Autocorrelation matrix eigenvalues and the power spectrum

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The distribution of the eigenvalues of an autocorrelation matrix approach the power spectrum asymptotically as the order of the matrix increases (this is known as Szegö's theorem[1] [1,2]). Some students are puzzled as to why the eigenvalues of a matrix would have *any* particular relationship to a power spectrum, which seems a disconnected concept. We explore and motivate why such a relationship is to be expected. The eigenvalues of a matrix and the transfer function of a linear-time-invariant (LTI) system are different expressions of the same fundamental idea. An explicit connection arises through the mathematical equivalence of periodic discrete-time signals and circulant matrices: multiplication by a circulant matrix is one expression of the circular convolution familiar in FFT theory.

LTI systems

LTI systems, both continuous-time and discrete-time varieties, have special properties. Although our primary concern later is with the periodic discrete-time signal, we carry along three types of signals to emphasize the parallels.

Signals

These signals models find wide use in signal processing:

Continuous-time: $x(t),\ -\infty < t < \infty$

Discrete-time: $x(k),\ -\infty < k < \infty$

Periodic discrete-time (with period N): $x(k+N) = x(k),\ -\infty < k < \infty$

Complex exponentials

Although slightly different notations are used for the three signal types, a complex-exponential signal plays an important role. In the continuous-time case,

[1] We do not explicitly prove Szegö's theorem here. There are many good references that have a proof, such as [2].

$$x(t) = e^{\sigma t} \cdot e^{j\omega t} = e^{st},\ -\infty < t < \infty$$

where σ is the rate of decay, ω is the radian frequency of oscillation, and $s = \sigma + j\omega$ is a complex variable. (Note that we use $j = \sqrt{-1}$, as is common in signal processing).

In the discrete-time case,

$$x(k) = e^{\sigma k} \cdot e^{j\omega k} = z^k,\ -\infty < k < \infty$$

where $z = e^{\sigma + j\omega}$ is a complex variable (equivalent to s through the transformation $z = e^s$). An additional wrinkle in the discrete-time case is that when we add 2π to the frequency, because the independent time variable k is an integer,

$$e^{j(\omega+2\pi)k} = e^{j2\pi k}e^{j\omega k} = e^{j\omega k}.$$

Thus, complex exponentials with frequencies ω and $\omega + 2\pi$ are equivalent, a reflection of aliasing in the sampling process. Without loss of generality, therefore, we can limit frequencies to the finite interval $-\pi < \omega \le \pi$.

The periodic discrete-time case adds another interesting wrinkle. The complex exponential for this case, like all other signals, must be periodic (with period N), so that $z^{k+N} = z^k$ or $z^N = 1$. Thus, to maintain periodicity the value of z is constrained to be one of the N roots of unity (points equally spaced on the unit circle). Define the principle N-th root of unity as $W_N = e^{j2\pi/N}$, and then the N roots become

$$z = W_N^n,\ 0 \le n \le N-1$$

Thus, we conclude in this case that $\sigma = 0$ (since otherwise would destroy periodicity) and the available frequencies are spaced at uniformly spaced multiples of $2\pi/N$, $\omega = 2\pi n/N,\ 0 \le n \le N-1$. Thus, complex exponentials in the periodic discrete-time case are limited to N possible frequencies:

$$x(k) = W_N^{nk},\ 0 \le n \le N-1,\ -\infty < k < \infty.$$

In summary, for the three cases the available frequencies of a complex exponential fall on the real line, in a finite interval, or are drawn from a finite set.

Eigenfunctions of an LTI system

Suppose we apply $x(t)$ or $x(k)$ as the input to an LTI system, and call the output $y(t)$ or $y(k)$. Then we can easily establish that complex exponentials are eigenfunctions of the LTI system; that is, the complexity of an LTI system withers in the face of a complex-exponential input, as the output is a complex exponential of the *same frequency*. An LTI system does not introduce new frequency components not present in the input. Use an arrow "$\rightarrow$" to denote the transformation of such a system. For continuous-time,

Complex-exponential input: $x(t) = e^{st} \rightarrow y(t)$

Time invariance: $x(t-\tau) = e^{s(t-\tau)} \rightarrow y(t-\tau)$

Linearity: $e^{-s\tau} \cdot x(t) = e^{-s\tau} \cdot e^{st} \rightarrow e^{-s\tau} \cdot y(t)$

Since the inputs are actually identical in the second and third cases, this establishes that

$$y(t-\tau) = e^{-s\tau} \cdot y(t) \text{ or } y(t) = H(s) \cdot e^{st},$$

where we have defined $H(s) = y(0)$ to emphasize that the "gain" of the system, a complex number $H(s)$, is dependent on the frequency of the input.

The same technique applies to discrete-time complex exponential inputs:

Complex-exponential input: $x(k) = z^k \to y(k)$

Time invariance: $x(k-T) = z^{k-T} \to y(k-T)$

Linearity: $z^{-T} \cdot x(t) = z^{-T} \cdot z^k \to z^{-T} \cdot y(k)$

and thus

$$y(k-T) = z^{-T} \cdot y(k) \text{ or } y(k) = H(z) \cdot z^k$$

For the periodic discrete-time case, we do not consider anything about the *system* to be special (the system itself may be identical to the general discrete-time case), but only that the input signals are periodic with period N. This implies that the output signal must also be periodic with the same period, as follows from time invariance:

Periodic input: $x(k) \to y(k)$

Time invariance: $x(k+N) \to y(k+N)$

Since these two inputs are identical by assumption, we get that $y(k+N) = y(k)$. An input complex exponential for the periodic discrete-time case is the same as the general discrete-time case, except that the input frequencies are limited to $z = W_N^n,\ 0 \le n \le N-1$. Thus, we can apply the previous result to conclude that

$$W_N^{nk} \to H(W_N^n) \cdot W_N^{nk}.$$

The functions $H(s)$ and $H(z)$ are called the *transfer function* of an LTI system. In each case, the effect of the system is to multiply the eigenfunction input by the transfer function (a complex number) yielding the output. The transfer function as a function of ω with $\sigma = 0$ has the interpretation as a frequency response of the system, characterizing the change in amplitude $|H|$ and phase $\angle H$ for a complex exponential passing through the system.

Impulse response

Another important input to an LTI system is the impulse (or unit-sample) response:

Continuous-time: $x(t) = \delta(t) \to h(t),\ -\infty < t < \infty$

Discrete-time: $x(k) = \delta(k) \to h(k),\ -\infty < k < \infty$

For the periodic discrete-time case, although it is not *necessary* to define the unit-sample response any differently, it is *convenient* to take advantage of the assumption that the system input is always periodic. Thus, we can define a periodic unit-sample response $h_N(k)$ as the response to a periodic train of impulses:

Periodic discrete-time: $x(k)=\sum_{l=-\infty}^{\infty}\delta(k-l\cdot N)\rightarrow h_N(k)$.

We know that this periodic unit-sample response must be periodic $h_N(k+N)=h_N(k)$, and thus it is fully characterized by one period, $h_N(k),\ 0\le k\le N-1$. The LTI properties give us a connection to the standard discrete-time unit-sample response,

Time invariance: $\delta(k-l\cdot N)\rightarrow h(k-l\cdot N)$

Linearity: $\sum_{l=-\infty}^{\infty}\delta(k-l\cdot N)\rightarrow\sum_{l=-\infty}^{\infty}h(k-l\cdot N)$

and thus

$$h_N(k)=\sum_{l=-\infty}^{\infty}h(k-l\cdot N),\ 0\le k\le N-1.$$

Note that in the last case that the system response, in the face of periodic inputs, is characterized by just N samples $h_N(k),\ 0\le k\le N-1$ in the time domain, just as it is characterized by the transfer function at N frequencies $W_N^n,\ 0\le n\le N-1$ in the frequency domain.

Given an impulse or unit-sample response, the input-output relationship for a general signal can be characterized by a convolution:

Continuous-time: $y(t)=\int h(\tau)\cdot x(t-\tau)\cdot d\tau\equiv h(t)\otimes x(t)$

Discrete-time: $y(k)=\sum_m h(m)\cdot x(k-m)\equiv h(k)\otimes x(k)$

Since the periodic discrete-time case is of primary interest later, the convolution sum for this case will be derived. A general input $x(k)$ can be expressed in terms of impulses over one period as

$$x(k)=\sum_{m=0}^{N-1}x(m)\cdot\delta(k-m),\ 0\le k\le N-1$$

and thus the entire periodic signal can be represented as

$$x(k)=\sum_{m=0}^{N-1}x(m)\cdot\sum_l\delta(k-m-l\cdot N),\ -\infty<k<\infty.$$

This leads to a convolution relationship between input and output:

Time invariance: $\sum_l\delta(k-m-l\cdot N)\rightarrow h_N(k-m)$

Linearity: $\sum_{m=0}^{N-1} x(m) \cdot \sum_{l} \delta(k-m-l \cdot N) \rightarrow \sum_{m=0}^{N\text{-}1} x(m) \cdot h_N(k-m)$

Thus

$$y(k) = \sum_{m=0}^{N-1} x(m) \cdot h_N(k-m) = \sum_{m=0}^{N-1} h_N(m) \cdot x(k-m) \equiv h_N(k) \otimes x(k),$$

which (because each of the signals in the convolution is periodic) is a *circular* convolution: as the index of $x(k-m)$ or $h_N(k-m)$ wanders outside the interval $[0, N-1]$ it is assumed that the signal is periodic. If it was originally specified only on $[0, N-1]$, it must be periodically extended.

A connection is easily formed between the impulse response and the transfer function, as the latter can be determined by inputting a complex exponential, verifying that the result of the convolution is a similar complex exponential, and observing the transfer function by inspection as:

Continuous-time: $H(s) = \int h(t) \cdot e^{-st} \cdot dt$

Discrete-time: $H(z) = \sum_{k} h(k) \cdot z^{-k}$

Periodic discrete-time: $H(W_N^n) = \sum_{k=0}^{N-1} h_N(k) \cdot W_N^{-nk}$

This last formula is of particular significance in what follows. It says that for any LTI discrete-time system, the transfer function at uniformly spaced frequencies on the unit circle must equal the DFT of one period of the periodic version of the unit-sample response.

Matrix eigenvectors and eigenvalues

Given an $N \times N$ matrix $\mathbf{A}$, and if

$$\mathbf{Av} = \lambda \cdot \mathbf{v}$$

then λ is an eigenvalue and $\mathbf{v}$ is an eigenvalue of $\mathbf{A}$. The eigenvalue and transfer functions are different expressions of the same concept. The linear transformation represented by $\mathbf{A}$ transforms $\mathbf{v}$ into a vector in the same direction, except that the length of the vector is multiplied by the complex number λ.

Hermitian matrices

For a Hermitian matrix $\mathbf{H}$, there exists an orthonormal set of eigenvectors $\mathbf{v_i}$, $0 \le i \le N-1$ with non-negative real-valued eigenvalues λ_i, $0 \le i \le N-1$ and, according to the spectral theorem

$$\mathbf{H} = \sum_{i=0}^{N-1} \lambda_i \cdot \mathbf{v_i v_i^H}.$$

(Our numbering from 0 to N is unconventional, but chosen to be consistent with the signal processing results earlier.) Thus, the linear transformation represented by $\mathbf{H}$ can be expressed differently as

$$\mathbf{Hx} = \sum_{i=0}^{N-1} \lambda_i \cdot \mathbf{v_i v_i^H x} = \sum_{i=0}^{N-1} \lambda_i (\mathbf{v_i^H x}) \cdot \mathbf{v_i}.$$

This expresses $\mathbf{Hx}$ in a new coordinate system, where the coordinates $\lambda_i(\mathbf{v_i^H x})$, $0 \le i \le N-1$ are determined by finding the inner product of $\mathbf{x}$ with each of the basis vectors, and then multiplying by the corresponding eigenvalue to stretch or shrink the coordinate in that dimension.

Circulant matrices

The circular convolution that characterizes a discrete-time LTI system with a periodic input signal can be expressed in matrix form as

$$\mathbf{Hx} = \mathbf{y}$$

where

$$\mathbf{H} = \begin{bmatrix} h_N(0) & h_N(N-1) & \dots & h_N(2) & h_N(1) \\ h_N(1) & h_N(0) & \dots & h_N(3) & h_N(2) \\ \dots & \dots & \dots & \dots & \dots \\ h_N(N-2) & h_N(N-3) & \dots & h_N(0) & h_N(N-1) \\ h_N(N-1) & h_N(N-2) & \dots & h_N(1) & h_N(0) \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \ldots \\ x(N-2) \\ x(N-1) \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ \ldots \\ y(N-2) \\ y(N-1) \end{bmatrix}$$

This is a general $N \times N$ *circulant* matrix, defined as a square matrix whose rows consist of all circular shifts of the first row. Just like a Toeplitz matrix, a circulant matrix has only N independently chosen elements (the first row), with all the remaining elements are determined by the circular shift property. In fact, $\mathbf{H}$ is always Toeplitz as well, since the (i, j) element of this matrix is $h_N(i-j)$ (taking into account the periodicity of $h_N(\cdot)$).

For the special case where $h_N(k) = h_N^*(-k)$, the circulant matrix is also Hermitian. This implies that $h_N(0)$ must be real-valued, as must $h_N(N/2)$ when N is even.

The eigenvectors of a matrix are normally not the samples of a complex exponential. The circulant matrix is an exception, however. From previous results we know that when

$$\mathbf{x} = \begin{bmatrix} 1 \\ W_N^n \\ \ldots \\ W_N^{n(N-2)} \\ W_N^{n(N-1)} \end{bmatrix} \quad \text{then} \quad \mathbf{y} = H(W_N^n) \cdot \mathbf{x}$$

Vectors of complex exponentials are eigenvectors of circulant matrices; a surprising property is that all circulant matrices have the same eigenvectors regardless of their actual elements. The eigenvalues, however, do depend on the elements, where

$$\lambda_n = H(W_N^n) = \sum_{k=0}^{N-1} h_N(k) \cdot W_N^{-nk}, \; 0 \le n \le N-1.$$

The eigenvalues of a circulant matrix coincide with the discrete Fourier transform of one period of $h_N(k)$. It is also true that the rows of a circulant matrix can be permuted in any fashion without destroying the circulant property, and that the complete set of eigenvalues can be obtained by taking the DFT of any row (the order, but not distribution, of values of a DFT is not affected by a time shift of the underlying periodic signal).

Autocorrelation matrix

Any function $r(k)$ that is Hermitian and positive semi-definite is an allowable autocorrelation function of a discrete-time wide-sense stationary random process. The power spectral density

$$P(z) = \sum_k r(k) \cdot z^{-k}$$

is real-valued and non-negative on the unit circle. The $N \times N$ autocorrelation matrix

$$\mathbf{R} = \begin{bmatrix} r(0) & r(-1) & \ldots & r(-N+2) & r(-N+1) \\ r(1) & r(0) & r(-1) & \ldots & r(-N+2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \\ r(N-1) & r(N-2) & \ldots & r(1) & r(0) \end{bmatrix}$$

arises in many finite-dimensional optimization problems in signal processing.

Since $r(k)$ is not usually a periodic function, $\mathbf{R}$ is not circulant. (Even if $r(k)$ were periodic, $\mathbf{R}$ would be circulant only when the size of the matrix matches the period or some integer multiple of the period.) However, we can create a new matrix $\mathbf{R_N}$ that is (a) circulant and (b) a good approximation to $\mathbf{R}$, especially as N gets large. For this purpose, define a new periodic version of the autocorrelation function,

$$r_N(k) = \sum_{l=-\infty}^{\infty} r(k + l \cdot N),$$

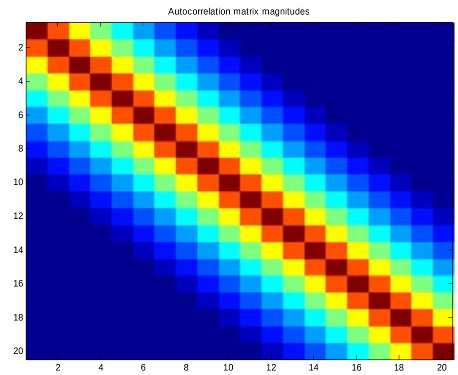
and populate $\mathbf{R_N}$ with this modified autocorrelation function. (This is the same as the transformation from a system unit-sample response to the new equivalent response in the face of periodic inputs.) From earlier results, we know that the eigenvalues of $\mathbf{R_N}$ are $P(W_N^n),\ 0 \le n \le N-1$; that is, the eigenvalues of $\mathbf{R_N}$ equal the power spectrum (z-transform of the autocorrelation function $r(k)$) sampled at uniformly spaced points about the unit circle.

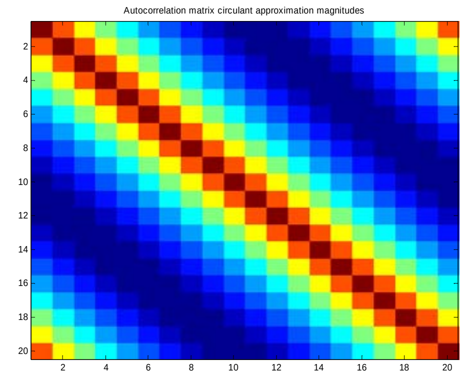
The important remaining issue is whether the eigenvalues of $\mathbf{R_N}$, which we have determined, are related to the eigenvalues of $\mathbf{R}$. It is plausible that they would have asymptotically the same distribution. However, $\mathbf{R_N}$ and $\mathbf{R}$ do differ significantly in the region of the upper right and lower left corners, since

$$r_N(N-1) = r_N(-1) \approx r(-1) \text{ and } r_N(-N+1) = r_N(1) \approx r(1).$$

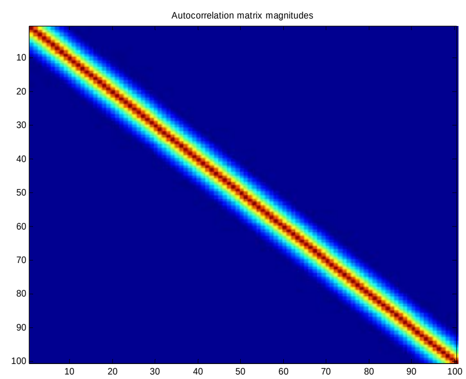
On the other hand, $\mathbf{R_N}$ and $\mathbf{R}$ are nearly the same near the diagonal, where the largest values reside in $\mathbf{R}$, especially for large N. The hope is that the values where the two matrices differ (far away from the diagonal) are relatively few in number and thus affect the eigenvalues relatively little. This is in fact the case, although the proof is omitted [2].

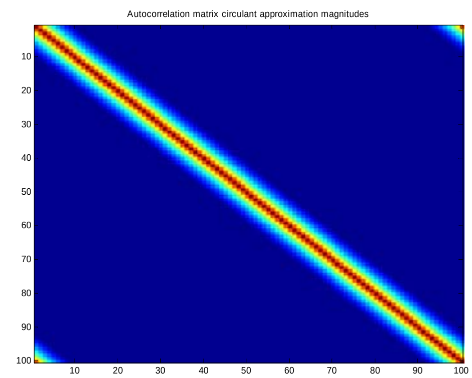
Example: Let the impulse response of an FIR filter be $f_k \leftrightarrow F(z)$. Then if unit variance white noise is filtered by $F(z)$, the autocorrelation values are $r_x(m) = f_m \otimes f_{-m}^*$ and the power spectrum is $P_x(e^{j\omega}) = \left|F(e^{j\omega})\right|^2$. Take for example $f_k = \alpha^k,\ 0 \le k \le M-1$ for $\alpha = 0.8$. Then the magnitudes of the autocorrelation matrix elements and the circulant approximation are shown below for an $N = 20$ order model. Note the significant differences in the upper right and lower left.



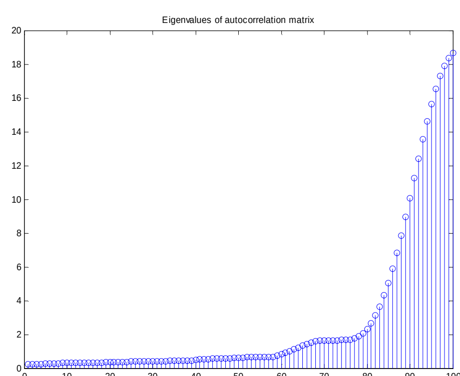


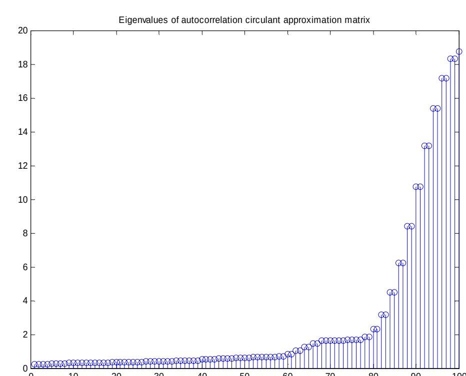
Here is the same example with $N = 100$. Note that the similar values along the diagonal dominate the shrinking (in relative terms) variation in the two corners:



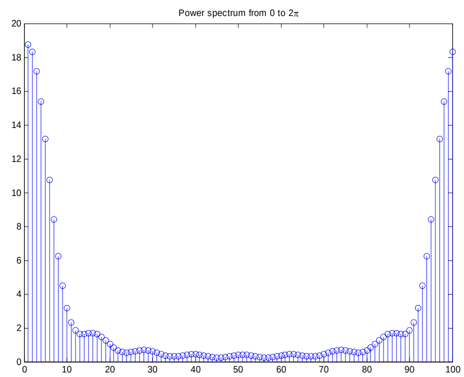


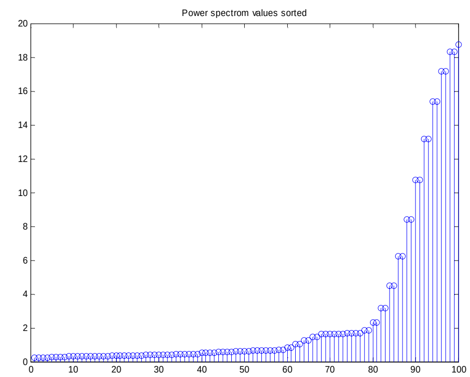
For the later case, the eigenvalues of the autocorrelation matrix (on the left, sorted from smallest to largest) can be compared to the eigenvalues of the circulant approximation (on the right). Note that the distributions are very similar:





Finally, we can examine the power spectrum (on the left) and its sorted values (on the right). Note that the latter are identical to the eigenvalues of the circulant approximation as expected:





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[2] Todd K. Moon, Wynn C. Stirling, *Mathematical Methods and Algorithms for Signal Processing*, Prentice Hall, 1999.