RESISTANCE TRIANGLE INEQUALITY

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Memorandum No. UCB/ERL M99/21

6 April 1999

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Abstract

This note proves that the 3 input resistances measured across *any* 3 nodes of a connected circuit made of linear positive resistors satisfy the *triangle inequality*.

1 Formal statement of triangle inequality

Let N be any *connected* circuit made of 2-terminal linear positive resistors, and choose any 3 nodes $\boxed{1}$, $\boxed{2}$, and $\boxed{3}$, as depicted in Fig.1(a). Connect 3 current sources $I_i, i = 1, 2, 3$, as shown in Fig.1(b). Let

$$R_{ii} = \left.\frac{V_i}{I_i}\right|_{I_j=0, j\neq i}, \quad i = 1, 2, 3 \tag{1}$$

denote the *input resistance* across the driving-point terminals formed by the node-pairs $\boxed{1}$-$\boxed{2}$, $\boxed{2}$-$\boxed{3}$, and $\boxed{3}$-$\boxed{1}$, respectively.

Theorem: *Resistance Triangle Inequality*

The resistances R_{ii}, $i = 1, 2, 3$, satisfy the following triangle inequality:

$$R_{ii} + R_{i+1,i+1} \geq R_{i+2,i+2} \tag{2}$$

$i \in \{$ positive integers mod 3$\}$.

Proof: Since N is connected and contains only 2-terminal linear positive resistors ($R > 0$), the 3-port in Fig.1(b) can be characterized uniquely by the following resistance matrix representation [1]:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12} & R_{22} & R_{23} \\ R_{13} & R_{32} & R_{33} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \tag{3}$$

where the 3×3 open-circuit resistance matrix $\mathbf{R}$ is *symmetric* and *positive semi-definite* [2]. Since nodes $\boxed{1}$, $\boxed{2}$, and $\boxed{3}$ form a loop,

$$V_1 + V_2 + V_3 = 0 \tag{4}$$

Substituting V_i from Eq.(3) into Eq.(4) and assigning $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ respectively to (I_1, I_2, I_3), we obtain

$$R_{11} + R_{12} + R_{13} = 0 \tag{5}$$

$$R_{12} + R_{22} + R_{23} = 0 \tag{6}$$

$$R_{13} + R_{23} + R_{33} = 0 \tag{7}$$

Subtracting Eq.(7) from the sum of Eqs.(5) and (6), we obtain

$$R_{11} + R_{22} = R_{33} - 2R_{12} \tag{8}$$

Now since

$$R_{12} = V_1|_{I_1=I_3=0,I_2=1} \tag{9}$$

it follows from the methods presented in [1] that[1]:

$$R_{12} \leq 0 \tag{10}$$

Applying inequality (10) to Eq.(8), we obtain the triangle inequality (2). □

2 Remark

1. By duality, the above triangle inequality also holds for the input *conductances* of connected circuits made of 2-terminal linear positive resistors.

2. The resistance triangle inequality (2) is a special case of a more general results presented in [2].

Acknowledgment

I would like to thank Professor David Gale for first posing to me the "Resistance triangle inequality" as a conjecture.

References

[1] L.O. Chua, Y.F. Lam and K.A. Stromsmoe, "Qualitative properties of resistive networks containing multi-terminal nonlinear elements: no gain properties," *IEEE Trans. on Circuits and Systems*, vol.CAS-24, no.3, March 1977, pp.93-118.

[2] D.J. Klein, "Resistance distance," J. of Mathematical Chemistry, **12**, pp.81-95, 1993.

[1]The inequality (10) can also be proved directly by the same procedure as in the proof the *maximum node-voltage property* on page 272-273 of [1].

Figure Caption

Fig.1: (a) A connected resistor circuit N with 3 arbitrarily chosen nodes $\boxed{1}$, $\boxed{2}$, and $\boxed{3}$. (b) Driving N with 3 current sources across the node-pairs $\boxed{1}$-$\boxed{2}$, $\boxed{2}$-$\boxed{3}$, and $\boxed{3}$-$\boxed{1}$.

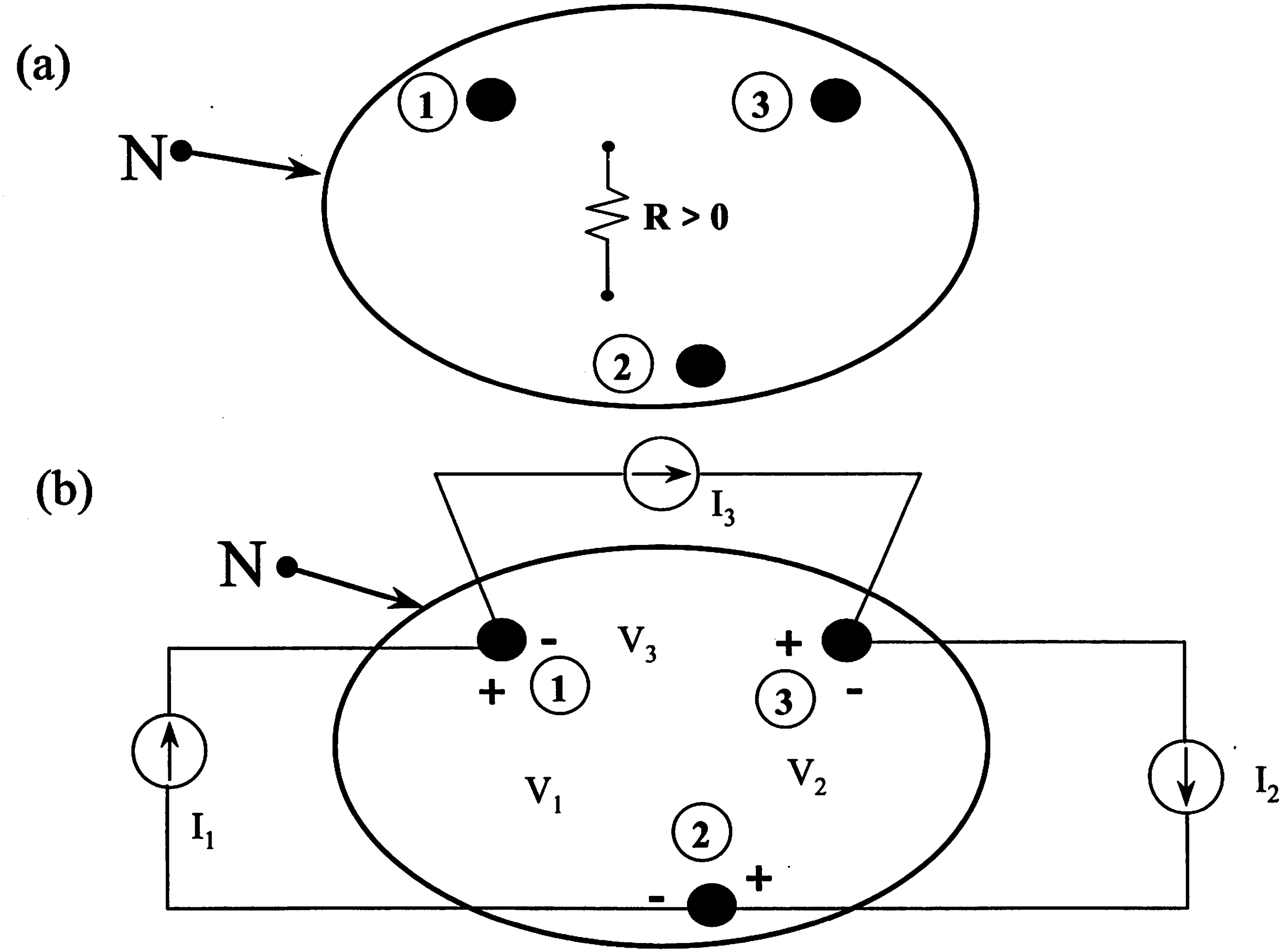

Fig.1 (a) A connected linear resistor circuit N with 3 arbitrarily chosen nodes ①, ②, and ③.
(b) Driving N with 3 current sources across the node-pairs,①-②, ②-③, and ③-①.