DESIGNING A CONTACT PROCESS: THE PIECEWISE-HOMOGENEOUS PROCESS ON A FINITE SET WITH APPLICATIONS

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Designing a Contact Process: The Piecewise-Homogeneous Process on a Finite Set with Applications

Aaron B. Wagner and Venkat Anantharam
Department of Electrical Engineering and Computer Sciences
University of California
Berkeley, CA 94720-1770 USA
{awagner,ananth}@eecs.berkeley.edu

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Abstract

We consider how to choose the reproduction rates in a one-dimensional contact process on a finite set to maximize the growth rate of the extinction time with the population size. The constraints are an upper bound on the average reproduction rate, and that the rate profile must be piecewise constant. We show that the optimum growth rate is achieved by a rate profile with at most two rates, and we characterize the solution in terms of a "spatial correlation length" of the supercritical process. We examine the analogous problem for the simpler biased voter model, for which we completely characterize the optimum profile. The contact process proofs make use of a planar-graph duality in the graphical representation, due to Durrett and Schonmann.

1 Introduction

The contact process (CP) on the one-dimensional integer lattice is the Markov process ξ_t with state space $2^{\mathbb{Z}}$ and transition rates

$$\begin{aligned} q(A, A\backslash\{j\}) &= 1 \quad \text{if } j \in A, \\ q(A, A \cup \{j\}) &= \lambda |A \cap \{j-1, j+1\}| \quad \text{if } j \notin A. \end{aligned} \tag{1}$$

Here $|\cdot|$ denotes cardinality. For this process and all others described in this paper, if A and B are subsets of $\mathbb{Z}$ such that $|A \Delta B| > 1$, then $q(A, B) = 0$. In words, each integer is either occupied by a member of some population or vacant; occupied sites become vacant at rate 1, while vacant sites become occupied at rate λ times the number of occupied neighbors. See Liggett [1, 2] for a construction of the process, additional information about it, and a proof of its phase transition: if $\xi_0 = \{0\}$, then there exists $\lambda_c \in (0, \infty)$ such that if $\lambda > \lambda_c$, then $P(\xi_t \neq \emptyset \text{ for all } t) > 0$, while if $\lambda \leq \lambda_c$, then $P(\xi_t \neq \emptyset \text{ for all } t) = 0$. We consider the contact process on a finite segment; let ζ_t^N be the Markov chain with state space $2^{\{1,\ldots,N\}}$ and transition rates given by (1) for $A \subset \{1, \ldots, N\}$ and $j \in \{1, \ldots, N\}$, with $\zeta_0^N = \{1, \ldots, N\}$. In the finite case, for all λ,

$$\sigma_N = \inf\{t \geq 0 : \zeta_t^N = \emptyset\} < \infty \text{ a.s.}$$

Durrett and Liu [3] and Durrett and Schonmann [4] show, via the following theorem, that the phase transition appears in the finite process in the limit as $N \to \infty$. Here γ_1 and γ_2 are deterministic functions of λ that are defined in the next section.

Theorem 1 ([3, 4]) *If $\lambda < \lambda_c$ then as $N \to \infty$,*

$$\frac{\sigma_N}{\log N} \to \frac{1}{\gamma_1(\lambda)}$$

in probability. If $\lambda > \lambda_c$ then as $N \to \infty$,

$$\frac{\log \sigma_N}{N} \to \gamma_2(\lambda)$$

in probability.

In words, σ_N grows logarithmically with N when $\lambda < \lambda_c$ and exponentially with N when $\lambda > \lambda_c$. When $\lambda = \lambda_c$, Durrett, Schonmann, and Tanaka [5] show that σ_N grows polynomially with N, but the correct power is unknown. We do not study the critical process here. Instead, we explore the following design question ensuing from this phenomenological result. Suppose one can vary the reproduction rate from point to point. How should this be done to maximize the asymptotic rate of growth of σ_N with N?

We restrict our attention to piecewise-constant rate profiles. That is, a *profile* (K, λ, α) consists of K rates $\lambda_1, \lambda_2, \dots, \lambda_K$, along with nonnegative constants $\alpha_1, \dots, \alpha_K$, such that $\sum_{j=1}^{K} \alpha_j = 1$. To construct the process of size N, we let $i_0 = 0$ and for $j \in \{1, \dots, K\}$, we let $i_j = \lfloor \sum_{k=1}^{j} \alpha_k N \rfloor$. Throughout we assume that N is sufficiently large that $i_{j-1} < i_j$ for all $j \in \{1, \dots, K\}$. Our contact process is then the Markov chain with state space $2^{\{1,\dots,N\}}$ and transition rates

$$\begin{aligned} q(A, A\backslash\{j\}) &= 1, \text{ if } j \in A \\ q(A, A \cup \{j\}) &= \lambda(j-1)|A \cap \{j-1\}| + \lambda(j+1)|A \cap \{j+1\}|, \text{ if } j \notin A, \end{aligned}$$

for $j \in \{1, \dots, N\}$, where (here and below) $\lambda(k) = \lambda_m$ where m satisfies $i_{m-1} < k \le i_m$.

We generalize Theorem 1 to these piecewise-homogeneous processes in Theorems 3 through 5. We then consider the optimization problem mentioned above. A simple coupling argument [2, p. 34] shows that increasing a $\lambda_i$incr eases the asymptotic growth rate of σ_N with N. We consider the problem of choosing the profile to maximize this growth rate subject to an upper bound on the average rate. Specifically, we consider the following optimization problem.

$$\begin{array}{ll} \text{maximize} & \liminf_{N\to\infty} (\log E(\sigma_N))/N \\ \text{over} & K, \lambda, \alpha \\ \text{subject to} & \sum_{j=1}^{K} \alpha_j \lambda_j \le \lambda_0 + \eta \\ & \lambda_j \ge \lambda_0 \text{ for all } j \in \{1, \dots, K\}. \end{array} \tag{2}$$

Here $\lambda_0 \ge 0$ and $\eta \ge 0$ are the data of the problem. We view λ_0 as the intrinsic rate endowed to each point, and η as the additional rate that we distribute over the points as we choose. We will prove that as long as $\eta > 0$, the maximum growth rate is exponential, and we write $R^*(\lambda_0, \eta)$ for the maximum achievable exponent, that is, for the supremum of $\liminf(\log E[\sigma_N])/N$ over the set of feasible profiles.

In the context of population-growth models, our optimization problem can be described as follows. Consider a population that lives on $\{1, \dots, N\}$ whose presence is desirable and that evolves as a contact process. The members of this population reproduce at a nominal rate λ_0, and we are provided with an amount η of "fertilizer" that we distribute over the points $\{1, \ldots, N\}$. Placing an amount ϵ of fertilizer at a point increases by ϵ the reproduction rate of members of the population who occupy that point. How then should the fertilizer be distributed to maximize the longevity of the population?

The optimization problem also arises in other contexts in which the contact process can be used. Consider, for example, the following caricature of a special kind of communication network, called a *wireless sensor network*. In order to track a vehicle moving in the 2-D plane, we drop an array of N radio-equipped sensors in a line near the vehicle. Each sensor detects a signal emitted by the vehicle and uses it to estimate the vehicle's bearing relative to the sensor. Periodically, the sensor broadcasts this information to a basestation, which uses the information received from all of the sensors to triangulate the position of the vehicle. The nodes broadcast asynchronously.

Occasionally, the signal received by a sensor becomes too noisy for the sensor to make a meaningful estimate of the vehicle's bearing. We assume that once this occurs, the sensor is unable to reacquire the signal on its own. We assume, however, that a broadcast by one of the neighboring nodes contains enough information about the vehicle's position for the node to reacquire the signal and continue tracking the vehicle. If we assume that a broadcast enables only one of the broadcasting node's neighbors to reacquire the signal, which would be the case if they used directed antennae, then we can model the randomness using a contact process, where the state refers to the set of nodes that are currently tracking the vehicle. Eventually, then, the network will reach the state in which every node has lost the target; the network designer seeks to maximize the time until this happens. Increasing a node's broadcast rate increases the power it consumes while it tracks the vehicle. Networks of this sort are typically power limited [6], so a rate constraint is a natural one. We arrive at our optimization problem.

Our solution to the optimization problem is as follows. We express $R^*(\lambda_0, \eta)$ in terms of the concave hull of $\gamma_2(\lambda)$ from Theorem 1, and we show that the optimum exponent is achieved by a profile with $K = 2$. We are unable to characterize the optimum profile further due to difficulty in characterizing γ_2. This difficulty is exemplified by a scaling theory conjecture combined with numerical simulations of critical exponents suggesting that $\gamma_2(\lambda)$ might have an inflection point to the right of λ_c. Section 4 contains additional details. We also consider the analogous optimization problem for the simpler biased voter model. For this process we provide a complete solution, which is given in Section 5.

Other interacting particle systems lend themselves to questions of this sort. In the context of the Ising model, consider a fixed volume of N magnetic materials, with varying magnetic strengths. These materials are combined in some way, then magnetized to store one bit of information. If the total volume of the materials is one, how should one arrange the materials within, say, the unit cube to maximize the time until the magnetization is lost?

Similar questions for two-dimensional site percolation have been studied by Carlson and Doyle [7, 8] in the context of power laws in complex systems. Robert, Carlson, and Doyle [9] consider, in the same context, the effect of design on a simple epidemic model in which infection spreads between three cells. There is a significant amount of work on the infinite contact process with inhomogeneous rates [2, p. 131]. Most of this work considers models in which the rates are random, and we are not aware of any work on the finite process.

The remainder of the paper is organized as follows. Section 2 contains the required background on the contact process, including the graphical construction and its planar-graph duality that is key to the later proofs. Section 3 describes the biased voter model. Section 4 contains our main results, which are the hitting time asymptotics and the solution to the optimization problem for the contact process.

Section 5 contains the analogous results for the biased voter model. Sections 6 and 7 contain the proofs of the contact process results and biased voter model results, respectively.

2 Contact Process Preliminaries

The functions γ_1 and γ_2 mentioned in the introduction are defined in terms of the infinite process. For more information about the following definitions and for proofs of the assertions see Liggett [2]. Let $(\xi_t^A)_{t\geq 0}$ be the contact process on $\mathbb{Z}$ with initial state A. The function γ_1 is defined as

$$\gamma_1(\lambda) = -\lim_{t\to\infty}\frac{1}{t}\log P\left(\xi_t^{\{0\}} \neq \emptyset\right).$$

The existence of the limit is proven using subadditivity: the process has at least one occupied point whenever it is alive, so

$$P(\xi_{t+s}^{\{0\}} \neq \emptyset | \xi_t^{\{0\}} \neq \emptyset) \geq P(\xi_s^{\{0\}} \neq \emptyset),$$

which implies

$$P(\xi_{t+s}^{\{0\}} \neq \emptyset) \geq P(\xi_s^{\{0\}} \neq \emptyset)P(\xi_t^{\{0\}} \neq \emptyset).$$

Thus $-\log P(\xi_t^{\{0\}} \neq \emptyset)$ is subadditive in t, which implies [2, Theorem B22] that

$$-\frac{1}{t}\log P(\xi_t^{\{0\}} \neq \emptyset)$$

converges to its infimum, which is positive if $\lambda < \lambda_c$. In particular,

$$P(\xi_t^{\{0\}} \neq \emptyset) \leq \exp(-\gamma_1 t).$$

Let

$$\tau^A = \inf\{t \geq 0 : \xi_t^A = \emptyset\}.$$

The function γ_2 is defined as

$$\gamma_2(\lambda) = -\lim_{N\to\infty}\frac{1}{N}\log P\left(\tau^{\{1,\ldots,N\}} < \infty\right).$$

This limit also exists for all λ by subadditivity, but it is positive if $\lambda > \lambda_c$.

Later we will use a third limiting function. Let $r_t = \sup \xi_t^{\{0\}}$ and let $R = \sup_{t\geq 0} r_t$. Then

$$\gamma_3(\lambda) = -\lim_{n\to\infty}\frac{1}{n}\log P(R \geq n)$$

exists, again by subadditivity, for all λ and it is positive if $\lambda < \lambda_c$. We often omit the explicit dependence of these limits on λ.

The key to the proof of Theorem 1 is the graphical representation of the contact process. Since we will make heavy use of it, we review it here.

The homogeneous contact process with a deterministic initial state can be constructed graphically from a countable number of Poisson processes: one with rate 1 and two with rate λ for each n. The vertical axis in this representation represents time while the horizontal axis represents space. We draw

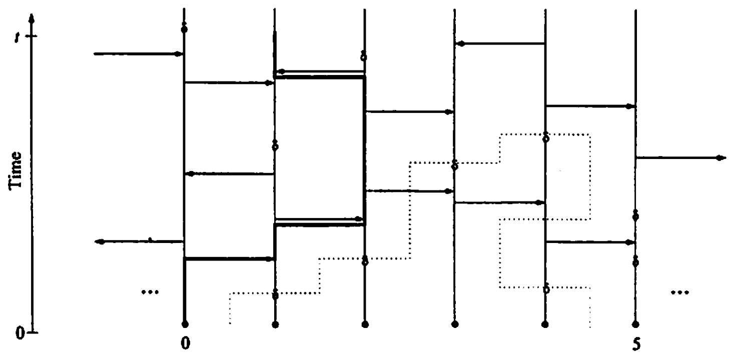


Figure 1: The graphical representation of the contact process.

(1) δ's above n at the arrival times of the rate-1 process, (2) arrows from n to $n-1$ at the arrival times of the first rate-λ process, and (3) arrows from n to $n+1$ at the arrival times of the second rate-λ process. Figure 1 shows a sample realization. The δ's represent potential deaths and the arrows represent potential births.

We say there is a contact process path from (i, s) to (j, t) if one can travel from (i, s) to (j, t) by combinations of (1) moving up while directly over integers without passing through a δ, and (2) moving horizontally from one integer to a neighboring one through an arrow. The bold line in Figure 1 is an example of a valid contact process path from $(0, 0)$ to $(1, t)$. We construct the contact process by setting

$$\xi_t^A = \{j \in \mathbb{Z} : \text{there is a contact process path from } (i, 0) \text{ to } (j, t) \text{ for some } i \in A\}.$$

In this paper, the graphical representation derives its utility from the notion of dual paths, due to Durrett and Schonmann [4]. Motivated by duality in percolation, we say there is a dual path from (i, s) to (j, t) if one can travel from (i, s) to (j, t) while observing the following rules:

1. The path may move upward over half integers but not through a right arrow.

2. The path may move downward over half integers but not through a left arrow.

3. The path may move horizontally from a half integer to the next lowest half integer only through δ's.

4. The path may move horizontally to the right between half-integer points without restriction.

The dotted line in Figure 1 is an example of a valid dual path from $(4.5, 0)$ to $(0.5, 0)$. That this is the appropriate way of defining dual paths can be seen by constructing the contact process from a sequence of increasingly-fine oriented percolations, and then allowing it to inherit their dual path rules [4]. Or one can verify Proposition 2 in Durrett and Schonmann [4].

Proposition 1 *There is a dual path from $(N + 1/2, 0)$ to $(1/2, 0)$ in $(-\infty, \infty) \times (0, T)$ if and only if there is no contact process path from $(n, 0)$ to (m, T) for all $n \in \{1, \ldots, N\}$ and all $m \in \mathbb{Z}$.*

We omit the proof but note that dual paths are defined so that a contact process path from $(n,0)$ to (m,T) and a dual path from $(N+1/2,0)$ to $(1/2,0)$ in $(-\infty,\infty)\times(0,T)$ can never intersect, and the boundary of the set of space-time points for which there is a contact process path from a point in $\{1,\ldots,N\}\times\{0\}$ is a valid dual path. The proof follows quickly from these observations.

By taking $T\to\infty$ in Proposition 1 we obtain a useful corollary,

$$\left\{\xi_t^{\{1,\ldots,N\}}=\emptyset \text{ for some } t\right\}=\{\exists \text{ a dual path from } (N+1/2,0) \text{ to } (1/2,0) \text{ in } (-\infty,\infty)\times(0,\infty)\}, \tag{3}$$

which implies

$$\lim_{N\to\infty}\frac{1}{N}\log P(\text{there is a dual path from } (N+1/2,0) \text{ to } (1/2,0) \text{ in } (-\infty,\infty)\times(0,\infty))=-\gamma_2. \tag{4}$$

Since we will be dealing with inhomogeneous processes, we note that we can construct a process $\tilde{\xi}_t^A$ in which point n has a lower reproduction rate, $\lambda'\le\lambda$, using the graphical representation by thinning the Poisson process of arrows leading from n with retention probability λ'/λ. Since removing arrows from the graphical representation does not create new contact process paths, we have $\tilde{\xi}_t^A\subset\xi_t^A$ for all $t\ge 0$.

3 The Biased Voter Model

The biased voter model is similar to the contact process except that the rate of a point transitioning from state 1 to 0 is equal to the number of neighbors in state 0, rather than constant. More precisely, it is the Markov chain with state space $2^{\{1,\ldots,N\}}$ and transition rates

$$\begin{aligned} q(A,A\backslash\{j\}) &= |(\mathbb{Z}\backslash A)\cap\{j-1,j+1\}| \quad \text{if } j\in A,\\ q(A,A\cup\{j\}) &= \lambda|A\cap\{j-1,j+1\}| \quad \text{if } j\notin A, \end{aligned}$$

for $j\in\{1,\ldots,N\}$. A point in state 1 with two neighbors in state 1 cannot change states, so we can construct the biased voter model on $\{1,\ldots,N\}$ with initial state $\{1,\ldots,N\}$ using two random walks: Let L_t be a random walk on $\{1,2,\ldots\}$ that moves to the left at rate λ and moves to the right at rate 1 with $L_0=1$. Let R_t be a random walk on $\{\ldots,N-1,N\}$ that moves to the right at rate λ and moves to the left at rate 1, with $R_0=N$. If we use ζ_t^N to denote the finite biased voter model at time t and define σ_N as before then we have

$$\zeta_t^N=\{L_t,\ldots,R_t\} \quad \text{for } t\le\sigma_N \tag{5}$$

and $\sigma_N=\inf\{t: R_t<L_t\}$. Similar to the contact process, L_t and R_t can be constructed graphically from Poisson processes. We construct $L_t$by placing arrows at rate λ from n to $n-1$ and arrows at rate 1 from n to $n+1$ for $n\ge 2$. Point 1 is similar except that we omit the arrows directed toward 0. Then L_t starts at $(1,0)$ and evolves in time by moving upward and following each arrow. We can construct R_t similarly.

For the biased voter model, we can state the analogue of Theorem 1 without resorting to definitions involving the process on $\mathbb{Z}$.

Theorem 2 ([3]) *If* $\lambda < 1$ *then as* $N \to \infty$,

$$\frac{\sigma_N}{N} \to \frac{1}{2(1-\lambda)}$$

in probability. If $\lambda > 1$ *then as* $N \to \infty$,

$$\frac{\log \sigma_N}{N} \to \log(\lambda)$$

in probability.

4 Contact Process Results

Our first step is to generalize Theorem 1 to piecewise-homogeneous processes. The case in which the entire process is subcritical is immediate; we provide it for completeness.

Theorem 3 *Let* (K, λ, α) *be a profile such that* $\lambda_j < \lambda_c$ *for all* $j \in \{1, \ldots, K\}$. *Then*

$$\frac{\sigma_N}{\log N} \to \frac{1}{\gamma_1(\max(\lambda_1, \ldots, \lambda_K))}$$

in probability as $N \to \infty$.

In the subcritical case, each partition dies before spreading very far into its neighboring partitions, so the partitions essentially evolve independently, and σ_N is determined by the extinction times of the partitions with the maximum rate. In the supercritical case, the partitions interact in a significant way.

Theorem 4 *Let* (K, λ, α) *be a profile such that* $\lambda_j > \lambda_c$ *for all* $j \in \{1, \ldots, K\}$. *Then*

$$\frac{\log \sigma_N}{N} \to \sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j)$$

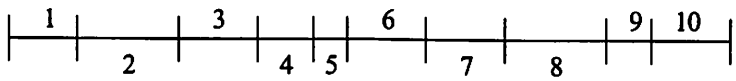
in probability as $N \to \infty$, *and*

$$\frac{\log E[\sigma_N]}{N} \to \sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j).$$

The proof essentially shows that the process dies out only when all of the individual partitions die out simultaneously. The chance that partition j evolving in isolation dies out in a short time interval is $\exp(-\alpha_j \gamma_2(\lambda_j) N + o(N))$. Over short time scales, the partitions are nearly independent, so the chance that the entire process dies out in a short time interval is $\exp(-\sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j) N + o(N))$. It then follows that the hitting time is $\exp(\sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j) N + o(N))$.

Our result about mixed profiles is incomplete. To state it, we require additional notation. Let F be the set of indices j such that i_j separates supercritical and nonsupercritical partitions,

$$F = \{j \in \{1, \ldots, K-1\} : (\lambda_j \wedge \lambda_{j+1}) \leq \lambda_c < (\lambda_j \vee \lambda_{j+1})\}.$$



$$C_1 = \{1\}, C_2 = \{2\}, C_3 = \{3\}, C_4 = \{4,5\}, C_5 = \{6\}, C_6 = \{7,8\}, C_7 = \{9,10\},$$
$$D_1 = \{1\}, D_2 = \{3\}, D_3 = \{6\}, D_4 = \{9,10\}$$

Figure 2: A sample mixed profile. The supercritical partitions have their index placed above the line. The subcritical, below.

Now $M = |F| + 1$ is the number of "aggregate partitions," sets of partitions that are connected, entirely supercritical or not, and maximal in that adding another partition either makes the set unconnected, or mixed. We denote these aggregate partitions by C_j for $j \in 1, \ldots, M$:

$$C_1 = \{1, \ldots, \inf F \cup \{K\}\}$$
$$C_j = \{\sup C_{j-1} + 1, \ldots, \inf\{k \in F : k > \sup C_{j-1}\} \cup \{K\}\}.$$

Let L be the number of aggregate partitions that are supercritical, so $L = \lceil M/2 \rceil$ if $\lambda_1 > \lambda_c$, otherwise $L = \lfloor M/2 \rfloor$. We call a C_j consisting of supercritical partitions an *island*, and a $C_j$consisti ng of nonsupercritical partitions a *sea*. Let D_j for $j \in \{1, \ldots, L\}$ denote the islands, which are the C_j's with even or odd indices depending on whether $\lambda_1 < \lambda_c$or $\lambda_1 > \lambda_c$, respectively. Figure 2 shows an example. Throughout, we interpret an empty sum as zero.

Theorem 5 *Let (K, λ, α) be a profile. Then*

$$P\left(\frac{\log \sigma_N}{N} < \max_{j \in \{1,\ldots,L\}} \left(\sum_{i \in D_j} \alpha_i \gamma_2(\lambda_i)\right) - \epsilon\right) \to 0,$$
$$P\left(\frac{\log \sigma_N}{N} > \sum_{j=1}^{L} \sum_{i \in D_j} \alpha_i \gamma_2(\lambda_i) + \epsilon\right) \to 0,$$

for all $\epsilon > 0$ as $N \to \infty$, and

$$\liminf_{N \to \infty} \frac{\log E[\sigma_N]}{N} \geq \max_{j \in \{1,\ldots,L\}} \left(\sum_{i \in D_j} \alpha_i \gamma_2(\lambda_i)\right)$$
$$\limsup_{N \to \infty} \frac{\log E[\sigma_N]}{N} \leq \sum_{j=1}^{L} \sum_{i \in D_j} \alpha_i \gamma_2(\lambda_i).$$

The difficulty is determining when the seas isolate the islands into separate processes. The lower and upper bounds in Theorem 5 correspond to two possible answers to this question, "always" and "never." If the islands are isolated then the extinction time of the process is just the extinction time of its longest-living island, giving an exponent of $\max_{j \in \{1,\ldots,L\}} \left(\sum_{i \in D_j} \alpha_i \gamma_2(\lambda_i)\right)$. If the population can spread from one island to another, a process we call *colonizing*, then the process dies out only when all of the islands die out simultaneously. By the discussion following Theorem 4, this gives an exponent of $\sum_{j=1}^{L}\left(\sum_{i\in D_j}\alpha_i\gamma_2(\lambda_i)\right)$.

We conjecture that the correct answer is "sometimes"; whether a sea prevents two islands from colonizing depends on their sizes and reproduction rates. To support this conjecture, consider the time the process takes to spread across a homogeneous subcritical region of width N. Bramson, Durrett, and Schonmann [10] prove the following.

Proposition 2 *Consider a modified subcritical contact process on $\mathbb{Z}$, $\bar{\xi}_t$, in which $\bar{\xi}_0 = \{0\}$ and 0 is always occupied. Let $\Delta_N = \inf\{t \geq 0 : N \in \bar{\xi}_t\}$. As $N \to \infty$,*

$$\frac{\log \Delta_N}{N} \to \gamma_3(\lambda)$$

in probability.

The intuition behind the result is that each time point 0 spreads to point 1, the process started with only $\{1\}$ occupied spreads to N before becoming extinct with probability $\exp(-\gamma_3(\lambda)N + o(N))$, and in an interval of length T, the number of chances for this to occur is proportional to T. For the piecewise-homogeneous process, we show that the chance of the process started with only $\{1\}$ occupied spreading to N before becoming extinct is $\exp(-\sum_{j=1}^{K}\alpha_j\gamma_3(\lambda_j)N + o(N))$, and thereby prove the following.

Proposition 3 *Let (K, λ, α) be a profile such that $\lambda_j < \lambda_c$ for all $j \in \{1, \ldots, K\}$. For each N, let $\tilde{\xi}_t^N$ be the piecewise-homogeneous contact process modified so that 1 is always occupied, and let*

$$\Delta_N = \inf\{t \geq 0 : N \in \tilde{\xi}_t^N\}.$$

Then as $N \to \infty$,

$$\frac{\log \Delta_N}{N} \to \sum_{j=1}^{K}\alpha_j\gamma_3(\lambda_j)$$

in probability.

Consider a profile with $K = 3$, $\alpha_i > 0$ for $i \in \{1, 2, 3\}$, and $\lambda_1 > \lambda_c$, $\lambda_3 > \lambda_c$, but $\lambda_2 < \lambda_c$. Theorem 1 gives the extinction times of the supercritical partitions when they evolve in isolation, namely $\exp(\alpha_1\gamma_2(\lambda_1)N + o(N))$ and $\exp(\alpha_3\gamma_2(\lambda_3)N + o(N))$. Proposition 2 gives the time until there is a contact process path across the subcritical region, namely $\exp(\alpha_2\gamma_3(\lambda_2)N + o(N))$.

If $\alpha_2\gamma_3(\lambda_2) > \max(\alpha_1\gamma_2(\lambda_1), \alpha_3\gamma_2(\lambda_3))$, then the chance that one of the supercritical partitions ever colonizes tends to zero as $N \to \infty$. In this case, we conjecture that the exponent for the extinction time of the entire process is $\max(\alpha_1\gamma_2(\lambda_1), \alpha_3\gamma_2(\lambda_3))$.

If $\alpha_2\gamma_3(\lambda_2) < \min(\alpha_1\gamma_2(\lambda_1), \alpha_3\gamma_2(\lambda_3))$, then if one of the supercritical partitions dies out, the other partition has infinitely many chances (in the limit) to restart it by colonizing. In this case we expect the population to die out only when both partitions die out simultaneously, giving the exponent $\alpha_1\gamma_2(\lambda_1) + \alpha_3\gamma_2(\lambda_3)$.

If $\min(\alpha_1\gamma_2(\lambda_1), \alpha_3\gamma_2(\lambda_3)) < \alpha_2\gamma_3(\lambda_2) < \max(\alpha_1\gamma_2(\lambda_1), \alpha_3\gamma_2(\lambda_3))$, then when the partition with exponent $\max(\alpha_1\gamma_2(\lambda_1), \alpha_3\gamma_2(\lambda_3))$ dies out, with probability approaching 1, the other will die out before colonizing. In this case we expect the exponent to be $\max(\alpha_1\gamma_2(\lambda_1), \alpha_3\gamma_2(\lambda_3))$.

$$\alpha_1\gamma_2(\lambda_1) = 2,\ \alpha_2\gamma_3(\lambda_2) = 1,\ \alpha_3\gamma_2(\lambda_3) = 2,$$
$$\alpha_4\gamma_3(\lambda_4) + \alpha_5\gamma_3(\lambda_5) = 3,\ \alpha_6\gamma_2(\lambda_6) = 4,$$
$$\alpha_7\gamma_3(\lambda_7) + \alpha_8\gamma_3(\lambda_8) = 5,\ \alpha_9\gamma_2(\lambda_9) + \alpha_{10}\gamma_2(\lambda_{10}) = 3.$$

Figure 3: Sample exponents.

Using Theorem 4 and Proposition 3, we can extend this reasoning to processes with more than three partitions. We illustrate the idea with an example. Consider again the process in Figure 2. Suppose, for the sake of discussion, that the equations in Figure 3 hold. Then our conjecture is that the first and third partitions will colonize to each other, so that the first and third will effectively act as a single island. This island will colonize across the sea $\{4,5\}$, as will the sixth partition, so $\{1,3,6\}$ will effectively act as a single island. Although this island can colonize across the sea $\{7,8\}$, the island $\{9,10\}$ cannot, so we conjecture that the exponent for this profile is $\alpha_1\gamma_2(\lambda_1) + \alpha_3\gamma_2(\lambda_3) + \alpha_6\gamma_2(\lambda_6)$. We hope that the reader can see how to extend the conjecture to an arbitrary profile.

Since we can resolve the optimization problem without knowing the validity of this conjecture, we will not investigate it further. Let $\hat{\gamma}_2^{\lambda_0}(\lambda)$ denote the concave hull of $\gamma_2(\lambda)$ on $[\lambda_0, \infty)$, i.e., for $x \geq \lambda_0$,

$$\hat{\gamma}_2^{\lambda_0}(x) = \sup\left\{\sum_{j=1}^{n}\alpha_j\gamma_2(\lambda_j) : \sum_{j=1}^{n}\alpha_j\lambda_j = x \text{ and } \lambda_j \geq \lambda_0 \text{ for } j = 1,\ldots,n\right\},$$

where the supremum is over n, α, and λ.

Theorem 6 *$R^*(\lambda_0,\eta) = \hat{\gamma}_2^{\lambda_0}(\lambda_0+\eta)$. Furthermore, $R^*(\lambda_0,\eta)$ is achieved by a profile with $K = 2$.*

The sufficiency of profiles with two rates follows from Carathéodory's Theorem [11, p. 155] and some continuity arguments. Clearly at least two rates are required in the case that $\lambda_0 + \eta < \lambda_c$, since then the constraint forbids us from making the entire process supercritical, so the optimum exponent is approached by making part of the process supercritical, and leaving the rest at λ_0. If γ_2 is concave on $[\lambda_c, \infty)$, then multiple partitions are required only if $\lambda_0 < \lambda_c$ and $\lambda_0 + \eta$ is sufficiently small. More precisely, if γ_2 is concave on $[\lambda_c, \infty)$ and $\lambda_0 \geq \lambda_c$, then $\hat{\gamma}_2^{\lambda_0} = \gamma_2$ on (λ_0, ∞) so $R^*(\lambda_0,\eta) = \gamma_2(\lambda_0+\eta)$, and an optimum profile would consist of a single partition with rate $\lambda_0+\eta$. And if $\lambda_0 < \lambda_c$, there would exist $\lambda^* > \lambda_c$ such that

$$\hat{\gamma}_2^{\lambda_0}(\lambda) = \begin{cases} \gamma_2(\lambda^*)\frac{\lambda-\lambda_0}{\lambda^*-\lambda_0} & \text{if } \lambda < \lambda^* \\ \gamma_2(\lambda) & \text{if } \lambda \geq \lambda^*, \end{cases} \tag{6}$$

and so if $\lambda_0 + \eta \geq \lambda^*$, an optimum profile would consist of a single partition with rate $\lambda_0 + \eta$, and if $\lambda_0+\eta < \lambda^*$, then an optimum profile would consist of two partitions, one with rate λ^* and another with rate λ_0.

One might expect γ_2 to be concave on (λ_c, ∞) since it is nondecreasing and depends on λ primarily through a comparison to the death rate, which is 1. Thus we expect the effect of increasing λ by Δ to diminish as λ increases. Indeed, γ_2 increases at most logarithmically: if all points in $\{1,\ldots,N\}$ die out before reproducing, then $\tau^{\{1,\ldots,N\}} < \infty$ so

$$P\left(\tau^{\{1,\ldots,N\}} < \infty\right) \geq \left(\frac{1}{1+2\lambda}\right)^N,$$

which gives $\gamma_2(\lambda) \le \log(1+2\lambda)$. But we are unable to prove that γ_2 is concave on $[\lambda_c, \infty)$; in fact we suspect that it is not.

Scaling theory predicts that the contact process has a natural length of scale, $L_\perp(\lambda)$, that tends to infinity as a power as $\lambda \to \lambda_c$,

$$\lim_{\lambda\to\lambda_c} \frac{\log L_\perp(\lambda)}{\log(\lambda-\lambda_c)} = -\alpha \tag{7}$$

for some $\alpha > 0$ [12]. We have expressed the convergence in the logarithmic sense but in reality the nature of the convergence is unclear. Accurate but nonrigorous simulations place α, assuming it exists, at 1.09681 [13] and 1.09684 [14]. The natural way of defining $L_\perp$ mathematically for the supercritical process is the following. Let ν_λ be the upper invariant measure of the homogeneous process on $\mathbb{Z}$ with rate λ, and let

$$C_\lambda(n,m) = \nu_\lambda(A : n \in A \text{ and } m \in A) - \nu_\lambda(A : n \in A)^2.$$

Assuming $C_\lambda(n,m) \to 0$ exponentially as $m - n \to \infty$, we define

$$(L_\perp(\lambda))^{-1} = -\lim_{n\to\infty} \frac{1}{n} \log C_\lambda(0,n) \tag{8}$$

The link to γ_2 is due to Durrett, Schonmann, and Tanaka [12], who show that for all $n \ge 0$ and $\lambda \ge 0$,

$$C_\lambda(0,n) \le \exp(-\gamma_2(\lambda)n). \tag{9}$$

If (7) and (8) hold with $\alpha > 1$, then (9) would imply that γ_2 is not concave near λ_c. We are unable to resolve this issue, but note that even if γ_2 is not concave near λ_c, we do not expect it to have more than one inflection point to the right of λ_c, and a single inflection point would not alter the solution to the optimization problem much over the concave case. Thus we conjecture the following.

Conjecture 1 *There exists $\lambda_{c_2} \ge \lambda_c$ such that γ_2 is convex on $[\lambda_c, \lambda_{c_2}]$ and concave on $[\lambda_{c_2}, \infty)$. If $\lambda_0 \ge \lambda_{c_2}$, then $R^*(\lambda_0, \eta) = \gamma_2(\lambda_0 + \eta)$ and $R^*(\lambda_0, \eta)$ is achieved by the profile $(1, \lambda_0 + \eta, 1)$. If $\lambda_0 < \lambda_{c_2}$ then R^* is achieved by a profile with at most two partitions, at most one of whose rate is not λ_0.*

Being unable to validate this conjecture, we cannot assert that two partitions are required only when $\lambda_0 < \lambda_c$ and $\lambda_0+\eta$ is sufficiently small, and in this case that at most one of the partitions is supercritical. However, for the simpler biased voter model, to which we turn next, we can make such an assertion.

5 Biased Voter Model Results

Again we consider piecewise-homogeneous processes. Our definition of a profile remains the same, but now given the profile (K, λ, α), we consider the Markov chain ζ_t^N with initial state $\{1, \ldots, N\}$ and transition rates

$$q(A, A\backslash\{j\}) = |(\mathbb{Z}\backslash A) \cap \{j-1, j+1\}| \quad \text{if } j \in A$$
$$q(A, A \cup \{j\}) = \lambda(j-1)|A \cap \{j-1\}| + \lambda(j+1)|A \cap \{j+1\}| \quad \text{if } j \notin A.$$

We can construct this process from two random walks by modifying the construction used for the homogeneous process. Let L_t be a random walk on $\mathbb{N}$, starting at 1, with transition rates

$$q(n,n+1) = 1$$

$$q(n,n-1) = \begin{cases} \lambda_1 & \text{if } 2 \le n \le i_1 \\ \lambda_2 & \text{if } i_1+1 \le n \le i_2 \\ \vdots & \\ \lambda_K & \text{if } i_{K-1}+1 \le n \end{cases},$$

and let R_t be a random walk on $\{\ldots, N-1, N\}$, starting at N, with transition rates

$$q(n,n-1) = 1$$

$$q(n,n+1) = \begin{cases} \lambda_1 & \text{if } n \le i_1 \\ \lambda_2 & \text{if } i_1+1 \le n \le i_2 \\ \vdots & \\ \lambda_K & \text{if } i_{K-1}+1 \le n \le N-1 \end{cases}.$$

Then we can construct ζ_t^N by (5) as with the homogeneous process. In the sequel we refer to this as the *edge construction* of the piecewise-homogeneous biased voter model.

Theorem 7 *If (K, λ, α) is a profile such that $\lambda_j < 1$ for all $j \in \{1, \ldots, K\}$, then*

$$\frac{\sigma_N}{N} \to \sum_{j=1}^{K} \frac{\alpha_j}{2(1-\lambda_j)}$$

in probability as $N \to \infty$. If $\lambda_j > 1$ for all $j \in \{1, \ldots, K\}$, then

$$\frac{\log \sigma_N}{N} \to \sum_{j=1}^{K} \alpha_j \log(\lambda_j)$$

in probability as $N \to \infty$, and

$$\frac{\log E[\sigma_N]}{N} \to \sum_{j=1}^{K} \alpha_j \log(\lambda_j).$$

Theorem 8 *Theorem 5 holds for the biased voter model if we replace λ_c with 1 and $\gamma_2(\lambda)$ with $\log^+ \lambda$.*

Determining the correct exponents for mixed profiles should be relatively easy for the biased voter model. We do not explore this here because our interest in the biased voter model is its solution to the optimization problem.

Theorem 9 *For the biased voter model, if $\lambda_0 \ge 1$, then $R^*(\lambda_0, \eta) = \log(\lambda_0 + \eta)$ is achieved by the profile $(1, \lambda_0 + \eta, 1)$. If $\lambda_0 < 1$, let λ_1 be the unique solution to*

$$1 - \frac{\lambda_0}{\lambda_1} = \log \lambda_1$$

that is greater than 1. If $\lambda_0 + \eta \ge \lambda_1$, then again $R^*(\lambda_0, \eta) = \log(\lambda_0 + \eta)$. If $\lambda_0 + \eta < \lambda_1$, then $R^*(\lambda_0, \eta) = \eta/\lambda_1$ is achieved by the profile $(2, (\lambda_1, \lambda_0), (\alpha, 1-\alpha))$, where $\alpha = \eta/(\lambda_1 - \lambda_0)$.

6 Proofs of Contact Process Results

Theorem 3 follows from two simple couplings.

Proof of Theorem 3. Consider the homogeneous process on $\{1, \dots, N\}$ with reproduction rate $\lambda = \max(\lambda_1, \dots, \lambda_K)$, and call its extinction time $\overline{\sigma}_N$. The piecewise-homogeneous process can be coupled to this homogeneous process such that $\sigma_N \leq \overline{\sigma}_N$, as described in Section 2. Then $\overline{\sigma}_N / \log N \to 1/\gamma_1(\lambda)$ by Theorem 1, which shows the upper bound. For the lower bound, choose $j \in \arg\max\{\lambda_1, \dots, \lambda_K\}$ and couple the piecewise-homogeneous process to the homogeneous process on $\{i_{j-1}, \dots, i_j\}$ formed by forbidding births to occur from i_{j-1} to $i_{j-1}+1$ and from i_j+1 to i_j in the piecewise-homogeneous process. Let $\underline{\sigma}_N$ denote the extinction time of this homogeneous process. Then $\underline{\sigma}_N \leq \sigma_N$, and

$$\begin{aligned} P\left(\frac{\sigma_N}{\log N} < \frac{1}{\gamma_1(\lambda)} - \epsilon\right) &\leq P\left(\frac{\underline{\sigma}_N}{\log N} < \frac{1}{\gamma_1(\lambda)} - \epsilon\right) \\ &\leq P\left(\frac{\underline{\sigma}_N}{\log(i_j - i_{j-1})} < \frac{1}{\gamma_1(\lambda)} - \frac{\epsilon}{2}\right) \\ &\quad + P\left(\frac{\underline{\sigma}_N}{\log(i_j - i_{j-1})} - \frac{\underline{\sigma}_N}{\log N} > \frac{\epsilon}{2}\right) \to 0. \end{aligned}$$

□

We turn to the more interesting supercritical case. Theorems 4 and 5 are proved as a sequence of lemmas. First we define some events. Borrowing from Durrett and Schonmann [4], we write $a \to b$ for "there is a dual path from a to b in the graphical representation." Some of these events require that the graphical representation be constructed for both positive and negative time.

$$\begin{aligned} A_N &= \{(N+1/2, 0) \to (1/2, 0) \text{ in } (-\infty, \infty) \times [0, \infty)\} \\ A_N^T &= \{(N+1/2, 0) \to (1/2, 0) \text{ in } (-\infty, \infty) \times [0, T]\} \\ B_N &= \{(N+1/2, 0) \to (1/2, 0) \text{ in } [1/2, N+1/2] \times [0, \infty)\} \\ B_N^T &= \{(N+1/2, 0) \to (1/2, 0) \text{ in } [1/2, N+1/2] \times [0, T]\} \\ C_N &= \{(N+1/2, 0) \to (1/2, t) \text{ in } [1/2, N+1/2] \times (-\infty, \infty) \text{ for some } t\} \\ C_N^T &= \{(N+1/2, 0) \to (1/2, t) \text{ in } [1/2, N+1/2] \times [-T, T] \text{f or some } t\} \\ D_N^T &= \{(N+1/2, s) \to (1/2, t) \text{ in } [1/2, N+1/2] \times [-T, T] \text{ for some } s \text{ and } t\} \end{aligned}$$

The scheme here is that the B events have both endpoints of the dual path fixed, while the C events have only one endpoint fixed and the D events have both free. When a superscript appears, it constrains the path vertically. Observe that $B_N^T \subset C_N^T \subset D_N^T$ and $B_N^T \subset A_N^T$. Note also that the B events depend on the birth arrows from N to $N+1$ and from 1 to 0, but not from $N+1$ to N or from 0 to 1. The C events depend on the birth arrows between N and $N+1$ in both directions, but not on the arrows between 0 and 1. The D events are independent of all birth arrows between N and $N+1$ and between 0 and 1. Note that A_N^T appeared in Proposition 1, while A_N appeared in its corollary (4).

All of these events refer to the homogeneous process on $\mathbb{Z}$. We will also find it convenient to use the B,C , and D events in the context of the piecewise-homogeneous process on $\{1, \dots, N\}$. When doing so, we place a tilde above the event (e.g. $\tilde{B}_N$) and we add arrows between N and $N+1$ at rate λ_K and between 0 and 1 at rate λ_1 to the graphical representation, since the B and C events use them.

Lemma 1 *There exist functions $\mu_k(\lambda)$, $\mu(\lambda)$, and $\nu_k(\lambda)$, which are positive on (λ_c, ∞), such that*

$$\lim_{N\to\infty} -\frac{1}{N}\log P(A_N^k) = \nu_k(\lambda) \quad \textit{and} \quad P(A_N^k) \le \exp(-\nu_k(\lambda)N) \tag{a}$$

$$\lim_{N\to\infty} -\frac{1}{N}\log P(B_N^k) = \mu_k(\lambda) \quad \textit{and} \quad P(B_N^k) \le \exp(-\mu_k(\lambda)N) \tag{b}$$

$$\lim_{N\to\infty} -\frac{1}{N}\log P(B_N) = \mu(\lambda) \quad \textit{and} \quad P(B_N) \le \exp(-\mu(\lambda)N) \tag{c}$$

The proof of this lemma is essentially the same as that of Lemma 1 in Durrett and Schonmann [4]. We provide a condensed version for completeness.

Proof. Consider (a). Let $A_{m,n}^k = \{(m+1/2,0) \to (n+1/2,0) \text{ in } (-\infty,\infty)\times[0,k]\}$. Then $A_N^k \cap A_{M+N,N}^k \subset A_{M+N}^k$ and A_N^k and $A_{M+N,N}^k$ are positively correlated by Harris' inequality [15] (see also [3]), so

$$P(A_{M+N}^k) \ge P(A_N^k \cap A_{M+N,N}^k) \ge P(A_N^k)P(A_{M+N,N}^k).$$

But $P(A_{M+N,N}^k) = P(A_M^k)$ so if we let $a_N = -\log P(A_N^k)$ then we see that a_N is subadditive, which proves all of (a) except that $\nu_k(\lambda) > 0$ if $\lambda > \lambda_c$. The convergence in parts (b) and (c) are proved similarly, although they do not require Harris' inequality since B_N and $B_{M+N,N}$ are independent as they depend on disjoint parts of the graphical representation. All of the limits are positive since $\gamma_2(\lambda) > 0$ on (λ_c, ∞) and $A_N^k \subset A_N$, $B_N^k \subset A_N$, and $B_N \subset A_N$. □

We wish to show that $\mu = \gamma_2$. We first show that $\mu_k \to \mu$ and $\nu_k \to \gamma_2$ as $k \to \infty$. The argument is from Durrett and Schonmann [4], who attribute it to J. Chayes and L. Chayes.

Lemma 2 *As $k \to \infty$, $\mu_k(\lambda) \to \mu(\lambda)$ and $\nu_k(\lambda) \to \gamma_2(\lambda)$ on $[0,\infty)$.*

Proof. Note that $\mu_k(\lambda)$ is decreasing in k and $\mu_k(\lambda) \ge \mu(\lambda)$ for all k. Fix $\epsilon > 0$ and note that for all sufficiently large N, we have

$$\exp(-(\mu(\lambda)+\epsilon)N) \le P(B_N).$$

Now $B_N^k \uparrow B_N$ as $k \to \infty$ so $P(B_N^k) \uparrow P(B_N)$ thus

$$\begin{aligned}\exp(-(\mu(\lambda)+\epsilon)N) &\le \lim_{k\to\infty} P(B_N^k)\\ &\le \lim_{k\to\infty} \exp(-\mu_k(\lambda)N)\\ &\le \exp\left(-\left(\lim_{k\to\infty}\mu_k(\lambda)\right)N\right).\end{aligned}$$

This implies

$$\mu(\lambda)+\epsilon \ge \lim_{k\to\infty}\mu_k(\lambda).$$

This shows that $\mu_k \to \mu$; the other is the same. □

To show that $\mu(\lambda) = \gamma_2(\lambda)$ on $[0,\infty)$, it suffices to show

Lemma 3 $\mu_k(\lambda) = \nu_k(\lambda)$ *on* $[0, \infty)$.

Proof. Note that $\mu_k \geq \nu_k$. Suppose that three events occur: A_N^k, {there is no arrow from N to $N+1$ during $[0, k]$}, and {there is no arrow from 1 to 0 during $[0, k]$}. Fix a dual path P from $(N + 1/2, 0)$ to $(1/2, 0)$, and without loss of generality, suppose that P is simple. Consider moving along P from $(N + 1/2, 0)$ to $(1/2, 0)$. Let S denote the point at which it crosses $\{N + 1/2\} \times [0, k]$ for the last time, and let T denote the point at which it crosses $\{1/2\} \times [0, k]$ for the first time. Then moving from $(N + 1/2, 0)$ to S along $\{N + 1/2\} \times [0, k]$, then S to T along P, then from T to $(1/2, 0)$ along $\{1/2\} \times [0, k]$ shows that there is a dual path from $(N + 1/2, 0)$ to $(1/2, 0)$ in $[1/2, N + 1/2] \times [0, k]$. Thus, B_N^k occurs. Since the three events are positively correlated by Harris' inequality,

$$e^{-2\lambda k} P(A_N^k) \leq P(B_N^k)$$
$$\nu_k \geq \mu_k.$$

□

Corollary 1

$$\lim_{N \to \infty} \frac{1}{N} \log P(B_N) = -\gamma_2.$$

Lemma 4

$$\lim_{N \to \infty} \frac{P\left(B_N^{N^2}\right)}{P(B_N)} = 1.$$

Proof. We will show that if $B_N \backslash B_N^{N^2}$ occurs, then the process on $\mathbb{Z}$ with initial state $\{1, \ldots, N\}$ must die out, but not before living at least N^2 time units, i.e., $N^2 < \tau^{\{1,\ldots,N\}} < \infty$. To see that $\tau^{\{1,\ldots,N\}} < \infty$, note that B_N implies A_N, then apply (3). To see that $N^2 < \tau^{\{1,\ldots,N\}}$, suppose B_N and $\tau^{\{1,\ldots,N\}} \leq N^2$ occur. We will show that $B_N^{N^2}$ must occur as well. There is a dual path, P_1, from $(N + 1/2, 0)$ to $(1/2, 0)$ in $[1/2, N + 1/2] \times [0, \infty)$. By Proposition 1 there is also a dual path, P_2, from $(N + 1/2, 0)$ to $(1/2, 0)$ in $(-\infty, \infty) \times [0, N^2]$. We may assume without loss of generality that P_1 and P_2 are simple. If P_1 or P_2 is contained in $[1/2, N + 1/2] \times [0, N^2]$, then clearly $B_N^{N^2}$ occurs, so suppose P_1 extends above N^2 and P_2 extends either to the right of $N + 1/2$ or to the left of 1/2. Consider moving along P_1 from $(N + 1/2, 0)$ to $(1/2, 0)$ and let p_1 and p_2 be the space-time points of the first and last times that the path intersects $[1/2, N + 1/2] \times \{N^2\}$. Let S_1 denote the part of P_1 between $(N + 1/2, 0)$ and p_1 and let S_2 denote the part of P_1 between p_2 and $(1/2, 0)$. Figure 4 shows an example.

Now S_1 and P_2 must intersect, since they both originate at $(N + 1/2, 0)$. Let p_3 be the space-time point of the last time that they intersect when one moves along P_2. Similarly, S_2 and P_2 must intersect, since they both end at $(1/2, 0)$. Let p_4 be the space-time point of the first time they intersect when one moves along P_2. Between points p_3 and p_4, P_2 must lie entirely in $[1/2, N + 1/2] \times [0, N^2]$, so moving from $(N + 1/2, 0)$ to p_3 along S_1, then p_3 to p_4 along P_2, then p_4 to $(1/2, 0)$ along S_2 exhibits a dual path from $(N + 1/2, 0)$ to $(1/2, 0)$ in $[1/2, N + 1/2] \times [0, N^2]$, which implies $B_N^{N^2}$. Thus

$$B_N \cap \{\tau^{\{1,\ldots,N\}} \leq N^2\} \subset B_N^{N^2}$$

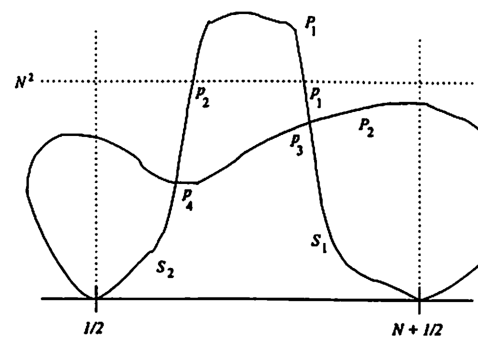


Figure 4: Sample dual paths for the proof of Lemma 4.

So

$$B_N \backslash B_N^{N^2} \subset \{N^2 < \tau^{\{1,\dots,N\}} < \infty\}.$$

It is known [2, Theorem 2.30] that there exist constants $C, \epsilon \in (0, \infty)$ such that for all $Z \subset \mathbb{Z}$,

$$P(t < \tau^Z < \infty) \le Ce^{-\epsilon t}.$$

Thus

$$P\left(B_N \backslash B_N^{N^2}\right) \le Ce^{-\epsilon N^2}.$$

So for all sufficiently large N,

$$0 \le 1 - \frac{P\left(B_N^{N^2}\right)}{P(B_N)} \le \frac{C\exp(-\epsilon N^2)}{\exp(-(\gamma_2+\epsilon)N)} \to 0.$$

□

Corollary 2

$$-\lim_{N\to\infty} \frac{1}{N} \log P\left(B_N^{N^2}\right) = \gamma_2.$$

In words, Corollaries 1 and 2 say that the chance that the process dies out before growing outside its original interval decays with the same exponent as the chance that the process dies out at all. The next lemma shows how to use this fact to bound the hitting time of the piecewise-homogeneous process.

Lemma 5 *Let (K, λ, α) be a profile such that $\lambda_j > \lambda_c$ for all $j \in \{1, \dots, K\}$. Then for all $\epsilon > 0$,*

$$P\left(\frac{\log \sigma_N}{N} > \sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j) + \epsilon\right) \to 0.$$

Proof. By comparing our process to a modified one in which at times kN^2, $k \in \mathbb{N}$, all nodes are made to be occupied if any of them are, we see that for all $t > 0$,

$$P(\sigma_N > t) \leq P(\sigma_N > N^2)^{\lfloor \frac{t}{N^2} \rfloor},$$

A modification of Proposition 1 shows that $\tilde{B}_N^{N^2}$ implies $\sigma_N \leq N^2$ so

$$P(\sigma_N > t) \leq \left(1 - P\left(\tilde{B}_N^{N^2}\right)\right)^{\lfloor \frac{t}{N^2} \rfloor}.$$

Now $\tilde{B}_N^{N^2}$ occurs if there is a dual path from $(i_j + 1/2, 0)$ to $(i_{j-1} + 1/2, 0)$ in $[i_{j-1} + 1/2, i_j + 1/2] \times [0, (i_j - i_{j-1})^2]$ for each $j \in \{1, \ldots, K\}$. Applying Corollary 2 to each partition, and noting that the dual paths across distinct partitions are independent, we have

$$\liminf_{N \to \infty} \frac{1}{N} \log P\left(\tilde{B}_N^{N^2}\right) \geq -\sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j). \tag{10}$$

Thus for all sufficiently large N,

$$P(\sigma_N > t) \leq \left(1 - \exp\left(-\left(\sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j) + \epsilon/2\right) N\right)\right)^{\lfloor t/N^2 \rfloor}. \tag{11}$$

Taking $t = \exp\left(\left(\sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j) + \epsilon\right) N\right)$ implies the result, since then the right-hand side converges to zero. □

Lemma 6

$$\lim_{N \to \infty} \frac{1}{N} \log P\left(D_N^{N^3}\right) = -\gamma_2.$$

Proof. Since $B_N^{N^2} \subset D_N^{N^3}$, we see that

$$\liminf_{N \to \infty} \frac{1}{N} \log P\left(D_N^{N^3}\right) \geq -\gamma_2.$$

For the upper bound, let

$$T_N = m(\{s \in [-N^3, N^3+1] : (N+1/2, s) \to (1/2, t) \text{ in } [1/2, N+1/2] \times [-N^3, N^3+1] \text{ for some } t).$$

Here and throughout, $m(\cdot)$ denotes Lebesgue measure. By Tonelli's theorem,

$$E[T_N] = \int_{-N^3}^{N^3+1} P((N + 1/2, s) \to (1/2, t) \text{ in } [1/2, N + 1/2] \times [-N^3, N^3 + 1] \text{ for some } t)\, ds.$$

Durrett and Schonmann [4, Lemma 4] show that if

$$G_N = \{(N + 1/2, 0) \to (1/2, t) \in \mathbb{R} \times \mathbb{R} \text{ for some } t\},$$

then

$$\lim_{N\to\infty}\frac{1}{N}\log P(G_N) = -\gamma_2.$$

Fix $\epsilon > 0$. Then for sufficiently large N,

$$P(G_N) \leq \exp(-(\gamma_2-\epsilon)N).$$

Since $P(G_N)$ upper bounds the integrand, this implies

$$E[T_N] \leq (2N^3+1)\exp(-(\gamma_2-\epsilon)N). \tag{12}$$

Define

$$U_N = \inf\{s : s = N^3 \text{ or } (N+1/2, s) \to (1/2, t) \text{ in } [1/2, N+1/2]\times[-N^3, N^3] \text{ for some } t\},$$

where the infimum is over $s \in [-N^3, N^3]$. Then U_N is independent of the arrows from $N+1$ to N, so the event that there are no arrows from $N+1$ to N during $[U_N, U_{N+1}]$is independent of U_N and has probability $\exp(-\lambda)$. If $U_N < N^3$, and there are no arrows from $N+1$ to N during $[U_N, U_{N+1}]$, then $T_N \geq 1$, so by Markov's inequality and (12),

$$\exp(-\lambda)P(U_N < N^3) \leq (2N^3+1)\exp(-(\gamma_2-\epsilon)N)$$

But $P(U_N < N^3) = P\left(D_N^{N^3}\right)$, so the previous inequality implies

$$\limsup_{N\to\infty}\frac{1}{N}\log P\left(D_N^{N^3}\right) \leq -\gamma_2.$$

□

The technique of relating events like G_N and $D_N^{N^3}$ using expectations and Tonelli's theorem will be used several times below. Having provided the complete argument in the previous proof, we will include less detail in the sequel.

Lemma 7

$$\lim_{N\to\infty}\frac{1}{N}\log P\left(\tilde{C}_N^{N^2}\right) = -\sum_{j=1}^{K}\alpha_j\gamma_2(\lambda_j).$$

Proof. Since $\tilde{B}_N^{N^2} \subset \tilde{C}_N^{N^2}$, the lower bound follows from (10). For the upper bound, define

$$D_{N,j}^{N^2} = \{(i_j+ 1/2, s) \to (i_{j-1}+1/2, t) \text{ in } [i_{j-1}+1/2, i_j+ 1/2]\times[-N^2, N^2] \text{ for some } s \text{ and } t\}.$$

For all sufficiently large N, $N^2 \leq (i_j - i_{j-1})^3$ for all $j \in \{1,\ldots,K\}$, so by Lemma 6,

$$\limsup_{N\to\infty}\frac{1}{\alpha_j N}\log P\left(D_{N,j}^{N^2}\right) \leq -\gamma_2(\lambda_j).$$

for all j. Since the $D_{N,j}^{N^2}$, $j \in \{1,\ldots,K\}$ are independent, and

$$\tilde{C}_N^{N^2} \subset \bigcap_{j=1}^{K} D_{N,j}^{N^2},$$

we have

$$\limsup_{N\to\infty} \frac{1}{N} \log P\left(\tilde{C}_N^{N^2}\right) \leq -\sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j).$$

□

We can complete the proof of Theorem 4 once we show that $\frac{1}{N}\log P(\tilde{C}_N) \to -\sum_{j=1}^{K}\alpha_j\gamma_2(\lambda_j)$. To relate $P(\tilde{C}_N)$ to $P(\tilde{C}_N^{N^2})$ we must generalize to inhomogeneous processes on $\mathbb{N}$ the fact used in Lemma 4 that for the homogeneous process on $\mathbb{Z}$, $P(t < \tau^Z < \infty) < C\exp(-\epsilon t)$ for all $Z \subset \mathbb{Z}$. We will apply our bound to a sequence of piecewise-homogeneous processes, and we require the same C and ϵ to work for each one. To accomplish this, we prove the bound for a general class of inhomogeneous processes. Let $\tilde{\xi}_t^Z$ be the contact process on $\mathbb{N}$ with $\tilde{\xi}_0^Z = Z$ and with transition rates

$$q(A, A\backslash\{j\}) = 1 \text{ if } j \in A$$

$$q(A, A\cup\{j\}) = \begin{cases} \theta_R(j-1)|A\cap\{j-1\}| + \theta_L(j+1)|A\cap\{j+1\}| & \text{if } j \notin A, j \neq 1 \\ \theta_L(2)|A\cap\{2\}| & \text{if } j \notin A, j = 1 \end{cases},$$

for $A \subset \mathbb{N}$ and $j \in \mathbb{N}$, where $\theta_L, \theta_R : \mathbb{N} \to [0,\infty)$. Let

$$\tilde{\tau}^Z = \inf\{t \geq 0 : \tilde{\xi}_t^Z = \emptyset\}.$$

Lemma 8 *For all $\lambda > \lambda_c$, there exists positive constants C and δ such that for all inhomogeneous contact processes on $\mathbb{N}$ such that $\min(\theta_L(i), \theta_R(i)) \geq \lambda$ for all $i \in \mathbb{N}$, all $Z \subset \mathbb{N}$, and all $t \geq 0$,*

$$P(t < \tilde{\tau}^Z < \infty) \leq C\exp(-\delta t).$$

Proof. We use a restart argument modeled after the one in Durrett [16, p. 1032].

We can couple $\tilde{\xi}_t^Z$ to a homogeneous process ξ_t^Z on $\mathbb{N}$ with rate λ by thinning the Poisson processes in the graphical representation. Let $\left(\zeta_t^{j,s}\right)_{t\geq s}$ denote the contact process on $[j,\infty)$ starting with $\{j\}$ infected, constructed from the graphical representation of $(\xi_t^Z)_{t\geq 0}$ restricted to $[j,\infty)\times[s,\infty)$.

Let $x_0 = \inf Z$, $T_0 = 0$, and $T_1 = \inf\{t \geq 0 : \zeta_t^{x_0,0} = \emptyset\}$. If $T_1 < \infty$ and $\tilde{\xi}_{T_1}^Z \neq \emptyset$, let $x_1 = \inf \tilde{\xi}_{T_1}^Z$. If $T_1 < \infty$ and $\tilde{\xi}_{T_1}^Z = \emptyset$, let $x_1 = 1$. Then let $T_2 = \inf\{t > T_1 : \zeta_t^{x_1,T_1} = \emptyset\}$, and repeat the procedure until we find a point (x_L, T_L) such that $\zeta_t^{x_L,T_L}$ survives forever. Such a point exists with probability 1, since each $\zeta_t^{x_i,T_i}$ has some probability $p > 0$ of surviving forever [3].

On $\tilde{\tau}^Z < \infty$, $T_L \geq \tilde{\tau}^Z$, so $t < \tilde{\tau}^Z < \infty$ implies $t < T_L$. There exists positive constants C and δ, independent of θ_L and θ_R, so that conditioned on $L \geq l$, $X_i = T_i - T_{i-1}$, $i = 1,\ldots,l$ are i.i.d. with

$$P(X_i \geq t) = P(t \leq \tau | \tau < \infty) \leq C\exp(-\delta t),$$

where

$$\tau = \inf\{t \geq 0 : \zeta_t^{0,0} = \emptyset\},$$

and the exponential bound is from Section 3 of Durrett and Liu [3]. Then there exists $\kappa > 0$ such that $\phi(\kappa) = E[\exp(\kappa X_i)] < \infty$, and $\epsilon > 0$ such that $e^{-\kappa}\phi(\kappa)^\epsilon < 1$. Then for all $t > 0$,

$$\begin{aligned} P(T_L > t) &\leq P(L > \lfloor \epsilon t \rfloor) + P(X_1 + \cdots X_{\lfloor \epsilon t \rfloor} \geq t) \\ &\leq (1-p)^{\lfloor \epsilon t \rfloor} + e^{-\kappa t}\phi(\kappa)^{\lfloor \epsilon t \rfloor}. \end{aligned}$$

□

Lemma 9

$$\lim_{N\to\infty} \frac{P\left(\tilde{C}_N^{N^2}\right)}{P(\tilde{C}_N)} = 1.$$

Proof. Construct $\tilde{C}_N$ and $\tilde{C}_N^{N^2}$ from the graphical representation of an inhomogeneous process on $\mathbb{N}$, $(\tilde{\xi}_t)_{t\geq 0}$, in which points $\{1,\ldots,N\}$ inherit their reproduction rates from the piecewise-homogeneous process, and the points in $\{N+1, N+2, \ldots\}$ reproduce at rate λ_K. Note that we require the graphical representation in both positive and negative time. On this graphical representation, define the following four events for all m and n such that $N \geq m \geq n > 1$,

$$\begin{aligned}
E^+_{m,n} &= \{(m+1/2, 0) \to (n-1/2, 0) \text{ in } [1/2, N+1/2] \times [0,\infty)\}\\
E^-_{m,n} &= \{(m+1/2, 0) \to (n-1/2, 0) \text{ in } [1/2, N+1/2] \times (-\infty,0]\}\\
F^+_{m,n} &= \{(m+1/2, 0) \to (n-1/2, 0) \text{ in } [1/2, N+1/2] \times [0,N^2]\}\\
F^-_{m,n} &= \{(m+1/2, 0) \to (n-1/2, 0) \text{ in } [1/2, N+1/2] \times [-N^2,0]\}
\end{aligned}$$

For $m \geq n = 1$, we allow the path to end at an arbitrary time. That is,

$$E^+_{m,1} = \{(m+1/2, 0) \to (1/2, s) \text{ in } [1/2, N+1/2] \times [0,\infty) \text{ for some } s\},$$

and similarly for the other three events. Let

$$\tau^m_n = \inf\{t \geq 0 : \tilde{\xi}_t^{\{n,\ldots,m\}} = \emptyset\},$$

where $\tilde{\xi}^A_t$ denotes $\tilde{\xi}_t$start ed in state A. For any $1 \leq n \leq m \leq N$, if $E^+_{m,n}$ occurs and $\{\tilde{\tau}^m_n \leq N^2\}$, then a modification of the argument used in Lemma 4 shows that $F^+_{m,n}$ must also occur. Thus

$$E^+_{m,n} \backslash F^+_{m,n} \subset \{N^2 < \tilde{\tau}^m_n < \infty\}.$$

So by the previous lemma, there exists $C > 0$ and $\delta > 0$ so that for all N, m, and n such that $1 \leq n \leq m \leq N$,

$$P\left(E^+_{m,n} \backslash F^+_{m,n}\right) \leq C\exp(-\delta N^2) \tag{13}$$

Let $\tilde{\Xi}_t$ be the contact process on $\mathbb{N}$ constructed from the graphical representation obtained by reflecting the portion of the original graphical representation that lies below $t = 0$ about the $t = 0$ axis, then reversing the direction of all of the arrows. Observe that there is a dual path from (i, s) to (j, t) in the graphical representation of $\tilde{\Xi}_t$ if and only if there is a dual path from $(i, -s)$ to $(j, -t)$ in $[1/2, \infty) \times (-\infty, 0]$ in the original graphical representation. Thus the argument leading up to (13) also shows that

$$P(E^-_{m,n} \backslash F^-_{m,n}) \leq C\exp(-\delta N^2).$$

Now suppose that $\tilde{C}_N$ occurs, and let P be any simple dual path from $(N+1/2, 0)$ to $\{1/2\} \times (-\infty, \infty)$ in $[1/2, N+1/2] \times (-\infty, \infty)$. Without loss of generality we can assume that P intersects $[1/2, N+1/2] \times \{0\}$ only at half-integers. Let $x_1 + 1/2, x_2 + 1/2, \ldots, x_{M-1} + 1/2$ be those half-integers, in the order in which it reaches them. Thus $x_1 = N$. Let $x_M = 0$. Again without loss of generality, we can assume that $x_1 > x_2 > \cdots > x_M$. Since P must live entirely above or below $t = 0$ between $x_i + 1/2$

and $x_{i+1} + 1/2$, either $E^+_{x_i,x_{i+1}+1}$ or $E^-_{x_i,x_{i+1}+1}$ must occur for each $i = 1, \ldots, M-1$. If $F^+_{m,n}$ occurs whenever $E^+_{m,n}$ does and $F^-_{n,m}$ occurs whenever $E^-_{m,n}$ does, then $\tilde{C}_N^{N^2}$ also occurs, so $\tilde{C}_N \backslash \tilde{C}_N^{N^2}$ implies that there exist m and n such that $1 \le n \le m \le N$, and $E^+_{m,n} \backslash F^+_{m,n}$ or $E^+_{m,n} \backslash F^+_{m,n}$ occur. Thus

$$P\left(\tilde{C}_N \backslash \tilde{C}_N^{N^2}\right) \le 2N^2 C \exp(-\delta N^2).$$

The result then follows as in the proof of Lemma 4. □

Corollary 3

$$\lim_{N\to\infty} \frac{1}{N} \log P\left(\tilde{C}_N\right) = -\sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j).$$

Lemma 10 *Let (K, λ, α) be a profile such that $\lambda_j > \lambda_c$ for all $j \in \{1, \ldots, K\}$. Then for all $\epsilon > 0$,*

$$P\left(\frac{\log \sigma_N}{N} < \sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j) - \epsilon\right) \to 0 \tag{14}$$

as $N \to \infty$, and

$$\frac{\log E[\sigma_N]}{N} \to \sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j).$$

Proof. We proceed as in Lemma 6 of Durrett and Schonmann [4]. Write $\gamma = \sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j)$. By Tonelli's theorem, for all sufficiently large N,

$$\begin{aligned} &E[m(\{s \in [0, T+1] : (N+1/2, s) \to (1/2, t) \text{ in } [1/2, N+1/2] \times (-\infty, \infty) \text{ for some } t\})] \\ &= \int_0^{T+1} P((N+1/2, s) \to (1/2, t) \text{ in } [1/2, N+1/2] \times (-\infty, \infty) \text{ for some } t)\, ds \\ &\le (T+1) \exp\left(-(\gamma - \epsilon/2) N\right). \end{aligned} \tag{15}$$

If there exists $s, t \in [0, T]$ such that $(N+1/2, s) \to (1/2, t)$ in $[1/2, N+1/2] \times [0, T]$, and there are no arrows from $N+1$ to N during $[s, s+1]$, then the measure in (15) is at least 1, so

$$\begin{aligned} &P((N+1/2, s) \to (1/2, t) \text{ in } [1/2, N+1/2] \times [0, T] \text{f or some } s \text{ and } t) \\ &\le e^{\lambda_K} (T+1) \exp\left(-(\gamma - \epsilon/2) N\right), \end{aligned}$$

which implies

$$P(\sigma_N \le T) \le e^{\lambda_K} (T+1) \exp(-(\gamma - \epsilon/2) N),$$

since $\sigma_N \le T$ implies the existence of such a path. Substituting $T = \exp((\gamma - \epsilon)N)$ shows (14). To show convergence of expectations, note that

$$\begin{aligned} E[\sigma_N] &= E[\sigma_N 1(\sigma_N < \exp((\gamma - \epsilon)N))] + E[\sigma_N 1(\sigma_N \ge \exp((\gamma - \epsilon)N))] \\ &\ge \exp((\gamma - \epsilon)N) P(\sigma_N \ge \exp((\gamma - \epsilon)N)). \end{aligned}$$

From this it follows that

$$\liminf_{N\to\infty} \frac{\log E[\sigma_N]}{N} \geq \gamma.$$

For the upper bound, write

$$E[\sigma_N] = \int_0^\infty P(\sigma_N \geq t)\, dt.$$

Then, using (11), we have for all sufficiently large N,

$$E[\sigma_N] \leq \exp((\gamma+\epsilon)N) + \int_{\exp((\gamma+\epsilon)N)}^\infty (1-\exp(-(\gamma+\epsilon/2)N))^{t/N^2-1}\, dt.$$

If we write $\theta(N)$ for $1-\exp(-(\gamma+\epsilon/2)N)$, then the integral evaluates to

$$\theta(N)^{-1} \cdot \frac{-N^2}{\log \theta(N)} \cdot \theta(N)^{\exp((\gamma+\epsilon)N)/N^2},$$

which converges to zero as $N \to \infty$. □

The proofs of Theorems 5 and 6 make use of the following fact.

Lemma 11 *$\gamma_2(\lambda)$ is continuous from the right on* $[0,\infty)$.

Proof. If $0 \leq \lambda \leq \lambda_c$, then

$$P\left(\tau^{\{1,\ldots,N\}} = \infty\right) \leq NP\left(\tau^{\{0\}} = \infty\right) = 0,$$

so $P(\tau^{\{1,\ldots,N\}} < \infty) = 1$ for all N and $\gamma_2(\lambda) = 0$. This proves the conclusion on $[0,\lambda_c)$. To verify it at λ_c, note that Durrett, Schonmann, and Tanaka [5] show that

$$\liminf_{\lambda\downarrow\lambda_c} \frac{(\lambda-\lambda_c)^{(1/5)}}{\gamma_2(\lambda)} > 0,$$

which implies that $\gamma_2(\lambda) \downarrow 0$ as $\lambda \downarrow \lambda_c$. To prove the conclusion on (λ_c,∞), write $B_N^{N^2}(\lambda)$ for the event $B_N^{N^2}$ in the graphical representation with rate λ, and note that if $\lambda_2 > \lambda_1$, then $B_N^{N^2}(\lambda_1)$ and $B_N^{N^2}(\lambda_2)$ can be coupled by constructing $B_N^{N^2}(\lambda_2)$ from the graphical representation of $B_N^{N^2}(\lambda_1)$ along with additional arrows with rate $\lambda_2 - \lambda_1$. Then

$$0 \leq P\left(B_N^{N^2}(\lambda_1)\right) - P\left(B_N^{N^2}(\lambda_2)\right) \leq 1 - \exp\left(2(\lambda_1-\lambda_2)N^3\right),$$

so $P(B_N^{N^2}(\lambda))$ is continuous. Define

$$f_N(\lambda) = -\frac{1}{N}\log P\left(B_N^{N^2}(\lambda)\right).$$

Then $f_N(\lambda)$ is continuous in λ for fixed N, and by subadditivity,

$$f_N(\lambda) \geq \gamma_2(\lambda).$$

Note that both f_N and γ_2 are nondecreasing. Now fix $\lambda > \lambda_c$ and $\epsilon > 0$. Choose N so that $f_N(\lambda) \le \gamma_2(\lambda) + \epsilon/2$. Then choose δ such that for all $\theta \in [\lambda, \lambda + \delta]$, $f_N(\theta) \le f_N(\lambda) + \epsilon/2$. Then

$$\gamma_2(\theta) \le f_N(\theta) \le \gamma_2(\lambda) + \epsilon.$$

□

Proof of Theorem 5. The lower bound in probability and expectation can be proven by coupling the process to the all-supercritical process consisting of one of the supercritical islands with a maximal hitting time exponent and then applying Theorem 4. We omit the details of this procedure and concentrate on the upper bounds.

Let C denote the indices of the nonsupercritical partitions. Choose $\lambda > \lambda_c$ such that $\gamma_2(\lambda) < \epsilon/2$, and let $\overline{\sigma}_N$ be the extinction time of the all-supercritical process obtained by making all of the partitions in C supercritical with rate λ. By coupling that process to the original one so that $\sigma_N \le \overline{\sigma}_N$, we have $E[\sigma_N] \le E[\overline{\sigma}_N]$, and

$$\begin{aligned} P\left(\frac{\log \sigma_N}{N} > \sum_{j \notin C} \alpha_j \gamma_2(\lambda_j) + \epsilon\right) &\le P\left(\frac{\log \overline{\sigma}_N}{N} > \sum_{j \notin C} \alpha_j \gamma_2(\lambda_j) + \epsilon\right) \\ &\le P\left(\frac{\log \overline{\sigma}_N}{N} > \sum_{j \notin C} \alpha_j \gamma_2(\lambda_j) + \sum_{j \in C} \alpha_j \gamma_2(\lambda) + \epsilon/2\right). \end{aligned}$$

Applying Theorem 4 completes the proof. □

Proof of Theorem 6. Let $\lambda_0 \ge 0$ and $\eta \ge 0$, and let (K, λ, α) be any feasible profile. By Theorem 5,

$$\liminf_{N \to \infty} \frac{1}{N} \log E[\sigma_N] \le \sum_{j=1}^{K} \alpha_j \gamma_2(\lambda_j) \le \hat{\gamma}_2^{\lambda_0}(\lambda_0 + \eta).$$

Thus $R^*(\lambda_0, \eta) \le \hat{\gamma}_2^{\lambda_0}(\lambda_0 + \eta)$. Observe that $\hat{\gamma}_2^{\lambda_0}(\lambda_0 + \eta) < \infty$ since $\gamma_2(\lambda) \le \log(1 + 2\lambda)$. By Carathéodory's Theorem, [11, p. 155], $\hat{\gamma}_2^{\lambda_0}$ is approached by profiles with two rates,

$$\hat{\gamma}_2^{\lambda_0}(x) = \sup\left\{\sum_{j=1}^{2} \alpha_j \gamma_2(\lambda_j) : \sum_{j=1}^{2} \alpha_i \lambda_j = x, \text{ and } \min(\lambda_1, \lambda_2) \ge \lambda_0\right\}.$$

Thus there exists sequences $(\alpha_j^k, \lambda_j^k, k \in \mathbb{N}, j = 1, 2)$ such that $\lambda_j^k \ge \lambda_0$ for all k and j, and

$$\sum_{j=1}^{2} \alpha_j^k \lambda_j^k \le \lambda_0 + \eta \ \ \forall k, \ \text{ and} \tag{16}$$

$$\sum_{j=1}^{2} \alpha_j^k \gamma_2(\lambda_j^k) \to \hat{\gamma}_2^{\lambda_0}(\lambda_0 + \eta), \tag{17}$$

as $k \to \infty$. By considering subsequences, we can assume that α_j^k and λ_j^k are monotonic. If $\lambda_j^k \uparrow \infty$ as $k \to \infty$, then

$$\alpha_j^k \gamma_2(\lambda_j^k) \leq \frac{\lambda_0 + \eta}{\lambda_j^k} \log(1 + 2\lambda_j^k) \to 0.$$

So setting $\lambda_j^k = \lambda_0$ for all k does not change (16) or (17). Thus we may assume that the sequences λ_1^k and λ_2^k are bounded, in which case there is a convergent subsequence of $(\alpha_j^k, \lambda_j^k, k \in \mathbb{N}, j = 1, 2)$ that converges to $\alpha_1, \alpha_2, \lambda_1, \lambda_2$. Since γ_2 is nondecreasing and right continuous, we have for any sequence $x_n \to x$,

$$\limsup_{n\to\infty} \gamma_2(x_n) \leq \gamma_2(x).$$

Thus

$$\hat{\gamma}_2^{\lambda_0}(\lambda_0 + \eta) \leq \alpha_1\gamma_2(\lambda_1) + \alpha_2\gamma_2(\lambda_2).$$

It follows that $R^*(\lambda_0, \eta) \geq \hat{\gamma}_2(\lambda_0 + \eta)$ and it is achieved by a profile with two rates. □

We will prove two more lemmas before proceeding with the proof of Proposition 3.

Lemma 12 *Let ξ_t^N be a piecewise-homogeneous contact process on $\{1, \dots, N\}$ with $\xi_0^N = \{2\}$ for which $\lambda_j < \lambda_c$ for all $j \in \{1, \dots, K\}$. Let $R = \sup_{t\geq 0} \sup \xi_t^N$ and $\Delta'_N = \inf\{t \geq 0 : N \in \xi_t^N\}$. Then for all $\beta > 0$,*

$$\lim_{N\to\infty} \frac{1}{N} \log P(R \geq N) - \frac{1}{N} \log P(\Delta'_N \leq \beta N^2) = 0.$$

The result also holds for a homogeneous subcritical contact process on $\mathbb{Z}$.

Proof. Consider the piecewise-homogeneous process. For all $\beta > 0$,

$$\begin{aligned} 0 \leq P(R \geq N) - P(\Delta'_N \leq \beta N^2) &= P(\beta N^2 < \Delta'_N < \infty) \\ &\leq \exp(-\gamma_1(\max(\lambda_1, \dots, \lambda_K))\beta N^2). \end{aligned}$$

Consider the homogeneous process on $\{0\} \cup \mathbb{N}$, ξ_t, with reproduction rate $\min(\lambda_1, \dots, \lambda_K)$ and initial state $\{0\}$, and let $R' = \sup_{t\geq 0} \sup \xi_t$. Then $P(R \geq N) \geq P(R' \geq N - 2)$, and $-\log P(R' \geq N)$ is subadditive, since by conditioning on the state of ξ_t when N first becomes occupied, it follows that

$$P(R' \geq M + N) \geq P(R' \geq N)P(R' \geq M).$$

Thus if we let

$$\theta = \inf_{N\in\mathbb{N}} -\frac{1}{N} \log P(R' \geq N),$$

then $\theta \geq \gamma_3(\min(\lambda_1, \dots, \lambda_K)) > 0$, and for all sufficiently large N,

$$P(R \geq N) \geq P(R' \geq N - 2) \geq \exp(-2\theta\ N - 2)),$$

in which case

$$0 \leq 1 - \frac{P(\Delta'_N \leq \beta N^2)}{P(R \geq N)} \leq \frac{\exp(-\gamma_1(\max(\lambda_1, \dots, \lambda_K))\beta N^2)}{\exp(-2\theta(N-2))} \to 0.$$

This shows that for all $\beta > 0$,

$$\lim_{N\to\infty} \frac{P(R \geq N)}{P(\Delta'_N \leq \beta N^2)} = 1,$$

and the result for the piecewise-homogeneous process follows. The proof for the homogeneous process is similar. □

Lemma 13 *Let ξ_t^N be a piecewise-homogeneous contact process on $\{1,\ldots,N\}$ with $\xi_0^N = \{2\}$ for which $\lambda_j < \lambda_c$ for all $j \in \{1,\ldots,K\}$. Let $R = \sup_{t\geq 0} \sup \xi_t^N$ and*

$$\hat{\Delta}_N = \inf\{t \geq 0 : \textit{there is a CP path from } (2,s) \textit{ to } (N,t) \textit{ in } [1,N]\times[s,t] \textit{ for some } s \geq 0\}.$$

Then

$$\lim_{N\to\infty} \frac{1}{N}\log P(R \geq N) - \frac{1}{N}\log P(\hat{\Delta}_N \leq \beta N^2) = 0.$$

The result also holds for a homogeneous subcritical contact process on $\mathbb{Z}$.

Proof. By Tonelli's theorem, for any $\beta > 0$,

$$\begin{aligned}
&E[m(\{s \in [0,\beta N^2+1] : \text{ there is a CP path from } (2,s) \text{ to } (N,t) \text{ for some } t \leq \beta N^2+1\})] \\
&= \int_0^{\beta N^2+1} P(\text{there is a CP path from } (2,s) \text{ to } (N,t) \text{ for some } t \leq \beta N^2+1)\, ds \\
&\leq \int_0^{\beta N^2+1} P(R \geq N)\, ds \\
&= (\beta N^2+1)P(R \geq N).
\end{aligned}$$

If there is a contact process path from $(2,s)$ to (N,t) for some $s \geq 1$ and $t \leq \beta N^2+1$ in $[1,N]\times[s,t]$, and there are no δ's at 1 during $[s-1,s]$, then the above measure is at least one, so

$$e^{-1}P(\hat{\Delta}_N \leq \beta N^2) \leq (\beta N^2+1)P(R \geq N).$$

Since $P(\Delta'_N \leq \beta N^2) \leq P(\hat{\Delta}_N \leq \beta N^2)$, the previous equation and Lemma 12 give the result. □

Proof of Proposition 3. For each $j \in \{1,\ldots,K\}$, let

$$\hat{\Delta}_N^j = \inf\{t \geq 0 : \exists \text{ a CP path from } (i_{j-1}+2,s) \text{ to } (i_j,t) \text{ in } [i_{j-1}+1,i_j]\times[s,t] \text{ for some } s \geq 0\}.$$

Constructing the jth partition from a homogeneous process on $\mathbb{Z}$ with rate λ_j, and applying Lemma 13 to the homogeneous process shows that

$$-\lim_{N\to\infty} \frac{1}{N}\log P(\hat{\Delta}_N^j \leq N^2) = \alpha_j\gamma_3(\lambda_j). \tag{18}$$

Let $\epsilon > 0$. For the piecewise-homogeneous process, using the notation from the previous Lemma,

$$P(\hat{\Delta}_N \leq N^2) \leq \prod_{j=1}^{K} P(\hat{\Delta}_N^j \leq N^2).$$

Then (18) implies

$$P(\hat{\Delta}_N \le N^2) \le \exp\left(-\left(\sum_{j=1}^{K} \alpha_j \gamma_3(\lambda_j) - \epsilon/4\right) N\right). \tag{19}$$

for all sufficiently large N, which implies, by Lemma 13,

$$P(R \ge N) \le \exp\left(-\left(\sum_{j=1}^{K} \alpha_j \gamma_3(\lambda_j) - \epsilon/2\right) N\right). \tag{20}$$

We can now proceed as in Bramson, Durrett, and Schonmann [10]. Let $T_1, T_2, \ldots,$ be the times at which there are arrows from 1 to 2, and let $T(t) = \sup\{k : T_k \le t\}$. If there is a path from $\{1\} \times [0,t]$ to $\{N\} \times [0,t]$ then there is a path from $\{2\} \times \{T_1, \ldots, T_{T(t)}\}$ to $\{N\} \times [0,t]$. Then by the union bound,

$$P(\Delta_N \le t) \le P(T(t) > 2\lambda_c t) + 2\lambda_c t \exp\left(-\left(\sum_{j=1}^{K} \alpha_j \gamma_3(\lambda_j) - \epsilon/2\right) N\right).$$

Substituting $t = \exp\left(\left(\sum_{j=1}^{K} \alpha_j \gamma_3(\lambda_j) - \epsilon\right) N\right)$ gives the lower bound, since then the first term tends to zero by the weak law of large numbers, and evidently the second term also tends to zero. Let

$$\Delta_N^j = \inf\{t \ge 0 : \text{there is a CP path from } (i_{j-1}+2, 0) \text{ to } (i_j, t) \text{ in } [i_{j-1}+1, i_j] \times [0,t]\}.$$

If we let

$$A_N^j = \{\text{the CP on } \{i_{j-1}+1, \ldots, i_j\} \text{ with initial state } \{i_{j-1}+2\} \text{ eventually occupies } i_j\},$$

then Lemma 12 implies

$$\lim_{N\to\infty} \frac{1}{\alpha_j N} \log P(\Delta_N^j \le N^2) - \frac{1}{\alpha_j N} \log P(A_N^j) = 0,$$

and Lemma 13 implies

$$\lim_{N\to\infty} \frac{1}{\alpha_j N} \log P(\hat{\Delta}_N^j \le N^2) - \frac{1}{\alpha_j N} \log P(A_N^j) = 0,$$

which when combined with (18) yields

$$\lim_{N\to\infty} \frac{1}{N} \log P(\Delta_N^j \le N^2) = -\alpha_j \gamma_3(\lambda_j). \tag{21}$$

There is a contact process path in the piecewise-homogeneous process from $(1,0)$ to (N,T) for some $T \le KN^2 + K$ if the following occur: (a) point 1 does not become vacant during $[0,1]$, (b) there is an arrow from 1 to 2 at time $s \in [0,1]$, (c) there is a contact process path from $(2,s)$ to $(i_1, s+t)$ with $t \le N^2$, (d) points i_1 and i_1+1 do not become vacant during $[s+t, s+t+1]$, (e) there is an arrow from i_1 to i_1+1 at time u and an arrow from i_1+1 to i_1+2 at time v with $s+t \le u \le v \le s+t+1$, (f) there is a contact process path from (i_1+2, v) to $(i_2, v+w)$ with $w \le N^2$ that never moves left of

$i_1 + 1$, etc. Let $\epsilon > 0$. Using (21) to lower bound the probabilities of the contact process paths, we have for some $p > 0$ and all sufficiently large N,

$$P(\Delta_N \leq KN^2 + K) \geq \exp\left(-\left(\sum_{j=1}^{K} \alpha_j \gamma_3(\lambda_j) + \epsilon/2\right) N\right) p^K$$

Now partition the time interval $[0, t]$ into subintervals of length $[0, KN^2+K]$. Each subinterval provides an independent chance of finding a contact process path from 1 to N, so

$$P(\Delta_N > t) \leq \left(1 - p^K \exp\left(-\left(\sum_{j=1}^{K} \alpha_j \gamma_3(\lambda_j) + \epsilon/2\right) N\right)\right)^{\lfloor t/(KN^2+K)\rfloor}.$$

Substituting $t = \exp((\sum_{j=1}^{K} \alpha_j \gamma_3(\lambda_j) + \epsilon)N)$ and noting that the right-hand side converges to zero completes the proof. □

7 Proofs of Biased Voter Model Results

The subcritical part of Theorem 7 follows from the next lemma, which is a calculation about the average drift of a random walk in an inhomogeneous environment.

Lemma 14 *Let X_t be a random walk on $\mathbb{Z}$ with $X_0 = 1$ and rates*

$$q(n, n+1) = 1$$
$$q(n, n-1) = \begin{cases} \lambda_n & \text{if } n \leq 1 \\ \lambda_2 & \text{if } n \geq 2 \end{cases},$$

with $\sup_{n\leq 2} \lambda_n < 1$. *Let* $T_n = \inf\{t \geq 0 : X_t = n\}$. *Then*

$$\frac{T_N}{N} \to \frac{1}{1-\lambda_2}$$

in probability as $N \to \infty$.

Proof. Let $S_n = T_{n+1} - T_n$. By a simple extension of the weak law of large numbers, it suffices to show that

$$\lim_{n\to\infty} E[S_n] = \frac{1}{1-\lambda_2} \tag{22}$$
$$\sup_{n\geq 1} E[S_n^2] < \infty. \tag{23}$$

Let Y_t be a random walk on $\mathbb{Z}$ with rates $q(n, n-1) = \lambda_2$ and $q(n, n+1) = 1$ and with $Y_0 = 1$. Let Z_t be a random walk on $\mathbb{Z}$ with $Z_0 = 1$ and with rates

$$q(n, n+1) = \frac{1 + \inf_{k\leq 2} \lambda_k}{1 + \sup_{k\leq 2} \lambda_k}$$
$$q(n, n-1) = \sup_{k\leq 2} \lambda_k \frac{1 + \inf_{k\leq 2} \lambda_k}{1 + \sup_{k\leq 2} \lambda_k}$$

Note that for each state $Z_t$has a larger average holding time and a smaller drift than either $X_t$or Y_t. Let

$$\begin{aligned} U_n &= \inf\{t \geq 0 : Y_t = n\} \\ V_n &= U_{n+1} - U_n \\ W_n &= \inf\{t : Z_t = n+1\} - \inf\{t : Z_t = n\}. \end{aligned}$$

Also define the events

$$\begin{aligned} A_n &= \{X_t = 1 \text{ for some } t \in (T_n, T_{n+1})\} \\ B_n &= \{Y_t = 1 \text{ for some } t \in (U_n, U_{n+1})\} \end{aligned}$$

Then $P(A_n) = P(B_n)$ for all $n \geq 2$, and

$$\begin{aligned} E[S_n] &= E[S_n 1(A_n^c)] + E[S_n 1(A_n)] \\ &= E[V_n 1(B_n^c)] + E[S_n 1(A_n)] \\ &= E[V_n] - E[V_n 1(B_n)] + E[S_n 1(A_n)] \\ |E[S_n] - E[V_n]| &\leq E[V_n 1(B_n)] + E[S_n 1(A_n)] \\ &\leq 2\sqrt{E[W_1^2] P(B_n)}. \end{aligned}$$

Now $E[V_n] = 1/(1-\lambda_2)$, $E[W_1^2] < \infty$, and the optional stopping theorem applied to the jump chain of $\lambda_2^{Y(t)}$, which is a discrete-time martingale, gives

$$P(B_n) = \frac{\lambda_2^{-1} - 1}{\lambda_2^{-n} - 1},$$

which shows (22). To see (23), note that $E[S_n^2] \leq E[W_1^2]$. □

We will also use the following elementary large deviations lemma.

Lemma 15 *Let X_t be a random walk on $\mathbb{Z}$ with $X_0 = 0$ and with rates $q(n, n+1) = \lambda$ and $q(n, n-1) = \mu$ with $\lambda > \mu$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$P(X_t - (\lambda - \mu)t \leq -\epsilon t) \leq \exp(-\delta t).$$

for all $t \geq 0$.

Proof. Let

$$f(\theta) = -\theta\epsilon + \theta(\lambda - \mu) - \lambda + \lambda e^{-\theta} - \mu + \mu e^{\theta}.$$

Then $f(0) = 0$, and $f'(0) = -\epsilon$, so fix $\theta > 0$ such that $f(\theta) < 0$. If Y_t and Z_t are independent Poisson processes with rates λ and μ, then X_t is identically distributed to $Y_t - Z_t$, so by the Chernoff bound,

$$\begin{aligned} P(X(t) - (\lambda - \mu)t \leq -\epsilon t) &\leq \exp(-\theta\epsilon t + \theta(\lambda - \mu)t) E[\exp(-\theta Y_t + \theta Z_t)] \\ &\leq \exp(-\theta\epsilon t + \theta(\lambda - \mu)t - \lambda t + \lambda t e^{-\theta} - \mu t + \mu t e^{\theta}) \\ &= \exp(f(\theta)t). \end{aligned}$$

□

Lemma 16 *If (K, λ, α) is a profile such that $\lambda_j < 1$ for all $j \in \{1, \dots, K\}$, then*

$$\frac{\sigma_N}{N} \to \sum_{j=1}^{K} \frac{\alpha_j}{2(1-\lambda_j)}$$

in probability as $N \to \infty$.

Proof. Write γ for $\sum_{j=1}^{K} \frac{\alpha_j}{2(1-\lambda_j)}$. Let L_t and R_t be the random walks in the edge construction of the process. Then

$$j^* = \sup\left\{1 \le j \le K : \sum_{k=1}^{j-1} \frac{\alpha_k}{1-\lambda_k} < \gamma\right\}$$

is the partition in which we expect L_t and R_t to meet. To pinpoint the location within j^*, choose β to satisfy

$$\sum_{k=1}^{j^*-1} \frac{\alpha_k}{1-\lambda_k} + \frac{\beta\alpha_{j^*}}{1-\lambda_{j^*}} = \sum_{k=j^*+1}^{K} \frac{\alpha_k}{1-\lambda_k} + \frac{(1-\beta)\alpha_{j^*}}{1-\lambda_{j^*}}, \tag{24}$$

and observe that the definition of j^* forces $\beta \in [0,1]$ and that both sides of this equation equal γ. We expect L_t and R_t to meet around

$$M = i_{j^*-1} + \lceil \beta\alpha_{j^*} N \rceil,$$

since the left-hand side of (24) is how long we expect L_t to take to reach M and the right-hand side of (24) is how long we expect R_t to take to reach M. For each $j \in \{1, \dots, K\}$, let $S_j = \inf\{t : L_t = i_j\}$ and let $T_j = \inf\{t : R_t = i_j\}$. Also let $S_0 = 0$, and

$$S = \inf\{t : L_t = M\}$$
$$T = \inf\{t : R_t = M\}.$$

Then

$$P\left(\frac{\sigma_N}{N} < \gamma - \epsilon\right) \le P\left(\frac{S}{N} < \gamma - \epsilon\right) + P\left(\frac{T}{N} < \gamma - \epsilon\right).$$

But

$$\begin{aligned} P\left(\frac{S}{N} < \gamma - \epsilon\right) &= P\left(\frac{S}{N} < \sum_{k=1}^{j^*-1} \frac{\alpha_k}{(1-\lambda_k)} + \frac{\beta\alpha_{j^*}}{1-\lambda_{j^*}} - \epsilon\right) \\ &\le \sum_{k=1}^{j^*-1} P\left(\frac{S_k - S_{k-1}}{N} < \frac{\alpha_k}{(1-\lambda_k)} - \epsilon/K\right) \\ &\quad + P\left(\frac{S - S_{j^*-1}}{N} < \frac{\beta\alpha_{j^*}}{1-\lambda_{j^*}} - \epsilon/K\right) \to 0, \end{aligned}$$

by Lemma 14. Similarly, $P(T/N < \gamma - \epsilon) \to 0$. For the upper bound, modify L_t and R_t so that at time S, L_t becomes a homogeneous random walk on $\mathbb{Z}$ with $q(n, n-1) = \max(\lambda_1, \dots, \lambda_K)$ and

$q(n, n+1) = 1$, and at time T, R_t becomes a homogeneous random walk on $\mathbb{Z}$ with $q(n, n-1) = 1$ and $q(n, n+1) = \max(\lambda_1, \ldots, \lambda_K)$. These walks can be coupled to the original process so that

$$\sigma_N \leq \inf\{t : R(t) < L(t)\}.$$

Then

$$P\left(\frac{\sigma_N}{N} > \gamma + \epsilon\right) \leq P\left(\frac{S}{N} > \gamma + \epsilon/2\right) + P\left(\frac{T}{N} > \gamma + \epsilon/2\right) + \\ P\left(\frac{S}{N} \leq \gamma + \epsilon/2, L_{(\gamma+\epsilon)N} \leq M\right) + P\left(\frac{T}{N} \leq \gamma + \epsilon/2, R_{(\gamma+\epsilon)N} \geq M\right).$$

The first two terms converge to zero as above. To see that the third and fourth terms also go to zero, consider $\tilde{L}_t = L_{t+S} - M$. Then $(\tilde{L}_t)_{t\geq 0}$ is a homogeneous random walk with positive drift, and

$$P\left(\frac{S}{N} < \gamma + \epsilon/2, L_{(\gamma+\epsilon)N} \leq M\right) = P\left(\frac{S}{N} < \gamma + \epsilon/2, \tilde{L}_{(\gamma+\epsilon)N-S} \leq 0\right)$$

Since $\tilde{L}_t$ and S are independent, Lemma 15 shows that there exists $\delta > 0$ such that

$$P\left(\frac{S}{N} < \gamma + \epsilon/2, \tilde{L}_{(\gamma+\epsilon)N-S} \leq 0\right) \leq \exp(-\delta\epsilon N/2).$$

□

Lemma 17 *Let X_t be a random walk on $\mathbb{Z}$ with $X_0 = -1$ and rates $q(n, n+1) = \lambda$ and $q(n, n-1) = 1$, with $\lambda > 1$. Let $T_n = \inf\{t \geq 0 : X_t = n\}$. Then for all $\beta > 0$,*

$$\lim_{N\to\infty} \frac{1}{N} \log P(T_{-N} = \min(T_{-N}, T_0, \beta N^2)) = -\log \lambda.$$

Proof. Applying the optional stopping theorem to the jump chain of λ^{-X_t}, which is a discrete-time martingale, gives for all $N \geq 1$,

$$P(T_{-N} = \min(T_{-N}, T_0)) = \frac{\lambda - 1}{\lambda^N - 1}.$$

It follows that

$$\lim_{N\to\infty} \frac{1}{N} \log P(T_{-N} = \min(T_{-N}, T_0)) = -\log \lambda.$$

Since

$$P(T_{-N} = \min(T_{-N}, T_0)) \leq P(T_{-N} = \min(T_{-N}, T_0), T_{-N} \leq \beta N^2) + P(\min(T_{-N}, T_0) > \beta N^2),$$

it suffices to show that $P(\min(T_{-N}, T_0) > \beta N^2) \leq \exp(-\delta N^2)$ for some $\delta > 0$. But this is true since for all sufficiently large N,

$$P(\min(T_{-N}, T_0) > \beta N^2) \leq P(T_0 > \beta N^2) \leq P(X_{\beta N^2} \leq 0) \\ \leq \exp(-\delta N^2),$$

where the large deviations bound is from Lemma 15. □

Lemma 18 *If (K, λ, α) is a profile such that $\lambda_j > 1$ for all $j \in \{1, \ldots, K\}$, then*

$$P\left(\frac{\log \sigma_N}{N} > \sum_{j=1}^{K} \alpha_j \log(\lambda_j) + \epsilon\right) \to 0$$

as $N \to \infty$ for all $\epsilon > 0$.

Proof. By comparing the process to one that is reset to $\{1, \ldots, N\}$ at times $N^2, 2N^2, \ldots$, if it is still alive, we obtain

$$P(\sigma_N > t) \leq [1 - P(\sigma_N < N^2)]^{\lfloor \frac{t}{N^2} \rfloor}.$$

Let L_t and R_t be the walks in the edge construction of the original (nonrestarted) process. Let $T_j = \inf\{t : R_t = i_j\}$ for $j \in \{0, \ldots, K\}$. Then

$$P(\sigma_N > t) \leq [1 - P(T_0 < N^2)]^{\lfloor \frac{t}{N^2} \rfloor}.$$

But

$$P(T_0 < N^2) \geq P\left(T_{j-1} - T_j < \frac{N^2}{K}, \text{ for all } j = 1, \ldots, K\right) = \prod_{j=1}^{K} P\left(T_{j-1} - T_j < \frac{N^2}{K}\right).$$

We can apply the previous lemma to bound the chance that after visiting i_j, R_t visits i_{j-1} before visiting $i_j + 1$ and before N^2/K time units have elapsed,

$$P\left(T_{j-1} - T_j < \frac{N^2}{K}\right) \geq \exp(-(\alpha_j \log \lambda_j + \epsilon/2K)N),$$

for all j and all sufficiently large N. Thus

$$P(\sigma_N > t) \leq \left[1 - \exp\left(-\left(\sum_{j=1}^{K} \alpha_j \log \lambda_j + \frac{\epsilon}{2}\right) N\right)\right]^{\lfloor \frac{t}{N^2} \rfloor}.$$

Taking $t = \exp((\sum_{j=1}^{K} \alpha_j \log \lambda_j + \epsilon)N)$ gives the result. □

Following Durrett and Schonmann [4], we prove the lower bound using the following estimate of the chance that a random walk makes an excursion against its drift. Our Lemma 19 is a generalization of their result (5.1) to walks in piecewise-homogeneous environments.

Lemma 19 *Let (K, λ, α) be a profile such that $\lambda_j > 1$ for all $j \in \{1, \ldots, K\}$. For each $\beta > 0$, there exist positive constants C and δ such that for all sufficiently large N, if L_t is the left edge in the edge construction of the piecewise-homogeneous biased voter model on $\{1, \ldots, N\}$ and $m \in \{1, \ldots, N\}$ and $t \leq \exp(\beta N)$,*

$$P\left(\max_{t-1 \leq s \leq t} L_s \geq m\right) \leq C N^{2K+2} \lambda_k^{-(m-i_{k-1})} \prod_{j=1}^{k-1} \lambda_j^{-\alpha_j N} + C e^{-\delta N^2},$$

where k satisfies $i_{k-1} < m \leq i_k$.

Proof. First we will find exponents that are uniform over N on the tail probabilities of the length of an excursion of L_t from 1, and the number of excursions that L_t makes from 1 during a finite time interval. Let $S_0 = 0$, and for $i \geq 1$ let

$$U_i = \inf\{t > S_{i-1} : L_t = 2\}$$
$$S_i = \inf\{t > U_i : L_t = 1\}.$$

Durrett and Liu [3] show that if L'_t is a random walk on $\mathbb{N}$ with rates $q(n, n+1) = 1$ and $q(n, n-1) = \min(\lambda_1, \ldots, \lambda_K)$ with $L'_0 = 1$, and

$$U = \inf\{t > 0 : L'_t = 2\}$$
$$S = \inf\{t > U : L'_t = 1\},$$

then there exists positive constants C and δ so that for all $t \geq 0$,

$$P(S > t) \leq C \exp(-\delta t).$$

This follows by first defining $V = S - U$ so that

$$P(S > t) \leq P(U > t/2) + P(V > t/2).$$

We then observe that U is exponentially distributed with mean 1, so $P(U > t/2) \leq \exp(-t/2)$. During $[U, S)$, L_t behaves like a random walk on $\mathbb{Z}$, starting at 2, which drifts to the left, so Lemma 15 provides an exponent to the tail of V. Durrett and Liu [3, p. 1171] supply additional details. Since L_t and L'_tcan be coupled so that $S_1 \leq S$, their argument also shows that

$$P(S_1 > t) \leq C \exp(-\delta t).$$

If Y_t is a Poisson process with rate 1, then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$P(Y_t > (1+\epsilon)t) \leq \exp(-\delta t)$$

for all $t \geq 0$. Let $T(t) = \sup\{n : U_n \leq t\}$. Then $T(t) \leq_{\text{st}} Y_t$, so $P(T(t) > 2t) \leq \exp(-\delta t)$. Here and below we redefine C and δ from line to line to simplify the notation.

Using these facts, we will find $C > 0$ and $\delta > 0$ such that for all sufficiently large N and all $m \in \{1, \ldots, N\}$,

$$P\left(\max_{0 \leq s \leq S_1} L_s \geq m, S_1 \leq N^2\right) \leq CN^{2K} \lambda_k^{-(m-i_{k-1})} \prod_{j=1}^{k-1} \lambda_j^{-\alpha_j N} + Ce^{-\delta N^2}. \tag{25}$$

To do this, form K independent random walks $X_t^1, \ldots, X_t^K$, the jth living on $\{i_{j-1}+1, \ldots, i_j\}$ with rate λ_j of moving left and rate 1 of moving right, and starting in state $i_{j-1}+1$ at time 0. Using the graphical construction described in Section 3, these walks can be coupled to L_t so that if L_t reaches m before time τ, then X_t^j reaches i_j before time τ for all $j < k$, and X_t^k reaches m before time τ. Then

$$P\left(\max_{0 \leq s \leq S_1} L_s \geq m, S_1 \leq N^2\right) \leq P\left(\max_{0 \leq s \leq N^2} X_s^k \geq m\right) \prod_{j=1}^{k-1} P\left(\max_{0 \leq s \leq N^2} X_s^j = i_j\right).$$

Fix $j \in \{1, \ldots, K\}$ and let M be the number of times that X_t^j leaves $i_{j-1}+1$ before time N^2. Let S be the first time that X_t^j returns to $i_{j-1}+1$. Then there exists $\epsilon > 0$ and $\delta > 0$ such that for all N and all $l \in \{i_{j-1}+1, \ldots, i_j\}$,

$$P\left(\max_{0 \le s \le N^2} X_s^j \ge l\right) \le P\left(M > 2N^2\right) + 2N^2 P\left(\max_{s \le S} X_t^j \ge l\right)$$
$$\le \exp(-\delta N^2) + 2N^2 \frac{\lambda_j^2 - \lambda_j}{\lambda_j^{l-i_{j-1}} - \lambda_j}.$$

There exists $C > 0$ such that for all $x \ge 1$ and all $j \in \{1, \ldots, K\}$,

$$\frac{\lambda_j - 1}{\lambda_j^x - 1} \le C\lambda_j^{-x}.$$

Thus

$$P\left(\max_{0 \le s \le N^2} X_s^k \ge m\right) \le \exp(-\delta N^2) + 2N^2 C \lambda_k^{-(m-i_{k-1})}.$$

and

$$P\left(\max_{0 \le s \le N^2} X_s^j = i_j\right) \le \exp(-\delta N^2) + 2N^2 C \lambda_j^{-(\alpha_j N - 2)}.$$

This establishes (25). Returning to L_t, let

$$A_N = \{S_i - S_{i-1} > N^2 \text{ for some } 1 \le i \le T(\exp(\beta N))\}$$
$$B_N^t = \{T(t) - T(t - N^2) > 2N^2\}.$$

Then

$$P(A_N) \le P(T(\exp(\beta N)) \ge 2\exp(\beta N)) + 2\exp(\beta N) P(S_1 \ge N^2)$$
$$\le C\exp(-\delta \exp(\beta N)) + C\exp(\beta N - \delta N^2)$$
$$\le C\exp(-\delta N^2), \tag{26}$$

and

$$P(B_N^t) \le \exp(-\delta N^2). \tag{27}$$

Thus

$$P\left(\max_{t-1 \le s \le t} L_s \ge m\right) \le P(A_N) + P(B_N^t) + 2N^2 P\left(\max_{0 \le s \le S_1} L_s = m, S_1 \le N^2\right)$$

Applying the bounds (25), (26), and (27) completes the proof. □

Lemma 20 *If (K, λ, α) is a profile such that $\lambda_j > 1$ for all $j \in \{1, \ldots, K\}$, then*

$$P\left(\frac{\log \sigma_N}{N} < \sum_{j=1}^{K} \alpha_j \log(\lambda_j) - \epsilon\right) \to 0$$

as $N \to \infty$ for all $\epsilon > 0$, and

$$\frac{\log E[\sigma_N]}{N} \to \sum_{j=1}^{K} \alpha_j \log \lambda_j.$$

Proof. We follow Durrett and Schonmann [4]. If $\sigma_N \leq T$ then there exists a $t < T$ and a $m \in \{1, \ldots, N\}$ such that $L_t = R_t = m$. Write $T = \exp((\sum_{j=1}^K \alpha_j \log(\lambda_j) - \epsilon)N)$ and fix $\beta > \sum_{j=1}^K \alpha_j \log(\lambda_j)$. Then by the union bound and the previous lemma,

$$P(\sigma_N \leq T) \leq \sum_{t=1}^{\lceil T \rceil} \sum_{m=1}^{N} P\left(\max_{t-1 \leq s \leq t} L_s \geq m\right) P\left(\min_{t-1 \leq s \leq t} R_s \leq m\right)$$

$$\leq (T+1)N \left(CN^{4K+4} \prod_{j=1}^{K} \lambda_j^{-\alpha_j N} + C\exp(-\delta N^2)\right),$$

which gives the first result since the right-hand side converges to zero. The proof that $(\log E[\sigma_N]/N)$ converges is the same as for the contact process, with $\log^+ \lambda$ in place of γ_2. □

We omit the proof of Theorem 8 because it is essentially the same as the proof of Theorem 5 for the contact process.

Proof of Theorem 9. Let (K, λ, α) be any feasible profile. Theorem 8 shows that

$$\liminf_{N\to\infty} \frac{\log E[\sigma_N]}{N} \leq \sum_{j:\lambda_j > 1} \alpha_j \log \lambda_j = \sum_{j=1}^{K} \alpha_j (\log \lambda_j)^+ \leq h(\lambda_0 + \eta),$$

where $h(\cdot)$ is the concave hull of $(\log x)^+$ on $[\lambda_0, \infty)$. Thus $R^*(\lambda_0, \eta) \leq h(\lambda_0 + \eta)$. If $\lambda_0 \geq 1$, then the concavity of log gives $h(\lambda_0 + \eta) = \log(\lambda_0 + \eta)$, so $R^*(\lambda_0, \eta) = \log(\lambda_0 + \eta)$, since it is achieved by the uniform profile. If $\lambda_0 < 1$, then observe that there is a unique $\lambda_1 > 1$ that solves

$$1 - \frac{\lambda_0}{\lambda_1} = \log \lambda_1,$$

and define

$$f(x) = \begin{cases} (x - \lambda_0)/\lambda_1 & \lambda_0 \leq x \leq \lambda_1 \\ \log x & x > \lambda_1 \end{cases}.$$

Observe that f is concave on $[\lambda_0, \infty)$ and $f(x) \geq (\log x)^+$ for $x \geq \lambda_0$. The latter claim is clear for $x > \lambda_1$ and $\lambda_0 \leq x \leq 1$. Observing that $f(x) - \log x$ is nonincreasing on $[1, \lambda_1]$ and equal to 0 at $x = \lambda_1$ shows it for $(1, \lambda_1]$.

Since f is concave and $f(x) \geq (\log x)^+$, we have $f(x) \geq h(x)$, so that $R^*(\lambda_0, \eta) \leq f(x)$. If $\lambda_0 + \eta \geq \lambda_1$, then the uniform profile has exponent $\log(\lambda_0 + \eta) = f(\lambda_0 + \eta)$, which implies that $R^*(\lambda_0, \eta) = \log(\lambda_0 + \eta)$. If $\lambda_0 + \eta < \lambda_1$, then some simple algebra shows that the profile stated in the theorem is feasible and has exponent $\eta/\lambda_1 = f(\lambda_0 + \eta)$, which implies that $R^*(\lambda_0, \eta) = \eta/\lambda_1$. □

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