Rigging Tournament Brackets for Weaker Players

Isabelle Stanton
Virginia Vassilevska Williams

Electrical Engineering and Computer Sciences
University of California at Berkeley

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Isabelle Stanton
Computer Science Department
University of California, Berkeley
isabelle@eecs.berkeley.edu

Virginia Vassilevska Williams
Computer Science Department
University of California, Berkeley
virgi@eecs.berkeley.edu

Abstract

Consider the following problem in game manipulation. A tournament designer who has full knowledge of the match outcomes between any possible pair of players would like to create a bracket for a balanced single-elimination tournament so that their favorite player will win. Although this problem has been studied in the areas of voting and tournament manipulation, it is still unknown whether it can be solved in polynomial time. We focus on identifying several general cases for which the tournament can always be rigged efficiently so that the given player wins. We give constructive proofs that, under some natural assumptions, if a player is ranked among the top K players, then one can efficiently rig the tournament for the given player, even when K is as large as 19% of the players.

1 Introduction

As a natural way to select a leader, competition is at the heart of life. It is intriguing, both for its participants, and its spectators. Society is riddled with organized competitions called *tournaments* with well-defined rules to select a winner from a pool of candidate players. Sports tournaments such as the FIFA World Cup and Wimbledon are immensely popular and generate huge amounts of revenue. Elections are another important type of tournaments: a leading party is selected according to some rules using votes from the population.

Two of the most common tournament formats employed in both sports and voting are *round-robin* and *single-elimination*. In the former, every pair of players are matched up, and a player's score is how many matches they won. If some player has beaten everyone else, then they are the clear (Condorcet) winner. Otherwise, the winner is not well-defined. However, given the outcomes of a round-robin tournament, there are various methods of producing rankings of the players. The most common definition of the optimal ranking is that it minimizes the number of wins of a lower-ranked player over a higher-ranked player [18]. Although finding such a ranking for a round-robin tournament is NP-hard [1], sorting the players according to their number of wins is a good approximation to the optimum ranking [7].

Single-elimination (SE) tournaments are played as follows. First, a permutation of the players, called the *bracket* or *schedule* is given. According to the bracket, the first two players are matched up, then the second pair of players etc. The winners of the matches move on to the next round. The bracket for this round is obtained by pairing up the remaining players according to the original bracket. If the number of players is a power of 2, the tournament is balanced. Otherwise, it is unbalanced and some players advance to the next round without playing a match. In practice, these *byes* are usually granted in the first round. Although the winner of an SE tournament is always well-defined, the chance of a particular player winning the tournament can vary immensely depending on the bracket. Arguably, this gives the tournament organizer a lot of power. The study of how much control an organizer has over the outcome of a tournament is called *agenda control* [3].

The most studied agenda control problem for balanced SE tournaments is to find a bracket which maximizes the probability that a given player will win the tournament. The tournament organizer is given the probability that i will beat j for every pair of players i, j. A major focus is to maximize the winning probability of the *strongest* player under some assumptions[1] (e.g., [2, 12, 23, 22]). Without assumptions on the probabilities, the agenda control problem for an

[1] A common assumption is monotonicity: the probability of beating a weaker player is at least as high as that of beating a stronger one.

arbitrary given player is NP-hard [14, 11], even when the probabilities are in $\{0, 1, 1/2\}$ [21]. Moreover, the maximum probability that a given player wins cannot be approximated within any constant factor unless P=NP [21]. When the probabilities are all either 0 or 1, the agenda control problem, then called the tournament fixing problem (TFP), is not well understood. One of the interesting open problems in computational social choice is whether a tournament fixing bracket can be efficiently found. Several variants of the problem are NP-hard – when some pairs of players cannot be matched [20], when some players must appear in given rounds [21], or when the most "interesting" tournament is to be computed [14].

Besides its natural connection to tournament manipulation, TFP studies the relationship between round-robin and single-elimination tournaments. The decision version of TFP asks, given the results of a round-robin tournament and a player $\mathcal{A}$, is $\mathcal{A}$ also the winner of some SE tournament, given the same match outcomes? In the area of voting, suppose all votes are in, can we simulate a win for a particular candidate, using single-elimination rules (binary cup)? In this work, we investigate the following question: if we consider a round-robin tournament and a ranking produced from it by sorting the players according to their number of wins, how many of the top players can actually win some SE tournament, given the same match outcomes? What conditions on the round-robin tournament suffice so that one can efficiently rig the SE tournament outcome for many of the top players?

Prior work has shown several intuitive results. For instance, if $\mathcal{A}$ is any player with the maximum number of wins in a round-robin tournament, then one can efficiently construct a winning (balanced) SE bracket for $\mathcal{A}$ [20]. We extend and strengthen many of the prior results.

Contributions. Let Π be an ordering of the players in nonincreasing order of their number of wins in the given round-robin tournament. We consider conditions under which, for large K, the SE tournament can be fixed efficiently for *any* of the first K players in Π. We are interested in natural and not too restrictive conditions under which a constant fraction of the players can be made to win. If the first player p_1 in Π beats everyone else, then p_1 wins all SE tournaments. We show that if *any* player can beat p_1, then we can also fix the tournament for the second player p_2. We show that for large enough tournaments, if there is a matching onto the top $K-1$ players $\{p_1, \ldots, p_{K-1}\}$ in Π from the rest of the players, then we can efficiently find a bracket for which p_K wins, where K is as large as 19% of the players.

Graph representation. The outcome of a round-robin tournament has a natural graph representation as a *tournament graph*: a directed graph in which for every pair of nodes a, b, there is an edge either from a to b, or from b to a. The nodes of a tournament graph represent the players in a round-robin tournament, and an edge (a, b) represents a win of a over b.

Notation and Definitions. Unless noted otherwise, all graphs in the paper are tournament graphs over n vertices, where n is a power of 2, and all SE tournaments are balanced. In Table 2, we define the notation that will be used in the rest of this paper. For the definitions, let $\mathcal{A} \in V$ be any node, let $X, Y \subseteq V$ be such that $X \cap Y = \emptyset$.

Consider a tournament graph $G = (V, E)$. We say that $\mathcal{A} \in V$ is a king over another node $x \in V$ if either $(\mathcal{A}, x) \in E$ or there exists $y \in V$ such that $(\mathcal{A}, y), (y, x) \in E$. A *king* in G is a node $\mathcal{A}$ which is a king over all $x \in V \setminus \{\mathcal{A}\}$. We say that set S covers a set T if for every $t \in T$ there is some $s \in S$ so that $(s, t) \in E$. Thus $N^{out}(\mathcal{A})$ covers the graph if and only if $\mathcal{A}$ is a king.

If one can efficiently construct a winning SE tournament bracket for a player $\mathcal{A}$, we say that $\mathcal{A}$ is an *SE winner*. We use the ranking Π formed by sorting the players in nondecreasing order of their outdegree.

We will construct SE tournaments as a series of matchings where each successive one will be over the winners of the previous one. A matching is defined as a set of pairs of vertices where each vertex appears in at most one pair. In our setting, these pairs are directed, so a matching from X to Y will consist only of edges that are directed from X to Y. If an edge is directed from x to y, then we refer to x as a *source*. Further, given a matching M from the sets X to Y, we will use the notation $X \setminus M$ to refer to the vertices in X that are not contained in the matching. A perfect matching from X to Y is one where every vertex of X is matched with a vertex of Y and $|X| = |Y|$. A perfect matching in a set S is a perfect matching from some $S' \subseteq S$ to $S \setminus S'$.

2 Motivation and Counterexamples

We will now discuss the motivation for our assumptions on the graph. We will look at some necessary and sufficient conditions for the top K players to win an SE tournament. We begin with an example.

Table 1: Notation

$N^{out}(\mathcal{A}) = \{v \vert (\mathcal{A}, v) \in E\}$	$N^{out}_X(\mathcal{A}) = N^{out}(\mathcal{A}) \cap X$
$N^{in}(\mathcal{A}) = \{v \vert (v, \mathcal{A}) \in E\}$	$N^{in}_X(\mathcal{A}) = N^{in}(\mathcal{A}) \cap X$
$out(\mathcal{A}) = \vert N^{out}(\mathcal{A})\vert$	$out_X(\mathcal{A}) = \vert N^{out}_X(\mathcal{A})\vert$
$in(\mathcal{A}) = \vert N^{in}(\mathcal{A})\vert$	$in_X(\mathcal{A}) = \vert N^{in}_X(\mathcal{A})\vert$
$\mathcal{H}^{in}(\mathcal{A}) = \{v \vert v \in N^{in}(\mathcal{A}), out(v) > out(\mathcal{A})\}$	$\mathcal{H}^{out}(\mathcal{A}) = \{v \vert v \in N^{out}(\mathcal{A}), out(v) > out(\mathcal{A})\}$
$\mathcal{H}(\mathcal{A}) = \mathcal{H}^{in}(\mathcal{A}) \cup \mathcal{H}^{out}(\mathcal{A})$	$E(X, Y) = \{(u, v) \vert (u, v) \in E, u \in X, v \in Y\}$
$\mathcal{M}(X, Y)$ is a maximal matching from X to Y	$\mathcal{CM}(X, Y)$ is the canonical matching (Section 5.1.1)

Table 2: A summary of the notation used in this paper.

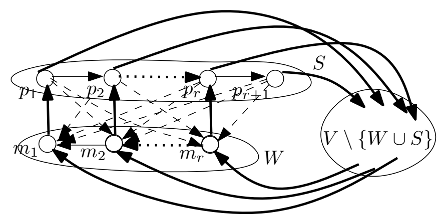

Figure 1: p_i only loses to m_i and p_j for $j < i$. No matter how the other edges of the tournament graph are placed, since the p_i beat everyone else and the m_i lose to everyone else, all SE tournament winners are in S.

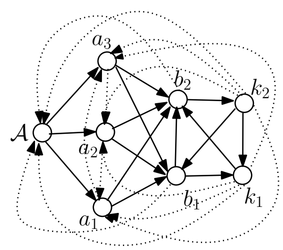

Figure 2: Example in which the two highest outdegree nodes, k_1 and k_2, have a matching into them but $\mathcal{A}$ cannot win an SE tournament.

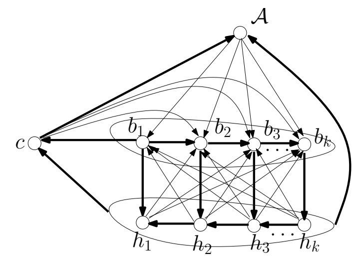

Figure 3: Example where there is a matching from $N^{out}(\mathcal{A})$ onto the k highest degree nodes but $\mathcal{A}$ can't win an SE tournament.

Consider the transitive tournament graph G with nodes $v_1, \ldots, v_n$, where v_i beats all nodes v_j for $j > i$. Then v_1 is the winner of all SE tournaments on G. Now, take any perfect matching from $\{v_1, \ldots, v_{n/2}\}$ to $\{v_{n/2+1}, \ldots, v_n\}$ and reverse these edges to create a *back-matching*. This gives each node from the weaker half of G a win against some node from the stronger half. The new outdegree ranking only swaps $v_{n/2}$ and $v_{n/2+1}$, however now the top $n/2 - 1$ players are SE winners: each of these nodes still beats at least $n/2$ other players, and the back-edges of the matching also make each one also a king. Prior work showed that this condition is sufficient for these players to be an SE tournament winner [20]. Thus, adding a back-matching to a transitive tournament can dramatically increase the set of winners. Our goal is to understand the impact of such back-edge matchings in general tournaments. As a warm-up, we consider the nodes of second and third highest outdegree. By case analysis, one can show the following theorem. The full proof is contained in the Appendix.

Theorem 1. *Let G be a tournament graph and let $\mathcal{A}$ be the node of second highest outdegree. Then $\mathcal{A}$ is an SE winner if and only if there is no Condorcet winner in G. If there is a matching onto the top 2 nodes, then the third highest outdegree node is also an SE winner.*

This simple result leads to a larger question. What are the necessary and sufficient conditions for the k^{th} ranked node to win an SE tournament? A natural conjecture is that if there is a perfect matching from $V \setminus \mathcal{H}(\mathcal{A})$ to $\mathcal{H}(\mathcal{A})$, then $\mathcal{A}$ should be able to win.

In Figure 1 we give an example of a tournament and a subset S consisting of the top $r + 1$ outdegree nodes such that there is a matching of size r from a subset $W = \{m_1, \ldots, m_r\}$ of $V \setminus S$ into S, but no matching of size $r + 1$ from $V \setminus S$ into S. Figure 1 only shows some of the graph edges. The edges within $V \setminus (W \cup S)$ are arbitrary, all nodes of S beat all nodes of $V \setminus (W \cup S)$, and all nodes of W lose to all nodes of $V \setminus (W \cup S)$. We can show that any node $\mathcal{A} \notin S$ cannot be an SE winner. p_1 only loses to m_1 and m_1 loses to everyone else so p_1 must be matched with m_1 in the first round if it is to ever be eliminated. Similarly, for any $i \leq r$, each p_i must be matched with m_i in the first round. Since all of the nodes that could possibly beat p_{r+1} lose in the first round, no one is left to beat p_{r+1} and $\mathcal{A}$ cannot win. Therefore, the only possible SE winners are contained in S. We have shown that for any r there exists a graph in which there is no matching onto the top r outdegree nodes and the $(r+1)$st outdegree node is not an SE winner. From this, we can conclude that the existence of a perfect matching from $V \setminus \mathcal{H}(\mathcal{A})$ into $\mathcal{H}(\mathcal{A})$ is, in a sense, necessary, in order for a node $\mathcal{A}$ to be an SE winner.

Now suppose that there is a perfect matching in G from $V \setminus \mathcal{H}(\mathcal{A})$ onto $\mathcal{H}(\mathcal{A})$. Can we conclude that the bracket can be fixed for $\mathcal{A}$? This turns out not to be true. Consider Figure 2. Here $\mathcal{H}(\mathcal{A})$ consists only of k_1 and k_2. These nodes are only beaten by b_1 and b_2 respectively, who lose to every other player except $\mathcal{A}$, so b_i and k_i must be matched in round 1. The a_i are symmetric, so without loss of generality we can match $\mathcal{A}$ to a_1 in round 1. The two remaining nodes, a_2 and a_3, must also be matched. After round 1 the nodes that survive are $\mathcal{A}, a_3, b_1, b_2$. However, $\mathcal{A}$ needs to have outdegree at least 2 to survive the next two rounds. As it only has outdegree 1, $\mathcal{A}$ cannot win an SE tournament.

A similar problem can arise when the matching comes from $N^{out}(\mathcal{A})$ instead of $N^{in}(\mathcal{A})$. Figure 3 gives an example of a graph construction for any $n \geq 8$ for which the node ranked $n/2$ cannot win any SE tournament even though there is a matching onto $\mathcal{H}(\mathcal{A}) = \cup_{i=1}^{k} h_i$. Each h_i only loses to b_i and $\cup_{j>i} h_j$. Each b_i only beats $\cup_{j>i} b_j$, except for b_1 who also beats c. The problem arises with who to match $\mathcal{A}$ to in the first round so that it can win the match. By induction, one can argue that every h_i for $i > 1$ must be matched to b_i in round 1. $\mathcal{A}$ must be matched to some node in $N^{out}(\mathcal{A})$, but only b_1 remains unmatched. This leaves h_1 and c who must be matched as well. However, in round 2, all nodes that beat h_1 have been eliminated and it is now a Condorcet winner in the induced subgraph. Therefore, it must be the winner of any SE tournament.

A common issue in the above counterexamples is that $\mathcal{H}(\mathcal{A})$ is too large while $out(\mathcal{A})$ is too small. Another commonality is that $\mathcal{H}(\mathcal{A}) = \mathcal{H}^{in}(\mathcal{A})$. Hence a better condition to look for is a matching from $V \setminus \mathcal{H}^{in}(\mathcal{A})$ onto $\mathcal{H}^{in}(\mathcal{A})$, and not necessarily onto $\mathcal{H}(\mathcal{A})$.

Finally, a natural question is, how reasonable is the assumption of the existence of a matching from lower ranked players to higher ranked players? Consider the Braverman-Mossel model [4] for generating tournament graphs. In this model, one assumes an underlying ranking $v_1 \cdots v_n$ of the players according to skill. The tournament is generated by adding an edge (v_i, v_j) with probability p if $j < i$ and $1-p$ if $j > i$ for $p < \frac{1}{2}$. This model can be viewed as a transitive tournament with each edge reversed with probability p. A classic result of [9] is that a bipartite graph with n nodes on each side with $2n \ln n$ edges selected uniformly at random contains a perfect matching with high probability. If a graph is generated by the Braverman-Mossel model with $p > \frac{4 \ln n}{n}$, then we expect there to be $n \ln n$ back edges from $v_{n/2} \cdots v_n$ to $v_1 \cdots v_{n/2-1}$. Therefore, in almost all such tournaments, a backedge matching exists.

3 Main Results

We are now ready to introduce our main result. As the proof is quite technical, we will first provide an intuitive sketch, some of the necessary Lemmas, and a more detailed account of the key part of our proof. All full proofs are included in the Appendix.

We present two main results. The first generalizes the idea of a king, and shows that if a node $\mathcal{A}$ is a king except for some subset and $\mathcal{A}$ beats many nodes that beat a king of that subset, then $\mathcal{A}$ is an SE winner.

Lemma 1 (Kings Except for a T subset). *Let $\mathcal{A}$ be a node in a tournament G and let T be a subset of $N^{in}(\mathcal{A})$ of size $|T| = 2^k$ for some k. Suppose that $\mathcal{A}$ is a king in $G \setminus T$ and $|N^{out}(\mathcal{A})| \geq |N^{in}(\mathcal{A})|$. Let t be a king in T with outdegree in T at least $\lfloor |T|/2 \rfloor$. Suppose that $|N^{in}(t) \cap N^{out}(\mathcal{A})| \geq |T|$. Then $\mathcal{A}$ is an SE winner.*

The key observation in proving Lemma 1 is that t can win an SE tournament over just the subgraph consisting of T in $\log |T|$ rounds. At the same time, there are at least $|T|$ nodes in $N^{out}(\mathcal{A})$ that beat t. In the worst case, these cannot eliminate any other nodes in $N^{in}(\mathcal{A})$ so they must be matched against each other for $\log |T|$ rounds as well. However, given the size, we are guaranteed that at least 1 will survive to eliminate t. At this point, $\mathcal{A}$ will be a king of high outdegree over the induced subgraph. The technical details of the proof proceed by induction on the size of T. Lemma 1 is used in the proof of our main theorem below. We highlight its use in the intuitive sketch.

We now address the main question of this paper - what can we show when a matching from $V \setminus \mathcal{H}^{in}(\mathcal{A})$ to $\mathcal{H}^{in}(\mathcal{A})$ exists?

Theorem 2 (Not a King but Matching into $\mathcal{H}^{in}(\mathcal{A})$). *There exists a constant n_0 such that for all $n \geq n_0$ the following holds. Let $G = (V, E)$ be a tournament graph on n nodes, $\mathcal{A} \in V$. Suppose there is a matching M from $V \setminus \mathcal{H}^{in}(\mathcal{A})$ onto $\mathcal{H}^{in}(\mathcal{A})$ of size K. If $K \leq (n-6)/7$, then $\mathcal{A}$ is an SE winner.*

The key restriction in this Theorem concerns the number of higher ranked players who beat a player, not the actual rank of that player. However, we are able to apply the fact that a player of rank k has outdegree at least $(n-k-1)/2$ to obtain a nice corollary for large tournament graphs: Any one of the top 19% of the nodes are SE winners, provided there is a matching onto the nodes of higher outdegree.

Corollary 1. *There exists a constant n_0 so that for all tournaments G on $n > n_0$ nodes the following holds. Let $\mathcal{A}$ be among the top $(6n+7)/31 \geq .19n$ highest outdegree nodes. If there is a matching from $V \setminus \mathcal{H}^{in}(\mathcal{A})$ onto $\mathcal{H}^{in}(\mathcal{A})$, then $\mathcal{A}$ is an SE winner.*

3.1 Intuition

We now give an intuitive sketch about how one might go about proving Theorem 2. The overall strategy of our proof is to set up the first round of the SE tournament, so that all of the high outdegree nodes that beat $\mathcal{A}$ are eliminated, and in the remaining tournament, $\mathcal{A}$ is a king over almost the entire graph, so that Lemma 1 can be applied.

At first glance, one might try to build the first round by using the existing matching, M, from $V \setminus \mathcal{H}^{in}(\mathcal{A})$ to $\mathcal{H}^{in}(\mathcal{A})$ and then finding some maximal matching M' from $N^{out}(\mathcal{A}) \setminus M$ to $N^{in}(\mathcal{A}) \setminus M$. The matching M' will guarantee that as many elements as possible of $N^{out}(\mathcal{A})$ will survive to compete in the second round. To complete round 1, the potentially remaining nodes in $N^{out}(\mathcal{A}) \setminus (M \cup M')$ should be matched amongst themselves, and the same for $N^{in}(\mathcal{A}) \setminus (M \cup M')$ in a matching called M''.

$\mathcal{A}$ is initially a king in G over any node with no larger outdegree than it (i.e. $V \setminus \mathcal{H}^{in}(\mathcal{A})$). However, if we do not create the matching M'' above carefully $\mathcal{A}$ may no longer be a king over the sources of M. Even worse, some source of M may lose all of the nodes that can potentially beat, and might become a Condorcet winner in the graph induced by the winners of round 1. This is demonstrated in Figure 4. In this example, we would like to fix the bracket for P_2, the second strongest player. P_3 can beat P_1, but only P_{n-1} and P_n beat P_3. If we use any matching of $N^{out}(P_2)$ that does not match P_n with P_{n-1}, P_3 will be a Condorcet winner in round 2, and P_2 cannot win.

The failure of this example motivates our approach. We begin our construction of round 1 as before. We use the perfect matching M from $V \setminus \mathcal{H}^{in}(\mathcal{A})$ to $\mathcal{H}^{in}(\mathcal{A})$ and M', a maximal matching M' from $N^{out}(\mathcal{A}) \setminus M$ to $N^{in}(\mathcal{A}) \setminus M$. At this point, we want to guarantee that as many of the sources of M as possible are still covered by winners of round 1. We start by finding the set T of sources of M that are not currently beaten by some source in M', or by $\mathcal{A}$. Because these nodes are all of lower outdegree than $\mathcal{A}$, we can argue that there is some subset S which is a subset of $N^{out}(\mathcal{A}) \setminus (M \cup M')$ that covers T. We use a greedy approach (Algorithm 1) to match up the nodes of S in round 1 so that the winners of this matching cover as many nodes of T as possible. We are able to show (in Lemma 2) that the set U of nodes of T that are not covered by the first round winners from S is very small: it has size at most $O(\sqrt{|T|})$. This will allow us to show that we can eliminate U in later rounds.

We design the next rounds using Lemma 1. To do this, we use the largest outdegree node t in T and find a set P, the size of which is a power of 2, that contains both t and U. The final requirements of Lemma 1 are that $\mathcal{A}$ beats at least as many first round winners outside P as it loses to (which we show using Theorem 3) and that the number of nodes from $N^{out}(A)$ that beat t and survive round 1 is at least $|P|$. To fulfill this last requirement, we add an extra iteration (for $q = 1$) in Algorithm 1 which constructs the first round matching of S so that enough nodes that beat t survive round 1.

Summary. We create the first round of the tournament by using M, a maximal matching M' from the remaining nodes of $N^{out}(\mathcal{A})$ to the remaining nodes of $N^{in}(\mathcal{A})$, and a greedily selected perfect matching M'' on S. Many sources of M'' beat t, and almost all of T is covered by the sources of M''. This does not fully specify the first round matching. A few nodes may remain unmatched, specifically $N^{in}(\mathcal{A}) \setminus (M \cup M')$, $N^{out}(\mathcal{A}) \setminus (M \cup M' \cup M'')$ and $\mathcal{A}$ itself. The final details are included in the proof sketch at the end of this section. The goal of the first round matching is to ensure that the requirements of Lemma 1 are met and the remaining rounds of the tournament can be completed so that $\mathcal{A}$ wins.

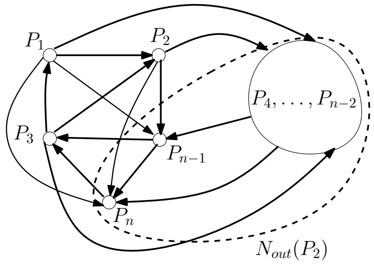

Figure 4: An example where an arbitrary matching of $N^{out}(P_2)$ is likely to fail.

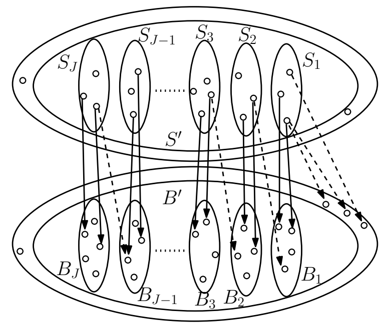

Figure 5: The construction of the sets S_i and B_i in Theorem 3.

3.2 Technical details.

With the above overview of the proof technique, we now introduce the necessary lemmas. As an SE tournament is a series of $\log n$ matchings, these lemmas are about the existence of matchings with desirable properties. The first is a very general result that can be specifically applied to lower-bound how large a matching can be found from $N^{out}(\mathcal{A})$ to $N^{in}(\mathcal{A}) \setminus \mathcal{H}^{in}(\mathcal{A})$.

Theorem 3 (Large Matching). *Let $h \in \mathbb{Z}$, possibly negative. Let S and B be disjoint sets such that $\forall X \subset B$, $|E(S,X)| \geq \binom{|X|}{2} - h|X|$. Then there exists a matching between S and B of size at least $\frac{|B|-2h-1}{2}$.*

Proof. Recall that M is a maximal matching from a set S to a set B if and only if there are no augmenting paths from the unmatched elements of S to the unmatched element of B. Our proof will proceed by using the large number of edges from S to any subset X of B to lower-bound the size of the matching.

Let M be a maximal matching from S to B. We refer to the sources of M as S' and the sinks as B'. We iteratively build up a family of sets S_j and B_j that consist of augmenting paths from the unmatched nodes in B.

Let S_1 be the subset of S' which contains all nodes with edges to $B \setminus B'$. Let B_1 be the nodes matched to S_1 by M. Now, we inductively define S_j as the nodes in $S' \setminus \cup_{i=1}^{j-1} S_i$ that have edges to B_{j-1}, where B_{j-1} are the nodes matched to S_{j-1} by M.

This process can be repeated up to some index $J+1$ such that there are no more nodes in $S' \setminus \cup_{i=1}^{J} S_i$ with edges to B_J. Let $\bar{S} = \cup_{i \leq J} S_i$ and $\bar{B} = (B \setminus B') \cup (\cup_{i \leq J} B_i)$.

First, note that there are no edges from $S \setminus S'$ to $\bar{B}$ since M is maximal. If there were, we would have an augmenting path. Therefore, all edges into $\bar{B}$ come from $\bar{S}$. The number of edges from $\bar{S}$ into $\bar{B}$ is at most $|\bar{B}||\bar{S}|$ (the number of edges in a complete bipartite graph) and at least $|\bar{B}|(|\bar{B}| - 1 - 2h)/2$ by the Theorem statement. Thus, we can conclude that

$$|M| = |B \setminus \bar{B}| + |\bar{S}| \geq$$
$$(|B| - |\bar{B}|) + \frac{(|\bar{B}| - 1 - 2h)}{2} \geq \frac{(|B| - 1 - 2h)}{2}.$$

□

Theorem 3 is used in the proof of Theorem 2 to argue about a lower bound on the size of $N^{out}(\mathcal{A})$ after the first round. An example application of this theorem is to set S to $N^{out}(\mathcal{A})$ and B to $N^{in}(\mathcal{A}) \setminus (M \cup \mathcal{H}^{in}(\mathcal{A}))$. Here, the conditions of the Theorem are met: we can show that for every subset X, $E(S,X) \geq \binom{|X|}{2} + |X|$ because every vertex in B beats $\mathcal{A}$ and is of lower outdegree than $\mathcal{A}$.

The other very important part of our proof is Algorithm 1. As mentioned earlier, it is a greedy way of creating a matching in a set S such that the sources cover many elements in a set T. It iteratively finds the source in S that covers the most uncovered elements of T and matches it with a vertex that it beats. The first iteration of the loop deals with an element t that is a king over T. This loop only considers the subset of S that beats t and guarantees that at least half of the nodes that beat t in S are preserved as sources. At any time in the algorithm, U_i is the set of the nodes that are

Algorithm 1 Greedy Matching

Input: $G = (V, E)$ a tournament and $S, T \subset V$, $t \in V$; Output: Matching M
Let $A_1 = N_S^{in}(t)$, $U_1 = T$, $i = 1$, $L_0 = \emptyset$, $M = \emptyset$.
for $q = 1, 2$ **do**
  while $|A_i| \geq 2$ **do**
    Let $x_i, y_i \in A_i$ have larger outdegree to U_i than all the other elements in A_i; x_i beats y_i.
    $M \leftarrow M \cup \{(x_i, y_i)\}$
    $L_i \leftarrow L_{i-1} \cup \{y_i\}$
    $U_{i+1} \leftarrow U_i \setminus N^{out}(x_i)$
    $A_{i+1} = \cup_{v \in U_{i+1}} N_{A_i}^{in}(v) \setminus L_i$
    $i \leftarrow i + 1$
  end while
  $A_i = \cup_{v \in U_i} N_S^{in}(U_i)$
end for

currently not covered by the sources of the matching M, A_i is the set of sources that beat any element in U_i, and L_i is the set of nodes that lose in M and are excluded from A_i.

We want to lower-bound the size of the generated cover. The main idea of the proof is that we initially have many edges from S to T, and specifically at least $\binom{|X|+1}{2}$ to each $X \subseteq T$. If we consider the first pair (x_1, y_1) added to M, then we can say x_1 covers k elements of T. Therefore, we now need to cover only a subset of size $|T| - k$ which has at least $\binom{|T|-k}{2}$ edges into it. However, this may include edges from y_1. When we remove y_1, we may lose up to $|T|$ edges. The key observation is that for the pair (x_2, y_2), y_2's outdegree is upper-bounded by x_1 so we are able to bound the number of edges lost by the matching as the number of vertices currently covered plus $|T|$. We then show that there will always be enough edges and sources to increase the size of the matching until at most $2\sqrt{|T|} + 1$ nodes remain uncovered.

Lemma 2. *Let $G = (V, E)$ be a tournament graph. Let $S \subseteq V$ and $T \subseteq V$ be disjoint sets such that for all $X \subseteq T$, the number of edges from S to X is at least $\binom{|X|+1}{2}$. Let $t \in V$ be given. Algorithm 1 generates a matching, M, in S such that at least $|T| - 1 - 2\sqrt{|T|}$ nodes in T are beaten by at least one source in M and at least $(in_S(t) - 2)/2$ of the sources also beat t.*

Proof. We need to define some additional concepts for the proof. The first is the set of covered nodes at iteration i, C_i, where $C_1 = \emptyset$. C_i is exactly $T \setminus U_i$ (so $|T| = |C_i| + |U_i|$). Let $d_i = |C_{i+1}| - |C_i|$ be the number of new nodes covered by iteration i. Our goal is to lower-bound the size of $|C_i|$ when the algorithm quits.

Consider the first execution of the WHILE loop. Let i_0 be the iteration at which the loop exits. This loop greedily covered T but only used vertices that also beat t. We will lower-bound the number of edges that remain from all unmatched sources in S (the set A_{i_0}) to U_{i_0}. At this point, $|C_{i_0}| = \sum_{j=1}^{i_0} d_j$. The number of edges from L_{i_0} to U_{i_0} is at most $|T| - |C_{i_0}| + \sum_{j=1}^{i_0-1} d_j \leq |T|$ since we picked the nodes so that $out_{U_i}(y_i) \leq out_{U_{i-1}}(x_{i-1}) = d_{i-1}$, and $out_{U_{i_0}}(y_1) \leq |U_{i_0}|$. Thus we can obtain a lower bound on the number of edges between A_{i_0} and U_{i_0}: $|E(A_{i_0}, U_{i_0})| \geq \binom{|U_{i_0}|+1}{2} - |T|$.

Let $j > i_0$ be any round in the second WHILE loop. As above, $|C_j| = |C_{i_0}| + \sum_{k=i_0+1}^{j} d_k$ and the number of edges from L_j to U_j is at most

$$|U_j| + |T| + \sum_{k=i_0+1}^{j-1} d_k = 2|T| - |C_j| + |C_j| - |C_{i_0}| \leq 2|T|.$$

We can lower-bound the number of usable edges from A_j to U_j as

$$|E(A_j, U_j)| \geq \binom{|U_j| + 1}{2} - 2|T| \geq$$

$$(|T|^2 + |C_j|^2 - (2|T|+1)|C_j| - 3|T|)/2.$$

The second WHILE loop exits when $|A_j| \leq 1$. Therefore, when the algorithm finishes, $|A_j| \leq 1$ and $|E(A_j, U_j)| \leq |U_j| = |T| - |C_j|$. We have:

$$(|T|^2 + |C_j|^2 - (2|T|+1)|C_j| - 3|T|)/2 \leq |T| - |C_j|,$$

This can be simplified as follows.

$$|C_j|^2 - (2|T|-1)|C_j| - 5|T| + |T|^2 \leq 0.$$

$$|C_j| \geq |T| - 1/2 - \sqrt{|T|^2 - |T| + 1/4 + 5|T| - |T|^2} =$$
$$|T| - 1/2 - \sqrt{4|T| + 1/4} \geq |T| - 1 - 2\sqrt{|T|}.$$

That is, the number of covered nodes is at least $|T| - 1 - 2\sqrt{|T|}$. After round i_0 we have at least i_0 sources in M covering t and at least $in_S(t) - 2i_0 - 1$ nodes of $N_S^{in}(t)$ that were not used in creating the rest of the matching because they did not cover any element of U_{i_0}. Match these among themselves to obtain at least $i_0 + \lfloor (in_S(t) - 1 - 2i_0)/2 \rfloor \geq (in_S(t) - 2)/2$ sources of the matching that are inneighbors of t. Complete the matching M from S to S by matching the rest of the nodes of S arbitrarily. □

The bounds on the greedy matching algorithm are only positive if $|T| > 5$. We don't want our bounds in Theorem 2 to depend on the size of the matching into $\mathcal{H}^{in}(\mathcal{A})$. We now present a sketch of the proof that ignores this difficulty. The full proof contained in the Appendix fixes this problem through the introduction of a technical Lemma, Lemma 6. This lemma allows one to artificially boost the size of T to guarantee that the above process will always work. Additionally, this proof sketch assumes that that the indegree of node $\mathcal{A}$ coming from the sources of M is large enough. This assumption is also lifted in the Appendix.

Proof sketch of Theorem 2: This proof proceeds by constructing the first round matching in stages. First, we will use M, the matching given by the theorem statement, and construct M', a maximal matching. Next, we show how to match $\mathcal{A}$ and construct the covering of the sources of M using Algorithm 1. Finally, we argue that the constructed first round matching satisfies the requirements of Lemma 1.

For simplicity, let $A = N^{out}(\mathcal{A})$ and $B = N^{in}(\mathcal{A})$. We divide the sources of M onto $\mathcal{H}^{in}(\mathcal{A})$ into two sets, A_T and B_T, where A_T are the sources of M in A while B_T are the sources in B. We can also divide $\mathcal{H}^{in}(\mathcal{A})$ into two sets, H_1 and H_2, where H_1 are the nodes matched to A_T and H_2 are matched to B_T by M. In order to later argue about the size of matchings, let $|A_T| = |H_1| = h$ and $|B_T| = |H_2| = k$. This means that K, the size of M is exactly $k + h$.

Let $B_{\text{rest}} = B \setminus (B_T \cup \mathcal{H}^{in}(\mathcal{A}))$ be the nodes who beat $\mathcal{A}$ and are not part of M. Take M' to be any maximal matching from $A \setminus A_T$ to B_{rest}. We want to argue about the size of M' by using Theorem 3. First, note that $|B_{\text{rest}}| = |B| - k - K$. Now, since we removed A_T, of size h, we can only say that every node b in B_{rest} has at least $out_B(b) + 1 - h$ inneighbors from $A \setminus A_T$. Therefore, by Theorem 3,

$$|M'| \geq (|B| - K - k - 2h + 2 - 1)/2 = (|B| - 2K - h + 1)/2.$$

We will use this fact later when arguing about the outdegree of $\mathcal{A}$ after the first round.

Finally, note that B_{rest} consists only of lower ranked nodes than $\mathcal{A}$, so every node in B_{rest} has some source of M' or A_T as an inneighbor.

(Matching $\mathcal{A}$ to some node.) Consider the currently unmatched portion of A. Call this $A_{\text{rest}} = A \setminus (A_T \cup M')$. If there is some $a' \in A_{\text{rest}}$, then match $\mathcal{A}$ to a'. If A_{rest} is empty, then we can argue that $|M'| > 1$ since

$$|A \setminus A_T| = |A| - h \geq (n - K)/3 - h \geq (n - 4K)/3 > 1.$$

Since $M' > 1$, we can dislodge any edge (a', b') from M' and match $\mathcal{A}$ to a'. After removing a', the lower bound for $|M'|$ goes down by 1: $|M'| \geq (|B| - 2K - h - 1)/2$.

(Creating a matching of $A_{\textbf{rest}} \setminus \{a'\}$).) We now use Algorithm 1 to cover B_T. Let $S = A_{\text{rest}} \setminus \{a'\}$ and T be the subset of B_T consisting of the nodes that do not have inneighbors among the sources of M' and A_T. For simplicity in this proof we assume that $|T|$ and hence $|B_T|$ is large enough.

Every subset X of the nodes of T has at least $\binom{|X|}{2} + 2|X| - |X| = \binom{|X|}{2} + |X|$ inneighbors in S since each node in X can have lost at most one inneighbor, a'. Let $t \in B_T$ be the node with highest outdegree in B_T. Run Algorithm 1 on S, T, t. This outputs a matching M'' on the nodes of S that covers all of T except for a subset, U, of size at most $1 + 2\sqrt{|T|}$. There are also at least $in_S(t)/2 - 1$ sources of M'' that beat t. This completes the first round matching. Let G' be the graph induced by the surviving nodes.

(Handling U.) We will construct P, a subset of T, such that P contains U, and t is a king over P who beats at least half of P.

We selected t so that it is a king in T. Therefore, there is a subset of at most $|U|$ nodes in its outneighborhood in T that cover U. We can add these nodes together with enough other nodes of $N_T^{out}(t)$ to P so that $|P|$ is a power of 2 and t is a king in P that beats at least half of P. This is possible since U is very small compared to T.

We can assume that the size of P is 2^c where 2^c is the closest power of 2 greater than $3 + 4\sqrt{|T|}$, as we may need as many as $|U| \leq 1 + 2\sqrt{|T|}$ extra nodes added to P to guarantee that t is a king over P. We can further conclude that $|P| \leq 5 + 8\sqrt{|T|}$ since we can at most double $3 + 4\sqrt{|T|}$ to make $|P|$ be a power of 2.

From Algorithm 1 we know that at least

$$in_S(t)/2 - 1 \geq (|B_T| - 1)/4 - 1$$

inneighbors of t from S are in G'. Since we assumed that B_T is large enough, we have

$$(|B_T| - 1)/4 - 1 \geq 5 + 8\sqrt{|T|}.$$

Hence there exists a subset of the surviving nodes of $N_S^{in}(t)$ of size at least $|P|$. The requirements of Lemma 1 are satisfied if $out_{G'}(\mathcal{A}) \geq in_{G'}(\mathcal{A})$. We prove this below and thus show that $\mathcal{A}$ is an SE winner.

(Showing that $out_{G'}(\mathcal{A}) \geq in_{G'}(\mathcal{A})$.) The number of nodes of $N^{out}(\mathcal{A})$ that survive the first round is at least

$$\lfloor (|A| + |M'| + |A_T| - 1)/2 \rfloor.$$

The number of nodes of $N^{in}(\mathcal{A})$ that survive is at most $\lceil (|B| - |A_T| - |M'|)/2 \rceil$. It suffices to show that

$$|A| + |M'| + |A_T| - 1 \geq |B| - |A_T| - |M'|.$$

Recall that $|M'| \geq (|B| - 2K - h - 1)/2$ so we must only show that $|A| + |B| - 2K - h + 2h - 2 \geq |B|$, or that $|A| - 2K + h - 2 \geq 0$. Since $|A| \geq (n - K)/3$ it suffices to show that $(n - K) \geq 6K + 6$, or that $K \leq (n - 6)/7$, which is true by assumption. $\square$

4 Conclusions

In this paper, we have shown that the existence of back-matchings can allow for an SE tournament to be manipulated in favor of any of the top 19% of players. The Braverman-Mossel model for tournament generation shows that back-matchings exist even when the noise in each match is very low ($O(\frac{\log n}{n})$). The question of the computational difficulty of manipulating SE tournaments in favor of a specific player remains open, but our result shows that many common examples can be efficiently manipulated in polynomial time using any algorithm for maximum matching. The fastest algorithms for matching run in $\tilde{O}(m\sqrt{n})$ time ([15] reimplementing [8]) and in $\tilde{O}(n^{2.376})$ time [16]. Possible future work includes finding even more general conditions under which a winning bracket can be found for a player, as well as trying to reduce the dependence our method has on the number of players.

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References

[1] N. Alon. Ranking tournaments. *SIAM J. Discret. Math.*, 20(1):137–142, 2006.

[2] D. R. Appleton. May the best man win? *The Statistician*, 44(4):529–538, 1995.

[3] J. Bartholdi, C. Tovey, and M. Trick. How hard is it to control an election? *Mathematical and Computer Modeling*, 16(8/9):27–40, 1992.

[4] M. Braverman and E. Mossel. Noisy sorting without resampling. In *Proc. SODA*, pages 268–276, 2008.

[5] Robert Chen and F. K. Hwang. Stronger players win more balanced knockout tournaments. *Graphs and Combinatorics*, 4(1):95–99, 1988.

[6] Y. Chevaleyre, U. Endriss, J. Lang, and N. Maudet. A short introduction to computational social choice. *SOFSEM*, 4362:51–69, 2007.

[7] D. Coppersmith, L. Fleischer, and A. Rudra. Ordering by weighted number of wins gives a good ranking for weighted tournaments. In *Proc. SODA*, pages 776–782, 2006.

[8] J. Edmonds. Paths, trees, and flowers. *Canad. J. Math.*, 17:449–467, 1965.

[9] P. Erdős and A. Rényi. On random matrices. *Publ. Math. Inst. Hung. Ac. Sci.*, 8:455–561, 1964.

[10] F. Fischer, A.D. Procaccia, and A. Samorodnitsky. A new perspective on implementation by voting trees. In *Proc. EC*, pages 31–40, 2009.

[11] N. Hazon, P.E. Dunne, S. Kraus, and M. Wooldridge. How to rig elections and competitions. In *Proc. COMSOC*, 2008.

[12] J. Horen and R. Riezman. Comparing draws for single elimination tournaments. *Operations Research*, 33(2):249–262, 1985.

[13] F. K. Hwang. New concepts in seeding knockout tournaments. *The American Mathematical Monthly*, 89(4):235–239, 1982.

[14] J. Lang, M. S. Pini, F. Rossi, K. B. Venable, and T. Walsh. Winner determination in sequential majority voting. In *Proc. IJCAI*, pages 1372–1377, 2007.

[15] S. Micali and V. Vazirani. An $O(\sqrt{|V|} \cdot |E|)$ algorithm for finding maximum matching in general graphs. In *Proc. FOCS*, pages 17–27, 1980.

[16] M. Mucha and P. Sankowski. Maximum matchings via gaussian elimination. In *FOCS*, pages 248–255, 2004.

[17] T. Russell and T. Walsh. Manipulating tournaments in cup and round robin competitions. In *Proc. ADT*, pages 26–37, 2009.

[18] P. Slater. Inconsistencies in a schedule of paired comparisons. *Biometrika*, 48(3/4):303–312, 1961.

[19] I. Stanton and V. Vassilevska Williams. Rigging tournament brackets for weaker players. *IJCAI*, 2011.

[20] V. Vassilevska Williams. Fixing a tournament. In *Proc. AAAI*, pages 895–900, 2010.

[21] T. Vu, N. Nazon, A. Altman, S. Kraus, Y. Shoham, and M. Wooldridge. On the complexity of schedule control problems for knock-out tournaments. *JAIR*, 2010.

[22] T. Vu and Y. Shoham. Fair seedings in knockout tournaments. *ACM Transactions on Intelligent Systems and Technology*, 2010.

[23] T. Vu and Y. Shoham. Optimal seeding in knockout tournaments. In *Proc. AAMAS*, pages 1579–1580, 2010.

5 Appendix

5.1 Additional Tools

For the full proofs of the main results, we rely on a few constructions and facts that were not mentioned in the main part of the paper. The first of these is the canonical matching. This is a generic matching construction that maximizes the surviving sources while minimizing the surviving sinks.

5.1.1 Canonical matching.

Let G be a tournament graph and let $A, B \subset V$ such that $|A| + |B|$ is even. A *canonical* matching, $\mathcal{CM}(A, B)$ is formed as follows: create a maximal matching M' from A to B. Match all of the nodes in A that are not in M' against each other, and all of the unmatched nodes in B against each other. If $|M'|$ is odd, then match the leftover node in A with the leftover node in B.

We now describe a canonical matching for a given king node $\mathcal{A}$. Let G be a tournament graph over an even number of nodes and $\mathcal{A}$ be a king in G with $out(\mathcal{A}) \geq in(\mathcal{A})$. In the following construction $\mathcal{CM}(\mathcal{A})$ we include $\mathcal{A}$ by modifying $\mathcal{CM}(N^{out}(\mathcal{A}), N^{in}(\mathcal{A}))$. Since $out(\mathcal{A}) \geq in(\mathcal{A})$ and n is even, $out(\mathcal{A})$ and $in(\mathcal{A})$ have different parity, and $out(\mathcal{A}) \geq 1 + in(\mathcal{A})$. Thus $|N^{out}(\mathcal{A}) \setminus M'| \geq 1$ and we can pick any node $a' \in N^{out}(\mathcal{A}) \setminus M'$ to match with $\mathcal{A}$. Match the nodes of $N^{out}(\mathcal{A}) \setminus M' \setminus \{a'\}$ amongst themselves. At most one node a'' is left over. Match the nodes of $N^{in}(\mathcal{A}) \setminus M'$ amongst themselves. At most one node b'' is left over, and it is left over iff a'' is. Match a'' and b'', completing $\mathcal{CM}(\mathcal{A})$. The proof of the following lemma follows from the maximality of M'.

Lemma 3. *Let $\mathcal{A}$ be a king such that $out(\mathcal{A}) \geq in(\mathcal{A})$. Let G' be the subtournament graph over the sources of $\mathcal{CM}(\mathcal{A})$. Then*

- *$\mathcal{A}$ is a king in G',*
- *$out_{G'}(\mathcal{A}) = \lfloor (out(\mathcal{A}) + |M'| - 1)/2 \rfloor$,*
- *$in_{G'}(\mathcal{A}) = \lfloor (in(\mathcal{A}) - |M'| + 1)/2 \rfloor$,*
- *$out_{G'}(\mathcal{A}) \geq in_{G'}(\mathcal{A})$.*

Proof. 1) follows since M' was maximal, and so any node in $N^{in}(a)$ that is not in M' must have some source of M' as an inneighbor. 2) follows since $|M'|$ of the nodes in $N^{out}(a)$ survive the matching M', one node is removed, and the rest are matched amongst themselves, possibly one losing to a node of $N^{in}(a)$. Hence, $out_{G'}(a) = |M'| + \lfloor (out(a) - 1 - |M'|)/2 \rfloor = \lfloor (out(a) - 1 + |M'|)/2$.

3) follows since $|M'|$ nodes of $N^{in}(a)$ are eliminated by M', and the rest are matched amongst themselves, possibly one beating a node of $N^{out}(a)$. Hence

$$in_{G'}(a) = \lceil (in(a) - |M'|)/2 \rceil = \lfloor (in(a) - |M'| + 1)/2 \rfloor.$$

4) If $|M'| \geq 1$, $in_{G'}(a) \leq \lfloor in(a)/2 \rfloor \leq out_{G'}(a)$. Otherwise, since a was a king and M' was maximal, $N^{in}(a) = \emptyset$. Then $out_{G'}(a) \geq 0 = in_{G'}(a)$. □

5.1.2 Bounds on $out(\mathcal{A})$.

We often need to argue about the size of a given set, given some constraints on the number of higher degree nodes that exist, or that a player beats. The following are useful facts of this type.

Fact 1. *For any tournament graph of size k, there exists a vertex with outdegree at least $\lfloor \frac{k}{2} \rfloor$.*

This follows directly from the fact that a tournament of size k has $\binom{k}{2}$ edges.

Lemma 4. *Let $\mathcal{A}$ be a node in a tournament graph $G = (V, E)$ with $|\mathcal{H}(\mathcal{A})| = k$. Then $out(\mathcal{A}) \geq \lfloor \frac{(n-k)}{2} \rfloor$.*

Proof. Let $|\mathcal{H}^{out}(a)| = k_1$, $|\mathcal{H}^{in}(a)| = k_2 = k - k_1$, and $out(a) = d$. Let $R = V \setminus \{a\} \setminus \mathcal{H}$. Then $|R \cap N^{in}(a)| = n - d - k_2 - 1$ and $|R| = n - k - 1$. Since for every $b \in R$, $out(b) \leq out(a) = d$, d is at least the average of the outdegrees of $R \cup \{a\}$ in G. The sum of these outdegrees is

$$\begin{array}{rc} & d + \binom{n-k-1}{2} + (n - d - k_2 - 1) + out_{\mathcal{H}}(R) \\ \geq & (n-k-1)(n-k-2)/2 + (n-k) - 1 \\ = & (n-k)(1 + (n-k-1)/2 - 1) + 1 - 1 \\ = & (n-k)(n-k-1)/2 \end{array}$$

Since $|R \cup \{a\}| = n - k$ and d is integral, $d \geq \lfloor (n-k)/2 \rfloor$. ☐

Lemma 5. *Let $\mathcal{A}$ be a node in a tournament graph such that $|\mathcal{H}^{in}(\mathcal{A})| = k$. Then $out(\mathcal{A}) \geq (n-k)/3$.*

Proof. Let $A = N^{out}(a)$ and $B = N^{in}(a)$. The number of edges from A to $B \setminus \mathcal{H}^{in}(a)$ is at most $|A|(|B| - k)$ and least $\binom{|B|-k}{2} + |B| - k = (|B| - k)(|B| - k + 1)/2$ since for every $b \in B \setminus \mathcal{H}^{in}(a)$, $1 + out_B(b) \leq in_A(b)$. Hence,

$$|A| \geq (|B| - k + 1)/2 = (n - 1 - |A| - k + 1)/2 \implies |A| \geq (n-k)/3.$$

☐

Fact 2. *Let x and y be nodes in a tournament graph such that $out(x) \geq out(y)$. Then the distance between x and y is at most 2. If $\mathcal{A}$ is a node such that for all $x \neq \mathcal{A}$, $out(\mathcal{A}) \geq out(x)$, then $\mathcal{A}$ is a king.*

5.1.3 Boosting set sizes

Our final additional tool is the technical lemma mentioned before the proof sketch of Theorem 2. Its application relies heavily on the greedy matching algorithm, Algorithm 1.

The bounds on the greedy matching algorithm given by Lemma 2 are only positive if $|T| > 5$. However, the way we will apply Lemma 2 in Theorem 2 will require that T be significantly larger than 5. We don't want our bounds in Theorem 2 to depend on the size of the matching into $\mathcal{H}^{in}(\mathcal{A})$, so we present the next lemma as a way of artificially boosting the size of T in order to guarantee that the above process will always work.

The intuition for the following technical lemma is that it is a method of picking a subset of nodes in $N^{out}(\mathcal{A})$, T, so that the requisite edges for the previous algorithm have no needed sources in T, and that $\forall X \subset T$, $|E(N^{out}(\mathcal{A}) \setminus T, X)| \geq \binom{|X|}{2} + 2|X|$.

Lemma 6. *Let C be a given constant. Let S and T be disjoint node sets of a tournament graph such that for every $t \in T$, $in_S(t) \geq out_T(t) + 2$, and $|T| < C$. Let $M \subseteq S$ such that $|S \setminus M| \geq (5C^2 + 17C + 4)/2$. Then there exists a subset $Z \subset S \setminus M$ such that $|Z| = 2(C - |T|)$ and $\forall Q \subseteq (Z \cup T)$, $|E(S \setminus Z, Q)| \geq \binom{|Q|}{2} + 2|Q|$.*

Proof. Form a subset $Y \subset S$ by including for every $t \in T$ exactly $out_T(t) + 2$ of its inneighbors from S. We can lower bound the size of Y as $|Y| \leq \binom{|T|}{2} + 2|T| \leq C(C+3)/2$. These are the sources needed to apply Lemma 2 to the set T. Let $R = S \setminus (M \cup Y)$. Hence

$$|R| \geq (5C^2 + 17C + 4)/2 - C(C+3)/2 = 2C^2 + 7C + 2.$$

Now we can create the set Z. While $|Z| < 2(C - |T|)$: pick $z \in R$ of largest indegree and add z to Z while removing it from R. Additionally, remove from R exactly $C + 2$ of the inneighbors of z.

We now want to bound the number of edges removed from R. Notice that

$$|R| - (2C - 2|T| - 1)(C + 3) \geq 2C^2 + 7C + 2 + 2|T|(C + 3) - (2C^2 + 5C - 3) \geq 1 + 2(C + 2).$$

Since we have removed at most $(2C - 2|T| - 1)(C + 3)$ nodes from R, at each step the indegree of z is at least $(|R| - (2C - 2|T| - 1)(C + 3) - 1)/2 \geq C + 2$ by Fact 1.

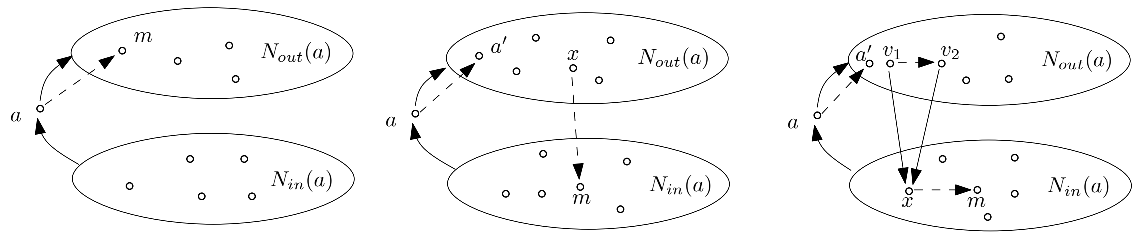

Figure 6: The three cases for the second strongest player, a where m is the strongest player.

Now consider $T \cup Z$. We will prove that $\forall Q \subseteq T \cup Z$, $|E(S \setminus Z, Q)| \geq \binom{|Q|}{2} + 2|Q|$ by induction on the number p of elements of Z contained in the subset $Q \subseteq T \cup Z$. The statement is clearly true when $p = 0$. Suppose it is true for all subsets with at most $p-1$ elements of Z. Consider a subset Q with p elements of Z and let $z \in Q \cap Z$. Then we know by the induction hypothesis that $|E(S \setminus Z, Q \setminus \{z\})| \geq \binom{|Q \setminus \{z\}|}{2} + 2|Q \setminus \{z\}|$. Since $in_{S \setminus Z}(z) \geq C + 2 \geq |Q| + 2$, we can conclude that

$$|E(S \setminus Z, Q)| \geq \binom{|Q|-1}{2} + 2(|Q|-1) + |Q| + 2 = \binom{|Q|}{2} + 2|Q|.$$

□

5.2 The top 3 players in a tournament

We now address the problem mentioned in the Motivations section.

Theorem 4. *Let $G = (V, E)$ be a tournament graph and let a be the node of second highest outdegree. Then a single-elimination tournament bracket can be fixed for a if and only if there is no Condorcet winner in G.*

Proof sketch. Let m be the node with the highest outdegree, a the second highest, and x is a node that beats m. The proof proceeds in 3 cases as demonstrated by Figure 6.

The first is that $x = a$. Here, a is a king that beats all nodes of higher outdegree. By [20], a can win a single-elimination tournament. The second case is that a beats x. Here, a is also a king. It can be shown that there exists a perfect matching that includes (x, m), a is among the sources of the matching and a has a maximal outdegree among the sources. Again, by [20], a can win a single-elimination tournament. The final case is that x beats a. Here, we note that since x's outdegree is no greater than a and beats both m and a, it must be beaten by two nodes, v_1 and v_2, that a also beats. In this case, there always exists a matching such that v_1 or v_2 is a source, a is a king over the sources, and a beats at least $n/4$ of the sources. □

We can generalize this theorem into the following result that for the top 3 nodes in the graph.

Theorem 5. (Top first, second and third node) *Let a be such that $|\mathcal{H}(a)| = K$ in a tournament $G = (V, E)$. Suppose there is a matching from $V \setminus \mathcal{H}(a)$ onto $\mathcal{H}(a)$. If $K \leq 1$, then for all n, a can win a single-elimination tournament. If $K = 2$, then for all $n \geq 16$, a can win a single-elimination tournament.*

The proof of this theorem is very similar to that of Theorem 4. The only change is that for the case of the third largest node, the two largest, m_1 and m_2 are beaten by some nodes a_1 and a_2. Since there is an edge between them (assume $a_1 a_2$ without loss of generality), a_1 must be either in $N^{out}(a)$ or beaten by 3 nodes in that set. Similarly, a_2 must be in $N^{out}(a)$ or beaten by 2 not necessarily distinct nodes. Simple case analysis shows that no matter the overlap of these extra nodes, there exists a matching that ensures a is a king over a_1 and a_2 (and all other nodes) in round 2 of the tournament.

Proof. The cases where $K = 0$ and $K = 1$ are covered by Theorem 4 so we will focus only on the case where $K = 2$, i.e. $\mathcal{H}(a) = \{m_1, m_2\}$. If $m_1 \in N^{out}(a)$, or $m_2 \in N^{out}(a)$ then the proof when $K = 1$ works. Therefore, we will assume that both m_1 and m_2 are in $N^{in}(a)$. By the Theorem statement, let the matching be $\{(x_1, m_1), (x_2, m_2)\}$.

Suppose first that $x_1, x_2 \in N^{out}(a)$. The obvious thing to do is to match x_1 to m_1 and x_2 to m_2 and then complete the matching as in the canonical matching. This works provided $out(a) > 2$ so that a can be matched to some node that isn't x_1 or x_2. This case can be avoided if $n \geq 8$ and $in(a) \geq 5$. This guarantees that there are at least 3 nodes other than m_1, m_2 in $N^{in}(a)$. The number of edges into those two nodes from $\{x_1, x_2\}$ is at least 3 and so one of x_1 or x_2 must have outdegree at least 3. 3 would be larger than $out(a)$ which is a contradiction to x_1 and x_2 not being m_1 or m_2.

Now consider the case when $x_1 \in N^{out}(a)$ and $x_2 \in N^{in}(a)$. Since x_2 is of lower outdegree than a, x_2 must have indegree at least 2 since it beats a and m_2. Let $z_1, z_2 \in N^{out}(a)$ both beat x_2. Let M' be a maximal matching from $N^{out}(a) \setminus \{x_1\}$ into $N^{in}(a) \setminus \{m_1, x_2, m_2\}$. If $M' \cup \{x_1\}$ contains either z_i for one of $i \in \{1, 2\}$, then we can pick any $a' \neq z_i, x_1$ to match to a. Just as in the canonical matching, if we unmatch a' from M', then the number of unmatched nodes from $N^{in}(a)$ is even. Therefore, the survivors both lose at most one inneighbor from $N^{out}(a)$ and at least one outneighbor from $N^{in}(a)$. If neither $z_1 \in M' \cup \{x_1\}$, nor $z_2 \in M' \cup \{x_1\}$, then match them to each other. In this case there is a counterexample on 8 nodes:

- $a : x_1, z_1, z_2,$
- $x_1 : m_1, x_2, y,$
- $z_1 : z_2, x_2, x_1,$
- $z_2 : x_1, x_2,$
- $y : z_1, a,$
- $x_2 : y, m_2, a,$
- $m_1 : y, x_2, z_1, z_2, a,$
- $m_2 : m_1, y, x_1, z_1, z_2, a.$

In order for a to win in this counterexample, x_2 must be matched to some node of $N^{out}(a)$ but then m_2 cannot lose.

Given that $n = 8$ doesn't necessarily work, assume $n \geq 16$. Now, $out(a) \geq \lceil (n-3)/2 \rceil = (n-2)/2 \geq 7$. We can extend $M' \cup \{(x_1, m_1), (x_2, m_2)\}$ just as in the canonical matching - since the outdegree of a is high enough, there is at least one node to match a' to. If $out(a) = l + n/2 - 1$, then

$$in(a) - 3 = n - 1 - 3 - (l + n/2 - 1) = n/2 - l - 3 \geq 5 - l.$$

If $5 - l \geq 2$, then $|M'| \geq 1$, and the number of nodes remaining after the first round is at least $\lfloor ((n-2)/2 - 1 + 2)/2 \rfloor = n/4$. Otherwise, $l \geq 4$, and $out(a) \geq n/2 + 3$. Then the number of nodes remaining after the first round is at least $\lfloor (n/2 + 3 - 1 + 1)/2 \rfloor \geq n/4$. Since a is a king it can win the tournament.

The final case is when $x_1, x_2 \in N^{in}(a)$. Without loss of generality, also let x_1 beat x_2. Here there is a counterexample for $n = 8$ as demonstrated in Figure 2.

Let $n \geq 16$, which implies that $out(a) \geq (n-2)/2 \geq (16-2)/2 = 7$. There are at least 3 inneighbors of x_1 and at least two inneighbors of x_2 in $N^{out}(a)$. We create a maximal matching M' from $N^{out}(a)$ into $N^{in}(a) \setminus \{x_1, x_2, m_1, m_2\}$. If for either x_1 or x_2 none of their inneighbors are sources of M' then there is a matching on their inneighbors in $N^{out}(a)$ so that the matching sources contain at least one inneighbor for each x_i. One can finish the matching just as in the canonical case. a can be matched since $out(a) > 4$. a will be a king in the remaining tournament. It remains to show that it has outdegree at least $n/4$. Let $out(a) = l + (n-2)/2 = n/2 + l - 1$. Then

$$|N^{in}(a) \setminus \{x_1, x_2, m_1, m_2\}| = n - 5 - n/2 - l + 1 = n/2 - 4 - l,$$

and

$$|M'| \geq n/4 - 2 - l/2.$$

The number of surviving outneighbors of a is

$$\lfloor (n/2 + l - 1 - 1 + n/2 - 2 - l/2)/2 \rfloor \geq n/2 - 2 \geq n/4$$

as desired. □

5.3 Full Proofs of Main Results

Reminder of Lemma 1 **[Kings Except for a T subset]** Let $\mathcal{A}$ be a node in a tournament G and let T be a subset of $N^{in}(\mathcal{A})$ of size $|T| = 2^k$ for some k. Suppose that $\mathcal{A}$ is a king in $G \setminus T$ and $|N^{out}(\mathcal{A})| \geq |N^{in}(\mathcal{A})|$. Let t be a king in T with outdegree in T at least $\lfloor |T|/2 \rfloor$. Suppose that $|N^{in}(t) \cap N^{out}(\mathcal{A})| \geq |T|$. Then $\mathcal{A}$ is an SE winner.

Proof of Lemma 1: This proof will proceed by induction on the size of T. As such, we establish the base case when $|T| = 1$. Here, $T = \{t\}$ and $\mathcal{A}$ is actually a king in G with outdegree at least half the graph. By [20] $\mathcal{A}$ can win a single-elimination tournament.

Now consider when $|T| > 1$. Our induction proceeds by assuming that $\mathcal{A}$ can win if $|T| < p$ for some p, provided that $|N^{out}(\mathcal{A})| \geq |N^{in}(\mathcal{A})|$, t is a king of outdegree at least $|T|/2$ in T, $|T|$ is a power of 2 and $|N^{in}(t) \cap N^{out}(\mathcal{A})| \geq |T|$. Now given a graph with $|T| = p$, we will give a perfect matching M_G of the graph such that the following is true of the tournament G' induced by the sources of M_G:

1. if T_r are the surviving nodes of T, then $t \in T_r$ and t is a king in T_r of outdegree at least $|T_r|/2$ and $|T_r| = |T|/2$ is a power of 2,
2. if A_r are the surviving nodes of $N^{out}(\mathcal{A})$, then $in_{A_r}(t) \geq |T_r|$,
3. $\mathcal{A}$ is a king in $G' \setminus T_r$, and
4. if B_r are the surviving nodes of $N^{in}(\mathcal{A})$, then $|A_r| \geq |B_r|$.

In order to create the necessary matching M_G, first create a canonical matching $\mathcal{CM}(t)$ for t in T. Let T_r be the sources of $\mathcal{CM}(t)$. Then by Lemma 3, Condition 1 follows.

Now, let S be a subset of $N^{in}_{N^{out}(\mathcal{A})}(t)$ of size $|T|$. Create $\mathcal{M}(N^{out}(\mathcal{A}), N^{in}(\mathcal{A}) \setminus T)$. Since $|N^{out}(\mathcal{A}) \setminus S| \geq 1 + |N^{in}(\mathcal{A}) \setminus T|$, there exists an unmatched node a' in $N^{out}(\mathcal{A}) \setminus S$. We can match $\mathcal{A}$ to a'.

Next, match any unmatched nodes of S, $N^{out}(\mathcal{A}) \setminus (M' \cup S)$ or $N^{in}(\mathcal{A})$ amongst their respective sets. Call this matching M''. The number of nodes of S that survive is at least $\lfloor (|S| + |M'' \cap S|)/2 \rfloor \geq |S|/2 = |T_r|$. This satisfies Condition 2. Since M'' was maximal, all nodes of $N^{in}(\mathcal{A}) \setminus T_r$ have surviving inneighbors in A_r. This shows that $\mathcal{A}$ is a king in $G' \setminus T_r$, or Condition 3.

It remains to show that $|A_r| \geq |B_r|$. We know that

$$|A_r| \geq \lfloor (|N^{out}(\mathcal{A})| + |M''| - 1)/2 \rfloor$$

and that

$$|B_r| \leq \lceil (|N^{in}(\mathcal{A})| - |T| - |M''|)/2 \rceil + |T|/2 = \lfloor (|N^{in}(\mathcal{A})| + 1 - |M''|)/2 \rfloor.$$

Now, since $|N^{out}(\mathcal{A})| \geq |N^{in}(\mathcal{A})|$ by the induction hypothesis

$$|A_r| \geq \lfloor (|N^{in}(\mathcal{A})| + |M''| - 1)/2 \rfloor.$$

If $|M''| \geq 1$, we immediately get $|A_r| \geq |B_r|$. If $M'' = \emptyset$, then $N^{in}(\mathcal{A}) = T$. But then both $|N^{in}(\mathcal{A})|$ and $|N^{out}(\mathcal{A}) \setminus \{a'\}|$ are even. Furthermore, $|N^{out}(\mathcal{A}) \setminus \{a'\}| \geq |T| = |N^{in}(\mathcal{A})|$. Hence, $|A_r| = |N^{out}(\mathcal{A}) \setminus \{a'\}|/2 \geq |N^{in}(\mathcal{A})|/2 = |B_r|$. This proves Condition 4 and concludes the proof of the lemma. □

Reminder of Theorem 2 **[Not a King but Matching into $\mathcal{H}^{in}(\mathcal{A})$]** There exists a constant n_0 such that for all $n \geq n_0$ the following holds. Let $G = (V, E)$ be a tournament graph on n nodes, $\mathcal{A} \in V$. Suppose there is a matching M from $V \setminus \mathcal{H}^{in}(\mathcal{A})$ onto $\mathcal{H}^{in}(\mathcal{A})$ of size K. If $K \leq (n-6)/7$, then $\mathcal{A}$ is an SE winner.

Proof of Theorem 2: This proof fleshes out the details that were ignored by the proof sketch given previously in the paper but follows the same structure. It will be useful to refer to Figures 7 and 8 as we proceed through the construction.

For simplicity, let $A = N^{out}(\mathcal{A})$ and $B = N^{in}(\mathcal{A})$. We divide the sources of M onto $\mathcal{H}^{in}(\mathcal{A})$ into two sets, M_1 and M_2, where M_1 are the sources of M in A while M_2 are the sources in B. We can also divide $\mathcal{H}^{in}(\mathcal{A})$ into two sets, H_1 and H_2, where H_1 are the nodes matched to M_1 and H_2 are matched to M_2 by M. In order to later argue about the size of matchings, let $|M_1| = |H_1| = h$ and $|M_2| = |H_2| = k$. This means that K, the size of M is exactly

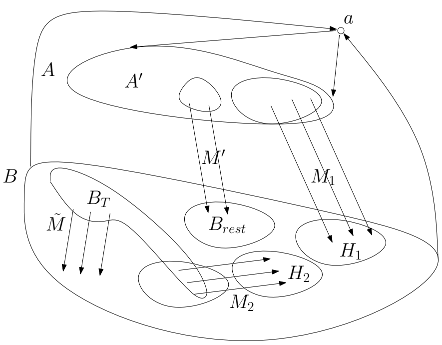

Figure 7: Situation in Theorem 2 when $Z = \emptyset$.

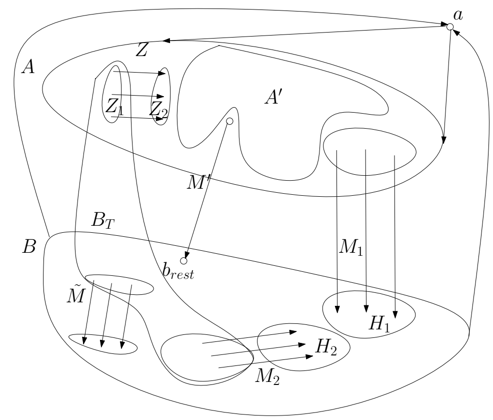

Figure 8: Situation in Theorem 2 when $Z \neq \emptyset$.

$k + h$. Further, let $n_0 = 10^6$ although we suspect the theorem is true for much smaller n_0 with more careful analysis. Let C be the constant 529 and $c = \max\{C - k, 0\}$.

Let $\tilde{M}$ be an arbitrary matching of $B \setminus (M_2 \cup H)$ of size $\min\{c, \lfloor |B \setminus (M_2 \cup H)|/2 \rfloor\}$. We will call the set of sources of M_2 and $\tilde{M}$ B_T. Let $B_{\text{rest}} = B \setminus (B_T \cup H)$. Because B_T does not contain any nodes ranked higher than $\mathcal{A}$, for every $b \in B_T$ we have that $out_{B_T} + 2 \leq in_A(b)$. For our proof we will require that $|B_T| \geq C$. If $|B_T| < C$, then we will show how to use nodes from A to artificially boost the size of B_T, while still preserving the properties we need.

(Boosting $|B_T|$ when $|B_T| < C$.) If $|B_T| < C$, then $\tilde{M}$ was maximal and $|B_{\text{rest}}| \leq 1$. We now show that we can apply Lemma 6.

If $B_{\text{rest}} = \{b_{\text{rest}}\}$ and some inneighbor a_{rest} of b_{rest} is in $A \setminus M_1$, then note that

$$\begin{aligned} |A \setminus (M_1 \cup \{a_{\text{rest}}\})| &\geq \\ (n-1) - 2|M_1| - 2|B_T| - 2 &\geq \\ n - 3 - 2h - 2k - 2C &\geq \\ (5C^2 + 17C + 4)/2 & \end{aligned}$$

since $n \geq 10^6 \geq 7(10 + 21C + 5C^2)/10$ and $K < n/7$. Therefore,

$$\begin{aligned} n - 2h - 2k = n - 2K > 5n/7 &\geq \\ (5/7) \cdot 7(10 + 21C + 5C^2)/10 &= \\ (5C^2 + 17C + 4)/2 + 2C + 3 & \end{aligned}$$

This satisfies the conditions so that we can apply Lemma 6 to the sets A, $(A \cap M_1) \cup \{a_{\text{rest}}\}$, and B_T with the value C. This will give us a set $Z \subset A \setminus (M_1 \cup \{a_{\text{rest}}\})$ of size $2(C - |B_T|)$ in Z. Let $Z_1 \to Z_2$ be a (perfect) matching of size $C - |B_T|$ in Z. Add Z to B and Z_1 to B_T. Now we can assume $|B_T| \geq C$ and that for every subset $Q \subseteq B_T$ there are at least $\binom{|Q|}{2} + 2|Q|$ inedges of Q from $A \setminus Z$.

Let $\tilde{A} = A \setminus Z$, $\tilde{B} = (B \setminus H) \cup Z \cup M_2$ and $B_{\text{rest}} = B \setminus (\tilde{M} \cup M_2 \cup H)$. Since we are defining many sets, refer to Figures 7 and 8 for clarity. The figures cover the cases where $Z = \emptyset$ and $Z \neq \emptyset$ separately.

(Covering some of B_{rest}.) Let M' be a maximal matching from $\tilde{A} \setminus M_1$ to B_{rest}. There are several cases for this construction.

- If $Z \neq \emptyset$, M' is either empty, or only consists of $(a_{\text{rest}}, b_{\text{rest}})$. If $k \geq C$, then $|\tilde{M}| = 0$, $|B_{\text{rest}}| = |B| - k - K$. Furthermore, since every node b in $B \setminus (H \cup M_2)$ has at least $out_B(b) + 1 - h = out_B(b) - (h-1)$ inneighbors from $A \setminus M_1$, by Theorem 3 we have that

$$\begin{aligned} |M'| &\geq (|B| - K - k - 2h + 2 - 1)/2 \\ &= (|B| - 2K - h + 1)/2. \end{aligned}$$

- If $Z = \emptyset$ and $k < C$, then $2(C - k)$ nodes of $B \setminus H \setminus M_2$ are matched to each other, and so by Theorem 3

$$\begin{aligned} |M'| &\geq (|B| - 2C + 2k - K - k - 2h + 2 - 1)/2 \\ &= (|B| - 2C - 3h + 1)/2. \end{aligned}$$

  Every node in B_{rest} has some source of M' or M_1 as an inneighbor.

(Matching $\mathcal{A}$ to some node.) Consider $A' = \tilde{A} \setminus (M_1 \cup M')$.

- If $Z \neq \emptyset$, then

$$\begin{aligned} |A'| &\geq n - 1 - |B| - |A \cap M_1| - 1 - |Z| \\ &\geq n - 2 - 2h - 2C \\ &\geq n - 2 - 2(n-6)/7 - 2C \\ &= (5n - 2 - 14C)/7 > 1 \end{aligned}$$

  Hence when $Z \neq \emptyset$, there is some $a' \in \tilde{A} \setminus (M_1 \cup M')$ that we can match $\mathcal{A}$ to.

- If $Z = \emptyset$. Then

$$\begin{aligned} |\tilde{A} \setminus M_1| &= \\ |A| - h &\geq \\ (n - K)/3 - h &\geq \\ (n - 4K)/3 &> 1 \end{aligned}$$

  If there is some $a' \in A'$, then match $\mathcal{A}$ to a'. Otherwise, $|M'| \geq 1$. Dislodge some edge (a', b') from M'. Since $out(\mathcal{A})$ and $in(\mathcal{A})$ have different parities and $A' = \emptyset$, the number of leftover unmatched elements of B after we add b' to them is even. Hence any matching we use to complete the first round of the tournament would be perfect on them. Even after removing a' from M' any surviving element b from the leftover B elements will have at least $out_B(b) \geq 1$ surviving inneighbors. The lower bounds we had computed for $|M'|$ go down by 1:

  - when $k < C$ and $Z = \emptyset$, $|M'| \geq (|B| - 2C - 3h - 1)/2$
  - when $k \geq C$, $|M'| \geq (|B| - 2K - h - 1)/2$ when $k \geq C$

Now let $S = A' \setminus \{a'\}$ and let T be the subset of B_T consisting of the nodes that do not have inneighbors among the sources of M' and M_1. Every subset Q of the nodes of T has at least $\binom{|Q|}{2} + 2|Q| - |Q| = \binom{|Q|}{2} + |Q|$ inneighbors in S since each node in Q can have lost at most one inneighbor, a'.

(Handling T' and completing round 1.) Let $t \in B_T$ be the node with highest outdegree in B_T. Running Algorithm 1 on S, T, t produces a matching M'' on the nodes of S so that almost all nodes of T are covered by sources S' of M'' except for a subset $T' \subset T$ with $|T'| \leq 1 + 2\sqrt{|T|}$. Further, there are at least $in_S(t)/2 - 1 \geq (|B_T| - 1)/4 - 1$ sources of M'' that beat t. The addition of M'' to the rest of our construction completes the first round matching. Call the graph induced by the surviving nodes G'.

Let P be the closest power of 2 greater than $3 + 4\sqrt{|T|}$. Then $P \leq 5 + 8\sqrt{|T|}$. Suppose that $|B_T| \geq 5 + 8\sqrt{|T|}$. This is true whenever $|B_T| \geq 81$, and since $|B_T| \geq C = 529 > 81$ the assumption is true. There exists a subtournament T_t of B_T such that $T' \cup \{t\} \in T_t$ and t is a king in T_t of outdegree at least $|T_t|/2$ and $|T_t| = P$, a power of 2.

If $t \in N^{out}(\mathcal{A})$, then we will not need its surviving inneighbors from S. In the following we handle the more complicated case when $t \in N^{in}(\mathcal{A})$, and so at least $in_S(t)/2 - 1$ inneighbors of t from S are in G'. We need that $(|B_T| - 1)/4 - 1 \geq 5 + 8\sqrt{|T|}$. This is true when $|B_T| \geq 529 = C$. Then there exists a subset of the surviving nodes of $N_S^{in}(t)$ of size at least $P = |T_t|$. Now we can apply Lemma 1 to show that $\mathcal{A}$ can win a single-elimination tournament, provided that $out_{G'}(\mathcal{A}) \geq in_{G'}(\mathcal{A})$.

(Showing $out_{G'}(\mathcal{A}) \geq in_{G'}(\mathcal{A})$.) Recall that $|M'| \geq (|B| - 2C - 3h - 1)/2$ when $k < C$ but $Z = \emptyset$ and $|M'| \geq (|B| - 2K - h - 1)/2$ when $k \geq C$. We have three cases.

1. $k \geq C$, and so $Z = \emptyset$.

   Here, the number of nodes of $N^{out}(a)$ that survive is at least $\lfloor(|A| + |M'| + |M_1| - 1)/2\rfloor$. Meanwhile, the number of nodes of $N^{in}(\mathcal{A})$ that survive is at most $\lceil(|B| - |M_1| - |M'|)/2\rceil$. It suffices to show that

   $$|A| + |M'| + |M_1| - 1 \geq |B| - |M_1| - |M'|.$$

   This happens when $|A| + 2|M'| + 2h - 1 \geq |B|$. By the assumptions of this case $|M'| \geq (|B| - 2K - h - 1)/2$, so we must show that

   $$|A| + |B| - 2K - h + 2h - 2 \geq |B|,$$

   or that

   $$|A| - 2K + h - 2 \geq 0.$$

   By Lemma 5, we know that $|A| \geq (n - K)/3$ so we just need that $(n - K) \geq 6K + 6$. This simplifies exactly to the assumption of the main theorem that $K \leq (n - 6)/7$.

2. $k < C$ and $Z = \emptyset$.

   In this situation, it still suffices to show that $|A| + 2|M'| + 2h - 1 \geq |B|$. However, now $|M'| \geq (|B| - 2C - 3h - 1)/2$. Combining these, we find we only need that $|A| - 2C - 3h + 2h - 2 \geq 0$, or equivalently that $n - 6C - 6 \geq K + 3h$. Simplifying this, we only need that $K \leq (n - 6C - 6)/4$, which is true since $(n - 6C - 6)/4 > (n - 6)/7$.

3. $Z \neq \emptyset$.

   If $Z \neq \emptyset$ then $|B| < h + 2C$, and at most C nodes of $B \cup Z$ survive. The number of nodes of $A \setminus Z$ that survive is at least

   $$\begin{aligned} \lfloor(|A| - |Z| + h + |M'| - 1)/2\rfloor &\geq \\ (|A| - C - 2 + K - C)/2 &= \\ (|A| + K)/2 - C - 1. \end{aligned}$$

   We need only that $(|A| + K) - 2C - 2 \geq 2C$. After applying Lemma 5, this becomes that $(n - K)/3 + K \geq 4C + 2$, and $n + 2K \geq 12C + 6$. It is true that $n \geq 12C + 6$ since $n_0 > 12C + 6$.

This covers all of the cases and concludes the proof. We have given the construction for a matching $M \cup \tilde{M} \cup M' \cup M''$ such that the conditions for Lemma 1 apply to the node $\mathcal{A}$ in the subtournament induced over the sources of our matching. □

We can state this result in terms of the size of $\mathcal{H}(\mathcal{A})$ instead of $\mathcal{H}^{in}(\mathcal{A})$ by applying Lemma 4 to lower bound the size of the initial set A.

Corollary 2. *There exists a constant n_0 so that for all tournaments G on $n > n_0$ nodes the following holds. Let $\mathcal{A}$ be among the top $(6n + 7)/31 \geq .19n$ highest outdegree nodes. If there is a matching from $V \setminus \mathcal{H}^{in}(\mathcal{A})$ onto $\mathcal{H}^{in}(\mathcal{A})$, then $\mathcal{A}$ is an SE winner.*