is $f = (f_1, f_2, f_3, f_4, f_5, f_5) = (1.5,1.5,38.5,0,0,4.857)$, which yields costs 3.81 and 0.944 for user 1 and 2, respectively. It is easy to see that this is the unique system flow configuration that achieves C' because in order for any system flow configuration to be optimum, it cannot send any flow on link 4 or link 5. Notice that even though the total system cost is smaller, the unique system-wide optimum flow configuration, f, requires user 2 to incur a cost that is greater than its reservation cost. Therefore, there exists no NEP of the repeated game that achieves C' and yet Pareto-dominates the reservation cost vector. This proves that it is not always possible to find an NEP of the repeated game that achieves the minimum total system cost in multiple source-destination pairs case.

Let us now consider the system flow configuration $(f^1, f^2) = ((1.33,1.33,35.368,3.302,0,3.302),$ $(0,0,3.428,0,3.428,1.429))$ which yields $\breve{J}^1 = 4.1839$ and $\breve{J}^2 = 0.5772$ for user 1 and 2, respectively. It is easy to see that $\exists\ \epsilon > 0$ such that all $\underline{J}$ with $\mid \underline{J}^i - \breve{J}^i \mid < \epsilon$ for $i \in \{1, 2\}$ are in the rational feasible cost region. This proves that the above example satisfies the full dimensionality condition.

Suppose that we are given a network with a finite number of users. Let $S = \{s_1, \cdots, s_m\}$ be the set of source-destination pairs, and I_k the set of users that have source-destination pair s_k. Class user k is a user whose source-destination pair is s_k and which has demand $r^k_{class} = \sum_{i \in I_k} r^i$. Each class user can be considered to represent the coalition of the users in the original network having a given source-destination pair. We will now consider the stage game NEPs of the network with only class users. Let $\hat{C}$ be the smallest among the total system costs achieved by such NEPs.

Theorem 5 *There is an NEP in the repeated game with the original users that achieves overall cost $\hat{C}$.*

Proof: Suppose $\hat{f}$ is a system flow configuration with class users that achieves $\hat{C}$. Then, given the flow configurations of other class users, class user k cannot reduce its cost by changing its own flow configuration. Thus, from class user k's point of view, all that the other class users do is to take away some of the capacities from each link used by them. Thus we are now back to a problem of routing with a single source-destination pair, with link l's capacity reduced by the total flow of the other class users, so that it is now $C_l - \hat{f}_l^{-k}$. Thus, if the cost incurred by class user k at the NEP is denoted $\hat{C}_k$ then, from the proof of Theorem 3, the reservation cost of each user i of class k in the original game is no smaller than the cost user i would incur when it used a flow configuration, $\frac{r^i}{r^k_{class}} \cdot \hat{f}^k$, and this would be $\frac{r^i}{r^k_{class}} \hat{C}_k$. The existence of an NEP of the original repeated game that achieves a total system cost of $\hat{C}$ now follows similarly as in the proof of Theorem 3. ■

An interesting question now would be whether $\hat{C}$ is the smallest total system cost that can be achieved by any NEP or if there is an even smaller cost achievable by some NEP. It proves to be very difficult to characterize the set of NEPs that achieve the minimum total system cost among the class user NEPs. However, it seems that in most cases there exists an NEP for the game among the original users that achieves a smaller cost than the minimum among the NEP costs with class users.

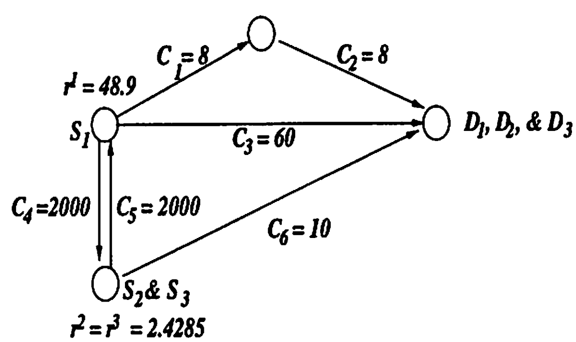

Figure 7: Multiple source-destination pairs case.

For an example of this, let us look at the network in Figure 7. There are three users, and user 2 and 3 have same source-destination pair and same demand while user 1 has a different source-destination pair. If we consider a global user who attempts to minimize the overall system cost, the unique system flow configuration that yields the minimum total system cost C' is $f' = (f'_1, ..., f'_6)$ = (1.5,1.5,47.4,0,0,4.857). Note that f' requires users 2 and 3 to use link 6 only, which has much higher cost per unit flow than path along link 3 and 5. Now, let us investigate the network with class users. Suppose class user 2 represents users 2 and 3. The reservation costs of user 1 and class user 2 are 4.778 and 0.5963, respectively. The minimum cost that is achievable among the NEPs in repeated games with class users can be shown to be 5.181 and the corresponding system flow configuration is $\hat{f} = (\hat{f}^1, \hat{f}^2)$ = ((1.2765,1.2765,44.15,3.4735,0,3.4735), (0,0,3.18,0,3.18,1.677)). Now, note that both links 4 and 5 are used in $\hat{f}$. Let us go back to the original game now. Suppose users 1 and 2 use flow configurations (0,0,48.9,0,0,0) and (0,0,1.76425,0,1.76425,0.66425) respectively. Then, user 3's best reply is (0, 0,1.21425,0,1.21425,1.21425), which yields a cost 0.299 for user 3. Since user 2 and 3 are identical, the reservation cost of each user 2 and 3 is greater than or equal to 0.299. Repeating $\check{f} = (\check{f}^1, \check{f}^2, \check{f}^3)$ = ((1.2765,1.2765,44.16,3.4635,0,3.4635), (0,0,1.585,0,1.585,0.8435), (0,0,1.585,0,1.585,0.8435)) yields a cost 4.579, 0.299, and 0.299 for users 1, 2, and 3, respectively, and a total system cost of 5.177. Hence, this is an NEP that achieves a total system cost smaller than 5.181. Note that this decrease in the total system cost comes from the reduction in the flow on link 4 and 5. This phenomenon is because users 2 and 3 can punish each other, and the presence of this threat drives the users to operate at an NEP closer to the system-wide optimum point. This proves that in some cases there is an NEP that achieves a smaller total system cost than the smallest NEP cost with class users.

5 Conclusion

Due to the steady increase in the demand for bandwidth, it is becoming increasingly important to find routing schemes that use resources efficiently. Since IPv6 allows routers to decide to some extent which path their packets will take, it is crucial to understand how network access providers will interact with each other given such capabilities. Because this is essentially a situation with selfish users who attempt to minimize their own costs, it is appropriate to model the network routing problem as a noncooperative game, where the agents playing the game are thought of as the network access providers. If the strategies of the users are such that no user finds it beneficial to deviate unilaterally, it is natural to believe that this represents an equilibrium situation for the network. The concept of an NEP captures exactly this idea. NEPs are, however, not necessarily efficient since users are interested in optimizing their own costs but not the total system cost. Since the network access providers will typically interact with each other several times before the structure of the game they are playing changes significantly, it is natural to investigate the problem of finding an NEP in the repeated game that achieves the system-wide optimum cost.

In this paper we have shown that in parallel link networks, there always exists an NEP that achieves the system-wide optimum cost and yet yields a cost for each user that is no greater than that of the unique stage game NEP. Further, this NEP is subgame-perfect (SPNEP), i.e., the strategies involved result in an NEP in every subgame of the overall game. This means that the strategies involved are based on credible threats and incentives. In general networks where every user has same source and destination nodes, we again show there exists an SPNEP that achieves the minimum total system cost, assuming that the network satisfies a mild technical condition. However, the existence of an SPNEP that not only achieves the minimum total system cost but also yields each user a cost no greater than that of any stage game NEP is still an open problem in this case. In more general networks where different agents have different source-destination pairs, we have shown that it is not always possible to find an NEP that achieves the system-wide optimum cost even when the full dimensionality condition holds. This is due to the fact that in order to achieve the minimum total system cost, some users are required to incur costs greater than their reservation costs. We have proved that there is an NEP that achieves the smallest cost among the NEP costs with class users. This, however, may not be the best one could do. We have given an example where there exists an NEP that yields a smaller total system cost than the smallest among the NEP costs with class users. This shows that it is not always good enough to consider the network where each source-destination pair is represented by a class user. More work needs to be done on characterizing the NEPs that achieve the smallest total system cost in general network with multiple source-destination pairs.

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