FIRST-ORDER ALGORITHMS FOR OPTIMIZATION PROBLEMS WITH A MAXIMUM EIGENVALUE/SINGULAR VALUE COST AND OR CONSTRAINTS

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First-Order Algorithms for Optimization Problems with a Maximum Eigenvalue/Singular Value Cost and or Constraints [1]

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Abstract

Optimization problems with maximum eigenvalue or singular eigenvalue cost or constraints occur in the design of linear feedback systems, signal processing, and polynomial interpolation on a sphere. Since the maximum eigenvalue of a positive definite matrix $Q(x)$ is given by $\max_{\|y\|=1}\langle y, Q(x)y\rangle$, we see that such problems are, in fact, semi-infinite optimization problems. We will show that the quadratic structure of these problems can be exploited in constructing specialized first-order algorithms for their solution that do not require the discretization of the unit sphere or the use of outer approximations techniques.

Keywords: maximum eigenvalue cost/constraints, singular value cost/constraints, min-max algorithms.

1 Introduction

Optimization problems with maximum eigenvalue or singular eigenvalue cost or constraints occur in a number of disciplines. For example, in the design of linear feedback systems, the suppression of disturbances can be modeled as the minimization of the norm of the disturbance transmission transfer function matrix $G_d(x, j\omega)$ over a specified range of frequencies, where $x \in \mathbb{R}^n$ is the design vector and $\omega \in \mathbb{R}$ is a frequency variable (see, e.g., [2], [1]). Since the norm of $G_d(x, j\omega)$ is its maximum singular

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value, we see that the minimization of this norm can be expressed as the semi-infinite optimization problem:

$$\min_{x\in X}\max_{\omega\in\Omega}\max_{\|y\|=1}\langle y, Q(x,j\omega)y\rangle, \tag{1}$$

where $X\subset\mathbb{R}^n$ is a constraint set, $\Omega=\{\omega_1,...,\omega_N\}$ is a grid of frequencies, and $Q(x,j\omega)=G_d(x,j\omega)^*G_d(x,j\omega)$.

In a wide variety of signal processing applications, such as beam forming [4] and radar imaging[7], it is desirable to form an estimate of a covariance matrix from samples of a process. The true covariance matrix is known to be positive definite and Hermitian Toeplitz.

Let $\{z_1\dots z_M\}$ denote the set of $N\times 1$ observation vectors of the process. Then the sample covariance matrix S is given by

$$S=\frac{1}{M}\sum_{m=1}^{M} z_m z_m^H \tag{2}$$

where z^H denotes the complex conjugate transpose of z. The desired estimate is then the $N\times N$ positive definite Hermitian Toeplitz matrix $\hat{R}$ given by

$$\hat{R}=\arg\max_{R\in\mathcal{T}^+}\{-\ln(\det R)-\mathrm{tr}(R^{-1}S)\} \tag{3}$$

where $\mathcal{T}^+$ is the set of all $N\times N$ positive definite Hermitian Toeplitz matrices, and $\mathrm{tr}(\cdot)$ is the trace operator.

Now, any Hermitian Toeplitz matrix R can be parametrized in terms of a pair of vectors $x=(x_R,x_I)\in\mathbb{R}^N\times\mathbb{R}^N$ as follows:

$$R(x)=\sum_{n=1}^{N}(x_{R,n}Q_{R,n}+jx_{I,n}Q_{I,n}) \tag{4}$$

where $[x_{R,1}+jx_{I,1},\dots,x_{R,N}+jx_{I,N}]$ is the first row of R (with real and imaginary parts shown explicitly), and $Q_{R,n}$ and $Q_{I,n}$ $(n=1,\dots,N)$ are symmetric matrices and skew symmetric matrices respectively.

Hence, problem (3) can be recast as a constrained semi-infinite optimization problem, as follows:

$$\hat{x}=\arg\max_{x=(x_R,x_I)\in\mathcal{R}^N\times\mathcal{R}^N}\Big\{-\ln(\det R(x))-\mathrm{tr}R(x)^{-1}S)\ \Big|\ \min_{\|y\|=1}\langle y,R(x)y\rangle\geq\sigma>0,\ R(x)=\sum_{n=1}^{N}(x_{R,n}Q_{R,n}+jx_{I,n}Q_{I,n})\Big\}. \tag{5}$$

The constrained maximum likelihood covariance estimate is then given by

$$\hat{R} = \sum_{n=1}^{N} \left(\hat{x}_{R,n} Q_{R,n} + j \hat{x}_{I,n} Q_{I,n} \right). \tag{6}$$

Our final example comes from the problem of choosing points on the unit sphere to minimize a bound on the norm of the polynomial interpollation operator [11], [9]. This bound is minimized by finding the m points on the unit sphere S^2 in $\mathbb{R}^3$ which maximizes the *smallest* eigenvalue of a symmetric gram matrix G, which is a nonlinear function of of the angles between these points. For polynomials of degree at most p, a fundamental system of points on the unit sphere in $\mathbb{R}^3$ consists of $(p+1)^2$ points for which the only polynomial of degree at most p vanishes at all points is the zero polynomial. For any fundamental system, G is a symmetric positive definite m by m matrix where $m = (p+1)^2$. Since a fundamental system can be parametrized using $n = 2m - 3$ variables, the problem of minimizing the *smallest* eigenvalue of G can be expressed as the following semi-infinite min-max problem

$$\min_{x \in \mathbb{R}^n} \max_{\|y\|=1} \langle y, -G(x)y \rangle. \tag{7}$$

In this paper we will present two new specialized algorithms for the types of problem described above. These algorithms appear to have serious advantages over existing algorithms (such as those described in [6], [10], for example) when the matrix in the quadratic form is very large (say at least 1000×1000)[2].

2 An Implementable First-Order Algorithm for Semi-Infinite Min-Max Problems.

We begin by considering problems of the form

$$\min_{x \in \mathbb{R}^n} \psi(x) \tag{8}$$

where

$$\psi(x) = \max_{y \in Y} \phi(x, y), \tag{9}$$

where $Y \subset \mathbb{R}^m$ is compact and $phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable. In particular, we will consider the case where

$$\phi(x, y) = \langle y, Q(x)y \rangle, \quad Y = \{ y \in \mathbb{R}^m | \|y\| = 1 \}. \tag{10}$$

[2]Private communication: Dr. R. S. Womersley, Dept. of Applied Mathematics, University of New South Wales, Sydney, Australia

Referring to [8], Section 3.1, we find a first-order optimality condition for (8) in terms of the set-valued map

$$\bar{G}\psi(x) \triangleq \underset{y\in Y}{\text{conv}} \left\{ \begin{array}{c} \psi(x) - \phi(x,y) \\ \\ \nabla_x \phi(x,y) \end{array} \right\}, \tag{11}$$

where conv denotes the convex hull of the indicated set.

Theorem 1[8].

(a) *The set-valued map $\bar{G}\psi(x)$, from $\mathbb{R}^n$ to the subsets of $\mathbb{R}^{n+1}$ is continuous in the Painlevé-Kuratowski sense.*

(b) *Let the elements of $\mathbb{R}^{n+1}$ be denoted by $\bar{\xi} \triangleq (\xi^0, \xi)$, with $\xi^0 \in \mathbb{R}$, and, with $\delta > 0$, let*

$$q(\bar{\xi}) = \xi^0 + \frac{1}{2\delta}\|\xi\|^2, \tag{12}$$

let the optimality function

$$\theta(x) \triangleq - \min_{\bar{\xi}\in\bar{G}\psi(x)} q(\bar{\xi}), \tag{13}$$

and let

$$\bar{h}(x) = (h^0(x), h(x)) \triangleq - \arg \min_{\bar{\xi}\in\bar{G}\psi(x)} q(\bar{\xi}). \tag{14}$$

Then

(i) *The functions $\theta(\cdot)$ and $\bar{h}(\cdot)$ are continuous, and for all $x \in \mathbb{R}^n$, $\theta(x) \leq 0$.*

(ii) *For any $x \in \mathbb{R}^n$, the directional derivative*

$$d\psi(x; h(x)) \leq \theta(x) - \frac{\delta}{2}\|h(x)|^2. \tag{15}$$

(iii) *If $\hat{x} \in \mathbb{R}^n$ is a local minimizer of $\psi(\cdot)$, then*

$$0 \in \bar{G}\psi(\hat{x}), \quad \theta(\hat{x}) = 0, \quad \bar{h}(\hat{x}) = 0. \tag{16}$$

Furthermore, (16) holds if and only if $0 \in \partial\psi(\hat{x})$.

In [8], Section 2.4.1, we find the Pshenichnyi-Pironneau-Polak minimax algorithm for finite min-max problems. This algorithm also has the following, non-implementable form for the problem (8).

Algorithm 1(Generalized Pshenichnyi-Pironneau-Polak Algorithm)

Parameters. $\alpha, \beta \in (0,1)$, $\delta > 0$

Data. $x_0 \in \mathbb{R}^n$.

Step 0. *Set* $i = 0$.

Step 1. *Compute* $\theta_i = \theta(x_i)$ *and* $h_i = h(x_i)$.

Step 2. *If* $\theta_i = 0$, *stop. Else, compute the step-size*

$$\lambda_i = \max_{k \in \mathbb{N}} \{\beta^k \mid \psi(x_i + \beta^k h_i) - \psi(x_i) - \beta^k \alpha \theta_i \leq 0\}, \tag{17}$$

where $\mathbb{N} = \{0, 1, 2, 3, ...\}$.

Step 3. *Set*

$$x_{i+1} = x_i + \lambda_i h_i, \tag{18}$$

replace i *by* $i+1$ *and go to* **Step 1**.

The reason Algorithm 1 is non-implementable for the problem (8) is that neither $\theta(x_i)$ nor $h(x_i)$ can be computed exactly in reasonable time. To obtain an implementable version of Algorithm 1, we must modify it so as to be able to use approximations to $\theta(x_i)$ and $h(x_i)$. We will now develop such an implementation which makes sense when $\phi(x,y)$ is of the form (10) and, possibly a few other cases as well. The success of the new algorithm depends on the following observation:

Theorem 2. *Suppose that* $x \in \mathbb{R}^n$ *is such that* $0 \notin \bar{G}\psi(x)$, $\gamma \in (0,1)$ *and* $\bar{\xi}_* \in \bar{G}\psi(x)$, $\bar{\xi}_{**} \notin \bar{G}\psi(x)$ *are such that*

(i) $\bar{\xi}^0_{**} > 0$;

(ii) $\langle \nabla q(\bar{\xi}_{**}), \bar{\xi} - \bar{\xi}_{**} \rangle \geq 0$ *for all* $\bar{\xi} \in \bar{G}\psi(x)$;

(iii) $q(\bar{\xi}_*) - q(\bar{\xi}_{**}) \leq \gamma q(\bar{\xi}_{**})$.

Then,

(a) $q(\bar{\xi}_{**}) \leq -\theta(x) \leq q(\bar{\xi}_*)$;

(b) $-\frac{1}{1+\gamma}\theta(x) \leq q(\bar{\xi}_{**}) \leq -\theta(x)$;

(c) *with* $h_{**} = -\xi_{**}$,

$$\tilde{\psi}(x; h_{**}) \triangleq \max_{y \in Y} [\phi(x,y) - \psi(x)] + \langle \nabla_x \phi(x,y), h \rangle + \frac{\delta}{2}\|h_{**}\|^2 \leq -q(\bar{\xi}_{**}). \tag{19}$$

and

$$d\psi(x; h_{**}) \leq -q(\bar{\xi}_{**}) - \frac{\delta}{2}\|h_{**}\|^2 \tag{20}$$

Proof. **(a)** Clearly, since $q(\cdot)$ is convex, it follows that

$$\bar{\xi}_{**} = \arg\min_{\bar{\xi}\in H} q(\bar{\xi}), \tag{21}$$

where

$$H \triangleq \{\bar{\xi} \in \mathbb{R}^{n+1} | \langle \nabla q(\bar{\xi}_{**}), \bar{\xi} - \bar{\xi}_{**}\rangle \geq 0\}. \tag{22}$$

Now, it follows from assumption **(ii)** that $\bar{G}\psi(x) \subset H$, and hence that

$$q(\bar{\xi}_{**}) \leq -\theta(x) = \min_{\bar{\xi}\in\bar{G}\psi(x)} q(\bar{\xi}) \leq q(\xi_*), \tag{23}$$

which proves **(a)**.

(b) It follows from assumption **(iii)** and (23) that

$$q(\bar{\xi}_{**}) \geq -\theta(x) - \gamma q(\bar{\xi}_{**}). \tag{24}$$

Hence it follows directly that

$$-\frac{1}{\gamma}\theta(x) \leq q(\bar{\xi}_{**}) \leq -\theta(x), \tag{25}$$

which proves **(b)**.

(c) Let $\rho > 0$ be such that $\bar{G}\psi(x) \subset B(0,\rho) \subset \mathbb{R}^{n+1}$ (where $B(0,\rho) \triangleq \{\bar{\xi} \mid \|\bar{\xi}\| \leq \rho\}$) and let

$$S = H \cap B(0,\rho). \tag{26}$$

Then,

$$\left.\begin{aligned}
\max_{\bar{\xi}\in S}\min_{h\in\mathbb{R}^n} -\xi^0 + \langle \xi, h\rangle + \frac{\delta}{2}\|h\|^2 &= \max_{\bar{\xi}\in S} -\xi^0 - \frac{1}{2\delta}\|\xi\|^2 \\
&= -\min_{\bar{\xi}\in S} \xi^0 + \frac{1}{2\delta}\|\xi\|^2 \\
&= -\min_{\bar{\xi}\in S} q(\bar{\xi}) \\
&= q(\bar{\xi}_{**}) \\
&= \max_{\bar{\xi}\in S} -\xi^0 + \langle \xi, h_{**}\rangle + \frac{\delta}{2}\|h_{**}\|^2
\end{aligned}\right\} \tag{27}$$

because the unconstrained min above yelds that

$$h = -\frac{1}{\delta}\xi, \tag{28}$$

and, by definition, $h_{**} = -\frac{1}{\delta}\bar{\xi}_{**}$.

Now, with h_{**} as above,

$$\left.\begin{aligned}
\max_{y\in Y}[\phi(x,y)-\psi(x)] &+ \langle \nabla_x\phi(x,y), h_{**}\rangle + \tfrac{\delta}{2}\|h_{**}\|^2 \\
&= \max_{\bar{\xi}\in \bar{G}\psi(x)} -\xi^0 + \langle \xi, h_{**}\rangle + \frac{\delta}{2}\|h_{**}\|^2 \\
&\leq \max_{\bar{\xi}\in S} -\xi^0 + \langle \xi, h_{**}\rangle + \frac{\delta}{2}\|h_{**}\|^2 \\
&= -q(\bar{\xi}_{**}),
\end{aligned}\right\} \tag{29}$$

which proves (19).

Next, let $\hat{Y}(x) \triangleq \{y \in Y \mid \phi(x,y) = \psi(x)\}$. Then we see that

$$\left.\begin{aligned}
d\psi(x,h_{**}) + \tfrac{\delta}{2}\|h_{**}\|^2 &= \max_{y\in\hat{Y}(x)} \langle \nabla_x\phi(x,y), h_{**}\rangle + \frac{\delta}{2}\|h_{**}\|^2 \\
&= \max_{y\in\hat{Y}(x)} [\phi(x,y)-\psi(x)] + \langle \nabla_x\phi(x,y), h_{**}\rangle + \frac{\delta}{2}\|h_{**}\|^2 \\
&\leq \max_{y\in Y}[\phi(x,y)-\psi(x)] + \langle \nabla_x\phi(x,y), h_{**}\rangle + \frac{\delta}{2}\|h_{**}\|^2 \\
&\leq -q(\bar{\xi}_{**}),
\end{aligned}\right\} \tag{30}$$

which proves (20). □

Later, we will show that given a point $x \in \mathbb{R}^n$, points $\bar{\xi}_*, \bar{\xi}_{**}$ as specified in Theorem 2, can be computed using either the Frank-Wolfe Algorithm [3] or the much more efficient Higgins-Polak Algorithm [5]. We will also show that for the case where $\phi(x,y)$ and Y are defined as in (10), these algorithms can be efficiently impementing using the fact that eigenvalues of a symmetric positive-semidefinite matrix are relatively easy to compute. However, first we state an implementable modification of Algorithm 1 which uses such points:

Algorithm 2(Modified Generalized Pshenichnyi-Pironneau-Polak Algorithm)

Parameters. $\alpha, \beta\gamma \in (0,1)$, $\delta > 0$

Data. $x_0 \in \mathbb{R}^n$.

Step 0. *Set* $i = 0$.

Step 1. *Use the Higgins-Polak or Frank-Wolfe algorithm to compute* $\bar{\xi}_{*i} \in \bar{G}\psi(x_i)$, $\bar{\xi}_{**i} \notin \bar{G}\psi(x_i)$ *are such that*

(i) $\bar{\xi}^0_{**i} > 0$;

(ii) $\langle \nabla q(\bar{\xi}_{**i}), \bar{\xi} - \bar{\xi}_{**i} \rangle \geq 0$ *for all* $\bar{\xi} \in \bar{G}\psi(x_i)$;

(iii) $q(\bar{\xi}_{*i}) - q(\bar{\xi}_{**i}) \leq \gamma q(\bar{\xi}_{**i})$.

and set $h_i = -\xi_{**}$.

Step 2. *Compute the step-size*

$$\lambda_i = \max_{k \in \mathbb{N}} \{ \beta^k \mid \psi(x_i + \beta^k h_i) - \psi(x_i) + \beta^k \alpha q(\bar{\xi}_{**i}) \leq 0 \}. \tag{31}$$

Step 3. *Set*

$$x_{i+1} = x_i + \lambda_i h_i, \tag{32}$$

replace i *by* $i+1$ *and go to* **Step 1.**

Lemma 1. *Suppose that* $\nabla\phi(\cdot,\cdot)$ *is Lipschitz continuous on bounded sets*[3]. *Then for all* $x_i \in \mathbb{R}^n$ *such that* $\theta(x_i) < 0$, λ_i *is well defined by (31).*

Proof. Suppose that $\theta(x_i) < 0$ and that $L < \infty$ is a Lipschitz constant for a sufficiently large neighborhood of x_i. then for any $\lambda \in [0, \min\{1, \delta/L\}$,

$$\left.\begin{aligned}
\psi(x_i + \lambda h_i) - \psi(x_i) &= \max_{y \in Y} \phi(x_i, y) - \psi(x_i) + \lambda \langle \nabla\phi(x_i, y), h_i \rangle \\
&\quad + \int_0^1 \langle \nabla\phi(x_i + s\lambda h_i, y) - \nabla\phi(x_i, y), h_i \rangle ds \\
&\leq \max_{y \in Y} \phi(x_i, y) - \psi(x_i) + \lambda \langle \nabla\phi(x_i, y), h_i \rangle + \frac{\lambda^2 L}{2} \|h_i\|^2 \\
&\leq \lambda \max_{y \in Y} \{ \phi(x_i, y) - \psi(x_i) + \langle \nabla\phi(x_i, y), h_i \rangle + \frac{\delta}{2} \|h_i\|^2 \\
&\leq -\lambda q(\bar{\xi}_{**i}).
\end{aligned}\right\} \tag{33}$$

[3]The assumption of local Lipschitz continuity can be relaxed to continuity, but the proof of the Lemma is then a bit more difficult.

Hence, for all $\lambda \in [0, \min\{1, \delta/L\}$,

$$\psi(x_i + \lambda h_i) - \psi(x_i) + \lambda \alpha q(\bar{\xi}_{**i}) \leq -\lambda(1-\alpha) q(\bar{\xi}_{**i}) \leq 0, \tag{34}$$

from which we deduce that the step-size $\lambda_i \geq \beta \min\{1, \delta/L\}$, which completes our proof. □

We will prove that Algorithm 2 is convergent, we will show that it has the MUD property (see [8], p. 21) and then make use of Theorem 1.2.8 in [8].

Lemma 2. *Suppose that $\nabla\phi(\cdot,\cdot)$ is Lipschitz continuous on bounded sets and that $\hat{x}$ is such that $\theta(\hat{x}) < 0$. Then there exists a $\rho > 0$ and a $\kappa > 0$, such that for all $x_i \in B(\hat{x}, \rho)$ and x_{i+1} constructed by Algorithm 2,*

$$\psi(x_{i+1}) - \psi(x_i) \leq -\kappa. \tag{35}$$

Proof. Let $\rho > 0$ be such that $\theta(x_i) \leq \theta(\hat{x})/2$ for all $x_i \in B(\hat{x}, \rho)$ and let $L < \infty$ be a Lipschitz constant for $\nabla\phi(\cdot,\cdot)$ on $B(\hat{x}, \rho)$. Then it follows from Lemma 1 and part (b) of Theorem 2 that for all $x_i \in B(\hat{x}, \rho)$

$$\left.\begin{aligned}
\psi(x_{i+1}) - \psi(x_i) &\leq \lambda_i \alpha q(\bar{\xi}_{**i}) \\
&\leq \alpha\beta \min\{1, \tfrac{\delta}{L}\} q(\bar{\xi}_{**i}) \\
&\leq \tfrac{\alpha\beta}{1+\gamma} \min\{1, \tfrac{\delta}{L}\} \theta(x_i) \\
&\leq \tfrac{\alpha\beta}{2(1+\gamma)} \min\{1, \tfrac{\delta}{L}\} \theta(\hat{x}) \\
&\triangleq -\kappa,
\end{aligned}\right\} \tag{36}$$

which completes our proof. □

The following theorem is a direct consequence of Lemma 1, Lemma 2 and Theorem 1.2.8 in [8].

Theorem 3. *If $\{x_i\}_{i=0}^{\infty}$ is a sequence constructed by Algorithm 2, then every accumulation point $\hat{x}$ of this sequence satisfies the first-order optimality condition $\theta(\hat{x}) = 0$.*

3 Rate of Convergence of Algorithm 2

To establish rate of convergence we will the following hypotheses:

Assumption 1. *We will assume that*

(a) *For every $y \in Y$, $\phi(\cdot, y)$ is convex.*

(b) *The second derivative matrix $\phi_{xx}(x, y)$ exists and is continous.*

(c) *There exists $0 < m \leq M$ such that for all x in a sufficiently large set, all $y \in Y$ and all $h \in \mathbb{R}^n$,*

$$m\|h\|^2 \leq \langle h, \phi_{xx}(x, y)y \rangle \leq M\|h\|^2. \tag{37}$$

(d) *$\delta \in [m, M]$ holds.*

In [8], p. 225, we find the following result:

Lemma 3. *Suppose that Assumption 1 is satisfied. Let $\hat{x}$ be the unique minimizer of $\psi(\cdot)$. Then, for any $x \in \mathbb{R}^n$,*

$$\psi(\hat{x}) - \psi(x) \geq \frac{\delta}{m}\theta(x). \tag{38}$$

Theorem 3. *Suppose that Assumption 1 is satisfied. If $\{x_i\}_{i=0}^{\infty}$ is a sequence constructed by Algorithm 2, then*

$$\frac{\psi(x_{i+1}) - \psi(\hat{x})}{\psi(x_i) - \psi(\hat{x})} \leq \left\{1 - \frac{\beta\alpha m}{M(1+\gamma)}\right\}. \tag{39}$$

Proof. First, it follows from Lemma 3 and Theorem 2 (b), that for all $i \in \mathbb{N}$,

$$\psi(\hat{x}) - \psi(x_i) \geq \frac{\delta}{m}\theta(x_i) \geq -\frac{\delta}{m}(1+\gamma)q(\bar{\xi}_{**i}). \tag{40}$$

Next, in view of (37) and (19), for any x_i and $\lambda \in [0, \delta/M]$,

$$\left.\begin{aligned} \psi(x_i + \lambda h_i) - \psi(x_i) &\leq \max_{y \in Y}[\phi(x_i, y) - \psi(x_i) + \lambda\langle \nabla_x \phi(x_i, y), h_i \rangle + \frac{\lambda^2 M}{2}\|h_i\|^2 \\ &\leq \lambda \max_{y \in Y}[\phi(x_i, y) - \psi(x_i) + \langle \nabla_x phi(x_i, y), h_i \rangle + \frac{\delta}{2}\|h_i\|^2 \\ &\leq -\lambda q(\bar{\xi}_{**i}). \end{aligned}\right\} \tag{41}$$

Hence, for any x_i and $\lambda \in [0, \delta/M]$,

$$\psi(x_i + \lambda h_i) - \psi(x_i) + \lambda\alpha q(\bar{\xi}_{**i}) \leq \lambda(1-\alpha)q(\bar{\xi}_{**i}) \leq 0, \tag{42}$$

which proves that $\lambda_i \geq \frac{\beta\delta}{M}$. Hence we have that

$$\psi(x_{i+1}) - \psi(x_i) \leq -\frac{\beta\delta\alpha}{M} q(\bar{\xi}_{**i}). \tag{43}$$

Now, it follows from (40) that

$$-q(\bar{\xi}_{**i}) \leq \frac{m}{\delta(1+\gamma)}[\psi(\hat{x}) - \psi(x_i)]. \tag{44}$$

Combining (43) and (44), we obtain that

$$\psi(x_{i+1}) - \psi(x_i) \leq \frac{\beta\delta\alpha}{M} \frac{m}{\delta(1+\gamma)}[\psi(\hat{x}) - \psi(x_i)]. \tag{45}$$

Subtracting $\psi(\hat{x}) - \psi(x_i)$ from both sides of (45), we finally obtain

$$\psi(x_{i+1}) - \psi(\hat{x}) \leq \left(1 - \frac{\beta\alpha m}{M(1+\gamma)}\right)[\psi(x_{i+1}) - \psi(\hat{x})], \tag{46}$$

which completes our proof. □

Remark 1. Note that when $\gamma = 0$, we revert to the conceptual form of the algorithm for which the rate of convergence is given in Theorem 2.4.5 of [8]. We see that when $\gamma = 0$, the expression (46) coincides with the corresponding expresssion given in Theorem 2.4.5 of [8]. □

4 Minimization of the Maximum Eigenvalue of a Symmetric Matrix

We now return to the special case where $\phi(\cdot,\cdot)$ and Y are as given in (10), with the matrix $Q(\cdot)$ at least once continously differentiable. To show that Algorithm 2 is implementable for this case, we only need to show how to compute the points $\bar{\xi}_{*i}$ and $\bar{\xi}_{**i}$ can be computed using either the Frank-Wolfe algorithm [3] or the much more efficient Higgins-Polak algorithm [5], [8] to minimize the function $q(\bar{\xi})$, defined in (12) over the set $\bar{G}\psi(x)$, defined in (11). Both of these algorithms depend on the computation of "support points" to the set $\bar{G}\psi(x)$ defined in (11), but the Frank-Wolfe algorithm is much simpler to explain, so we will restrict itself to it.

Modified Frank-Wolfe Algorithm (Computes points $\bar{\xi}_*$ and $\bar{\xi}_{**}$)

Parameters. $\gamma \in (0,1)$.

Data. $\bar{\xi}_0 \in \bar{G}\psi(x)$.

Step 0. *Set* $i = 0$.

Step 1. *Compute* a support point $\bar{\zeta}_i \in \bar{G}\psi(x)$ *according to*

$$\bar{\zeta}_i \in \arg\min\{\langle \nabla q(\bar{\xi}_i), \bar{\zeta} - \bar{\xi}_i\rangle \mid \bar{\zeta} \in \bar{G}\psi(x)\}. \tag{47}$$

Step 2. *compute the point*

$$\bar{\zeta}_i' = \arg\min\{q(\bar{\zeta}) \mid \bar{\zeta} \in \mathcal{H}(\zeta_i)\}, \tag{48}$$

where

$$\mathcal{H}(\zeta_i) = \{\bar{\zeta} \in \mathbb{R}^{n+1} \mid \langle \bar{\zeta} - \bar{\zeta}_i, \nabla q(\bar{\xi}_i)\rangle = 0\}. \tag{49}$$

Step 3. If $\zeta_i'^0 > 0$ *and*

$$q(\bar{\xi}_i) - q(\bar{\zeta}_i') \le \gamma q(\bar{\zeta}_i'), \tag{50}$$

set $\bar{\xi}_* = \bar{\xi}_i$, $\bar{\xi}_{**} = \bar{\zeta}_i'$ *and exit.*

Else, *set* $\bar{\eta}_i = \bar{\zeta}_i - \bar{\xi}_i$ *and go to* **Step 4**.

Step 4. *Compute the step-length*

$$\lambda_i = \arg\min\{q(\bar{\xi}_i + \lambda\bar{\eta}_i) \mid \lambda \in [0,1]\}. \tag{51}$$

Step 5. *Update: Set*

$$\bar{\xi}_{i+1} = \bar{\xi}_i + \lambda_i\bar{\eta}_i \tag{52}$$

and go to **Step 1**.

The following result is a direct consequence of the fact (see Theorem 2.4.9 in [8]) that if the Frank-Wolfe does not exit in Step 3, above, then the sequence $\{\bar{\xi}_i\}_{i=0}^{\infty}$ converges to the unique minimizer of $q(\cdot)$ on $\bar{G}\psi(x)$.

Theorem 4. *Suppose that $x \in \mathbb{R}^n$ is such that $\theta(x) < 0$, then the Modified Frank-Wolfe Algorithm will compute the required points $\bar{\xi}_*$, $\bar{\xi}_{**}$ in a finite number of iterations.*

Proof. Suppose that $x \in \mathbb{R}^n$ is such that $\theta(x) < 0$ and that the Modified Frank-Wolfe Algorithm does not exit in Step 3 after a finite number of iterations. Then it follows from Theorem 2.4.9, in [8] that the sequence $\{\bar{\xi}_i\}_{i=0}^{\infty}$ converges to the unique minimizer $\bar{\xi}^*$ of $q(\cdot)$ on $\bar{G}\psi(x)$ and the same holds for the sequence $\{\bar{\zeta}_i\}_{i=0}^{\infty}$. Hence the hyperlanes $\mathcal{H}(\bar{\zeta}_i)$ converge to the hyperplane $\mathcal{H}(\bar{\xi}^*)$ and therefore the sequence $\{\bar{\zeta}_i'\}_{i=0}^{\infty}$ also converges to $\bar{\xi}^*$. Since this implies that $q(\bar{\xi}_i) - q(\bar{\zeta}_i') \to 0$, as $i \to \infty$, we have a contradiction, which completes our proof. □

Clearly, neither the minimization of the quadratic function $q(\cdot)$ on the hyperplane $\mathcal{H}(\bar{\zeta}_i)$ in (48), nor the step-length calculation in ((51) pause any difficulty. The only difficult operation in the Modified Frank-Wolfe Algorithm seems to be the computation of the support (contact) point $\bar{\zeta}_i$, according to (47). We will now show that because of the quadratic form of the function $\phi(x,y)$ this computation is quite simple.

Now, when $\phi(x,y)$ and Y are as in (10), the set $\bar{G}\psi(x)$ assumes the specific form:

$$\bar{G}\psi(x) = \underset{\|y\|^2=1}{\text{conv}} \left\{ \begin{pmatrix} \psi(x) - \langle y, Q(x)y \rangle \\ \langle y, Q_1(x)y \rangle \\ \vdots \\ \langle y, Q_n(x)y \rangle \end{pmatrix} \right\}, \tag{53}$$

where $Q_j(x) = \partial Q(x)/\partial x^j$.

5 A Numerical Example

6 Conclusion

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