Performance Analysis of a 1-bit Feedback Beamforming Algorithm

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Technical Report No. UCB/EECS-2009-161
http://www.eecs.berkeley.edu/Pubs/TechRpts/2009/EECS-2009-161.html

November 30, 2009

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I. Introduction

The signal processing technique of beamforming is commonly used to increase the efficiency of a communication network. While a variety of techniques like adaptive filtering have been used in beamforming, the 1-bit feedback algorithm used in the implementation of beamforming is fairly new and has been proved to have an efficient running time [3]. Currently, a 1-bit feedback beamforming algorithm that switches all of the transmitters' phases on each iteration to choose a new received signal has been developed and proven to have a running time that is proportional to the number of transmitters in the system [1]. Another 1-bit feedback beamforming algorithm that involves switching a few of the transmitters' phases on each iteration and has a running time that is proportional to the number of transmitters has been devised [2], but this analysis was done on only binary signals (signals -1 and 1). Our work shows that the 1-bit feedback beamforming algorithm involving few transmitters switching each iteration has a running time with a linear lower bound even if the signals the individual transmitters send out can be any complex signal on the unit circle ($e^{j\theta}, \theta \in [-\pi, \pi]$).

II. Problem Statement

A. Model Setup

In the communication system at iteration k, there are N transmitters, with each node sending $X_i[k] = e^{j\theta_i[k]}$. Each $X_i[k]$ is sent through a channel represented by $h_i = e^{j\phi_i}$. The total received signal is $Y[k] = \sum_{i=1}^{N} e^{j(\theta_i[k]+\phi_i)} + N[k]$, where $N[k]$ is circularly symmetric complex Gaussian noise with variance σ^2. The receiver stores the magnitude of the received signal with the best magnitude so far: $\|Y_{best}[k]\| = \max(\|Y[1]\|, \|Y[2]\|, \dots \|Y[k]\|)$. The receiver then sends a feedback bit $F[k]$, where $F[k] \in \{-1, 1\}$. $F[k] = -1$ indicates that the received signal's magnitude decreased, while $F[k] = 1$ indicates that the received signal's magnitude increased. When $\|Y_{best}[k]\| \geq \beta N$ for a given parameter $\beta \in [0, 1]$, the algorithm terminates.

B. Proposed Algorithm

At time k = 0, the signal transmitters send initial signals with random phases. Each transmitter i transmits $X_i[0] = e^{j(\theta_i[0])}$ through a channel h_i. The received signal $Y_i[0] = \sum_{i=1}^{N} e^{j(\theta_i[0]+\phi_i)} + N[k]$. In the proceeding iterations, a common feedback bit $F[k]$ at time k is sent back over a noiseless channel to the transmitters indicating whether this current signal's magnitude is larger or smaller than the largest received signal magnitude so far. At the transmitters, two sets of signals are stored: $X[c]$ and $X[k]$, where c denotes the iteration where the transmitted signals produced the greatest received signal magnitude and k denotes the current iteration. After receiving the feedback bit from the receiver, the set of transmitted signals $X[c]$ is kept if a -1 bit is received and $X[k]$ is kept if the feedback bit is $+1$. After throwing out either $X[c]$ or $X[k]$, the transmitters must transmit a new set of signals. The new set of signals would be chosen like this:

1) Each transmitter chooses a number 1 with probability $p = \frac{\alpha}{N}$ and 0 with probability $p' = 1 - \frac{\alpha}{N}$. We refer to p as the "switch probability".
2) The transmitters choosing 1 will perturb their phases by a random amount chosen from the distribution Unif$[-\gamma, \gamma]$. These new perturbed signals will be the signals that these transmitters transmit in the next iteration, while the transmitters choosing 0 will transmit the same signals that were transmitted in the previous iteration. After the transmitters choosing 1 toggles their phases, the algorithm repeats until the received signal strength is greater than βN.

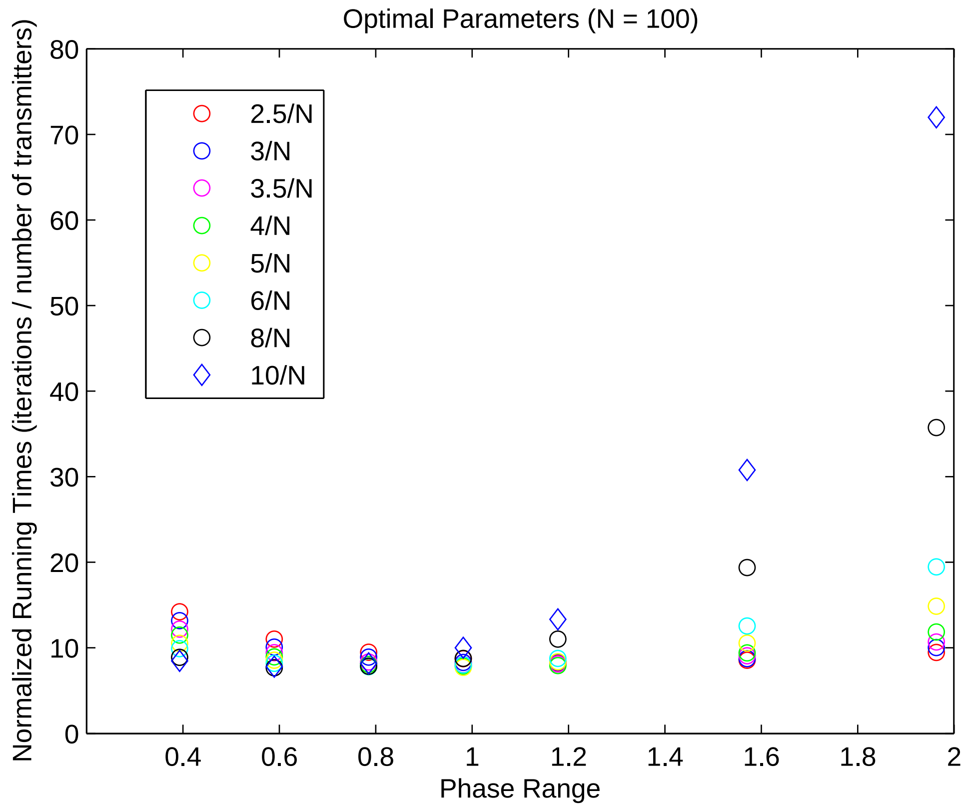

Fig. 1. This plot demonstrates the estimation of the optimal parameters of the algorithm using simulations.

III. Numerical Simulations for the Noiseless Case

In order to maximize the performance of the proposed algorithm, we ran simulations to optimize the two parameters, phase range γ and switch probability p in the case where noise is absent in the system. As seen in Figure 1, the optimal phase range is around $\frac{3\pi}{16}$, and for that particular phase range, the different transmitter switch probabilities appear to yield similar performances, with the switch probability $\frac{8}{N}$ producing slightly lower average running times.

To evaluate our algorithm, we compared it with the algorithm developed by Mudumbai [1]. His algorithm is very similar to the algorithm we devised, except that after every iteration, all of the transmitters toggle their phases and transmit a different signal, whereas our algorithm toggles only a few of the transmitters. In comparing the two algorithms, we ran our algorithm with the optimal parameters found in the previous section (switch probability = $\frac{8}{N}$, phase range = $\frac{3\pi}{16}$) and ran Mudumbai's algorithm with the optimal phase range parameter we discovered (phase range = $\frac{\pi}{32}$). From Figure 2, it is clear that although Mudumbai's algorithm performs better for a small number of transmitters in the network, our algorithm outperforms it as the number of transmitters in the network grows to larger numbers.

In addition to evaluating our algorithm under the ideal, noise-free conditions, we measured the algorithm's performance in situations with different levels of noise. As shown in Figure 3, the algorithm works relatively well for SNR values above 40 dB, but once the SNR decreases to below 40 dB, the average running time of the algorithm deviates from its linear characteristic present in ideal, noise-free circumstances.

IV. Mathematical Analysis of our Proposed Algorithm

To do running time analysis, we approximate our algorithm by analyzing it as a phase synchronization problem. Instead of analyzing the time required for the received signal to reach a certain magnitude, we use a probabilistic model to analyze the time required for the individual transmitted signals to attain a phase within a certain range,

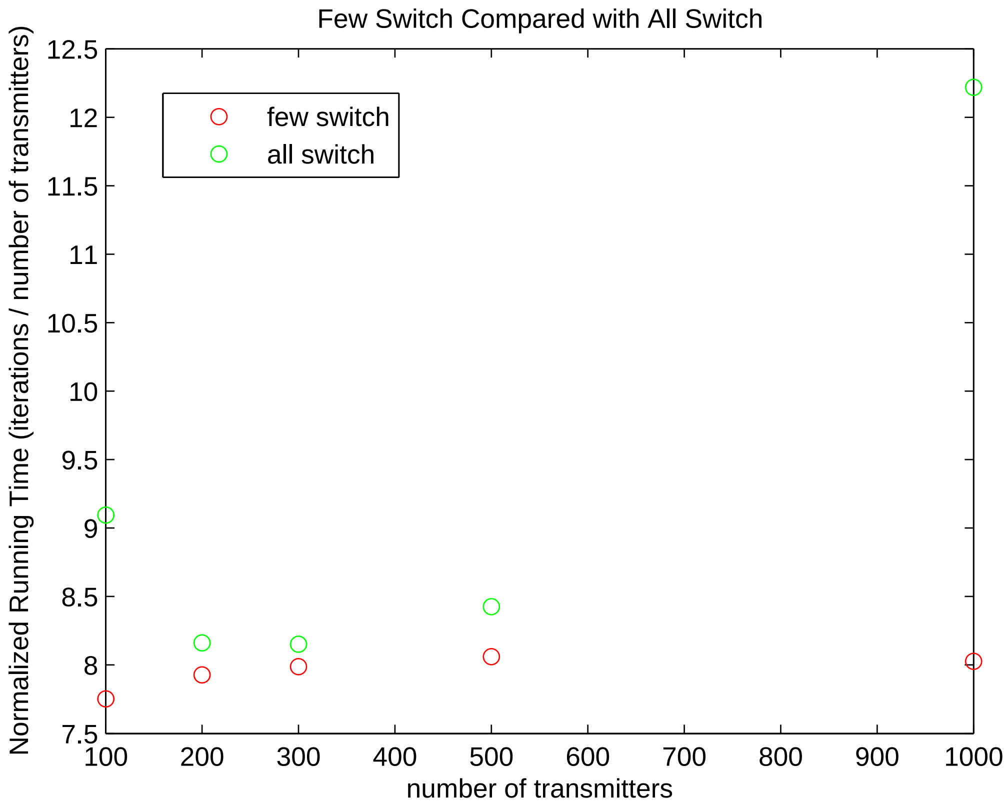

Fig. 2. This plot compares the running time of the all switch and few switch algorithms

which translates to the received signal attaining a certain required magnitude. To validate this proof, we also show the correspondence between phase synchronization and magnitude maximization. In this proof, we assume that α is small enough so that only one transmitter toggles its phase in any iteration.

A. Proof of Linearity with respect to the Number of Transmitters

Because our proposed distributed beamforming algorithm is a probabilistic algorithm, we demonstrate the linearity of the running time's lower bound by showing that the lower bound on the expected value of the running time of the algorithm scales with the number of transmitters in the network. To demonstrate the linearity of the lower bound of the running time's expected value, we must first formulate the definition of the termination of the algorithm.

We define the termination of the algorithm as the iteration when the probability of the phase of any transmitter having a value between $\arccos(\beta)$ and $-\arccos(\beta)$ rises above $1-\epsilon$, where ϵ is a very small positive number and β is the percentage of total transmit power required for the termination of the algorithm. When this condition occurs, virtually all of the transmitters have phases between $\arccos(\beta)$ and $-\arccos(\beta)$, which means that $\cos(X_i) \geq \beta$ for virtually every i. Since there are N transmitters:

$$\sum_{i=1}^{N} \cos(X_i) \geq \beta N$$

This is the desired termination condition as the total strength of the received signal is greater than βN. For the probability of the phase of any transmitter to lie between $\arccos(\beta)$ and $-\arccos(\beta)$, that transmitter must have on average toggled a certain number of times x.

$$x = \frac{4}{\gamma}(\frac{\pi}{2} - \arccos(\beta))$$

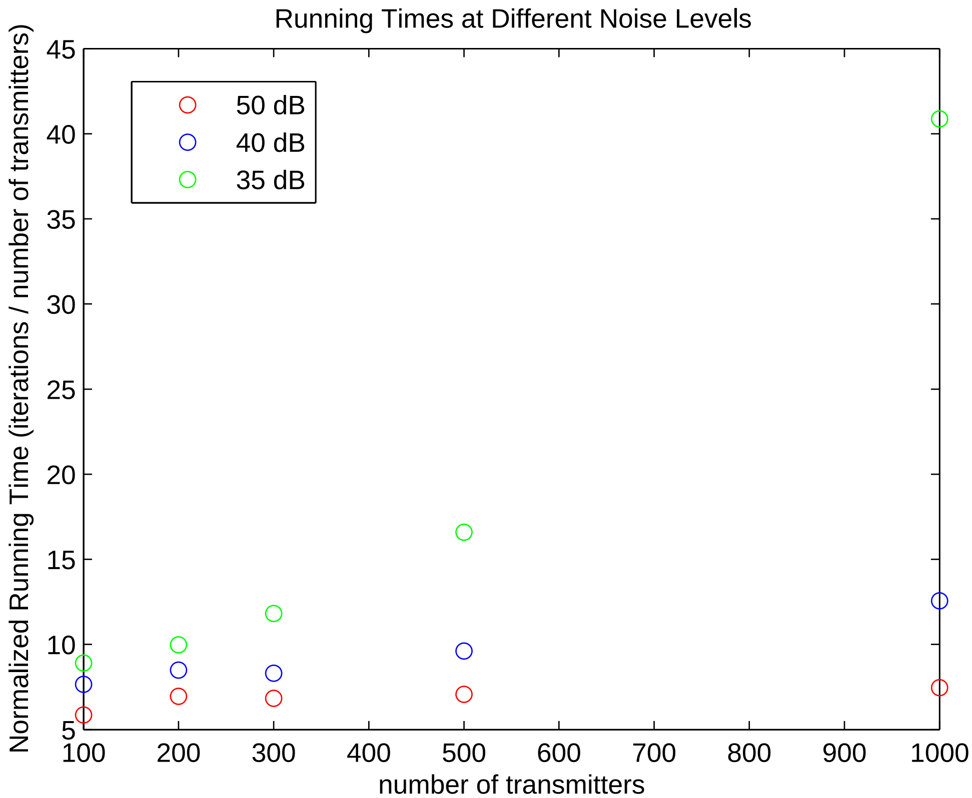

Fig. 3. This is a plot of the running times of the few switch algorithm at different noise levels

γ is the improvement phase range. In the average case, the average initial phase difference between any given transmitter and the boundaries of the target phase range $\arccos(\beta)$ and $-\arccos(\beta)$ is $(\frac{\pi}{2} - \arccos(\beta))$ since initially, a transmitter's phase differs from 0 by $\pi/2$ in the average case. The probability that a transmitter's phase improves on any iteration is approximately $\frac{1}{2}$, provided that γ is sufficiently small, and if the transmitter does improve its phase, the expected amount of phase improvement is $\frac{\gamma}{2}$. This means that the expected value for the phase improvement is $\frac{\gamma}{4}$ if a transmitter is toggled. Since the average phase improvement per toggle is $\frac{\gamma}{4}$ and the goal is to improve the initial phase by an average phase of $(\frac{\pi}{2} - \arccos(\beta))$, the average number of toggles needed to reach the goal would be $x = \frac{4}{\gamma}(\frac{\pi}{2} - \arccos(\beta))$.

The probability that a given transmitter has toggled at least x times after T total toggles is:

$$\sum_{i=x}^{T} \binom{T}{i} (\frac{1}{N})^i (1 - \frac{1}{N})^{T-i} = 1 - \sum_{i=0}^{x-1} \binom{T}{i} (\frac{1}{N})^i (1 - \frac{1}{N})^{T-i}$$

To reach termination condition,

$$1 - \sum_{i=0}^{x-1} \binom{T}{i} (\frac{1}{N})^i (1 - \frac{1}{N})^{T-i} \geq 1 - \epsilon,$$

or

$$\sum_{i=0}^{x-1} \binom{T}{i} (\frac{1}{N})^i (1 - \frac{1}{N})^{T-i} \leq \epsilon$$

Using Chernoff's inequality:

$$\sum_{i=0}^{x-1} \binom{T}{i} (\frac{1}{N})^i (1 - \frac{1}{N})^{T-i} \leq e^{-\frac{N}{2T}(\frac{T}{N} - x + 1)^2}$$

Let $C = x - 1$. The previous inequality implies:

$$e^{-\frac{N}{2T}(\frac{T}{N}-C)^2} \leq \epsilon$$

which implies $-\frac{N}{2T}(\frac{T}{N} - C)^2 \leq \ln \epsilon$. Solve the quadratic inequality for T:

$$T \geq N(C - \ln \epsilon + \sqrt{(\ln \epsilon)^2 - 2C \ln \epsilon})$$

Since $\epsilon < 1$ and $C > 0$, the expression inside the square root is always positive, which means that the expression above is always real. This result implies that the lower bound on the number of iterations taken on average to toggle $(1 - \epsilon)N$ transmitters x times is linear with respect to the number of transmitters in the system.

B. Correspondence of phase synchronization and magnitude

To justify decomposing the magnitude maximization problem into a phase synchronization problem, we now prove that moving the phase of one transmitter away from the phase of the total summed signal almost always results in a decrease in magnitude and vice versa. This proof is an assumption that only one transmitter is chosen to toggle on a particular iteration. Without loss of generality, assign a phase of 0 radians to the initial resultant received signal R. Let θ be the initial phase of the one transmitter chosen to toggle in the next iteration and ϕ be the phase of that same transmitter after toggling. Therefore, the initial signal of the transmitter selected to toggle is $t = \cos(\theta) + j\sin(\theta)$ and $t' = \cos(\phi) + j\sin(\phi)$ is the signal of the same transmitter after toggling (see Figure 4). The difference vector between the total received signal before and after toggling is represented as $z = t' - t$, or breaking t' and t into components, $z = \cos\phi - \cos\theta + j\sin(\phi) - j\sin(\theta)$ since all of the signals of the transmitters lie on the unit circle. $\|R'\|$ represents the magnitude of the new received signal while $\|R\|$ is the magnitude of the old received signal.

$$\|R'\| = \sqrt{(\|R\| + \cos(\phi) - \cos(\theta))^2 + (\sin(\phi) - \sin(\theta))^2}$$

or

$$\|R'\| = \sqrt{\|R\|^2 + 2\|R\|(\cos(\phi) - \cos(\theta)) + (\cos(\phi) - \cos(\theta))^2 + (\sin(\phi) - \sin(\theta))^2}$$

For $\|R'\| < \|R\|$,

$$2\|R\|(\cos(\phi) - \cos(\theta)) + (\cos(\phi) - \cos(\theta))^2 + (\sin(\phi) - \sin(\theta))^2 < 0$$

or

$$2\|R\|(\cos(\phi) - \cos(\theta)) + (\cos(\phi))^2 - 2\cos(\phi)\cos(\theta) + (\cos(\theta))^2 + (\sin(\phi))^2 - 2\sin(\phi)\sin(\theta) - (\sin(\theta))^2 < 0$$

which is equal to

$$2\|R\|(\cos(\phi) - \cos(\theta)) + 2(1 - \cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta)) < 0$$

or

$$2\|R\|(\cos(\phi) - \cos(\theta)) + 2(1 - \cos(\phi - \theta)) < 0$$

Working out the algebra, this expression is equal to:

$$\cos\phi - \cos\theta < \frac{\cos(\phi - \theta) - 1}{\|R\|}$$

If $\|R\|$ is large enough, this inequality can be approximated by: $\cos\phi - \cos\theta < 0$. This happens only if $|\theta| < |\phi|$, i.e. the new phase ϕ is further from the phase of the resultant than the old phase θ, since cosine is monotonically decreasing from 0 to π. This means that perturbing a transmitter's phase away from the phase of the total received signal results in a decrease in magnitude of the total received signal.

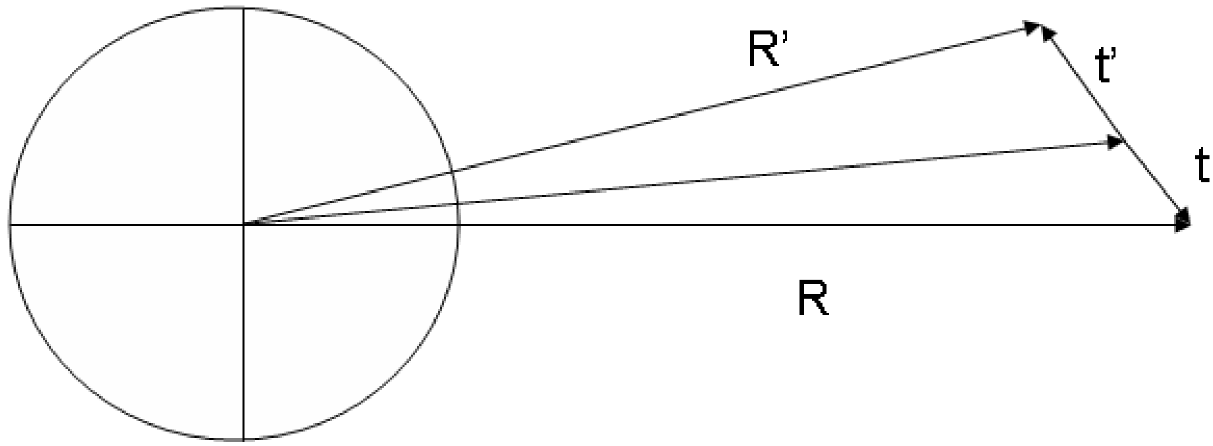

Fig. 4. This is a figure of the signals in an iteration of the algorithm with one transmitter toggling

V. Conclusion and Future Work

In our work, we have devised a novel beamforming method for transmitters transmitting complex signals with unit power and shown that the lower bound on the time required for the algorithm to complete is proportional to the number of transmitters. Linearity is hard to achieve for an inherently nonlinear problem like beamforming, and creating a beamforming technique that exhibits linearity for a relatively general class of signals represents a breakthrough in the development of efficient beamforming algorithms. In addition to the linearity of the lower bound of the convergence time of our beamforming technique, the 1-bit feedback nature of the algorithm greatly reduces the computational complexity and power usage of the transmitters in the communication network. One problem that remains to be solved is the issue with the robustness of the algorithm to noise. Currently, this beamforming technique exhibits linear running time provided that the channels through which transmitters transmit signals are noise free. However, the algorithm loses the ability to converge in linear time when the SNR of the channels drop below 50 dB. The solution to this robustness issue remains an open topic for further research.

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[3] C. Lin, V. Veeravalli, and S. Meyn, “Distributed Beamforming with Feedback: Convergence Analysis,” *IEEE International Symposium on Information Theory (ISIT)*, Toronto, Canada, July 2008.