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Kalman Filtering with Intermittent Observations

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Abstract

Motivated by our experience in building sensor networks for navigation as part of the Networked Embedded Systems Technology (NEST) project at Berkeley, we consider the problem of performing Kalman filtering with intermittent observations. When data travel along unreliable communication channels in a large, wireless, multi-hop sensor network, the effect of communication delays and loss of information in the control loop cannot be neglected. We address this problem starting from the discrete Kalman filtering formulation, and modeling the arrival of the observation as a random process. We study the statistical convergence properties of the estimation error covariance, showing the existence of a critical value for the arrival rate of the observations, beyond which a transition to an unbounded error occurs.

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I. Introduction

Advances in VLSI and MEMS technology have boosted the development of micro sensor integrated systems. Such systems combine computing, storage, radio technology, and energy source on a single chip [1], [2]. When distributed over a wide area, networks of sensors can perform a variety of tasks that range from environmental monitoring and military surveillance, to navigation and control of a moving vehicle [3] [4] [5]. A common feature of these systems is the presence of significant communication delays and data loss across the network. From the point of view of control theory, significant delay is equivalent to loss, as data needs to arrive at its destination in time to be used for control. In short, communication and control become tightly coupled such that the two issues cannot be addressed independently.

Consider, for example, the problem of navigating a vehicle based on the estimate from a sensor web of its current position and velocity. The measurements underlying this estimate can be lost or delayed due to the unreliability of the wireless links. What is the amount of data loss that the control loop can tolerate to reliably perform the navigation task? Can communication protocols be designed to satisfy this constraint? At Berkeley, we have faced these kinds of questions in building sensor networks for pursuit evasion games as part of the Network Embedded Systems Technology (NEST) project [2]. Practical advances in the design of these systems are described in [6]. The goal of this paper is to examine some control-theoretic implications of using sensor networks for control. These require a generalization of classical control techniques that explicitly take into account the stochastic nature of the communication channel.

In our setting, the sensor network provides observed data that are used to estimate the state of a controlled system, and this estimate is then used for control. We study the effect of data losses due to the unreliability of the network links. We generalize the most ubiquitous recursive estimation technique in control—the discrete Kalman filter [7]—modeling the arrival of an observation as a random process whose parameters are related to the characteristics of the communication channel. We characterize the statistical convergence of the expected estimation error covariance in this setting.

The classical theory relies on several prior assumptions that guarantee convergence of the

Kalman filter. Consider the following discrete time linear dynamical system:

$$\begin{aligned} x_{t+1} &= Ax_t + w_t \\ y_t &= Cx_t + v_t, \end{aligned} \quad (1)$$

where $x_t \in \Re^n$ is the state vector, $y_t \in \Re^m$ the output vector, $w_t \in \Re^p$ and $v_t \in \Re^m$ are Gaussian random vectors with zero mean and covariance matrices $Q \geq 0$ and $R > 0$, respectively. w_t is independent of w_s for $s < t$. We assume that the initial state, x_0, is also a Gaussian vector of zero mean and covariance Σ_0. It is well known that, under the hypothesis of stabilizability of the pair (A, Q) and detectability of the pair (A, C), the estimation error covariance of the Kalman filter converges to a unique value from any initial condition [8].

These classical assumptions have been relaxed in various ways [8]. Extended Kalman filtering attempts to cope with nonlinearities in the model; particle filtering is also appropriate for nonlinear models, and additionally does not require that the noise model be Gaussian. More recently, more general observation processes have been studied. In particular, in [9], [10] the case in which observations are randomly spaced in time according to a Poisson process has been studied, where the underlying dynamics evolve in continuous time. These authors showed the existence of a lower bound on the arrival rate of the observations below which it is possible to maintain the estimation error covariance below a fixed value, with high probability. The results were restricted to scalar SISO systems.

We approach a similar problem within the framework of discrete time, and provide results for general n-dimensional MIMO systems. In particular, we consider a discrete-time system in which the arrival of an observation is a Bernoulli process with parameter $0 < \lambda < 1$, and, rather than asking for the estimation error covariance to be bounded with high probability, we study the asymptotic behavior (in time) of its average. Our main contribution is to show that, depending on the eigenvalues of the matrix A, and on the structure of the matrix C, there exists a critical value λ_c, such that if the probability of arrival of an observation at time t is $\lambda > \lambda_c$, then the expectation of the estimation error covariance is always finite (provided that the usual stabilizability and detectability hypotheses are satisfied). If $\lambda \leq \lambda_c$, then the expectation of the estimation error covariance tends to infinity. We give explicit upper and lower bounds on λ_c, and show that they are tight in some special cases.

Philosophically this result can be seen as another manifestation of the well known *uncertainty* *threshold principle* [11], [12]. This principle states that optimum long-range control of a dynamical system with uncertainty parameters is possible if and only if the uncertainty does not exceed a given threshold. The uncertainty is modelled as white noise scalar sequences acting on the system and control matrices. In our case, the result is for optimal estimation, rather than optimal control, and the uncertainty is due to the random arrival of the observation, with the randomness arising from losses in the network.

The paper is organized as follows. In section II we formalize the problem of Kalman filtering with intermittent observations. In section III we provide upper and lower bounds on the average estimation error covariance of the Kalman filter, and find the conditions on the observation arrival probability λ for which the upper bound converges to a fixed point, and for which the lower bound diverges. Section IV describes the scalar case and gives an intuitive understanding of the results. Finally, in section V, we state our conclusions and give directions for future work.

II. Problem Formulation

Consider the canonical state estimation problem. We define the arrival of the observation at time t as a binary random variable γ_t, with probability distribution $p_{\gamma_t}(1) = \lambda$, and with γ_t independent of γ_s if $t \neq s$. The output noise v_t is defined in the following way:

$$p(v_t|\gamma_t) = \begin{cases} \mathcal{N}(0, R) & : \quad \gamma_t = 1 \\ \mathcal{N}(0, \sigma^2 I) & : \quad \gamma_t = 0, \end{cases}$$

for some σ^2 . Therefore, the variance of the observation at time t is R if γ_t is 1, and $\sigma^2 I$ otherwise. In reality the absence of observation corresponds to the limiting case of $\sigma \to \infty$. Our approach is to re-derive the Kalman filter equations using a "dummy" observation with a given variance when the real observation does not arrive, and then take the limit as $\sigma \to \infty$.

First let us define:

$$\hat{x}_{t|t} \triangleq \mathbb{E}[x_t|\mathbf{y}_t, \gamma_t] \tag{2}$$

$$P_{t|t} \triangleq \mathbb{E}[(x_t x_t'|\mathbf{y}_t, \gamma_t] \tag{3}$$

$$\hat{x}_{t+1|t} \triangleq \mathbb{E}[x_{t+1}|\mathbf{y}_t, \gamma_{t+1}] \tag{4}$$

$$P_{t+1|t} \triangleq \mathbb{E}[x_{t+1} x_{t+1}'|\mathbf{y}_t, \gamma_{t+1}] \tag{5}$$

$$\hat{y}_{t+1|t} \triangleq \mathbb{E}[y_{t+1}|\mathbf{y}_t, \gamma_{t+1}], \tag{6}$$

where we have defined the vectors $\mathbf{y_t} \triangleq [y_0, \ldots, y_t]'$ and $\gamma_t \triangleq [\gamma_0, \ldots, \gamma_t]'$. Using the Dirac delta $\delta(\cdot)$ we have:

$$\mathbb{E}[(y_{t+1} - \hat{y}_{t+1|t})(x_{t+1} - \hat{x}_{t+1|t})'|\mathbf{y_t}, \gamma_{t+1}] = CP_{t+1|t} \tag{7}$$

$$\mathbb{E}[(y_{t+1} - \hat{y}_{t+1|t})(y_{t+1} - \hat{y}_{t+1|t})'|\mathbf{y_t}, \gamma_{t+1}] = CP_{t+1|t}C' + \delta(\gamma_{t+1} - 1)R + \delta(\gamma_{t+1})\sigma^2 I, \tag{8}$$

and it follows that the random variables x_{t+1} and y_{t+1}, conditioned on the output $\mathbf{y_t}$ and on the arrivals γ_{t+1}, are jointly gaussian with mean

$$\mathbb{E}[x_{t+1}, y_{t+1}|\mathbf{y_t}, \gamma_{t+1}] = \begin{pmatrix} \hat{x}_{t+1|t} \\ C\hat{x}_{t+1|t} \end{pmatrix},$$

and covariance

$$COV(x_{t+1}, y_{t+1}|\mathbf{y_t}, \gamma_{t+1}) = \begin{pmatrix} P_{t+1|t} & P_{t+1|t}C' \\ CP_{t+1|t} & CP_{t+1|t}C' + \delta(\gamma_{t+1} - 1)R + \delta(\gamma_{t+1})\sigma^2 I \end{pmatrix}. \tag{9}$$

Hence, the Kalman filter equations are modified as follows:

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t} \tag{10}$$

$$P_{t+1|t} = AP_{t|t}A' + Q \tag{11}$$

$$\hat{x}_{t+1|t+1} = \hat{x}_{t+1|t} + P_{t+1|t}C'(CP_{t+1|t}C' + \delta(\gamma_{t+1} - 1)R + \delta(\gamma_{t+1})\sigma^2 I)^{-1}(y_{t+1} - C\hat{x}_{t+1|t}) \tag{12}$$

$$P_{t+1|t+1} = P_{t+1|t} - P_{t+1|t}C'((CP_{t+1|t}C' + \delta(\gamma_{t+1} - 1)R + \delta(\gamma_{t+1})\sigma^2 I)^{-1}CP_{t+1|t}. \tag{13}$$

Taking the limit as $\sigma \to \infty$, the update equations (12) and (13) can be rewritten as follows:

$$\hat{x}_{t+1|t+1} = \hat{x}_{t+1|t} + \gamma_{t+1}P_{t+1|t}C'(CP_{t+1|t}C' + R)^{-1}(y_{t+1} - C\hat{x}_{t+1|t}) \tag{14}$$

$$P_{t+1|t+1} = P_{t+1|t} - \gamma_{t+1}P_{t+1|t}C'(CP_{t+1|t}C' + R)^{-1}CP_{t+1|t}. \tag{15}$$

Note that performing this limit corresponds *exactly* to propagating the previous state when there is no observation update available at time t. We also point out the main difference from the standard Kalman filter formulation: Both $\hat{x}_{t+1|t+1}$ and $P_{t+1|t+1}$ are now random variables, being a function of γ_{t+1}, which is itself random.

Given the new formulation, we now study the Riccati equation of the state error covariance matrix in this generalized setting and provide deterministic upper and lower bounds on its expectation. We then characterize the convergence of these upper and lower bounds, as a function of the arrival probability λ of the observation.

III. Convergence conditions and phase transition

It is easy to verify that the modified Kalman filter Equations (11) and (15), can be rewritten as follows:

$$P_{t+1} = AP_tA' + Q - \gamma_{t+1}\, AP_tC'(CP_tC' + R)^{-1}CP_tA', \tag{16}$$

where we use the simplified notation $P_t = P_{t|t-1}$. Since the sequence $\{\gamma_t\}_0^\infty$ is random, the modified Kalman filter iteration is stochastic and cannot be determined off-line. Therefore, only statistical properties can be deduced. In this section we show the existence of a critical value λ_c for the arrival probability of the observation update, such that for $\lambda > \lambda_c$ the mean state covariance $\mathbb{E}[P_t]$ is bounded for all initial conditions, and for $\lambda \leq \lambda_c$ the mean state covariance diverges for some initial condition. We also find a lower bound $\underline{\lambda}$, and upper bound $\overline{\lambda}$, for the critical probability λ_c, i.e., $\underline{\lambda} \leq \lambda_c \leq \overline{\lambda}$. The lower bound is expressed in closed form, the upper bound is the solution of a linear matrix inequality (LMI). In some special cases the two bounds coincide, giving a tight estimate. Finally, we show numerical algorithms to compute a lower bound $\bar{S}$, and upper bound $\bar{V}$, for $\lim_{t\to\infty}\mathbb{E}[P_t]$, when it is bounded.

First, we define the modified algebraic Riccati equation (MARE) for the Kalman filter with intermittent observations as follows,

$$g_\lambda(X) = AXA' + Q - \lambda\, AXC'(CXC' + R)^{-1}CXA'. \tag{17}$$

Our results derive from two principal facts: the first is that concavity of the modified algebraic Riccati equation for our filter with intermittent observations allows use of Jensen's inequality to find an upper bound on the mean state covariance; the second is that all the operators we use to estimate upper and lower bounds are monotonically increasing, therefore if a fixed point exists, it is also stable.

We formally state all main results in form of theorems. Omitted proofs appear in the Appendix. The first theorem expresses convergence properties of the MARE.

Theorem 1. *Consider the operator* $\phi(K,X) = (1-\lambda)(AXA' + Q) + \lambda(FXF' + V)$, *where* $F = A + KC$, $V = Q + KRK'$. *Suppose there exists a matrix* $\tilde{K}$ *and a positive definite matrix* $\tilde{P}$ *such that*

$$\tilde{P} > 0 \quad and \quad \tilde{P} > \phi(\tilde{K}, \tilde{P})$$

Then,

(a) for any initial condition $P_0 \geq 0$, the MARE converges, and the limit is independent of the initial condition:

$$\lim_{t\to\infty} P_t = \lim_{t\to\infty} g_\lambda^t(P_0) = \overline{P}$$

(b) $\overline{P}$ is the unique positive semi-definite solution of the MARE.

The next theorem states the existence of a phase transition.

Theorem 2. *If $(A, Q^{\frac{1}{2}})$ is controllable, (A, C) is detectable, and A is unstable, then there exists a $\lambda_c \in [0, 1)$ such that*

$$\lim_{t\to\infty} \mathbb{E}[P_t] = +\infty \qquad \text{for } 0 \leq \lambda \leq \lambda_c \text{ and some initial condition } P_0 \geq 0 \tag{18}$$

$$\mathbb{E}[P_t] \leq M_{P_0} \ \ \forall t \qquad \text{for } \lambda_c < \lambda \leq 1 \text{ and any initial condition } P_0 \geq 0 \tag{19}$$

where $M_{P_0} > 0$ depends on the initial condition $P_0 \geq 0$.

The next theorem gives upper and lower bounds for the critical probability λ_c.

Theorem 3. *Let*

$$\underline{\lambda} = \operatorname{arginf}_\lambda[\exists \hat{S} \mid \hat{S} = (1-\lambda)A\hat{S}A' + Q] = 1 - \frac{1}{\alpha^2} \tag{20}$$

$$\overline{\lambda} = \operatorname{arginf}_\lambda[\exists(\hat{K}, \hat{X}) \mid \hat{X} > \phi(\hat{K}, \hat{X})] \tag{21}$$

where $\alpha = \max_i |\sigma_i|$ and σ_i are the eigenvalues of A. Then

$$\underline{\lambda} \leq \lambda_c \leq \overline{\lambda}. \tag{22}$$

Finally, the following theorem gives an estimate of the limit of the mean covariance matrix $\mathbb{E}[P_t]$, when this is bounded.

Theorem 4. *Assume that $(A, Q^{\frac{1}{2}})$ is controllable, (A, C) is detectable and $\lambda > \overline{\lambda}$, where $\overline{\lambda}$ is defined in Theorem 4. Then*

$$0 \leq \bar{S} \leq \lim_{t\to\infty} \mathbb{E}[P_t] \leq \bar{V} \qquad \forall\, \mathbb{E}[P_0] \geq 0 \tag{23}$$

where $\bar{S} = (1-\lambda)A\bar{S}A' + Q$ and $\bar{V} = g_\lambda(\bar{V})$.

The previous theorems give lower and upper bounds for both the critical probability λ_c and for the mean error covariance $\mathbb{E}[P_t]$. The lower bound $\underline{\lambda}$ is expressed in closed form. We now resort to numerical algorithms for the computation of the remaining bounds $\overline{\lambda}, \bar{S}, \bar{V}$.

The computation of the upper bound $\overline{\lambda}$ can be reformulated as the iteration an LMI feasibility problem. To do so we need the following theorem:

Theorem 5. *If $(A, Q^{\frac{1}{2}})$ is controllable and (A, C) is detectable,then following statements are equivalent:*

(a) $\exists \bar{X}$ *such that* $\bar{X} > g_\lambda(\bar{X})$

(b) $\exists \bar{K}, \bar{X} > 0$ *such that* $\bar{X} > \phi(\bar{K}, \bar{X})$

(c) $\exists \bar{Z}$ *and* $0 < \bar{Y} \leq I$ *such that*

$$\Psi_\lambda(Y, Z) = \begin{bmatrix} Y & \sqrt{\lambda}(YA + ZC) & \sqrt{1-\lambda}YA \\ \sqrt{\lambda}(A'Y + C'Z') & Y & 0 \\ \sqrt{1-\lambda}A'Y & 0 & Y \end{bmatrix} > 0.$$

Proof: (a)$\Longrightarrow$(b) If $\bar{X} > g_\lambda(\bar{X})$ exists, then $\bar{X} > 0$ by Lemma 1(g). Let $\bar{K} = K_{\bar{X}}$. Then $\bar{X} > g_\lambda(\bar{X}) = \phi(\bar{K}, \bar{X})$ which proves the statement. (b)$\Longrightarrow$(a) Clearly $\bar{X} > \phi(\bar{K}, \bar{X}) \geq g_\lambda(\bar{X})$ which proves the statement. (b)$\Longleftrightarrow$(c) Let $F = A + KC$, then:

$$X > (1-\lambda)AXA' + \lambda FXF' + Q + \lambda KRK'$$

is equivalent to

$$\begin{bmatrix} X - (1-\lambda)AXA' & \sqrt{\lambda}F \\ \sqrt{\lambda}F' & X^{-1} \end{bmatrix} > 0,$$

where we used the Schur complement decomposition and the fact that $X - (1-\lambda)AXA' \geq \lambda FXF' + Q + \lambda KRK' \geq Q > 0$. Using one more time the Schur complement decomposition on the first element of the matrix we obtain

$$\Theta = \begin{bmatrix} X & \sqrt{\lambda}F & \sqrt{1-\lambda}A \\ \sqrt{\lambda}F' & X^{-1} & 0 \\ \sqrt{1-\lambda}A' & 0 & X^{-1} \end{bmatrix} > 0.$$

This is equivalent to

$$\Lambda = \begin{bmatrix} X^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \Theta \begin{bmatrix} X^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} > 0$$

$$= \begin{bmatrix} X^{-1} & \sqrt{\lambda} X^{-1} F & \sqrt{1-\lambda} X^{-1} A \\ \sqrt{\lambda} F' X^{-1} & X^{-1} & 0 \\ \sqrt{1-\lambda} A' X^{-1} & 0 & X^{-1} \end{bmatrix} > 0.$$

Let us consider the following change of variable $Y = X^{-1} > 0$ and $Z = X^{-1}K$, then the previous LMI is equivalent to:

$$\Psi(Y,K) = \begin{bmatrix} Y & \sqrt{\lambda}(YA + ZC) & \sqrt{1-\lambda} YA \\ \sqrt{\lambda}(A'Y + C'Z') & Y & 0 \\ \sqrt{1-\lambda} A'Y & 0 & Y \end{bmatrix} > 0.$$

Since $\Psi(Y,K)$ is linear, i.e. $\Psi(\alpha Y, \alpha K) = \alpha \Psi(Y,K)$, then Y can be restricted to $Y \leq I$, which completes the theorem. ■

Combining theorems 3 and 5 we immediately have the following corollary

Corollary 1. *The upper bound $\overline{\lambda}$ is given by the solution of the following optimization problem,*

$$\overline{\lambda} = \text{argmin}_{\lambda} \Psi(Y,Z) > 0, \quad 0 \leq Y \leq I.$$

The one above is a quasi-convex optimization problem in the variables (λ, Y, Z) and the solution can be obtained by iterating LMI feasibility problems and using bisection for the variable λ.

The lower bound $\tilde{S}$ for the mean covariance matrix can be easily obtained via standard Lyapunov Equation solvers. The upper bound $\bar{V}$ can be found by iterating the MARE or by solving an semi-definite programming (SDP) problem as shown in the following.

Theorem 6. *If $\lambda > \overline{\lambda}$, then the matrix $\bar{V} = g_{\lambda}(V)$ is given by:*

(a) $\lim_{t\to\infty} V_{t+1} = V_t$ *where* $V_0 \geq 0$

(b)

$$\begin{aligned} & \operatorname{argmax}_V \quad Trace(V) \\ \textit{subject to} \quad & \begin{bmatrix} AVA' - V & \sqrt{\lambda}AVC' \\ \sqrt{\lambda}CVA' & CVC' + R \end{bmatrix} \geq 0, \quad V \geq 0 \end{aligned}$$

Proof: (a) It follows directly from Theorem 1.

(b) It can be obtained by using the Schur complement decomposition on Equation $V \leq g_\lambda(V)$. Clearly the solution $\bar{V} = g_\lambda(\bar{V})$ belongs to the feasible set of the optimization problem. We now show that the solution of the optimization problem is the fixed point of the mare. Suppose it is not, i.e. $\hat{V}$ solves the optimization problem but $\hat{V} \neq g_\lambda(\hat{V})$. Since $\hat{V}$ is a feasible point of the optimization problem, then $\hat{V} < g_\lambda(\hat{V}) = \hat{\hat{V}}$. However, this implies that $Trace(\hat{V}) < Trace(\hat{\hat{V}})$, which contradicts the hypothesis of optimality of matrix $\hat{V}$. Therefore $\hat{V} = g_\lambda(\hat{V})$ and this concludes the theorem. ∎

IV. Special Cases and Examples

In this section we present some special cases in which upper and lower bounds on the critical value λ_c coincide and give some examples. From Theorem 1 follows that if there exists a $\tilde{K}$ such that F is the zero matrix, then the convergence condition of the MARE is for $\lambda > \lambda_c = 1 - 1/\alpha^2$, where $\alpha = \max_i |\sigma_i|$, and σ_i are the eigenvalues of A.

- **C is invertible.** In this case a choice of $K = -AC^{-1}$ makes $F = 0$. Note that the scalar case also falls under this category. Figure (1) shows a plot of the steady state of the upper and lower bounds versus λ in the scalar case. The discrete time LTI system used in this simulation is $A = -1.25$, $C = 1$, and v_t , w_t with zero mean and variance $R = 2.5$, $Q = 1$ respectively. For this system we have $\lambda_c = 0.36$. The transition clearly appears in the figure, where we see that the steady state value of both upper and lower bound tends to infinity as λ approaches λ_c. The dashed line shows the upper bound, the solid line the lower bound, and the dash-dot line shows the asymptote.
- **A has a single unstable eigenvalue.** In this case, regardless of the dimension of C (and as long as the couple(A, C) is detectable), we can use Kalman decomposition to bring to zero the unstable part of F and therefore to obtain tight bounds. Figure (2) shows a plot

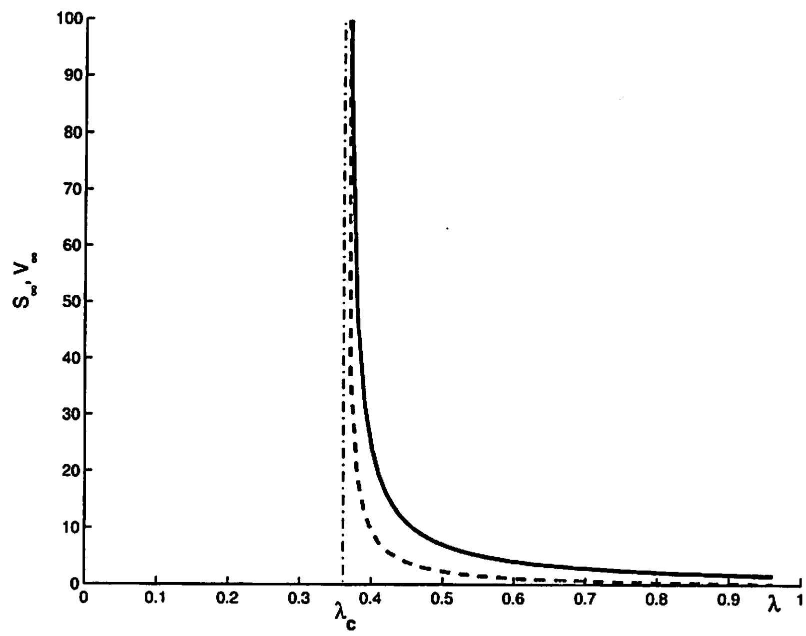

Fig. 1. Phase Transition, scalar case.

for the system $A = \begin{pmatrix} 1.25 & 1 & 0 \\ 0 & .9 & 7 \\ 0 & 0 & .60 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$

and v_t, w_t with zero mean and variance $R = 2.5$, $Q = 20 * I_{3x3}$ respectively. Once again $\lambda_c = 0.36$.

In general F cannot always be made zero and we have shown that while a lower bound on λ_c can be written in closed form, an upper bound on λ_c is the result of a LMI. Figure(3) shows an example where upper an lower bounds have different convergence conditions. The system used for this simulation is $A = \begin{pmatrix} 1.25 & 0 \\ 1 & 1.1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \end{pmatrix}$

and v_t, w_t with zero mean and variance $R = 2.5$, $Q = 20 * I_{2x2}$ respectively.

Finally, in Figure (4) we report results of another experiment, plotting the state estimation error of another system at two similar values of λ, one being below and one above the critical value. We note a dramatic change in the error at $\lambda_c \approx 0.125$. The figure on the left shows the estimation error with $\lambda = 0.1$. The figure on the right shows the estimation error for the same system evolution with $\lambda = 0.15$. In the first case the estimation error grows dramatically, making it practically useless for control purposes. In the second case, a small increase in λ reduces the

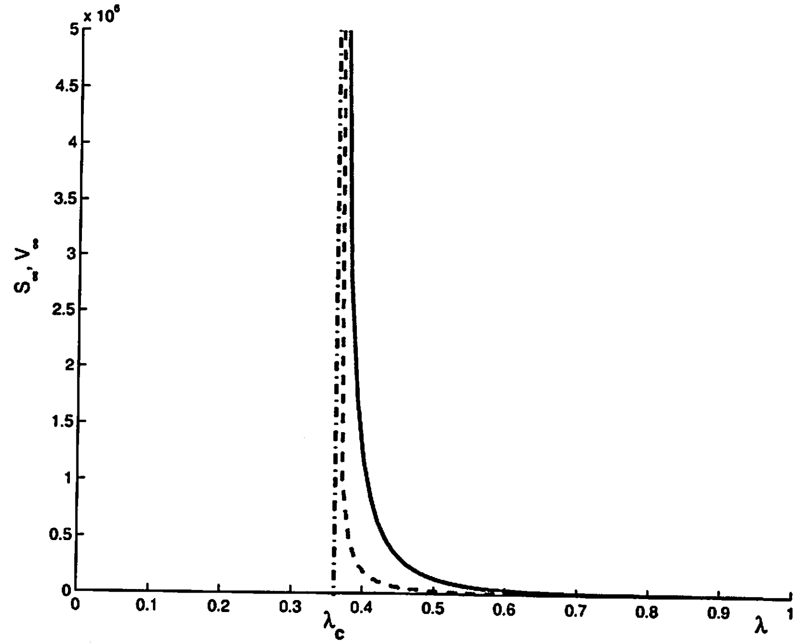

Fig. 2. Phase Transition, one unstable eigenvalue.

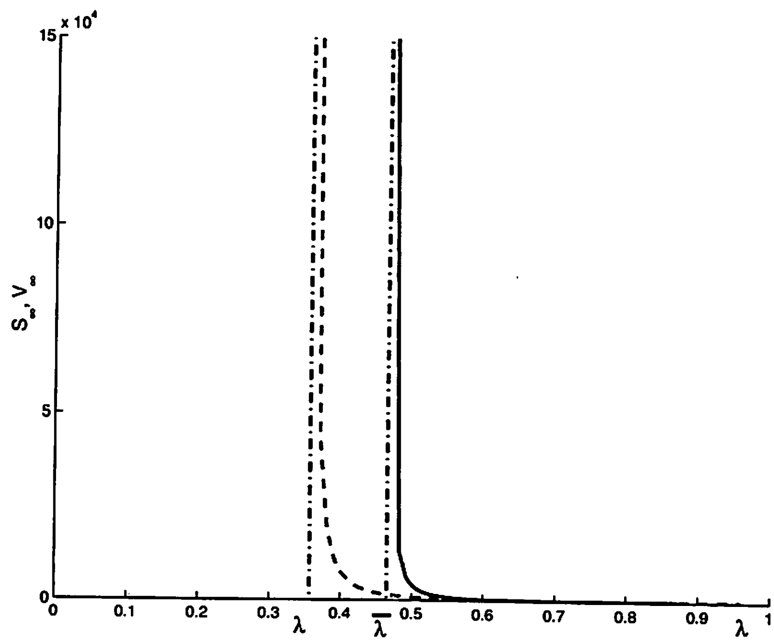

Fig. 3. Phase Transition, general case

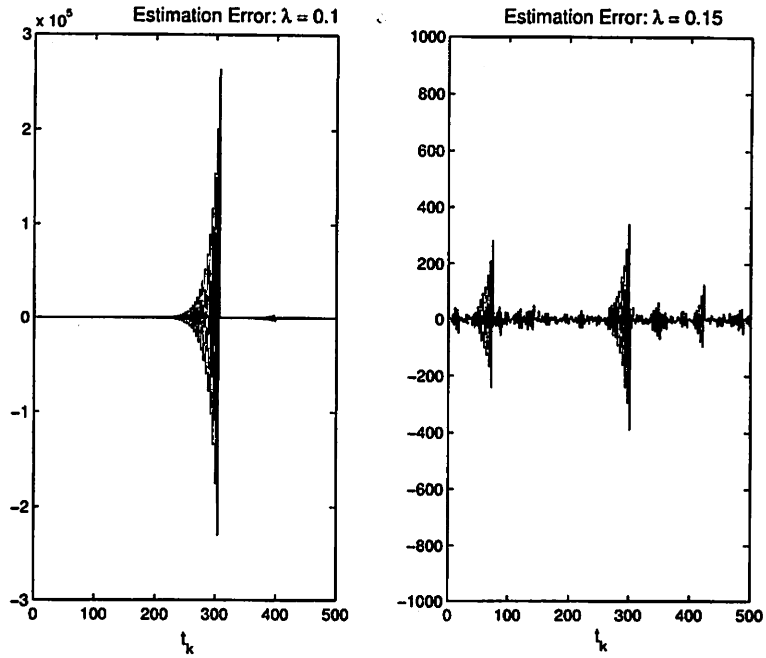

Fig. 4. Estimation error for λ below (left) and above (right) the critical value

estimation error of approximately three orders of magnitude.

V. Conclusions

In this paper we have presented an analysis of Kalman filtering in the setting of intermittent observations. We have shown how the expected estimation error covariance depends on the tradeoff between loss probability and the system dynamics. Such a result is useful to the system designer who must assess the relationship between the dynamics of the system whose state is to be estimated and the reliability of the communication channel through which that system is measured.

Our motivating application is a distributed sensor network that collects observations and sends them to one or more central units that are responsible for estimation and control. For example, in a pursuit evasion game in which mobile pursuers perform their control actions based on the current estimate of the positions of both pursuers and evaders, the sensing capability of each pursuer is generally limited, and an embedded sensor network is essential for providing a larger overall view of the terrain. The results that we have presented here can aid the designer of the sensor network in the choice of the number and disposition of the sensors.

This application also suggests a number of interesting directions for further work. For example, although we have assumed independent Bernoulli probabilities for the observation events, in the sensor network there will generally be temporal and spatial sources of variability that lead to correlations among these events. While it is possible to compute posterior state estimates in such a setting, it would be of interest to see if a priori bounds of the kind that we have obtained here can be obtained in this case. Similarly, in many situations there may be correlations between the states and the observation events; for example, such correlations will arise in the pursuit evasion game when the evaders move near the boundaries of the sensor network. Finally, the sensor network setting also suggests the use of smoothing algorithms in addition to the filtering algorithms that have been our focus here. In particular, we may be willing to tolerate a small amount of additional delay to wait for the arrival of a sensor measurement, if that measurement is expected to provide a significant reduction in uncertainty. Thus we would expect that the tradeoff that we have studied here between loss probability and the system dynamics should also be modulated in interesting ways by the delay due to smoothing.

VI. Appendix A

In order to give complete proofs of our main theorems, we need to prove some preliminary lemmas. The first one shows some useful properties of the MARE.

Lemma 1. *Let the operator*

$$\phi(K,X) = (1-\lambda)(AXA'+Q) + \lambda(FXF'+V) \tag{24}$$

where $F = A + KC, \quad V = Q + KRK'$. *Assume* $X \in \mathbb{S} = \{S \in \mathbb{R}^{n\times n} | S \geq 0\}$, $R > 0$, $Q \geq 0$, *and* $(A, Q^{\frac{1}{2}})$ *is controllable. Then the following facts are true:*

(a) *With* $K_X = -AXC'(CXC'+R)^{-1}$, $g_\lambda(X) = \phi(K_X, X)$

(b) $g_\lambda(X) = \min_K \phi(K,X) \leq \phi(K,X) \forall K$

(c) *If* $X \leq Y$, *then* $g_\lambda(X) \leq g_\lambda(Y)$

(d) *If* $\lambda_1 \leq \lambda_2$ *then* $g_{\lambda_1}(X) \geq g_{\lambda_2}(X)$

(e) *If* $\alpha \in [0,1]$, *then* $g_\lambda(\alpha X + (1-\alpha)Y) \geq \alpha g_\lambda(X) + (1-\alpha) g_\lambda(Y)$

(f) $g_\lambda(X) \geq (1-\lambda)AXA' + Q$

(g) *If* $\bar{X} \geq g_\lambda(\bar{X})$, *then* $\bar{X} > 0$

(h) If X is a random variable, then $(1-\lambda)A\mathbb{E}[X]A' + Q \leq \mathbb{E}[g_\lambda(X)] \leq g_\lambda(\mathbb{E}[X])$

Proof: (a) Define $F_X = A + K_X C$, and observe that

$$F_X X C' + K_X R = (A + K_X C) X C' + K_X R = AXC' + K_X(CXC' + R) = 0.$$

Next, we have

$$\begin{aligned} f(X) &= (1-\lambda)(AXA' + Q) + \lambda(AXA' + Q - AXC'(CXC' + R)^{-1}CXA') \\ &= (1-\lambda)(AXA' + Q) + \lambda(AXA' + Q + K_X CXA') \\ &= (1-\lambda)(AXA' + Q) + \lambda(F_X XA' + Q) \\ &= (1-\lambda)(AXA' + Q) + \lambda(F_X XA' + Q) + (F_X XC' + K_X R)K' \\ &= \phi(K_X, X) \end{aligned}$$

(b) Let $\psi(K, X) = (A + KC)X(A + KC)' + KRK' + Q$. Note that

$$\text{argmin}_K \phi(K, X) = \text{argmin}_K FXF' + V = \text{argmin}_K \psi(X, K).$$

Since $X, R \geq 0$, $\phi(K, X)$ is quadratic and convex in the variable K, therefore the minimizer can be found by solving $\frac{\partial\psi(K,X)}{\partial K} = 0$, which gives:

$$2(A + KC)XC' + 2KR = 0 \Longrightarrow K = -AXC'(CXC' + R)^{-1}.$$

Since the minimizer correspond to K_X defined above, the fact follows from fact (1)

(c) Note that $\phi(K, X)$ is affine in X. Suppose $X \leq Y$. Then

$$g_\lambda(X) = \phi(K_X, X) \leq \phi(K_Y, X) \leq \phi(K_Y, Y) = g_\lambda(Y).$$

This completes the proof.

(d) Note that $AXC'(CXC' + R)^{-1}CXA \geq 0$. Then

$$\begin{aligned} g_{\lambda_1}(X) &= AXA' + Q - \lambda_1 AXC'(CXC' + R)^{-1}CXA \\ &\geq AXA' + Q - \lambda_2 AXC'(CXC' + R)^{-1}CXA = g_{\lambda_2}(X) \end{aligned}$$

(e) Let $Z = \alpha X + (1-\alpha)Y$ where $\alpha \in [0,1]$. Then we have

$$\begin{aligned}
g_\lambda(Z) &= \phi(K_Z, Z) \\
&= \alpha(A + K_Z\, C)X(A + K_Z\, C)' + (1-\alpha)(A + K_Z\, C)Y(A + K_Z\, C)' + \\
&\quad +(\alpha + 1 - \alpha)(K_Z\, R\, K_Z' + Q) \\
&= \alpha\phi(K_Z, X) + (1-\alpha)\phi(K_Z, Y) \\
&\geq \alpha\phi(K_X, X) + (1-\alpha)\phi(K_Y, Y) \\
&= \alpha g_\lambda(X) + (1-\alpha) g_\lambda(Y).
\end{aligned} \tag{25}$$

(f) Note that $F_X X F_X' \geq 0$ and $KRK' \geq 0$ for all K and X. Then

$$\begin{aligned}
g_{\lambda_1}(X) &= \phi(K_X, X) = (1-\lambda)(AXA' + Q) + \lambda(F_X X F_X' + K_X R K_X' + Q) \\
&\geq (1-\lambda)(AXA' + Q) + \lambda Q = (1-\lambda)AXA' + Q.
\end{aligned}$$

(g) From fact (f) follows that $\bar{X} \geq g_{\lambda_1}(\bar{X}) \geq (1-\lambda)A\bar{X}A' + Q$. Let $\hat{X}$ such that $\hat{X} = (1-\lambda)A\hat{X}A' + Q$. Such $\hat{X}$ must clearly exist. Therefore $\bar{X} - \hat{X} \geq (1-\lambda)A(\bar{X} - \hat{X})A' \geq 0$. Moreover the matrix $\hat{X}$ solves the Lyapunov Equation $\hat{X} = \tilde{A}\hat{X}\tilde{A}' + Q$ where $\tilde{A} = \sqrt{1-\lambda}A$. Since $(\tilde{A}, Q^{\frac{1}{2}})$ is detectable, it follows that $\hat{X} > 0$ and so $\bar{X} > 0$, which proves the fact.

(h) Using fact (f) and linearity of expectation we have

$$\mathbb{E}[g_\lambda(X)] \geq \mathbb{E}[(1-\lambda)AXA' + Q] = (1-\lambda)A\mathbb{E}[X]A' + Q,$$

fact (e) implies that the operator $g_\lambda()$ is concave, therefore by Jensen's Inequality we have $\mathbb{E}[g_\lambda(X)] \leq g_\lambda(\mathbb{E}[X])$. ■

Lemma 2. *Let $X_{t+1} = h(X_t)$ and $Y_{t+1} = h(Y_t)$. If $h(X)$ is a monotonically increasing function then:*

$$\begin{aligned}
X_1 \geq X_0 &\Rightarrow X_{t+1} \geq X_t, \quad \forall t \geq 0 \\
X_1 \leq X_0 &\Rightarrow X_{t+1} \leq X_t, \quad \forall t \geq 0 \\
X_0 \leq Y_0 &\Rightarrow X_t \leq Y_t, \qquad \forall t \geq 0
\end{aligned}$$

Proof: This lemma can be readily proved by induction. It is true for $t = 0$, since $X_1 \geq X_0$ by definition. Now assume that $X_{t+1} \geq X_t$, then $X_{t+2} = h(X_{t+1}) \geq h(X_{t+1}) = X_{t+1}$ because of monotonicity of $h(\cdot)$. The proof for the other two cases is analogous. ■

It is important to note that while in the scalar case $X \in \mathbb{R}$ either $h(X) \leq X$ or $h(X) \geq X$; in the matrix case $X \in \mathbb{R}^{n \times n}$, it is not generally true that either $h(X) \geq X$ or $h(X) \leq X$.

This is the source of the major technical difficulty for the proof of convergence of sequences in higher dimensions. In this case convergence of a sequence $\{X_t\}_0^\infty$ is obtained by finding two other sequences, $\{Y_t\}_0^\infty, \{Z_t\}_0^\infty$ that bound X_t, i.e., $Y_t \leq X_t \leq Z_t, \forall t$, and then by showing that these two sequences converge to the same point.

The next two Lemmas show that when the MARE has a solution $\bar{P}$, this solution is also stable, i.e., every sequence based on the difference Riccati equation $P_{t+1} = g_\lambda(P_t)$ converges to $\bar{P}$ for all initial positive semidefinite conditions $P \geq 0$.

Lemma 3. *Define the linear operator*

$$\mathcal{L}(Y) = (1-\lambda)(AYA') + \lambda(FYF')$$

Suppose there exists $\overline{Y} > 0$ *such that* $\overline{Y} > \mathcal{L}(\overline{Y})$.

(a) For all $W \geq 0$,

$$\lim_{k\to\infty} \mathcal{L}^k(W) = 0$$

(b) Let $V \geq 0$ *and consider the linear system*

$$Y_{k+1} = \mathcal{L}(Y_k) + V \quad \textit{initialized at} \quad Y_0.$$

Then, the sequence Y_k *is bounded.*

Proof: (a) First observe that $0 \leq \mathcal{L}(Y)$ for all $0 \leq Y$. Also, $X \leq Y$ implies $\mathcal{L}(X) \leq \mathcal{L}(Y)$. Choose $0 \leq r < 1$ such that $\mathcal{L}(\overline{Y}) < r\overline{Y}$. Choose $0 \leq m$ such that $W \leq m\overline{Y}$. Then,

$$0 \leq \mathcal{L}^k(W) \leq m\mathcal{L}^k(\overline{Y}) < mr^k\overline{Y}$$

The assertion follows when we take the limit $r \to \infty$, on noticing that $0 \leq r < 1$.

(b) The solution of the linear iteration is

$$\begin{aligned} Y_k &= \mathcal{L}^k(Y_0) + \sum_{t=0}^{k-1} \mathcal{L}^t(V) \\ &\leq \left(m_{Y_0} r^k + \sum_{t=0}^{k-1} m_V r^t \right) \overline{Y} \\ &\leq \left(m_{Y_0} r^k + \frac{m_V}{1-r} \right) \overline{Y} \\ &\leq \left(m_{Y_0} + \frac{m_V}{1-r} \right) \overline{Y} \end{aligned}$$

proving the claim. ■

Lemma 4. *Consider the operator $\phi(K,X)$ defined in Equation (24). Suppose there exists a matrix $\overline{K}$ and a positive definite matrix $\overline{P}$ such that*

$$\overline{P} > 0 \quad \textit{and} \quad \overline{P} > \phi(\overline{K},\overline{P}).$$

Then, for any P_0, the sequence $P_t = g_\lambda^t(P_0)$ is bounded, i.e. there exists $M_{P_0} \geq 0$ dependent of P_0 such that

$$P_t \leq M \quad \textit{for all} \quad t.$$

Proof: First define the matrices $\overline{F} = A + \overline{K}C$ and consider the linear operator

$$\mathcal{L}(Y) = (1-\lambda)(AYA') + \lambda(\overline{F}Y\overline{F}')$$

Observe that

$$\overline{P} > \phi(\overline{K},\overline{P}) = \mathcal{L}(\overline{P}) + Q + \overline{K}R\overline{K}' \geq \mathcal{L}(\overline{P}).$$

Thus, $\mathcal{L}$ meets the condition of Lemma 3. Finally, using fact (b) in Lemma 1 we have

$$P_{t+1} = g_\lambda(P_t) \leq \phi(\overline{K},P_t) = \mathcal{L}(P_t) + \overline{V}.$$

Using Lemma 3, we conclude that the sequence P_t is bounded. ■

We are now ready to give proofs for Theorems 1-4.

A. *Proof of Theorem 1*

(a) We first show that the modified Riccati difference equation initialized at $Q_0 = 0$ converges. Let $Q_k = g_\lambda^k(0)$. Note that $0 = Q_0 \leq Q_1$. It follows from Lemma 1(c) that

$$Q_1 = g_\lambda(Q_0) \leq g_\lambda(Q_1) = Q_2.$$

A simple inductive argument establishes that

$$0 = Q_0 \leq Q_1 \leq Q_2 \leq \cdots \leq M_{Q_0}.$$

Here, we have used Lemma 4 to bound the trajectory. We now have a monotone non-decreasing sequence of matrices bounded above. It is a simple matter to show that the sequence converges, i.e.

$$\lim_{k\to\infty} Q_k = \overline{P}.$$

Also, we see that $\overline{P}$ is a fixed point of the modified Riccati iteration:

$$\overline{P} = g_\lambda(\overline{P}),$$

which establishes that it is *a* positive semi-definite solution of the MARE.

Next, we show that the Riccati iteration initialized at $R_0 \geq \overline{P}$ also converges, and to the same limit $\overline{P}$. First define the matrices

$$\overline{K} = -A\overline{P}C'\left(C\overline{P}C' + R\right)^{-1}, \quad \overline{F} = A + \overline{K}C$$

and consider the linear operator

$$\hat{\mathcal{L}}(Y) = (1-\lambda)(AYA') + \lambda(\overline{F}Y\overline{F}').$$

Observe that

$$\overline{P} = g_\lambda(\overline{P}) = \mathcal{L}(\overline{P}) + Q + \overline{K}R\overline{K}' > \hat{\mathcal{L}}(\overline{P}).$$

Thus, $\hat{\mathcal{L}}$ meets the condition of Lemma 3. Consequently, for all $Y \geq 0$,

$$\lim_{k\to\infty} \hat{\mathcal{L}}^k(Y) = 0.$$

Now suppose $R_0 \geq \overline{P}$. Then,

$$R_1 = g_\lambda(R_0) \geq g_\lambda(\overline{P}) = \overline{P}.$$

A simple inductive argument establishes that

$$R_k \geq \overline{P} \quad \text{for all } k.$$

Observe that

$$\begin{aligned} 0 \leq (R_{k+1} - \overline{P}) &= g_\lambda(R_k) - g_\lambda(\overline{P}) \\ &= \phi(K_{R_k}, R_k) - \phi(K_{\overline{P}}, \overline{P}) \\ &\leq \phi(K_{\overline{P}}, R_k) - \phi(K_{\overline{P}}, \overline{P}) \\ &= (1-\lambda)A(R_k - \overline{P})A' + \lambda F_{\overline{P}}(R_k - \overline{P})F_{\overline{P}}' \\ &= \hat{\mathcal{L}}(R_k - \overline{P}). \end{aligned}$$

Then, $0 \leq \lim_{k\to\infty}(R_{k+1} - \overline{P}) \leq 0$, proving the claim.

We now establish that the Riccati iteration converges to $\overline{P}$ for all initial conditions $P_0 \geq 0$. Define $Q_0 = 0$ and $R_0 = P_0 + \overline{P}$. Consider three Riccati iterations, initialized at Q_0, P_0, and R_0. Note that

$$Q_0 \leq P_0 \leq R_0.$$

It then follows from Lemma 2 that

$$Q_k \leq P_k \leq R_k \quad \text{for all } k.$$

We have already established that the Riccati equations P_k and R_k converge to $\overline{P}$. As a result, we have

$$\overline{P} = \lim_{k\to\infty} P_k \leq \lim_{k\to\infty} Q_k \leq \lim_{k\to\infty} R_k = \overline{P},$$

proving the claim.

(b) Finally, we establish that the MARE has a unique positive semi-definite solution. To this end, consider $\hat{P} = g_\lambda(\hat{P})$ and the Riccati iteration initialized at $P_0 = \hat{P}$. This yields the constant sequence

$$\hat{P}, \hat{P}, \cdots$$

However, we have shown that every Riccati iteration converges to $\overline{P}$. Thus $\overline{P} = \hat{P}$.

B. *Proof of Theorem 2*

First we note that the two cases expressed by the theorem are indeed possible. If $\lambda = 1$ the modified Riccati difference equation reduces to the standard Riccati difference equation, which is known to converge to a fixed point, under the theorem's hypotheses. Hence, the covariance matrix is always bounded in this case, for any initial condition $P_0 \geq 0$. If $\lambda = 0$ then we reduce to open loop prediction, and if the matrix A is unstable, then the covariance matrix diverges for some initial condition $P_0 \geq 0$. Next, we show the existence of a single point of transition between the two cases. Fix a $0 < \lambda_1 \leq 1$ such that $\mathbb{E}_{\lambda_1}[P_t]$ is bounded for any initial condition

$P_0 \geq 0$. Then, for any $\lambda_2 \geq \lambda_1$ $\mathbb{E}_{\lambda_2}[P_t]$ is also bounded for all $P_0 \geq 0$. In fact we have

$$\begin{aligned} \mathbb{E}_{\lambda_1}[P_{t+1}] &= \mathbb{E}_{\lambda_1}[AP_tA' + Q - \gamma_{t+1}AP_tC'(CP_tC' + R)^{-1}CP_tA] \\ &= \mathbb{E}[AP_tA' + Q - \lambda_1 AP_tC'(CP_tC' + R)^{-1}CP_tA] \\ &= \mathbb{E}[g_{\lambda_1}(P_t)] \\ &\geq \mathbb{E}[g_{\lambda_2}(P_t)] \\ &= \mathbb{E}_{\lambda_2}[P_{t+1}], \end{aligned}$$

where we exploited fact (d) of Lemma 1 to write the above inequality . We can now choose

$$\lambda_c = \{\inf \lambda^* : \lambda > \lambda^* \Rightarrow \mathbb{E}_\lambda[P_t] \text{is bounded, for all } P_0 \geq 0\},$$

completing the proof.

C. *Proof of Theorem 3*

Define the Lyapunov operator $m(X) = \tilde{A}X\tilde{A}' + Q$ where $\tilde{A} = \sqrt{1-\lambda}A$. If $(A, Q^{\frac{1}{2}})$ is controllable, also $(\tilde{A}, Q^{\frac{1}{2}})$ is controllable. Therefore, it is well known that $\hat{S} = m(\hat{S})$ has a unique strictly positive definite solution $\hat{S} > 0$ if and only if $\max_i |\sigma_i(\tilde{A})| < 1$, i.e. $\sqrt{1-\lambda}\max_i |\sigma_i(A)| < 1$, from which follows $\underline{\lambda} = 1 - \frac{1}{\alpha^2}$.

Let us consider the difference equation $S_{t+1} = m(S_t)$, $S_0 = 0$. It is clear that $S_0 = 0 \leq Q = S_1$. Since the operator $m()$ is monotonic increasing, by Lemma 2 it follows that the sequence $\{S_t\}_0^\infty$ is monotonically increasing, i.e. $S_{t+1} \geq S_t$ for all t. If $\lambda < \underline{\lambda}$ this sequence cannot be bounded, otherwise it would converge to a finite matrix $\bar{S}$, and by continuity $\bar{S} = m(\bar{S})$, which is not possible. Therefore

$$\lim_{t\to\infty} S_t = \infty.$$

Let us consider now the mean covariance matrix $\mathbb{E}[P_t]$ initialized at $\mathbb{E}[P_0] \geq 0$. Clearly $0 = S_0 \leq \mathbb{E}[P_0]$. Moreover it is also true

$$S_t \leq \mathbb{E}[P_t] \Longrightarrow S_{t+1} = (1-\lambda)AS_tA' + Q \leq (1-\lambda)A\mathbb{E}[P_t]A' + Q \leq \mathbb{E}[g_\lambda(P_t)] = \mathbb{E}[P_{t+1}],$$

where we used fact (h) from Lemma 1. By induction, it is easy to show that

$$S_t \leq \mathbb{E}[P_t] \ \forall t, \ \forall \mathbb{E}[P_0] \geq 0 \Longrightarrow \lim_{t\to\infty} \mathbb{E}[P_t] \geq \lim_{t\to\infty} S_t = \infty.$$

This implies that for any initial condition $\mathbb{E}[P_t]$ is unbounded for any $\lambda < \underline{\lambda}$, therefore $\underline{\lambda} \leq \lambda_c$, which proves the first part of the Theorem.

Now consider the sequence $V_{t+1} = g_\lambda(V_t),\ V_0 = \mathbb{E}[P_0] \geq 0$. Clearly

$$\mathbb{E}[P_t] \leq V_t \implies \mathbb{E}[P_{t+1}] = \mathbb{E}[g_\lambda(P_t)] \leq g_\lambda(\mathbb{E}[P_t]) \leq [g_\lambda(V_t)] = V_{t+1}],$$

where we used facts (c) and (h) from Lemma 1. Then a simple induction argument shows that $V_t \geq \mathbb{E}[P_t]$ for all t. Let us consider the case $\lambda > \overline{\lambda}$, therefore there exists $\hat{X}$ such that $\hat{X} \geq g_\lambda(\hat{X})$. By Lemma 1(g) $\bar{X} > 0$, therefore all hypotheses of Lemma 3 are satisfied, which implies that

$$\mathbb{E}[P_t] \leq V_t \leq M_{V_0} \ \forall t.$$

This shows that $\lambda_c \leq \overline{\lambda}$ and concludes the proof of the Theorem.

D. *Proof of Theorem 4*

Let consider the sequences $S_{t+1} = (1-\lambda)AS_tA' + Q,\ S_0 = 0$ and $V_{t+1} = g_\lambda(V_t),\ V_0 = \mathbb{E}[P_0] \geq 0$. Using the same induction arguments in Theorem 3 it is easy to show that

$$S_t \leq \mathbb{E}[P_t] \leq V_t \ \forall t.$$

From Theorem 1 follows that $\lim_{t\to\infty} V_t = \bar{V}$, where $\bar{V} = g_\lambda V$. As shown before the sequence S_t is monotonically increasing. Also it is bounded since $S_t \leq V_t \leq M$. Therefore $\lim_{t\to\infty} S_t = \bar{S}$, and by continuity $\bar{S} = (1-\lambda)A\bar{S}A' + Q$, which is a Lyapunov equation. Since $\sqrt{1-\lambda}A$ is stable and $(A, Q^{\frac{1}{2}})$ is controllable, then the solution of the Lyapunov equation is strictly positive definite, i.e. $\bar{S} > 0$. Adding all the results together we get

$$0 < \bar{S} = \lim_{t\to\infty} S_t \leq \lim_{t\to\infty} \mathbb{E}[P_t] \leq \lim_{t\to\infty} V_t = \bar{V},$$

which concludes the proof.

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