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Optimal Sequences, Power Control and Capacity of Spread-Spectrum Systems with Multiuser Linear Receivers*

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Abstract

To meet the increasing demand on wireless networks, there has been intense effort in the past decade on developing *multi-user* receiver structures which mitigate interference between users in spread-spectrum systems. While much of this research is performed at the physical layer, the appropriate power control and choice of signature sequences in conjunction with multiuser receivers and the resulting network capacity is not well understood. In this paper we will focus on a single cell and consider both the uplink and downlink scenarios and assume a synchronous CDMA (S-CDMA) system. We characterize the *capacity* of a single cell with both the optimal linear receiver (MMSE receiver) and matched filter receiver structures. The capacity of the system is the maximum number of users per unit processing gain admissible in the system such that each user has its quality-of-service (QoS) requirement (expressed in terms of its desired signal-to-interference ratio) met. We also identify the "optimal" signature sequences and power control strategies so that the users meet their QoS requirement. We propose a simple construction scheme for the "optimal" signature sequences that we identify. We also characterize the effect of transmit power constraints on the capacity.

1 Introduction

A central problem in the design of wireless networks is how to use the limited resources such as bandwidth and power most efficiently in order to meet the quality-of-service requirements

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of applications in terms of bitrate and loss. To meet these challenges, there have been intense efforts in developing more sophisticated physical layer communication techniques. A significant thrust of work has been on developing *multi-user* receiver structures which mitigate the interference between users in spread spectrum systems. (See for example [19, 6, 7, 21, 8, 12, 14].)

Despite significant work done in the area, there is still much debate about the network capacity of the various approaches to deal with multi-user interference in spread-spectrum and multi-antenna systems. One reason is that the networking level problems of resource allocation and power control are less well understood in the context of multi-user techniques than with more traditional multi-access schemes, such as TDMA, FDMA and conventional CDMA systems. For example, much of the previous work on performance evaluation of multiuser receivers focus on their ability to reject worst-case interference (so-called *near-far resistance* [6]) rather than on their performance in a power-controlled system.

In this paper, we would like to study the capacity of a power-controlled spread-spectrum system with multiuser receivers. We consider a single cell synchronous CDMA (S-CDMA) system. Although such complete synchronism is rarely achieved in practice, the study of such systems is a first step towards analysing asynchronous models. The users are distinguished from each other by their signature sequences or codes. The processing gain represents the *degrees of freedom* in the system. We assume that at the receiver, each user is demodulated using a linear receiver structure. We are interested in the *capacity* of both the uplink (mobiles to base station) and the downlink (base station to mobiles) of this system equipped with multi-user linear receiver structures. We say that a set of users is *admissible* in the uplink system with processing gain L if one can allot signature sequences to the users and control their transmit power such that the signal-to-interference (SIR) of each user is greater than its SIR requirement. We are interested in the problem of characterizing the maximum number of users per degree of freedom, called the capacity of the uplink system. Analogous definitions of admissibility and capacity can be made for the downlink.

Though any choice of linear receiver is allowed in this framework, in practice, two linear receivers are of particular interest: the MMSE receiver which is the best linear receiver (in the sense of maximizing the SIR of each user, see [21, 8, 12, 14]),and the conventional matched filter receiver. We consider each receiver separately and characterize the capacity of the system. Our main results, which hold both for the uplink and downlink, are as follows:

1. With the MMSE linear receiver structure for each user, M users each having a SIR requirement of β are admissible in the system with processing gain L if and only if $\frac{M}{L} < 1 + \frac{1}{\beta}$.
2. With the matched filter receiver structure for each user, M users each having a SIR requirement of β are admissible in the system with processing gain L if and only if $\frac{M}{L} < 1 + \frac{1}{\beta}$. Thus we have the unexpected result that by a choice of "good" signature sequences there is no loss in capacity by using the, a priori inferior, matched filter receiver.

3. Voice and data users may well be in the same system and it is important to have a level of generality in the model that caters to users having different SIR requirements. We say that a user is of class l if it has SIR requirement β_l and we assume that there is a finite number, say N, of classes. The following statements are true with both MMSE and matched filter receivers.

   - If M_l users of class l are admissible in the system with processing gain L then
   $$\sum_{l=1}^{N} M_l \frac{\beta_l}{1+\beta_l} < L.$$
   - Conversely, if
   $$\sum_{l=1}^{L} \alpha_l \frac{\beta_l}{1+\beta_l} < 1,$$
   then for every class l, α_l users per unit processing gain of that class are asymptotically admissible in the system as $L \to \infty$.

   This allows us to characterize the admissibility of users via a notion of *effective bandwidth.* If we consider $\frac{\beta_l}{1+\beta_l}$ as the effective bandwidth of a user of class l, then M_l users of class l are admissible in the system with processing gain L implies that the sum of the effective bandwidths of all users is less than or equal to L.

4. We identify the "optimal" signature sequences and the "minimal" transmit powers of the users to admit the maximum number of users in the system. We also provide a simple algorithm to construct the "optimal" signature sequences that we have identified.

In [16], the authors consider the scenario when the signature sequences of the users are independent and randomly chosen. They show that the SIR of the users of such a large system converges (in probability) and analyze the capacity of the system based on the value to which the SIR converges. It is interesting to compare the performance of that system with the one considered here when the sequences are optimally chosen:

1. Under the MMSE receiver, the capacity of a system using random sequences is asymptotically *identical* to that of a system with optimally chosen sequences. This holds when there are no transmit power constraints, or equivalently, when the background noise power is low. We will provide an explanation for this phenomenon.

2. Under the conventional matched-filter receiver, a system using random sequences admits 1 user per degree of freedom less than when the sequences are optimally chosen. This shows that while the MMSE and the matched-filter receivers have the same performance when the sequences are optimally chosen, the MMSE receiver is much more robust to the choice of spreading sequences.

3. Under transmit power constraints, systems employing random sequences admit strictly less users than the corresponding systems with optimal sequences. We quantify precisely the gap in performance.

In related work, there has been a great deal of research studying the problem of power control of the users for conventional CDMA systems. Distributed iterative algorithms that achieve power control of the users is discussed in for example [1] and [2]. These ideas were extended subsequently to systems with MMSE receivers [18], [5], but they focused on deriving convergent power control algorithms rather than analysing the achievable capacity. The problem of identifying good signature sequences has been studied in [11] in the context of a spread-spectrum system with conventional receiver and equal received power for all users, and in [10] in an information theoretic setting. Different from these works, we here study the joint optimization problem of designing multiuser receiver structure, power control and spreading sequences, and obtain simple characterizations of the resulting system capacity in various scenarios.

An important special case subsumed by our framework is when the signature sequences are constrained to be chosen from an orthogonal sequence set. This corresponds to dividing the entire bandwidth into frequency slots (or channels), i.e., a joint FDMA/CDMA system. In this case the receiver is trivial and both MMSE and matched filter receiver structures coincide. Our main result in this framework is as follows: M users each with SIR requirement β are admissible in the system with processing gain L if and only if $M < L\lfloor 1+\frac{1}{\beta}\rfloor$. We observe that the maximum number of users admissible per unit processing gain differs from the earlier results by just an integer part. Thus, we identify the gain by using non-orthogonal codes and multi-user linear receivers; the difference depends on the factor $\frac{1}{\beta}$ and the processing gain L. In the scenario when users are differentiated by their SIR requirement, we identify the capacity of the system.

This paper is organized as follows: In Section 2, we give a precise definition of the uplink model and of the admissibility of the users. Capacity of the uplink system is identified in Section 3 and Section 4 deals with the situation when the receiver structure is fixed to be the matched filter. Section 5 discusses the situation when users have different SIR requirements. In a physical system, the power transmitted by a user is constrained naturally. In Section 6, we adopt a model of mobility of the users and appropriately define capacity of the system with a transmit power constraint. We characterize this capacity for both MMSE and matched filter receiver structures, which turn out to be identical. In Section 7, we will focus on the downlink. We can ask our admissibility and capacity region questions in this setup too. As can be expected, there is a lot of connection between the downlink and uplink scenarios and we summarize the results. Section 8 focuses on the joint FDMA/CDMA setup that corresponds to the restriction of signature sequences to be chosen from an orthogonal sequence set and identifies the capacity of the system under various settings. Section 9 contains our conclusions.

2 Model and Definitions

We consider a multi-access symbol-synchronous spread spectrum system and focus on the uplink. Each user spreads its information on a common channel through modulation using its signature sequence. Let the processing gain of the system be L. Traditional spread spectrum systems choose their signature sequences from $\{-1,+1\}^L$ and a simple multiplicative demodulator followed by a low pass filter. With a choice of a general linear receiver, we assume that the signature sequences come from $\mathcal{S}_1^L$, the unit sphere in $\mathcal{R}^L$. Suppose there are M users in the system and each has signature sequences $s_1, s_2, \dots, s_M$. We can model the information transmitted by each user as zero mean, independent random variables $X_1, X_2, \dots, X_M$. The variance $E\left[X_i^2\right]$ is the power at which user i is received. We denote the received power of user i as p_i, the product of the transmit power of user i and the path gain from user i to the receiver (base station). We assume an ambient white gaussian noise, denoted by $W \sim N\left(0, \sigma^2 I\right)$ independent of the transmitted symbols. Then the received signal at the receiver, represented by Y, can be written as:

$$Y = \sum_{i=1}^{M} s_i X_i + W$$

Suppose the symbol of user i is decoded using a linear receiver, denoted by c_i(a vector in $\mathcal{R}^L$), then the signal to interference ratio of user i (SIR_i) is

$$SIR_i = \frac{\left(c_i, s_i\right)^2 p_i}{\sigma^2\left(c_i, c_i\right) + \sum_{j \neq i}\left(c_i, s_j\right)^2 p_j} \tag{1}$$

We say that M users are *admissible* in the system if there is a choice of positive powers $p_1, \dots, p_M$, signature sequences $s_1, \dots, s_M \in \mathcal{S}_1^L$ and linear receiver structures $c_1, \dots, c_M$ such that

$$SIR_i \geq \beta \ \ \forall i = 1 \dots M$$

Here $\beta > 0$ is some fixed SIR requirement of each user that has to be met for satisfactory performance.

2.1 Structure of Optimum linear receiver

In the framework above, the choice of the linear receiver structure is left open. It is well known that the MMSE receiver is the optimum linear receiver structure, optimum in the sense of maximizing the SIR of each user. While there are many derivations of the structure of the MMSE receiver (see [8], [16] for example) $c_1, \dots, c_M$, we give an elementary derivation of the same as the argument of a problem of minimizing a convex function over a convex set below (this will also aid us in developing notation to be used in the characterization of the capacity regions):

Fix the user powers $p_1, \dots, p_M$, and the signature sequences $s_1, \dots, s_M$. The optimum receiver c_i is one that maximizes SIR_i. Now, let $S = [s_1, s_2, \dots, s_M]$ and $D = \operatorname{diag}(p_1, \dots, p_M)$

and $S_i = [s_1 \ldots, s_{i-1}, s_{i+1} \ldots, s_M]$ and $D_i = \operatorname{diag}(p_1, \ldots, p_{i-1}, p_{i+1} \ldots, p_M)$. Let $Z_i = S_i D_i S_i^t + \sigma^2 I$ and $Z = SDS^t + \sigma^2 I$ and we note that they are positive definite. Let $Z_i = U_i \Lambda_i U_i^t$ for a positive diagonal matrix Λ_i and unitary U_i. Also, let $SDS^t = U\Lambda U^t$. Then,

$$\begin{aligned}
\max_{c_i \neq 0} SIR_i &= \max_{c_i \neq 0} \frac{(c_i, s_i)^2 p_i}{c_i^t Z_i c_i} \\
&= p_i \max_{x_i \neq 0} \frac{x_i^t \Lambda_i^{-\frac{1}{2}} U_i^t s_i s_i^t U_i \Lambda_i^{-\frac{1}{2}} x_i}{x_i^t x_i} \text{ where } x_i = \Lambda_i^{\frac{1}{2}} U_i^t c_i
\end{aligned}$$

Thus the argmax is given by $x_i = \Lambda_i^{-\frac{1}{2}} U_i^t s_i$ and the optimal receiver structure is

$$c_i = Z_i^{-1} s_i \tag{2}$$

Hence, under the MMSE receiver,

$$SIR_i = s_i^t Z_i^{-1} s_i p_i \tag{3}$$

3 Characterization of Capacity

In this section we derive the first main result of this paper: the identification of the capacity of a single cell S-CDMA system equipped with the MMSE receiver. We assume that each user has the same SIR requirement β. Observe that if the number of users is less than or equal to the processing gain, the trivial choice of orthogonal signature sequences for the users ensures arbitrary SIR requirements to be met if we can scale up the power of the users. Hence, without of loss of generality we shall henceforth assume that the number of users is greater than the processing gain.

Theorem 3.1 *M users are admissible in the system with processing gain L if and only if*

$$M < L\left(1 + \frac{1}{\beta}\right).$$

Proof Step 1: Upper Bound on the number of users
Suppose M users are admissible in the system with processing gain L. Then, by definition, there exist sequences $s_1, \ldots, s_M \in \mathcal{S}_1^L$, positive powers $p_1, \ldots, p_M \ni$ for every user i, we have $SIR_i \geq \beta$, where the receiver structure is as in (2). Now, $\forall i = 1 \ldots M$, from (3),

$$\begin{aligned}
SIR_i &= s_i^t Z_i^{-1} s_i p_i \\
&= s_i^t \left(Z - p_i s_i s_i^t\right)^{-1} s_i p_i \\
&= s_i^t \left(Z^{-1} + \frac{Z^{-1} s_i s_i^t Z^{-1} p_i}{1 - s_i^t Z^{-1} s_i p_i}\right) s_i p_i \\
&= \frac{s_i^t Z^{-1} s_i p_i}{1 - s_i^t Z^{-1} s_i p_i} \qquad (4)
\end{aligned}$$

where we used the following formula in the second step:

$$\left(A - xy^t\right)^{-1} = A^{-1} + \frac{A^{-1}xy^tA^{-1}}{1 - y^tA^{-1}x}$$

whenever the terms exist. Here, every eigenvalue of Z is strictly bigger than p_i and thus all the terms are well defined. Thus, we have

$$\min_{i=1\ldots M} s_i^t Z^{-1} s_i p_i \geq \frac{\beta}{1+\beta} \tag{5}$$

Now, summing up all the terms, we have

$$\begin{aligned}
\sum_{i=1}^{M} s_i^t Z^{-1} s_i p_i &= \operatorname{tr}\left(S^t Z^{-1} S D\right) \\
&= \operatorname{tr}\left(S D S^t Z^{-1}\right) \\
&= \operatorname{tr}\left(\Lambda\left(\Lambda + \sigma^2 I\right)^{-1}\right) \\
&= \sum_{i=1}^{L} \frac{\lambda_i}{\lambda_i + \sigma^2} \quad \text{where } \Lambda_{ii} = \lambda_i \\
&< L
\end{aligned} \tag{6}$$

where we used the elementary fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all matrices A, B of dimensions $M \times L$ and $L \times M$ respectively and for all M, L. Using (5) and (6), we have

$$\begin{aligned}
\frac{\beta}{1+\beta} &\leq \min_{i=1\ldots M} s_i^t Z^{-1} s_i p_i \\
&\leq \frac{1}{M} \sum_{i=1}^{M} s_i^t Z^{-1} s_i p_i \\
&< \frac{L}{M}
\end{aligned}$$

This completes the proof of the upper bound. ■

Step 2: Achievability of the bound
We shall need the following Lemma first.

Lemma 3.1 *Fix $M \geq L$. Then, there exists a $L \times M$ real matrix $S = [s_1 s_2 \ldots, s_M]$ where $s_i \in S_1^L$ and the rows of S are orthogonal and have l_2 norm $\sqrt{\frac{M}{L}}$.*

Thus, $\forall M \geq L$, Lemma 3.1 gives a real $L \times M$ matrix $S = [s_1 s_2 \ldots, s_M]$ such that $SS^t = \frac{M}{L} I$. We prove the lemma in Appendix B and also provide a method of construction of such vectors $s_1, \ldots, s_M$ $\forall M \geq L$. These sequences were first identified in [10] (but in their context the sequences were in $\{1,-1\}^L$) and the authors referred to such $s_1, \ldots, s_M$ as WBE sequences, sequences that meet the so-called *Welch Bound Equality* (see [20]). WBE sequences have also appeared in [11] in an information theoretic setting. The proof of the existence of WBE sequences is missing and construction of WBE sequences is not exhibited for every $M \geq L$ in [11] and [10]. We shall henceforth assume that the WBE sequences for the pair (M, L) are in $\mathcal{S}_1^L$.

Suppose $M < L\left(1+\frac{1}{\beta}\right)$. Choose the signature sequences for the users as the WBE sequences for the pair (M, L). Choose the powers to be

$$p_i = p = \frac{\sigma^2 \beta L}{L(1+\beta) - M\beta} \quad \forall i = 1 \ldots M \tag{7}$$

Then, by our particular choice of signature sequences, $SS^t = \frac{M}{L}I$. Using (4), $\forall i = 1 \ldots M$,

$$\begin{aligned} \frac{SIR_i}{1+SIR_i} &= s_i^t \left(pSS^t + \sigma^2 I\right)^{-1} s_i p \\ &= s_i^t \left(\frac{M}{L} pI + \sigma^2 I\right)^{-1} s_i p \\ &= \frac{Lp}{Mp + L\sigma^2} \\ &= \frac{\beta}{1+\beta} \end{aligned}$$

Hence for each user i, we have $SIR_i = \beta$ and the M users are admissible in the system with processing gain L. ■

Suppose M users are admissible in the system with processing gain L and suppose the choice of the sequences can be made to be $s_1, \ldots, s_M$. Then, among all the choice of powers p_i for the users that can be made so that SIR_i of each user is greater than or equal to β, there exists a component-wise minimal power choice (see [18]). With the choice of WBE sequences for the pair (M, L) as the signature sequences of the users, the choice of powers for the users in the proof above, namely $p_i = p = \frac{\sigma^2 \beta L}{L(1+\beta) - M\beta}$ represents the component-wise minimal power solution. We shall prove this in Section 6 when we revisit capacity with transmit power constraints. In general, if the sequences chosen are $s_1, \ldots, s_M$ let $S = [s_1 \ldots s_M]$ and denote $\bar{p}_S$ as the component-wise minimal power choice.

It is interesting to compare this result with the corresponding one in [16]. The results in [16] are asymptotic and are valid for a large system (i.e., a system with a large processing gain and large number of users). Then, it is shown that for a system with M users (each user having SIR requirement β) and processing gain L as $M \to \infty$ and $L \to \infty$ and $\frac{M}{L} \to \alpha$, the users have their SIR requirements met if and only if $\alpha < 1 + \frac{1}{\beta}$. Thus for a large system this suggests that using random sequences is as good as using the optimal WBE sequences for the signature sequences of the users.

To gain more insights about why random sequences are as good as optimal sequences, let us give a different interpretation to Theorem 3.1. We first give a formula for the MMSE receiver and the associated SIR under the MMSE receiver, alternative but equivalent to (2) and (3). First recall the channel model in matrix form:

$$Y = SX + W$$

where S is the matrix the columns of which are the signature sequences of the users and X is the vector of transmitted symbols from the users. If $\hat{X}$ is the vector MMSE estimate of X, a direct application of the orthogonality principle $E[(\hat{X}-X)^t Y] = 0$ yields

$$\hat{X} = DS^t \left[SDS^t + \sigma^2 I\right]^{-1} Y$$

and the covariance matrix of the error $\epsilon \equiv \hat{X} - X$ is given by

$$K_\epsilon = D - DS^t \left[SDS^t + \sigma^2 I\right]^{-1} SD \tag{8}$$

where $D \equiv \text{diag}(p_1, \ldots, p_M)$ is the covariance matrix of X. Right multiplying the above equation with D^{-1} and taking the trace of both sides, we get:

$$\begin{aligned} & \text{trace}(K_\epsilon D^{-1}) & (9)\\ = \;& M - \text{trace}\left(DS^t \left[SDS^t + \sigma^2 I\right]^{-1} S\right) & \\ = \;& M - \text{trace}\left(SDS^t \left[SDS^t + \sigma^2 I\right]^{-1}\right) \qquad \text{using the fact trace}(AB) = \text{trace}(BA) & \\ = \;& M - \sum_{i=1}^{L} \frac{\lambda_i}{\lambda_i + \sigma^2} & (10) \end{aligned}$$

where λ_i's are the eigenvalues of the matrix SDS^t. If we let

$$\text{MMSE}_i \equiv \frac{E[(\hat{X} - X_i)^2]}{p_i}$$

be the (normalized) minimum mean-square error for user i, then (10) says that

$$\sum_{i=1}^{M} \text{MMSE}_i = M - \sum_{i=1}^{L} \frac{\lambda_i}{\lambda_i + \sigma^2} \tag{11}$$

It is known that there is a simple one-to-one relationship between the SIR and the MMSE error for each user (see eg. [8]):

$$\text{MMSE}_i = \frac{1}{1 + \text{SIR}_i}. \tag{12}$$

When there are no maximum power constraint, the background noise power can be made negligible by boosting up the power of all the users. In this case, (11) simplifies to:

$$\sum_{i=1}^{M} \text{MMSE}_i = M - \text{rank}\,(S) \tag{13}$$

Note that the total MMSE of the users is a constant, independent of the relative powers of the users and depending very weakly on their signature sequences. To minimize the maximum MMSE among all users (or equivalently, to maximize the minimum SIRs), it is therefore optimal to have symmetry among the users such that they have the same MMSE. This was achieved using equal received power and the WBE sequences described earlier. However, this "symmetrization" can also be achieved asymptotically when random sequences are used, since it is shown in [16] that the SIR's of all users will converge to the same number. But maximizing the minimum SIR's is equivalent to maximizing the number of users in a system with given equal SIR requirements. Hence, random sequences and the WBE sequences yield the same capacity asymptotically.

4 Matched Filter Receiver and Capacity

The matched filter receiver is a traditional choice among designers and is computationally very simple: just project the received signal onto the direction of transmission. In this section we shall analyze the capacity of the system equipped with such a receiver structure for every user. A priori, by choosing an inferior linear receiver (in the sense of maximizing the SIRs of the users) we could be reducing the capacity; however we obtain the unexpected result that the capacity is identical to the one obtained with the optimal linear receiver. This result is now stated below; we shall make some comments on this at the end of proof. We assume that each user has the same SIR requirement β.

Theorem 4.1 *M users are admissible in the system with processing gain L and equipped with matched filter receivers if and only if*

$$M < L\left(1+\frac{1}{\beta}\right).$$

Proof Step 1: Upper Bound on the number of users
Since for any given choice of signature sequences and powers the achieved SIRs of the users with matched filter receiver is less than that achieved with the MMSE receiver structure, an appeal to Theorem 3.1 suffices. However the following direct proof will aid us in developing notation and intuition in Section 6 when we consider capacity of the system with transmit power constraints.

Let M users be admissible in the system with processing gain L. Then there exist sequences $s_1, \ldots, s_M$ and powers $p_1, \ldots, p_M$ such that, analogous to (1),

$$SIR_i = \frac{p_i}{\sigma^2 + \sum_{j\neq i} (s_i, s_j)^2 p_j} \geq \beta \ \forall i = 1 \ldots M \tag{14}$$

The linear inequalities (14) can be rewritten in matrix notation as

$$\left(I - \frac{\beta}{1+\beta}A\right)\bar{p} \geq \frac{\beta\sigma^2}{1+\beta}e \tag{15}$$

where $\bar{p} = (p_1, \ldots p_M)^t$ and $e = (1, 1 \ldots, 1)^t$ and $A_{ij} = (s_i, s_j)^2$ Suppose that the sequences $s_1, \ldots, s_M$ are such that A is irreducible. Then, the existence of $\bar{p}$ is equivalent, by an appeal to Theorem 2.1 in [15], to

$$r(A) < 1 + \frac{1}{\beta} \tag{16}$$

As before, let $S = [s_1, s_2, \ldots, s_M]$. We may then write $A = S^t S \circ S^t S$, where $\circ$ represents the Hadamard product defined as $(A \circ B)_{ij} = A_{ij} B_{ij}$ (see Chapter 5 in [4]). Since A is a symmetric matrix,

$$\begin{aligned} r(A) &\geq \frac{e^t A e}{e^t e} \quad \text{; the classical Rayleigh quotient} \\ &= \frac{\operatorname{tr}[S^t S S^t S]}{M} \quad \text{; see Lemma 5.1.5 in [4]} \\ &= \frac{\operatorname{tr}\left[(S S^t)^2\right]}{M} \\ &\geq \frac{(\operatorname{tr}[S S^t])^2}{ML} \qquad (17) \\ &= \frac{M}{L} \text{ since } \operatorname{tr}\left[S S^t\right] = \operatorname{tr}\left[S^t S\right] = M \qquad (18) \end{aligned}$$

where, in (17), we used the fact (which can be shown easily), that for any symmetric matrix A, we have $\operatorname{tr}[A^2] \geq \frac{(\operatorname{tr}[A])^2}{\operatorname{rank}(A)}$. Using (16) and (18), we have $M < L\left(1 + \frac{1}{\beta}\right)$. If A is not irreducible, suppose it has C irreducible components each component having, say M_l elements. Let us denote lth irreducible component by $\mathcal{I}_l = \left\{i_1^l, \ldots, i_{M_l}^l\right\}$. Let $\forall l = 1 \ldots C$, the sequences in the lth irreducible component, namely, $s_{i_1^l}, \ldots, s_{i_{M_l}^l}$ span a subspace of dimension n_l. Then, necessarily, $\sum_{l=1}^{L} M_l = M$ and $\sum_{l=1}^{C} n_l \leq L$ and the sequences belonging to different irreducible components are orthogonal. In this case, the M linear inequalities in (14) split into C disjoint sets of linear inequalities each having M_l terms. By an argument identical to the irreducible case, we now have,

$$\frac{M_l}{n_l} < 1 + \frac{1}{\beta} \quad \forall l = 1 \ldots C \tag{19}$$

A simple induction argument shows that

$$\max_{l=1 \ldots C} \frac{M_l}{n_l} \geq \frac{\sum_{l=1}^{C} M_l}{\sum_{l=1}^{C} n_l} \geq \frac{M}{L}$$

Using (19), we then have $M < L\left(1 + \frac{1}{\beta}\right)$. ∎

Step 2: Achievability of the bound
Suppose $M < L\left(1 + \frac{1}{\beta}\right)$. Choose the signature sequences for the users as the WBE sequences for the pair (M, L). Choose the powers to be

$$p_i = p = \frac{\sigma^2 \beta L}{L(1 + \beta) - M\beta} \quad \forall i = 1 \ldots M \tag{20}$$

Then, by our particular choice of signature sequences, $SS^t = \frac{M}{L}I$ and thus $\sum_{j=1}^{M}(s_i, s_j)^2 = \frac{M}{L}$ for every user j. We now have for every user i,

$$\begin{aligned} SIR_i &= \frac{p_i}{\sigma^2 - p_i + \sum_{j=1}^{N} p_j (s_i, s_j)^2} \\ &= \frac{Lp}{L(\sigma^2 - p) + Mp} \\ &= \beta \end{aligned}$$

Hence M users are admissible in the system with processing gain L. ■

This unexpected result of the performance of the system with matched filter receivers can be explained by the intuition gained in the achievability parts of the proofs of Theorems 3.1 and 4.1. We observe that the achievability proofs in both the two theorems used the same signature sequences for the users, namely, WBE sequences for the pair (M, L). Furthermore, the powers chosen were the same for each user. In this situation when the signature sequences are the WBE sequences, and the powers chosen for the users are identical (equal to $p = \frac{\sigma^2 \beta L}{L(1+\beta) - M\beta}$ as in (7)), the MMSE receiver for user i is given by, following (2),

$$\begin{aligned} c_i &= Z_i^{-1} s_i \\ &= \left(\sigma^2 I + \sum_{j \neq i} p_j s_j s_j^t\right)^{-1} s_i \\ &= \left(\sigma^2 I - p s_i s_i^t + p S S^t\right)^{-1} s_i \\ &= \left(\sigma^2 I - p s_i s_i^t + \frac{pM}{L} I\right)^{-1} s_i \\ &= K s_i \end{aligned}$$

where K is a constant (which can be shown to be equal to $\frac{\beta}{p}$). Thus the optimal linear filter in this situation is just a scaled version of the matched filter and this explains the system with matched filter receivers having the same capacity as the one equipped with MMSE receivers even when there are more users than the processing gain (when the number of users are less than the processing gain, the trivial choice of orthogonal signature sequences for the users ensures identical capacities for the systems with MMSE and matched filter receivers).

Suppose the signature sequences chosen are $s_1, \ldots, s_M$ and denote the $[s_1 \ldots s_M]$, a $L \times M$ matrix by S. Then, among all the choices of power for the users such that the SIR requirements of the users are met, there is a component-wise minimal choice (see Theorem 2.1 in [15]) given by $\bar{p}_S = \frac{\sigma^2 \beta}{1+\beta}\left(I - \frac{\beta}{1+\beta} S^t S \circ S^t S\right)^{-1} e$ where e is a vector of all ones. When the choice of signature sequences are the WBE sequences for the pair (M, L) then, it is easy to verify that the component-wise minimal power choice is the one made above, namely, $p_i = p = \frac{\sigma^2 \beta L}{L(1+\beta) - M\beta}$.

As before, it is interesting to compare the analogous results obtained with random signature sequences. In [16] the authors show that in a large system, α users per unit processing gain have their SIR requirements met if and only if $\alpha < \frac{1}{\beta}$. Thus 1 user per degree of freedom is lost asymptotically when random sequences are used.

5 Multiple Classes and Capacity

Data and voice users may be sharing the common system and it is important to have a level of generality in the model that allows users to have different SIR requirements. To quantify this notion, we introduce different "classes" of users. A user of class l has a QoS requirement β_l and we assume that there are a finite number (fixed, in practice one can imagine about 2-3 classes), say N, of classes. In this section we focus on the capacity of a system with a given processing gain in both the cases when the system is equipped with matched filter and MMSE receivers. We shall first focus on the system equipped with MMSE receiver structure.

As in Section 2, we define admissibility of M users (with SIR requirements $\beta_1, \beta_2, \ldots, \beta_M$) in the system with processing gain L and equipped with MMSE receiver structure as being able to allot signature sequences and powers for the users such that for each user i the achieved SIR (as in (3)) is greater than or equal to β_i. We first derive a necessary condition on admissibility and then consider the achievability issue. As before, note that if the number of users is less than or equal to the processing gain, then the users can be given orthogonal signature sequences and trivially they are admissible in the system. Thus, with no loss of generality, we assume that the number of users is greater than the processing gain.

Theorem 5.1 *Suppose M users (with SIR requirements $\beta_1, \beta_2, \ldots, \beta_M$) are admissible in the system with processing gain L and equipped with MMSE receivers for each user. Then*

$$\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} < L.$$

Proof Since the M users are admissible, by definition there exist for each user signature sequences s_i and powers p_i such that SIR_i (as in (3)) is greater than or equal to β_i. Then, as in (4), we have

$$\begin{aligned} SIR_i &= \frac{s_i^t Z^{-1} s_i p_i}{1 - s_i^t Z^{-1} s_i p_i} \geq \beta_i \\ \frac{SIR_i}{1+SIR_i} &= s_i^t Z^{-1} s_i p_i && (21) \\ &\geq \frac{\beta_i}{1+\beta_i} && (22) \end{aligned}$$

Also, as in (6), we have the upper bound

$$\sum_{i=1}^{M} s_i^t Z^{-1} s_i p_i < L \qquad (23)$$

Now using (21), (22) and (23) we have $\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} < L$. ∎

This allows us to consider the quantity $\frac{\beta_l}{1+\beta_l}$ as the *effective bandwidth* of a user of class l. Thus if M users are admissible in a system then the above result shows that the sum of their effective bandwidths is less than the processing gain of the system. We know that L degrees of freedom can support $\lfloor\left(1+\frac{1}{\beta}\right) L\rfloor$ users each with SIR requirement β. This suggests that we could "channelize" the system such that users of different classes do not interfere with each other and achieve asymptotically achieve α_l users of class l per degree of freedom whenever $\sum_{l=1}^{N} \frac{\alpha_l\beta_l}{1+\beta_l} < 1$. This is indeed true and we make this precise in the following theorem. We assume the system is equipped with MMSE receiver structure and has large processing gain (large compared to the number of classes N).

Theorem 5.2 *Given $\alpha_1, \ldots, \alpha_N$ positive such that*

$$\sum_{l=1}^{N} \frac{\alpha_l \beta_l}{1+\beta_l} < 1, \tag{24}$$

we can admit at least $k_l = \lfloor \alpha_l L - 1 - \frac{1}{\beta_l} \rfloor$ users of class l in a system with processing gain L

Proof We shall proceed by "channelizing" the system by dividing the processing gain L into N "parts". For every class l, we shall give users of class l, $n_l = \lfloor \frac{L\alpha_l\beta_l}{1+\beta_l} \rfloor$ degrees of freedom (note that $\sum_{l=1}^{N} n_l \leq L$ from (24) and due to our conservative choice and hence such a division can be achieved). Now, in each "part" l, by an appeal to Theorem 3.1, we can admit $\lfloor n_l \left(1+\frac{1}{\beta_l}\right)\rfloor$ users of class l. Thus, we can admit at least

$$\begin{aligned} k_l &= \lfloor n_l \left(1+\frac{1}{\beta_l}\right) \rfloor \\ &\geq \lfloor \left(\frac{\alpha_l\beta_l L}{1+\beta_l} - 1\right)\left(1+\frac{1}{\beta_l}\right) \rfloor \\ &= \lfloor \alpha_l L - 1 - \frac{1}{\beta_l} \rfloor \end{aligned}$$

users of class l. This completes the proof. ∎

This result shows that by "channelizing" the system, we can admit α_l users per unit processing gain of class l asymptotically whenever $\sum_{l=1}^{N} \frac{\alpha_l\beta_l}{1+\beta_l} < 1$. This result tends to be weak for small processing gains, and our next result is true for any value of the processing gain. We saw in Theorem 3.1 that when the M users have the same SIR requirement β they are admissible in the system with processing gain L if $\frac{M\beta}{1+\beta} < L$. It is reasonable to expect that if the users have SIR requirements $\beta_1, \ldots, \beta_M$ that are not too different from each other, then the M users should be admissible if $\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} < L$. We make the notion "not too different" precise below. Again we assume that the system is equipped with MMSE receiver structure.

Theorem 5.3 *M users having SIR requirements $\beta_1, \ldots, \beta_M$ (without loss of generality let $\beta_1 \geq \cdots \geq \beta_M$) are admissible in a system with processing gain L if*

$$\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} < L$$

and

$$\frac{1}{L}\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} \geq \frac{1}{n}\sum_{i=1}^{n} \frac{\beta_i}{1+\beta_i} \quad \forall n = 1 \ldots L \tag{25}$$

We prove the theorem in Appendix C and also identify the powers and signature sequences to be alloted to the users so that their SIR requirements are met. This is a sufficient condition on the SIR requirements and the processing gain of the system so that the users are admissible. We identify two important situations when (25) is satisfied:

1. When all the SIR requirements are identical, then (25) is trivially satisfied. This is the result contained in Theorem 3.1.

2. When there are at least as many users in each class as the processing gain of the system then it is straightforward to see that (25) is satisfied. Suppose class l has $M_l \geq L$ users and $\sum_{l=1}^{N} \frac{M_l \beta_l}{1+\beta_l} < L$. Then, we can make a familiar choice of signature sequences for the users. Choose signature sequences for users of class l to be WBE sequences for the pair (M_l, L). Choose powers the same for every user i of class l to be $p_i^l = p^l = \frac{\beta_l}{1+\beta_l} \frac{L\sigma^2}{L - \sum_{j=1}^{N} \frac{M_j \beta_j}{1+\beta_j}}$. The SIR of user i of class l is, as in (4),

$$SIR_i^l = \frac{\left(s_i^l\right)^t Z^{-1} s_i^l p_i^l}{1 - \left(s_i^l\right)^t Z^{-1} s_i^l p_i^l} \quad \forall l = 1 \ldots N,\ \forall i = 1 \ldots M_l \tag{26}$$

where

$$Z = \sum_{l=1}^{N} \sum_{i=1}^{M_l} p_i^l s_i^l \left(s_i^l\right)^t + \sigma^2 I \tag{27}$$

Now by our choice of signature sequences, Z from (27) is $\left(\sigma^2 + \sum_{l=1}^{N} \frac{M_l p^l}{L}\right) I$. Hence, substituting in (26), for every user i of every class l,

$$\begin{aligned} \frac{SIR_i^l}{1+SIR_i^l} &= \left(s_i^l\right)^t Z^{-1} s_i^l p_i^l \\ &= \frac{p^l}{\sigma^2 + \sum_{l=1}^{N} \frac{M_l p^l}{L}} \\ &= \frac{\beta_l}{1+\beta_l} \end{aligned}$$

This ensures that for every class l, M_l users of that class are admissible in the system.

It is interesting to observe that the linearity of the boundary of the capacity is a consequence of (13), that the total minimum mean-square errors of the users is a constant independent of the received powers and dependent very weakly on the signature sequences. This also explains why here, as in the single class case, random sequences achieve asymptotically (as the processing gain gets large) the same performance as optimal sequences.

It is remarkable that we are able to make the same statements (as for MMSE receivers) for the a priori inferior matched filter receiver. As earlier, fix the receiver to be the matched filter. As in the situation of MMSE receiver structures, we shall first derive a necessary condition for the admissibility of the users and then discuss achievability of the capacity identified.

Theorem 5.4 *Suppose M users (with SIR requirements $\beta_1, \ldots, \beta_M$) are admissible in the system with processing gain L and equipped with matched filter receivers. Then,*

$$\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} < L.$$

Proof Since the achieved SIR with the matched filter receiver can only be less than that achieved with the MMSE receiver for the same signature sequences and powers of the users, an appeal to Theorem 5.1 suffices. ■

This result allows us to identify the quantity $\frac{\beta_l}{1+\beta_l}$ as the *effective bandwidth* of a user of class l just as in the situation when we had MMSE receiver structure. We shall first consider achievability in an asymptotic sense: for a large system, we show now, as in the MMSE receiver situation, α_l users per unit processing gain of class l are admissible in the system if $\sum_{l=1}^{L} \frac{\alpha_l \beta_l}{1+\beta_l} < 1$. We make a sharper statement that is exactly identical to that in Theorem 5.2. The assumption below is that the system is equipped with matched filter receiver structure and has a large processing gain.

Theorem 5.5 *Given $\alpha_1, \ldots, \alpha_N$ positive such that*

$$\sum_{l=1}^{N} \frac{\alpha_l \beta_l}{1+\beta_l} < 1,$$

we can admit at least $k_l = \lfloor \alpha_l L - 2 - \frac{1}{\beta_l} \rfloor$ users of class l in a system with processing gain L

We observe that the proof follows identically as the proof of Theorem 5.2 and by an appeal to Theorem 4.1 and hence omit the proof. Thus by "channelizing" a system with large processing gain L we can admit α_l users of class l per unit processing gain if $\sum_{l=1}^{L} \frac{\alpha_l \beta_l}{1+\beta_l} < 1$.

The statement above is weak for a system with a small processing gain and we make a statement analogous to Theorem 5.3 that deals with systems with any processing gain L.

Theorem 5.6 *M users having SIR requirements $\beta_1, \ldots, \beta_M$ (without loss of generality let $\beta_1 \geq \cdots \geq \beta_M$) are admissible in a system with processing gain L if*

$$\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} < L$$

and

$$\frac{n}{L}\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} \geq \sum_{i=1}^{n} \frac{\beta_i}{1+\beta_i} \quad \forall n = 1 \ldots L \tag{28}$$

We shall outline the proof in Appendix C where we also identify the powers and signature sequences to be used so that the users achieve their SIR requirements. However, we shall identify two important situations when (28) is easily satisfied:

1. When all the SIR requirements are identical, then (28) is trivially satisfied (see Appendix A for the details). This is the result contained in Theorem 4.1.
2. When there are at least as many users in each class as the processing gain of the system then it is straightforward to see that (25) is satisfied. Suppose class l has $M_l \geq L$ users and $\sum_{l=1}^{N} \frac{M_l \beta_l}{1+\beta_l} < L$. Then, we can make a familiar choice of signature sequences for the users. Choose signature sequences for users of class l to be WBE sequences for the pair (M_l, L). Choose powers the same for every user i of class l to be $p_i^l = p^l = \frac{\beta_l}{1+\beta_l} \frac{L\sigma^2}{L - \sum_{j=1}^{N} \frac{M_j \beta_j}{1+\beta_j}}$. By our choice of signature sequences, note that we have $\sum_{k=1}^{M_j} \left(s, s_k^j\right)^2 = \frac{M_j}{L}$ for every $s \in \mathcal{S}_1^L$. Then the achieved SIR of user i of class l is, as in (14),

$$\begin{aligned} SIR_i^l &= \frac{p_i^l}{\sigma^2 - p_i^l + \sum_{j=1}^{L} \sum_{k=1}^{M_j} \left(s_i^l, s_k^j\right)^2 p_k^j} \\ &= \frac{p^l}{\sigma^2 - p^l + \sum_{j=1}^{L} p^j \frac{M_j}{L}} \\ &= \beta_l \end{aligned}$$

which shows that for every class l, M_l users of class l are admissible in the system.

6 Power Constraint and Capacity

Our model has not included any constraints on the received power of the users. However, in a cellular system, there is a natural constraint on the transmit power of the mobile. In particular, one could consider two types of constraints: peak power constraint and average power constraint. In this section, we shall focus on the average power constraint on the transmit power of the mobiles. Before being able to define such an "average" constraint, we will need to have a model of mobility of the users in the system. We will adopt the following model: for each user i the attenuation process (denoted by $\{g_i(t)\}_{t\geq 0}$) is stationary and ergodic with the mean of the stationary distribution, being equal to, say, G (the same for every user). This is a fairly common model, for example see [17]. Now suppose M users each with SIR requirement β are admissible in the system with processing gain L. Suppose the signature sequences can be chosen to be $s_1, \ldots, s_M$ and denote the $L \times M$ matrix $[s_1 \ldots s_M]$ by S and the received powers chosen to be $\bar{p}_S$. Note that our assumption of perfect power control implies that we combat the fading by keeping the received powers fixed (to $\bar{p}_S$) at all times. The transmit power of user i at time t is $p_i^{\mathrm{tr}} = p_i^S g_i(t)$. Then, the average transmit power of user i is defined as

$$p_i^{\mathrm{avg}} \triangleq \lim_{t\to\infty} \frac{1}{t} \int_0^t p_i^{\mathrm{tr}}(\tau)\,\mathrm{d}\tau = p_i^S \lim_{t\to\infty} \frac{1}{t} \int_0^t g_i(\tau)\mathrm{d}\tau$$

The limit exists due to our assumption of stationarity of the process $\{g_i(t)\}_{t\geq 0}$ and hence, $p_i^{\mathrm{avg}} = G p_i^S$. Thus, the average transmit power constraint, say P, translates to the received power constraint of each user i as $p_i^S \leq \frac{P}{G}$.

If the transmit power constraint P (G is fixed) is small, then that might lead to a pathological situation when even a single user in the system cannot meet its SIR requirement without exceeding the transmit power constraint. Suppose there is just one user in the system. Then to meet a SIR requirement of β it should have a received power at least $\beta\sigma^2$. Thus, to avoid the situation when even a single user is not admissible in the system with any processing gain L, we shall henceforth assume that

$$\frac{P}{G} \geq \sigma^2 \beta \tag{29}$$

Now, we can define the admissibility of M users (each having SIR requirement β) in the system with processing gain L and average transmit power constraint P as equivalent to being able to allot for every user i signature sequence s_i and power $p_i \leq \frac{P}{G}$ such that the achieved SIR_i (given in (3) for MMSE receiver and in (14) for matched filter receiver) is greater than or equal to β. Our main result is to precisely identify the loss in capacity by including such a power constraint. As before, we assume that the number of users is greater than the processing gain. We state our main result below for the system with MMSE receiver structure:

Theorem 6.1 *M users (each having SIR requirement β) are admissible in the system with processing gain L and average transmit power constraint P if and only if*

$$M < L\left(1 + \frac{1}{\beta} - \frac{G\sigma^2}{P}\right).$$

Proof Step 1: Upper Bound for the number of users
Suppose M users are admissible. Then, there exist signature sequences $s_1, \ldots, s_M \in S_1^L$,

positive powers $p_1, \ldots, p_M \leq \frac{P}{G} \ni$ for every user i, we have $SIR_i \geq \beta$. Now, proceeding as in (4), we have for each user i,

$$SIR_i = \frac{s_i^t Z^{-1} s_i p_i}{1 - s_i^t Z^{-1} s_i p_i} \geq \beta \tag{30}$$

where $Z = \sigma^2 I + \sum_{j=1}^{M} p_j s_j s_j^t$. Summing up the terms we have,

$$\begin{aligned}
\sum_{i=1}^{M} s_i^t Z^{-1} s_i p_i &= \operatorname{tr}\left(S^t Z^{-1} S D\right) \\
&= \operatorname{tr}\left(S D S^t Z^{-1}\right) \\
&= \operatorname{tr}\left(\Lambda\left(\Lambda + \sigma^2 I\right)^{-1}\right) \quad \text{where } SDS^t = U\Lambda U^t \\
&= \sum_{i=1}^{L} \frac{\lambda_i}{\lambda_i + \sigma^2} \text{ where } \Lambda_{ii} = \lambda_i
\end{aligned}$$

We note that

$$\begin{aligned}
\operatorname{tr}(\Lambda) &= \operatorname{tr}\left(SDS^t\right) \\
&= \operatorname{tr}(D)
\end{aligned}$$

where the second equality follows by some algebra and noting that the columns of S have unit l_2 norm. Now, if we let $p^* = \frac{1}{L}\sum_{i=1}^{M} p_i$, then the vector $(p^*, p^*, \ldots, p^*)$ is majorized by the vector $(\lambda_1, \ldots, \lambda_L)$ (for the details see Appendix A). We now observe that the map $x \mapsto \frac{x}{x+\sigma^2}$ is concave in x and hence the symmetric concave map

$$(\lambda_1, \ldots, \lambda_L) \mapsto \sum_{i=1}^{L} \frac{\lambda_i}{\lambda_i + \sigma^2}$$

is Schur-concave (see Appendix A for the definition and notation). Then we have

$$\begin{aligned}
\sum_{i=1}^{L} \frac{\lambda_i}{\lambda_i + \sigma^2} &\leq \sum_{i=1}^{L} \frac{p^*}{p^* + \sigma^2} \\
&= \frac{Lp^*}{p^* + \sigma^2}
\end{aligned}$$

Using (30), we have

$$M\frac{\beta}{1+\beta} \leq \frac{Lp^*}{p^* + \sigma^2} \tag{31}$$

which implies that $p^* \geq \frac{M\beta\sigma^2}{L(1+\beta) - M\beta}$. Since each $p_i \leq \frac{P}{G}$ it follows that $p^* \leq \frac{MG}{LP}$. We then conclude that $M < L\left(1 + \frac{1}{\beta} - \frac{G\sigma^2}{P}\right)$. ■

Step 2: Achievability of the bound
Suppose $M < L\left(1+\frac{1}{\beta}-\frac{G\sigma^2}{P}\right)$. Choose the signature sequences to be WBE sequences for the pair (M, L). For each user i, choose the powers

$$p_i = p = \frac{L\beta\sigma^2}{L(1+\beta) - M\beta}$$

Since the choice is identical to the one made in the proof of Theorem 3.1, we have as in the proof of Theorem 3.1, that for each user i, the achieved SIR is equal to β. We only need to verify that our choice of powers does not exceed the constraint $\frac{P}{G}$. Using the prior $M < L\left(1+\frac{1}{\beta}-\frac{G\sigma^2}{P}\right)$ we have for each user power

$$p_i = \frac{L\beta\sigma^2}{L(1+\beta) - M\beta} \leq \frac{P}{G}$$

Thus the M users are admissible in the system. ∎

Suppose M users are admissible in the system with processing gain L. Then for any choice of signature sequences as the columns of a $L \times M$ matrix S and some user powers (denoted by say p^S) such that the SIR requirements of the users are met, from (31), we have that

$$\sum_{i=1}^{M} p_i^S \geq \frac{ML\beta\sigma^2}{L(1+\beta) - M\beta} \tag{32}$$

The choice made in the proof of Theorem 6.1, namely the signature sequences being the WBE sequences for the pair (M, L) and the powers all equal to $p = \frac{L\beta\sigma^2}{L(1+\beta)-M\beta}$, we have that the lower bound in (32) is met with equality. We conclude that:

1. The choice $p_i = p = \frac{L\beta\sigma^2}{L(1+\beta)-M\beta}$ is the component-wise minimal choice of powers when the sequences are chosen as WBE sequences for the pair (M, L).
2. The choice of signature sequences as the WBE sequences for the pair (M, L) and the corresponding minimal power choice gives the lowest possible (among all choices of signature sequences and powers for the users) sum of received powers. In this sense, the choice of signature sequences to be the WBE sequences is an "optimal" choice. The simple scheme of constructing WBE sequences for any $M \geq L$ outlined in Appendix B makes this choice of signature sequences more appealing.

Again, it is interesting to compare the results with the performance under random sequences. It is shown in [16] that with a power constraint P, the asymptotic capacity per degree of freedom achieved by the MMSE receiver is:

$$1 + \frac{1}{\beta} - (1+\beta)\frac{G\sigma^2}{P}$$

which is strictly less than that with optimal sequences, when the power constraint is finite. To understand why, we can appeal to (11):

$$\sum_{i=1}^{M} \mathrm{MMSE}_i = M - \sum_{i=1}^{L} \frac{\lambda_i}{\lambda_i + \sigma^2} \tag{33}$$

where λ_i's are the eigenvalues of the matrix SDS^t. When the background noise power is small compared to the received powers, any set of sequences which symmetrizes the MMSE for all users are optimal. When σ^2 is non-negligible, good sequences should also minimize the right hand side as well. The WBE sequences achieve that by making the eigenvalues least "spread out", i.e. all the same. Using random sequences, the eigenvalues are more spread out, resulting in a capacity penalty when σ^2 is non-negligible.

In Section 4 we had the unexpected result that the capacity of the system is unchanged even when we made a restriction of using an a priori inferior matched filter receiver. It is interesting to see if a power constraint changes this result. We show that the capacity of the system with power constraints is unchanged even when we make the restriction to using matched filter receiver.

Theorem 6.2 *M users (each having SIR requirement β) are admissible in the system with processing gain L and average transmit power constraint P and equipped with matched filter receivers if and only if*

$$M < L\left(1 + \frac{1}{\beta} - \frac{G\sigma^2}{P}\right).$$

Proof Step 1: Upper Bound on the number of users
Since for any given choice of signature sequences and powers satisfying the power constraint the achieved SIRs of the users with matched filter receiver is less than that achieved with the MMSE receivers, an appeal to Theorem 6.1 suffices. However the following direct proof will aid us in identifying "optimal" signature sequences. Let M users be admissible in the system with processing gain L equipped with matched filter receivers. Then there exist signature sequences $s_1, \ldots, s_M$ and powers $p_1, \ldots, p_M \leq \frac{P}{G}$ for the users such that (14) is met. As noted in Section 4, the component-wise power choice is given by

$$\bar{p} = \frac{\sigma^2\beta}{1+\beta}\left(I - \frac{\beta}{1+\beta}A\right)^{-1} e \tag{34}$$

where A and e are as defined in the proof of Theorem 4.1. Note that we have $1 > \frac{\beta}{1+\beta} r(A) \geq \frac{M\beta}{L(1+\beta)}$ (since, for a solution (34) to exist, it is necessary and sufficient that $\frac{\beta}{1+\beta} r(A) < 1$ and we have $r(A) \geq \frac{M}{L}$ by an appeal to (18)). Since for every user $\bar{p}_i \leq \frac{P}{G}$ we have $\frac{\sum_{i=1}^{M} \bar{p}_i}{M} \leq \frac{P}{G}$. Now, rewriting (34) (see Lemma B.1 in [15]) , we have

$$\bar{p} = \frac{\sigma^2\beta}{1+\beta} \sum_{m=0}^{\infty} \left(\frac{\beta}{1+\beta}\right)^m A^m e$$

Now,

$$\frac{\sum_{i=1}^{M}\bar{p}_i}{M} = \frac{e^t\bar{p}}{M} = \frac{\sigma^2\beta}{1+\beta}\sum_{m=0}^{\infty}\left(\frac{\beta}{1+\beta}\right)^m \frac{e^tA^me}{M} \tag{35}$$

We make the following conjecture:

$$\frac{e^tA^me}{M} \geq \left(\frac{M}{L}\right)^m \quad \forall m \geq 1 \tag{36}$$

Suppose (36) is true. Then, using (35),

$$\begin{aligned}
\frac{P}{G} &\geq \frac{e^t\bar{p}}{M} \\
&\geq \frac{\sigma^2\beta}{1+\beta}\sum_{m=0}^{\infty}\left(\frac{M\beta}{L(1+\beta)}\right)^m \\
&= \frac{L\sigma^2\beta}{L(1+\beta)-M\beta}
\end{aligned} \tag{37}$$

Thus, from (37), $M < L\left(1+\frac{1}{\beta}-\frac{G\sigma^2}{P}\right)$. We now only need to prove our conjecture in (36). We need the following lemma first:

Lemma 6.1 *For any positive semidefinite A (of dimension, say $M \times M$) and for any vector $x \in \mathcal{R}^M$ we have $\forall m \geq 1$,*

$$x^tA^mx \leq \left(x^tA^{m+1}x\right)^{\frac{1}{2}}\left(x^tA^{m-1}x\right)^{\frac{1}{2}}$$

Proof A can be written as $U^t\Lambda U$ for some unitary U and a non-negative diagonal matrix Λ. Let $\Lambda_{ii} = \lambda_i$ and $Ux = y$. Then,

$$\begin{aligned}
x^tA^mx &= (Ux)^t\Lambda^mUx \\
&= y^t\Lambda^my \\
&= \sum_{i=1}^{M} y_i^2\lambda_i^m \\
&= \sum_{i=1}^{M}\left(|\, y_i \,|\,\lambda_i^{\frac{m-1}{2}}\right)\left(|\, y_i \,|\,\lambda_i^{\frac{m+1}{2}}\right) \\
&\leq \left(\sum_{i=1}^{M} y_i^2\lambda_i^{m-1}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{M} y_i^2\lambda_i^{m+1}\right)^{\frac{1}{2}} \quad \text{using Cauchy-Schwarz inequality} \\
&= \left(y^t\Lambda^{m-1}y\right)^{\frac{1}{2}}\left(y^t\Lambda^{m+1}y\right)^{\frac{1}{2}} \\
&= \left(x^tA^{m-1}x\right)^{\frac{1}{2}}\left(x^tA^{m+1}x\right)^{\frac{1}{2}}
\end{aligned}$$

This completes the proof of Lemma 6.1. ■

From Lemma 6.1 it follows that for all $x \ni Ax \neq 0$, we have $\frac{x^t A^{m+1} x}{x^t A^m x} \geq \frac{x^t A^m x}{x^t A^{m-1} x}$ for all $m \geq 1$. We shall prove our conjecture (36) by induction using the above observation. We first observe that the matrix $A = S^t S \circ S^t S$ is positive semidefinite by appealing to the Schur product theorem (see Theorem 5.2.1 in [4]). For $m = 1$, (36) is true by the argument used in the proof of Theorem 4.1 (see (18)). Suppose it is true for all $2 \leq m < q$. Since each component of the vector Ae is at least 1,

$$\begin{aligned}
\frac{e^t A^q e}{M} &= \frac{e^t A^q e}{e^t e} \\
&\geq \frac{\left(e^t A^{q-1} e\right)^2}{\left(e^t A^{q-2} e\right)\left(e^t e\right)} \\
&= \left(\frac{e^t A^{q-1} e}{e^t e}\right)\left(\frac{e^t A^{q-1} e}{e^t A^{q-2} e}\right) \\
&\geq \left(\frac{e^t A^{q-1} e}{e^t e}\right)\left(\frac{e^t A^{q-2} e}{e^t A^{q-3} e}\right) \\
&\geq \left(\frac{e^t A^{q-1} e}{e^t e}\right)\left(\frac{e^t A e}{e^t e}\right)
\end{aligned}$$

Hence, $\frac{e^t A^q e}{M} \geq \left(\frac{M}{n}\right)^{q-1} \frac{M}{n}$ and (36) is true for $m = q$ also. This verifies our conjecture (36) and completes the proof of the upper bound. ■

Step 2: Achievability of the bound

Suppose $M < L\left(1 + \frac{1}{\beta} - \frac{G\sigma^2}{P}\right)$. Choose the signature sequences to be WBE sequences for the pair (M, L). For each user i, choose the powers

$$p_i = p = \frac{L\beta\sigma^2}{L(1+\beta) - M\beta}$$

Since the choice is identical to the one made in the proof of Theorem 4.1, we have as in the proof of Theorem 4.1, that for each user i, the achieved SIR is equal to β. We only need to verify that our choice of powers does not exceed the constraint $\frac{P}{G}$. Using the prior $M < L\left(1 + \frac{1}{\beta} - \frac{G\sigma^2}{P}\right)$ we have for each user power

$$p_i = \frac{L\beta\sigma^2}{L(1+\beta) - M\beta} \leq \frac{P}{G}$$

Thus the M users are admissible in the system. ■

Suppose M users are admissible in the system with processing gain L and equipped with matched filter receivers. Then for any choice of signature sequences as the columns of a $L \times M$ matrix S and some user powers (denoted by say p^S) such that the SIR requirements of the users are met, from (35) and (36), we have that

$$\sum_{i=1}^{M} p_i^S \geq \frac{ML\beta\sigma^2}{L(1+\beta) - M\beta} \tag{38}$$

The choice made in the proof of Theorem 6.2, namely the signature sequences being the WBE sequences for the pair (M, L) and the powers all equal to $p = \frac{L\beta\sigma^2}{L(1+\beta)-M\beta}$, we have that the lower bound in (38) is met with equality. This allows us to make the same conclusions as in the previous subsection: the choice of WBE sequences as the signature sequences is "optimal" in the sense of minimizing the received power of the users and their simple construction scheme developed in Appendix B makes their choice more attractive.

7 Downlink and Capacity

Until now, we have been considering the uplink of the cellular system. In the downlink of this system there is a single transmitter (the base station) and there are multiple receivers (the users). The path gains from the base station to the users distinguishes the users. We shall first formally define our model and then consider the capacity of the downlink.

7.1 Definitions and Model

Suppose there are M users in the downlink of the system. Let the path gain from the base station (interchangeably referred to as transmitter) to user i be h_i. We suppose that the noise at the receivers is additive white gaussian with the same variance σ^2 per degree of freedom for each user (there is no loss of generality in this assumption since we can incorporate this into the path gain parameter h_i). We say that M users (with path gains from the base station being $h_1, \ldots, h_M$ and each having the same SIR requirement of β) are admissible in the downlink of the system with processing gain L if we can allot transmit power p_i and signature sequence s_i at the transmitter corresponding to user i and the MMSE receiver at the user i such that its achieved SIR

$$SIR_i = \frac{p_i h_i (s_i, c_i)^2}{\sigma^2 (c_i, c_i) + \sum_{j \neq i} p_j h_i (s_j, c_i)^2} \geq \beta \tag{39}$$

where c_i is the MMSE receiver given the signature sequences and the powers. Proceeding as in Section 2.1, it is easy to verify that the optimal (in the sense of maximizing SIR for each user) linear receiver c_i for user i is

$$c_i = \tilde{Z}_i^{-1} s_i \tag{40}$$

where $\tilde{Z}_i = \frac{\sigma^2}{h_i} I + \sum_{j \neq i} p_j s_j s_j^t$ and the corresponding SIR_i with the optimal receiver is

$$SIR_i = s_i^t \tilde{Z}_i^{-1} s_i p_i \tag{41}$$

It is clear that we can make a similar definition of admissibility of the users when the receiver structure is fixed to be the matched filter. The similarity of the achieved SIR equation (39) to the corresponding one in the uplink in (1) is apparent. Only the noise variance σ^2 in (1) is replaced now by $\frac{\sigma^2}{h_i}$. Since the derivation of the main results of Sections 3 and 4, which were the identification of capacity of the system with MMSE and matched filter receivers respectively, did not explicitly utilize the fact that the noise variance term σ^2 in (1) was identical for all users i, we may hope to achieve identical characterizations of the capacity. This was essentially due to the fact that we had no constraints on the transmit power and by choosing high enough transmit powers one could null out the additive noise. Indeed, this turns out to be the case and we state the main results below.

7.2 MMSE receiver and Capacity

Our main result is the following. We assume that the system is equipped with MMSE receivers.

Theorem 7.1 *M users with path gains $h_1, \ldots, h_M$ from the base station and each having the same SIR requirement β are admissible in the downlink of the system with processing gain L if and only if*

$$M < L\left(1 + \frac{1}{\beta}\right).$$

Proof Step 1: Upper Bound for the number of users
Suppose the M users are admissible in the downlink. Then, for each user i, there exists signature sequence s_i and transmit power at the base station p_i (as a function of the path gains $h_1, \ldots, h_M$) such that the achieved SIR of user i (as in (39)) is greater than or equal to β. Using (41)

$$s_i^t \tilde{Z}_i^{-1} s_i p_i \geq \beta \tag{42}$$

Proceeding as in (4), we have, for each user i,

$$s_i^t \tilde{Z}_i^{-1} s_i p_i = \frac{s_i^t \hat{Z}_i^{-1} s_i p_i}{1 - s_i^t \hat{Z}_i^{-1} s_i p_i} \tag{43}$$

where $\hat{Z}_i = \tilde{Z}_i - p_i s_i s_i^t$. Recalling the notation developed in Section 2, $S = [s_1, \ldots, s_M]$ and $D = \text{diag}\{p_1, \ldots, p_M\}$ and $SDS^t = U \Lambda U$. Then we can rewrite $\hat{Z}_i = \frac{\sigma^2}{h_i} I + SDS^t$. We have, from (42), that for each user i,

$$s_i^t \hat{Z}_i^{-1} s_i p_i \geq \frac{\beta}{1+\beta}$$

Equivalently, we have for each user i,

$$(Us_i)^t \left(\frac{\sigma^2}{h_i} I + \Lambda\right)^{-1} (Us_i) p_i \geq \frac{\beta}{1+\beta}$$

Denoting $\hat{h} = \max_{i=1}^{M} h_i$, we have for each user i,

$$\frac{\beta}{1+\beta} \leq (Us_i)^t \left(\frac{\sigma^2}{\hat{h}} I + \Lambda\right)^{-1} (Us_i) p_i$$

$$= s_i^t \left(\frac{\sigma^2}{\hat{h}} I + SDS^t\right)^{-1} s_i p_i$$

Summing up the terms, we have

$$\begin{aligned}
\frac{M\beta}{1+\beta} &\leq \operatorname{tr}\left[S^t \left(\frac{\sigma^2}{\hat{h}} I + SDS^t\right)^{-1} SD\right] \\
&= \operatorname{tr}\left[SDS^t \left(\frac{\sigma^2}{\hat{h}} I + SDS^t\right)^{-1}\right] \\
&< L \quad \text{as in (6)}
\end{aligned}$$

This completes the proof. ■

Step 2: Achievability of the bound
Suppose $M < L\left(1 + \frac{1}{\beta}\right)$ and $h_1, \ldots, h_M$ be arbitrary positive real numbers. Choose the signature sequences for the users as WBE sequences for the pair (M, L) and powers

$$p_i = p = \frac{\sigma^2 \beta}{\check{h}\,(1 + \beta - k\beta)} \quad \forall i = 1 \ldots M \tag{44}$$

where $\check{h} = \min_{i=1}^{M} h_i$. Then, by our particular choice of signature sequences, $SS^t = \frac{M}{L} I$ and hence for every user i we have $\hat{Z}_i = \left(\frac{\sigma^2}{h_i} + \frac{Mp}{L}\right) I$. Using (43), $\forall i = 1 \ldots, M$,

$$\begin{aligned}
SIR_i &= \frac{p\left(\frac{\sigma^2}{h_i} + \frac{Mp}{L}\right)^{-1}}{1 - p\left(\frac{\sigma^2}{h_i} + \frac{Mp}{L}\right)^{-1}} \\
&= \frac{p}{\frac{\sigma^2}{h_i} + \frac{Mp}{L} - p} \qquad (45) \\
&= \frac{\beta}{\frac{\check{h}}{h_i}\left(1 + \beta - \frac{M\beta}{L}\right) + \frac{M\beta}{L} - \beta} \\
&\geq \beta \qquad (46)
\end{aligned}$$

where we used the fact that $\frac{\check{h}}{h_i}\left(1 + \beta - \frac{M\beta}{L}\right) + \frac{M\beta}{L} - \beta \leq 1$ since by hypothesis we have that $M < L\left(1 + \frac{1}{\beta}\right)$ and $\check{h} \leq h_i$. Hence the M users are admissible. ■

We have identified the WBE sequences as the choice of the signature sequences of the users. But the choice of user powers in (44) is not the corresponding component-wise minimal one. A simple closed form expression of the component-wise minimal power choice seems unattainable. But it is worth emphasizing that the choice of signature sequences is independent of the path gains and only the powers are chosen as a function of the path gain. If we compare the proof of Theorem 7.1 with that of Theorem 3.1, we notice that the upper bound part is identical while the achievability part used the same signature sequences (namely the WBE sequences) and only the choice of users powers was different. We shall appeal to this similarity in the proofs of the uplink and the downlink case and just state our capacity results for the other settings. However, to emphasize this, we shall detail the proof of the capacity of the downlink system with matched filter receivers.

Let the system be equipped with matched filter receiver structures at the base station. Then,

Theorem 7.2 *M users with path gains $h_1, \ldots, h_M$ from the base station and each having the same SIR requirement β are admissible in the downlink of the system with processing gain L if and only if*

$$M < L\left(1 + \frac{1}{\beta}\right).$$

Proof Step 1: Upper Bound for the number of users
Since for a given choice of signature sequences and powers to the users the achieved SIR of each user is less than or equal to that achieved with the MMSE receiver, an appeal to Theorem 7.1 suffices.

Step 2: Achievability
Let $M < L\left(1 + \frac{1}{\beta}\right)$ and $h_1, \ldots, h_M$ be arbitrary positive real numbers. Choose the signature sequences for the users as WBE sequences for the pair (M, L) and powers $p_i = p = \frac{\sigma^2 \beta}{h(1+\beta-k\beta)}$. Then, since $SS^t = \frac{M}{L}I$, we have for every user i,

$$\begin{aligned} SIR_i &= \frac{p_i}{\sigma^2 - p_i + \sum_{j=1}^{N} p_j \left(s_i, s_j\right)^2} \\ &= \frac{p}{\sigma^2 - p + \frac{Mp}{L}} \\ &\geq \beta \end{aligned}$$

due to an argument as in (46). Hence the M users are admissible in the downlink of the system. ■

We now state the following results without proof. As mentioned earlier, these results can be verified by arguments similar to those used in the corresponding uplink situation.

1. If M users with path gains from the base station $h_1, \ldots, h_M$ and having SIR requirements $\beta_1, \ldots, \beta_M$ (without loss of generality, let $\beta_1 \geq \ldots \geq \beta_M$) are admissible in the downlink of the system with processing gain L then $\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} < L$. This result is true for both the situations when the users are equipped with MMSE and matched filter receiver structures.

2. For every class l, α_l users per unit processing gain of that class are admissible in the downlink of a large enough system if $\sum_{l=1}^{N} \frac{\alpha_l \beta_l}{1+\beta_l} < 1$. This result is true in both the situations when the users are equipped with MMSE and matched filter receiver structures.

3. If

   - $L < \sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i}$ and
   - $\frac{n}{L} \sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} \geq \sum_{i=1}^{n} \frac{\beta_i}{1+\beta_i} \ \forall n = 1 \ldots L$

   then M users with path gains $h_1, \ldots, h_M$ from the base station are admissible in the downlink of the system with processing gain L. This result is true in both the situations when the users have MMSE and matched filter receiver structures.

8 Joint FDMA/CDMA Case and Capacity

Traditional multiple-access schemes divide the channel into slots and it is important to note that we can incorporate a slotted system into our framework. We achieve this by forcing the signature sequences to be chosen only from an orthogonal sequence set. Then users that have the same signature sequence are in the same "slot" or "channel" and do not cause any interference to users in different "channels" due to the orthogonality of the signature sequences. In this case the receiver is trivial and the MMSE and matched filter receivers coincide. It is interesting to identify the capacity in this situation and this exercise will enable us to explicitly identify the gain in capacity by using non-orthogonal signature sequences. In this section, we identify the capacity of the slotted system in the variety of settings. The conclusion we draw from the results is that the capacity in this case differs from the earlier one by an integer part of a function of the SIR requirement. The assumption below is that the signature sequences are now constrained to be chosen from an orthogonal sequence set (whose linear span has dimension equal to the processing gain of the system). We first focus on the uplink.

Theorem 8.1 *M users each with SIR requirement β are admissible in the system having processing gain L if and only if*

$$\begin{array}{ll} M < L\left(1 + \frac{1}{\beta}\right) & \textit{if } \lfloor \frac{1}{\beta} \rfloor = \frac{1}{\beta} \\ M \leq L \lfloor 1 + \frac{1}{\beta} \rfloor & \textit{else} \end{array}$$

Proof Since the sequences are chosen from an orthogonal set, only users having the same sequence (we shall refer to them as users in the same channel) cause interference to each other. We shall hence focus on the capacity for a single channel. $\tilde{M}$ users are admissible into a channel with SIR requirement β if there exist positive powers $p_1, \ldots, p_{\tilde{M}}$ such that, analogous to (1),

$$SIR_i = \frac{p_i}{\sigma^2 + \sum_{j \neq i} p_j} \geq \beta \;\; \forall i = 1 \ldots \tilde{M} \tag{47}$$

The existence of such powers can be seen to be equivalent to (see Theorem 2.1 in [15])

$$r(A) < 1 + \frac{1}{\beta} \tag{48}$$

where $r(\cdot)$ is the Perron-Frobenius eigenvalue of the argument (which is a non-negative irreducible matrix; for notation and definition see Chapter 1 in [15]) and A is a $\tilde{M} \times \tilde{M}$ matrix with all entries being equal to 1. Since $r(A) = \tilde{M}$, the existence of powers satisfying (47) is equivalent to the number of users $\tilde{M} < 1 + \frac{1}{\beta}$. Since we have L channels available, this is equivalent to the total number of users $M < L\left(1 + \frac{1}{\beta}\right)$ if $\lfloor \frac{1}{\beta} \rfloor = \frac{1}{\beta}$ and $M \leq L \lfloor 1 + \frac{1}{\beta} \rfloor$ else. ■

We shall focus on a single channel first.

Theorem 8.2 *For every class l, M_l users of that class are admissible in the downlink of a single channel if and only if*

$$\sum_{l=1}^{N} \frac{M_l \beta_l}{1 + \beta_l} < 1.$$

Proof M_l users of class l are admissible in the single channel iff there exist positive power for every user i of class l denoted by p_i^l such that (analogous to (47)),

$$SIR_i^l = \frac{p_i^l}{\sigma^2 - p_i^l + \sum_{j=1}^{N} \sum_{t=1}^{M_j} p_t^j} \geq \beta_l$$

These linear inequalities can be rewritten in matrix notation as

$$\left(I - D_{\bar{\beta}} A\right) \bar{p} \geq \sigma^2 D_{\bar{\beta}} e \tag{49}$$

where $\bar{p} = \left(p^1, \ldots, p^N\right)^t$ and for every class l, $p^l = \left(p_1^l, \ldots p_{M_l}^l\right)$ and A is a $\sum_{l=1}^{N} M_l \times \sum_{l=1}^{N} M_l$ matrix of all ones and e is a $\sum_{l=1}^{N} M_l \times 1$ vector of all ones and $D_{\bar{\beta}}$ is a diagonal matrix with first M_1 diagonal elements equal to $\frac{\beta_1}{1+\beta_1}$ and next M_2 diagonal elements equal to $\frac{\beta_2}{1+\beta_2}$ and the last M_N elements equal to $\frac{\beta_N}{1+\beta_N}$. An appeal to Theorem 2.1 in [15] shows that the existence of a positive vector $\bar{p}$ satisfying (49) is equivalent to the condition $r(D_{\bar{\beta}} A) < 1$. Now, it is straightforward to verify that $r(D_{\bar{\beta}} A) = \sum_{l=1}^{N} \frac{M_l \beta_l}{1+\beta_l}$. Hence the existence of positive powers satisfying (49) is equivalent to $\sum_{l=1}^{N} \frac{M_l \beta_l}{1+\beta_l} < 1$. Since there are L channels, for every class l, total of M_l users of that class are admissible in the system is equivalent to $\sum_{l=1}^{N} \frac{M_l \beta_l}{1+\beta_l} < L$. ■

The capacity of the slotted system with L slots now follows from the characterization of the capacity of a single channel. M_l users of class l are admissible if there is a way of dividing the users into the L channels such that within each channel the division does not exceed the capacity.

Theorem 8.3 *M users each with SIR requirement β are admissible in the system having processing gain L and power constraint P if and only if*

$$M \leq L\lfloor 1 + \frac{1}{\beta} - \frac{G\sigma^2}{P} \rfloor.$$

Let us consider the single channel first. M users are admissible in the single channel if there exist positive powers $p_1, \ldots, p_M$ each upper bounded by $\frac{P}{G}$ such that for each user i

$$SIR_i = \frac{p_i}{\sigma^2 + \sum_{j \neq i} p_j} \geq \beta$$

We can rewrite this in matrix notation as $\left(I - \frac{\beta ee^t}{1+\beta}\right) p \geq \frac{\beta\sigma^2}{1+\beta} e$ where e is a $M \times 1$ vectors of all ones. As in (48), the existence of such positive powers is seen to be equivalent to $r(ee^t) = M < 1 + \frac{1}{\beta}$. Furthermore, under this condition, there is a component wise minimal power solution, (see Theorem 2.1 in [15]) given by

$$\begin{aligned} \bar{p} &= \frac{\beta\sigma^2}{1+\beta}\left(I - \frac{\beta ee^t}{1+\beta}\right)^{-1} e \\ &= \frac{\beta\sigma^2}{1+\beta - M\beta} e \quad \text{after some elementary algebra} \end{aligned}$$

Thus, M users are admissible in the single channel with power constraint P if and only if $M < 1 + \frac{1}{\beta}$ and $\frac{\beta\sigma^2}{1+\beta-M\beta} \leq \frac{P}{G}$. This is equivalent to

$$M \leq 1 + \frac{1}{\beta} - \frac{G\sigma^2}{P}$$

Since there are L channels, we conclude that this is equivalent to the total number of users (in all the L channels) $M \leq L\lfloor 1 + \frac{1}{\beta} - \frac{G\sigma^2}{P} \rfloor$. ■

As earlier, we shall state the capacity for the downlink system in this situation without detailing the proofs. The proofs can be obtained by an argument similar to the ones made above.

1. M users (with path gains from the base station being $h_1, \ldots, h_M$) each with SIR requirement β are admissible in the downlink of the system having processing gain L if and only if

$$\begin{array}{ll} M < L\left(1 + \frac{1}{\beta}\right) & \text{if } \lfloor \frac{1}{\beta} \rfloor = \frac{1}{\beta} \\ M \leq L\lfloor 1 + \frac{1}{\beta} \rfloor & \text{else} \end{array}$$

2. M users (with path gains from the base station being $h_1, \ldots, h_M$) with SIR requirements $\beta_1, \ldots, \beta_M$ are admissible in a single channel in the downlink if and only if $\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} < 1$.

9 Conclusions

We have characterized the capacity in a S-CDMA system using linear receiver structures for both the uplink and the downlink. The effect on capacity by limitations such as choice of receiver structure (restriction to matched filter) and transmit power has also been characterized. Capacity when there are multiple classes of users has been discussed. We also identified the signature sequences and the appropriate received powers to choose so that every point in the capacity region is attained and provided a simple explicit means of constructing the optimal signature sequences. In a practical system one has asynchronous reception of the users' symbols by the base station and multi-paths are present. Furthermore, the assumption of perfect power control made in this study is also not valid in practical systems. However, these results give some insights to the best one can achieve. We are currently studying the capacity of systems with asynchronous reception, multi-paths and fading.

It also must be emphasized that these results are for the case of a single base station. In a cellular system with many base stations, the characterization of the capacity region (now a mobile will have to be distinguished by its path gains to the different base stations) is an important problem. In particular questions such as how does the multi-base station receiver capacity compare with the single receiver capacity will be answered by such a characterization. Our preliminary results in this direction show that the answer depends on the path gains of the mobiles to the base stations and we discover an important property of these systems - the need to do a kind of "channel-sharing". We have an explicit characterization of the capacity region in the multi-receiver setup when the sequences are chosen from an orthogonal sequence set and we conjecture that the capacity regions with no restriction on the sequences (for both matched filter and MMSE receivers) differ from the earlier one only by integer parts. This conjecture is based on the strength of the results obtained in the single receiver case. We are currently working on verifying this conjecture.

A Definitions and Relevant Results from Theory of Majorization

In this appendix we collect together relevant definitions and results from the theory of majorization. All of these results can be found in the comprehensive reference on majorization [9]. Majorization makes precise the vague notion that the components of a vector x are "less spread out" or "more nearly equal" than are the components of a vector y by the statement x is majorized by y.

For any $x = (x_1, \dots, x_n) \in \mathcal{R}^n$, let

$$x_{[1]} \geq \cdots \geq x_{[n]}$$

denote the components of x in decreasing order. For $x, y \in \mathcal{R}^n$, define

$$x \prec y \quad \text{if} \quad \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, & k = 1 \dots n-1 \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \end{cases}$$

When $x \prec y$ say that x is majorized by y (or y majorizes x). An important though trivial example of majorization is

$$\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right) \prec (a_1, \dots, a_n) \tag{50}$$

for every $a \in \mathcal{R}^n$ such that $\sum_{i=1}^{n} a_i = 1$. An important characterization of majorization is the result that $x \prec y$ if and only if there exists a doubly stochastic matrix P such that $x = yP$.

A real valued function $\phi : \mathcal{R}^n \to \mathcal{R}$ is said to be Schur-concave if for all $x, y \in \mathcal{R}^n$ such that $x \prec y$ we have $\phi(x) \geq \phi(y)$. ϕ is said to be Schur-convex if $-\phi$ is Schur-concave. Using the observation in (50), for any Schur-concave function ϕ and for any vector $x \in \mathcal{R}^n$

$$\phi(\bar{x}) \geq \phi(x)$$

where $\bar{x} = \left(\frac{\sum_{i=1}^{n} x_i}{n}, \frac{\sum_{i=1}^{n} x_i}{n}, \dots, \frac{\sum_{i=1}^{n} x_i}{n}\right)$. A well known structure of Schur-convex functions is the following result (Theorem 3.C.1 in [9]): If $g : \mathcal{R} \to \mathcal{R}$ is convex then the symmetric convex function $\phi(x) = \sum_{i=1}^{n} g(x_i)$ is Schur-convex. It is obvious that if $g : \mathcal{R} \to \mathcal{R}$ is concave then the symmetric concave function $\phi(x) = \sum_{i=1}^{n} g(x_i)$ is Schur-concave.

It is well known that the sum of diagonal elements of a matrix is equal to the sum of its eigenvalues. When the matrix is symmetric the *precise* relationship between the diagonal elements and the eigenvalues is that of majorization: Let H be a $n \times n$ symmetric matrix with diagonal elements $h_1, \dots, h_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$. Then $h \prec \lambda$ (Theorem 9.B.1, [9]). That h and λ cannot be compared by an ordering stronger than majorization is the consequence of the following converse (Theorem 9.B.2, [9]):

Theorem A.1 *If $h_1 \geq \cdots \geq h_n$ and $\lambda_1 \geq \cdots \lambda_n$ are $2n$ numbers satisfying $h \prec \lambda$ in $\mathcal{R}^n$, then there exists a real symmetric matrix H with diagonal elements $h_1, \dots, h_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$.*

The proof of Theorem A.1 suggests a way to construct such a symmetric matrix H and this is of importance in constructing the WBE sequences. Hence, we outline the proof below. We need the following lemma first (Lemma 9.B.3 in [9]).

Lemma A.1 *Given real numbers $c_1, \dots, c_{n-1}$ and $\lambda_1, \dots, \lambda_n$ satisfying the interlacing property*

$$\lambda_1 \geq c_1 \geq \lambda_2 \geq c_2 \geq \cdots \geq c_{n-1} \geq \lambda_n,$$

there exists a real symmetric $n \times n$ matrix of the form

$$W = \begin{bmatrix} D_c & v^t \\ v & v_n \end{bmatrix}$$

with eigenvalues $\lambda_1, \ldots, \lambda_n$ where D_c is a diagonal matrix with diagonal elements $c_1, \ldots, c_{n-1}$.

Before we proceed to outline the proof of Theorem A.1, we need the following technical lemma (Theorem 5.A.10 in [9]):

Lemma A.2 *If $x \prec y$ then there exist $c_1, \ldots, c_{n-1}$ such that $y_{[1]} \geq c_1 \geq y_{[2]} \geq \cdots \geq c_{n-1} \geq y_{[n]}$ and*

$$\left(x_{[1]}, x_{[2]}, \ldots, x_{[n-1]}\right) \prec (c_1, c_2, \ldots, c_{n-1})$$

We are now ready to outline the proof of Theorem A.1.

Proof of Theorem A.1: The proof is by induction. First observe that the result clearly holds for $n = 1$. From $h \prec \lambda$ and Lemma A.2, there exist numbers $c_1, \ldots, c_{n-1}$ such that

$$\begin{gathered} \lambda_1 \geq c_1 \geq \cdots \geq c_{n-1} \geq \lambda_n \quad \text{and} \\ (h_1, \ldots, h_{n-1}) \prec (c_1, \ldots, c_{n-1}) \end{gathered}$$

By the inductive hypothesis there exists a $n-1 \times n-1$ real symmetric matrix H_1 with diagonal elements $h_1, \ldots, h_{n-1}$ and eigenvalues $c_1, \ldots, c_{n-1}$. Let Q be an orthogonal matrix such that

$$Q^t H_1 Q = D_c = \operatorname{diag}(c_1, \ldots, c_{n-1})$$

By Lemma A.1, there exists an $n \times n$ symmetric matrix

$$W = \begin{bmatrix} D_c & v^t \\ v & v_n \end{bmatrix}$$

with eigenvalues $\lambda_1, \ldots, \lambda_n$. Now form the matrix

$$\begin{aligned} H = \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D_c & v^t \\ v & v_n \end{bmatrix} \begin{bmatrix} Q^t & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} QD_cQ^t & Qv^t \\ vQ^t & v_n \end{bmatrix} \\ &= \begin{bmatrix} H_1 & Qv^t \\ vQ^t & v_n \end{bmatrix} \end{aligned}$$

Then H has characteristic roots $\lambda_1, \ldots, \lambda_n$ and diagonal elements $h_1, \ldots, h_{n-1}, v_n$. But $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n-1} h_i + v_n$ and by hypothesis $h \prec \lambda$. Hence $v_n = h_n$ which completes the proof. ∎

This provides a constructive method of arriving at the matrix H given its diagonal elements and its eigenvalues. This general scheme, though constructive, is non-algorithmic in nature.

B Existence and Construction of WBE Sequences

In this section we shall prove Lemma 3.1, thereby showing the existence of WBE sequences. As a byproduct of the proof, we shall obtain a means of constructing these sequences. As a reprise, we repeat here the definition of WBE sequences. Fix $M \geq L$ henceforth. Let $s_1, \ldots, s_M \in \mathcal{S}_1^L$, the unit sphere in $\mathcal{R}^L$. Let $S_{(M,L)} = [s_1, s_2, \ldots, s_M]$. Then, $s_1, \ldots, s_M$ are WBE sequences for the pair (M, L) and $S_{(M,L)}$ the WBE matrix for the pair (M, L), if the following three conditions are satisfied:

1. The rows of $S_{(M,L)}$ have the same l_2 norm, equal to $\sqrt{\frac{M}{L}}$.
2. The columns of $S_{(M,L)}$ have unit l_2 norm.
3. The rows of $S_{(M,L)}$ are orthogonal to each other.

Properties 1 and 3 can also be succinctly expressed as $SS^t = \frac{M}{L}I$. When $M = L$, orthonormal matrices are the only matrices satisfying the above 3 properties. Hence, without loss of generality, we can take $M > L$. We shall now define some matrices that are central to the proof of Lemma 3.1 and the construction of WBE sequences.

Let $\lambda_{(M,L)}$ be a $M \times 1$ vector with any L entries equal to $\frac{M}{L}$ and the other entries being zero. Let e_M be the $M \times 1$ vector of all entries being unity. Then, $\lambda_{(M,L)}$ majorizes e_M. For notation, see Appendix A. Then, appealing to Theorem B.2, Chapter 9 in [9],(also see Appendix A for a restatement of this classical theorem) there exists a real symmetric matrix, say $P_{(M,L)}$ with unit diagonal entries and eigenvalues $\frac{M}{L}$ and 0 with multiplicities (both geometric and algebraic) L and $M - L$ respectively . Let us denote the set of such matrices by $\mathcal{P}_{(M,L)}$ and $P_{(M,L)}$ as an element of this set. Also, let $v_1^{(M,L)}, \ldots, v_L^{(M,L)}$ be the normalized eigenvectors of $P_{(M,L)}$ corresponding to the eigenvalue $\frac{M}{L}$(written as elements of $\mathcal{R}^{1\times M}$. Let $u_1^{(M,L)}, \ldots u_{M-L}^{(M,L)}$ be the eigenvectors of $P_{(M,L)}$ corresponding to the eigenvalue 0, written as elements of $\mathcal{R}^{1\times M}$. Let

$$V_{(M,L)} = \begin{bmatrix} v_1^{(M,L)} \\ v_2^{(M,L)} \\ \vdots \\ v_L^{(M,L)} \end{bmatrix}, \quad U_{(M,L)} = \begin{bmatrix} u_1^{(M,L)} \\ u_2^{(M,L)} \\ \vdots \\ u_{M-L}^{(M,L)} \end{bmatrix} \text{ and } Q_{(M,L)} = \begin{bmatrix} V_{(M,L)} \\ U_{(M,L)} \end{bmatrix}$$

Then $Q_{(M,L)}$ is an orthonormal matrix.

B.1 Existence of WBE Sequences

<u>*Proof of Lemma 3.1*</u>: We claim that the choice $\sqrt{\frac{M}{L}}V_{(M,L)}$ for the WBE matrix satisfies all the three properties. Properties (1) and (3) of WBE sequences are satisfied by definition.

Also,

$$\begin{aligned}\frac{M}{L}V^t_{(M,L)}V_{(M,L)} &= Q^t_{(M,L)}\begin{bmatrix}\frac{M}{L}I & 0\\ 0 & 0\end{bmatrix}Q_{(M,L)}\\ &= P_{(M,L)}\end{aligned}$$

Since the square of the l_2 norm of the columns of $\sqrt{\frac{M}{L}}V_{(M,L)}$ i s equal to the diagonal elements of $P_{(M,L)}$ which by definition are equal to unity, this verifies property (2) also. If $S_{(M,L)}$ is any WBE matrix for (M,L) then, we observe that rows of $V_{(M,L)} \triangleq \sqrt{\frac{L}{M}}S_{(M,L)}$ serves as the eigenvectors corresponding to the eigenvalue $\frac{M}{L}$ of some matrix $P_{(M,L)} \in \mathcal{P}_{(M,L)}$. Thus, the claim that every WBE matrix $S_{(M,L)}$ has to be of this form for some choice of $P_{(M,L)} \in \mathcal{P}_{(M,L)}$ is verified. ■

B.2 Construction of WBE Sequences

In Appendix A we indicated a recursive method of construction of the matrix $P_{(M,L)}$ and $V_{(M,L)}$ can then be extracted from this. However, this is computationally cumbersome and not algorithmic and it would be helpful to have a more explicit algorithmic computational procedure. Below, we obtain an explicit form of $V_{(M,L)}$ which is an instance of all possible $V_{(M,L)}$. Our construction is also recursive and we shall consider two cases $M < 2L$ and $M \geq 2L$.

Theorem B.1 *Let $M < 2L$. Given any $P_{(L,M-L)} \in \mathcal{P}_{(L,M-L)}$ (and hence given $U_{(L,M-L)}$ and $V_{(L,M-L)}$ the corresponding matrices constructed from the eigenvectors of $P_{(L,M-L)}$ as above), define*

$$\begin{aligned}V_{(M,L)} &= \begin{bmatrix}U_{(L,M-L)} & 0\\ \sqrt{\frac{M-L}{M}}V_{(L,M-L)} & \sqrt{\frac{L}{M}}I\end{bmatrix} \quad \textit{and}\\ U_{(M,L)} &= \begin{bmatrix}\sqrt{\frac{L}{M}}V_{(L,M-L)} & -\sqrt{\frac{M-L}{M}}I\end{bmatrix}\end{aligned}$$

Then, $\sqrt{\frac{M}{L}}V_{(M,L)}$ is a WBE matrix for the pair (M,L).

Proof It is easy to see that $Q^t_{(M,L)} \triangleq \left[V^t_{(M,L)}U^t_{(M,L)}\right]$ is an orthonormal matrix. We only need to verify that $\frac{M}{L}V^t_{(M,L)}V_{(M,L)}$ has unit diagonal entries. By hypothesis,

$$\frac{M}{L}V^t_{(M,L)}V_{(M,L)} = \begin{bmatrix}\frac{M}{L}U^t_{(L,M-L)}U_{(L,M-L)} + \frac{M-L}{L}V^t_{(L,M-L)}V_{(L,M-L)} & \sqrt{\frac{M-L}{L}}V^t_{(L,M-L)}\\ \sqrt{\frac{M-L}{L}}V_{(L,M-L)} & I\end{bmatrix} \quad (51)$$

Now, by definition,

$$\left[V^t_{(L,M-L)}\ U^t_{(L,M-L)}\right]\begin{bmatrix}\frac{L}{M-L}I & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_{(L,M-L)}\\ U_{(L,M-L)}\end{bmatrix} \text{ has unit diagonal entries}$$

Equivalently,

$$\left[V^t_{(L,M-L)}\ U^t_{(L,M-L)}\right]\begin{bmatrix}-I & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_{(L,M-L)}\\ U_{(L,M-L)}\end{bmatrix} \text{ has diagonal entries all } \left(1-\frac{M}{L}\right)$$

Equivalently,

$$\left[V^t_{(L,M-L)}\ U^t_{(L,M-L)}\right]\begin{bmatrix}\left(\frac{M}{L}-1\right)I & 0\\ 0 & \frac{M}{L}I\end{bmatrix}\begin{bmatrix}V_{(L,M-L)}\\ U_{(L,M-L)}\end{bmatrix} \text{ has unit diagonal entries}$$

Using this in (51), we obtain that $\frac{M}{L}V^t_{(M,L)}V_{(M,L)}$ has unit diagonal entries, verifying property 2 also. This completes the proof. ∎

Thus, when $M < 2L$, Theorem B.1 describes a way to construct $U_{(M,L)}$ and $V_{(M,L)}$ given a choice of the matrices $U_{(L,M-L)}$ and $V_{(L,M-L)}$. For the case when $M \geq 2L$, the following theorem gives a way to construct $U_{(M,L)}$ and $V_{(M,L)}$ given a choice of the matrices $U_{(M-L,L)}$ and $V_{(M-L,L)}$. The proof is similar to the previous one and is omitted.

Theorem B.2 *Let $M \geq 2L$. Given a choice of $U_{(M-L,L)}$ and $V_{(M-L,L)}$ define*

$$\begin{aligned} U_{(M,L)} &= \begin{bmatrix}\sqrt{\frac{L}{M}}V_{(M-L,L)} & -\sqrt{\frac{M-L}{M}}I\\ U_{(M-L,L)} & 0\end{bmatrix} \quad \textit{and}\\ V_{(M,L)} &= \begin{bmatrix}\sqrt{\frac{M-L}{M}}V_{(M-L,L)} & \sqrt{\frac{L}{M}}I\end{bmatrix}\end{aligned}$$

Then $\sqrt{\frac{M}{L}}V_{(M,L)}$ is a WBE matrix for the pair (M, L).

B.3 Observations and Examples

The following observations are straightforward to verify:

1. If $S_{(M,L)}$ is a WBE matrix for (M, L), then so is $QS_{(M,L)}$ for any $L \times L$ orthonormal matrix Q. Hence, rotating all the WBE vectors together does not change the three properties above. It is tempting to conjecture that the WBE matrix $S_{(M,L)}$ is unique under such rotations; however, later in this section we will see that this is not true in general.

2. If $M_1 \geq L$ and $M_2 \geq n$, given WBE matrices $S_{(M_1,L)}$ and $S_{(M_2,L)}$ for the pairs (M_1, L) and (M_2, L) respectively, the matrix $\left[S_{(M_1,L)} \;\; S_{(M_2,L)}\right]$ is a WBE matrix for the pair $(M_1 + M_2, L)$. This shows that we need restrict our attention only to $M < 2L$. When the number of users is more than $2L$, then the signature sequences can be "reused".

3. If $s_1, s_2, \ldots, s_M$ are WBE sequences for (M, L) then so are $\epsilon_1 s_1, \epsilon_2 s_2, \ldots, \epsilon_M s_M$ where $\forall i = 1 \ldots M$, $\epsilon_i \in \{1, -1\}$. Thus the WBE sequences are sign independent.

We now consider the base cases and simple examples of constructing WBE sequences using our general results above. It must be emphasized that these are some particular choices of the WBE sequences.

Example B.1 *$L = 1$ and M arbitrary*

Here, $P_{(M,1)}$ is a matrix of all ones. Thus,

$$V_{(M,1)} = \left[\sqrt{\frac{1}{M}}, \sqrt{\frac{1}{M}}, \ldots, \sqrt{\frac{1}{M}}\right]$$

and rows of $U_{(M,1)}$ form a basis for the $M-1$ dimensional subspace $\{k\,(1,1,\ldots,1) : k \in \mathcal{R}\}^{\perp}$. The WBE matrix is trivial, given by $(1, 1, \ldots, 1)$.

The case $M = L + 1$ closely follows from the results for $L = 1$, by an appeal to Theorem B.1.

Example B.2 *$L = 2$ and M arbitrary*

First, consider $M = 3$. Then,

$$U_{(3,2)} = \left[\sqrt{\frac{2}{3}} V_{(2,1)}, -\sqrt{\frac{1}{3}}\right] = \left(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}\right)$$

$$V_{(3,2)} = \begin{bmatrix} \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{2}{3}} \end{bmatrix}$$

For general $M \geq 2$, it is easy to verify that the WBE vectors are $s_i^t = \left(\sin\left(\frac{2i\pi}{M}\right), \cos\left(\frac{2i\pi}{M}\right)\right)$ for all $i = 1 \ldots M$.

Example B.3 $L = 3$, $M = 5$

In this case,

$$V_{(5,3)} = \begin{bmatrix} U_{(3,2)} & 0 \\ \sqrt{\frac{2}{5}}V_{(3,2)} & \sqrt{\frac{3}{5}}I \end{bmatrix}$$

Thus,

$$S_{(5,3)} = S_1 = \sqrt{\frac{5}{3}}V_{(5,3)} = \begin{bmatrix} \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} & 0 & 0 \\ \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 & 1 \end{bmatrix}$$

Example B.4 $L = 3$, $M = 5$ *example revisited.*

In this example, we shall attempt to follow the recursive procedure indicated in Appendix A and arrive at $V_{(5,3)}$.

Step 1: Find a real 4×4 symmetric matrix, say P_1, with unit diagonal entries and eigenvalues $\frac{2}{3}, \frac{5}{3}, 0$ with multiplicities (both algebraic and geometric) 1,2,1 respectively. Let Q_1 be an orthonormal matrix such that $Q_1 \operatorname{diag}\left\{\frac{5}{3}, \frac{5}{3}, \frac{2}{3}, 0\right\} Q_1^t = P_1$.
Step 2: Notice that

$$P_{(5,3)} = \begin{bmatrix} Q_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_1^t & 0 \\ 0 & 1 \end{bmatrix}$$

¿From this, $V_{(5,3)}$ can be found and hence $S_{(5,3)}$.

To complete Step 1, we need to go through another recursive step.
SubStep 1: Find a real, 3×3 symmetric matrix, say P_2, with unit diagonal entries and eigenvalues $\frac{5}{3}, \frac{7}{6}, \frac{1}{6}$. Let Q_2 be an orthonormal matrix such that $Q_2 \operatorname{diag}\left\{\frac{5}{3}, \frac{7}{6}, \frac{1}{6}\right\} Q_2^t = P_2$.
SubStep 2: Notice that

$$P_1 = \begin{bmatrix} Q_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{3} & 0 & 0 & 0 \\ 0 & \frac{7}{6} & 0 & \sqrt{\frac{1}{8}} \\ 0 & 0 & \frac{1}{6} & \sqrt{\frac{7}{24}} \\ 0 & \sqrt{\frac{1}{8}} & \sqrt{\frac{7}{24}} & 1 \end{bmatrix} \begin{bmatrix} Q_2^T & 0 \\ 0 & 1 \end{bmatrix}$$

To complete SubStep 1, we will need to go one further recursive step. We shall skip this step and just note that P_2 can be chosen to be:

$$P_2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{a^*+b^*}{2} \\ \frac{1}{2} & 1 & \frac{a^*-b^*}{2} \\ \frac{a^*+b^*}{2} & \frac{a^*-b^*}{2} & 1 \end{bmatrix} \text{ where } a^* = \sqrt{\frac{2}{27}}, \text{ and } b^* = \sqrt{\frac{7}{27}}$$

Using this in SubStep 2, we have an expression for P_1. We can now use this expression in Step 2, to obtain $P_{(5,3)}$ and hence $V_{(5,3)}$. Then,

$$S_{(5,3)} = \sqrt{\frac{5}{3}} V_{(5,3)} = \begin{bmatrix} 0.7408 & 0.1693 & 0.9426 & -0.4480 & 0 \\ -0.6339 & -0.7951 & 0.2888 & -0.7142 & 0 \\ -0.2222 & 0.5824 & -0.1676 & -0.5 & 1 \end{bmatrix}$$

Let us denote this matrix by S_2 and it is easy to verify that $S_2 S_2^t = \frac{5}{3} I$ and the l_2 norms of the columns of S_2 are unity. Thus S_2 is indeed a WBE matrix for $(5,3)$. Now, suppose there exists an orthonormal matrix Q such that $S_1 = QS_2$ where S_1 was obtained in the previous example. Then, $S_1^t S_1$ would have to equal $S_2^t S_2$. It is easily verified that this is not the case; in particular the (4,5) element of $S_1^t S_1$ is zero while the corresponding element of $S_2^t S_2$ is -0.5 . This shows that the WBE matrix $S_{(M,L)}$ is, in general, *not* unique up to orthonormal transformations of all sequences.

C Proofs of Theorems 5.3 and 5.6

Proof of Theorem 5.3:
Let $x \in \mathcal{R}^M$ have L entries equal to $\frac{1}{L}\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i}$ and the other $M - L$ entries equal to 0. Since we are given that $\beta_1 \geq \cdots \geq \beta_M$ and that

$$\frac{n}{L}\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i} \geq \sum_{i=1}^{n} \frac{\beta_i}{1+\beta_i} \quad \forall n = 1 \ldots L$$

we have $x \succ \left(\frac{\beta_1}{1+\beta_1}, \ldots \frac{\beta_M}{1+\beta_M}\right)$ by an appeal to the definition of majorization in Appendix A. Now, appealing to Theorem A.1, there exists a symmetric matrix, say P, with diagonal entries $\frac{\beta_1}{1+\beta_1}, \ldots, \frac{\beta_M}{1+\beta_M}$ and eigenvalues $\frac{1}{L}\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i}$ and 0 with multiplicities (both algebraic and geometric) equal to L and $M - L$ respectively. Let $v_1, \ldots, v_L \in \mathcal{R}^{1\times M}$ be orthonormal eigenvectors of P corresponding to the eigenvalue $\frac{1}{L}\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i}$. Following the notation developed in Section 2, let $D = \operatorname{diag}\{p_1, \ldots, p_M\}$ and $S = [s_1, \ldots, s_M]$. Now choose powers for the users $p_i = c\frac{\beta_i}{1+\beta_i}$ where c is a constant equal to $\frac{L\sigma^2}{L - \sum_{j=1}^{M} \frac{\beta_j}{1+\beta_j}}$. Choose the signature sequences for the users as

$$S = \sqrt{\frac{c\sum_{j=1}^{M} \frac{\beta_j}{1+\beta_j}}{L}} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_L \end{bmatrix} D^{-0.5}$$

Then, note that $SDS^t = \frac{c\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i}}{L} I$. Hence,

$$Z = SDS^t + \sigma^2 I = \left(\sigma^2 + \frac{c\sum_{j=1}^{M} \frac{\beta_j}{1+\beta_j}}{L}\right) I$$

Substituting in (4),

$$
\begin{aligned}
SIR_i &= \frac{s_i^t Z^{-1} s_i p_i}{1 - s_i^t Z^{-1} s_i p_i} \\
&= \frac{L p_i}{L\sigma^2 + c\sum_{j=1}^{M} \frac{\beta_j}{1+\beta_j} - L p_i} \\
&= \beta_i
\end{aligned} \tag{52}
$$

Thus each user has its SIR requirement met which completes the proof. ■

Proof of Theorem 5.6:
We shall follow a line of proof that highlights the reason why Theorems 5.3 and 5.6 are identical though one system is equipped with MMSE receiver structure while the other with matched filter receiver structure. We shall choose the same signature sequences and powers as in the earlier proof of Theorem 5.3. Note that with this choice of powers and signature sequences the MMSE receiver structure for every user i is, as in (2),

$$
\begin{aligned}
c_i &= Z_i^{-1} s_i \\
&= \left(\sigma^2 I + \sum_{j \neq i} p_j s_j s_j^t\right)^{-1} s_i \\
&= \left(\sigma^2 I - p_i s_i s_i^t + S D S^t\right)^{-1} s_i \\
&= \left(\sigma^2 I - p_i s_i s_i^t + \frac{c\sum_{i=1}^{M} \frac{\beta_i}{1+\beta_i}}{L} I\right)^{-1} s_i \\
&= K_i s_i
\end{aligned}
$$

where K_i is a constant (which is easily seen to be $(1+\beta_i)\left(1 - \frac{1}{L}\sum_{j=1}^{M} \frac{\beta_j}{1+\beta_j}\right)$). Thus the MMSE receiver structure is just the scaled matched filter receiver structure and hence the achieved SIR of every user i with matched filter receiver is identical to that with the MMSE receiver which as seen in (52) is β_i. This completes the proof. ■

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