Interlinked Isohedral Tilings of 3D Space

Roman Fuchs
Carlo H. Séquin

Electrical Engineering and Computer Sciences
University of California at Berkeley

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Roman Fuchs
University of California, Berkeley
fuchsro@berkeley.edu

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Abstract

Different isohedral tilings of the Euclidian Space are studied in this paper. Using the Voronoi zone in various lattices, we derived toroidal shapes that interlink with each other to fill space completely. The tile of main interest was the 3-segment ring-tile, for which we found several linear approximations.

1 Introduction

Uniform tiling of the Euclidian plane is a well studied problem in geometry. There are three *regular* tilings of the plane that are made up of congruent regular polygons: Triangular tiling, Square tiling and Hexagonal tiling. Furthermore, there are eight semiregular tilings that use a variety of regular polygons. For those tilings the arrangement of polygons at every vertex point is identical. These eleven tilings are said to be *1-uniform* and are also called *Archimedean* tilings [5].

Other types of tesselations exist using irregular patterns. Dutch artist M.C. Escher for instance created many irregular shapes that interlock completely and fill the plane without leaving space. His tiling patterns oftern included animals and other non-trivial shapes.

Tiling the three dimensional space by various shapes without voids is an strongly related problem in geometric modeling. Several solutions can be found by extending the plane tilings. Grünbaum enumerated 1994 all 28 uniform tilings of 3D space using regular and semiregular polyhedral cells [6]. Also, different non-trivial tiles have been found [1]. Some toroidal tiles (genus 1) can be derived from crystal lattices such as the cubic or diamond lattice or the (10,3)-a net [3] (triamond lattice [4]). To fill space, these often highly symmetrical ring-tiles interlink with each other, e.g. in the cubic lattice each tile is interlinked with four other tiles. We are especially interested in isohedral tilings, where every tile lies the same way in the lattice. Monohedral tilings on the other hand solely require that only one tile is used.

It is generally desirable that these tiles resemble the Voronoi zones around the geometrical shapes lying in the crystal lattices. A first step was therefore done to study the 3D Voronoi diagram of line segments. Voronoi tesselations have already been studied in order to model protein structure [7]. There however the structure was modeled by Voronoi cells around single points in space, i.e. the atoms of the protein.

2 3D Voronoi diagram

Generating the Voronoi diagram in 3D space is clearly more complex than in the 2D case. The direct extension is to use points in 3D space. However, once we're using line segments, it gets even more complicated. One of the following three cases can occur:

2.1 Point vs. point

If we have two points in 3D space, the shape of the Voronoi zone between them is simply a plane. There exist already tools to generate 3D Voronoi diagrams for a set of points in space [1]. Similiarly to the 2D case, one can construct the Delaunay triangulation and construct the planes dividing the connecting lines.

2.2 Line segment vs. point

If we have a line segment and a point in 3D space, the surface of equidistance is shaped as a cylindrical parabola. The parabola gets steeper as the distance of the point and the line segments gets smaller. While the simplest linear approximation of this shape as a plane is very inaccurate for small distances, it is acceptable for large distances between point and line segment. Arbitrary accurate linear approximations can be achieved by increasing the number of planes.

2.3 Line segment vs. line segment

Two skew line segments in 3D space generate a hyperbolic paraboloid as a surface of equidistance. The shape can be characterized by 2 parameters: the distance between the two line segments and the angle of twist between the lines. Again, as the distance between the lines decreases, the paraboloid gets steeper.

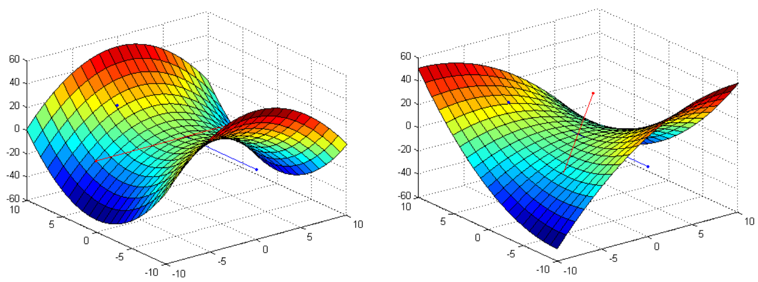

Figure 1: Hyperbolic paraboloid for 90° and 45° angle

[1] `http://home.scarlet.be/zoetrope/voronoi3d/`

While first order approximations replace this bicubic surface simply with a plane, a more detailed approximation uses 4 quadrilaterals. Approximations with high accuracy can be generated using 9 support points and Catmull-Clark subdivision [8] (Figure 2).

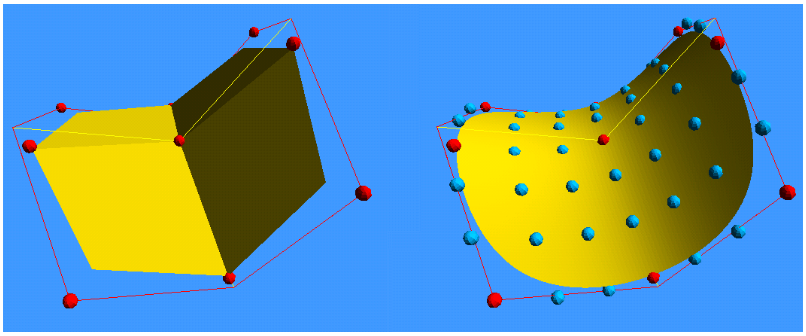

Figure 2: Hyperbolic paraboloid generated by Catmull-Clark subdivision

3 Interlinking Tiles

In oder to get isohedral tilings, we need to investigate uninodal nets. These networks guarantee a regular structure and provide simple means to generate interlinking tiles.

3.1 4-Segment Ring-Tile

The simplest toroidal ring-tiles that interlinks with each other can be derived from the cubic lattice. If we put square wire frames into the lattice and shrink them by just a small epsilon to avoid ambiguities, the Voronoi zones around the wire frames define a 4-segment ring-tile.

On the outer side of the rings, we have four parallel line segments which tile the space into four 90° segments that become the outer faces of the ring-tile. In the inside we have four adjacent lines passing perpendicular to the base frame. This leads to four hyperbolic parabolas as inner faces. Linear approximations can either replace those faces by a plane or use four quadrilaterals as shown in figure 3.

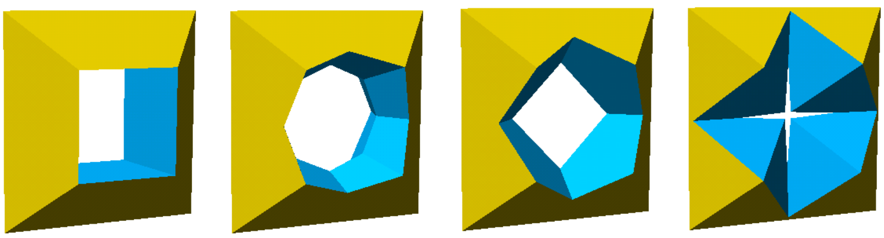

Figure 3: Quadrilateral rings with different developments of the saddle surface

The size of the square frames can be adjusted. While in one extreme case the neighboring square frames that lie in same plane touch each other, in the other extreme case the square frames are shrunk until the lines passing perpendicular to the base frame touch that frame from the inside. As the distance between the perpendicular lines is getting closer, the hyperbolic parabola gets steeper.

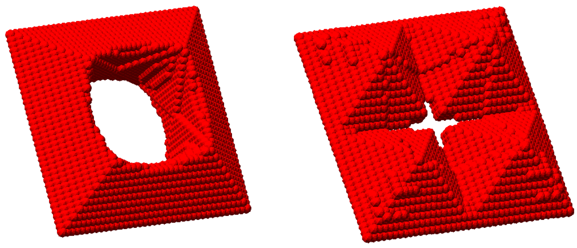

Figure 4: Exact Voronoi zone for the two extreme cases

3.2 3-Segment Ring-Tile

Deriving a ring-tile consisting of three segments is not a trivial problem. Here we will study several tilings that result from triangular frames.

3.2.1 Straight Tile

If we use a simple 3-segment-ring with the cross section of an obtuse (30-120-30 degree) triangle, we can interlink each tile with 3 other tiles to fill its hole completely. But after several generations of interlinking, this shape leads to inconsistencies.

Figure 5: Straight 3-segment rings (shrunk by 50%) that interlink compactly

A space-filling solution can be derived by utilizing the (10,3)-a net and calculating the Voronoi zone around the triangular node links. The Schläfli symbol (10,3) describes that in this network there are ten nodes in the fundamental ring and three links at each node. It has a torsion angle ω, that we set to 70.5 degrees (the dihedral angle of the tetrahedron), to get a regular lattice.

3.2.2 Basic Twisted Tile

The first linear approximation has been found by Carlo H. Séquin [1] and uses 15 planar faces (Figure 6). We strongly believe, this is the linear approximation with minimal number of planar faces.

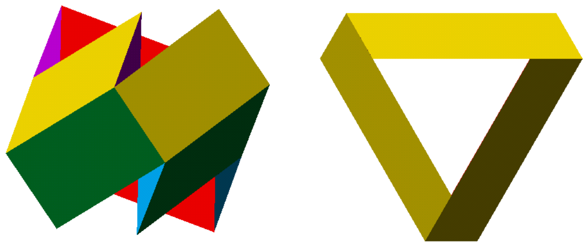

Figure 6: Side view (a) and top view (b) of the basic 3-segment ring-tile

As this tile has some pointy angles at the corners, we weren't really satisfied and tried to find some "nicer" tiles, that comes closer to the real Voronoi shape. There is however not a unique solution and different metrics can be applied to get the "simplest" tile, or the most "beautiful" one. One approach could be to get the minimal number of planar faces. Another objective is to get the tile with the largest minimal dihedral angle to avoid pointy angles. Accordingly we can try to use as few quadric surfaces as possible or to get the minimal maximal curvature of these shapes.

The used crystal lattice leads to a highly symmetrical structure. The 3-Segment Ring tile has a 3-fold rotational symmetry axis through its midpoint and three C_2 symmetry axis through the corners of the generating triangle. Using the symmetry structure we only have to define a 1/6 of the surface of the tile.

Using a brute-force approach where I uniformly sampled 3D space and calculated the distance to each single line segment, I generated an visualization of the exact shape of the 3D Voronoi zone being defined by a set of line segments. The underlying network (derived from the (10,3)-a net) is shown in figure 7.

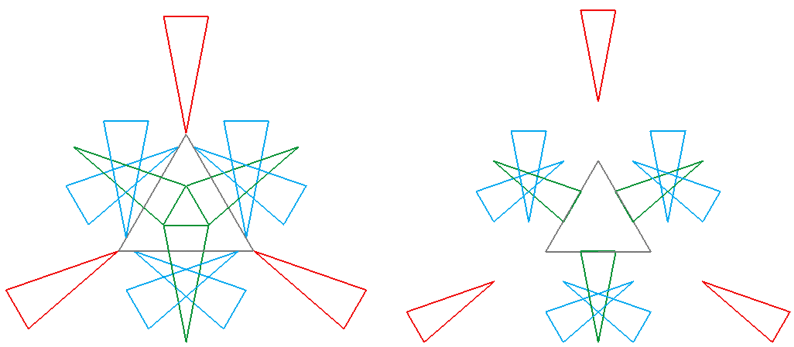

Figure 7: Network with full size triangles (a) and shrunk triangles (b)

The size of the triangles that we insert into the (10,3)-a net is not fixed. Figure 7a shows the largest setting where the corners of the grey and red triangles touch. The triangles can be shrunk until the grey and green triangles tear apart at the midpoints of their adjacent lines (Figure 7b).

As the hyperbolic parabola gets steeper as the distance of two line segments decreases, the associated Voronoi zone looks quite different for the two extreme cases (Figure 8). The surface consists of colored patches that show, which adjacent line segment has determined that area.

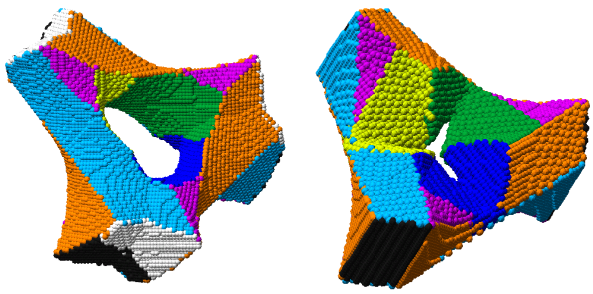

Figure 8: Generated tiles for the two network settings

3.2.3 Further Refinements

Adding several new planar faces, I was able to get rid of the corner that we had in the basic twisted tria-tile. Using a slider, there are arbitrary many intermediate configurations that we can generate with either 30 or 36 planar faces (Figure 9).

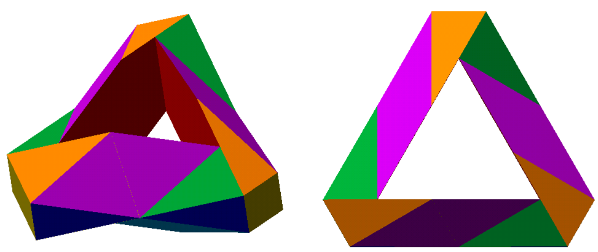

Figure 9: Improved ring-tile with cut corners

So far we haven't touched the planar faces on the inside. As we have seen in case of the 4-segment ring-tiles, the hyperbolic parabola can be approximated by four quadrilaterals. Accordingly, I applied the same approximation to the basic 3-segment ring-tile, which resulted in another tile with 30 planar faces (Figure 10).

These two approximations combined lead to a tile with a total of 69 planar faces (Figure 11). This tile comes closest to the true Voronoi zone defined by the triangular skeletons.

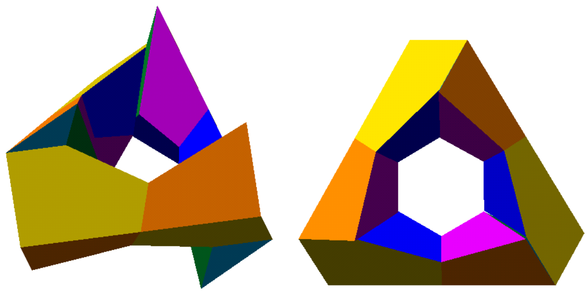

Figure 10: Ring-tile with improved hyperbolic paraboloid approximation

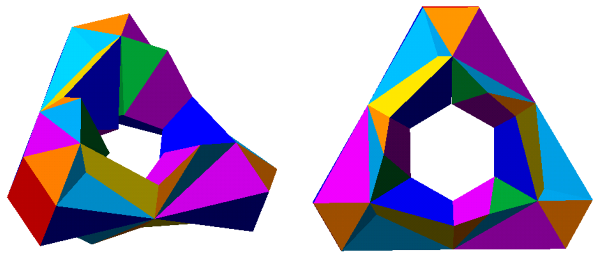

Figure 11: Ring-tile with both improvements incorporated

There are several other uninodal three-connected nets mentioned in the literature, that we could derive differing ring-tiles from. Another net with genus 3, which is very common, is the *ths net* (also called *(10,3)-b net*). Furthermore, the *bto net* (*(10,3)-c net*), where the fundamental 10-ring is twisted rather than chair-formed, has genus 4 and the *utp net* (*(10,3)-d net*) has genus 5. However, as the nomenclature orders the nets from high to low symmetry, the *srs net* (*(10,3)-a net*) provides us with the most symmetrical structure.

3.3 Trefoil knot

A more complicated configuration can be obtained from interlinking knots [2]. The simplest knot-tile is the trefoil knot. If we use again the (10,3)-a net as underlying lattice, every trefoil knot interlinks with three other trefoils. As a skeleton to generate the Voronoi diagram in 3D space we used a 6-segment approximation of the trefoil knot (Figure 12).

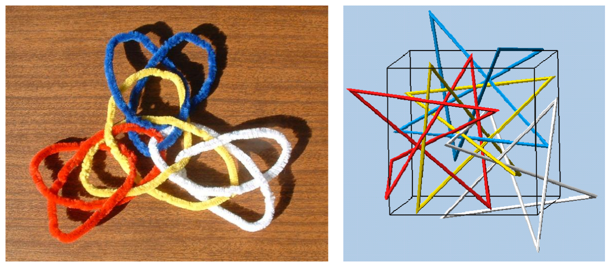

Figure 12: Interlinked trefoil knots (a) and 6-segment approximation (b)

The geometry of the interlinked knot strands can be differently specified. The goal is to create a space filling Voronoi partition of space that will preserve the knotted genus-1 shape of each trefoil knot. Therefore, strands from the same knot should not cross close to each other, otherwise their Voronoi zones aren't separated.

I tried several configurations for which I visualized the true Voronoi diagram (Figure 13). However, with the natural arrangement of the trefoils in the (10,3)-a net there weren't enough strands of neighboring knots that pass through the inner regions of each knot, so that different segments of the same trefoil merged. That's apparently a problem that needs some special handling.

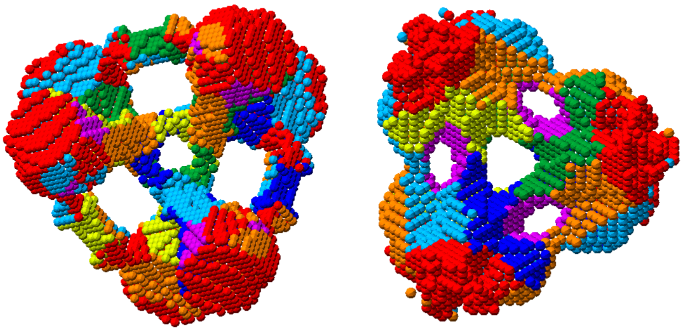

Figure 13: Voronoi diagrams for 2 configurations of the trefoil knot skeleton

Furthermore, getting a linear approximation of this structure using planar faces is quite a challenging task. The use of some automated tools seems to be essential.

4 Conclusion

There still remain a number of tiles and network topologies to investigate. For the trefoil knot we were not yet able to derive a simple approximation with planar faces. I was however able to get some more linear approximations of the 3-segment ring-tile that get rid of the pointy angles and fit closer to the true shape of the Voronoi diagram. The geometry has been visualized in SLIDE [9] and using rapid prototyping, we build some physical models of these 3D tiles.

The geometric complexity was however harder than expected. After visualizing the Voronoi zone of different networks, extracting the exact bicubic shape was surprisingly difficult. And compared to the trefoil knot and other complicated structures, the used triangle network was still relatively simple. Therefore, the first step would be to use a linear approximation for all three cases (point vs. point, point vs. line segment, line segment vs. line segment). This first approximation however doesn't lead to a space-filling tiling. There are some hollow corners left that have to be assigned to some adjacent zone. To implement such a tool still remains an open task. Some interesting insights into different structures can be expected.

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