Games with Non-Probabilistic Uncertainty

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A keen reader may notice that this dissertation pages have more white space than black ink. I wrote in black, but all the white spaces are filled by my cherished family.

Games with Non-Probabilistic Uncertainty

by

JI WOONG LEE

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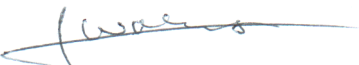University of California, Berkeley

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University of California, Berkeley

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Abstract

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The thesis studies games with non-probabilistic uncertainty about some parameters that affect the rewards of the players. The goal is to understand whether players should be optimistic or pessimistic in such situations.

The first chapter provides a brief overview of the standard solution concepts in Game Theory.

The second chapter proposes a model where the players are strategic in choosing their attitude (degree of optimism), instead of having an intrinsic risk aversion. The idea is that Alice may be able to take advantage of Bob's pessimism, in which case Bob should not be pessimistic. The chapter presents a few examples of two-player non-cooperative games where the agents have a dominant attitude (e.g., optimism), regardless of the unknown private information of the opponent. The chapter also analyses a Cournot duopoly game where each firm has confidential knowledge of its production cost. In the symmetric case, it is shown that pessimism is never a dominant attitude. Finally, the chapter defines a robust attitude and the price of uncertainty, and analyzes them in the Cournot duopoly game.

The third chapter studies a simple wireless network with two relay nodes that cooperate to forward information to a common destination. For a range of success probabilities, only the node with the largest success probability should relay packets to avoid collisions at the destination. However, the success probability of a node is known initially only to that node. To improve the performance of the network, the nodes exchange link state messages through a control channel that is not fully reliable. The chapter studies a protocol where each node tries to protect the performance against the worst possible choice of the other node. The performance of that protocol does not converge as the relays exchange more and more link state messages. Essentially, the relay nodes use an excess of caution. The chapter studies another protocol where each relay node ignores the possible states of knowledge of the other node. The throughput of this less cautious protocol converges to the maximum possible value.

To my dearly loved wife, Yoonsuk

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A keen reader may notice that this dissertation pages have more white space than black ink. I wrote in black, but all the white spaces are filled by my cherished family. My beloved wife Yoonsuk Choi poured overflowing love and support every second, in all form. My two boys, Seojin Roy Lee and Seohyun Eden Lee are undeniable source of joy. They taught me an invaluable skill: how-to-change-diapers-while-doing-research-and-teaching. I am not anymore me. Indeed, I am them.

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Education

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Chapter 1

Introduction

Decision making is choosing among alternatives to try to maximize a given objective function. Decision making is deeply embedded in every aspect of life. Most science, engineering, society, politics, and psychology problems with economic objectives demand a suitable methodology for problems of this kind.

The process of choosing the most preferable alternative among all possible ones is called *optimization* when the problem involves a single decision maker. However, many important decision problems involve multiple autonomous decision makers acting simultaneously. In such problems, decision making is substantially more complicated than in a single agent optimization problem since each decision maker's best decision generally depends on what the others decide. Game Theory is a field of applied mathematics that addresses such multi-agent decision making problems. The decision making problem may be either with full or uncertain information. When the information is uncertain, it may or may not be characterized by a known probability distribution.

This dissertation proposes a novel way of solving games associated with non-probabilistic uncertainty, studies the properties of this approach, and discusses insights that might be applied to other similar problems. In particular, Chapter 2 studies a model of optimism as an augmented degree of rationalization under uncertainty when agents are non-cooperative. Chapter 3 studies a limitation of imperfect communication in establishing common knowledge and robust performance under uncertainty when agents are cooperative.

Chapter 1 provides preliminary background. The first section describes probabilistic and non-probabilistic single agent decision criteria. The second section defines a game and Nash equilibrium concepts. The third section introduces a list of historically important game theory solution concepts. Finally the last section discusses the issue of common knowledge and communication.

1.1 Single-agent Decision Making with Uncertainty

A mathematical formulation of an optimization problem should specify the following: optimization objective, optimization variables, and constraint sets that define feasibility of the optimization variables.

Consider a single agent problem, whose optimization variable x belongs to some feasible set $\mathcal{X}$, and with objective function $u : \mathcal{X} \to \Re$. Then the mathematical optimization problem has the form

$$\arg\max_{x \in \mathcal{X}} u(x). \tag{1.1}$$

A variable $x^* \in \mathcal{X}$ is called a solution of the problem (1.1) if

$$u(x^*) \geq u(x) \text{ for all } x \in \mathcal{X}.$$

A solution exists, for instance, if $\mathcal{X}$ is a non-empty compact set and $u(x)$ is continuous. Problem (1.1) is called a full information optimization problem since u and $\mathcal{X}$ are fully known.

However, in many engineering and economic problems, unknown variables affect the objective function. Suppose these unknown variables are represented by $\theta \in \Theta$. Then consider a modified objective function $u : \mathcal{X} \times \Theta \to \Re$, and a modified optimization problem:

$$\arg\max_{x \in \mathcal{X}} u(x, \theta). \tag{1.2}$$

Generally, the solution of problem (1.2) depends on θ, and cannot be solved without knowing θ.

1.1.1 Alternative Objectives

Instead of using the original objective function $u(x, \theta)$, the designer selects an alternative objective function $f(x)$ that does not depend on the unknown variable θ. This alternative objective function should be chosen to capture the physical, and useful meaning of the underlying context, and should be computable with only available information. We explain the standard choices for $f(x)$. They are classified into probabilistic and non-probabilistic choices.

1.1.2 Probabilistic Criteria

Expected Value

If θ is a random variable (or a vector of random variables) following a probability distribution P, then one alternative objective is the *expected value*.

$$f(x, P) := E_P[u(x, \theta)].$$

This is a popular, and well acceptable alternative in a vast number of engineering problems if: (i) P is known with some accuracy, and (ii) the expected value is a useful performance metric. The second condition is satisfied when many independent and identically distributed copies of θ occur in repeated instances of the decision problem and one is interested in the average value of the objective function.

Expected Utility

The expected value criterion takes into account only the average value of objective function. It does not capture the human player's *true valuation* that may incorporate *risk aversion*. In order to overcome this shortfall, John von Neumann and Oska Morgenstern [41] proposed the use of a concave utility function $\mathcal{T}$ following the school of Daniel Bernoulli [8]. The alternative objective function is then defined as

$$f(x, P, \mathcal{T}) := E_P[\mathcal{T}(u(x, \theta))].$$

Cumulative Prospect Criterion

Experiments in Behavioral Economics show that human subjects do not behave according to the expected utility theory. Allais' paradox [2] and Ellsberg's paradox [18] are prominent examples of this observation. The cumulative prospect criterion [39] is an attempt to describe human behavior, taking into consideration the following major issues: (i) one's valuation is relative to a certain reference point; (ii) one weights more losses than gains; (iii) one overweighs extreme but unlikely events and underweighs average events. In a general form, this criterion is an expected value with transformations of the objective function ($\mathcal{T}$) and of the probability distribution ($\mathcal{S}$) where $\mathcal{T}$ and $\mathcal{S}$ are chosen in accordance with the behavioral observations above. Thus, the criterion corresponds to the following function:

$$f(x, P, \mathcal{T}, \mathcal{S}) := E_{\mathcal{S}(P)}[\mathcal{T}(u(x, \theta))].$$

1.1.3 Non-probabilistic Criteria

Any probabilistic criteria assumes a prior knowledge of the probability distribution P of the uncertainty θ. However, this assumption may not be appropriate and, in some situations, one may know only the set Θ of possible values of θ. This subsection covers a few popular non-probabilistic criteria based on only that information.

Pessimism Criterion

The pessimism, or robust, criterion, considers the worst case scenario. Treating the uncertainty as *adversary*, one seeks to maximize the worst case performance. That is, the objective is defined as

$$f(x, \Theta) := \min_{\theta \in \Theta} u(x, \theta). \tag{1.3}$$

A solution of (1.3) might be appropriate in some applications.

Optimism Criterion

This criterion assumes that the uncertainty is favorable. It corresponds to the following objective function:

$$f(x, \Theta) := \max_{\theta \in \Theta} u(x, \theta).$$

We combine this criterion with pessimism later.

Regret Criterion

The regret is the loss in performance due to the uncertainty. Define

$$x^*(\theta) = \arg \max_{x \in \mathcal{X}} u(x, \theta; \theta)$$

Then the alternative objective function is defined as the minimum regret defined as follows:

$$f(x, \Theta) := \min_{\theta \in \Theta} \{u(x, \theta) - u(x^*(\theta), \theta)\}$$

Instead of the difference, an alternative definition of the regret is the ratio $u(x^*(\theta), \theta)/u(x, \theta)$.

Laplace Criterion

Pierre-Simon Laplace proposed to use the uniform distribution over uncertainty when no other information is available. This approach, called Laplace's principle of indifference,

corresponds to the following objective function:

$$f(x, U) := E_U[u(x, \theta)]$$

where U is the uniform distribution over all possible θ's.

Hurwicz Criterion

Hurwicz criterion [22] is a generalization of pessimism and optimism, with the control parameter $\alpha \in [0, 1]$:

$$f(x, \alpha, \Theta) := \alpha \max_{\theta \in \Theta} u(x, \theta) + (1 - \alpha) \min_{\theta \in \Theta} u(x, \theta)$$

One caveat of this criterion is that the choice of α is arbitrary.

1.1.4   Criteria for Super-Problem

Another modeling approach is to combine probabilistic and non-probabilistic criteria in a super-problem. A super-problem is defined by constructing a two-fold alternative objective. The inner alternative objective uses the expected value criterion over a probability distribution P over θ. The outer alternative objective considers P as uncertain within a collection of probability distributions $\mathcal{P}$, and then applies one of the above-mentioned non-probabilistic criteria.

As one example, a pessimism criterion for expected value when one knows the underlying distribution belongs to $\mathcal{P}$ is
$$f(x, \mathcal{P}) := \min_{P \in \mathcal{P}} E_P[u(x, \theta)].$$

Similarly, one can define other alternative objectives for optimism, regret, Laplace, and Hurwicz.

1.2   Complete Information Games

Many decision problems involve more than one decision maker. Decision makers are autonomous, rational, and independent from each other in making decisions, but they are tightly connected to each other in the sense that the reward of one agent depends on the others' decisions. We use Game Theory to address such multi-agent decision making problems. Formally, a strategic form game Γ is defined as a triplet

$$\Gamma = (\mathcal{N}, \mathcal{X}, u) \tag{1.4}$$

where $\mathcal{N}$ is a set of decision makers (or agents, or players), $\mathcal{X} := \Pi_i \mathcal{X}_i$ is a product space of each agent i's strategy space $\mathcal{X}_i$, and $u := (u_i)$ is a list of each agent's objective function $u_i : \mathcal{X}_i \to \Re$.

Together with (1.5), traditional game theory further makes the following assumptions:

Axiom 1.

1. *Instrumentally rational individual action;*

2. *Common knowledge on rationality;*

3. *Consistent alignment on beliefs.*

This dissertation is going to address a few issues when those assumptions are challenged in some game situations.

An agent i is said to be instrumentally rational if she has a well-defined preference ordering that is represented by u_i. The game is said to have common knowledge on rationality if each agent is instrumentally rational, each agent knows that each agent is instrumentally rational, each agent knows that each agent knows that each agent is instrumentally rational, ad infinitum. Also, each agent knows the set of strategies and the utility function of every other agent. Consistent alignment on beliefs means no agent expects that an agent with the same information can develop a different thought process [21].

There exists a rich literature studying solution concepts, their existence, uniqueness, refinement, stability, and convergence of algorithms in many different contexts. We focus attention to the decision criteria.

First consider a full information game where all agents are aware of Γ and Axiom 1.

Definition 1 (Nash Equilibrium in Full information game). *Let $\triangle \mathcal{X}_i$ be the set of probability measures on $\mathcal{X}_i$. For $\sigma \in \Pi_i \triangle \mathcal{X}_i$, one defines*

$$u_i(\sigma) = E_\sigma(u_i(X))$$

where E_σ is the expectation when X has the distribution σ.

Then $\sigma^* \in \Pi_i \triangle \mathcal{X}_i$ is called a Nash equilibrium *if, for all i,*

$$u_i(\sigma^*) \geq u_i(\sigma_i, \sigma^*_{-i}) \text{ for all } \sigma_i \in \triangle \mathcal{X}_i.$$

When σ^* assigns probability one to some $x^* \in \Pi_i \mathcal{X}_i$, the equilibrium is called a pure *Nash equilibrium. Otherwise, it is called a* mixed strategy *Nash equilibrium.*

To find a Nash equilibrium, every agent i independently identifies her best response σ_i^* to the choices of the other agents. That is, she determines

$$\sigma_i^* \in \arg \max_{\sigma_i \in \triangle \mathcal{X}_i} u_i(\sigma_i, \sigma_{-i}^*).$$

This solution assumes that all the agents correctly compute the others' best strategy, which requires Axiom 1. This search for the fixed point is discussed by Lismont and Mongin [27], and argued as an interactive rationality by Aumann [5]. Nash has shown that any game with a finite number of players with finite strategy sets has at least one mixed strategy Nash equilibrium [33]. When the utility functions are diagonally concave and the sets of strategies are compact, Rosen [35] has shown that the game has a pure Nash equilibrium.

When the Nash equilibrium is not unique, the meaning of each equilibrium becomes questionable and the selection among equilibria requires some care. The next sections review some of those issues.

1.2.1 Iterated Strict Dominance

Iterated strict dominance is a *survival strategy* solution concept, and introduced by Luce and Raiffa [28]. The survival process is infinitely repeated. More precisely, let $\mathcal{X}_i$ be agent i's pure strategy space and let $\triangle \mathcal{X}_i$ be her mixed strategy space.

Definition 2. *Initialize sets $S_i^0 := \mathcal{X}_i$ and $\Sigma_i^0 := \triangle \mathcal{X}_i$. Recursively define*

$$S_i^n := \{ s_i \in S_i^{n-1} \mid \nexists \sigma_i \in \Sigma_i^{n-1} \text{ s.t. } u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^{n-1} \}$$

$$\Sigma_i^n := \{ \sigma_i \in \Sigma_i \mid \sigma_i(s_i) > 0 \text{ only if } s_i \in S_i^n \}.$$

Finally we define

$$S_i^\infty := \bigcap_{n=0}^{\infty} S_i^n$$

$$\Sigma_i^\infty := \{ \sigma_i \mid \nexists \sigma_i' \text{ s.t. } u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i}), \text{ for all } s_{-i} \in S_{-i}^\infty, \sigma_i', \sigma_i \in \triangle S_i^\infty \}$$

Then S_i^∞ and Σ_i^∞ are agent i's pure/mixed strategies that survive iterated strict dominance.

1.2.2 Focal Points

This is a view argued by Shelling [37] that the strategic form game definition washed away too much information by which otherwise agents in a coordination game may be able

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to coordinate better than Nash equilibrium. Shelling's one experiment was when two players with no communication are asked to meet at New York City on a fixed day, but not instructed about neither a time nor a location, most of the participants choose Grand Central Station at noon. A focal point is a concept, which allows a multitude of different mathematical definitions, about a feature of such a coordination game that provides to a combination of actions a distinction from others. While universally acknowledged by game theorists, it has not been assimilated info formal game theory [14].

1.2.3   Selection Theory (Trembling Hand)

Shelten's equilibria selection theory [38] is a method for selecting among multiple Nash equilibria. This theory considers the possibility of small operational error, called a *trembling hand*. Shelten argues that agents should reject Nash equilibria when the reward deteriorates significantly if an agent makes a small error.

1.2.4   Correlated Equilibrium

The correlated equilibrium proposed by R. Aumann [3] is a generalized Nash equilibrium that introduces *a preplay discussion* and *a public signal* from nature that obeys a *common prior*. During the preplay discussion, agents agree to a *correlating device*. A correlating device is a triple

$$(\Omega, \{H_i\}, P),$$

where Ω is the sample space of the device, P is a probability measure on Ω, and H_i is agent i's information partition on Ω. The correlating device notifies the agent i of $h_i(\omega) \in H_i$ upon ω occuring. Recall that $\mathcal{X}_i$ is i's pure strategy space. Let $\mathcal{C}_i$ be the collection of maps $r_i : H_i \to \mathcal{X}_i$.

Definition 3 (Correlated Equilibrium). *A correlated equilibrium $r^* = (r_i^*) \in \Pi_i \mathcal{C}_i$ relative to the correlating device $(\Omega, \{H_i\}, P)$ is a Nash equilibrium in r-strategies. That is, for all i,*

$$E\left[u_i(r_i^*(h_i(\omega)), r_{-i}^*(h_{-i}(\omega)))\right] \geq E\left[u_i(r_i(h_i(\omega)), r_{-i}^*(h_{-i}(\omega)))\right] \ \text{for all } r_i \in \mathcal{C}_i$$

The set of correlated equilibria is at least as large as the set of mixed strategy Nash equilibria since the coordination signals can correspond to independent randomizations of the strategies by the different agents [19]. However, in many games, if such a coordination is possible, it can improve the agents' utility.

1.2.5 Rationalizable Strategies

Rationalizable strategies are all the strategies that a rational player could play. This concept is complementary to iterated strict dominance, and introduced by Bernheim [7] Pearce [34].

Definition 4. *The rationalizable strategies for agent i are $\bigcup_{n=0}^{\infty} \Sigma_i^n$, where for each i one defines recursively*

$$\Sigma_i^0 = \triangle \mathcal{X}_i$$

$$\Sigma_i^n := \{\sigma_i \in \Sigma_i^{n-1} \mid \exists \sigma_{-i} \in \times_{j \neq i} \triangle \Sigma_j^{n-1} \ s.t. \ u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \ for \ all \ \sigma_i' \in \Sigma_i^{n-1}\}$$

In general, the set of rationalizable strategies is contained in the set that survives iterated strict dominance. In two agent games, they are identical.

1.2.6 Rational Expectation

Similarly to Shelling's focal point motivation, R. Aumann [5] argues that the strategic form game does not carry enough information. He redefines a game as *game situation* $\mathcal{G} = (\Gamma, \beta)$, where Γ is a strategic form game as we defined earlier, and β is called a *belief system* that defines a player strategy for her each type, and represents how each *type* player believes how others will behave. A player's type uniquely determines the whole hierarchy of her beliefs.

The expectation of a player is her expected payoff with respect to her belief given her type. If the belief system agrees to a consistent common prior, and if the strategy the type prescribes maximizes the player's expected payoff, then the players expectation is called rational. Still, in general, there exist infinitely many consistent belief systems which can be paired with Γ, though a game situation is defined with a single consistent belief system. Now consider a view from the opposite direction: suppose nature governs a random event and provides a private signal to each player as a function of that event. Given that signal, a player has a posterior view on how others will behave. Carefully designed, this set of signals provides a coordination among players from which no one wants to unilaterally deviate. This is precisely the definition of a correlated equilibrium, for which a player can compute her conditional payoff. Aumanns contribution is to show a relation between rational expectations in (Γ, β) and conditional payoffs of correlated equilibria in *a game which is closely related to* Γ. This closed related game is 2Γ to be defined shortly.

We first redefine Γ as follows: A strategic form game Γ is defined as a triplet

$$\Gamma = (\mathcal{N}, \mathcal{L}, u) \tag{1.5}$$

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where $\mathcal{N}$ is a set of decision makers (or agents, or players), $\mathcal{L} := \Pi_i \mathcal{L}_i$ is a product space of each agent i's finite strategy list $\mathcal{L}_i$. Let $\{\mathcal{L}_i\}$ denote the largest set of elements of $\mathcal{L}_i$, that is, a set of i's feasible strategies without redundancy. $u := (u_i)$ is a list of each agent's objective function $u_i : \{\mathcal{L}\} \to \Re$.

A doubled list $2\mathcal{L}_i$ is $\mathcal{L}_i \times \{1, 2\}$, where the first copy and the second copy of a strategy are identical. Finally define a doubled game

$$2\Gamma = (\mathcal{N}, 2\mathcal{L}, u) \tag{1.6}$$

where $u : \{2\mathcal{L}\} \to \Re$ giving the same payoff as in u of Γ, no matter which copy of a strategy is used. Note $\{2\mathcal{L}\} \equiv \{\mathcal{L}\}$.

The introduction 2Γ is a *trick* to assign any correlated equilibrium that will lead to *any* consistent belief system in $\mathcal{G}$, by splitting weights of correlated equilibrium distribution.

Theorem 1. *The rational expectations in a game* (Γ, β) *are precisely the conditional payoffs to correlated equilibria in the doubled game* 2Γ.

Aumann provides two intuitions (See [5] Section VI) underlying this theorem:

(i) The common prior probability of a consistent belief system β in a game Γ is essentially the same thing as a correlated equilibrium of a game Γ_B closely related to Γ – that in which each strategy of each player appears as many times as there are types that play that strategy in β.

(ii) The conditional expectation of a strategy in a correlated equilibrium does not change when *other* strategies that are identical are amalgamated. Amalgamation is replacing identical strategies by a single strategy, by adding prior probabilities over added strategies.

Then, define 2Γ by amalgamating in Γ_B all identical strategies into two. By (ii), a conditional correlated equilibrium payoff in Γ_B for a particular strategy is also a conditional correlated equilibrium payoff in 2Γ. Together with (i), this yields above theorem.

The goal of this solution concept is somewhat different from traditional ones in two ways. First, instead of focusing on the recommendation to agents, this formulation focuses on what rational players should expect to get, or the *value* of the game. It is named as *rational expectation*. Second, as far as the recommendation is concerned, it does not deal with 'equilibria'. It simply suggests an agent to do single-agent-like maximization against one's subjective probabilities over others' strategies.

One should note also that the level of consistency Aumann requires is weak. The belief system is required to be consistent to a common prior that exists, but it is not required to be consistent to feasibility of outcomes. Rational expectations can be outside of the convex hull of pure strategy payoffs. This possibility is called the inconsistency of assessment. An existence of a common prior is not sufficient to guarantee the feasibility of rational expectations.

1.3 Incomplete Information Game

As in the case of a single-agent optimization problem, many games of interest have incomplete information. The uncertainty game we consider is one where each agent has her own private information which is not known to the others, and all agents know this as common knowledge. Agent i's private information influences i's utility, best response, and ultimately the solution of the game. Thus, in contrast to the full information game case, an agent does not know the exact preference ordering of other agents. We summarize the private information of agent i by a parameter $\theta_i \in \Theta_i$.

Bayesian Nash Equilibrium

The notion of Bayesian Nash equilibrium is Harsanyi's proposal to model and understand an incomplete information game. This model assumes that there exists a common prior probability distribution P with which nature randomly chooses each agent's private information or *type*. The utility function of of agent i u_i is

$$u_i : \mathcal{X} \times \Theta \to \Re.$$

The probability distribution of $\theta = (\theta_i) \in \Pi_i \Theta_i$ is P and $(\mathcal{N}, \mathcal{X}, u, P, \Theta)$ is common knowledge.

Definition 5 (Mixed Strategy Bayesian Nash Equilibrium)**.** *We define $\triangle \mathcal{X}_i$ and $u_i(\sigma, \theta_i)$ as before. Then $\sigma^* \in \Pi_i \triangle \mathcal{X}_i$ is called a* mixed strategy Bayesian Nash equilibrium *if, for all i,*

$$\sigma_i^*(\theta_i) \in \arg \max_{\sigma_i \in \triangle \mathcal{X}_i} E[u_i(\sigma_i, \sigma_{-i}^*(\theta_{-i}), \theta) | \theta_i].$$

1.3.1 Motivation for Non-Probabilistic Solution Concept

The concept of Bayesian Nash equilibrium requires some strong assumptions: (i) existence of common prior that governs nature's move; (ii) common knowledge on the prior; (iii) error-free observation of nature's selection of private information. As in the case of a single-agent optimization problem, there are many situations where these assumptions are not satisfied.

Many researchers have explored non-Bayesian models of uncertainty. Knight [23] raised questions about the suitability of probabilistic characterizations of uncertainty in some situations. Allais' parodox [2] and Ellsberg's paradox [18] are examples of situations where decision makers violate the expected utility hypothesis. More recently, Binmore [12] and Lec and Leroux [25] explored more philosophical questions on inaccuracy, arbitrariness, and illegitimacy of Bayesianism in games. The behavioral sociology literature also reports that

Bayesian strategies fail to occur in some real world games [40]. A few noteworthy experiments demonstrate a *certainty effect* where people prefer less uncertain events, a *refection effect* where people respond differently to gain and loss [6], and *preference reversals* where people show different valuations when they buy and when they sell the same lottery [13]. See also [26] for a related discussion of the modeling of uncertainty through a family of probability distributions.

For such situations, one may consider one of the non-probabilistic criteria developed in the preceding section: pessimism, optimism, regret, etc. In a single-agent decision making problem, one is free to choose a criterion as long as it captures the physical meaning of the problem context. In multi-agent decision making problem, however, that is not sufficient. The choice should be *strategic*.

Let's take pessimism for example. Suppose an agent cares the robust performance (for herself). However, if this fact is known to other agents, they can exploit this and may *strategically* select the optimism criterion in order to achieve higher performance for themselves. Knowing this possibility, one should be careful in choosing the pessimism criterion.

More generally, it should be questioned if it is purely in the best interest of an agent to follow a particular decision criterion. Based on this argument, the model in Chapter 2 considers that the choice of decision criterion is strategic. The proposed model has a form similar to Hurwicz's criterion in single-agent decision making, but now the parameter α will be chosen strategically.

1.3.2   Common Knowledge

Since optimally coordinated outcome of the game cannot be worse than uncoordinated outcome, in some game situations, such as cooperative games, agents are willing to commonly share their private information so as to achieve the most efficient outcome for all. For this purpose, communication, or message exchange is the most natural way of information sharing among autonomous agents.

Agents are said to have information consensus if all know that information. Agent are said to have common information (knowledge) if all know the information, all know that all know the information, all know that all know that all know the information, ad infinitum. To reach a coordinated actions, it is obvious that agents need to reach common knowledge first.

If the communication is perfect and error-free, once a message sender sends a message to a receiver, the former is very sure the latter gets the message. If the communication is error-prone however, no matter how small the chance of error is, one needs an additional protective mechanism to make sure of the message delivery. Consider any in-band mechanism (using message exchange as a way of protection) of that kind. In 1978, J. Gray maintained there exists no such mechanism using finite number of message exchanges. His note is simple and worth to quote here as it is.

The Generals Paradox [20]

"There are two generals on campaign. They have an objective (a hill) which they want to capture. If they simultaneously march on the objective they are assured of success. If only one marches, he will be annihilated. The generals are encamped only a short distance apart, but due to technical difficulties, they can communicate only via runners. These messengers have a flaw, every time they venture out of camp they stand some chance of getting lost (they are not very smart.) The problem is to find some protocol which allows the generals to march together even though some messengers get lost. There is a simple proof that no fixed length protocol exists: Let P be the shortest such protocol. Suppose the last messenger in P gets lost. Then either this messenger is useless or one of the generals doesn't get a needed message. By the minimality of P, the last message is not useless so one of the general doesn't march if the last message is lost. This contradiction proves that no such protocol P exists."

We can construct a discrete knowledge hierarchy as the m, the number of message exchanges, increases. We expect at infinity m, generals reach common knowledge. Intuition says if $m < \infty$ is large, generals are likely to reach common knowledge. The Generals Paradox asserts however, that no matter how large m is, if it is finite, generals never reach the common knowledge.

The cost of failure to reach common knowledge is high (death) in this problem. Thus, we can deduce that if we define an agent's strategy as a function of m, or $s(m)$, then

$$\lim_{m \to \infty} s(m) \neq s(\infty).$$

In other words, the sequence of strategies does not converge as the knowledge hierarchy builds up. (An essentially same observation is made by A. Rubinstein in 1989 in incomplete information game and Bayesian Nash equilibrium context [36].) This observation triggered sequels of many research interests that attempt to understand the topological properties of belief spaces. Good pieces of work along this line include [17, 32, 16, 1, 30].

In Chapter 3, we study a cooperative game with private information with robust decision criterion. Not only we provide an impossibility result aligned with the General Paradox, but also we provide a way of coordination to achieve guaranteed efficient outcome.

Chapter 2

Non-cooperative Game with Non-probabilistic Uncertainty

This chapter studies one-shot two-player games with non-Bayesian uncertainty. The players have an *attitude* that ranges from optimism to pessimism in the face of uncertainty. Given the attitudes, each player forms a belief about the set of possible strategies of the other player. If these beliefs are consistent, one says that they form an *uncertainty equilibrium*. One then considers a two-phase game where the players first choose their attitude and then play the resulting game. The chapter illustrates these notions with a number of games where the approach provides a new insight into the plausible strategies of the players.

2.1   Introduction

We study a one-shot non-cooperative game of two rational players with non-probabilistic information uncertainty. Specifically, we assume that the set of possible values of the uncertain parameter is known, but that no prior distribution is available. Thus, instead of the more traditional Bayesian approach where user maximize their expected reward, here, players have an *attitude* that models their risk-aversion. An optimistic (respectively, pessimistic) player assumes that the other player will choose a strategy that is beneficial (respectively, detrimental) to her. A moderately optimistic player makes an intermediate assumption. However, in contrast with other approaches, we assume that the players choose their attitude by analyzing the consequences of their choice, instead of assuming that their risk-aversion is pre-determined.

Different players may have a different objective in the face of uncertainty. Some popular

choices include minimax regret, maximin pessimism or maximax optimism. Instead of a fixing a player's optimization objective, we allow a rational player to choose somewhere between worst case and best case. We parametrize a player's subjective decision criterion as a convex combination of pessimism and optimism with parameter π, and we call it a player's *attitude* against uncertainty. As we explained in the Introduction, Hurwicz (1951) [22] proposed a similar convex combination criterion for a single agent decision making problem. However, one crucial aspect of this study is that the attitude is not fixed ahead of time. Instead, the players choose their attitude *strategically*. For instance, the players may realize that the only rational attitude is to be optimistic because it is the only Nash equilibrium in a two-stage game where the first stage is to choose the attitude. More generally, there may be a set of attitudes for each player from which it is not rational to deviate unilaterally. In such a case, the model provides some information about how to behave rationally in the face of uncertainty.

Section 2.2 develops a model of two non-cooperative players with non-probabilistic parameter uncertainty, and introduces the notions of attitude and uncertainty equilibrium. Section 2.4 presents examples for which the approach provides a new insight into the strategies. Section 2.3 proves the existence condition of an uncertainty equilibrium and relates it to a Nash equilibrium of the corresponding full information game. Section 2.5 proves that at least one player should not be pessimistic. Section 2.6 concludes the chapter.

2.2 Uncertainty Equilibrium

The section defines the model of game with uncertainty. It then introduces the notion of uncertainty equilibrium for players that have specific attitudes. The section then defines the two-phase game. First, we define a reference game with full information.

Definition 6 (Certainty Game $\mathcal{G}_o$)**.**

Two non-cooperative, selfish and rational players $i = 1, 2$ and $j = 3 - i$ play a game with strategies $x := (x_1, x_2) \in X_{1,o} \times X_{2,o}$, where $X_{i,o} \subset \mathbb{R}$ is i's closed bounded strategy interval. Player i has type $\theta_i \in \mathbb{R}$. The reward of player i is real-valued $u_i(x, \theta_i)$. This is a full information game with common knowledge about u_i, $X_{i,o}$, and θ_i for all i. We assume that this game is such that $u_i(x, \theta_i)$ is continuous in (x, θ_i), has a unique maximizer $x_i(x_j, \theta_i)$ for every (x_j, θ_i), and has at least one pure Nash equilibrium.

We now consider the game with uncertainty about the opponent's type.

Definition 7 (Uncertainty Game $\mathcal{G}$)**.**

Player i knows her own true type θ_i but only that $\theta_j \in \Theta_j$ for $j = 3 - i$, where Θ_j is a closed bounded real interval, and this is common knowledge. To avoid triviality, Θ_j is assumed to be of non-zero length unless specified otherwise.

The goal of the chapter is to study the notion of equilibrium in such a situation. Our approach is non-Bayesian. That is, we do assume neither a known posterior distribution of the parameters nor the existence of a common prior distribution.

We start with a simple approach to refine the set of rational strategies. Assume that it is known that player i chooses $x_i \in X_i$. It may be reasonable to believe that player j will choose a strategy $x_j(x_i, \theta_j)$ for some $x_i \in X_i$. This best response x_j is defined in the strategy space $X_{j,o}$. Since player i does not know θ_j, she may then believe that player j chooses $x_j \in \phi_j(X_i)$ where

$$\phi_j(X_i) := \{x_j(x_i, \theta_j) \mid x_i \in X_i, \theta_j \in \Theta_j\}. \tag{2.1}$$

These considerations lead to the following definition.

Definition 8. *The sets $X_1^\dagger, X_2^\dagger$ are* consistent *if $X_j^\dagger = \phi_j(X_i^\dagger)$ for $i = 1, 2$ and $j = 3 - i$.*

Since the best response $x_i(x_j, \theta_i)$ is defined in $X_{i,o}$, $\phi_i(X_j)$ for any X_j is also defined in $X_{i,o}$. Thus for consistent sets $X_1^\dagger, X_2^\dagger$,

$$X_i^\dagger \subseteq X_{i,o}, \text{ for all } i.$$

The consistent sets form a product space of strategies beyond which no rational player plays. Although the sets $X_i^\dagger$ are smaller than the original strategy spaces $X_{i,o}$, they may be large and provide little recommendation on the strategies the players should choose. Moreover, one may question whether the players will choose strategies in the consistent sets.

2.2.1 Optimism and Pessimism

We now develop a different formulation of the game that considers the *attitudes* $\pi = (\pi_1, \pi_2) \in [0, 1]^2$ of players in the face of uncertainty.

Definition 9 (Game with Attitudes π: $\mathcal{G}(\pi)$)**.**

If it is known that player j chooses $x_j \in X_j$, then player i chooses $x_i \in X_{i,o}$ to maximize

$$f_i(x_i, X_j, \theta_i, 1) := \max_{x_j \in X_j} u_i(x, \theta_i)$$

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if she is optimistic and to maximize

$$f_i(x_i, X_j, \theta_i, 0) := \min_{x_j \in X_j} u_i(x, \theta_i)$$

if she is pessimistic. In general, for $0 \leq \pi_i \leq 1$, if player i has attitude π_i, she chooses $x_i \in X_{i,o}$ to maximize

$$f_i(x_i, X_j, \theta_i, \pi_i)$$
$$:= \pi_i \max_{x_j \in X_j} u_i(x, \theta_i) + (1 - \pi_i) \min_{x_j \in X_j} u_i(x, \theta_i). \tag{2.2}$$

We primarily study a discrete attitude space $\pi_i \in \{0, 1\}$, and later use the continuous attitude space $\pi_i \in [0, 1]$ in developing the notion of robust attitude.

Designate by $r_i(X_j, \theta_i; \pi_i)$ the set of maximizers of $f_i(x_i, X_j, \theta_i, \pi_i)$. That is,

$$r_i(X_j, \theta_i, \pi_i) := \arg \max_{x_i \in X_{i,o}} f_i(x_i, X_j, \theta_i, \pi_i). \tag{2.3}$$

Since player j does not know θ_i, she assumes that $x_i \in \psi_i(X_j; \pi_i)$ where

$$\psi_i(X_j; \pi_i) := \bigcup_{\theta_i \in \Theta_i} r_i(X_j, \theta_i; \pi_i). \tag{2.4}$$

2.2.2 Uncertainty Equilibrium

We then have the following definition.

Definition 10 (Uncertainty Equilibrium of $\mathcal{G}(\pi)$).

The pair of sets (X_1, X_2) is an uncertainty equilibrium *for players with attitudes π, if $X_i = \psi_i(X_j; \pi_i)$ for $i = 1, 2$ and $j = 3 - i$.*

Moreover, if the uncertainty equilibrium is unique, we consider that player i plays $x_i \in r_i(X_j, \theta_i; \pi_i)$ to maximize her *interim anticipated reward $f_i(x_i, X_j, \theta_i, \pi_i)$*. If the corresponding x_i is unique and equal to $x_i(\theta_i, \pi)$, it results in *actual (ex-post) rewards $U_i := u_i(x_i(\theta_i, \pi), x_j(\theta_j, \pi), \theta_i)$*. If the context is clear, we simplify as $U_i(\pi) := u_i(x_i(\pi), x_j(\pi), \theta_i)$ where $x_i(\pi) = x_i(\theta_i, \pi)$.

2.2.3 Attitude Game

Is it preferable to be optimistic or pessimistic? To answer this question, we consider a two-stage game.

Definition 11 (Attitude Game $\mathcal{A}$)**.**

In the first stage, the players choose their attitudes $(\pi_1, \pi_2) \in \{0, 1\}^2$. In the second stage, they play $\mathcal{G}(\pi)$ and get the rewards $U_i(\pi)$.

If $\pi = (0, 0)$ is a unique Nash equilibrium for the two-stage game, we conclude that the players should be pessimistic. Moreover, the analysis then specifies precisely how they should choose their second stage strategy. The situation is similar if any $\pi \in [0, 1]^2$ is a unique Nash equilibrium attitude. A player i's attitude π_i^* is said to be *dominant* if for any π_j and θ_j, $j = 3 - i$,

$$U_i(\pi_i^*, \pi_j) \geq U_i(\pi_i, \pi_j)$$

for all π_i.

In contrast with traditional approaches, we do not consider that players have a fixed attitude (as a *type*). Instead, they choose their attitude by analyzing the game instead of being driven by a preordained risk aversion.

As we show in the following sections, there are games where this approach enables to rationalize specific strategies under uncertainty.

2.3 Existence of Uncertainty Equilibrium and its Relation to Nash Equilibrium

This section provides a condition for the existence of an uncertainty equilibrium.

Theorem 2 (Existence of Uncertainty Equilibrium)**.**

Assume $r_i(X_j, \theta_i, \pi_i)$ is single-valued and continuous in X_j, θ_i and π_i. Then there exists an uncertainty equilibrium $(X_1^*(\pi), X_2^*(\pi))$.

At an uncertainty equilibrium $(X_1^*(\pi), X_2^*(\pi))$, i's best response is

$$x_i^*(\pi) = r_i(X_j^*(\pi), \theta_i, \pi_i).$$

From the proof of Theorem 2, note there is one-to-one correspondence between $x_i^*(\pi)$'s and $X_i^*(\pi)$'s via r_i's. In particular, if Θ_i is a singleton, then $X_i^*(\pi) = x_i^*(\pi)$. This (obvious) observation is stated in the next theorem.

Theorem 3. *Under the assumptions of Theorem 2, $\mathcal{G}(\pi)$'s uncertainty equilibrium $(X_1^*(\pi), X_2^*(\pi))$ coincides with game $\mathcal{G}_o$'s Nash equilibrium $(x_1^\star, x_2^\star)$ if $\Theta_i = \{\theta_i\}$ for $i = 1, 2$, irrespective of π.*

2.4 Examples

The first example is a game with negative externality. In this game, the players should be optimistic even when they are uncertain about the opponent's type. The second example is a game with positive externality. In this game the players are better off when they both are pessimistic than when they both are optimistic. However, we will see that players are inconclusive in the choice of attitudes because there are two pure Nash attitudes. The third example is a Cournot duopoly game [15] with uncertainty. For this game, we study conditions for the existence of dominant attitudes, and robust attitudes. For clarity, the algebraic derivations are in the appendix.

2.4.1 A Game with Negative Externality

Consider two agents $i = 1, 2$ who consume resources $x_i \in [0, 1]$ to gain some benefit. The consumption degrades the quality of the environment which affects both players. The agent's reward is defined to be the benefit minus the degradation of the environment. The benefit is assumed to be proportional to the consumption. The environment degrades exponentially in sum of players' consumption ($\exp\{x_1 + x_2\}$), via scaling factor $\exp\{-\theta_i\}$, where θ_i^{-1} captures i's susceptibility to the environmental degradation. Here, θ_i is private information for agent i. $x_i \in [0, 1], \theta_i \in [\alpha, \beta]$ for some $0 < \alpha < 2\alpha < \beta < 1$. Agent i's reward is

$$u_i(x, \theta_i) = x_i - \exp\{-\theta_i + x_i + x_j\}.$$

(One may add a constant to make the rewards positive.)

Theorem 4. *Agents should be optimistic and choose the consumption levels $x_i = \theta_i - \alpha/2$ for $i = 1, 2$. In contrast, if θ_1, θ_2 are fully known and $\theta_1 < \theta_2$, then the only Nash strategy is $(x_1, x_2) = (0, \theta_2)$.*

For this game, the only consistent sets (see Definition 8) are $X_1 = X_2 = [0, \beta]$, which provides little information about the strategies of the agents.

2.4.2 A Game with Positive Externality

Consider two agents $i = 1, 2$ and $j = 3 - i$ who spend the effort $x_i \geq 0$ to gain some benefit. There is a positive externality in benefit: the opponent agent's effort spills over and affects positively the agent's benefit. The sensitivity of agent i to the spill-over is her private information θ_i. Agent i's utility is

$$u_i = \sqrt{x_i + \theta_i x_j} - x_i.$$

Let, for all i, $\Theta_i = [1/4, 1/2]$.

Certainty game

It is easy to see this game has a free-riding effect. Both agents free ride on each other, to some degree. A social planner would make agents invest more than they do at the Nash equilibrium. First, we study a Nash equilibrium for the full information game. From the first order condition, we find

$$\frac{\partial u_i}{\partial x_i} = \frac{1}{2} \frac{1}{\sqrt{x_i + \theta_i x_j}} - 1 = 0.$$

Thus i's best response to x_j is

$$x_i = [\frac{1}{4} - \theta_i x_j]^+.$$

This game has only one pure Nash equilibrium

$$x_i = \frac{1}{1 - \theta_i \theta_j} \frac{1 - \theta_i}{4}$$

with corresponding utility

$$u_i = \frac{1}{2} - \frac{1 - \theta_i}{4(1 - \theta_i \theta_j)}.$$

Uncertainty game: Optimism case

Assume that both agents are optimistic and that both know this: they choose the attitudes $\pi = (O, O)$. Starting with $X_i = [c_i, d_i]$ at equilibrium, since u_i is increasing in x_j, we see that an optimistic agent assumes that the other agent selects her largest strategy. As a result,

$$x_i = [\frac{1}{4} - \theta_i d_j]^+.$$

Thus

$$X_i = [\frac{1}{4} - \frac{1}{2}d_j, \frac{1}{4} - \frac{1}{4}d_j] := [c_i, d_i].$$

The unique equilibrium is $X_i = [3/20, 1/5]$. Accordingly, i's strategy becomes

$$x_i^*(O, O) = \frac{1}{4}(1 - \frac{4}{5}\theta_i).$$

Uncertainty game: Pessimism case

Assume now that both agents are pessimistic and that both know this: they choose the attitudes $\pi = (P, P)$. Arguing as before, a pessimistic agent assumes that the other agent selects her minimum strategy. Hence, starting with $X_i = [c_i, d_i]$, we find

$$x_i = [\frac{1}{4} - \theta_i c_j]^+.$$

Thus

$$X_i = [\frac{1}{4} - \frac{1}{2}c_j, \frac{1}{4} - \frac{1}{4}c_j] := [c_i, d_i].$$

The unique equilibrium is $X_i = [1/6, 5/24]$. Accordingly, i's strategy becomes

$$x_i^*(P, P) = \frac{1}{4}(1 - \frac{2}{3}\theta_i).$$

We can see that a pessimistic agent invests more than an optimistic one.

A right attitude

We can continue similar analysis for $\pi = (O, P)$ and $\pi = (P, O)$. For any θ_i, one can easily show that the following inequalities hold:

$$U_{1,OP} > U_{1,PP} > U_{1,PO} > U_{1,OO} \quad \text{for all} \quad \theta_2 \in \Theta_2,$$

and

$$U_{2,PO} > U_{2,PP} > U_{2,OP} > U_{2,OO} \quad \text{for all} \quad \theta_1 \in \Theta_1.$$

Thus, in this positive externality game, the ex-post utilities when both agents are pessimistic are larger than when both of them are optimistic. It is easy to see this game has two pure Nash equilibria : (O,P) and (P,O). Thus, in terms of picking one single attitude, players are inconclusive.

2.4.3 Cournot Duopoly Game

Full Information Case

For $i = 1, 2$, selfish and rational player i produces a non-negative quantity x_i of homogeneous items with a non-negative production cost $\theta_i \in [0, 1/2]$ per item. The selling price per item is $(1 - x_1 - x_2)^+$ where $y^+ = \max\{y, 0\}$ for $y \in \Re$. Accordingly, the reward (profit) of player i is $u_i(x, \theta_i)$ defined as follows:

$$u_i(x, \theta_i) := x_i(1 - x_1 - x_2)^+ - \theta_i x_i \tag{2.5}$$

where $x = (x_1, x_2)$.

Player i's strategy is the quantity x_i to produce. The value of x_i that maximizes $u_i(x, \theta_i)$ is $x_i = (1 - \theta_i - x_j)/2$, for $i = 1, 2$ and $j = 3 - i$. The unique solution of these equations is the Nash equilibrium $x^\star := (x_1^\star, x_2^\star)$ where

$$x_i^\star = (1 - 2\theta_i + \theta_j)/3. \tag{2.6}$$

The corresponding utilities are

$$u_i^\star = x_i^{\star 2}. \tag{2.7}$$

Note that the pair $x = (x_1, x_2)$ that maximizes $u_{social} := \sum_{i=1,2} u_i(x, \theta_i)$ is $((1 - \theta_1)/2, 0)$ when $\theta_1 < \theta_2$. This "social optimum" is quite different from the Nash equilibrium. There

$$u_{social} = (1 - \theta_1)^2/4. \tag{2.8}$$

Bayesian Uncertainty Case

In a Bayesian model, one assumes that θ_1 and θ_2 are independent with known distributions; each player $i = 1, 2$ knows θ_i and only the distribution of θ_j for $j = 3 - i$, and this is common knowledge. In that case,

$$E[u_1(x, \theta_1)|x_1, \theta_1] = x_1(1 - x_1 - E[x_2|x_1, \theta_1]) - \theta_1 x_1$$

and this expression is maximized by

$$x_1 = (1 - E[x_2|x_1, \theta_1] - \theta_1)/2 = (1 - E(x_2) - \theta_1)/2.$$

The last expression follows from the observation that x_2 is only a function of θ_2 which is independent of θ_1. Consequently, for $i = 1, 2$,

$$E(x_i) = (1 - E(x_j) - \mu_i)/2 \text{ where } \mu_i := E(\theta_i).$$

Solving this system of two equations, we find

$$E(x_1) = (1 - 2\mu_1 + \mu_2)/3 \text{ and } E(x_2) = (1 - 2\mu_2 + \mu_1)/3.$$

Accordingly, for $i = 1, 2$,

$$x_{i,B} = (2 - 3\theta_i - \mu_i + 2\mu_j)/6 \text{ where } j = 3 - i. \tag{2.9}$$

This solution is a unique Bayesian Nash equilibrium. Note that player i's strategy maximizes her *interim expected utility* $E[u_i(x_B, \theta_i)|x_{i,B}, \theta_i]$, rather than the *ex post utility* $u_i(x_B, \theta_i)$, which i cannot compute.

Consistent Set

Recall the definition of a consistent set. It provides a strategy bound beyond which a rational player should not play when non-probabilistic uncertainties prevail.

Consider that player i's best response to x_j at θ_i, $x_i(x_j, \theta_i)$, which is real and continuous in x_j and θ_i. Then $\phi_j(X_i)$ is a continuous and compact interval for any continuous and compact X_i. Suppose $X_1^\dagger = [a, b]$ and $X_2^\dagger = [c, d]$. Using definition $\Theta_i = [\alpha_i, \beta_i]$, we find

Theorem 5. *The consistent set for Cournot duopoly game is unique and is given as*

$$X_1^\dagger = [(1 - 2\beta_1 + \alpha_2)/3, (1 - 2\alpha_1 + \beta_2)/3] \text{ and}$$

$$X_2^\dagger = [(1 - 2\beta_2 + \alpha_1)/3, (1 - 2\alpha_2 + \beta_1)/3].$$

Proof. Recall $x_i(x_j, \theta_i) = (1 - x_j - \theta_i)/2$. Thus

$$\phi_1(X_2) = [(1 - d - \beta_1)/2, (1 - c - \alpha_1)/2] := [a, b].$$

Similarly

$$\phi_2(X_1) = [(1 - b - \beta_2)/2, (1 - a - \alpha_2)/2] := [c, d].$$

Solving the two equations above yields the result. □

Game with Attitudes

One assumes that, for $i = 1, 2$, player i knows θ_i but only that $\theta_j \in \Theta_j := [\alpha_j, \beta_j]$ for $j = 3 - i$ where $\beta_j \le 1/2$. This is common knowledge. Moreover, player i has attitude $\pi_i \in [0, 1]$. The following result is shown in the appendix.

Theorem 6. *The unique uncertainty equilibrium with attitudes π is the pair of intervals* $\mathcal{B}[s_i, t_i] := [s_i - t_i/2, s_i + t_i/2]$ *for $i = 1, 2$, where*

$$s_i = \frac{1}{3}\Delta_j \pi_i - \frac{1}{6}\Delta_i \pi_j + \frac{1}{12}(4 - 3\beta_i - 5\alpha_i + 4\alpha_j) \tag{2.10}$$

and $\Delta_i := \beta_i - \alpha_i$ and $t_i = (\beta_i - \alpha_i)/4$. The strategies that maximize the interim anticipated rewards are

$$x_i^*(\pi) = \frac{1}{3}\Delta_j \pi_i - \frac{1}{6}\Delta_i \pi_j + \lambda_i, \tag{2.11}$$

where $\lambda_i = (2 - \alpha_i + 2\alpha_j - 3\theta_i)/6$.

In fact, the *attitudes* are added degree of freedom that enables rationalization in choosing a strategy out of the consistent set. That is, the recommended strategy does swing in the entire consistent sets by varying π. That result is expressed in the following theorem.

Theorem 7. *The consistent set $(X_1^\dagger, X_2^\dagger)$ and the uncertainty equilibrium $(X_1(\pi), X_2(\pi))$ at π establish the following relation:*

$$X_i^\dagger = \left[\inf_{\pi \in \{0,1\}^2} X_i(\pi), \sup_{\pi \in \{0,1\}^2} X_i(\pi)\right], \tag{2.12}$$

for $i = 1, 2$.

Therefore, the attitude structure is exhaustively descriptive in expressing any feasible rational strategy in consistent sets. Note that the consistent sets do not require imposition on any form of knowledge about uncertainties except they are defined within certain ranges. Thus the same consistent sets are obtained when one considers a set of rational strategies over a family of probability distributions of uncertainties whose supports are the same ranges. As a result, one can expect that there always exists a pair of attitudes that corresponds to a particular choice of probability distributions of uncertainties. Indeed, when a Bayesian Cournot game assumes distributions of (θ_1, θ_2) with mean (μ_1, μ_2), the corresponding attitudes are

$$\pi_i = \frac{\mu_j - \alpha_j}{\beta_j - \alpha_j}, \text{ for all } i.$$

This result is obtained by equating (2.9) and (2.11) for $i = 1, 2$ and solving these equations.

Rationalization

The strategies (2.11) are rational when π is known. In an attitude game $\mathcal{A}$, a player first decides her attitude and then chooses the strategy (2.11). An immediate question is what a rational attitude is. First of all, there are situations where a player can choose a dominant attitude based on its private information and common knowledge, but not on the opponent's private information. The following lemma is proved in the appendix.

Lemma 1 (Dominant attitude).

Let $\underline{\theta}_i := \frac{1}{3}(2 - \beta_i + 4\alpha_j - 2\beta_j)$ and $\overline{\theta}_i := \frac{1}{3}(2 - \alpha_i + 4\beta_j - 2\alpha_j)$. Assume that the attitude space is discrete $\Pi = \{0, 1\}$. Then the following properties hold:

1. *If $\theta_i \leq \underline{\theta}_i$, then optimism is a dominant strategy for player i;*

2. *If $\theta_i \geq \overline{\theta}_i$, then pessimism is a dominant strategy for player i;*

3. *If $\underline{\theta}_i < \theta_i < \overline{\theta}_i$, then there is no dominant strategy for player i.*

In particular, if $\beta_i < 1/3$ for $i = 1, 2$ (i.e., if the unit production costs are sufficiently low), both players should be optimistic.

The game is said to be *symmetric* if $u_1 = u_2$ and $\Theta_1 = \Theta_2$.

There is a connection between a symmetric attitude game and a Prisoner's Dilemma game when the attitude space is discrete with $\Pi = \{0, 1\}$.

Theorem 8. *Consider the symmetric game $\mathcal{A}$ with $\Theta_1 = \Theta_2 = [\alpha, \beta]$ where $\beta > \alpha$. Then the following properties hold:*

1. *(PP) is never a Nash equilibrium;*

2. *(PP) is pareto efficient;*

3. *(PP) is pareto superior to (OO);*

4. *O is the dominant strategy if $\beta \leq \max(1/3, 2\alpha)$, so that (OO) is the only Nash equilibrium.*

Together with 1), 2), and 3), the condition in the last part makes the attitude game a Prisoner's Dilemma. The last condition requires that the costs are not too large.

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Robust attitude

As we observed from the previous example, game $\mathcal{A}$ may not have a dominant attitude for player i. In such a case, player i may prefer a strategy that guarantees the largest minimum ex-post reward. That is, player i might seek the *robust attitude* $\pi_i^\sharp \in [0, 1]$ defined by

$$\pi_i^\sharp := \arg \max_{\pi_i} \min_{\pi_j} u_i(x_i(\theta_i, \pi), x_j(\theta_j, \pi), \theta_i).$$

Theorem 9. *The robust attitude of Cournot duopoly does not coincide with pessimism and is given by*

$$\pi_i^\sharp = \min(1, (2 - 3\theta_i - \beta_i + 2\alpha_j)/4\Delta_j)$$

for $\Delta_j > 0$. Consequently, $\pi_i^\sharp > 0$, except for a singular case $\alpha_j = 0$ and $\theta_i = \beta_i = 1/2$.

Example 1. *Let $\beta := \max(\beta_i, \beta_j)$. Then if $\beta \le 1/4$, $\pi_i^\sharp = \pi_j^\sharp = 1$. That is, when costs are sufficiently small, the robust strategy is optimism. To see this, note that $\pi_i^\sharp = \min(1, (2 - 3\theta_i - \beta_i + 2\alpha_j)/4(\beta_j - \alpha_j)) \ge \min(1, (2 - 4\beta)/4\beta)) = 1$.*

Rationalization in a general attitude space $\Pi = [0, 1]$

In real economic situations demanding decisions, a human player does not necessarily take an extreme attitude - complete optimism or complete pessimism. It is more natural to think that one should take some combination of optimism and pessimism. That is, one can be π_i-optimistic, for some $\pi_i \in [0, 1]$.

Define the bounds of a rational attitude for player i by

$$\underline{\pi_i} \le \pi_i \le \overline{\pi_i}, \tag{2.13}$$

implying that it is not rational for player i to choose an attitude outside of this interval. In other words, for any possible true value of the uncertainty that i has and any possible rational attitude of player j, player i's best response attitude is neither $\pi_i < \underline{\pi_i}$ nor $\pi_i > \overline{\pi_i}$. If any attitude in $[0, 1]$ is rational, then $\underline{\pi_i} = 0$ and $\overline{\pi_i} = 1$. If a particular attitude is dominant, then $\underline{\pi_i} = \overline{\pi_i}$.

Recall the player i's ex-post utility of the attitude game when two players choose $\pi = (\pi_1, \pi_2)$.

$$U_i(\pi) = u_i(x_i^*(\pi), x_j^*(\pi), \theta_i), \text{ for } j = 3 - i.$$

Using

$$\frac{\partial x_i^*(\pi)}{\partial \pi_i} = \frac{1}{3}\Delta_j \text{ and } \frac{\partial x_j^*(\pi)}{\partial \pi_i} = -\frac{1}{6}\Delta_i,$$

we find

$$36\frac{\partial U_i(\pi)}{\partial \pi_i} = 2 - 3\theta_i - 4\Delta_j\pi_i - \Delta_i\pi_j - a_i - 4a_j + 6\theta_j.$$

In order to achieve the largest ex-post utility, player i should select the largest attitude as long as $\frac{\partial U_i(\pi)}{\partial \pi_i} \geq 0$. Also she should select the smallest attitude as long as $\frac{\partial U_i(\pi)}{\partial \pi_i} \leq 0$. To find $\underline{\pi_i}$ and $\overline{\pi_i}$, player i needs to consider all possible values of θ_j and π_j.

Assume $\theta_j = b_j$. Then,

$$36\frac{\partial U_i(\pi)}{\partial \pi_i} = (2 - 3\theta_i - b_i\pi_i) + 4\Delta_j(1 - \pi_i) + (2b_j + a_i\pi_j - a_i) \geq 0.$$

Thus, $\pi_i = 1$ can always be a rational attitude. Therefore $\overline{\pi_i} = 1$ for all i.

Also $\frac{\partial U_i(\pi)}{\partial \pi_i}$ is minimized at $\pi_j = 1$ and $\theta_j = a_j$. Considering π_i is defined in $[0, 1]$, the ex-post utility is maximized by

$$\pi_i = \frac{1}{4\Delta_j}\min((2 - 3\theta_i - b_i + 2a_j), 1),$$

which is the smallest possible rational attitude. These observations lead to the following theorem.

Theorem 10. *Player i should not select an attitude beyond the interval*

$$[\underline{\pi_i}, \overline{\pi_i}] = \left[\frac{1}{4\Delta_j}\min((2 - 3\theta_i - b_i + 2a_j), 1), 1\right].$$

Finally, there is a similar dominant attitude criterion for a continuous attitude space. The proof is immediate from the above theorem, and thus omitted.

Theorem 11 (Dominant attitude)**.**

Let $\underline{\theta_i} := \frac{1}{3}(2 - \beta_i + 6\alpha_j - 4\beta_j)$ and $\overline{\theta_i} := \frac{1}{3}(2 - \beta_i + 2\alpha_j)$. Assume that the attitude space is continuous $\Pi = [0, 1]$. Then the following properties hold:

1. *If $\theta_i \leq \underline{\theta_i}$, then $\pi_i = 1$ is a dominant strategy for player i;*

2. *If $\theta_i \geq \overline{\theta_i}$, then $\pi_i = 0$ is a dominant strategy for player i;*

3. *If $\underline{\theta_i} < \theta_i < \overline{\theta_i}$, then there is no dominant strategy for player i.*

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Price of Uncertainty

This subsection studies the effect of uncertainty on the social welfare. The social welfare is defined as the sum of ex-post utilities of players. A basic question is how bad an uncertainty is to social welfare. To measure this, we define Price of Uncertainty (PoU) as the ratio of sum utilities of an uncertainty game with respect to the sum utilities of a full information game at its Nash , assuming that the latter is unique.

Definition 12 (Price of Uncertainty).

The price of uncertainty of the Cournot duopoly game with attitudes π is defined as follows:

$$PoU := \frac{\sum_{i=1,2} u_i}{\sum_{i=1,2} u_i^\star}, \tag{2.14}$$

where $u_i^\star$ is defined in (2.7).

A different notion, *Price of Anarchy* captures how bad the lack of coordination in a game affects the social welfare in comparison to the case where a single designer optimizes the society. The motivation for the Price of Uncertainty is different: one cannot avoid a game situation, so that the optimum social welfare cannot be achieved.

All the numerical bound of the Price of Uncertainty can be found by simple algebra or numerical analysis, and thus the derivations are omitted for simplicity.

As a reference, we start with the PoU of social optimum (SO) case. (This is just a reciprocal of Price of Anarchy.)

$$PoU(SO) := \frac{\sum_{i=1,2} u_{i,SO}}{\sum_{i=1,2} u_i^\star}, \tag{2.15}$$

$u_{i,SO}$ is found at (2.8). Together with (2.7),

$$1 \leq PoU(SO) \leq \frac{5}{4}. \tag{2.16}$$

The lower bound is obtained when $(\theta_1, \theta_2) = (0, 1/2)$, and the upper bound is obtained when $\theta_2 = \frac{1}{5}(1 + 4\theta_1)$ for any $0 \leq \theta_1 \leq \frac{3}{8}$.

For the second reference, we consider the Price of Uncertainty of Bayesian game. Consider a family of distributions over θ_1, θ_2 with support of Θ_1, Θ_2 respectively. Designate $\mu = (\mu_1, \mu_2)$ as the means of the uncertainties. Then define

$$PoU(Bayes) := \frac{\sum_{i=1,2} u_{i,Bayes}}{\sum_{i=1,2} u_i^\star}, \tag{2.17}$$

where $u_{i,\text{Bayes}} := u(x_B, \theta_i)$ where in turn x_B is found at (2.9).

Then

$$\frac{1}{2} \leq \text{PoU(Bayes)} \leq \frac{5}{4} \tag{2.18}$$

and the lower bound is obtained at $(\theta_1, \theta_2) = (0, 1/2)$ and $(\mu_1, \mu_2) = (1/2, 0)$, and the upper bound is obtained at $(\theta_1, \theta_2) = (3/8, 1/2)$ and $(\mu_1, \mu_2) = (0, 1/2)$.

Although these bounds are feasible, it does not make sense for a player to willingly change the distributions of uncertainties.

Finally, define the Price of Uncertainty of an attitude game

$$\text{PoU(Attitude)} := \frac{\sum_{i=1,2} U_i}{\sum_{i=1,2} u_i^\star}. \tag{2.19}$$

Even for the same (θ_1, θ_2), players can willingly change their attitudes as long as their private information does not yield a particular dominant strategy. It is found

$$\frac{39}{64} \leq \text{PoU(Attitude)} \leq \frac{5}{4}. \tag{2.20}$$

The lowerbound is obtained when $(\theta_1, \theta_2) = (0, 1/2)$ and $(\pi_1, \pi_2) \to (3/4, 1)$. The upperbound is obtained when $(\theta_1, \theta_2) = (3/8, 1/2)$ and $(\pi_1, \pi_2) = (1, 0)$.

Note the lower bound of PoU(Attitude) and that of PoU(Bayes) are obtained at the same $\theta_i's$. However, at the worst case, the first is larger than the second. One implication follows: Consider a hypothetical scenario where players are not given distributions for the uncertainties. Suppose a system designer seek the worst case social welfare. If he assumes the use of Bayesian game mechanism, he would search for the distributions that will yield the least PoU, which will be 1/2. Now instead, if he assume the use of Attitude game mechanism, he would found the least PoU larger than 1/2. Obviously the designer would prefer attitude game mechanism.

One intuitive explanation behind this is optimism framework provides an additional degree of freedom in rational strategy set (consistent set), and thus enables players to play *more* strategically.

2.5 At least one player does not prefer pessimism

We identify conditions when pessimism cannot be dominant for both players.

The first theorem proves this for the non-symmetric Cournot duopoly game. The following theorem is for a more general utility structure of symmetric games.

Theorem 12. *Both Cournot duopoly players cannot simultaneously have pessimism as their dominant attitude.*

Now we consider a more general utility function case.

Theorem 13. *Consider a symmetric game where u_i is strictly monotonic in x_j and $r_i(x_j, \theta_i)$ is single valued and strictly monotonic in x_j and θ_i. Then pessimism cannot be a dominant attitude for any of the two players.*

2.6 Conclusions

This chapter proposes a framework to analyze two-player games with non-probabilisitc information uncertainty. The formulation allows a rational player to choose an attitude against uncertainty characterized by a degree of optimism. Corresponding to a pair of attitudes, we define an uncertainty equilibrium as a pair of sets of strategies from which rational players would not depart unilaterally. This concept coincides with the traditional Nash equilibrium when there is no uncertainty. We then define a two-phase game where players first choose their attitude. Finally, we illustrate the framework with a consumption game and a Cournot duopoly game with uncertainty. We show that the framework may identify uniquely the strategies of the players.

2.7 Proofs

This section presents the proofs of the results of this chapter.

2.7.1 Proof of Theorem 2

Since r_i is continuous in θ_i, and Θ_i is a bounded and closed interval, X_i is a closed interval. Let $X_i = [\underline{x}_i, \overline{x}_i] \subset X_{i,o}$, $\underline{x}_i \leq \overline{x}_i$. We define a map $\phi(\underline{x}_i, \overline{x}_i) = (\underline{x}_i', \overline{x}_i')$ such that

$$
\underline{x}_i' = \arg \min_{\theta_i \in \Theta_i} r_i([\underline{x}_j, \overline{x}_j], \theta_i, \pi_i)
$$
$$
\overline{x}_i' = \arg \max_{\theta_i \in \Theta_i} r_i([\underline{x}_j, \overline{x}_j], \theta_i, \pi_i)
$$

where

$$X_i \xrightarrow{\quad r_j \quad} r_j(X_i, \theta_j, \pi_j)$$

$$\theta_i \in \Theta_i \qquad \phi_i \qquad \theta_j \in \Theta_j$$

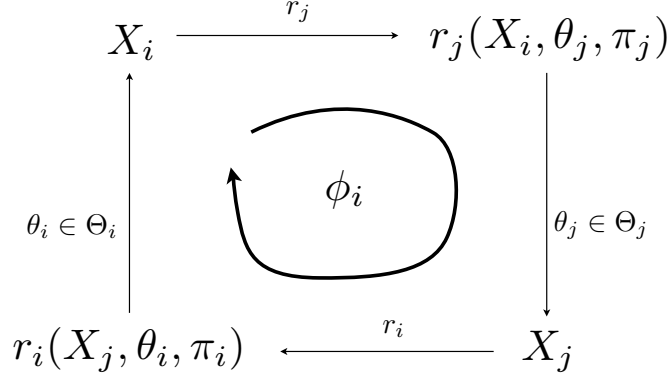$$r_i(X_j, \theta_i, \pi_i) \xleftarrow{\quad r_i \quad} X_j$$

Figure 2.1. ϕ_i mapping

$$\underline{x}_j = \arg \min_{\theta_j \in \Theta_j} r_j([\underline{x}_i, \overline{x}_i], \theta_j, \pi_j)$$

$$\overline{x}_j = \arg \max_{\theta_j \in \Theta_j} r_j([\underline{x}_i, \overline{x}_i], \theta_j, \pi_j).$$

From construction $\underline{x}_i' \leq \overline{x}_i'$. If ϕ_i is a continuous mapping, then by Brouwer's fixed point theorem, there exists $(\underline{x}_i^*, \overline{x}_i^*) \in X_{i,o}^2$ such that

$$\phi_i(\underline{x}_i^*, \overline{x}_i^*) = (\underline{x}_i^*, \overline{x}_i^*).$$

Then $X_i = [\underline{x}_i^*, \overline{x}_i^*]$ is, by definition, an uncertainty equilibrium. Now we show that ϕ_i is continuous in $\underline{x}_i, \overline{x}_i$.

Let $v := y(\underline{x}_i, \overline{x}_i) := \arg \sup_{x_i \in [\underline{x}_i, \overline{x}_i]} u_j(x_j, x_i, \theta_j)$ and define z such that $\underline{x}_i - \epsilon \leq z \leq \underline{x}_i + \epsilon$. Then $\lim_{\epsilon \to 0} u_j(x_j, z, \theta_j) = u_j(x_j, \underline{x}_i, \theta_j)$ from u_j's continuity. There are two cases: (1) $y(\underline{x}_i, \overline{x}_i) > \underline{x}_i$. Then $y(z, \overline{x}_i) = y(\underline{x}_i, \overline{x}_i)$ as for small ϵ. (2) $y(\underline{x}_i, \overline{x}_i) = \underline{x}_i$. Then $\underline{x}_i - \epsilon \leq w := y(z, \overline{x}_i) \leq \underline{x}_i + \epsilon$. As a result

$$\sup_{x_i \in [z, \overline{x}_i]} u_j(x_j, x_i, \theta_j) - \sup_{x_i \in [\underline{x}_i, \overline{x}_i]} u_j(x_j, x_i, \theta_j)$$
$$= u_j(x_j, w, \theta_j) - u_j(x_j, x_i, \theta_j) \to 0$$

as $\epsilon \to 0$ from u_j's continuity.

Therefore $\sup_{x_i \in [\underline{x}_i, \overline{x}_i]} u_j(x_j, x_i, \theta_j)$ is continuous in $\underline{x}_i$. Similarly we can show it is continuous in $\overline{x}_i$. These steps can be repeated for $\inf_{x_i \in [\underline{x}_i, \overline{x}_i]} u_j(x_j, x_i, \theta_j)$. As a result f_j and r_j are continuous in $\underline{x}_i, \overline{x}_i$. Since r_j is continuous in θ_j and Θ_j is a closed and bounded interval, $X_j := [\underline{x}_j, \overline{x}_j] := \{r_j([\underline{x}_i, \overline{x}_i], \tilde{\theta}_j, \pi_j | \tilde{\theta}_j \in \Theta_j\}$ is a closed interval too. Using the same procedure, $\underline{x}_i'$ and $\overline{x}_i'$ are continuous in $\underline{x}_j, \overline{x}_j$. Since ϕ_i is a composite function of continuous functions in $\underline{x}_i, \overline{x}_i$, ϕ_i is therefore continuous in $(\underline{x}_i, \overline{x}_i)$. This completes the proof.

2.7.2 Proof of Theorem 3

Let $\Theta_i = \{\theta_i\}$ for all i. Then for arbitrary X_j, $X_i := \{r_i(X_j, \theta_i, \pi_i) | \theta_i \in \Theta_i\}$ is a singleton. Let $X_i = \{x_i^\dagger\}$. Then

$$x_j^\dagger := r_j(X_i, \theta_j, \pi_j) = \arg \max_{x_j \in X_{j,o}} u_j(x_j, x_i^\dagger, \theta_j)$$

is j's best response function of game $\mathcal{G}_o$ when j predicts i plays $x_i^\dagger$. By assumption an equilibrium of this is a $(x_1^\star, x_2^\star)$. And by construction, it is also an uncertainty equilibrium $(X_1^*(\pi), X_2^*(\pi))$ of $\mathcal{G}(\pi)$, and it does not depend on π.

2.7.3 Proof of Theorem 4

The partial derivative with respect to x_i is $1 - \exp\{-\theta_i + x_i + x_j\}$, which is positive for $x_i < \theta_i - x_j$ and negative for $x_i > \theta_i - x_j$. Accordingly, the best response $x_i(x_j)$ is $x_i(x_j) = [\theta_i - x_j]^+$. If $\theta_i < \theta_j$, the only Nash equilibrium is then $x_i = 0, x_j = \theta_j$. The outcome of the game is very sensitive to the order of the parameters.

Assume i knows that $x_j \in X_j$. For $z \in \mathbb{R}$, define $[z]_0^1 := \min\{\max\{z, 0\}, 1\}$. Then, if i is optimistic, she maximizes $x_i - \exp\{-\theta_i + x_i + \alpha_j\}$ where $\alpha_j = \min X_j$. Thus,

$$x_i = [\theta_i - \alpha_j]_0^1 \in [[\alpha - \alpha_j]_0^1, [\beta - \alpha_j]_0^1].$$

Also, if i is pessimistic, she maximizes $x_i - \exp\{-\theta_i + x_i + \beta_j\}$ where $\beta_j = \max X_j$. Thus,

$$x_i = [\theta_i - \beta_j]_0^1 \in [[\alpha - \beta_j]_0^1, [\beta - \beta_j]_0^1].$$

Suppose both players are optimistic. Then the only uncertainty equilibrium is $X_i = X_j = [a, b]$ where $a = \alpha - a$ and $b = \beta - \alpha$. Hence $X_i = X_j = [\frac{\alpha}{2}, \beta - \frac{\alpha}{2}]$. Consequently, $x_i = \theta_i - \frac{\alpha}{2}$ and

$$U_i(1, 1) := \theta_i - \frac{\alpha}{2} - \exp\{\theta_j - \alpha\}.$$

Second, suppose both players are pessimistic. Then the only consistent sets are $X_i = X_j = [a, b]$ where $a = \alpha - b$ and $b = \beta - b$. Hence, $X_i = X_j = [\alpha - \frac{\beta}{2}, \frac{\beta}{2}]$. Consequently, $x_i = \theta_i - \frac{\beta}{2}$ and

$$U_i(0, 0) := \theta_i - \frac{\beta}{2} - \exp\{\theta_j - \beta\}.$$

Third, suppose that player 1 is optimistic and player 2 is pessimistic. In that case, the only consistent sets are $X_1 = [a_1, b_1]$ and $X_2 = [a_2, b_2]$ where $a_1 = [\alpha - a_2]_0^1, b_1 = [\beta - a_2]_0^1, a_2 = [\alpha - b_1]_0^1, b_2 = [\beta - b_1]_0^1$. Hence, $X_1 = [\alpha, \beta]$ and $X_2 = \{0\}$. Consequently, $x_1 = \theta_1$ and $x_2 = \theta_2 - \beta$, so that

$$U_1(1, 0) := \theta_1 - \exp\{\theta_2 - \beta\}.$$

By symmetry,
$$U_1(0,1) := \theta_1 - \beta - \exp\{\theta_2 - \beta\}.$$

By inspection, we see
$$U_1(1,0) \geq U_1(0,0) \text{ and } U_1(1,1) > U_1(0,1).$$

Thus, optimism is a dominant strategy for player 1. By symmetry, it is also dominant for player 2.

2.7.4 Proof of Theorem 6

The proof goes in following steps: First we define the uncertainty set as a ball. Then we show the ball's radius is constant. Finally we show the center of the ball is fixed at equilibrium. Note u_i is negatively affine in x_j. Let $X_o = [0, 1/2]$ be the strategy space. Thus $\inf X_j = \arg\sup_{x_j \in X_j} u_i(x, \theta_i)$ and $\sup X_j = \arg\inf_{x_j \in X_j} u_i(x, \theta_i)$. Define
$$h_i(X_j, \pi_i) = \pi_i \inf X_j + (1 - \pi_i) \sup X_j.$$

Then $f_i(x_i, X_j, \theta_i, \pi_i) = u_i(x_i, h_i(X_j, \pi_i), \theta_i)$. From the first order condition and definition, i's best response to X_j becomes
$$r_i(X_j, \theta_i, \pi_i) = (1 - h_i(X_j, \pi_i) - \theta_i)/2.$$

This yields
$$\begin{aligned} \sup X_i &= (1 - r_i(X_j, \pi_i) - \alpha_i)/2 \\ \inf X_i &= (1 - r_i(X_j, \pi_i) - \beta_i)/2. \end{aligned}$$

Now let $X_i^* = \mathcal{B}[s_i, t_i]$ for $i = 1, 2$ and $j \neq i$ where $\mathcal{B}[s, t]$ is a closed ball or radius t centered at s. Then
$$t_i = (\sup X_i^* - \inf X_i^*)/2 = \Delta_i/4,$$

where $\Delta_i := \beta_i - \alpha_i$. This is independent of $X_i^*, X_j^*, \theta_i, \theta_j$. Now since $\sup X_j^* = s_j + t_j$ and $\inf X_j^* = s_j - t_j$,
$$h_i(X_j^*, \pi_i) = s_j + t_j(1 - 2\pi_i).$$

Define $\sigma_i := (\alpha_i + \beta_i)/4,$. Then
$$\begin{aligned} s_i &= (\sup X_i^* + \inf X_i^*)/2 \\ &= (1 - r_i(X_j^*))/2 - \sigma_i = (1 - s_j - t_j(1 - 2\pi_i))/2 - \sigma_i \end{aligned}$$

for $i = 1, 2$. We have two equations relating s_i and s_j. By solving algebra, we get (2.10). i's best response at uncertainty equilibrium $x_i = r_i(X_j^*, \theta_i, \pi_i)$ becomes (2.11). $\mathcal{B}[s_i, t_i]$ then is uniquely determined by given $(\pi, \theta_i, \Theta_i, \Theta_j)$. To show its existence, it is sufficient to show $\mathcal{B}[s_i, t_i] \subset X_o$. To see this, it is straightforward to verify $\min s_i + t_i \leq \sup X_o$ and $\max s_i - t_i \geq \inf X_o$ for all combinations of π, Θ_1, Θ_2.

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2.7.5 Proof of Lemma 1

1. We need to find a condition, without loss of generality, such that (i) $u_1(OO) \geq u_1(PO)$ and (ii) $u_1(OP) \geq u_1(PP)$ for every $\theta_2 \in [\alpha_2, \beta_2]$. By algebra,

$$u_1(OO) - u_1(PO) \geq \Delta_2[\underline{\theta}_1 - \theta_1]/12,$$

which is non-negative for all $\theta_1 \leq \underline{\theta}_1 := \frac{1}{3}(2 - \beta_1 + 4\alpha_2 - 2\beta_2)$ and for all θ_2. (ii) is immediate because

$$u_1(OP) - u_1(PP) \geq u_1(OO) - u_1(PO).$$

2. Similar development yields $\theta_i \geq \overline{\theta}_i$ where $\overline{\theta}_i := \frac{1}{3}(2 - \alpha_i - 2\alpha_j + 4\alpha_i)$.

2.7.6 Proof of Theorem 8

1. We show at least one player always have an incentive to deviate from (PP). This part of the theorem is true even for non-symmetric Θ_1 and Θ_2. Define $u_i(\pi) := u_i(x_i^*(\pi), x_j^*(\pi), \theta_i)$ and $\Delta = \beta - \alpha$. Suppose player 1 does not have the incentive to deviate from (PP). That is, $u_1(PP) \geq u_1(OP)$. Then we prove by showing $u_2(OP) > u_2(PP)$. From the proof of Lemma 1, $u_1(PP) \geq u_1(OP)$ is equivalent to $3\theta_1 \geq 2 - 3\alpha - 2\beta + 6\theta_2$. Then, $u_2(PO) - u_2(PP) = \frac{\Delta_1}{36}(2 - 3\alpha - 2\beta + 6\theta_1 - 3\theta_2) \geq \frac{\Delta_1}{36}(6 - 9\alpha - 6\beta + 9\theta_2) > 0$. The last inequality comes from the boundary condition $0 \leq \alpha \leq \theta_i \leq \beta \leq 1/2$.

2. We show that a rival player's optimistic attitude is always detrimental: $36(u_1(PP) - u_1(PO)) = \Delta(6x_1(PP)) + \Delta(6 - 6\theta_1 - 6x_1(PP) - 6x_2(PP) - \Delta) > 0$. We can similarly show $36(u_1(OP) - u_1(OO)) > 0$. At (PP), suppose one player has incentive to change to O. That change hurts the ex post utility of the other player. This concludes (PP) is pareto efficient.

3. We need to show $u_i(PP) > u_i(OO)$. To see this, $36(u_i(PP) - u_i(OO)) = 12\Delta x_1(PP) - \Delta(6 - 6\theta_1 - 6x_1(PP) - 6x_2(PP) - 2\Delta) = \Delta(2 + 2\alpha + 2\beta - 3\theta_1 - 3\theta_2) \geq 0$.

4. If $\beta \leq \max(1/3, 2\alpha)$, then $\underline{\theta}_i \geq \beta \geq \theta_i$ for all i, and importantly, this fact becomes a common knowledge. From Lemma 1, O is the dominant strategy. Together with 1), 2) and 3), this constitutes a Prisoner's Dilemma game.

2.7.7 Proof of Theorem 9

$u_i(q^*(\pi), \theta_i)$ is non-increasing in π_j for all possible combinations of parameters. Thus u_i is minimized at $\pi_j = 1$. u_i is convex in π_i. From the first order condition, the result is immediately obtained.

2.7.8 Proof of Theorem 12

Suppose player 1's dominant attitude is pessimism. From Lemma 1, this implies

$$\beta_1 \geq \theta_1 \geq \bar{\theta}_1 = (2 - \alpha_1 + 4\beta_2 - 2\alpha_2)/3.$$

Now then,

$$
\begin{aligned}
\bar{\theta}_2 &= (2 - \alpha_2 + 4\beta_1 - 2\alpha_1)/3 \\
&\geq (14 - 10\alpha_1 - 11\alpha_2 + 7\beta_2)/9 + \beta_2 > \beta_2.
\end{aligned}
$$

Thus $\theta_2 \leq \beta_2 < \bar{\theta}_2$. Therefore pessimism cannot be player 2's dominant strategy.

2.7.9 Proof of Theorem 13

Consider player 1 representatively. We will show $U_1(OO) > U_1(PO)$ for some $\theta_2 \in \Theta_2$. Let $u := u_i$, $r := r_i$ and $\Theta := [\alpha, \beta] = \Theta_i$ for $i = 1, 2$. $\alpha < \beta$. As one case, assume u_i is strictly decreasing in x_j, r_i is decreasing in x_j and θ_i both. The conclusion is the same if any of 'decreasing' condition is changed to 'increasing' condition. Define equilibrium sets for each π as follows:

$$
\begin{aligned}
X_1 = X_2 = [a, b] \text{ for } \pi = (OO); \\
X_1 = X_2 = [c, d] \text{ for } \pi = (PP); \\
X_1 = [e, f], X_2 = [g, h] \text{ for } \pi = (OP); \\
X_1 = [g, h], X_2 = [e, f] \text{ for } \pi = (PO).
\end{aligned}
$$

Then

$$
\begin{aligned}
a = r(a, \beta) \text{ and } b = r(a, \alpha); \\
c = r(d, \beta) \text{ and } d = r(d, \alpha); \\
e = r(g, \beta) \text{ and } f = r(g, \alpha); \\
g = r(f, \beta) \text{ and } h = r(f, \alpha).
\end{aligned}
$$

From monotonicity of r, we draw relation one by one: From $a = r(a, \beta)$ and $d = r(d, \beta)$, it is immediate to see $a < d$. Noting $d = r(r(d, \alpha), \alpha)$ and $g = r(r(g, \alpha), \beta)$, we get $g < d$. Thus $d < f$ from $d = r(d, \alpha)$ and $f = r(g, \alpha)$. From $a < f$, we get $g < a$. Finally we get $a < e$. Take $\theta_2 = \beta$. Then,

$$
\begin{aligned}
U_1(OO) &= u(x_1(\theta_1, OO), x_2(\theta_2, OO), \theta_1) \\
&= u(r(a, \theta_1), r(a, \theta_2), \theta_1) \\
&= u(r(a, \theta_1), r(a, \beta), \theta_1) \\
&= u(r(a, \theta_1), a, \theta_1) \\
&> u(r(f, \theta_1), a, \theta_1) \\
&> u(r(f, \theta_1), e, \theta_1) = U_1(PO).
\end{aligned}
$$

Therefore pessimism cannot be a dominant attitude in a symmetric game.

Chapter 3

A Cooperative Game with Non-probabilistic Uncertainty

The motivation underlying this chapter is to analyze the effect of uncertainty on the design and performance of protocols. The chapter considers two types of situation. The first is when different nodes in the network have bounded knowledge about what other nodes know. The second, called common knowledge about inconsistent beliefs, is when the information is inconsistent but everyone knows it. Situations of bounded or inconsistent information arise naturally in networks because the state of these systems changes and nodes take time to learn of those changes.

The specific problem that this chapter explores is the relaying of packets in a simple butterfly network. Despite its apparent simplicity, this problem enables to illustrate key features of situations of uncertain knowledge that arise in networks. This chapter presents two impossibility facts and one possibility fact. In the latter, we introduce a scheme that enables optimal coordination given persisting imperfection in knowledge.

3.1 Introduction

This chapter studies the impact of bounded or inconsistent information on the performance of a simple relay network.

In a network, nodes typically implement distributed protocols for routing, relaying, discovery, leader election, congestion control, and other operations. Generally, the nodes have

delayed and incomplete information about the state of the network. It is therefore natural to question the impact of this incomplete information on the performance of the protocols.

A first line of inquiry considers delays and lack of synchrony among the nodes. A representative result is that a distributed Bellman-Ford protocol converges to the shortest paths if messages are eventually delivered between nodes, assuming that the network topology does not change [10]. More general results concern the convergence of parallel and distributed algorithms [9].

A second tread of investigation addresses impossibility theorems for distributed applications. An early result is the impossibility of two generals to agree with certainty when messages they exchange have some probability of not being delivered [20, 29]. Another well-known result is the Byzantine general problem where loyal generals cannot agree on whether to attack or retreat if at least one third of the generals are traitors [24, 31].

In game theory, a related formulation of the imperfection of information has received considerable attention after the publication of Rubinstein's electronic mail paper [36]. In that paper, two friends exchange lossy messages to decide whether to go out for coffee. One friend knows that the weather is bad and tries to agree with his friend that they should postpone their going out. Even after a large number of messages, they may end up not making the correct joint decisions.

This chapter examines similar situations where different nodes should coordinate their actions to prevent a bad outcome. However, because of imperfection of knowledge, the nodes may choose the wrong actions. We focus on a simple example where only one of two nodes in a network should relay a packet to prevent a collision. The difficulty is that the nodes do not know perfectly the two probabilities of success nor what the other node knows. Even after exchanging an arbitrarily large number of 'link state messages' the nodes may end up making the same decision of either relaying the packet or not. The goal is to explore protocols that avoid such pitfalls and are robust with respect to imperfect knowledge.

The first part focuses on the impact of bounded knowledge. The second part studies the situations where the nodes have inconsistent beliefs but they know it as a common knowledge.

Many other protocol design problems face similar difficulties, such as leader election, routing, and forwarding. We hope that the discussion here will increase awareness of this issue.

3.2 Problem Formulation

Consider the network shown in Figure 3.1. There are four wireless nodes: S, A, B, and D. At time 0, node S broadcasts a packet to increase the chance of delivery to D, and relay nodes A and B receive it correctly. At time 1, the nodes A and B decide to forward the packet with probability a and b, respectively. If node A forwards the packet, it gets to
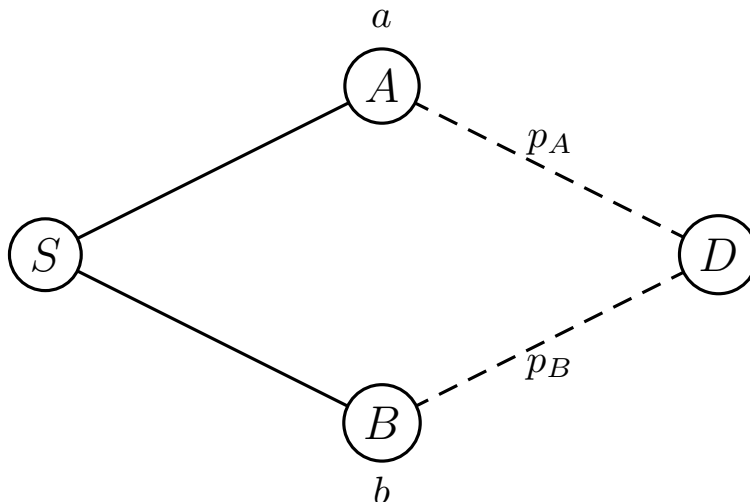
Figure 3.1. Butterfly relay network

node D with probability p_A. Otherwise, the link from A to D is in deep fade and no energy reaches node D. The situation is similar for node B, but with probability p_B instead of p_A. The assumption is that if the packet reaches D both from A and from B, then the two copies of the packet collide and D does not get the packet correctly. The question of interest is how A and B should choose the probabilities a and b to maximize the probability $\pi(a, b)$ that D gets the packet. (One can think of a more general scenario where A and B receive the packet from S with some probability or where simultaneous forwarding may not yield packet loss. It does not change the conclusions of the chapter.)

From the description of the system, one finds $\pi(a, b)$ is given by

$$\pi(a, b) = p_A a + p_B b - 2 p_A p_B a b. \tag{3.1}$$

If the nodes A and B both know $p := (p_A, p_B)$ and share that knowledge as a common fact, they can choose the values a^* and b^* such that

$$\pi(a^*, b^*) = \pi^* := \max_{a,b} \pi(a, b).$$

We call this situation *perfect knowledge*. Thus, both nodes know p and know that both know it. The knowledge is *common* and *exact*: the nodes know the precise state of the network and they both know that precise knowledge is common to both.

It is easy to verify that

$$(a^*, b^*) = \begin{cases} (1, 1), & \text{if } p \in [0, \frac{1}{2}]^2 \\ (1_{\{p_A \geq p_B\}}, 1_{\{p_A < p_B\}}), & \text{otherwise} \end{cases} \tag{3.2}$$

with the corresponding optimal performance

$$\pi^* = \begin{cases} p_A + p_B - 2 p_A p_B, & \text{if } p \in [0, \frac{1}{2}]^2 \\ \max\{p_A, p_B\}, & \text{otherwise.} \end{cases} \tag{3.3}$$

Roughly speaking, if none of links AD and BD is good (p_A and p_B are small), both nodes should relay. Otherwise, only the node with the best link should relay.

However, the success probabilities p of the links change over time and the nodes can observe only their local link directly. Thus, in practice, the nodes never have a perfect knowledge. One practical approach is for the nodes to exchange 'link state' messages to improve their knowledge. A key aspect of the formulation is to model precisely the knowledge of the nodes A and B and to understand how this knowledge affects their decisions and the resulting performance measures of the network.

The nodes A and B communicate somehow to increase their knowledge about the network. Their communication path is not explicitly shown in the figure. The nodes exchange lossy messages and we examine what they know after n messages. We call that knowledge 'Level-n knowledge.'

Initially, before they exchange messages, we assume that A knows p_A and B knows p_B. This is Level-0 knowledge. Now synchronously each node sends a message to the other. Node A sends a message to B saying 'I know p_A.' At the same time, B sends a message to A saying 'I know p_B.' (The synchronous assumption is relaxed later.) When it gets that message, B knows p_B, and that A knows p_A. However, B is not sure that A knows that B knows p_A. Similarly, A knows p_A, and that B knows p_B but A is not sure that B knows that A knows p_B. This is level-1 knowledge. After the next exchange of messages, the nodes have Level-2 knowledge, and so on. Note that Level-n knowledge is defined when the nodes receive the n messages, even though the nodes assume that these messages can get lost.

These levels of knowledge can be formalized as follows. Let the notation $K_A(0)$ mean 'A knows p_A.' Similarly, $K_B(0)$ means 'B knows p_B.' Let then $K_A(1)$ mean 'A knows p_A and $K_B(0)$.' That is, $K_A(1)$ means 'A knows p_A and that B knows p_B.' Inductively, define $K_A(n+1)$ to mean 'A knows p_A and $K_B(n)$' and similarly $K_B(n+1)$ to mean 'B knows p_B and $K_A(n)$' for $n \geq 0$. The interpretation is that the $(n+1)^{th}$ message from A to B carries $K_A(n)$, so that upon receiving it node B knows p_B and $K_A(n)$, which is $K_B(n+1)$. The situation is similar with A and B interchanged.

Of course, the nth message from A may get lost, in which case the knowledge of B remains what it was previously. The discussion on this case is postponed to Section 3.4

One expects that, as they exchange more and more messages, the nodes' knowledge approaches perfect knowledge. However, it turns out that the values of the relaying probabilities a_n and b_n that the nodes choose with Level-n knowledge may result in a probability of success $\pi(a_n, b_n)$ that does not approach π^*.

3.3   Analysis

To study the impact of imperfect knowledge on node decisions, we explore the strategies of the two nodes A and B under different levels of knowledge.

$$K_A(1) : A \text{ knows} \begin{cases} p_A \\ B \text{ knows} \quad p_B \end{cases}$$

(a)

$$K_B(1) : B \text{ knows} \begin{cases} p_B \\ A \text{ knows} \quad p_A \end{cases}$$

$$K_A(n+1) : A \text{ knows} \begin{cases} p_A \\ K_B(n) \end{cases}$$

(b)

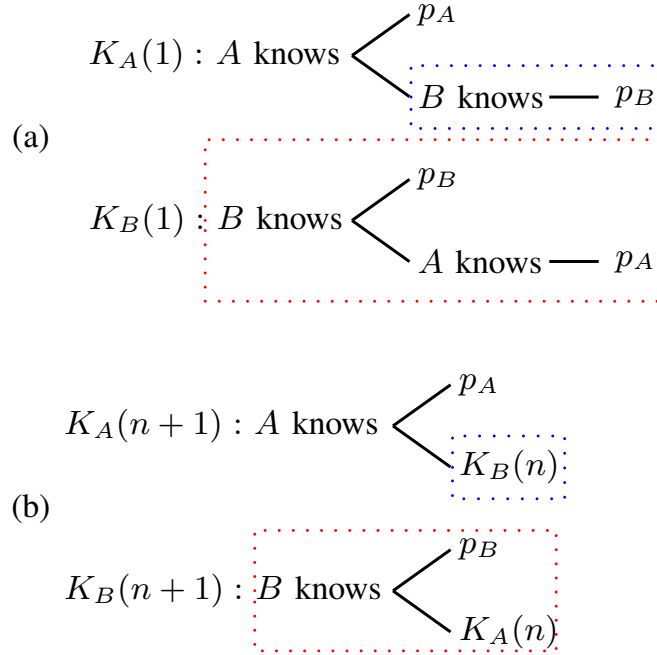$$K_B(n+1) : B \text{ knows} \begin{cases} p_B \\ K_A(n) \end{cases}$$

Figure 3.2. (a) Level-1 (b) Level-$n+1$ knowledge structure. Upper part for A, lower part for B. Note two dotted boxes contain disparate knowledge for B.

3.3.1   Level-0

Consider first the case of Level-0 knowledge where node A knows p_A and node B knows p_B but not more than that. Since node A does not know anything about p_B and what B knows, it is sensible for that node to choose a value of its relaying probability a that guarantees a good probability of success, no matter what p_B is and what the choice of node B is. That is, node A chooses the reliable value a_0 of a given by

$$a_0 = \arg\max_a \min_{b, p_B} \pi(a, b).$$

Similarly, node B chooses the value b_0 of b given by

$$b_0 = \arg\max_b \min_{a, p_A} \pi(a, b).$$

From (3.1), one finds that

$$\pi(a, b) = p_A a + p_B b(1 - 2p_A a),$$

so that

$$\min_{b, p_B} \pi(a, b) = \begin{cases} 1 - p_A a, & \text{if } 1 \le 2p_A a \\ p_A a, & \text{otherwise.} \end{cases}$$

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Consequently, the maximizing value a_0 of $\min_{b,p_B} \pi(a,b)$ is given by

$$a_0 = \min\{\frac{1}{2p_A}, 1\}. \tag{3.4}$$

Similarly,

$$b_0 = \min\{\frac{1}{2p_B}, 1\}.$$

The resulting probability of success is

$$
\begin{aligned}
\pi_0 \quad := \quad & \pi(a_0, b_0) = p_A a_0 + p_B b_0 - 2 p_A p_B a_0 b_0 \\
= \quad & \min\{\frac{1}{2}, p_A\} + \min\{\frac{1}{2}, p_B\} \\
& -2 \min\{\frac{1}{2}, p_A\} \min\{\frac{1}{2}, p_B\}.
\end{aligned}
$$

We find that

$$\frac{\pi_0}{\pi^*} = \begin{cases} 1, & \text{if } p \in [0, \frac{1}{2}]^2 \\ \frac{1}{2\max(p_A, p_B)}, & \text{otherwise.} \end{cases} \tag{3.5}$$

Note that $\frac{1}{2}\pi^* \leq \pi_0 \leq \pi^*$. Therefore the network is guaranteed not to lose more than half of the performance when relays have Level-0 knowledge.

3.3.2   Level-1

After exchanging the first messages, the nodes reach Level-1 knowledge $K_A(1)$ and $K_B(1)$. That is, A has learned that B knows $K_B(0)$ and, consequently, that B will base the choice of b on $K_B(0)$. That is A considers that B will choose $b = b_0$. Accordingly, A chooses the value $a = a_1$ such that

$$a_1 = \arg\max_a \pi(a, b_0).$$

Since $b_0 = \min\{\frac{1}{2p_B}, 1\}$, one finds

$$
\begin{aligned}
\pi(a, b_0) \quad = \quad & p_A a + \min\{\frac{1}{2}, p_B\} - p_A a \min\{1, 2p_B\} \\
= \quad & p_A a (1 - \min\{1, 2p_B\}) + \min\{\frac{1}{2}, p_B\}.
\end{aligned}
$$

Consequently the maximizing value a_1 is given by

$$a_1 = \begin{cases} 1, & \text{if } p_B \leq \frac{1}{2} \\ \text{any in } [0,1], & \text{otherwise.} \end{cases} \tag{3.6}$$

Similarly,

$$b_1 = \begin{cases} 1, & \text{if } p_A \leq \frac{1}{2} \\ \text{any in } [0,1], & \text{otherwise.} \end{cases}$$

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For certain set of p, multiple choices are possible for a_1 and/or b_1, which in turn correspond to different values of the probability of success $\pi(a_1, b_1)$. The worst case is a possible cost of the lack of knowledge. Define $\pi_1 := \min \pi(a_1, b_1)$. One finds

$$
\pi_1 = \begin{cases} 0, & \text{for } p \in (\frac{1}{2}, 1]^2 \\ p_A + p_B - 2p_A p_B, & \text{otherwise.} \end{cases}
$$

Consequently,

$$
\frac{\pi_1}{\pi^*} = \begin{cases} 1, & \text{for } p \in [0, \frac{1}{2}]^2 \\ 0, & \text{for } p \in (\frac{1}{2}, 1]^2 \\ \frac{p_A + p_B - 2p_A p_B}{\max(p_A, p_B)}, & \text{otherwise,} \end{cases} \tag{3.7}
$$

which shows that the imperfect knowledge can reduce the probability of success to zero.

3.3.3   Level-n and Failure of Convergence

After exchanging $(n+1)^{th}$ messages, and reaching Level-$(n+1)$ knowledge, node A chooses a_{n+1} as the best response to its belief about the node B's decision, and similarly for B. That is,

$$
a_{n+1} = \arg\max_a \min_{b_n} \pi(a, b_n)
$$
$$
b_{n+1} = \arg\max_b \min_{a_n} \pi(a_n, b).
$$

It is easy to verify that $a_{2k} = a_0$, $b_{2k} = b_0$ and $a_{2k+1} = a_1$, $b_{2k+1} = b_1$ for all $k \in \mathbb{Z}^+$. Since $a_0 \neq a_1$ and $b_0 \neq b_1$, one sees that the solution does not converge as the level of knowledge increases. In general, for $(p_A, p_B) \notin [0, \frac{1}{2}]^2$

$$
\limsup_{n \to \infty} \pi_n \leq \pi_o < \pi^*. \tag{3.8}
$$

Thus, robust optimization against uncertainties never leads to the optimal performance regardless of the number of messages exchanged.

The failure to converge is due to an excess of caution. At step n, the relays try to maximize the worst-case success probability over the possible choices of the other relay, based on what the other relay night know at that time. Even if the nodes exchange n messages successfully, the possibility that a message gets lost suffices to prevent the nodes from making optimal decisions. Figure 3.3 summarizes the results of this section.

More generally, the node may have only imprecise Level-0 knowledge. For example, node A knows p belongs to a set Z, or $K_A(0) = \{p \in Z\}$, $Z \subset [0, 1]^2$. The imprecise knowledge situation is widespread because of imperfection of observation or estimation on state of the nature. As a result, the nodes know only a rough range containing the true state.
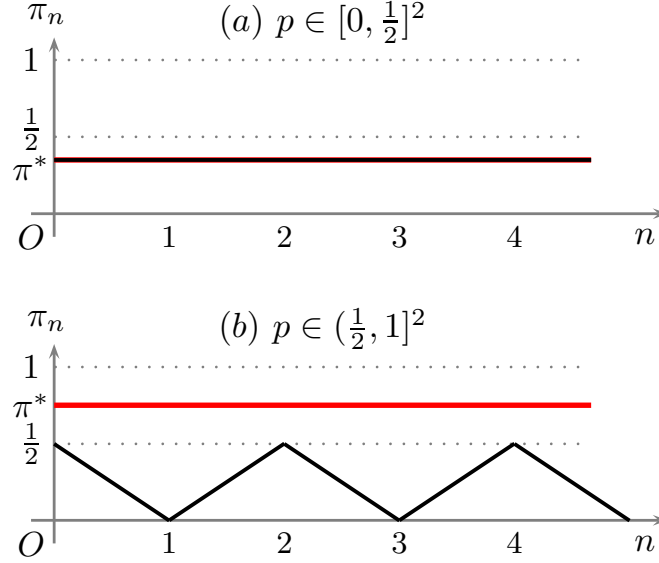
Figure 3.3. Evolution of the network performance (a) $p \in [0, \frac{1}{2}]^2$ (b) $p \in (\frac{1}{2}, 1]^2$

Lemma 2. *Define $Z := [p'_A, p''_A] \times [p'_B, p''_B]$. The reliable solution for a at Level 0 is*

$$
a_0(Z) = \begin{cases}
1, & \text{for } p''_B \leq \frac{1}{2} \\
1, & \text{for } p''_B(2p''_A - 1) \leq (p''_A - p'_A) \\
\frac{p''_B}{p'_A + p''_A(2p''_B - 1)}, & \text{otherwise.}
\end{cases}
$$

See Appendix for proof. An interesting special case is when A has no clue about the exact link states. That is, $Z = [0, 1]^2$. The result suggests $a_0(Z) = 1$, or to forward always. This general solution does not change the result of failure of convergence.

3.4   Knowledge with Message Loss

In the previous section, we implicitly assumed the nodes A and B synchronously increase their knowledge level despite the possibility that messages might get lost. Figure 3.4 illustrates a more realistic scenario: At some time t_0, nodes A and B have knowledge $K_A(n)$ and $K_B(n)$ respectively. At time t_0 both nodes send messages to each other. Node A's message gets lost and node B does not receive it, whereas node B's message reaches node A. At time t_1, node A reaches an additional level of knowledge while node B's knowledge does not change. At the next time step, node B's knowledge level jumps by 2, to level $n + 2$, whereas node A's knowledge does not change and remains at level $n + 1$, and so on. Note
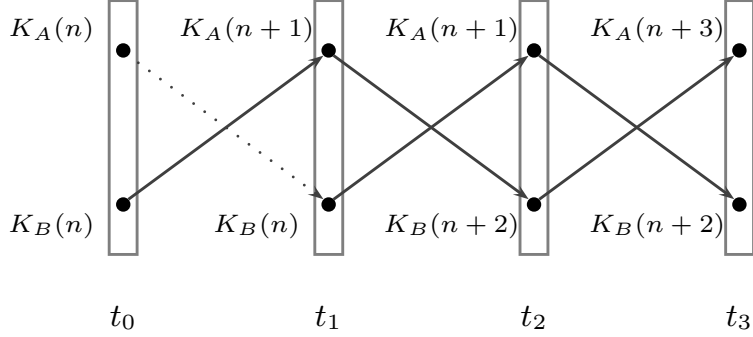
Figure 3.4. Assumption of synchronous message exchange is relaxed: Knowledge evolution when a message from A to B is lost at time t_0.

that the nodes' knowledge level may lose synchronization when a message is lost. However, their knowledge level gap is never more than one because one node's next knowledge level depends on the other node's current knowledge level.

Thus, at any given time t, the network performance can be either

$$\pi(a_{n+1}, b_n) \text{ or } \pi(a_n, b_n) \text{ or } \pi(a_n, b_{n+1}).$$

Consequently, in addition to the lossless case, it suffices to consider $\pi(a_1, b_0)$ and $\pi(a_0, b_1)$. From (3.3.1) and (3.6), one finds

$$\pi(a_1, b_0) = \begin{cases} p_A + p_B - 2p_A p_B, & \text{if } p_B \leq \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

By symmetry, $\pi(a_0, b_1)$ is similarly found.

This completes the claim that the relays cannot reach optimal coordination via lossy message exchange no matter how high a knowledge level they obtain.

We conclude this section with a summarizing fact.

Fact 1. *A distributed system cannot reach the optimal coordination by building higher level of bounded knowledge via lossy message exchange.*

3.5 Achieving Optimal Coordination

The previous section shows that the lack of certainty in message delivery prevents the nodes from coordinating their actions optimally. This failure of optimal coordination persists regardless of the level of knowledge. This observation is similar to the conclusions of

the electronic mail game [36] which led many researchers to study the continuity of belief structure. Hopefully, the model of this paper shows the relevance of such considerations to network protocols.

The practical question is to how find a mechanism that achieves the optimal coordination based on local knowledge $K_A(n)$ or $K_B(n)$ among distributed agents. The preceding analysis shows that the set of parameters of the system determines the evolution of the performance as n increases. For instance, the analysis shows that if $p := (p_A, p_B) \in [0, \frac{1}{2}]^2$, then $a_n = b_n = 1$ is optimal for $n \geq 0$. Also, once node A knows $K_A(n)$ for some $n \geq 1$, it knows that $\pi = \pi^*$ regardless of the level of knowledge that node B has reached. The same is true for node B. That is, if $p \in [0, \frac{1}{2}]^2$, the nodes know that Level-1 knowledge suffices to enable optimal decisions.

However, the solution does not converge for $p \notin [0, \frac{1}{2}]^2$. In that case, the nodes know that basing their decisions on Level-n knowledge does not lead to optimal coordination. With this observation, they can choose to deviate from the myopic max-min strategy and follow instead the following mechanism:

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Upon receiving message $K_B(0)$ from B, A updates its knowledge and obtains $K_A(1)$. However node A does not send $K_A(1)$ to node B. Instead, node A keeps sending $K_A(0)$ to node B. Similarly, node B sends $K_B(0)$ to node A, even though node B actually knows $K_B(1)$.

Once node A obtains $K_A(1)$, it learns if $p \in [0, \frac{1}{2}]^2$. In that case, the relaying solution based on $K_A(1)$ is known to be optimal. If $p \notin [0, \frac{1}{2}]^2$, node A knows that gaining a higher level of knowledge results in oscillations that can in turn yield a zero probability of success. Accordingly, node A does not follow the myopic max-min algorithm based on an additional level of knowledge. Instead, it assumes that $K_A(1)$ is the global information. At that time, although node B still has knowledge $K_B(0)$, the network performance is guaranteed to be at least half the optimal level, because of the Level-0 result. If node B has knowledge $K_B(1)$, then both nodes A and B agree to the optimal coordination. Since the message is lossy, node A keeps sending $K_A(0)$, so that node B eventually reaches knowledge $K_B(1)$ with probability one.

This is a strategy combining pessimism and optimism with restriction on knowledge propagation. At level 0 knowledge, one plays a robust strategy (pessimism). At level 1, one keep sending level 0 knowledge to each other, although she has built level 1 knowledge, and *pretends* to have common knowledge. That is, the relays throw caution to the wind by restricting knowledge propagation (KR) and hope for the best.

Let C_n be the event that two relays reach the consensus about the complete network parameters within nth round of message exchange. The probability $P(C_n)$ of the event C_n

is given by
$$P(C_n) = 1 - (1 - (1-\epsilon)^2)^n \approx 1 - (2\epsilon)^n,$$
where ϵ is the small probability that a message may get lost between two relays. This probability approaches to 1 at exponential rate. Also, using this modified protocol, the two relays choose the probabilities a^* and b^* when the event C occurs. Let $\pi_{KR}(n)$ be the performance of the protocol after the relays have sent n messages. We find that
$$\pi^* P(C_n) \leq \pi_{KR}(n) \leq \pi^*$$
and $\pi^* P(C_n) \to \pi^*$, as $n \to \infty$. Therefore
$$\lim_{n \to \infty} \pi_{KR}(n) = \pi^*. \tag{3.9}$$

It is worth mentioning some differences with the Electronic Mail Game result where no finite sequence of message exchanges can result in optimal coordination. First, in the current problem, the payoff is defined as the max-min performance rather than the von Neumann-Morgenstern form. Second, there is no negative payoff biasing the players' decision. Third, the different message exchange protocol is pivotal because it does not assume an automatic acknowledgment that is one of the main causes making the coordination impossible as pointed out in [11].

Fact 2. *A distributed system with lossy message exchange can asymptotically reach the optimal coordination by restricting the information propagation.*

3.6 Throughput

We analyze the throughput performance of 2-stage protocols and 3 stage protocols such as repetition coding and time division multiplexing (TDM) that we describe next. The throughput is defined as average rate of successful deliveries per unit time.

Time cost plays a critical role. If learning takes negligible time or no further learning is needed after initial learning, then throughput of the scheme with learning may outperform that of oblivious one. However if the scheme requires continuous learning at non-negligible time cost, then we will see some oblivious scheme in fact outperforms the one with learning.

The first 3-stage protocol is repetition coding. In this protocol, the relays transmit twice with the probabilities a_0 and b_0, respectively. The throughput of this protocol is
$$\pi_{REP} = 1 - (1 - \pi_0)^2 = 2\pi_0 - \pi_0^2.$$
Since $\pi_0 \leq \frac{1}{2}$, $\pi_{REP} \leq \frac{3}{4}$. Considering the fact that KR may reach $\pi_{KR} = 1$ depending on the value of p, the repetition coding may not be close to optimal. Note that the throughput

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of repetition coding is always better than or equal to the throughput of the 2-stage oblivious protocol. That is, define $T_0 = \pi_0/2$ and $T_{REP} = \pi_{REP}/3$. Since $6(T_{REP} - T_0) = 4\pi_0 - 2\pi_0^2 - 3\pi_0 = \pi_0(1 - 2\pi_0) \geq 0$,

$$T_0 \leq T_{REP}. \tag{3.10}$$

The second 3-stage protocol utilizes time division multiplexing. In this protocol, the relays take turns to forward. Since there is no collision, both relays forward with probability 1. The corresponding probability of success is

$$\begin{aligned} \pi_{TDM} &= 1 - (1 - p_A)(1 - p_B) \\ &= p_A + p_B - p_A p_B. \end{aligned}$$

Now let us consider when learning time cost is negligible. Indeed, when $p \in [0, \frac{1}{2}]^2$, $\pi^* = \pi_0$ and thus $T^* = T_0$. Therefore

$$T^* \leq \max(T_{REP}, T_{TDM}), \text{ when } p \in [0, 1/2]^2,$$

otherwise T^* can be greater than $\max(T_{REP}, T_{TDM})$.

Let us consider when learning time cost is non-negligible. Suppose learning is required per source packet. For convenience, assume that knowledge exchange takes one time unit until the relays reach the optimal decisions. That is, assume $\pi_{KR} = \pi^*$ at the cost of time unit.

Under this assumption, since both KR and TDM are 3-stage protocols, their throughput is proportional to the probability of successful delivery and one finds

$$\pi_{KR} \leq \pi^* \leq \pi_{TDM},$$

The left inequality is immediate. For the right inequality, we consider two cases. If $p \in [0, \frac{1}{2}]^2$, $\pi^* = p_A + p_B - 2p_A p_B = \pi_{TDM} - p_A p_B \leq \pi_{TDM}$. Otherwise, $\pi^* = \max(p_A, p_B) \leq \max(p_A + p_B(1 - p_A), p_B) \leq \max(p_A + p_B(1 - p_A), p_B + p_A(1 - p_B)) = p_A + p_B - p_A p_B = \pi_{TDM}$. Thus, even with an idealized learning time cost (one unit time) and performance ($\pi_{KR} = \pi^*$), an oblivious scheme that requires no learning outperforms the ideal scheme requiring learning.

$$T_{KR} \leq T_{TDM}.$$

3.7   Common Knowledge about Inconsistent Beliefs

Information is not knowledge but belief when it does not guarantee the inclusion of the true state. Suppose node i has a belief about p's possible values: $B_i := \{p \in Z_i\}$. A different node may have a different belief. One key observation is however, i does think B_i as a knowledge rather than a belief since otherwise it would modify Z_i to make it include

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broader values. By exchanging B_i, distributed nodes can build common knowledge about beliefs. Unless Z_i and Z_j conflict, they cannot distinguish knowledge from belief. We say they reach the common knowledge state about consistent beliefs, or simply a common belief. When they discover Z_i and Z_j conflict, we say they reach the common knowledge state about inconsistent beliefs. It is of a practical challenge to make a strategic decision in a coordination game while players have common knowledge about inconsistent beliefs. [4] explained that the distributed players with the same prior cannot agree to disagree. We study the game where the prior is not defined.

This situation frequently occurs in many practical games. For an example, consider a double tennis match game in which two players see each other and need to decide who returns a ball. Due to different experience, two may have inconsistent views on the game. Further they know it as a common knowledge. It is likely that they try to hit or leave the ball simultaneously, failing to coordinate.

Similarly, due to imperfection or randomness of observation, two relays in relaying network may obtain different beliefs about the network state. After information exchange step, they build common knowledge about inconsistent beliefs. $K_A = \{p \in Z_A, K_B\}$ and $K_B = \{p \in Z_B, K_A\}$.

Upon facing inconsistency, an issue about trust arises - trust about information but not about the intent of the information source. A trust is a meta information of the coordination game how the distributed information should be interpreted. It is seldom explicitly stated in the game description. When the way of trust is specified however, it helps the nodes to reach the conciliation, if required, from inconsistency.

In a coordination game, it is obvious that the nodes with common belief cannot outperform those with common knowledge. However, it is not clear if consistent belief will be always better than inconsistent belief.

3.7.1   Distrust

It is called distrust (in others) when the node trusts only its belief. Then nodes will fail to reach the coordination. Suppose $Z_A = \{(0.9, 0.2)\}$ and $Z_B = \{(0.2, 0.9)\}$. Under distrust, knowing B's best response, A's best response is $a = 1$. Similarly B's best response is $b = 1$. As a result, both know $(a, b) = (1, 1)$ will be played. Note that the true network state p can be neither of Z_A nor Z_B, but can be something else. Interestingly, it is not the case the incoordination from distrust always underperforms; depending on the true network state p, the failure of coordination may prove to be good.

An example may suffice to convince readers. Let $p_{true} = (0.4, 0.4)$, $Z = Z_A = \{(0.9, 0.2)\}$ and $Z_B = \{(0.2, 0.9)\}$. The coordination solution is $(a, b) = (1, 0)$ based on Z, whose choice is to be explained shortly. Then the coordination performance based on Z is $\pi(a^*(Z), b^*(Z)) = 0.4$ while the incoordination performance is $\pi(a^*(Z_A), b^*(Z_B)) = 0.48$.

If the game is such that the price of the failure of coordination is significant however,

players may elect to rely on an external conciliation rule. This rule should be performance ignorant; since they cannot agree on the range of true network state, there is no common measure to compare the performance of one rule to other. A simplest way is to adopt a single belief from the node whose lexicographical order is the highest. Then each node's decision will be based on that single belief.

3.7.2 Partial Trust

In some situation the nature of the game suggests that a node give up some trust in its initial belief and take some beliefs from others. The partial trust may arise in various forms. We discuss a few cases: Locality trust, Meet type trust and Join type trust.

Here the trust is a way of constructing a new common belief from the distributed and inconsistent information. The choice of a trust form should be mandated across the players at the time of the game design, if any conciliation is to be needed.

Define z_i^k to be the set of possible values for p_k that node i initially believes. z_i^i is a belief about its local link and $z_i^k, k \neq i$ about its foreign link. Then

$$Z_i := \prod_k z_i^k.$$

Locality trust

It is possible that link i can be best known to node i. That is, each node has trust in everyone's local link belief but not foreign link. In this case, the node with common knowledge about inconsistent beliefs are willing to agree on a common belief that is constructed with most trusting elements.

$$Z_{Local} := \prod_i z_i^i.$$

Meet type trust

In Meet type trust, the node accepts other's information too and construct a new common belief as the smallest set containing all beliefs.

$$Z_{Meet} := \prod_k \bigcup_i z_i^k.$$

Join type trust

In Join type trust, the node constructs a new common belief as the intersection of all beliefs.

$$Z_{Join} := \prod_k \bigcap_i z_i^k.$$

Note that Z_{Join} can be degenerate when $\bigcap_i z_i^k = \emptyset$ for some k.

Let's restrict ourselves to the case where the result of construction indeed constitutes a knowledge. Let $Z_A = [0.2, 0.9] \times \{0.2\}$ and $Z_B = \{0.2\} \times [0.2, 0.9]$ with $p_{true} = (0.4, 0.4)$. Then $Z_{Local} = [0.2, 0.9]^2 = Z_{Meet}$. A simple exercise shows that the distrust solution is $(a, b) = (1, 1)$ and a coordinated solution with local/meet type conciliation is $(a, b) = (1, 0)$. At p_{true}, the distrust solution outperforms coordinated solution after conciliation.

We conclude this section with the a summarizing fact.

Fact 3. *Neither belief consistency nor a common knowledge about beliefs is sufficient to achieve an optimal coordination in a distributed system.*

3.8 Conclusion

The distributed system with a common goal often faces the issue of independent decision with limited knowledge where the price of coordination failure can be significant. In this chapter we focused to understand the impact of information uncertainties. In particular we studied bounded knowledge about other's knowledge and common knowledge about inconsistent beliefs.

To make the problem down-to-earth, we adopted a simple butterfly relaying network which has been a popular platform in communication networking area. In the first problem, we showed that two relay nodes with bounded level of knowledge about other's knowledge cannot reach the state of global coordination regardless the depth of the level. However, we also provided a scheme in which the network can asymptotically achieve the optimal coordination via the intentional restriction of knowledge propagation. Finally we showed that belief consistency or the state of common knowledge about beliefs in general is not a sufficient condition for optimal coordination.

3.9 Proofs

3.9.1 Proof of Lemma 2

$$a_0 = \arg\max_a \min_{b,p \in Z} \pi(a,b).$$

with $Z := [p'_A, p''_A] \times [p'_B, p''_B]$. A key observation is that inside minimization problem, nature and the node B jointly choose either $p_B b = 0$ or $p_B b = p''_B$. Then for a fixed a,

$$\min_{b,p \in Z} \pi(a,b) = \min_{p_A \in [p'_A, p''_A]} (p_A a, p_A a(1 - 2p''_B) + p''_B).$$

When drawn with respect to a, the result of minimization is the lower boundary of the area spanned by two linear curves $p_A a$ and $p_A a(1 - 2p''_B) + p''_B$ for all feasible values of p_A and p_B. a_0 is found where this lower boundary is maximized.

When $1 - 2p''_B \geq 0$, both curves are increasing in a. Thus $a_0 = 1$. When $1 - 2p''_B < 0$, the lower boundary is decided by two curves $p'_A a$ and $p''_A a(1 - 2p''_B) + p''_B$. If the former is not greater than the latter for $a \in [0,1]$, then the lower boundary is maximized at $a_0 = 1$. Otherwise, their intersection is the maximum. Thus $a_0 = \frac{p''_B}{p'_A + p''_A(2p''_B - 1)}$.

Another simpler case is when $K_A(0) = \{p_A \in [p'_A, p''_A]\}$.

$$a_0 = \begin{cases} 1, & \text{for } p'_A \leq 1 - p''_A \\ \frac{1}{p'_A + p''_A}, & \text{otherwise.} \end{cases}$$

One way to view this result is to think p'_A as the base state and $p''_A - p'_A \geq 0$ as its maximum departure. One can see that a_0 with the imprecise knowledge is always less than a_0 with the precise knowledge where $p''_A = p'_A$. In this view, the node should forward more cautiously when its knowledge is less precise. (The interpretation is the other way when p''_A is regarded as the base state.) Irrespectively, the guaranteed performance is always lower in imprecise knowledge.

When this knowledge is built up in a higher level, the solution and the performance do not converge in general.
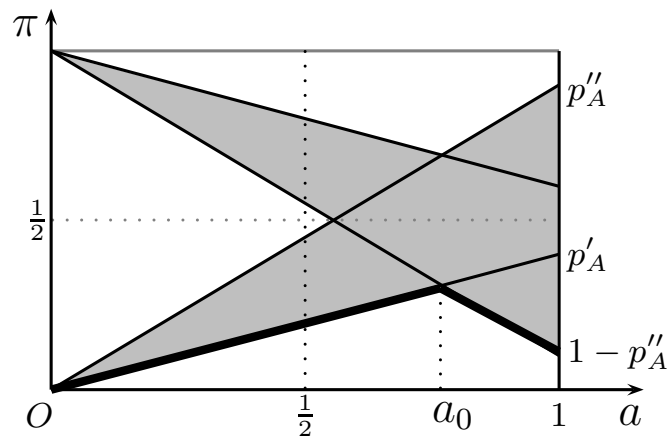
Figure 3.5. Feasible region of π is shaded when $1 - p''_A < p'_A < \frac{1}{2}$, with respect to a. Minimum is the lower boundary. Its maximum is obtained at a_0

Chapter 4

Conclusion and Future Work

4.1 Conclusion

Game theory with incomplete information usually assumes preplay knowledge about uncertainty in the form of a probability distribution of unknown parameters. This dissertation explores games where no such distribution is assumed, only the knowledge of the support of the unknown parameters.

This dissertation studies one-shot two-agent non-cooperative and cooperative games. For the non-cooperative games, we define consistent sets as a product space of rational strategies under uncertainty, and then introduce the optimism - pessimism attitude as an additional degree of strategy for the players. Corresponding to given attitudes, we define consistent sets of strategies from which rational players should not depart. We then consider a two-stage game where the players first strategically choose their attitude to maximize the ex-post utilities they receive after the second stage where they play the game with known attitudes. This formulation sometimes results in a specific strategy such as being optimistic or pessimistic. In such cases, the agents may have a uniquely specified strategy despite the uncertainty.

Next, we study a cooperative game where relay nodes collaborate to maximize the probability of successful delivery of a packet in a wireless network. For this model, the nodes exchange error prone link state messages to inform each other of their link characteristics. We show that, in this model, the nodes should not be overly cautious in trying to protect the throughput against the failure of delivery of a link state message. If they are, the throughput does not converge to a high value as the number of link state messages increases. We also show that a more optimistic protocol that does not consider the worst case behavior of the other node has a throughput that converges to the maximum possible value.

4.2   Future Work

In this thesis, we considered that agents choose their attitude in the face of uncertainty. We parametrized this attitude by a degree of optimism and then considered a two-stage game. Other schemes are conceivable for describing a player's attitude. Desirable schemes should be able to describe exhaustively the consistent sets. Its rational equilibrium should exist under a wide class of utility structures. Also, there should be a close connection between that equilibrium and the Nash equilibrium under full information, when the uncertainties decrease.

The relationship between Nash equilibria under full information and the uncertainty equilibria of attitudes deserves some further exploration. We imagine that, as the uncertainty grows, multiple possible distinct Nash equilibria may merge into a single uncertainty equilibrium. Shelten's trembling hand selection theory avoided this puzzle by considering only very small uncertainties.

This dissertation provides a novel computation methodology for two-agent problems, but has not explored more general n-agent problems. The expansion of the methodology should be studied, with careful understanding about existence, convergence, uniqueness and beyond.

Finally, the validation of this methodology through behavioral game theory should be studied.

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