Computational Tool for Cyber-Physical Systems

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by

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Abstract

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Cyber–Physical systems, which is the class of dynamical systems where physical and computational components interact in a tight coordination, are found in many applications, from large–scale distributed systems, such as the electric power grid, to micro–robotic platforms based on legged locomotion, among many others. Due to their mixed nature between physical and computational components, Cyber–Physical systems are well modeled using hybrid dynamical models, which incorporate both continuous and discrete valued state variables. Also, thanks to the flexibility and great variety of optimal control formulations, it is natural to apply optimal control algorithms to solve complex problems in the context of Cyber–Physical systems, such as the verification of a given specification, or the robust identification of parameters under state constraints.

This thesis presents three new computational tools that bring the strength of hybrid dynamical models and optimal control to applications in Cyber–Physical systems. The first tool is an algorithm that finds the optimal control of a switched hybrid dynamical system under state constraints, the second tool is an algorithm that approximates the trajectories of autonomous hybrid dynamical systems, and the third tool is an algorithm that computes the optimal control of a nonlinear dynamical system using pseudospectral approximations.

These results achieve several goals. They extend widely used algorithms to new classes of dynamical systems. They also present novel mathematical techniques that can be applied to develop new, computationally efficient, tools in the context of hybrid dynamical systems. More importantly, they enable the use of control theory in new exciting applications, that because of their number of variables or complexity of their models, cannot be addressed using existing tools.

Dedicado a Carolina.

Pero si ella vuelve, si ella vuelve,
que hermoso, que alocado.

Pues hay menos peces que nadan en el mar,
que los besos que le daré en su boca.

Dentro de mis brazos los abrazos han
de ser millones de abrazos.
Apretando, pegados, callados,
abrazos y besitos y caricias sin fin.

—Vinícius de Moraes, *Chega de Saudade*.

Contents

List of Figures

List of Tables

List of Algorithms

Acknowledgments

My experience as a graduate student at UC Berkeley was nothing less than an adventure. I arrived to Berkeley six years ago certain that this was a great opportunity, but with little knowledge about what I wanted to do in the future, or what research area would better fit my interests and abilities. Now I can say that the community at Berkeley not only received me with open arms, in addition it offered me some of the best years of my life, guided me towards my future in academia as a professor, and more importantly, pushed me to achieve my greatest. The list of people that I want to acknowledge due to their great influence during these six years is very long, so I will summarize it in the next few paragraphs.

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Chapter 1

Introduction

Control theory lives in the boundary between mathematics and engineering, taking problems from engineering to then solve them using tools obtained from mathematics. As such, the research in this field is therefore affected by changes in either of these branches of science. Indeed, for many years the community focused most of its efforts in the analysis of linear dynamical systems using frequency domain tools, such as the Laplace transform and the Z–transform, until major developments in state-space models complemented, and in many cases outperformed, the existing results. Similarly, linear models dominated the literature, due to their wide applications to electrical circuits and chemical processes, until it became apparent that new, inherently nonlinear, models from aerospace applications needed completely new tools, opening the doors to the development of control theory for nonlinear systems.

From this perspective, we can argue that the latest developments in control theory have been inspired by several factors. For example, convex optimization introduced numerical tools that transformed the computation of some problems, such as robust H_∞–control and finite time LQ–control, into trivial exercises. Also, the surge of networked computer platforms introduced variables beyond the scope of the dominant dichotomy between linear and nonlinear models, such as the influence in communication channels, delayed signals, and distributed behavior, among others. In general, we can say that the latest developments in control theory have in common that, either they consider dynamical systems with discrete–valued states induced by computational components in them, or they apply computational tools to solve problems that cannot be clearly solved using pure analytical tools. Moreover, these new trends greatly enlarged the spectrum of applications where control theory could be used, mostly in the areas of robotics and machine learning.

Nevertheless, there exists a wide range of problems where the tools that control theory can offer are still in their infancy. This thesis is immersed within this framework, presenting three new computational tools that aim to increase the number and types of applications where control theory can provide solutions. Our results are motivated by applications in Cyber–Physical systems, which is the class of systems where physical and computational components work in a tight coordination, with two goals in mind: impose as few assumptions as possible in the dynamical model of the system, and produce solutions that can be used

in real–time applications.

The following sections present contextual information that is fundamental to the understanding and analysis of the different computational tools described in this document. We begin with the definition of a Cyber–Physical system in Section 1.1. We continue with the definition of a Hybrid Dynamical system in Section 1.2. Then, we define an Optimal Control Problem and we present some of its well known properties in Section 1.3. Finally, we enumerate the main contributions of this thesis in Section 1.4.

1.1 Cyber–Physical Systems

As we mention above, a Cyber–Physical system is commonly defined as a system where its physical and computational components work in a tight coordination. This definition aims to differentiate Cyber–Physical systems from both classical dynamical systems, modeled using ordinary differential equations, and purely computational systems, modeled using finite–state machines. But this definition goes beyond the modeling tools used to mathematically describe the system. Cyber–Physical systems are present in every intersection between computer technology and physical dynamics, ranging from the electrical power grid to autonomous vehicles, including prosthetic devices, air–traffic control systems, and HVAC installations, among many others. Therefore the definition of Cyber–Physical system accounts for most dynamical systems that involve modern technology and are immersed in the society.

A very particular property of all Cyber–Physical systems is their dependence on data produced by sensors in real–time. The interface between the cybernetic part and the physical part of every Cyber–Physical system is governed by sensors, transforming physical information into cybernetic data, and actuators, modulating physical energy from cybernetic commands. From a control perspective, sensor data comes from many different sources, unlike in classical applications where sensors only measured signals produced by the system itself. From a computational perspective, the calculations have to be synchronized with the physical system, and the results of a given actuation can only be understood as they interact with a physical process (for example, in path planning for autonomous vehicles a computational algorithm can produce trajectories that the physical process cannot follow). Hence, Cyber–Physical systems not only introduce a new type of classification, but also introduce exciting new problems that need to be addressed with new fundamental results by the control theory community.

A few examples of Cyber–Physical systems are:

Electric Power Grid: This is probably one of the most important Cyber–Physical system currently in existence. Ranging from its description as a coupled circuit between generators and loads via distributed parameter transmission lines, to the scheduling problem that decides who and how the circuit is connected, there exists a great number of dynamical problems in the context of the electric grid that fall within the scope of Cyber–Physical systems.

Perhaps the most interesting problems in the electric power grid today, from a Cyber–Physical perspective, are the security issues in a mostly networked system, i.e. whether it is possible to turn the electric grid unsafe by affecting the networked software in its controlling network of computers, and the incorporation of distributed sources of energy, such as wind and solar energy, using smart–meters and modern power–electronic devices.

Robotic Legged Locomotion: Most of the ground vehicles developed by the robotics community are based in wheeled locomotion due to its simplicity and robustness across many applications. But legged locomotion has clear benefits in some scenarios, the most important being micro–robots and prosthetic devices. Even though some tools from classical control theory can be applied to legged locomotion, its dynamics are inherently non–continuous due to the instantaneous change of speed at the impacts, hence embedded computers and new mathematical tools are required to handle real–time sensing and control in these platforms.

Autonomous Vehicles: Vehicles of all kinds, either aerial, ground, or underwater, can be controlled using classical control theory, applied to relatively simple models, when they are in isolation. But whenever obstacles, static or dynamic, are considered, the problem of controlling an autonomous vehicle becomes much harder. Today we have the technology to acquire and process data from heterogeneous classes of sensors in real–time, enabling us to explore new control strategies. Interesting problems that arise from autonomous vehicles are path planning under uncertainty, security of the computer platforms in the vehicle to malicious software, and the interaction between automation and humans, among others.

The present thesis addresses the challenge of controlling and analyzing Cyber–Physical systems in two ways: mathematically, modeling Cyber–Physical systems using hybrid dynamical models, and computationally, developing new optimal control algorithms to solve problems involving hybrid dynamical models. The next sections are dedicated to present more details about hybrid dynamical models and the optimal control of dynamical systems.

1.2 Hybrid Dynamical Models

We say that a dynamical system is hybrid when its state contains both continuous–valued and discrete–valued variables. In classical systems, the dynamics of continuous variables are usually modeled using ordinary differential equations, and the dynamics of discrete variables are modeled using finite–state machines. Hence, it is natural that hybrid dynamical models incorporate both of these theories in a single framework.

Among all the possible classes of hybrid dynamical models, two types are of particular interest in this thesis:

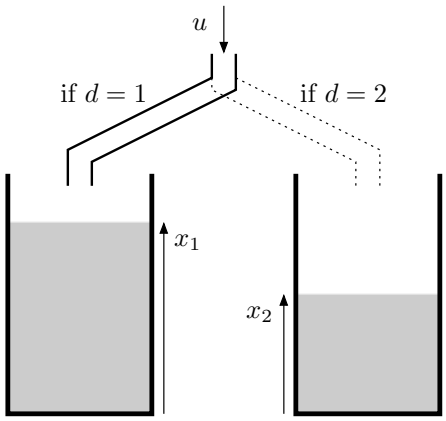


Figure 1.1: Double water tank system illustration, where x_1 and x_2 are the water height of tanks 1 and 2, u is the input flow of water, and d is the decision variable that chooses which tank receives the water.

- We say that a hybrid model is *switched* if its discrete variables are completely controlled, i.e. if the discrete variables can change arbitrarily regardless of the values of all the other variables. In this case, the discrete variables behave as an input that, for each instant of time, map to a discrete set.

  For example, consider a double water tank system, as shown in Figure 1.1. The input flow of water is denoted by u, and the height of the water in each tank is denoted x_1 and x_2, respectively. The system has two discrete modes: either the input flow is directed to tank 1, or it is directed to tank 2. Hence, the vector field is:

$$f(x, u, 1) = \begin{pmatrix} u \\ 0 \end{pmatrix}, \qquad f(x, u, 2) = \begin{pmatrix} 0 \\ u \end{pmatrix}, \tag{1.1}$$

  where the last argument is the discrete input indicating which tank receives the input flow.

- The dual of a switched hybrid model is a hybrid model where the discrete variable changes only as a functions of the state of the system. We say that this is an *autonomous* hybrid system, and in this case the discrete variables evolve according to the transitions in a directed graph, where the nodes of such graph are all the possible discrete modes of operation.

  The most used example for this class of hybrid system is the bouncing ball, as shown in Figure 1.2. Consider a ball with height p and velocity v under a gravitational field g. Whenever $p > 0$, the dynamics are governed by the following differential equation:

$$\begin{pmatrix} \dot{p} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -g \end{pmatrix}. \tag{1.2}$$

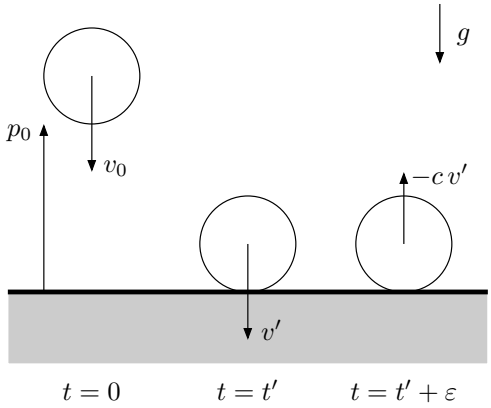


Figure 1.2: Bouncing ball illustration at three instants of time, with a bounce at $t = t'$. Left: initial condition. Center: before the bounce at $p = 0$. Right: after the bounce at $p = 0$.

When $p = 0$ and $v < 0$, the ball bounces by changing its velocity instantaneously to $-c\,v$, where $c \in [0, 1]$ is the coefficient of restitution that models the loss of energy due to the impact. The bouncing ball is a particular type of hybrid model because it has only one discrete mode of operation, similar to classical differential equation based models, but since the velocity is discontinuous it cannot be modeled using classical tools.

Hybrid dynamical models provide a rich new framework to describe phenomena that cannot be modeled only using differential equations, but at the same time it introduces many new theoretical challenges. Just to name a few, the trajectories produced by hybrid systems are not necessarily unique, nor they are continuous with respect to initial conditions and continuous inputs, all of which are standard properties for nonlinear dynamical systems. These challenges are one of the main forces behind the surge of new tools specifically designed for hybrid systems, as the ones presented in this thesis.

In Chapter 2 we develop an algorithm for the optimal control of switched hybrid systems with nonlinear dynamics and state constraints. In Chapter 3 we develop an algorithm for the approximation of trajectories of autonomous hybrid systems, whenever the state variables evolve in a Riemannian manifold.

1.3 Optimal Control of Dynamical Systems

The problem of finding the optimal control of a dynamical system can be regarded as a particular case in the field of calculus of variations, which deals with finding the extrema of mappings whose domain are functional spaces. In particular, the problem of optimal control deals with finding a control law for a given dynamical system such that certain criteria, defined by constraints, are satisfied and an objective function, whose range is the reals, is

minimized. This formulation is fairly general, and its applications to dynamical systems range from control to the identification of unknown parameters, as well as the verification of desired performance guarantees.

The history of the calculus of variations can be dated back to the 3rd century BC, when Dido, who later became the first Queen of Carthage, was told to take as much land as could be covered by an oxhide. She proceeded to cut the oxhide into tiny strips, and she used them to encircle an entire hill. By doing that, Queen Dido solved the Isoperimetric Problem, that is, she found the plane figure of the largest possible area given a fixed perimeter. The solution to this problem was known by the ancient Greeks, but a formal proof did not appear until the 19th century with the use of the calculus of variations.

It is commonly agreed that the modern history of the calculus of variations begins with the Brachistochrome Problem, originally stated by Johann Bernoulli in the late 17th century. This problem can be stated as follows: find the shape of a wire connecting two points such that a frictionless bead on this wire, starting from the higher point and under the action of gravity, can cover the distance to the second point in minimum time. The same Johann Bernoulli solved the problem showing that the optimal shape for the wire is the Cycloid Curve.

These are just two examples of the versatility of the calculus of variations. It is worth noting that using present–time techniques, both of these problems can be easily formulated as optimal control problems.

Optimal control emerged as a distinct field of research during the 1950s, with aerospace engineering being the main source of problems that could not be addressed with the existing tools at that time. But its mathematical foundations were not set into place until Lev Pontryagin published his book in 1962, stating the most general form of optimality condition for the solutions of optimal control problems known today.

Within the scope of Cyber–Physical systems, the use of optimal control algorithms can lead to a jump in the number of problems that can be solve. For example, using optimal control we can formulate the scheduling problem of the electric power grid with guarantees about safety and performance, we can also formulate the parameter identification problem for dynamical models of robotic legged locomotion, and we can formulate the path planning problem for autonomous vehicles embedded in uncontrolled environments. But there exists a gap between the formulation of these problems and our ability to solve them, since the algorithms that currently exist either are not compatible with hybrid dynamical models, or they do not scale to sizes where they become useful in practice.

1.4 Our Contribution

This thesis lives in the intersection between the concept of Cyber–Physical systems, the principles of hybrid dynamical modeling, and the tools used to solve optimal control problems. Our main goal is to provide new tools that can solve problems relevant to the community of Cyber–Physical systems using hybrid dynamical models and optimal control.

In Chapter 2 we propose a new algorithm that finds the optimal control of switched dynamical systems under state constraints. The main feature of this algorithm is that, based on our results, it improves the speed of computation by at least an order of magnitude when compared with currently used algorithms. Moreover, together with this great speed improvement, we prove that the solutions of our numerical implementation converge to the solutions of the original, infinite dimensional, problem. In Chapter 3 we present an algorithm for the simulation of autonomous hybrid dynamical systems. This algorithm is an extension of the well–known Forward Euler discretization, from which it retains its simple implementation. We show that, under general assumptions, the trajectories produced by our algorithm converge to the real trajectories of the hybrid system. To the best of our knowledge, it is the first time that an algorithm with this property is published. Finally, in Chapter 4 we present an algorithm that finds the optimal control of nonlinear dynamical systems using pseudospectral approximations. Pseudospectral approximations have been shown to greatly increase the speed of computation in practical experiments, but their theoretical properties are still an open area for research. In our result we show that it is possible to create a numerical algorithm with desirable properties in terms of convergence, but a price must be paid in the way the approximation is formulated.

Chapter 2

Optimal Control of Switched Dynamical Systems

Hybrid dynamical models arise naturally in systems in which discrete modes of operation interact with continuous state evolution. Such systems have been used in a variety of modeling applications including automobiles and locomotives employing different gears [HR99; Rin+08], biological systems [GT01], situations where a control module has to switch its attention among a number of subsystems [LR01; RS04; WYB02], manufacturing systems [CPW01] and situations where a control module has to collect data sequentially from a number of sensory sources [Bro95; EW02]. In addition, many complex nonlinear dynamical systems can be decomposed into simpler linear modes of operation that are more amenable to analysis and controller design [FDF00; Gil+11].

Given their utility, there has been considerable interest in devising algorithms to perform optimal control of such systems. In fact, even Branicky et al.'s seminal work which presented many of the theoretical underpinnings of hybrid systems included a set of sufficient conditions for the optimal control of such systems using quasi–variational inequalities [BBM98]. Though compelling from a theoretical perspective, the application of this set of conditions to the construction of a numerical optimal control algorithm for hybrid dynamical systems requires the application of value iterations which is particularly difficult in the context of switched systems, wherein the switching between different discrete modes is specified by a discrete–valued input signal. The control parameter for such systems has both a discrete component corresponding to the schedule of discrete modes visited and two continuous components corresponding to the duration of time spent in each mode in the mode schedule and the continuous input. The determination of an optimal control for this class of hybrid systems is particularly challenging due to the combinatorial nature of calculating an optimal mode schedule.

Related Work

The algorithms to solve this switched system optimal control problem can be divided into two distinct groups according to whether they do or do not rely on the Maximum Principle [Pic98; Pon+62; Sus99a]. Given the difficulty of the problem, both groups of approaches sometimes employ similar tactics during algorithm construction. A popular such tactic is one formalized by Xu et al. [XA02] who proposed a bi-level optimization scheme that at a low level optimized the continuous components of the problem while keeping the mode schedule fixed and at a high level modified the mode schedule.

We begin by describing the algorithms for switched system optimal control that rely on the Maximum Principle. One of the first such algorithms, presented by Alamir et al. [AA04], applied the Maximum Principle directly to a discrete time switched dynamical system. In order to construct such an algorithm for a continuous time switched dynamical system, Shaikh et al. [SC03] employed the bi-level optimization scheme proposed by Xu et al. and applied the Maximum Principle to perform optimization at the lower level and applied the Hamming distance to compare different possible nearby mode schedules.

Given the algorithm that we construct in this chapter, the most relevant of the approaches that rely on the Maximum Principle is the one proposed by Bengea et al. [BD05] who relax the discrete–valued input and treat it as a continuous-valued input over which they can apply the Maximum Principle to perform optimal control. A search through all possible discrete valued inputs is required in order to find one that approximates the trajectory of the switched system due to the application of the constructed relaxed discrete–valued input. Though such a search is expensive, the existence of a discrete–valued input that approximates the behavior of the constructed relaxed discrete–valued input is proven by the Chattering Lemma [Ber74]. Moreover, this combinatorial search is unavoidable by employing the Chattering Lemma since it provides no means to construct a discrete–valued input that approximates a relaxed discrete–valued input with respect the trajectory of the switched system. Unfortunately their numerical implementation for nonlinear switched systems is fundamentally restricted due to their reliance on approximating strong or needle variations with arbitrary precision as explained in [MP75].

Next, we describe the algorithms that do not rely on the Maximum Principle but rather employ weak variations. Several have focused on the optimization of autonomous switched dynamical systems (i.e. systems without a continuous input) by fixing the mode sequence and working on devising first [EWA06] and second order [JM11] numerical optimal control algorithms to optimize the amount of time spent in each mode. In order to extend these optimization techniques, Axelsson et al. [Axe+08] employed the bi-level optimization strategy proposed by Xu et al., and after performing optimization at the lower-level by employing a first order numerical optimal control algorithm to optimize the amount of time spent in each mode while keeping the mode schedule fixed, they modified the mode sequence by employing a single mode insertion technique.

There have been two major extensions to Axelsson et al.'s algorithm. First, Wardi et al. [WE12b], extend the approach by performing several single mode insertions at each

iteration. Second, Gonzalez et al. [Gon+10a; Gon+10b], extend the approach to make it applicable to constrained switched dynamical systems with a continuous-valued input. Though these single mode insertion techniques avoid the computational expense of considering all possible mode schedules during the high-level optimization, this improvement comes at the expense of restricting the possible modifications of the existing mode schedule, which may introduce undue local minimizers, and at the expense of requiring a separate optimization for each of the potential mode schedule modifications, which is time consuming.

Our Contribution and Organization

Inspired by the potential of the Chattering Lemma, in this chapter, we devise and implement a first order numerical optimal control algorithm for the optimal control of constrained nonlinear switched systems. The contents of this chapter are based on the results presented in [Vas+12]. In Section 2.1, we introduce the notation and assumptions used throughout the chapter and formalize the the optimal control for constrained nonlinear switched systems. Our approach to solve this problem, which is formulated in Section 2.2, first relaxes the optimal control problem by treating the discrete–valued input to be continuous-valued. Next, a first order numerical optimal control algorithm is devised for this relaxed problem. After this optimization is complete, an extension of the Chattering Lemma that we construct, allows us to design a projection that takes the computed relaxed discrete–valued input back to a "pure" discrete–valued input while controlling the quality of approximation of the trajectory of the switched dynamical system generated by applying the projected discrete–valued input rather than the relaxed discrete–valued input. In Section 2.3, we prove that the sequence of points generated by recursive application of our first order numerical optimal control algorithm converge to a point that satisfies a necessary condition for optimality of the constrained nonlinear switched system optimal control problem.

We then describe in Section 2.4 how our algorithm can be formulated in order to make numerical implementation feasible. In fact, in Section 2.5, we prove that the this computationally implementable algorithm is a *consistent approximation* of our original algorithm. This ensures that the sequence of points generated by the recursive application of this numerically implementable algorithm converge to a point that satisfies a necessary condition for optimality of the constrained nonlinear switched system optimal control problem. In Section 2.6, we implement this algorithm and compare its performance to a commercial mixed integer optimization algorithm on four separate problems to illustrate its superior performance with respect to speed and quality of constructed minimizer.

2.1 Preliminaries

In this section, we formalize the problem we solve in this chapter. Before describing this problem, we define the function spaces and norms used throughout this chapter.

Norms and Functional Spaces

This chapter focuses on the optimization of functions with finite L^2-norm and finite bounded variation. To formalize this notion, we require a norm. For each $x \in \mathbb{R}^n$, $p \in \mathbb{N}$, and $p > 0$, we let $\|x\|_p$ denote the p–norm of x. For each $A \in \mathbb{R}^{n\times m}$, $p \in \mathbb{N}$, and $p > 0$, we let $\|A\|_{i,p}$ denote the induced p–norm of A.

Given these definitions, we say a function, $f : [0,1] \to \mathcal{Y}$, where $\mathcal{Y} \subset \mathbb{R}^n$, belongs to $L^2([0,1],\mathcal{Y})$ with respect to the Lebesgue measure on $[0,1]$ if:

$$\|f\|_{L^2} = \left(\int_0^1 \|f(t)\|_2^2\, dt\right)^{\frac{1}{2}} < \infty. \tag{2.1}$$

We say a function, $f : [0,1] \to \mathcal{Y}$, where $\mathcal{Y} \subset \mathbb{R}^n$, belongs to $L^\infty([0,1],\mathcal{Y})$ with respect to the Lebesgue measure on $[0,1]$ if:

$$\|f\|_{L^\infty} = \inf\big\{\alpha \in [0,\infty) \mid \|f(x)\|_2 \le \alpha \text{ for almost every } x \in [0,1]\big\} < \infty. \tag{2.2}$$

In order to define the space of functions of finite bounded variation, we first define the total variation of a function. Given P, the set of all finite partitions of $[0,1]$, we define the *total variation* of $f : [0,1] \to \mathcal{Y}$ by:

$$\mathrm{V}(f) = \sup\left\{\sum_{j=0}^{m-1} \|f(t_{j+1}) - f(t_j)\|_1 \mid k \in \mathbb{N},\ \{t_k\}_{k=0}^m \in P\right\}. \tag{2.3}$$

We say that f is of *bounded variation* if $\|f\|_{BV} < \infty$, and we define $BV([0,1],\mathcal{Y})$ to be the set of all functions of bounded variation from $[0,1]$ to $\mathcal{Y}$.

There is an important connection between the functions of bounded variation and weak derivatives, which we rely on throughout this chapter. Given $f : [0,1] \to \mathcal{Y}$, we say that f has a *weak derivative* if there exists a Radon signed measure μ over $[0,1]$ such that, for each smooth bounded function v with $v(0) = v(1) = 0$,

$$\int_0^1 f(t)\dot{v}(t)dt = -\int_0^1 v(t)d\mu(t). \tag{2.4}$$

Moreover, we say that $\dot{f} = \frac{d\mu(t)}{dt}$, where the derivative is taken in the Radon–Nikodym sense, is the weak derivative of f. Note that $\dot{f}$ is in general a distribution, thus it only makes sense as an element in the dual space of L^1. Perhaps the most common example of weak derivative is the Dirac Delta distribution, which is the weak derivative of the Step Function. The following result is fundamental in our analysis of functions of bounded variation:

Theorem 2.1 (Exercise 5.1 in [Zie89])**.** *If $f \in BV([0,1],\mathcal{Y})$, then f has a weak derivative, denoted by $\dot{f}$. Moreover,*

$$\mathrm{V}(f) = \int_0^1 \left\|\dot{f}(t)\right\|_1 dt. \tag{2.5}$$

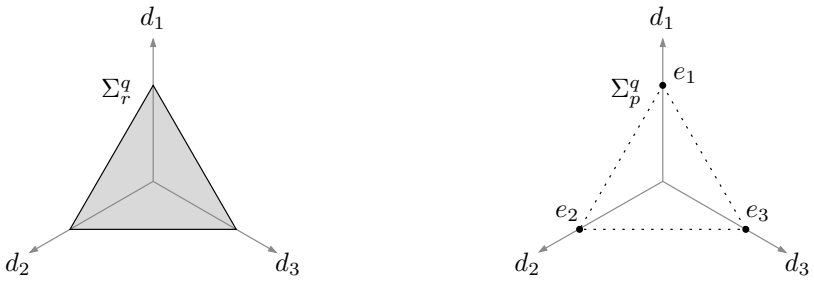


Figure 2.1: Illustration of Σ_r^q and Σ_p^q for the case $q = 3$.

We omit the proof of this result since it is beyond the scope of this chapter. More details about the functions of bounded variation and weak derivatives can be found in Sections 3.5 and 9 in [Fol99] and Section 5 in [Zie89].

Optimization Spaces

We are interested in the control of systems whose trajectory is governed by a set of vector fields $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{Q} \to \mathbb{R}^n$, indexed by their last argument where $\mathcal{Q} = \{1, 2, \ldots, q\}$. Each of these distinct vector fields is called a *mode* of the switched system. To formalize the optimal control problem, we define three spaces: the *pure discrete input space*, $\mathcal{D}_p$, the *relaxed discrete input space*, $\mathcal{D}_r$, and the *continuous input space*, $\mathcal{U}$. Throughout the document, we employ the following convention: given the pure or relaxed discrete input d, we denote its i–th coordinate by d_i.

Before formally defining each space, we require some notation. Let the q–simplex, Σ_r^q, be defined as:

$$\Sigma_r^q = \left\{ (d_1, \ldots, d_q) \in [0,1]^q \mid \sum_{i=1}^{q} d_i = 1 \right\}, \tag{2.6}$$

and let the corners of the q-simplex, Σ_p^q, be defined as:

$$\Sigma_p^q = \left\{ (d_1, \ldots, d_q) \in \{0,1\}^q \mid \sum_{i=1}^{q} d_i = 1 \right\}. \tag{2.7}$$

Note that $\Sigma_p^q \subset \Sigma_r^q$. Also, there are exactly as many corners, denoted e_i for $i \in \mathcal{Q}$, of the q–simplex as there are distinct vector fields. Thus, $\Sigma_p^q = \{e_1, \ldots, e_q\}$. An illustration of the sets Σ_r^q and Σ_p^q is presented in Figure 2.1.

Using this notation, we define the pure discrete input space, $\mathcal{D}_p$, as:

$$\mathcal{D}_p = L^2([0,1], \Sigma_p^q) \cap BV([0,1], \Sigma_p^q). \tag{2.8}$$

Next, we define the relaxed discrete input space, $\mathcal{D}_r$:

$$\mathcal{D}_r = L^2([0,1], \Sigma_r^q) \cap BV([0,1], \Sigma_r^q). \tag{2.9}$$

Notice that the discrete input at each instance in time can be written as the linear combination of the corners of the simplex. Given this observation, we employ these corners to index the vector fields, i.e. for each $i \in \mathcal{Q}$ we write $f(\cdot,\cdot,\cdot,e_i)$ for $f(\cdot,\cdot,\cdot,i)$. Finally, we define the continuous input space, $\mathcal{U}$:

$$\mathcal{U} = L^2([0,1],U) \cap BV([0,1],U), \tag{2.10}$$

where $U \subset \mathbb{R}^m$ is a bounded, convex set.

Let $\mathcal{X} = L^\infty([0,1],\mathbb{R}^m) \times L^\infty([0,1],\mathbb{R}^q)$ be endowed with the following norm for each $\xi = (u,d) \in \mathcal{X}$:

$$\|\xi\|_{\mathcal{X}} = \|u\|_{L^2} + \|d\|_{L^2}, \tag{2.11}$$

where the L^2–norm is as defined in Equation (2.1). We combine $\mathcal{U}$ and $\mathcal{D}_p$ to define our *pure optimization space*, $\mathcal{X}_p = \mathcal{U} \times \mathcal{D}_p$, and we endow it with the same norm as $\mathcal{X}$. Similarly, we combine $\mathcal{U}$ and $\mathcal{D}_r$ to define our *relaxed optimization space*, $\mathcal{X}_r = \mathcal{U} \times \mathcal{D}_r$, and endow it with the $\mathcal{X}$–norm too. Note that $\mathcal{X}_p \subset \mathcal{X}_r \subset \mathcal{X}$.

Trajectories, Cost, Constraint, and the Optimal Control Problem

Given $\xi = (u,d) \in \mathcal{X}_r$, for convenience throughout the chapter we let:

$$f\big(t,x(t),u(t),d(t)\big) = \sum_{i=1}^{q} d_i(t) f\big(t,x(t),u(t),e_i\big), \tag{2.12}$$

where $d(t) = \sum_{i=1}^{q} d_i(t) e_i$. We employ the same convention when we consider the partial derivatives of f. Given $x_0 \in \mathbb{R}^n$, we say that a *trajectory of the system* corresponding to $\xi \in \mathcal{X}_r$ is the solution to:

$$\dot{x}(t) = f\big(t,x(t),u(t),d(t)\big), \quad \forall t \in [0,1], \quad x(0) = x_0, \tag{2.13}$$

and denote it by $x^{(\xi)} : [0,1] \to \mathbb{R}^n$, where we suppress the dependence on x_0 in $x^{(\xi)}$ since it is assumed given. To ensure the clarity of the ensuing analysis, it is useful to sometimes emphasize the dependence of $x^{(\xi)}(t)$ on ξ. Therefore, we define the *flow of the system*, $\varphi_t : \mathcal{X}_r \to \mathbb{R}^n$ for each $t \in [0,1]$ as:

$$\varphi_t(\xi) = x^{(\xi)}(t). \tag{2.14}$$

To define the cost function, we assume that we are given a *terminal cost*, $h_0 : \mathbb{R}^n \to \mathbb{R}$. The *cost function*, $J : \mathcal{X}_r \to \mathbb{R}$, for the optimal control problem is then defined as:

$$J(\xi) = h_0\big(x^{(\xi)}(1)\big). \tag{2.15}$$

Notice that if the problem formulation includes a running cost, then one can extend the existing state vector by introducing a new state, and modifying the cost function to evaluate

this new state at the final time, as shown in Section 4.1.2 in [Pol97]. By performing this type of modification, observe that each mode of the switched system can have a different running cost associated with it.

Next, we define a family of functions, $h_j : \mathbb{R}^n \to \mathbb{R}$ for $j \in \mathcal{J} = \{1, \ldots, N_c\}$. Given a $\xi \in \mathcal{X}_r$, the state $x^{(\xi)}$ is said to satisfy the constraint if $h_j(x^{(\xi)}(t)) \leq 0$ for each $t \in [0,1]$ and for each $j \in \mathcal{J}$. We compactly describe all the constraints by defining the *constraint function* $\Psi : \mathcal{X}_r \to \mathbb{R}$, by:

$$\Psi(\xi) = \max_{j \in \mathcal{J},\, t \in [0,1]} h_j\big(x^{(\xi)}(t)\big), \tag{2.16}$$

since $h_j\big(x^{(\xi)}(t)\big) \leq 0$ for each t and j if and only if $\psi(\xi) \leq 0$. To ensure the clarity of the ensuing analysis, it is useful to sometimes emphasize the dependence of $h_j\big(x^{(\xi)}(t)\big)$ on ξ. Therefore, we define *component constraint functions*, $\psi_{j,t} : \mathcal{X}_r \to \mathbb{R}$ for each $t \in [0,1]$ and $j \in \mathcal{J}$ as:

$$\psi_{j,t}(\xi) = h_j\left(\varphi_t(\xi)\right). \tag{2.17}$$

With these definitions, we can state the Switched System Optimal Control Problem:

$$\text{(SSOCP)} \qquad \min_{\xi \in \mathcal{X}_p} \left\{ J(\xi) \mid \Psi(\xi) \leq 0 \right\}. \tag{2.18}$$

Assumptions and Uniqueness

In order to devise an algorithm to solve Switched System Optimal Control Problem, we make the following assumptions about the dynamics, cost, and constraints:

Assumption 2.2. *For each $i \in \mathcal{Q}$, $f(\cdot,\cdot,\cdot,e_i)$ is differentiable in both x and u. Also, each $f(\cdot,\cdot,\cdot,e_i)$ and its partial derivatives are Lipschitz continuous with constant $L > 0$, i.e. given $t_1, t_2 \in [0,1]$, $x_1, x_2 \in \mathbb{R}^n$, and $u_1, u_2 \in U$:*

(1) $\|f(t_1,x_1,u_1,e_i) - f(t_2,x_2,u_2,e_i)\|_2 \leq L\left(|t_1 - t_2| + \|x_1 - x_2\|_2 + \|u_1 - u_2\|_2\right)$,

(2) $\left\|\frac{\partial f}{\partial x}(t_1,x_1,u_1,e_i) - \frac{\partial f}{\partial x}(t_2,x_2,u_2,e_i)\right\|_{i,2} \leq L\left(|t_1 - t_2| + \|x_1 - x_2\|_2 + \|u_1 - u_2\|_2\right)$,

(3) $\left\|\frac{\partial f}{\partial u}(t_1,x_1,u_1,e_i) - \frac{\partial f}{\partial u}(t_2,x_2,u_2,e_i)\right\|_{i,2} \leq L\left(|t_1 - t_2| + \|x_1 - x_2\|_2 + \|u_1 - u_2\|_2\right)$.

Assumption 2.3. *The functions h_0 and h_j are Lipschitz continuous and differentiable in x for all $j \in \mathcal{J}$. In addition, the derivatives of these functions with respect to x are also Lipschitz continuous with constant $L > 0$, i.e. given $x_1, x_2 \in \mathbb{R}^n$, for each $j \in \mathcal{J}$:*

(1) $|h_0(x_1) - h_0(x_2)| \leq L\|x_1 - x_2\|_2$,

(2) $\left\|\frac{\partial h_0}{\partial x}(x_1) - \frac{\partial h_0}{\partial x}(x_2)\right\|_2 \leq L\|x_1 - x_2\|_2$,

(3) $|h_j(x_1) - h_j(x_2)| \leq L\|x_1 - x_2\|_2$,

(4) $\left\|\frac{\partial h_j}{\partial x}(x_1) - \frac{\partial h_j}{\partial x}(x_2)\right\|_2 \leq L\|x_1 - x_2\|_2$.

If a running cost is included in the problem statement (i.e. if the cost also depends on the integral of a function), then this function must also satisfy Assumption 2.2. Assumption 2.3 is a standard assumption on the objectives and constraints and is used to prove the convergence properties of the algorithm defined in the next section. These assumptions lead to the following result:

Lemma 2.4. *There exists a constant $C > 0$ such that, for each $\xi \in \mathcal{X}_r$ and $t \in [0,1]$,*

$$\left\|x^{(\xi)}(t)\right\|_2 \leq C, \tag{2.19}$$

where $x^{(\xi)}$ is a solution of Differential Equation (2.13).

Proof. Given $\xi = (u,d) \in \mathcal{X}_r$ and noticing that $|d_i(t)| \leq 1$ for all $i \in \mathcal{Q}$ and $t \in [0,1]$, we have:

$$\left\|x^{(\xi)}(t)\right\|_2 \leq \|x_0\|_2 + \sum_{i=1}^{q} \int_0^t \left\|f\big(s, x^{(\xi)}(s), u(s), e_i\big)\right\|_2 ds. \tag{2.20}$$

Next, observe that $\|f(0, x_0, 0, e_i)\|_2$ is bounded for all $i \in \mathcal{Q}$ and $u(s)$ is bounded for each $s \in [0,1]$ since U is bounded. Then by Assumption 2.2, we know there exists a $K > 0$ such that for each $s \in [0,1]$, $i \in \mathcal{Q}$, and $\xi \in \mathcal{X}_r$,

$$\left\|f\big(s, x^{(\xi)}(s), u(s), e_i\big)\right\|_2 \leq K\Big(\left\|x^{(\xi)}(s)\right\|_2 + 1\Big). \tag{2.21}$$

Applying the Bellman-Gronwall Inequality (Lemma 5.6.4 in [Pol97]) to Equation (2.20), we have $\left\|x^{(\xi)}(t)\right\|_2 \leq e^{qK}\big(1 + \|x_0\|_2\big)$ for each $t \in [0,1]$. Since x_0 is assumed given and bounded, we have our result. □

In fact, this implies that the dynamics, cost, constraints, and their derivatives are all bounded:

Corollary 2.5. *There exists a constant $C > 0$ such that for each $\xi = (u,d) \in \mathcal{X}_r$, $t \in [0,1]$, and $j \in \mathcal{J}$:*

(1) $\left\|f\big(t, x^{(\xi)}(t), u(t), d(t)\big)\right\|_2 \leq C$, $\left\|\dfrac{\partial f}{\partial x}\big(t, x^{(\xi)}(t), u(t), d(t)\big)\right\|_{i,2} \leq C$, *and* $\left\|\dfrac{\partial f}{\partial u}\big(t, x^{(\xi)}(t), u(t), d(t)\big)\right\|_{i,2} \leq C$.

(2) $\left|h_0\big(x^{(\xi)}(t)\big)\right| \leq C$, *and* $\left\|\dfrac{\partial h_0}{\partial x}\big(x^{(\xi)}(t)\big)\right\|_2 \leq C$.

(3) $\left|h_j\big(x^{(\xi)}(t)\big)\right| \leq C$, *and* $\left\|\dfrac{\partial h_j}{\partial x}\big(x^{(\xi)}(t)\big)\right\|_2 \leq C$.

Where $x^{(\xi)}$ is a solution of Differential Equation (2.13).

Proof. The result follows immediately from the continuity of f, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial u}$, h_0, $\frac{\partial h_0}{\partial x}$, h_j, and $\frac{\partial h_j}{\partial x}$ for each $j \in \mathcal{J}$, as stated in Assumptions 2.2 and 2.3, and the fact that each of the arguments to these functions can be constrained to a compact domain, which follows from Lemma 2.4 and the compactness of U and Σ_r^q. □

An application of this corollary leads to a fundamental result:

Theorem 2.6. *For each $\xi \in \mathcal{X}_r$ Differential Equation* (2.13) *has a unique solution.*

Proof. First let us note that f, as defined in Equation (2.12), is also Lipschitz with respect to its fourth argument. Indeed, given $t \in [0,1]$, $x \in \mathbb{R}^n$, $u \in U$, and $d_1, d_2 \in \Sigma_r^q$,

$$\begin{aligned} \big\| f(t,x,u,d_1) - f(t,x,u,d_2) \big\|_2 &= \left\| \sum_{i=1}^{q} \big(d_{1,i} - d_{2,i}\big) f(t,x,u,e_i) \right\|_2 \\ &\leq Cq \|d_1 - d_2\|_2, \end{aligned} \tag{2.22}$$

where $C > 0$ is as in Corollary 2.5.

Given that f is Lipschitz with respect to all its arguments, the result follows as a direct extension of the classical existence and uniqueness theorem for nonlinear differential equations (see Section 2.4.1 in [Vid02] for a standard version of this theorem). □

Therefore, since $x^{(\xi)}$ is unique, it is not an abuse of notation to denote the solution of Differential Equation (2.13) by $x^{(\xi)}$. Next, we develop an algorithm to solve the Switched System Optimal Control Problem.

2.2 Optimization Algorithm

In this section, we describe our optimization algorithm. Our approach proceeds as follows: first, we treat a given pure discrete input as a relaxed discrete input by allowing it to belong $\mathcal{D}_r$; second, we perform optimal control over the relaxed optimization space; and finally, we project the computed relaxed input into a pure input. Before describing our algorithm in detail, we begin with a brief digression to motivate why such a roundabout construction is required in order to devise a first order numerical optimal control scheme for the Switched System Optimal Control Problem defined in Equation (2.18).

Directional Derivatives

To appreciate why the construction of a numerical scheme to find the local minima of the Switched System Optimal Control Problem defined in Equation (2.18) is difficult, suppose that the optimization in the problem took place over the relaxed optimization space rather

than the pure optimization space. The Relaxed Switched System Optimal Control Problem is then defined as:

$$\text{(RSSOCP)} \qquad \min_{\xi \in \mathcal{X}_r} \{J(\xi) \mid \Psi(\xi) \leq 0\}. \tag{2.23}$$

The local minimizers of this problem are then defined as follows:

Definition 2.7. *Let us denote an ε–ball in the $\mathcal{X}$–norm centered at ξ by:*

$$\mathcal{N}_{\mathcal{X}}(\xi, \varepsilon) = \left\{\bar{\xi} \in \mathcal{X}_r \mid \left\|\xi - \bar{\xi}\right\|_{\mathcal{X}} < \varepsilon\right\}. \tag{2.24}$$

We say that a point $\xi \in \mathcal{X}_r$ is a local minimizer of the Relaxed Switched System Optimal Control Problem *defined in Equation* (2.23) *if $\Psi(\xi) \leq 0$ and there exists $\varepsilon > 0$ such that $J(\hat{\xi}) \geq J(\xi)$ for each $\hat{\xi} \in \mathcal{N}_{\mathcal{X}}(\xi, \varepsilon) \cap \left\{\bar{\xi} \in \mathcal{X}_r \mid \Psi(\bar{\xi}) \leq 0\right\}$.*

Given this definition, a first order numerical optimal control scheme can exploit the vector space structure of the relaxed optimization space in order to define directional derivatives that find local minimizers for this Relaxed Switched System Optimal Control Problem.

To concretize how such an algorithm would work, we introduce some additional notation. Given $\xi \in \mathcal{X}_r$, $\mathcal{Y}$ a Euclidean space, and any function $G : \mathcal{X}_r \to \mathcal{Y}$, the directional derivative of G at ξ, denoted $\mathrm{D}G(\xi; \cdot) : \mathcal{X} \to \mathcal{Y}$, is computed as:

$$\mathrm{D}G(\xi; \xi') = \lim_{\lambda \downarrow 0} \frac{G(\xi + \lambda\xi') - G(\xi)}{\lambda}. \tag{2.25}$$

To understand the connection between directional derivatives and local minimizers, suppose the Relaxed Switched System Optimal Control Problem is unconstrained and consider the first order approximation of the cost J at a point $\xi \in \mathcal{X}_r$ in the $\xi' \in \mathcal{X}$ direction by employing the directional derivative $\mathrm{D}J(\xi; \xi')$:

$$J(\xi + \lambda\xi') \approx J(\xi) + \lambda \mathrm{D}J(\xi; \xi'), \tag{2.26}$$

where $0 \leq \lambda \ll 1$. It follows that if $\mathrm{D}J(\xi; \xi')$, whose existence is proven in Lemma 2.24, is negative, then it is possible to decrease the cost by moving in the ξ' direction. That is if the directional derivative of the cost at a point ξ is negative along a certain direction, then for each $\varepsilon > 0$ there exists a $\hat{\xi} \in \mathcal{N}_{\mathcal{X}}(\xi, \varepsilon)$ such that $J(\hat{\xi}) < J(\xi)$. Therefore if $\mathrm{D}J(\xi; \xi')$ is negative, then ξ is not a local minimizer of the unconstrained Relaxed Switched System Optimal Control Problem.

Similarly, for the general Relaxed Switched System Optimal Control Problem, consider the first order approximation of each of the component constraint functions, $\psi_{j,t}$ for each $j \in \mathcal{J}$ and $t \in [0, 1]$ at a point $\xi \in \mathcal{X}_r$ in the $\xi \in \mathcal{X}$ direction by employing the directional derivative $\mathrm{D}\psi_{j,t}(\xi; \xi')$:

$$\psi_{j,t}(\xi + \lambda\xi') \approx \psi_{j,t}(\xi) + \lambda \mathrm{D}\psi_{j,t}(\xi; \xi'), \tag{2.27}$$

where $0 \leq \lambda \ll 1$. It follows that if $\mathrm{D}\psi_{j,t}(\xi;\xi')$, whose existence is proven in Lemma 2.27, is negative, then it is possible to decrease the infeasibility of $\varphi_t(\xi)$ with respect to h_j by moving in the ξ' direction. That is if the directional derivatives of the cost and all of the component constraints for all $t \in [0,1]$ at a point ξ are negative along a certain direction and $\Psi(\xi) = 0$, then for each $\varepsilon > 0$ there exists a $\hat{\xi} \in \{\bar{\xi} \in \mathcal{X}_r \mid \Psi(\bar{\xi}) \leq 0\} \cap \mathcal{N}_{\mathcal{X}}(\xi, \varepsilon)$ such that $J(\hat{\xi}) < J(\xi)$. Therefore, if $\Psi(\xi) = 0$ and $\mathrm{D}J(\xi;\xi')$ and $\mathrm{D}\psi_{j,t}(\xi;\xi')$ are negative for all $j \in \mathcal{J}$ and $t \in [0,1]$, then ξ is not a local minimizer of the Relaxed Hybrid Optimal Control Problem. Similarly, if $\Psi(\xi) < 0$ and $\mathrm{D}J(\xi;\xi')$ is negative, then ξ is not a local minimizer of the Relaxed Hybrid Optimal Control Problem, even if $\mathrm{D}\psi_{j,t}(\xi;\xi')$ is greater than zero for all $j \in \mathcal{J}$ and $t \in [0,1]$.

Returning to the Switched System Optimal Control Problem, it is unclear how to define a directional derivative for the pure discrete input space since it is not a vector space. Therefore, in contrast to the relaxed discrete and continuous input spaces, the construction of a first order numerical scheme for the optimization of the pure discrete input is non-trivial. One could imagine trying to exploit the directional derivatives in the relaxed optimization space in order to construct a first order numerical optimal control algorithm for the Switched System Optimal Control Problem, but this would require devising some type of connection between points belonging to the pure and relaxed optimization spaces.

The Weak Topology on the Optimization Space and Local Minimizers

To motivate the type of relationship required between the pure and relaxed optimization space in order to construct a first order numerical optimal control scheme, we begin by describing the Chattering Lemma:

Theorem 2.8 (Theorem 1 in [BD05])**.** *For each $\xi_r \in \mathcal{X}_r$ and $\varepsilon > 0$ there exists a $\xi_p \in \mathcal{X}_p$ such that for each $t \in [0,1]$:*

$$\|\varphi_t(\xi_r) - \varphi_t(\xi_p)\|_2 \leq \varepsilon, \tag{2.28}$$

where $\varphi_t(\xi_r)$ and $\varphi_t(\xi_p)$ are solutions to Differential Equation (2.13) *corresponding to ξ_r and ξ_p, respectively.*

The theorem as is proven in [Ber74] is not immediately applicable to switched systems, but a straightforward extension as is proven in Theorem 1 in [BD05] makes that feasible. Note that the theorem as stated in [BD05], considers only two vector fields (i.e. $q = 2$), but as the author's of the theorem remark, their proof can be generalized to an arbitrary number of vector fields. A particular version of this existence theorem can also be found in Lemma 1 in [Sus72].

Theorem 2.8 says that the behavior of any element of the relaxed optimization space with respect to the trajectory of switched system can be approximated arbitrarily well by a point in the pure optimization space. Unfortunately, the relaxed and pure point as in Theorem 2.8 need not be near one another in the metric induced by the $\mathcal{X}$-norm. Therefore, though there exists a relationship between the pure and relaxed optimization spaces, this connection is

not reflected in the topology induced by the $\mathcal{X}$-norm; however, in a particular topology over the relaxed optimization space, a relaxed point and the pure point that approximates it as in Theorem 2.8 can be made arbitrarily close:

Definition 2.9. *We define the* weak topology on $\mathcal{X}_r$ induced by Differential Equation (2.13) *as the smallest topology on $\mathcal{X}_r$ such that the map $\xi \mapsto x^{(\xi)}$ is continuous. Moreover, an ε-ball in the weak topology centered at ξ is denoted by:*

$$\mathcal{N}_w(\xi, \varepsilon) = \left\{ \bar{\xi} \in \mathcal{X}_r \mid \left\| x^{(\xi)} - x^{(\bar{\xi})} \right\|_{L^2} < \varepsilon \right\}. \tag{2.29}$$

A longer introduction to weak topology can be found in Section 3.8 in [Rud91] or Section 2.3 in [KZ05], but before continuing we make an important observation that aids in motivating the ensuing analysis. In order to understand the relationship between the topology generated by the $\mathcal{X}$-norm on $\mathcal{X}_r$ and the weak topology on $\mathcal{X}_r$, observe that φ_t is Lipschitz continuous for all $t \in [0, 1]$ (this is proven in Corollary 2.13). Therefore, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if a pair of points of the relaxed optimization space belong to the same δ–ball in the $\mathcal{X}$–norm, then the pair of points belong to the same ε–ball in the weak topology on $\mathcal{X}_r$.

Notice, however, that it is not possible to show that for every $\varepsilon > 0$ that there exists a $\delta > 0$ such that if a pair of points of the relaxed optimization space belong to the same δ–ball in the weak topology on $\mathcal{X}_r$, then the pair of points belong to the same ε–ball in the $\mathcal{X}$–norm. More informally, a pair of points may generate trajectories that are near one another in the L^2–norm while not being near one another in the $\mathcal{X}$–norm. Since the weak topology, in contrast to the $\mathcal{X}$–norm induced topology, naturally places points that generate nearby trajectories next to one another, we extend Definition 2.9 in order to define a weak topology on $\mathcal{X}_p$ which we then use to define a notion of local minimizer for the Switched System Optimal Control Problem:

Definition 2.10. *We say that a point $\xi \in \mathcal{X}_p$ is a* local minimizers of the Switched System Optimal Control Problem *defined in Equation* (2.18) *if $\Psi(\xi) \leq 0$ and there exists $\varepsilon > 0$ such that $J(\hat{\xi}) \geq J(\xi)$ for each $\hat{\xi} \in \mathcal{N}_w(\xi, \varepsilon) \cap \left\{ \bar{\xi} \in \mathcal{X}_p \mid \Psi(\bar{\xi}) \leq 0 \right\}$, where $\mathcal{N}_w$ is as defined in Equation* (2.29).

With this definition of local minimizer, we can exploit Theorem 2.8, even just as an existence result, along with the notion of directional derivative over the relaxed optimization space to construct a necessary condition for optimality for the Switched System Optimal Control Problem.

An Optimality Condition

Motivated by the approach undertaken in [Pol97], we define an *optimality function*, denoted by $\theta : \mathcal{X}_p \to (-\infty, 0]$, that determines whether a given point is a local minimizer of the

Switched System Optimal Control Problem and a corresponding *descent direction*, $g : \mathcal{X}_p \to \mathcal{X}_r$:

$$\theta(\xi) = \min_{\xi' \in \mathcal{X}_r} \zeta(\xi, \xi'), \qquad g(\xi) = \arg\min_{\xi' \in \mathcal{X}_r} \zeta(\xi, \xi'), \tag{2.30}$$

where

$$\zeta(\xi, \xi') = \begin{cases} \max \left\{ \mathrm{D}J(\xi; \xi' - \xi), \max\limits_{\substack{j \in \mathcal{J} \\ t \in [0,1]}} \mathrm{D}\psi_{j,t}(\xi; \xi' - \xi) + \gamma \Psi(\xi) \right\} + \|\xi' - \xi\|_{\mathcal{X}} & \text{if } \Psi(\xi) \leq 0, \\ \max \left\{ \mathrm{D}J(\xi; \xi' - \xi) - \Psi(\xi), \max\limits_{\substack{j \in \mathcal{J} \\ t \in [0,1]}} \mathrm{D}\psi_{j,t}(\xi; \xi' - \xi) \right\} + \|\xi' - \xi\|_{\mathcal{X}} & \text{if } \Psi(\xi) > 0, \end{cases} \tag{2.31}$$

where $\gamma > 0$ is a design parameters. For notational convenience in the previous equation we have left out the natural inclusion of ξ from $\mathcal{X}_p$ to $\mathcal{X}_r$. Before proceeding, we make two observations. First, note that $\theta(\xi) \leq 0$ for each $\xi \in \mathcal{X}_p$, since we can always choose $\xi' = \xi$ which leaves the trajectory unmodified. Second, note that at a point $\xi \in \mathcal{X}_p$ the directional derivatives in the optimality function consider directions $\xi' - \xi$ with $\xi' \in \mathcal{X}_r$ in order to ensure that first order approximations constructed as in Equations (2.26) and (2.27) belong to the relaxed optimization space $\mathcal{X}_r$ which is convex (e.g. for $0 < \lambda \ll 1$, $J(\xi) + \lambda \mathrm{D}J(\xi; \xi' - \xi) \approx J((1 - \lambda)\xi + \lambda\xi')$ where $(1 - \lambda)\xi + \lambda\xi' \in \mathcal{X}_r$).

To understand how the optimality function behaves, consider several cases. First, if $\theta(\xi) < 0$ and $\Psi(\xi) = 0$, then there exists a $\xi' \in \mathcal{X}_r$ such that both $\mathrm{D}J(\xi; \xi' - \xi)$ and $\mathrm{D}\psi_{j,t}(\xi; \xi' - \xi)$ are negative for all $j \in \mathcal{J}$ and $t \in [0, 1]$. By employing the aforementioned first order approximation, we can show that for each $\varepsilon > 0$ there exists an ε–ball in the $\mathcal{X}$-norm centered at ξ such that $J(\hat{\xi}) < J(\xi)$ for some $\hat{\xi} \in \{\bar{\xi} \in \mathcal{X}_r \mid \Psi(\bar{\xi}) \leq 0\} \cap \mathcal{N}_{\mathcal{X}}(\xi, \varepsilon)$. As a result and because the cost and each of the component constraint functions are assumed Lipschitz continuous and φ_t for all $t \in [0, 1]$ is Lipschitz continuous as is proven in Corollary 2.13, an application of Theorem 2.8 allows us to show that for each $\varepsilon > 0$ there exists an ε–ball in the weak topology on $\mathcal{X}_p$ centered at ξ such that $J(\xi_p) < J(\xi)$ for some $\xi_p \in \{\bar{\xi} \in \mathcal{X}_p \mid \Psi(\bar{\xi}) \leq 0\} \cap \mathcal{N}_w(\xi, \varepsilon)$. Therefore, it follows that if $\theta(\xi) < 0$ and $\Psi(\xi) = 0$, then ξ is not a local minimizer of the Switched System Optimal Control Problem.

Second, if $\theta(\xi) < 0$ and $\Psi(\xi) < 0$, then there exists a $\xi' \in \mathcal{X}_r$ such that $\mathrm{D}J(\xi; \xi' - \xi)$ is negative. Though $\mathrm{D}\psi_{j,t}(\xi; \xi' - \xi)$ maybe positive for some $j \in \mathcal{J}$ and $t \in [0, 1]$, by employing the aforementioned first order approximation, we can show that for each $\varepsilon > 0$ there exists an ε–ball in the $\mathcal{X}$-norm centered at ξ such that $J(\hat{\xi}) < J(\xi)$ for some $\hat{\xi} \in \{\bar{\xi} \in \mathcal{X}_r \mid \Psi(\bar{\xi}) \leq 0\} \cap \mathcal{N}_{\mathcal{X}}(\xi, \varepsilon)$. As a result and because the cost and each of the constraint functions are assumed Lipschitz continuous and φ_t for all $t \in [0, 1]$ is Lipschitz continuous as is proven in Corollary 2.13, an application of Theorem 2.8 allows us to show that for each $\varepsilon > 0$ there exists an ε–ball in the weak topology on $\mathcal{X}_p$ centered at ξ such that $J(\xi_p) < J(\xi)$ for some $\xi_p \in \{\bar{\xi} \in \mathcal{X}_p \mid \Psi(\bar{\xi}) \leq 0\} \cap \mathcal{N}_w(\xi, \varepsilon)$. Therefore, it follows that if $\theta(\xi) < 0$ and $\Psi(\xi) < 0$, then ξ is not a local minimizer of the Switched System Optimal Control Problem. In this

case, the addition of the Ψ term in ζ ensures that a direction that reduces the cost does not simultaneously require a decrease in the infeasibility in order to be considered as a potential descent direction.

Third, if $\theta(\xi) < 0$ and $\Psi(\xi) > 0$, then there exists a $\xi' \in \mathcal{X}_r$ such that $\mathrm{D}\psi_{j,t}(\xi; \xi' - \xi)$ is negative for all $j \in \mathcal{J}$ and $t \in [0,1]$. By employing the aforementioned first order approximation, we can show for each $\varepsilon > 0$ there exists an ε–ball in the $\mathcal{X}$-norm centered at ξ such that $\Psi(\hat{\xi}) < \Psi(\xi)$ for some $\hat{\xi} \in \mathcal{N}_{\mathcal{X}}(\xi, \varepsilon)$. As a result and because each of the constraint functions are assumed Lipschitz continuous and φ_t for all $t \in [0,1]$ is Lipschitz continuous as is proven in Corollary 2.13, an application of Theorem 2.8 allows us to show that for each $\varepsilon > 0$ there exists an ε–ball in the weak topology on $\mathcal{X}_p$ centered at ξ such that $\Psi(\xi_p) < \Psi(\xi)$ for some $\xi_p \in \mathcal{N}_w(\xi, \varepsilon)$. Therefore, though it is clear that ξ is not a local minimizer of the Switched System Optimal Control Problem since $\Psi(\xi) > 0$, it follows that if $\theta(\xi) < 0$ and $\Psi(\xi) > 0$, then it is possible to locally reduce the infeasibility of ξ. In this case, the addition of the $\mathrm{D}J$ term in ζ serves as a heuristic to ensure that the reduction in infeasibility does not come at the price of an undue increase in the cost.

These observations are formalized in Theorem 2.34 where we prove that if ξ is a local minimizer of the Switched System Optimal Control Problem, then $\theta(\xi) = 0$, or that $\theta(\xi) = 0$ is a necessary condition for the optimality of ξ. To illustrate the importance of θ satisfying this property, recall how the directional derivative of a cost function is employed during unconstrained finite dimensional optimization. Since the directional derivative of the cost function at a point being equal to zero in all directions is a necessary condition for optimality for an unconstrained finite dimensional optimization problem, it is used as a stopping criterion by first order numerical algorithms (Corollary 1.1.3 and Algorithm Model 1.2.23 in [Pol97]). Similarly, by satisfying Theorem 2.34, θ is a necessary condition for optimality for the Switched System Optimal Control Problem and can therefore be used as a stopping criterion for a first order numerical optimal control algorithm trying to solve the Switched System Optimal Control Problem. Given θ's importance, we say *a point, $\xi \in \mathcal{X}_p$, satisfies the optimality condition* if $\theta(\xi) = 0$.

Choosing a Step Size and Projecting the Relaxed Discrete Input

Impressively, Theorem 2.8 just as an existence result is sufficient to allow for the construction of an optimality function that encapsulates a necessary condition for optimality for the Switched System Optimal Control Problem. Unfortunately, Theorem 2.8 is unable to describe how to exploit the descent direction, $g(\xi)$, since its proof provides no means to construct a pure input that approximates the behavior of a relaxed input while controlling the quality of the approximation. In this chapter, we extend Theorem 2.8 by devising a scheme that remedies this shortcoming. This allows for the development of a numerical optimal control algorithm for the Switched System Optimal Control Problem that first, performs optimal control over the relaxed optimization space and then projects the computed relaxed control into a pure control.

Before describing the construction of this projection, we describe how the descent direction, $g(\xi)$, can be exploited to construct a point in the relaxed optimization space that either reduces the cost (if the ξ is feasible) or the infeasibility (if ξ is infeasible). Comparing our approach to finite dimensional optimization, the argument that minimizes ζ is a "direction" along which to move the inputs in order to reduce the cost in the relaxed optimization space, but we require an algorithm to choose a step size. We employ a line search algorithm similar to the traditional Armijo algorithm used during finite dimensional optimization in order to choose a step size (Algorithm Model 1.2.23 in [Pol97]). Fixing $\alpha \in (0,1)$ and $\beta \in (0,1)$, a step size for a point $\xi \in \mathcal{X}_p$ is chosen by solving the following optimization problem:

$$\mu(\xi) = \begin{cases} \min\Big\{k \in \mathbb{N} \mid J\big(\xi + \beta^k(g(\xi) - \xi)\big) - J(\xi) \leq \alpha\beta^k\theta(\xi), \\ \qquad\qquad \Psi\big(\xi + \beta^k(g(\xi) - \xi)\big) \leq \alpha\beta^k\theta(\xi)\Big\} & \text{if } \Psi(\xi) \leq 0, \\ \min\Big\{k \in \mathbb{N} \mid \Psi\big(\xi + \beta^k(g(\xi) - \xi)\big) - \Psi(\xi) \leq \alpha\beta^k\theta(\xi)\Big\} & \text{if } \Psi(\xi) > 0. \end{cases} \tag{2.32}$$

In Lemma 2.43, we prove that for $\xi \in \mathcal{X}_p$, if $\theta(\xi) < 0$, then $\mu(\xi) < \infty$. Therefore, if $\theta(\xi) < 0$ for some $\xi \in \mathcal{X}_p$, then we can construct a descent direction, $g(\xi)$, and a step size, $\mu(\xi)$, and a new point $\big(\xi + \beta^{\mu(\xi)}(g(\xi) - \xi)\big) \in \mathcal{X}_r$ that produces a reduction in the cost (if ξ is feasible) or a reduction in the infeasibility (if ξ is infeasible).

We define the projection that takes this constructed point to a point belonging the pure optimization space while controlling the quality of approximation in two steps. First, we approximate the relaxed input by its N–th partial sum approximation via the Haar wavelet basis. To define this operation, $\mathcal{F}_N : L^2([0,1],\mathbb{R}) \cap BV([0,1],\mathbb{R}) \to L^2([0,1],\mathbb{R}) \cap BV([0,1],\mathbb{R})$, we employ the Haar wavelet (Section 7.2.2 in [Mal99]):

$$\lambda(t) = \begin{cases} 1 & \text{if } t \in \left[0, \frac{1}{2}\right), \\ -1 & \text{if } t \in \left[\frac{1}{2}, 1\right), \\ 0 & \text{otherwise.} \end{cases} \tag{2.33}$$

Letting $\mathbb{1} : \mathbb{R} \to \mathbb{R}$ be the constant function equal to one and $b_{kj} : [0,1] \to \mathbb{R}$ for $k \in \mathbb{N}$ and $j \in \{0, \ldots, 2^k - 1\}$, be defined as $b_{kj}(t) = \lambda\big(2^k t - j\big)$, the projection $\mathcal{F}_N$ for some $c \in L^2([0,1],\mathbb{R}) \cap BV([0,1],\mathbb{R}) \to L^2([0,1],\mathbb{R}) \cap BV([0,1],\mathbb{R})$ is defined as:

$$[\mathcal{F}_N(c)](t) = \langle c, \mathbb{1}\rangle + \sum_{k=0}^{N} \sum_{j=0}^{2^k-1} \langle c, b_{kj}\rangle \frac{b_{kj}(t)}{\|b_{kj}\|_{L^2}^2}. \tag{2.34}$$

Note that the inner product here is the traditional Hilbert space inner product.

This projection is then applied to each of the coordinates of an element in the relaxed optimization space. To avoid introducing additional notation, we let the coordinate-wise application of $\mathcal{F}_N$ to some relaxed discrete input $d \in \mathcal{D}_r$ be denoted as $\mathcal{F}_N(d)$ and similarly for some continuous input $u \in \mathcal{U}$. Lemma 2.35 proves that for each $N \in \mathbb{N}$, each $t \in [0,1]$,

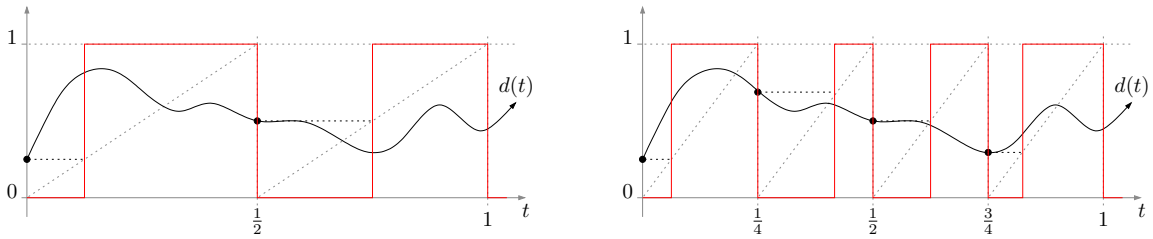


Figure 2.2: Illustration of the projection operation $\mathcal{P}_N$ of a function $d : [0,1] \to \mathbb{R}$. Left: $[\mathcal{P}_1(d)](t)$. Right: $[\mathcal{P}_2(d)](t)$.

and each $i \in \{1,\dots,q\}$, $[\mathcal{F}_N(d)]_i(t) \in [0,1]$ and $\sum_{i=1}^{q} [\mathcal{F}_N(d)]_i(t) = 1$ for the projection $\mathcal{F}_N(d)$. Therefore it follows that for each $d \in \mathcal{D}_r$, $\mathcal{F}_N(d) \in \mathcal{D}_r$.

Second, we project the output of $\mathcal{F}_N(d)$ to a pure discrete input by employing the function $\mathcal{P}_N : \mathcal{D}_r \to \mathcal{D}_p$ which computes the pulse width modulation of its argument with frequency 2^{-N}:

$$[\mathcal{P}_N(d)]_i(t) = \begin{cases} 1 & \text{if } t \in \left[\frac{\left(k+\sum_{j=1}^{i-1} d_j\left(\frac{k}{2^N}\right)\right)}{2^N}, \frac{\left(k+\sum_{j=1}^{i} d_j\left(\frac{k}{2^N}\right)\right)}{2^N} \right), \ k \in \left\{0,1,\dots,2^N-1\right\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.35}$$

Lemma 2.35 proves that for each $N \in \mathbb{N}$, each $t \in [0,1]$, and each $i \in \{1,\dots,q\}$,

$$\left[\mathcal{P}_N\left(\mathcal{F}_N(d)\right)\right]_i(t) \in \{0,1\}, \quad \text{and} \quad \sum_{i=1}^{q} \left[\mathcal{P}_N\left(\mathcal{F}_N(d)\right)\right]_i(t) = 1. \tag{2.36}$$

This proves that $\mathcal{P}_N\left(\mathcal{F}_N(d)\right) \in \mathcal{D}_p$ for each $d \in \mathcal{D}_r$. Figure 2.2 illustrates how $\mathcal{P}_N$ transform a function defined on $[0,1]$ into a function defined on $\{0,1\}$. Note that, in implementation terms, $\mathcal{P}_N$ is simply the pulse-width modulation operator.

Fixing $N \in \mathbb{N}$, we compose the two projections and define $\rho_N : \mathcal{X}_r \to \mathcal{X}_p$ as:

$$\rho_N(u,d) = \left(\mathcal{F}_N(u), \mathcal{P}_N\left(\mathcal{F}_N(d)\right)\right). \tag{2.37}$$

Critically, as shown in Theorem 2.38, this projection allows us to extend Theorem 2.8 by constructing an upper bound that goes to zero as N goes infinity between the error of employing the relaxed control rather than its projection in the solution of Differential Equation (2.13). Therefore in a fashion similar to applying the Armijo algorithm, we choose an $N \in \mathbb{N}$ at which to perform pulse width modulation by performing a line search. Fixing $\bar{\alpha} \in (0,\infty)$, $\bar{\beta} \in \left(\frac{1}{\sqrt{2}}, 1\right)$, and $\omega \in (0,1)$, a frequency at which to perform pulse width modulation for a

point $\xi \in \mathcal{X}_p$ is computed by solving the following optimization problem:

$$\nu(\xi) = \begin{cases} \min\Big\{k \in \mathbb{N} \mid \xi' = \rho_k\big(\xi + \beta^{\mu(\xi)}(g(\xi) - \xi)\big), \\ \qquad J(\xi') - J(\xi) \leq \big(\alpha\beta^{\mu(\xi)} - \bar{\alpha}\bar{\beta}^k\big)\theta(\xi),\ \Psi(\xi') \leq 0, \\ \qquad \bar{\alpha}\bar{\beta}^k \leq (1-\omega)\alpha\beta^{\mu(\xi)}\Big\} & \text{if } \Psi(\xi) \leq 0, \\ \min\Big\{k \in \mathbb{N} \mid \xi' = \rho_k\big(\xi + \beta^{\mu(\xi)}(g(\xi) - \xi)\big), \\ \qquad \Psi(\xi') - \Psi(\xi) \leq \big(\alpha\beta^{\mu(\xi)} - \bar{\alpha}\bar{\beta}^k\big)\theta(\xi), \\ \qquad \bar{\alpha}\bar{\beta}^k \leq (1-\omega)\alpha\beta^{\mu(\xi)}\Big\} & \text{if } \Psi(\xi) > 0. \end{cases} \tag{2.38}$$

In Lemma 2.44, we prove that for $\xi \in \mathcal{X}_p$, if $\theta(\xi) < 0$, then $\nu(\xi) < \infty$. Therefore, if $\theta(\xi) < 0$ for some $\xi \in \mathcal{X}_p$, then we can construct a descent direction, $g(\xi)$, a step size, $\mu(\xi)$, a frequency at which to perform pulse width modulation, $\nu\,(\xi)$, and a new point $\rho_{\nu(\xi)}\big(\xi + \beta^{\mu(\xi)}(g(\xi) - \xi)\big) \in \mathcal{X}_p$ that produces a reduction in the cost (if ξ is feasible) or a reduction in the infeasibility (if ξ is infeasible).

Switched System Optimal Control Algorithm

Consolidating our definitions, Algorithm 2.1 describes our numerical method to solve the Switched System Optimal Control Problem. For analysis purposes, we define $\Gamma : \mathcal{X}_p \to \mathcal{X}_p$ by

$$\Gamma(\xi) = \rho_{\nu(\xi)}\big(\xi + \beta^{\mu(\xi)}(g(\xi) - \xi)\big). \tag{2.39}$$

We say $\{\xi_j\}_{j\in\mathbb{N}}$ is *a sequence generated by Algorithm 2.1* if $\xi_{j+1} = \Gamma(\xi_j)$ for each $j \in \mathbb{N}$. We can prove several important properties about the sequence generated by Algorithm 2.1. First, in Lemma 2.45, we prove that if there exists $i_0 \in \mathbb{N}$ such that $\Psi(\xi_{i_0}) \leq 0$, then $\Psi(\xi_i) \leq 0$ for each $i \geq i_0$. That is, if the Algorithm constructs a feasible point, then the sequence of points generated after this feasible point are always feasible. Second, in Theorem 2.46, we prove $\lim_{j\to\infty} \theta(\xi_j) = 0$ or that Algorithm 2.1 converges to a point that satisfies the optimality condition.

2.3 Algorithm Analysis

In this section, we derive the various components of Algorithm 2.1 and prove that Algorithm 2.1 converges to a point that satisfies our optimality condition. Our argument proceeds as follows: first, we prove the continuity of the state, cost, and constraint, which we employ in latter arguments; second, we construct the components of the optimality function and prove that these components satisfy various desired properties; third, we prove that we can control the quality of approximation between the trajectories generated by a relaxed discrete input and its projection by ρ_N as a function of N; finally, we prove the convergence of our algorithm.

Require: $\xi_0 \in \mathcal{X}_p$, $\alpha \in (0,1)$, $\bar{\alpha} \in (0,\infty)$, $\beta \in (0,1)$, $\bar{\beta} \in \left(\frac{1}{\sqrt{2}}, 1\right)$, $\gamma \in (0,\infty)$, $\omega \in (0,1)$.
1: Set $j = 0$.
2: **loop**
3: Compute $\theta(\xi_j)$ as defined in Equation (2.30).
4: **if** $\theta(\xi_j) = 0$ **then**
5: **return** ξ_j.
6: **end if**
7: Compute $g(\xi_j)$ as defined in Equation (2.30).
8: Compute $\mu(\xi_j)$ as defined in Equation (2.32).
9: Compute $\nu(\xi_j)$ as defined in Equation (2.38).
10: Set $\xi_{j+1} = \rho_{\nu(\xi_j)}\big(\xi_j + \beta^{\mu(\xi_j)}(g(\xi_j) - \xi_j)\big)$, as defined in Equation (2.37).
11: Replace j by $j+1$.
12: **end loop**

Algorithm 2.1 Optimization Algorithm for the Switched System Optimal Control Problem

Continuity

In this subsection, we prove the continuity of the state, cost, and constraint. We begin by proving the continuity of the solution to Differential Equation (2.13) with respect to ξ by proving that this mapping is sequentially continuous:

Lemma 2.11. *Let $\{\xi_j\}_{j=1}^{\infty} \subset \mathcal{X}_r$ be a convergent sequence with limit $\xi \in \mathcal{X}_r$. Then the corresponding sequence of trajectories $\{x^{(\xi_j)}\}_{j=1}^{\infty}$, as defined in Equation (2.13), converges uniformly to $x^{(\xi)}$.*

Proof. For notational convenience, let $\xi_j = (u_j, d_j)$, $\xi = (u, d)$, and φ_t as defined in Equation (2.14). We begin by proving the convergence of $\{\varphi_t(\xi)\}_{j=1}^{\infty}$ to $\varphi_t(\xi)$ for each $t \in [0,1]$. Consider

$$\|\varphi_t(\xi_j) - \varphi_t(\xi)\|_2 = \left\| \int_0^t \sum_{i=1}^{q} [d_j]_i(\tau) f\big(\tau, \varphi_\tau(\xi_j), u_j(\tau), e_i\big) - d_i(\tau) f\big(\tau, \varphi_\tau(\xi), u(\tau), e_i\big) d\tau \right\|_2. \tag{2.40}$$

Therefore,

$$\begin{aligned} \|\varphi_t(\xi_j) - \varphi_t(\xi)\|_2 = \Bigg\| \int_0^t \sum_{i=1}^{q} &\big([d_j]_i(\tau) - d_i(\tau)\big) f\big(\tau, \varphi_\tau(\xi_j), u_j(\tau), e_i\big) + \\ &+ d_i(\tau)\big(f\big(\tau, \varphi_\tau(\xi_j), u_j(\tau), e_i\big) - f\big(\tau, \varphi_\tau(\xi), u_j(\tau), e_i\big)\big) + \\ &+ d_i(\tau)\big(f\big(\tau, \varphi_\tau(\xi), u_j(\tau), e_i\big) - f\big(\tau, \varphi_\tau(\xi), u(\tau), e_i\big)\big) d\tau \Bigg\|_2. \end{aligned} \tag{2.41}$$

Applying the Triangle Inequality, Assumption 2.2, Condition 1 in Corollary 2.5, and the boundedness of d, we have that there exists a $C > 0$ such that

$$\|\varphi_t(\xi_j) - \varphi_t(\xi)\|_2 \leq \int_0^1 \sum_{i=1}^{q} C\big|[d_j]_i(\tau) - d_i(\tau)\big| + L\,\|\varphi_\tau(\xi_j) - \varphi_\tau(\xi)\|_2 + L\,\|u_j(\tau) - u(\tau)\|_2\,d\tau. \tag{2.42}$$

Applying the Bellman-Gronwall Inequality (Lemma 5.6.4 in [Pol97]), we have that

$$\|\varphi_t(\xi_j) - \varphi_t(\xi)\|_2 \leq e^L \left(\int_0^1 C\,\|d_j(\tau) - d(\tau)\|_1 + L\,\|u_j(\tau) - u(\tau)\|_2\,d\tau \right). \tag{2.43}$$

Note that $\|u\|_2 \leq \|u\|_1$ for each $u \in \mathbb{R}^m$. Then applying Holder's inequality (Proposition 6.2 in [Fol99]) to the vector valued function, we have:

$$\int_0^1 \|d_j(\tau) - d(\tau)\|_1\,d\tau \leq \|d_j - d\|_{L^2}, \quad \text{and} \quad \int_0^1 \|u_j(\tau) - u(\tau)\|_1\,d\tau \leq \|u_j - u\|_{L^2}. \tag{2.44}$$

Since the sequence ξ_j converges to ξ, for every $\varepsilon > 0$ we know there exists some j_0 such that for all j greater than j_0, $\|\xi_j - \xi\|_{\mathcal{X}} \leq \varepsilon$. Therefore $\|\varphi_t(\xi_j) - \varphi_t(\xi)\|_2 \leq e^L(L + C)\varepsilon$, which proves the convergence of $\{\varphi_t(\xi_j)\}_{j=1}^{\infty}$ to $\varphi_t(\xi)$ for each $t \in [0, 1]$ as $j \to \infty$. Since this bound does not depend on t, we in fact have the uniform convergence of $\{x^{(\xi_j)}\}_{j=1}^{\infty}$ to $x^{(\xi)}$ as $j \to \infty$, hence obtaining our desired result. □

Notice that since $\mathcal{X}_r$ is a metric space, the previous result proves that the function φ_t which assigns $\xi \in \mathcal{X}_r$ to $\varphi_t(\xi)$ as the solution of Differential Equation (2.13) employing the notation defined in Equation (2.14) is continuous.

Corollary 2.12. *The function φ_t that maps $\xi \in \mathcal{X}_r$ to $\varphi_t(\xi)$ as the solution of Differential Equation* (2.13) *where we employ the notation defined in Equation* (2.14) *is continuous for all $t \in [0, 1]$.*

In fact, our arguments have shown that this mapping is Lipschitz continuous:

Corollary 2.13. *There exists a constant $L > 0$ such that for each $\xi_1, \xi_2 \in \mathcal{X}_r$ and $t \in [0, 1]$:*

$$\|\varphi_t(\xi_1) - \varphi_t(\xi_2)\|_2 \leq L\|\xi_1 - \xi_2\|_{\mathcal{X}}, \tag{2.45}$$

where $\varphi_t(\xi)$ is as defined in Equation (2.14).

As a result of this corollary, we immediately have the following results:

Corollary 2.14. *There exists a constant $L > 0$ such that for each $\xi_1 = (u_1, d_1) \in \mathcal{X}_r$, $\xi_2 = (u_2, d_2) \in \mathcal{X}_r$, and $t \in [0, 1]$:*

(1) $\big\|f\big(t, \varphi_t(\xi_1), u_1(t), d_1(t)\big) - f\big(t, \varphi_t(\xi_2), u_2(t), d_2(t)\big)\big\|_2 \leq$
$$\leq L\big(\|\xi_1 - \xi_2\|_{\mathcal{X}} + \|u_1(t) - u_2(t)\|_2 + \|d_1(t) - d_2(t)\|_2\big),$$

(2) $$\left\|\frac{\partial f}{\partial x}\big(t,\varphi_t(\xi_1),u_1(t),d_1(t)\big)-\frac{\partial f}{\partial x}\big(t,\varphi_t(\xi_2),u_2(t),d_2(t)\big)\right\|_{i,2}\leq$$
$$\leq L\big(\|\xi_1-\xi_2\|_{\mathcal{X}}+\|u_1(t)-u_2(t)\|_2+\|d_1(t)-d_2(t)\|_2\big),$$

(3) $$\left\|\frac{\partial f}{\partial u}\big(t,\varphi_t(\xi_1),u_1(t),d_1(t)\big)-\frac{\partial f}{\partial u}\big(t,\varphi_t(\xi_2),u_2(t),d_2(t)\big)\right\|_{i,2}\leq$$
$$\leq L\big(\|\xi_1-\xi_2\|_{\mathcal{X}}+\|u_1(t)-u_2(t)\|_2+\|d_1(t)-d_2(t)\|_2\big),$$

where $\varphi_t(\xi)$ is as defined in Equation (2.14).

Proof. The proof of Condition 1 follows by the fact that the vector field f is Lipschitz in all its arguments, as shown in the proof of Theorem 2.6, and applying Corollary 2.13. The remaining conditions follow in a similar fashion. □

Corollary 2.15. *There exists a constant $L>0$ such that for each $\xi_1,\xi_2\in\mathcal{X}_r$, $j\in\mathcal{J}$, and $t\in[0,1]$:*

(1) $\big|h_0\big(\varphi_1(\xi_1)\big)-h_0\big(\varphi_1(\xi_2)\big)\big|\leq L\,\|\xi_1-\xi_2\|_{\mathcal{X}}$,

(2) $\left\|\frac{\partial h_0}{\partial x}\big(\varphi_1(\xi_1)\big)-\frac{\partial h_0}{\partial x}\big(\varphi_1(\xi_2)\big)\right\|_2\leq L\,\|\xi_1-\xi_2\|_{\mathcal{X}}$,

(3) $\big|h_j\big(\varphi_t(\xi_1)\big)-h_j\big(\varphi_t(\xi_2)\big)\big|\leq L\,\|\xi_1-\xi_2\|_{\mathcal{X}}$,

(4) $\left\|\frac{\partial h_j}{\partial x}\big(\varphi_t(\xi_1)\big)-\frac{\partial h_j}{\partial x}\big(\varphi_t(\xi_2)\big)\right\|_2\leq L\,\|\xi_1-\xi_2\|_{\mathcal{X}}$,

where $\varphi_t(\xi)$ is as defined in Equation (2.14).

Proof. This result follows by Assumption 2.3 and Corollary 2.13. □

Even though it is a straightforward consequence of Condition 1 in Corollary 2.15, we write the following result to stress its importance.

Corollary 2.16. *There exists a constant $L>0$ such that, for each $\xi_1,\xi_2\in\mathcal{X}_r$:*

$$|J(\xi_1)-J(\xi_2)|\leq L\,\|\xi_1-\xi_2\|_{\mathcal{X}} \tag{2.46}$$

where J is as defined in Equation (2.15).

In fact, the Ψ is also Lipschitz continuous:

Lemma 2.17. *There exists a constant $L>0$ such that, for each $\xi_1,\xi_2\in\mathcal{X}_r$:*

$$|\Psi(\xi_1)-\Psi(\xi_2)|\leq L\,\|\xi_1-\xi_2\|_{\mathcal{X}} \tag{2.47}$$

where Ψ is as defined in Equation (2.16).

Proof. Since the maximum in Ψ is taken over $\mathcal{J} \times [0,1]$, which is compact, and the maps $(j,t) \mapsto \psi_{j,t}(\xi)$ are continuous for each $\xi \in \mathcal{X}$, we know from Condition 3 in Corollary 2.15 that there exists $L > 0$ such that,

$$\begin{aligned} \Psi(\xi_1) - \Psi(\xi_2) &= \max_{(j,t)\in\mathcal{J}\times[0,1]} \psi_{j,t}(\xi_1) - \max_{(j,t)\in\mathcal{J}\times[0,1]} \psi_{j,t}(\xi_2) \\ &\leq \max_{(j,t)\in\mathcal{J}\times[0,1]} \psi_{j,t}(\xi_1) - \psi_{j,t}(\xi_2) \\ &\leq L \left\|\xi_1 - \xi_2\right\|_{\mathcal{X}}. \end{aligned} \tag{2.48}$$

By reversing ξ_1 and ξ_2, and applying the same argument we get the desired result. □

Derivation of Algorithm Terms

Next, we formally derive the components of the optimality function and prove that it is properly defined. We begin by deriving the formal expression for the directional derivative of the trajectory of the switched system.

Lemma 2.18. *Let $\xi = (u,d) \in \mathcal{X}_r$, $\xi' = (u',d') \in \mathcal{X}$, and let $\varphi_t : \mathcal{X}_r \to \mathbb{R}^n$ be as defined in Equation* (2.14)*. Then the directional derivative of φ_t, as defined in Equation* (2.25)*, is given by*

$$\mathrm{D}\varphi_t(\xi;\xi') = \int_0^t \Phi^{(\xi)}(t,\tau)\left(\frac{\partial f}{\partial u}\big(\tau,\varphi_\tau(\xi),u(\tau),d(\tau)\big)u'(\tau) + \sum_{i=1}^{q} f\big(\tau,\varphi_\tau(\xi),u(\tau),e_i\big)d_i'(\tau)\right)d\tau, \tag{2.49}$$

where $\Phi^{(\xi)}(t,\tau)$ is the unique solution of the following matrix differential equation:

$$\frac{\partial \Phi}{\partial t}(t,\tau) = \frac{\partial f}{\partial x}\big(t,\varphi_t(\xi),u(t),d(t)\big)\Phi(t,\tau), \quad t \in [0,1], \quad \Phi(\tau,\tau) = I. \tag{2.50}$$

Proof. For notational convenience, let $x^{(\lambda)} = x^{(\xi+\lambda\xi')}$, $u^{(\lambda)} = u + \lambda u'$, and $d^{(\lambda)} = d + \lambda d'$. Then, if we define $\Delta x^{(\lambda)} = x^{(\lambda)} - x^{(\xi)}$,

$$\Delta x^{(\lambda)}(t) = \int_0^t f\big(\tau,x^{(\lambda)}(\tau),u^{(\lambda)}(\tau),d^{(\lambda)}(\tau)\big) - f\big(\tau,x^{(\xi)}(t),u(\tau),d(\tau)\big)d\tau, \tag{2.51}$$

thus,

$$\begin{aligned} \Delta x^{(\lambda)}(t) = &\int_0^t f\big(\tau,x^{(\lambda)}(\tau),u^{(\lambda)}(\tau),d^{(\lambda)}(\tau)\big) - f\big(\tau,x^{(\lambda)}(t),u^{(\lambda)}(\tau),d(\tau)\big)d\tau + \\ &+ \int_0^t f\big(\tau,x^{(\lambda)}(\tau),u^{(\lambda)}(\tau),d(\tau)\big) - f\big(\tau,x^{(\xi)}(t),u^{(\lambda)}(\tau),d(\tau)\big)d\tau + \\ &+ \int_0^t f\big(\tau,x^{(\xi)}(\tau),u^{(\lambda)}(\tau),d(\tau)\big) - f\big(\tau,x^{(\xi)}(t),u(\tau),d(\tau)\big)d\tau, \end{aligned} \tag{2.52}$$

and applying the Mean Value Theorem,

$$\Delta x^{(\lambda)}(t) = \int_0^t \lambda \sum_{i=1}^{q} d_i'(\tau) f\big(\tau, x^{(\lambda)}(\tau), u^{(\lambda)}(\tau), e_i\big) + \\ + \int_0^t \frac{\partial f}{\partial x}\big(\tau, x^{(\xi)}(\tau) + \nu_x(\tau)\Delta x^{(\lambda)}(\tau), u^{(\lambda)}(\tau), d(\tau)\big)\Delta x^{(\lambda)}(\tau) + \\ + \int_0^t \lambda \frac{\partial f}{\partial u}\big(\tau, x^{(\xi)}(\tau), u(\tau) + \nu_u(\tau)\lambda u'(\tau), d(\tau)\big)u'(\tau)dt, \quad (2.53)$$

where $\nu_u, \nu_x : [0,t] \to [0,1]$.

Let $z(t)$ be the unique solution of the following differential equation:

$$\dot{z}(\tau) = \frac{\partial f}{\partial x}\big(\tau, x^{(\xi)}(\tau), u(\tau), d(\tau)\big)z(\tau) + \frac{\partial f}{\partial u}\big(\tau, x^{(\xi)}(\tau), u(\tau), d(\tau)\big)u'(\tau) + \\ + \sum_{i=1}^{q} d_i'(\tau) f\big(\tau, x^{(\xi)}(\tau), u(\tau), e_i\big), \quad \tau \in [0,t], \quad z(0) = 0. \quad (2.54)$$

We want to show that $\lim_{\lambda \downarrow 0} \left\| \frac{\Delta x^{(\lambda)}(t)}{\lambda} - z(t) \right\|_2 = 0$. To prove this, consider the following inequalities that follow from Condition 2 in Assumption 2.2:

$$\begin{aligned} &\left\| \int_0^t \frac{\partial f}{\partial x}\big(\tau, x^{(\xi)}(\tau), u(\tau), d(\tau)\big)z(\tau) + \right. \\ &\qquad \left. - \frac{\partial f}{\partial x}\big(\tau, x^{(\xi)}(\tau) + \nu_x(\tau)\Delta x^{(\lambda)}(\tau), u^{(\lambda)}(\tau), d(\tau)\big)\frac{\Delta x^{(\lambda)}(\tau)}{\lambda} d\tau \right\|_2 \leq \\ &\leq \int_0^t \left\| \frac{\partial f}{\partial x} \right\|_{L^\infty} \left\| z(\tau) - \frac{\Delta x^{(\lambda)}(\tau)}{\lambda} \right\|_2 d\tau + \int_0^t L\left(\left\| \nu_x(\tau)\Delta x^{(\lambda)}(\tau) \right\|_2 + \lambda \left\| u'(\tau) \right\|_2 \right) \left\| z(t) \right\|_2 d\tau \\ &\leq L \int_0^t \left\| z(\tau) - \frac{\Delta x^{(\lambda)}(\tau)}{\lambda} \right\|_2 d\tau + L \int_0^t \left(\left\| \Delta x^{(\lambda)}(\tau) \right\|_2 + \lambda \left\| u'(\tau) \right\|_2 \right) \left\| z(t) \right\|_2 d\tau, \end{aligned} \quad (2.55)$$

also from Condition 3 in Assumption 2.2:

$$\left\| \int_0^t \left(\frac{\partial f}{\partial u}\big(\tau, x^{(\xi)}(\tau), u(\tau), d(\tau)\big) - \frac{\partial f}{\partial u}\big(\tau, x^{(\xi)}(\tau), u(\tau) + \nu_u(\tau)\lambda u'(\tau), d(\tau)\big) \right) u'(\tau) \right\|_2 \leq \\ \leq L \int_0^t \lambda \left\| \nu_u(\tau) u'(\tau) \right\|_2 \left\| u'(\tau) \right\|_2 d\tau \leq L \int_0^t \lambda \left\| u'(\tau) \right\|_2^2 d\tau, \quad (2.56)$$

and from Condition 1 in Assumption 2.2:

$$\left\| \int_0^t \sum_{i=1}^{q} d_i'(\tau) \left(f\big(\tau, x^{(\xi)}(\tau), u(\tau), e_i\big) - f\big(\tau, x^{(\lambda)}(\tau), u^{(\lambda)}(\tau), e_i\big) \right) d\tau \right\|_2 \leq$$
$$\leq L \int_0^t \sum_{i=1}^{q} d_i'(\tau) \left(\left\| \Delta x^{(\lambda)}(\tau) \right\|_2 + \lambda \left\| u'(\tau) \right\|_2 \right) d\tau. \quad (2.57)$$

Now, using the Bellman-Gronwall Inequality (Lemma 5.6.4 in [Pol97]) and the inequalities above,

$$\left\| \frac{\Delta x^{(\lambda)}(t)}{\lambda} - z(t) \right\|_2 \leq e^{Lt} L \Bigg(\int_0^t \left(\left\| \Delta x^{(\lambda)}(\tau) \right\|_2 + \lambda \left\| u'(\tau) \right\|_2 \right) \left\| z(t) \right\|_2 + \lambda \left\| u'(\tau) \right\|_2^2 +$$
$$+ \sum_{i=1}^{q} d_i'(\tau) \left(\left\| \Delta x^{(\lambda)}(\tau) \right\|_2 + \lambda \left\| u'(\tau) \right\|_2 \right) d\tau \Bigg), \quad (2.58)$$

but note that every term in the integral above is bounded, and $\Delta x^{(\lambda)}(\tau) \to 0$ for each $\tau \in [0,t]$ since $x^{(\lambda)} \to x^{(\xi)}$ uniformly as shown in Lemma 2.11, thus by the Dominated Convergence Theorem (Theorem 2.24 in [Fol99]) and by noting that $\mathrm{D}\varphi_t(\xi;\xi')$, as defined in Equation (2.49), is exactly the solution of Differential Equation (2.54) we get:

$$\lim_{\lambda \downarrow 0} \left\| \frac{\Delta x^{(\lambda)}(t)}{\lambda} - z(t) \right\|_2 = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left\| x^{(\xi+\lambda\xi')}(t) - x^{(\xi)}(t) - D\varphi_t(\xi;\lambda\xi') \right\|_2 = 0. \quad (2.59)$$

The result of the Lemma then follows. □

Next, we prove that $\mathrm{D}\varphi_t$ is bounded by proving that $\Phi^{(\xi)}$ is bounded:

Corollary 2.19. *There exists a constant $C > 0$ such that for each $t, \tau \in [0,1]$ and $\xi \in \mathcal{X}_r$:*

$$\left\| \Phi^{(\xi)}(t,\tau) \right\|_{i,2} \leq C, \quad (2.60)$$

where $\Phi^{(\xi)}(t,\tau)$ is the solution to Differential Equation (2.50).

Proof. Notice that, since the induced matrix norm is submultiplicative,

$$\begin{aligned} \left\| \Phi^{(\xi)}(t,\tau) \right\|_{i,2} &= \left\| \Phi^{(\xi)}(t,\tau) + \int_\tau^t \left(\frac{\partial f}{\partial x}\big(s, x^{(\xi)}(s), u(s), d(s)\big) \Phi^{(\xi)}(s,\tau) \right) ds \right\|_{i,2} \\ &\leq 1 + \int_\tau^t \left\| \frac{\partial f}{\partial x}\big(s, x^{(\xi)}(s), u(s), d(s)\big) \right\|_{i,2} \left\| \Phi^{(\xi)}(t,s) \right\|_{i,2} ds \\ &\leq e^{qC}, \end{aligned} \quad (2.61)$$

where in the last step we employed Condition 1 from Corollary 2.5 with a constant $C > 0$ and the Bellman-Gronwall Inequality. □

Corollary 2.20. *There exists a constant $C > 0$ such that for all $\xi \in \mathcal{X}_r$, $\xi' \in \mathcal{X}$, and $t \in [0,1]$:*

$$\|\mathrm{D}\varphi_t(\xi;\xi')\|_2 \leq C \|\xi'\|_{\mathcal{X}}, \tag{2.62}$$

where $\mathrm{D}\varphi_t$ is as defined in Equation (2.49).

Proof. This result follows by employing the Cauchy-Schwarz Inequality, Corollary 2.5 and Corollary 2.19. □

In fact, we can actually prove the Lipschitz continuity of $\Phi^{(\xi)}$:

Lemma 2.21. *There exists a constant $L > 0$ such that for each $\xi_1, \xi_2 \in \mathcal{X}_r$ and each $t, \tau \in [0,1]$:*

$$\left\|\Phi^{(\xi_1)}(t,\tau) - \Phi^{(\xi_2)}(t,\tau)\right\|_{i,2} \leq L \|\xi_1 - \xi_2\|_{\mathcal{X}}, \tag{2.63}$$

where $\Phi^{(\xi)}$ is the solution to Differential Equation (2.50).

Proof. Letting $\xi_1 = (u_1, d_1) \in \mathcal{X}_r$ and $\xi_2 = (u_2, d_2) \in \mathcal{X}_r$ and by applying the Triangle Inequality and noticing the induced matrix norm is compatible, observe:

$$\begin{aligned}
&\left\|\Phi^{(\xi_1)}(t,\tau) - \Phi^{(\xi_2)}(t,\tau)\right\|_{i,2} \leq \\
&\qquad \leq \int_\tau^t \left(\left\| \frac{\partial f}{\partial x}\left(s, x^{(\xi_2)}(s), u_2(s), d_2(s)\right)\right\|_{i,2} \left\|\Phi^{(\xi_1)}(s,\tau) - \Phi^{(\xi_2)}(s,\tau)\right\|_{i,2} \right) ds + \\
&+ \int_\tau^t \left(\left\| \frac{\partial f}{\partial x}\left(s, x^{(\xi_1)}(s), u_1(s), d_1(s)\right) - \frac{\partial f}{\partial x}\left(s, x^{(\xi_2)}(s), u_2(s), d_2(s)\right)\right\|_{i,2} \left\|\Phi^{(\xi_1)}(s,\tau)\right\|_{i,2} \right) ds.
\end{aligned} \tag{2.64}$$

By applying Condition 1 in Corollary 2.5, Condition 2 in Corollary 2.14, Corollary 2.19, the same argument as in Equation (2.44), and the Bellman-Gronwall Inequality (Lemma 5.6.4 in [Pol97]), our desired result follows. □

A simple extension of our previous argument shows that for all $t \in [0,1]$, $\mathrm{D}\varphi_t(\xi;\cdot)$ is Lipschitz continuous with respect to its point of evaluation, ξ.

Lemma 2.22. *There exists a constant $L > 0$ such that for each $\xi_1, \xi_2 \in \mathcal{X}_r$, $\xi' \in \mathcal{X}$, and $t \in [0,1]$:*

$$\|\mathrm{D}\varphi_t(\xi_1;\xi') - \mathrm{D}\varphi_t(\xi_2;\xi')\|_2 \leq L\|\xi_1 - \xi_2\|_{\mathcal{X}} \|\xi'\|_{\mathcal{X}} \tag{2.65}$$

where $\mathrm{D}\varphi_t$ is as defined in Equation (2.49).

Proof. Let $\xi_1 = (u_1, d_1)$, $\xi_2 = (u_2, d_2)$, and $\xi' = (u', d')$. Then, by applying the Triangle Inequality, and noticing that the induced matrix norm is compatible, observe:

$$\begin{aligned}
&\|\mathrm{D}\varphi_t(\xi_1;\xi') - \mathrm{D}\varphi_t(\xi_2;\xi')\|_2 \leq \\
&\qquad \leq \int_0^t \Bigg(\left\|\Phi^{(\xi_1)}(t,s) - \Phi^{(\xi_2)}(t,s)\right\|_{i,2} \left\|\frac{\partial f}{\partial u}\big(s, x^{(\xi_1)}(s), u_1(s), d_1(s)\big)\right\|_{i,2} + \\
&+\left\|\Phi^{(\xi_2)}(t,s)\right\|_{i,2} \left\|\frac{\partial f}{\partial u}\big(s, x^{(\xi_1)}(s), u_1(s), d_1(s)\big) - \frac{\partial f}{\partial u}\big(s, x^{(\xi_2)}(s), u_2(s), d_2(s)\big)\right\|_{i,2} \Bigg) \|u'(s)\|_2 \, ds + \\
&\qquad + \int_0^t \sum_{i=1}^{q} \Bigg(\left\|\Phi^{(\xi_1)}(t,s) - \Phi^{(\xi_2)}(t,s)\right\|_{i,2} \left\|f\big(s, x^{(\xi_1)}(s), u_1(s), e_i\big)\right\|_2 + \\
&\quad + \left\|\Phi^{(\xi_2)}(t,s)\right\|_{i,2} \left\|f\big(s, x^{(\xi_1)}(s), u_1(s), e_i\big) - f\big(s, x^{(\xi_2)}(s), u_2(s), e_i\big)\right\|_2 \Bigg) \|d'(s)\| \, ds. \quad (2.66)
\end{aligned}$$

By applying Corollary 2.19, Condition 1 in Corollary 2.5, Lemma 2.21, Conditions 1 and 3 in Corollary 2.14, together with the boundedness of $u'(s)$ and $d'(s)$, and an argument identical to the one used in Equation (2.44), our desired result follows. □

Next, we prove that $\mathrm{D}\varphi_t$ is simultaneously continuous with respect to both of its arguments.

Lemma 2.23. *For each $t \in [0,1]$, $\xi \in \mathcal{X}_r$, and $\xi' \in \mathcal{X}$, the map $(\xi, \xi') \mapsto \mathrm{D}\varphi_t(\xi;\xi')$, as defined in Equation* (2.49)*, is continuous.*

Proof. To prove this result, we can employ an argument identical to the one used in the proof of Lemma 2.11. First, note that $u(t) \in U$ for each $t \in [0,1]$. Second, note that $\Phi^{(\xi)}$, f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial u}$ are bounded, as shown in Corollary 2.19 and Condition 1 in Corollary 2.5. Third, recall that $\Phi^{(\xi)}$, f, and $\frac{\partial f}{\partial u}$ are Lipschitz continuous, as proven in Lemma 2.21 and Conditions 1 and 3 in Corollary 2.14, respectively. Finally, the result follows after using an argument identical to the one used in Equation (2.44). □

We can now construct the directional derivative of the cost J and prove it is Lipschitz continuous.

Lemma 2.24. *Let $\xi \in \mathcal{X}_r$, $\xi' \in \mathcal{X}$, and J be as defined in Equation* (2.15)*. Then the directional derivative of the cost J in the ξ' direction is:*

$$\mathrm{D}J(\xi;\xi') = \frac{\partial h_0}{\partial x}\big(\varphi_1(\xi)\big)\mathrm{D}\varphi_1(\xi;\xi'). \quad (2.67)$$

Proof. The result follows directly by the Chain Rule and Lemma 2.18. □

Corollary 2.25. *There exists a constant $L > 0$ such that for each $\xi_1, \xi_2 \in \mathcal{X}_r$ and $\xi' \in \mathcal{X}$:*

$$|\mathrm{D}J(\xi_1;\xi') - \mathrm{D}J(\xi_2;\xi')| \leq L \left\|\xi_1 - \xi_2\right\|_{\mathcal{X}} \left\|\xi'\right\|_{\mathcal{X}}, \tag{2.68}$$

where $\mathrm{D}J$ *is as defined in Equation* (2.67)*.*

Proof. Notice by the Triangle Inequality and the Cauchy-Schwartz Inequality:

$$\begin{aligned}|\mathrm{D}J(\xi_1;\xi') - \mathrm{D}J(\xi_2;\xi')| \leq \left\|\frac{\partial h_0}{\partial x}\big(\varphi_1(\xi_1)\big)\right\|_2 \left\|\mathrm{D}\varphi_1(\xi_1;\xi') - \mathrm{D}\varphi_1(\xi_2;\xi')\right\|_2 + \\ + \left\|\frac{\partial h_0}{\partial x}\big(\varphi_1(\xi_1)\big) - \frac{\partial h_0}{\partial x}\big(\varphi_1(\xi_2)\big)\right\|_2 \left\|\mathrm{D}\varphi_1(\xi_2;\xi')\right\|_2. \end{aligned} \tag{2.69}$$

The result then follows by applying Condition 2 in Corollary 2.5, Condition 2 in Corollary 2.15, Corollary 2.20, and Lemma 2.22. □

Next, we prove that $\mathrm{D}J$ is simultaneously continuous with respect to both of its arguments, which is a direct consequence of Lemma 2.23.

Corollary 2.26. *For each $\xi \in \mathcal{X}_r$ and $\xi' \in \mathcal{X}$, the map $(\xi, \xi') \mapsto \mathrm{D}J(\xi;\xi')$, as defined in Equation* (2.67)*, is continuous.*

Next, we construct the directional derivative of each of the component constraint functions $\psi_{j,t}$ and prove that each of the component constraints is Lipschitz continuous.

Lemma 2.27. *Let $\xi \in \mathcal{X}_r$, $\xi' \in \mathcal{X}$, and $\psi_{j,t}$ defined as in Equation* (2.17)*. Then for each $j \in \mathcal{J}$ and $t \in [0,1]$, the directional derivative of $\psi_{j,t}$, denoted $\mathrm{D}\psi_{j,t}$, is given by:*

$$\mathrm{D}\psi_{j,t}(\xi;\xi') = \frac{\partial h_j}{\partial x}\big(\varphi_t(\xi)\big)\mathrm{D}\varphi_t(\xi;\xi'). \tag{2.70}$$

Proof. The result follows using the Chain Rule and Lemma 2.18. □

Corollary 2.28. *There exists a constant $L > 0$ such that for each $\xi_1, \xi_2 \in \mathcal{X}_r$, $\xi' \in \mathcal{X}$, and $t \in [0,1]$:*

$$|\mathrm{D}\psi_{j,t}(\xi_1;\xi') - \mathrm{D}\psi_{j,t}(\xi_2;\xi')| \leq L \left\|\xi_1 - \xi_2\right\|_{\mathcal{X}} \left\|\xi'\right\|_{\mathcal{X}}, \tag{2.71}$$

where $\mathrm{D}\psi_{j,t}$ is as defined in Equation (2.70)*.*

Proof. Notice by the Triangle Inequality and the Cauchy Schwartz Inequality:

$$\begin{aligned}|\mathrm{D}\psi_{j,t}(\xi_1;\xi') - \mathrm{D}\psi_{j,t}(\xi_2;\xi')| \leq \left\|\frac{\partial h_j}{\partial x}\big(\varphi_t(\xi_1)\big)\right\|_2 \left\|\mathrm{D}\varphi_t(\xi_1;\xi') - \mathrm{D}\varphi_t(\xi_2;\xi')\right\|_2 + \\ + \left\|\frac{\partial h_j}{\partial x}\big(\varphi_t(\xi_1)\big) - \frac{\partial h_j}{\partial x}\big(\varphi_t(\xi_2)\big)\right\|_2 \left\|\mathrm{D}\varphi_t(\xi_2;\xi')\right\|_2. \end{aligned} \tag{2.72}$$

The result then follows by applying Condition 3 in Corollary 2.5, Condition 4 in Corollary 2.15, Corollary 2.20, and Lemma 2.22. □

Next, we prove that $\mathrm{D}\psi_{j,t}$ is simultaneously continuous with respect to both of its arguments, which follows directly from Lemma 2.23:

Corollary 2.29. *For each $\xi \in \mathcal{X}_r$, $\xi' \in \mathcal{X}$, and $t \in [0,1]$, the map $(\xi, \xi') \mapsto \mathrm{D}\psi_{j,t}(\xi; \xi')$, as defined in Equation (2.70), is continuous.*

Given these results, we can begin describing the properties satisfied by the optimality function:

Lemma 2.30. *Let ζ be defined as in Equation (2.31). Then there exists a constant $L > 0$ such that, for each $\xi_1, \xi_2, \xi' \in \mathcal{X}_r$,*

$$|\zeta(\xi_1, \xi') - \zeta(\xi_2, \xi')| \leq L \left\|\xi_1 - \xi_2\right\|_{\mathcal{X}}. \tag{2.73}$$

Proof. To prove the result, first notice that for $\{x_i\}_{i\in\mathcal{I}}, \{y_i\}_{i\in\mathcal{I}} \subset \mathbb{R}$:

$$\left|\max_{i\in\mathcal{I}} x_i\right| \leq \max_{i\in\mathcal{I}} |x_i|, \quad \text{and} \quad \max_{i\in\mathcal{I}} x_i - \max_{i\in\mathcal{I}} y_i \leq \max_{i\in\mathcal{I}} \{x_i - y_i\}. \tag{2.74}$$

Therefore,

$$\left|\max_{i\in\mathcal{I}} x_i - \max_{i\in\mathcal{I}} y_i\right| \leq \max_{i\in\mathcal{I}} |x_i - y_i|. \tag{2.75}$$

Letting $\Psi^+(\xi) = \max\{0, \Psi(\xi)\}$ and $\Psi^-(\xi) = \max\{0, -\Psi(\xi)\}$, observe:

$$\zeta(\xi, \xi') = \max\left\{\mathrm{D}J(\xi; \xi' - \xi) - \Psi^+(\xi), \max_{j\in\mathcal{J},\, t\in[0,1]} \mathrm{D}\psi_{j,t}(\xi; \xi' - \xi) - \gamma\Psi^-(\xi)\right\} + \|\xi' - \xi\|_{\mathcal{X}}. \tag{2.76}$$

Employing Equation (2.75):

$$\begin{aligned}\left|\zeta(\xi_1, \xi') - \zeta(\xi_2, \xi')\right| \leq \max\Bigg\{&\left|\mathrm{D}J(\xi_1; \xi' - \xi_1) - \mathrm{D}J(\xi_2; \xi' - \xi_2)\right| + \left|\Psi^+(\xi_2) - \Psi^+(\xi_1)\right|, \\ &\max_{j\in\mathcal{J},\, t\in[0,1]} \left|\mathrm{D}\psi_{j,t}(\xi_1; \xi' - \xi_1) - \mathrm{D}\psi_{j,t}(\xi_2; \xi' - \xi_2)\right| + \gamma\left|\Psi^-(\xi_2) - \Psi^-(\xi_1)\right|\Bigg\} + \left|\|\xi' - \xi_1\|_{\mathcal{X}} - \|\xi' - \xi_2\|_{\mathcal{X}}\right|.\end{aligned} \tag{2.77}$$

We show three results that taken together with the Triangle Inequality prove the desired result. First, by applying the reverse Triangle Inequality:

$$\left|\|\xi' - \xi_1\|_{\mathcal{X}} - \|\xi' - \xi_2\|_{\mathcal{X}}\right| \leq \|\xi_1 - \xi_2\|_{\mathcal{X}}. \tag{2.78}$$

Second,

$$\begin{aligned}\left|\mathrm{D}J(\xi_1; \xi' - \xi_1) - \mathrm{D}J(\xi_2; \xi' - \xi_2)\right| &\leq \left|\mathrm{D}J(\xi_1; \xi') - \mathrm{D}J(\xi_2; \xi')\right| + \left|\mathrm{D}J(\xi_1; \xi_1) - \mathrm{D}J(\xi_2; \xi_1)\right| + \\ &\quad + \left|\frac{\partial h_0}{\partial x}\big(\varphi_1(\xi_2)\big)\mathrm{D}\varphi_1(\xi_2; \xi_2 - \xi_1)\right| \\ &\leq L \left\|\xi_1 - \xi_2\right\|_{\mathcal{X}},\end{aligned} \tag{2.79}$$

where $L > 0$ and we employed the linearity of $\mathrm{D}J$, Corollary 2.25, the fact that ξ' and ξ_1 are bounded since $\xi', \xi_1 \in \mathcal{X}_r$, the Cauchy-Schwartz Inequality, Condition 2 in Corollary 2.5, and Corollary 2.20. Notice that by employing an argument identical to Equation (2.79) and Corollary 2.28, we can assume without loss of generality that $\left|\mathrm{D}\psi_{j,t}(\xi_1;\xi'-\xi_1)-\mathrm{D}\psi_{j,t}(\xi_2;\xi'-\xi_2)\right| \leq L\left\|\xi_1-\xi_2\right\|_{\mathcal{X}}$. Finally, notice that by applying Lemma 2.17, $\Psi^+(\xi)$ and $\Psi^-(\xi)$ are Lipschitz continuous. □

In fact, ζ satisfies an even more important property:

Lemma 2.31. *For each $\xi \in \mathcal{X}_p$, the map $\xi' \mapsto \zeta(\xi, \xi')$, as defined in Equation (2.31), is strictly convex.*

Proof. The proof follows after noting that the maps $\xi' \mapsto \mathrm{D}J(\xi;\xi'-\xi)$ and $\xi' \mapsto \mathrm{D}\psi_{j,t}(\xi;\xi'-\xi)$ are affine, hence any maximum among these function is convex, and the map $\xi' \mapsto \|\xi'-\xi\|_{\mathcal{X}}$ is strictly convex since we chose the 2–norm as our finite dimensional norm. □

The following theorem, which follows as a result of the previous lemma, is fundamental to our result since it shows that g, as defined in Equation (2.30), is a well-defined function. We omit the proof since it is a particular case of a well known result regarding the existence of unique minimizers of strictly convex functions over bounded sets in Hilbert spaces (Proposition II.1.2 in [ET87]).

Theorem 2.32. *For each $\xi \in \mathcal{X}_p$, the map $\xi' \mapsto \zeta(\xi, \xi')$, as defined in Equation (2.31), has a unique minimizer.*

Employing these results we can prove the continuity of the optimality function. This result is not strictly required in order to prove the convergence of Algorithm 2.1 or in order to prove that the optimality function encodes local minimizers of the Switched System Optimal Control Problem, but is useful when we describe the implementation of our algorithm.

Lemma 2.33. *The function θ, as defined in Equation (2.30), is continuous.*

Proof. First, we show that θ is upper semi-continuous. Consider a sequence $\{\xi_i\}_{i=1}^{\infty} \subset \mathcal{X}_r$ converging to ξ, and $\xi' \in \mathcal{X}_r$ such that $\theta(\xi) = \zeta(\xi, \xi')$, i.e. $\xi' = g(\xi)$, where g is defined as in Equation (2.30). Since $\theta(\xi_i) \leq \zeta(\xi_i, \xi')$ for all $i \in \mathbb{N}$,

$$\limsup_{i\to\infty} \theta(\xi_i) \leq \limsup_{i\to\infty} \zeta(\xi_i, \xi') = \zeta(\xi, \xi') = \theta(\xi), \tag{2.80}$$

which proves the upper semi-continuity of θ.

Second, we show that θ is lower semi-continuous. Let $\{\xi_i'\}_{i\in\mathbb{N}}$ such that $\theta(\xi_i) = \zeta(\xi_i, \xi_i')$, i.e. $\xi_i' = g(\xi_i)$. From Lemma 2.30, we know there exists a Lipschitz constant $L > 0$ such that for each $i \in \mathbb{N}$, $|\zeta(\xi, \xi_i') - \zeta(\xi_i, \xi_i')| \leq L\left\|\xi - \xi_i\right\|_{\mathcal{X}}$. Consequently,

$$\theta(\xi) \leq \big(\zeta(\xi, \xi_i') - \zeta(\xi_i, \xi_i')\big) + \zeta(\xi_i, \xi_i') \leq L\|\xi - \xi_i\|_{\mathcal{X}} + \theta(\xi_i). \tag{2.81}$$

Taking limits we conclude that

$$\theta(\xi) \leq \liminf_{i\to\infty} \theta(\xi_i), \tag{2.82}$$

which proves the lower semi-continuity of θ, and our desired result. □

Finally, we can prove that θ encodes a necessary condition for optimality:

Theorem 2.34. *Let θ be as defined in Equation (2.30), then:*

(1) *θ is non-positive valued, and*

(2) *If $\xi \in \mathcal{X}_p$ is a local minimizer of the Switched System Optimal Control Problem as in Definition 2.10, then $\theta(\xi) = 0$.*

Proof. Notice that $\zeta(\xi,\xi) = 0$, therefore $\theta(\xi) = \min_{\xi'\in\mathcal{X}_r} \zeta(\xi,\xi') \leq \zeta(\xi,\xi) = 0$. This proves Condition 1.

To prove Condition 2, we begin by making several observations. Given $\xi' \in \mathcal{X}_r$ and $\lambda \in [0,1]$, using the Mean Value Theorem and Corollary 2.25 we have that there exists $s \in (0,1)$ and $L > 0$ such that

$$\begin{aligned} J\big(\xi + \lambda(\xi'-\xi)\big) - J(\xi) &= \mathrm{D}J\big(\xi + s\lambda(\xi'-\xi); \lambda(\xi'-\xi)\big) \\ &\leq \lambda \mathrm{D}J\big(\xi; \xi'-\xi\big) + L\lambda^2 \|\xi'-\xi\|_{\mathcal{X}}^2. \end{aligned} \tag{2.83}$$

Letting $\mathcal{A}(\xi) = \big\{(j,t) \in \mathcal{J}\times[0,1] \mid \Psi(\xi) = h_j\big(x^{(\xi)}(t)\big)\big\}$, similar to the equation above, there exists a pair $(j,t) \in \mathcal{A}\big(\xi + \lambda(\xi'-\xi)\big)$ and $s \in (0,1)$ such that, using Corollary 2.28,

$$\begin{aligned} \Psi\big(\xi + \lambda(\xi'-\xi)\big) - \Psi(\xi) &\leq \psi_{j,t}\big(\xi + \lambda(\xi'-\xi)\big) - \Psi(\xi) \\ &\leq \psi_{j,t}\big(\xi + \lambda(\xi'-\xi)\big) - \psi_{j,t}(\xi) \\ &= \mathrm{D}\psi_{j,t}\big(\xi + s\lambda(\xi'-\xi); \lambda(\xi'-\xi)\big) \\ &\leq \lambda \mathrm{D}\psi_{j,t}\big(\xi; \xi'-\xi\big) + L\lambda^2 \|\xi'-\xi\|_{\mathcal{X}}^2. \end{aligned} \tag{2.84}$$

Finally, letting L denote the Lipschitz constant as in Condition 1 in Assumption 2.3, notice:

$$\begin{aligned} \Psi\big(\xi + \lambda(\xi'-\xi)\big) - \Psi(\xi) &= \max_{(j,t)\in\mathcal{J}\times[0,1]} \psi_{j,t}\big(\xi + \lambda(\xi'-\xi)\big) - \max_{(j,t)\in\mathcal{J}\times[0,1]} \psi_{j,t}(\xi) \\ &\leq \max_{(j,t)\in\mathcal{J}\times[0,1]} \psi_{j,t}\big(\xi + \lambda(\xi'-\xi)\big) - \psi_{j,t}(\xi) \\ &\leq L \max_{t\in[0,1]} \left\|\varphi_t(\xi + \lambda(\xi'-\xi)) - \varphi_t(\xi)\right\|_2. \end{aligned} \tag{2.85}$$

We prove Condition 2 by contradiction. That is, using Definition 2.10, we assume that $\theta(\xi) < 0$ and show that for each $\varepsilon > 0$ there exists $\hat{\xi} \in \mathcal{N}_w(\xi,\varepsilon) \cap \big\{\bar{\xi} \in \mathcal{X}_p \mid \Psi(\bar{\xi}) \leq 0\big\}$ such that $J(\hat{\xi}) < J(\xi)$, where $\mathcal{N}_w(\xi,\varepsilon)$ is as defined in Equation (2.29), hence arriving at a contradiction.

Before arriving at this contradiction, we make three initial observations. First, notice that since $\xi \in \mathcal{X}_p$ is a local minimizer of the Switched System Optimal Control Problem, $\Psi(\xi) \leq 0$. Second, consider g as defined in Equation (2.30), which exists by Theorem 2.32, and notice that since $\theta(\xi) < 0$, $g(\xi) \neq \xi$. Third, notice that, as a result of Theorem 2.8, for each $(\xi + \lambda(g(\xi) - \xi)) \in \mathcal{X}_r$ and $\varepsilon' > 0$ there exists a $\xi_\lambda \in \mathcal{X}_p$ such that

$$\left\| x^{(\xi_\lambda)} - x^{(\xi+\lambda(g(\xi)-\xi))} \right\|_{L^\infty} < \varepsilon' \tag{2.86}$$

where $x^{(\xi)}$ is the solution to Differential Equation (2.13).

Now, letting $\varepsilon' = -\frac{\lambda\theta(\xi)}{2L} > 0$ and using Corollary 2.13:

$$\begin{aligned} \left\| x^{(\xi_\lambda)} - x^{(\xi)} \right\|_{L^2} &\leq \left\| x^{(\xi_\lambda)} - x^{(\xi+\lambda(g(\xi)-\xi))} \right\|_{L^2} + \left\| x^{(\xi+\lambda(g(\xi)-\xi))} - x^{(\xi)} \right\|_{L^2} \\ &\leq \left(-\frac{\theta(\xi)}{2L} + L \left\| g(\xi) - \xi \right\|_{\mathcal{X}} \right) \lambda. \end{aligned} \tag{2.87}$$

Next, observe that:

$$\theta(\xi) = \max \left\{ \mathrm{D}J(\xi; g(\xi) - \xi), \max_{(j,t)\in\mathcal{J}\times[0,1]} \mathrm{D}\psi_{j,t}(\xi; g(\xi) - \xi) + \gamma\Psi(\xi) \right\} + \left\| g(\xi) - \xi \right\|_{\mathcal{X}} < 0. \tag{2.88}$$

Also, by Equations (2.83), (2.86), and (2.88), together with Condition 1 in Assumption 2.3 and Corollary 2.13:

$$\begin{aligned} J(\xi_\lambda) - J(\xi) &\leq J(\xi_\lambda) - J\big(\xi + \lambda(g(\xi) - \xi)\big) + J\big(\xi + \lambda(g(\xi) - \xi)\big) - J(\xi) \\ &\leq L \left\| \varphi_1(\xi_\lambda) - \varphi_1(\xi + \lambda(g(\xi) - \xi)) \right\|_2 + \theta(\xi)\lambda + 4A^2L\lambda^2 \\ &\leq L\varepsilon' + \theta(\xi)\lambda + 4A^2L\lambda^2 \\ &\leq \frac{\theta(\xi)\lambda}{2} + 4A^2L\lambda^2, \end{aligned} \tag{2.89}$$

where $A = \max\left\{ \|u\|_2 + 1 \mid u \in U \right\}$ and we used the fact that $\|\xi - \xi'\|_{\mathcal{X}}^2 \leq 4A^2$ and $\mathrm{D}J(\xi; \xi' - \xi) \leq \theta(\xi)$. Hence for each $\lambda \in \left(0, \frac{-\theta(\xi)}{8A^2L}\right)$,

$$J(\xi_\lambda) - J(\xi) < 0. \tag{2.90}$$

Similarly, using Condition 1 in Assumption 2.3, together with Equations (2.84), (2.85), and (2.88), we have:

$$\begin{aligned} \Psi(\xi_\lambda) &\leq \Psi(\xi_\lambda) - \Psi\big(\xi + \lambda(g(\xi) - \xi)\big) + \Psi\big(\xi + \lambda(g(\xi) - \xi)\big) \\ &\leq L \max_{t\in[0,1]} \left\| \varphi_t(\xi_\lambda) - \varphi_t(\xi + \lambda(g(\xi) - \xi)) \right\|_2 + \Psi(\xi) + \big(\theta(\xi) - \gamma\Psi(\xi)\big)\lambda + 4A^2L\lambda^2 \\ &\leq L\varepsilon' + \theta(\xi)\lambda + 4A^2L\lambda^2 + (1 - \gamma\lambda)\Psi(\xi) \\ &\leq \frac{\theta(\xi)\lambda}{2} + 4A^2L\lambda^2 + (1 - \gamma\lambda)\Psi(\xi), \end{aligned} \tag{2.91}$$

where $A = \max\left\{\|u\|_2+1 \mid u \in U\right\}$ and we used the fact that $\|\xi-\xi'\|_{\mathcal{X}}^2 \leq 4A^2$ and $\mathrm{D}\psi_{j,t}(\xi;\xi'-\xi) \leq \theta(\xi) - \gamma\Psi(\xi)$ for each $(j,t) \in \mathcal{J} \times [0,1]$. Hence for each $\lambda \in \left(0, \min\left\{\frac{-\theta(\xi)}{8A^2L}, \frac{1}{\gamma}\right\}\right)$:

$$\Psi(\xi_\lambda) \leq (1-\gamma\lambda)\Psi(\xi) \leq 0. \tag{2.92}$$

Summarizing, suppose $\xi \in \mathcal{X}_p$ is a local minimizer of the Switched System Optimal Control Problem and $\theta(\xi) < 0$. For each $\varepsilon > 0$, by choosing any

$$\lambda \in \left(0, \min\left\{\frac{-\theta(\xi)}{8A^2L}, \frac{1}{\gamma}, \frac{2L\varepsilon}{2L^2\left\|g(\xi)-\xi\right\|_{\mathcal{X}} - \theta(\xi)}\right\}\right), \tag{2.93}$$

we can construct a $\xi_\lambda \in \mathcal{X}_p$ such that $\xi_\lambda \in \mathcal{N}_w(\xi,\varepsilon)$, by Equation (2.87), such that $J(\xi_\lambda) < J(\xi)$, by Equation (2.90), and $\Psi(\xi_\lambda) \leq 0$, by Equation (2.92). Therefore, ξ is not a local minimizer of the Switched System Optimal Control Problem, which is a contradiction and proves Condition 2. □

Approximating Relaxed Inputs

In this subsection, we prove that the projection operation, ρ_N, allows us to control the quality of approximation between the trajectories generated by a relaxed discrete input and its projection. First, we prove for $d \in \mathcal{D}_r$, $\mathcal{F}_N(d) \in \mathcal{D}_r$ and $\mathcal{P}_N\big(\mathcal{F}_N(d)\big) \in \mathcal{D}_p$:

Lemma 2.35. *Let $d \in \mathcal{D}_r$, $\mathcal{F}_N$ be as defined in Equation (2.34), and $\mathcal{P}_N$ be as defined in Equation (2.35). Then for each $N \in \mathbb{N}$ and $t \in [0,1]$:*

(1) $[\mathcal{F}_N(d)]_i(t) \in [0,1]$,

(2) $\sum_{i=1}^q [\mathcal{F}_N(d)]_i(t) = 1$,

(3) $\big[\mathcal{P}_N\big(\mathcal{F}_N(d)\big)\big]_i(t) \in \{0,1\}$,

(4) $\sum_{i=1}^q \big[\mathcal{P}_N\big(\mathcal{F}_N(d)\big)\big]_i(t) = 1$.

Proof. Condition 1 follows due to the result in Section 3.3 in [Haa10]. Condition 2 follows since the wavelet approximation is linear, thus,

$$\begin{aligned}\sum_{i=1}^q [\mathcal{F}_N(d)]_i &= \sum_{i=1}^q \left(\langle d_i, \mathbb{1}\rangle + \sum_{k=0}^N \sum_{j=0}^{2^k-1} \langle d_i, b_{kj}\rangle \frac{b_{kj}}{\|b_{kj}\|_{L^2}^2}\right) \\ &= \langle \mathbb{1}, \mathbb{1}\rangle + \sum_{k=0}^N \sum_{j=0}^{2^k-1} \langle \mathbb{1}, b_{kj}\rangle \frac{b_{kj}}{\|b_{kj}\|_{L^2}^2} = \mathbb{1},\end{aligned} \tag{2.94}$$

where the last equality holds since $\langle \mathbb{1}, b_{kj}\rangle = 0$ for each k, j.

Conditions 3 and 4 are direct consequences of the definition of $\mathcal{P}_N$, since $\mathcal{P}_N$ can only take the values 0 or 1, and only one coordinate is equal to 1 at any given time $t \in [0,1]$. □

Recall that in order to avoid the introduction of additional notation, we let the coordinate-wise application of $\mathcal{F}_N$ to some relaxed discrete input $d \in \mathcal{D}_r$ be denoted as $\mathcal{F}_N(d)$ and similarly for some continuous input $u \in \mathcal{U}$, but in fact $\mathcal{F}_N$ as originally defined took $L^2([0,1],\mathbb{R}) \cap BV([0,1],\mathbb{R})$ to $L^2([0,1],\mathbb{R}) \cap BV([0,1],\mathbb{R})$. Next, we prove that the wavelet approximation allows us to control the quality of approximation:

Lemma 2.36. *Let $f \in L^2([0,1],\mathbb{R}) \cap BV([0,1],\mathbb{R})$, then*

$$\|f - \mathcal{F}_N(f)\|_{L^2} \leq \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^N \mathrm{V}(f)\,, \tag{2.95}$$

where $\mathcal{F}_N$ is as defined in Equation (2.34) *and* $\mathrm{V}(\cdot)$ *is as defined in Equation* (2.3).

Proof. Since L^2 is a Hilbert space and the collection $\{b_{kj}\}_{k,j}$ is a basis, then

$$f = \langle f, \mathbb{1}\rangle + \sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1}\langle f, b_{kj}\rangle \frac{b_{kj}}{\|b_{kj}\|_{L^2}^2}. \tag{2.96}$$

Note that $\|b_{kj}\|_{L^2}^2 = 2^{-k}$ and that

$$v_{kj}(t) = \int_0^t b_{kj}(s)ds = \begin{cases} t - j2^{-k} & \text{if } t \in \left[j2^{-k}, \left(j+\frac{1}{2}\right)2^{-k}\right), \\ -t + (j+1)2^{-k} & \text{if } t \in \left[\left(j+\frac{1}{2}\right)2^{-k}, (j+1)\,2^{-k}\right), \\ 0 & \text{otherwise,} \end{cases} \tag{2.97}$$

thus $\|v_{kj}\|_{L^\infty} = 2^{-k-1}$. Now, using integration by parts, and since $f \in BV([0,1],\mathbb{R})$,

$$|\langle f, b_{kj}\rangle| = \left|\int_{j2^{-k}}^{(j+1)2^{-k}} \dot{f}(t)v_{kj}(t)dt\right| \leq 2^{-k-1}\int_{j2^{-k}}^{(j+1)2^{-k}} \left|\dot{f}(t)\right|dt \tag{2.98}$$

Finally, Parseval's Identity for Hilbert spaces (Theorem 5.27 in [Fol99]) implies that

$$\begin{aligned} \left\|f - \mathcal{F}_N(f)\right\|_{L^2}^2 &= \sum_{k=N+1}^{\infty}\sum_{j=0}^{2^k-1}\frac{|\langle f, b_{kj}\rangle|^2}{\|b_{kj}\|_{L^2}^2} \\ &\leq \sum_{k=N+1}^{\infty} 2^{-k-2}\sum_{j=0}^{2^k-1}\left(\int_{j2^{-k}}^{(j+1)2^{-k}} \left|\dot{f}(t)\right|dt\right)^2 \\ &\leq 2^{-N-2}\,\mathrm{V}(f)^2\,, \end{aligned} \tag{2.99}$$

as desired, where in the last inequality we used Theorem 2.1. □

The following lemma is fundamental to find a rate of convergence for the approximation of the solution of differential equations using relaxed inputs:

Lemma 2.37. *There exists $K > 0$ such that for each $d \in \mathcal{D}_r$ and $f \in L^2([0,1],\mathbb{R}^q) \cap BV([0,1],\mathbb{R}^q)$,*

$$\left|\left\langle d - \mathcal{P}_N\big(\mathcal{F}_N(d)\big), f\right\rangle\right| \leq K\left(\left(\frac{1}{\sqrt{2}}\right)^N \|f\|_{L^2}\,\mathrm{V}(d) + \left(\frac{1}{2}\right)^N \mathrm{V}(f)\right), \tag{2.100}$$

where $\mathcal{F}_N$ is as defined Equation (2.34), $\mathcal{P}_N$ is as defined in Equation (2.35), and $\mathrm{V}(\cdot)$ is as defined in Equation (2.3).

Proof. To simplify our notation, let $t_k = \frac{k}{2^N}$, $p_{ik} = [\mathcal{F}_N(d)]_i(t_k)$, $S_{ik} = \sum_{j=1}^{i} p_{jk}$, and

$$A_{ik} = \left[t_k + \frac{1}{2^N}S_{(i-1)k}, t_k + \frac{1}{2^N}S_{ik}\right). \tag{2.101}$$

Also let us denote the indicator function of the set A_{ik} by $\mathbb{1}_{A_{ik}}$. Consider

$$\left\langle \mathcal{F}_N(d) - \mathcal{P}_N\big(\mathcal{F}_N(d)\big), f\right\rangle = \sum_{k=0}^{2^N-1}\sum_{i=1}^{q}\int_{t_k}^{t_{k+1}} \big(p_{ik} - \mathbb{1}_{A_{ik}}(t)\big) f_i(t)dt. \tag{2.102}$$

Let $w_{ik} : [0,1] \to \mathbb{R}$ be defined by

$$\begin{aligned} w_{ik}(t) &= \int_{t_k}^{t} p_{ik} - \mathbb{1}_{A_{ik}}(s)ds \\ &= \begin{cases} p_{ik}(t - t_k) & \text{if } t \in \left[t_k, t_k + \frac{1}{2^N}S_{(i-1)k}\right), \\ \frac{1}{2^N}p_{ik}S_{(i-1)k} + (p_{ik} - 1)\left(t - t_k - \frac{1}{2^N}S_{(i-1)k}\right) & \text{if } t \in A_{ik}, \\ \frac{1}{2^N}p_{ik}(S_{ik} - 1) + p_{ik}\left(t - t_k - \frac{1}{2^N}S_{ik}\right) & \text{if } t \in \left[t_k + \frac{1}{2^N}S_{ik}, t_{k+1}\right), \end{cases} \end{aligned} \tag{2.103}$$

when $t \in [t_k, t_{k+1}]$, and $w_{ik}(t) = 0$ otherwise. Note that $\|w_{ik}\|_{L^\infty} \leq \frac{p_{ik}}{2^N}$. Thus, using integration by parts,

$$\left|\int_{t_k}^{t_{k+1}} \big(p_{ik} - \mathbb{1}_{A_{ik}}(t)\big) f_i(t)dt\right| = \left|\int_{t_k}^{t_{k+1}} w(t)\dot{f}_i(t)dt\right| \leq \frac{p_{ik}}{2^N}\int_{t_k}^{t_{k+1}} \big|\dot{f}_i(t)\big|dt, \tag{2.104}$$

and

$$\begin{aligned} \left|\left\langle \mathcal{F}_N(d) - \mathcal{P}_N\big(\mathcal{F}_N(d)\big), f\right\rangle\right| &\leq \frac{1}{2^N}\sum_{k=0}^{2^N-1}\int_{t_k}^{t_{k+1}}\sum_{i=1}^{q} p_{ik}\big|\dot{f}_i(t)\big|dt \\ &\leq \frac{1}{2^N}\,\mathrm{V}(f)\,. \end{aligned} \tag{2.105}$$

where the last inequality follows by Hölder's Inequality and Theorem 2.1.

Also, by Lemma 2.36 we have that

$$\big\|d_i - [\mathcal{F}(d)]_i\big\|_{L^2} \leq \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^N \mathrm{V}(d_i)\,. \tag{2.106}$$

Hence, using Cauchy-Schwartz's Inequality,

$$\left|\left\langle d - \mathcal{P}_N\big(\mathcal{F}_N(d)\big), f\right\rangle\right| \leq \|d - \mathcal{F}_N(d)\|_{L^2}\|f\|_{L^2} + \left|\left\langle \mathcal{F}_N(d) - \mathcal{P}_N\big(\mathcal{F}_N(d)\big), f\right\rangle\right|, \tag{2.107}$$

and the desired result follows from Equations (2.99) and (2.105). □

Note that Lemma 2.37 does not prove convergence of $\mathcal{P}_N\big(\mathcal{F}_N(d)\big)$ to d in the weak topology on $\mathcal{D}_r$. Such a result is indeed true, i.e. $\mathcal{P}_N\big(\mathcal{F}_N(d)\big)$ does converge in the weak topology to d, and it can be shown using an argument similar to the one used in Lemma 1 in [Sus72]. The reason we chose to prove a weaker result is because in this case we get an explicit rate of convergence, which is fundamental to the construction of our optimization algorithm because it allows us to bound the quality of approximation of the state trajectory.

Theorem 2.38. *Let ρ_N be defined as in Equation (2.37) and φ_t be defined as in Equation (2.14). Then there exists $K > 0$ such that for each $\xi = (u, d) \in \mathcal{X}_r$ and for each $t \in [0, 1]$,*

$$\big\|\varphi_t\big(\rho_N(\xi)\big) - \varphi_t(\xi)\big\|_2 \leq K\left(\frac{1}{\sqrt{2}}\right)^N \big(\mathrm{V}(\xi) + 1\big), \tag{2.108}$$

where $\mathrm{V}(\cdot)$ is as defined in Equation (2.3).

Proof. To simplify our notation, let us denote $u_N = \mathcal{F}_N(u)$ and $d_N = \mathcal{P}_N\big(\mathcal{F}_N(d)\big)$, thus $\rho_N(\xi) = (u_N, d_N)$. Consider

$$\big\|x^{(u_N,d_N)}(t) - x^{(u,d)}(t)\big\|_2 \leq \big\|x^{(u_N,d_N)}(t) - x^{(u,d_N)}(t)\big\|_2 + \big\|x^{(u,d_N)}(t) - x^{(u,d)}(t)\big\|_2. \tag{2.109}$$

The main result of the theorem will follow from upper bounds from each of these two parts.

Note that

$$\begin{aligned}\big\|x^{(u_N,d_N)}(t) - x^{(u,d_N)}(t)\big\|_2 \leq \\ &\leq \int_0^1 \big\|f\big(s, x^{(u_N,d_N)}(s), u_N(s), d_N(s)\big) - f\big(s, x^{(u,d_N)}(s), u(s), d_N(s)\big)\big\|_2 ds \\ &\leq L\int_0^1 \big\|x^{(u_N,d_N)}(s) - x^{(u,d_N)}(s)\big\|_2 + \big\|u_N(s) - u(s)\big\|_2 ds,\end{aligned} \tag{2.110}$$

thus, using the Bellman-Gronwall Inequality (Lemma 5.6.4 in [Pol97]) together with the result in Lemma 2.36 we get

$$\big\|x^{(u_N,d_N)}(t) - x^{(u,d_N)}(t)\big\|_2 \leq \frac{Le^L\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}}\right)^N \mathrm{V}(u) \tag{2.111}$$

On the other hand,

$$\begin{aligned}x^{(u,d_N)}(t) - x^{(u,d)}(t) = \int_0^t \sum_{i=1}^q \big([d_N]_i(s) - d_i(s)\big) f\big(s, x^{(u,d)}(s), u(s), e_i\big) ds + \\ + \int_0^t \sum_{i=1}^q [d_N]_i(s)\left(f\big(s, x^{(u,d_N)}(s), u(s), e_i\big) - f\big(s, x^{(u,d)}(s), u(s), e_i\big)\right) ds,\end{aligned} \tag{2.112}$$

thus,

$$\left\|x^{(u,d_N)}(t) - x^{(u,d)}(t)\right\|_2 \leq \left\| \int_0^1 \sum_{i=1}^{q} \left([d_N]_i(s) - d_i(s)\right) f\left(s, x^{(u,d)}(s), u(s), e_i\right) ds \right\|_2 + \\ + L \int_0^1 \left\|x^{(u,d_N)}(s) - x^{(u,d)}(s)\right\|_2 ds. \quad (2.113)$$

Using the Bellman-Gronwall Inequality we get

$$\left\|x^{(u,d_N)}(t) - x^{(u,d)}(t)\right\|_2 \leq e^L \left\| \int_0^1 \sum_{i=1}^{q} \left([d_N]_i(s) - d_i(s)\right) f\left(t, x^{(u,d)}(s), u(s), e_i\right) ds \right\|_2 . \quad (2.114)$$

Recall that f maps to $\mathbb{R}^n$, so let us denote the k–th coordinate of f by f_k. Let $v_{ki}(t) = f_k\left(t, x^{(u,d)}(t), u(t), e_i\right)$ and $v_k = (v_{k1}, \ldots, v_{kq})$, then v_k is of bounded variation. Indeed, by Theorem 2.1 and Condition 1 in Corollary 2.5, we have that $\mathrm{V}\left(x^{(\xi)}\right) \leq C$. Thus, by Condition 1 in Assumption 2.2 and again using Theorem 2.1, we get that, for each $i \in \mathcal{Q}$,

$$\mathrm{V}(v_{ki}) \leq L\left(1 + C + \mathrm{V}(u)\right). \quad (2.115)$$

Moreover, Condition 1 in Corollary 2.5 directly imply that $\|v_{ki}\|_{L^2} \leq C$. Hence, Lemma 2.37 implies that there exists $K > 0$ such that

$$|\langle d - d_N, v_k \rangle| \leq K \left(\left(\frac{1}{\sqrt{2}}\right)^N C\, \mathrm{V}(d) + q \left(\frac{1}{2}\right)^N (1 + C + \mathrm{V}(u)) \right). \quad (2.116)$$

Since Equation (2.116) is satisfied for each $k \in \{1, \ldots, n\}$, then after ordering the constants and noting that $2^N \geq 2^{\frac{N}{2}}$ for each $N \in \mathbb{N}$, together with Equation (2.111) we get the desired result. □

Convergence of the Algorithm

To prove the convergence of our algorithm, we employ a technique similar to the one prescribed in Section 1.2 in [Pol97]. Summarizing the technique, one can think of an algorithm as discrete-time dynamical system, whose desired stable equilibria are characterized by the stationary points of its optimality function, i.e. points $\xi \in \mathcal{X}_p$ where $\theta(\xi) = 0$, since we know from Theorem 2.34 that all local minimizers are stationary.

Before applying this line of reasoning to our algorithm, we present a simplified version of this argument for a general unconstrained optimization problem defined over a metric space $\mathcal{S}$. This is done in the interest of clarity. Inspired by the stability analysis of dynamical systems, a sufficient condition for the convergence of optimization algorithms can be formulated by requiring that the cost function satisfy a notion of sufficient descent with respect to an optimality function:

Definition 2.39. *Let $\mathcal{S}$ be a metric space, and consider the problem of minimizing the cost function $J : \mathcal{S} \to \mathbb{R}$. We say that a function $\Gamma : \mathcal{S} \to \mathcal{S}$ has the* sufficient descent property *with respect to an optimality function $\theta : \mathcal{S} \to (-\infty, 0]$ if for each $x \in \mathcal{S}$ with $\theta(x) < 0$, there exists a $\delta_x > 0$ and $O_x \subset \mathcal{S}$, a neighborhood of x, such that:*

$$J\big(\Gamma(x')\big) - J(x') \leq -\delta_x, \quad \forall x' \in O_x. \tag{2.117}$$

Importantly, a function satisfying the sufficient property can be proven to approach the zeros of the optimality function:

Theorem 2.40 (Theorem 1.2.8 in [Pol97])**.** *Let $\mathcal{S}$ be a metric space. Consider the problem of minimizing the cost function $J : \mathcal{S} \to \mathbb{R}$. Suppose that $\mathcal{S}$ is a a metric space and a function $\Gamma : \mathcal{S} \to \mathcal{S}$ has the sufficient descent property with respect to an optimality function $\theta : \mathcal{S} \to (-\infty, 0]$, as in Definition 2.39. Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence such that, for each $j \in \mathbb{N}$:*

$$x_{j+1} = \begin{cases} \Gamma(x_j) & \text{if } \theta(x_j) < 0, \\ x_j & \text{if } \theta(x_j) = 0. \end{cases} \tag{2.118}$$

Then every accumulation point of $\{x_j\}_{j\in\mathbb{N}}$ belongs to the set of zeros of the optimality function θ.

Theorem 2.40, as originally stated in [Pol97], requires $\mathcal{S}$ to be a Euclidean space, but the result as presented here can be proven without requiring this property using the same original argument. Though Theorem 2.40 proves that the accumulation point of a sequence generated by Γ converges to a stationary point of the optimality function, it does not prove the existence of the accumulation point. This is in general not a problem for finite-dimensional optimization problems since the level sets of the cost function are usually compact, thus every sequence produced by a descent method has at least one accumulation point. On the other hand, infinite-dimensional problems, such as optimal control problems, do not have this property, since bounded sets may not be compact in infinite-dimensional vector spaces. Thus, even though Theorem 2.40 can be applied to both finite-dimensional and infinite-dimensional optimization problems, the result is much weaker in the latter case.

The issue mentioned above has been addressed several times in the literature [Axe+08; PW84; WE12a; WE12b], by formulating a stronger version of sufficient descent:

Definition 2.41 (Definition 2.1 in [Axe+08])**.** *Let $\mathcal{S}$ be a metric space, and consider the problem of minimizing the cost function $J : \mathcal{S} \to \mathbb{R}$. A function $\Gamma : \mathcal{S} \to \mathcal{S}$ has the* uniform sufficient descent property *with respect to an optimality function $\theta : \mathcal{S} \to (-\infty, 0]$ if for each $C > 0$ there exists a $\delta_C > 0$ such that, for every $x \in \mathcal{S}$ with $\theta(x) < 0$,*

$$J\big(\Gamma(x)\big) - J(x) \leq -\delta_C. \tag{2.119}$$

A sequence of points generated by an algorithm satisfying this property, under mild assumptions, can be shown to approach the zeros of the optimality function:

Theorem 2.42 (Proposition 2.1 in [Axe+08])**.** *Let $\mathcal{S}$ be a metric space. Consider the problem of minimizing a lower bounded cost function $J : \mathcal{S} \to [\alpha, \infty)$. Suppose that $\mathcal{S}$ is a a metric space and $\Gamma : \mathcal{S} \to \mathcal{S}$ satisfies the uniform sufficient descent property with respect to an optimality function $\theta : \mathcal{S} \to (-\infty, 0]$, as stated in Definition 2.41. Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence such that, for each $j \in \mathbb{N}$:*

$$x_{j+1} = \begin{cases} \Gamma(x_j) & \text{if } \theta(x_j) < 0, \\ x_j & \text{if } \theta(x_j) = 0. \end{cases} \tag{2.120}$$

Then,

$$\lim_{j\to\infty} \theta(x_j) = 0. \tag{2.121}$$

Proof. Suppose that $\liminf_{j\to\infty} \theta(x_j) = -2\varepsilon < 0$. Then there exists a subsequence $\{x_{j_k}\}_{k\in\mathbb{N}}$ such that $\theta(x_{j_k}) < -\varepsilon$ for each $k \in \mathbb{N}$. Definition 2.41 implies that there exists δ_ε such that

$$J(x_{j_k+1}) - J(x_{j_k}) \leq -\delta_\varepsilon, \quad \forall k \in \mathbb{N}. \tag{2.122}$$

But this is a contradiction, since $J(x_{j+1}) \leq J(x_j)$ for each $j \in \mathbb{N}$, thus $J(x_j) \to -\infty$ as $j \to \infty$, contrary to the assumption that J is lower bounded. □

Note that Theorem 2.42 does not assume the existence of accumulation points of the sequence $\{x_j\}_{j\in\mathbb{N}}$. Thus, this Theorem remains valid even when the sequence generated by Γ does not have accumulation points. This becomes tremendously useful in infinite-dimensional problems where the level sets of the cost function may not be compact. Though we include these results for the sake of completeness of presentation, our proof of convergence of the sequence of points generated by Algorithm 2.1 does not make explicit use of Theorem 2.42. The line of argument is similar, but our approach, as described in Theorem 2.46, requires special treatment due to the projection operation, ρ_N, as defined in Equation (2.37) and the existence of constraints.

Now, we begin the convergence proof of Algorithm 2.1 by showing that the Armijo algorithm, as defined in Equation (2.32), terminates after a finite number of steps and its value is bounded.

Lemma 2.43. *Let $\alpha \in (0, 1)$ and $\beta \in (0, 1)$. For every $\delta > 0$ there exists an $M^*_\delta < \infty$ such that if $\theta(\xi) \leq -\delta$ for $\xi \in \mathcal{X}_p$, then $\mu(\xi) \leq M^*_\delta$, where θ is as defined in Equation* (2.30) *and μ is as defined in Equation* (2.32)*.*

Proof. Given $\xi' \in \mathcal{X}$ and $\lambda \in [0, 1]$, using the Mean Value Theorem and Corollary 2.25 we have that there exists $s \in (0, 1)$ such that

$$\begin{aligned} J\big(\xi + \lambda(\xi' - \xi)\big) - J(\xi) &= \mathrm{D}J\big(\xi + s\lambda(\xi' - \xi); \lambda(\xi' - \xi)\big) \\ &\leq \lambda \mathrm{D}J\big(\xi; \xi' - \xi\big) + L\lambda^2 \|\xi' - \xi\|^2_{\mathcal{X}}. \end{aligned} \tag{2.123}$$

Letting $\mathcal{A}(\xi) = \big\{(j,t) \in \mathcal{J} \times [0,1] \mid \Psi(\xi) = h_j\big(x^{(\xi)}(t)\big)\big\}$, then there exists a pair $(j,t) \in \mathcal{A}\big(\xi + \lambda(\xi' - \xi)\big)$ and $s \in (0,1)$ such that, using Corollary 2.28,

$$\begin{aligned}\Psi\big(\xi + \lambda(\xi' - \xi)\big) - \Psi(\xi) &\leq \psi_{j,t}\big(\xi + \lambda(\xi' - \xi)\big) - \Psi(\xi) \\ &\leq \psi_{j,t}\big(\xi + \lambda(\xi' - \xi)\big) - \psi_{j,t}(\xi) \\ &= \mathrm{D}\psi_{j,t}\big(\xi + s\lambda(\xi' - \xi); \lambda(\xi' - \xi)\big) \\ &\leq \lambda \mathrm{D}\psi_{j,t}\big(\xi; \xi' - \xi\big) + L\lambda^2 \|\xi' - \xi\|_{\mathcal{X}}^2.\end{aligned} \tag{2.124}$$

Now let us assume that $\Psi(\xi) \leq 0$, and consider g as defined in Equation (2.30). Then

$$\theta(\xi) = \max\left\{\mathrm{D}J(\xi; g(\xi) - \xi), \max_{(j,t) \in \mathcal{J} \times [0,1]} \mathrm{D}\psi_{j,t}(\xi; g(\xi) - \xi) + \gamma\Psi(\xi)\right\} \leq -\delta, \tag{2.125}$$

and using Equation (2.123),

$$J\big(\xi + \beta^k(g(\xi) - \xi)\big) - J(\xi) - \alpha\beta^k\theta(\xi) \leq -(1-\alpha)\delta\beta^k + 4A^2L\beta^{2k}, \tag{2.126}$$

where $A = \max\big\{\|u\|_2 + 1 \mid u \in U\big\}$. Hence, for each $k \in \mathbb{N}$ such that $\beta^k \leq \frac{(1-\alpha)\delta}{4A^2L}$ we have that

$$J\big(\xi + \beta^k(g(\xi) - \xi)\big) - J(\xi) \leq \alpha\beta^k\theta(\xi). \tag{2.127}$$

Similarly, using Equations (2.124) and (2.125),

$$\Psi\big(\xi + \beta^k(g(\xi) - \xi)\big) - \Psi(\xi) + \beta^k\big(\gamma\Psi(\xi) - \alpha\theta(\xi)\big) \leq -\delta\beta^k + 4A^2L\beta^{2k}, \tag{2.128}$$

hence for each $k \in \mathbb{N}$ such that $\beta^k \leq \min\left\{\frac{(1-\alpha)\delta}{4A^2L}, \frac{1}{\gamma}\right\}$ we have that

$$\Psi\big(\xi + \beta^k(g(\xi) - \xi)\big) - \alpha\beta^k\theta(\xi) \leq \big(1 - \beta^k\gamma\big)\,\Psi(\xi) \leq 0. \tag{2.129}$$

If $\Psi(\xi) > 0$ then

$$\max_{(j,t) \in \mathcal{J} \times [0,1]} \mathrm{D}\psi_{j,t}(\xi; g(\xi) - \xi) \leq \theta(\xi) \leq -\delta, \tag{2.130}$$

thus, from Equation (2.124),

$$\Psi\big(\xi + \beta^k(g(\xi) - \xi)\big) - \Psi(\xi) - \alpha\beta^k\theta(\xi) \leq -(1-\alpha)\delta\beta^k + 4A^2L\beta^{2k}. \tag{2.131}$$

Hence, for each $k \in \mathbb{N}$ such that $\beta^k \leq \frac{(1-\alpha)\delta}{4A^2L}$ we have that

$$\Psi\big(\xi + \beta^k(g(\xi) - \xi)\big) - \Psi(\xi) \leq \alpha\beta^k\theta(\xi). \tag{2.132}$$

Finally, let

$$M_\delta^* = 1 + \max\left\{\log_\beta\left(\frac{(1-\alpha)\delta}{4A^2L}\right), \log_\beta\left(\frac{1}{\gamma}\right)\right\}, \tag{2.133}$$

then from Equations (2.127), (2.129), and (2.132), we get that $\mu(\xi) \leq M_\delta^*$ as desired. □

Next, we show that the determination of the frequency at which to perform pulse width modulation as defined in Equation (2.38) terminates after a finite number of steps.

Lemma 2.44. *Let $\alpha \in (0,1)$, $\bar{\alpha} \in (0,\infty)$, $\beta \in (0,1)$, $\bar{\beta} \in \left(\frac{1}{\sqrt{2}},1\right)$, and $\xi \in \mathcal{X}_p$. If $\theta(\xi) < 0$, then $\nu(\xi) < \infty$, where θ is as defined in Equation* (2.30) *and ν is as defined in Equation* (2.38)*.*

Proof. Throughout the proof, we leave out the natural inclusion taking $\xi \in \mathcal{X}_p$ to $\xi \in \mathcal{X}_r$. To simplify our notation let us denote $M = \mu(\xi)$ and $\xi' = \xi + \beta^M\big(g(\xi) - \xi\big)$. Theorem 2.38 implies that there exists $K > 0$ such that

$$J\big(\rho_N(\xi')\big) - J(\xi') \leq KL\left(\frac{1}{\sqrt{2}}\right)^N \big(\mathrm{V}(\xi') + 1\big), \tag{2.134}$$

where L is the constant defined in Assumption 2.3 and $\mathrm{V}(\cdot)$ is as defined in Equation (2.3).

Let $\mathcal{A}(\xi) = \big\{(j,t) \in \{1,\dots,N_c\} \times [0,1] \mid \Psi(\xi) = h_j\big(x^{(\xi)}(t)\big)\big\}$, then for each pair $(j,t) \in \mathcal{A}\big(\rho_N(\xi')\big)$ we have that

$$\begin{aligned} \Psi\big(\rho_N(\xi')\big) - \Psi(\xi') &= \psi_{j,t}\big(\rho_N(\xi')\big) - \Psi(\xi') \\ &\leq \psi_{j,t}\big(\rho_N(\xi')\big) - \psi_{j,t}(\xi') \\ &\leq KL\left(\frac{1}{\sqrt{2}}\right)^N \big(\mathrm{V}(\xi') + 1\big). \end{aligned} \tag{2.135}$$

Recall that $\bar{\alpha} \in (0,\infty)$, $\bar{\beta} \in \left(\frac{1}{\sqrt{2}},1\right)$, and $\omega \in (0,1)$, hence there exists $N_0 \in \mathbb{N}$ such that, for each $N \geq N_0$,

$$KL\left(\frac{1}{\sqrt{2}}\right)^N \big(\mathrm{V}(\xi') + 1\big) \leq -\bar{\alpha}\bar{\beta}^N\theta(\xi). \tag{2.136}$$

Also, there exists $N_1 \geq N_0$ such that, for each $N \geq N_1$,

$$\bar{\alpha}\bar{\beta}^N \leq (1-\omega)\alpha\beta^M. \tag{2.137}$$

Now suppose that $\Psi(\xi) \leq 0$, then, for each $N \geq N_1$,

$$\begin{aligned} J\big(\rho_N(\xi')\big) - J(\xi) &= J\big(\rho_N(\xi')\big) - J(\xi') + J(\xi') - J(\xi) \\ &\leq \big(\alpha\beta^M - \bar{\alpha}\bar{\beta}^N\big)\,\theta(\xi), \end{aligned} \tag{2.138}$$

and

$$\begin{aligned} \Psi\big(\rho_N(\xi')\big) &= \Psi\big(\rho_N(\xi')\big) - \Psi(\xi') + \Psi(\xi') \\ &\leq \big(\alpha\beta^M - \bar{\alpha}\bar{\beta}^N\big)\,\theta(\xi) \\ &\leq 0. \end{aligned} \tag{2.139}$$

Similarly, if $\Psi(\xi) > 0$ then, using the same argument as above, we have that

$$\Psi\big(\rho_N(\xi')\big) - \Psi(\xi) \leq \big(\alpha\beta^M - \bar{\alpha}\bar{\beta}^N\big)\,\theta(\xi). \tag{2.140}$$

Therefore, from Equations (2.138), (2.139), and (2.140), it follows that $\nu(\xi) \leq N_1$ as desired. □

The following lemma proves that, once Algorithm 2.1 finds a feasible point, every point generated afterwards is also feasible. We omit the proof since it follows directly from the definition of ν in Equation (2.38).

Lemma 2.45. *Let Γ be defined as in Equation (2.39) and let Ψ be as defined in Equation (2.16). Let $\{\xi_i\}_{i\in\mathbb{N}}$ be a sequence generated by Algorithm 2.1. If there exists $i_0 \in \mathbb{N}$ such that $\Psi(\xi_{i_0}) \leq 0$, then $\Psi(\xi_i) \leq 0$ for each $i \geq i_0$.*

Employing these preceding results, we can prove the convergence of Algorithm 2.1 to a point that satisfies our optimality condition by employing an argument similar to the one used in the proof of Theorem 2.42:

Theorem 2.46. *Let θ be defined as in Equation (2.30). If $\{\xi_i\}_{i\in\mathbb{N}}$ is a sequence generated by Algorithm 2.1, then $\lim_{i\to\infty} \theta(\xi_i) = 0$.*

Proof. If the sequence produced by Algorithm 2.1 is finite, then the theorem is trivially satisfied, so we assume that the sequence is infinite.

Suppose the theorem is not true, then $\liminf_{i\to\infty} \theta(\xi_i) = -2\delta < 0$ and therefore there exists $k_0 \in \mathbb{N}$ and a subsequence $\{\xi_{i_k}\}_{k\in\mathbb{N}}$ such that $\theta(\xi_{i_k}) \leq -\delta$ for each $k \geq k_0$. Also, recall that $\nu(\xi)$ was chosen such that, given $\mu(\xi)$,

$$\alpha\beta^{\mu(\xi)} - \bar{\alpha}\bar{\beta}^{\nu(\xi)} \geq \omega\alpha\beta^{\mu(\xi)}, \tag{2.141}$$

where $\omega \in (0, 1)$ is a parameter.

From Lemma 2.43 we know that there exists M^*_δ, which depends on δ, such that $\beta^{\mu(\xi)} \geq \beta^{M^*_\delta}$. Suppose that the subsequence $\{\xi_{i_k}\}_{k\in\mathbb{N}}$ is eventually feasible, then, by Lemma 2.45, without loss of generality we can assume that the sequence is always feasible. Thus, given Γ as defined in Equation (2.39),

$$\begin{aligned} J\big(\Gamma(\xi_{i_k})\big) - J(\xi_{i_k}) &\leq \big(\alpha\beta^{\mu(\xi)} - \bar{\alpha}\bar{\beta}^{\nu(\xi)}\big)\theta(\xi_{i_k}) \\ &\leq -\omega\alpha\beta^{\mu(\xi)}\delta \\ &\leq -\omega\alpha\beta^{M^*_\delta}\delta. \end{aligned} \tag{2.142}$$

This inequality, together with the fact that $J(\xi_{i+1}) \leq J(\xi_i)$ for each $i \in \mathbb{N}$, implies that $\liminf_{k\to\infty} J(\xi_{i_k}) = -\infty$, but this is a contradiction since J is lower bounded, which follows from Condition 1 in Corollary 2.15.

The case when the sequence is never feasible is analogous after noting that, since the subsequence is infeasible, then $\Psi(\xi_{i_k}) > 0$ for each $k \in \mathbb{N}$, establishing a similar contradiction. □

2.4 Implementable Algorithm

In this section, we describe how to implement Algorithm 2.1 given the various algorithmic components derived in the Section 2.3. Numerically computing a solution to the Switched System Optimal Control Problem defined as in Equation (2.18) demands employing some form of discretization. When numerical integration is introduced, the original infinite-dimensional optimization problem defined over function spaces is replaced by a finite-dimensional discrete-time optimal control problem. Changing the discretization precision results in an infinite sequence of such approximating problems.

Our goal is the construction of an implementable algorithm that generates a sequence of points by recursive application that converge to a point that satisfies the optimality condition defined in Equation (2.30). Given a particular choice of discretization precision, at a high level, our algorithm solves a finite dimensional optimization problem and terminates its operation when a discretization improvement test is satisfied. At this point, a finer discretization precision is chosen, and the whole process is repeated, using the last iterate, obtained with the coarser discretization precision as a "warm start."

In this section, we begin by describing our discretization strategy, which allows us to define our discretized optimization spaces. Next, we describe how to construct discretized trajectories, cost, constraints, and optimal control problems. This allows us to define a discretized optimality function, and a notion of *consistent approximation* between the optimality function and its discretized counterpart. We conclude by constructing our numerically implementable optimal control algorithm for constrained switched systems.

Discretized Optimization Space

To define our discretization strategy, for any positive integer N we first define the *N–th switching time space* as:

$$\mathcal{T}_N = \left\{ \{\tau_0, \ldots, \tau_k\} \subset [0,1] \mid 0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_k = 1,\ |\tau_i - \tau_{i-1}| \leq \frac{1}{2^N}\ \forall i \in \{1, \ldots, k\} \right\}, \tag{2.143}$$

i.e. $\mathcal{T}_N$ is the collection of finite partitions of $[0,1]$ whose samples have a maximum distance of $\frac{1}{2^N}$. For notational convenience, given $\tau \in \mathcal{T}_N$, we define $|\tau|$ as the cardinality of τ. Importantly, notice that the sets $\mathcal{T}_N$ are nested, i.e. for each $N \in \mathbb{N}$, $\mathcal{T}_{N+1} \subset \mathcal{T}_N$.

We utilize the switching time spaces to define a sequence of finite dimensional subspaces of $\mathcal{X}_p$ and $\mathcal{X}_r$. Given $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, and $k \in \{0, \ldots, |\tau| - 1\}$, we define $\pi_{\tau,k} : [0,1] \to \mathbb{R}$ that scales the discretization:

$$\pi_{\tau,k}(t) = \begin{cases} 1 & \text{if } t \in [\tau_k, \tau_{k+1}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.144}$$

Using this definition, we define $\mathcal{D}_{\tau,p}$, a subspace of the discrete input space, as:

$$\mathcal{D}_{\tau,p} = \left\{ d \in \mathcal{D}_p \mid d = \sum_{k=0}^{|\tau|-1} \bar{d}_k \pi_{\tau,k},\ \bar{d}_k \in \Sigma_p^q\ \forall k \right\}. \tag{2.145}$$

Similarly, we define $\mathcal{D}_{\tau,r}$, a subspace of the relaxed discrete input space, as:

$$\mathcal{D}_{\tau,r} = \left\{ d \in \mathcal{D}_r \mid d = \sum_{k=0}^{|\tau|-1} \bar{d}_k \pi_{\tau,k},\ \bar{d}_k \in \Sigma_r^q\ \forall k \right\}. \tag{2.146}$$

Finally, we define $\mathcal{U}_\tau$, a subspace of the continuous input space, as:

$$\mathcal{U}_\tau = \left\{ u \in \mathcal{U} \mid u = \sum_{k=0}^{|\tau|-1} \bar{u}_k \pi_{\tau,k},\ \bar{u}_k \in U\ \forall k \right\}. \tag{2.147}$$

Now, we can define the *N–th discretized pure optimization space induced by switching vector τ* as $\mathcal{X}_{\tau,p} = \mathcal{U}_\tau \times \mathcal{D}_{\tau,p}$, and the *$N$–th discretized relaxed optimization space induced by switching vector τ* as $\mathcal{X}_{\tau,r} = \mathcal{U}_\tau \times \mathcal{D}_{\tau,r}$. Similarly, we define a subspace of $\mathcal{X}$:

$$\mathcal{X}_\tau = \left\{ (u,d) \in \mathcal{X} \mid u = \sum_{k=0}^{|\tau|-1} \bar{u}_k \pi_{\tau,k},\ \bar{u}_k \in \mathbb{R}^m\ \forall k,\ \text{and } d = \sum_{k=0}^{|\tau|-1} \bar{d}_k \pi_{\tau,k},\ \bar{d}_k \in \mathbb{R}^q\ \forall k \right\}. \tag{2.148}$$

In order for these discretized optimization spaces to be useful, we need to know to show that we can use a sequence of functions belonging to these finite-dimensional subspaces to approximate any infinite dimensional function. The following lemma proves this result and validates our choice of discretized spaces:

Lemma 2.47. *Let $\{\tau_k\}_{k\in\mathbb{N}}$ with $\tau_k \in \mathcal{T}_k$.*

(1) *For each $\xi \in \mathcal{X}_p$ there exists a sequence $\{\xi_k\}_{k\in\mathbb{N}}$, with $\xi_k \in \mathcal{X}_{\tau_k,p}$, such that $\xi_k \to \xi$ as $k \to \infty$.*

(2) *For each $\xi \in \mathcal{X}_r$ there exists a sequence $\{\xi_k\}_{k\in\mathbb{N}}$, with $\xi_k \in \mathcal{X}_{\tau_k,r}$, such that $\xi_k \to \xi$ as $k \to \infty$.*

Proof. We only present an outline of the proof, since the argument is outside the scope of this chapter. First, every Lebesgue measurable set in $[0,1]$ can be arbitrarily approximated by intervals (Theorem 2.40 in [Fol99]). Second, the sequence of partitions $\{\tau_k\}_{k\in\mathbb{N}}$ can clearly approximate any interval. Finally, the result follows since every measurable function can be approximated in the L^2–norm by integrable simple functions, which are the finite linear combination of indicator functions defined on Borel sets (Theorem 2.10 in [Fol99]). □

Discretized Trajectories, Cost, Constraint, and Optimal Control Problem

For a positive integer N, given a switching vector, $\tau \in \mathcal{T}_N$, a relaxed control $\xi = (u, d) \in \mathcal{X}_{\tau,r}$, and an initial condition $x_0 \in \mathbb{R}^n$, the discrete dynamics, denoted by $\left\{z_\tau^{(\xi)}(\tau_k)\right\}_{k=0}^{|\tau|} \subset \mathbb{R}^n$, are computed via the Forward Euler Integration Formula:

$$z_\tau^{(\xi)}(\tau_{k+1}) = z_\tau^{(\xi)}(\tau_k) + (\tau_{k+1} - \tau_k) f\big(\tau_k, z_\tau^{(\xi)}(\tau_k), u(\tau_k), d(\tau_k)\big),\ \forall k \in \{0, \ldots, |\tau|-1\},\ z_\tau^{(\xi)}(0) = x_0. \tag{2.149}$$

Employing these discrete dynamics we can define the *discretized trajectory*, $z_\tau^{(\xi)} : [0,1] \to \mathbb{R}^n$, by performing linear interpolation over the discrete dynamics:

$$z_\tau^{(\xi)}(t) = \sum_{k=0}^{|\tau|-1} \left(z_\tau^{(\xi)}(\tau_k) + \frac{t - \tau_k}{\tau_{k+1} - \tau_k} \big(z_\tau^{(\xi)}(\tau_{k+1}) - z_\tau^{(\xi)}(\tau_k) \big) \right) \pi_{\tau,k}(t), \tag{2.150}$$

where $\pi_{\tau,k}$ are as defined in Equation (2.144). Note that the definition in Equation (2.150) is valid even if $\tau_k = \tau_{k+1}$ for some $k \in \{0, \ldots, |\tau|\}$, which becomes clear after replacing Equation (2.149) in Equation (2.150). For notational convenience, we suppress the dependence on τ in $z_\tau^{(\xi)}$ when it is clear in context.

Employing the trajectory computed via Euler integration, we define the *discretized cost function*, $J_\tau : \mathcal{X}_{\tau,r} \to \mathbb{R}$:

$$J_\tau(\xi) = h_0\big(z^{(\xi)}(1)\big). \tag{2.151}$$

Similarly, we define the *discretized constraint function*, $\psi_\tau : \mathcal{X}_{\tau,r} \to \mathbb{R}$:

$$\Psi_\tau(\xi) = \max_{j \in \mathcal{J},\ k \in \{0,\ldots,|\tau|\}} h_j\big(z^{(\xi)}(\tau_k)\big). \tag{2.152}$$

Note that these definitions extend easily to points belonging to $\mathcal{X}_{\tau,p}$.

As we did in Section 2.1, we now introduce some additional notation to ensure the clarity of the ensuing analysis. First, for any positive integer N and $\tau \in \mathcal{T}_N$, we define the *discretized flow of the system*, $\varphi_{\tau,t} : \mathcal{X}_r \to \mathbb{R}^n$ for each $t \in [0,1]$ as:

$$\varphi_{\tau,t}(\xi) = z_\tau^{(\xi)}(t). \tag{2.153}$$

Second, for any positive integer N and $\tau \in \mathcal{T}_N$, we define *component constraint functions*, $\psi_{\tau,j,t} : \mathcal{X}_r \to \mathbb{R}$ for each $t \in [0,1]$ and each $j \in \mathcal{J}$ as:

$$\psi_{\tau,j,t}(\xi) = h_j\big(\varphi_{\tau,t}(\xi)\big). \tag{2.154}$$

Notice that the discretized cost function and the discretized constraint function become

$$J_\tau(\xi) = h_0\big(\varphi_{\tau,1}(\xi)\big), \quad \text{and} \quad \Psi_\tau(\xi) = \max_{j \in \mathcal{J},\ k \in \{0,\ldots,|\tau|\}} \psi_{\tau,j,\tau_k}(\xi), \tag{2.155}$$

respectively. This notation change is made to emphasize the dependence on ξ.

Local Minimizers and a Discretized Optimality Condition

Before proceeding further, we make an observation that dictates the construction of our implementable algorithm. Recall how we employ directional derivatives and Theorem 2.8 in order to construct a necessary condition for optimality for the Switched System Optimal Control Problem. In particular, if at a particular point belonging to the pure optimization space the appropriate directional derivatives are negative, then the point is not a local minimizer of the Relaxed Switched System Optimal Control Problem. An application of Theorem 2.8 to this point proves that it is not a local minimizer of the Switched System Optimal Control Problem.

Proceeding in a similar fashion, for any positive integer $N \in \mathbb{N}$ and $\tau \in \mathcal{T}_N$, we can define a Discretized Relaxed Switched System Optimal Control Problem:

$$\text{(DRSSOCP)} \qquad \min_{\xi \in \mathcal{X}_{\tau,r}} \left\{ J_\tau(\xi) \mid \Psi_\tau(\xi) \leq 0 \right\}. \tag{2.156}$$

The local minimizers of this problem are then defined as follows:

Definition 2.48. *Fix $N \in \mathbb{N}$, and $\tau \in \mathcal{T}_N$. Let us denote an ε–ball in the $\mathcal{X}$–norm centered at ξ induced by switching vector τ by:*

$$\mathcal{N}_{\tau,\mathcal{X}}(\xi, \varepsilon) = \left\{ \bar{\xi} \in \mathcal{X}_{\tau,r} \mid \left\| \xi - \bar{\xi} \right\|_{\mathcal{X}} < \varepsilon \right\}. \tag{2.157}$$

We say that a point $\xi \in \mathcal{X}_{\tau,r}$ is a local minimizer of the Relaxed Switched System Optimal Control Problem Induced by Switching Vector τ *defined in Equation* (2.156) *if $\Psi_\tau(\xi) \leq 0$ and there exists $\varepsilon > 0$ such that $J_\tau(\hat{\xi}) \geq J_\tau(\xi)$ for each $\hat{\xi} \in \mathcal{N}_{\tau,\mathcal{X}}(\xi, \varepsilon) \cap \left\{ \bar{\xi} \in \mathcal{X}_{\tau,r} \mid \Psi_\tau(\bar{\xi}) \leq 0 \right\}$.*

Given this definition, a first order numerical optimal control scheme can exploit the vector space structure of the discretized relaxed optimization space in order to define discretized directional derivatives that find local minimizers for this Discretized Relaxed Switched System Optimal Control Problem. Just as in Section 2.2, we can employ a first order approximation argument and the existence of the directional derivative of the cost, $\mathrm{D}J_\tau$ (proven in Lemma 2.67), and of each of the component constraints, $\mathrm{D}\psi_{\tau,j,\tau_k}$ (proven in Lemma 2.70), for each $j \in \mathcal{J}$ and $k \in \{0, \ldots, |\tau|\}$ in order to elucidate this fact.

Employing these directional derivatives, we can define a discretized optimality function. Fixing a positive integer N and $\tau \in \mathcal{T}_N$, we define a *discretized optimality function*, $\theta_\tau : \mathcal{X}_{\tau,p} \to (-\infty, 0]$ and a corresponding *discretized descent direction*, $g_\tau : \mathcal{X}_{\tau,p} \to \mathcal{X}_{\tau,r}$:

$$\theta_\tau(\xi) = \min_{\xi' \in \mathcal{X}_{\tau,r}} \zeta_\tau(\xi, \xi'), \qquad g_\tau(\xi) = \operatorname*{arg\,min}_{\xi' \in \mathcal{X}_{\tau,r}} \zeta_\tau(\xi, \xi'), \tag{2.158}$$

where

$$\zeta_\tau(\xi,\xi') = \begin{cases} \max\Big\{ \mathrm{D}J_\tau(\xi;\xi'-\xi), \\ \qquad \max\limits_{j\in\mathcal{J},\, k\in\{0,\ldots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi) + \gamma\Psi_\tau(\xi) \Big\} + \|\xi'-\xi\|_{\mathcal{X}} & \text{if } \Psi_\tau(\xi) \leq 0, \\ \max\Big\{ \mathrm{D}J_\tau(\xi;\xi'-\xi) - \Psi_\tau(\xi), \\ \qquad \max\limits_{j\in\mathcal{J},\, k\in\{0,\ldots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi) \Big\} + \|\xi'-\xi\|_{\mathcal{X}} & \text{if } \Psi_\tau(\xi) > 0, \end{cases} \tag{2.159}$$

and $\gamma > 0$ is a design parameter as in the original optimality function θ, defined in Equation (2.30). Before proceeding, we make two observations. First, note that $\theta_\tau(\xi) \leq 0$ for each $\xi \in \mathcal{X}_{\tau,p}$, since we can always choose $\xi' = \xi$ which leaves the trajectory unmodified. Second, note that at a point $\xi \in \mathcal{X}_{\tau,p}$ the directional derivatives in the optimality function consider directions $\xi' - \xi$ with $\xi' \in \mathcal{X}_{\tau,r}$ in order to ensure that first order approximations belong to the discretized relaxed optimization space $\mathcal{X}_{\tau,r}$ which is convex (e.g. for $0 < \lambda \ll 1$, $J_\tau(\xi) + \lambda \mathrm{D}J_\tau(\xi;\xi'-\xi) \approx J_\tau((1-\lambda)\xi + \lambda\xi')$ where $(1-\lambda)\xi + \lambda\xi' \in \mathcal{X}_{\tau,r}$).

Just as we argued in the infinite dimensional case, we can prove, as we do in Theorem 2.78, that if $\theta_\tau(\xi) < 0$ for some $\xi \in \mathcal{X}_{\tau,p}$, then ξ is not a local minimizer of the Discretized Relaxed Switched System Optimal Control Problem. Proceeding as we did in Section 2.2, we can attempt to apply Theorem 2.8 to prove that θ encodes local minimizers by employing the weak topology over the discretized pure optimization space. Unfortunately, Theorem 2.8 does not prove that the element in the pure optimization space, $\xi_p \in \mathcal{X}_p$, that approximates a particular relaxed control $\xi_r \in \mathcal{X}_{\tau,r} \subset \mathcal{X}_r$ at a particular quality of approximation $\varepsilon > 0$ with respect to the trajectory of the switched system, belongs to $\mathcal{X}_{\tau,p}$. Though the point in the pure optimization space that approximates a particular discretized relaxed control at a particular quality of approximation exists, it may exist at a different discretization precision.

This deficiency of Theorem 2.8 which is shared by our extension to it, Theorem 2.38, means that our computationally tractable algorithm, in contrast to our conceptual algorithm, requires an additional step where the discretization precision is allowed to improve. Nevertheless, if we prove that the Discretized Switched System Optimal Control Problem consistently approximates the Switched System Optimal Control Problem in a manner that is formalized next, then an algorithm that generates a sequence of points by recursive application that converge to a point that is a zero of the discretized optimality function is also converging to a point that is a zero of the original optimality function.

Formally, motivated by the approach taken in [Pol97], we define consistent approximation as:

Definition 2.49 (Definition 3.3.6 in [Pol97])**.** *The Discretized Relaxed Switched System Optimal Control Problem as defined in Equation* (2.156) *is a* consistent approximation *of the Switched System Optimal Control Problem as defined in Equation* (2.18) *if for any infinite sequence* $\{\tau_i\}_{i\in\mathbb{N}}$ *and* $\{\xi_i\}_{i\in\mathbb{N}}$ *such that* $\tau_i \in \mathcal{T}_i$ *and* $\xi_i \in \mathcal{X}_{\tau_i,p}$ *for each* $i \in \mathbb{N}$*, then*

$\lim_{i\to\infty} |\theta_{\tau_i}(\xi_i) - \theta(\xi_i)| = 0$, *where* θ *is as defined in Equation* (2.30) *and* θ_τ *is as defined in Equation* (2.158).

Importantly, if this notion of consistent approximation is satisfied, then a critical result follows:

Theorem 2.50. *Suppose the Discretized Relaxed Switched System Optimal Control Problem, as defined in Equation* (2.156)*, is a consistent approximation, as in Definition 2.49, of the Switched System Optimal Control Problem, as defined in Equation* (2.18)*. Let* $\{\tau_i\}_{i\in\mathbb{N}}$ *and* $\{\xi_i\}_{i\in\mathbb{N}}$ *be such that* $\tau_i \in \mathcal{T}_i$ *and* $\xi_i \in \mathcal{X}_{\tau_i,p}$ *for each* $i \in \mathbb{N}$*. In this case, if* $\lim_{i\to\infty} \theta_{\tau_i}(\xi_i) = 0$*, then* $\lim_{i\to\infty} \theta(\xi_i) = 0$.

Proof. Arguing by contradiction, suppose there exists a $\delta > 0$ such that $\liminf_{i\to\infty} \theta(\xi_i) < -\delta$. Then by the super-additivity of the lim inf,

$$\liminf_{i\to\infty} \theta_{\tau_i}(\xi_i) - \liminf_{i\to\infty} \theta(\xi_i) \leq \liminf_{i\to\infty} \theta_{\tau_i}(\xi_i) - \theta(\xi_i). \tag{2.160}$$

Rearranging terms and applying Definition 2.49, we have that:

$$\liminf_{i\to\infty} \theta_{\tau_i}(\xi_i) \leq \liminf_{i\to\infty} (\theta_{\tau_i}(\xi_i) - \theta(\xi_i)) + \liminf_{i\to\infty} \theta(\xi_i) < -\delta, \tag{2.161}$$

which contradicts the fact that $\lim_{i\to\infty} \theta_{\tau_i}(\xi_i) = 0$. Since by Condition 1 in Theorem 2.34, $\liminf_{i\to\infty} \theta(\xi_i) \leq \limsup_{i\to\infty} \theta(\xi_i) \leq 0$, we have our result. □

To appreciate the importance of this result, observe that if we prove that the Discretized Relaxed Switched System Optimal Control Problem is a consistent approximation of the Switched System Optimal Control Problem, as we do in Theorem 2.79, and devise an algorithm for the Discretized Relaxed Switched System Optimal Control Problem that generates a sequence of discretized points that converge to a point that is a zero of the discretized optimality function, then the sequence of points generated actually converges to a point that also satisfies the necessary condition for optimality for the Switched System Optimal Control Problem.

Choosing a Discretized Step Size and Projecting the Discretized Relaxed Discrete Input

Before describing the step in our algorithm where the discretization precision is allowed to increase, we describe how the descent direction can be exploited in order to construct a point in the discretized relaxed optimization space that either reduces the cost (if the original point is feasible) or the infeasibility (if the original point is infeasible). Just as we did in Section 2.2, we employ a line search algorithm similar to the traditional Armijo algorithm used during finite dimensional optimization in order to choose a step size (Algorithm

Model 1.2.23 in [Pol97]). Given $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\alpha \in (0,1)$, and $\beta \in (0,1)$, a step size for a point $\xi \in \mathcal{X}_{\tau,p}$, is chosen by solving the following optimization problem:

$$\mu_\tau(\xi) = \begin{cases} \min\big\{k \in \mathbb{N} \mid J_\tau\big(\xi + \beta^k(g_\tau(\xi) - \xi)\big) - J_\tau(\xi) \leq \alpha\beta^k\theta_\tau(\xi), \\ \qquad\qquad \Psi_\tau\big(\xi + \beta^k(g_\tau(\xi) - \xi)\big) \leq \alpha\beta^k\theta_\tau(\xi)\big\} & \text{if } \Psi_\tau(\xi) \leq 0, \\ \min\big\{k \in \mathbb{N} \mid \Psi_\tau\big(\xi + \beta^k(g_\tau(\xi) - \xi)\big) - \Psi_\tau(\xi) \leq \alpha\beta^k\theta_\tau(\xi)\big\} & \text{if } \Psi_\tau(\xi) > 0. \end{cases} \tag{2.162}$$

Continuing as we did in Section 2.2, given $N \in \mathbb{N}$ we can apply $\mathcal{F}_N$ defined in Equation (2.34) and $\mathcal{P}_N$ defined in Equation (2.35) to the constructed discretized relaxed discrete input. The pulse width modulation at a particular frequency induces a partition in $\mathcal{T}_N$ according to the times at which the constructed pure discrete input switched. That is, let $\sigma_N : \mathcal{X}_r \to \mathcal{T}_N$ be defined by

$$\sigma_N(u,d) = \big\{0\big\} \cup \left\{ \frac{k}{2^N} + \frac{1}{2^N} \sum_{j=1}^{i} \big[\mathcal{F}_N(d)\big]_j \left(\frac{k}{2^N}\right) \right\}_{\substack{i \in \{1,\ldots,q\} \\ k \in \{0,\ldots,2^N-1\}}}. \tag{2.163}$$

Employing this induced partition, we can be more explicit about the range of ρ_N by stating that $\rho_N(\xi) \in \mathcal{X}_{\sigma_N(\xi),p}$ for each $\xi \in \mathcal{X}_r$.

Now, given given $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\bar{\alpha} \in (0,\infty)$, $\bar{\beta} \in \left(\frac{1}{\sqrt{2}}, 1\right)$, $\omega \in (0,1)$, and $k_M \in \mathbb{N}$, a frequency at which to perform pulse width modulation for a point $\xi \in \mathcal{X}_{\tau,p}$ is computed by solving the following optimization problem::

$$\nu_\tau(\xi, k_M) = \begin{cases} \min\big\{k \leq k_M \mid \xi' = \xi + \beta^{\mu_\tau(\xi)}\big(g_\tau(\xi) - \xi\big), \\ \qquad J_{\sigma_k(\xi')}\big(\rho_k(\xi')\big) - J_\tau(\xi) \leq \big(\alpha\beta^{\mu_\tau(\xi)} - \bar{\alpha}\bar{\beta}^k\big)\theta_\tau(\xi), \\ \qquad \Psi_{\sigma_k(\xi')}\big(\rho_k(\xi')\big) \leq \big(\alpha\beta^{\mu_\tau(\xi)} - \bar{\alpha}\bar{\beta}^k\big)\theta_\tau(\xi), \\ \qquad \bar{\alpha}\bar{\beta}^k \leq (1-\omega)\alpha\beta^{\mu_\tau(\xi)}, \big\} & \text{if } \Psi_\tau(\xi) \leq 0, \\ \min\big\{k \leq k_M \mid \xi' = \xi + \beta^{\mu_\tau(\xi)}\big(g_\tau(\xi) - \xi\big), \\ \qquad \Psi_{\sigma_k(\xi')}\big(\rho_k(\xi')\big) - \Psi_\tau(\xi) \leq \big(\alpha\beta^{\mu_\tau(\xi)} - \bar{\alpha}\bar{\beta}^k\big)\theta_\tau(\xi), \\ \qquad \bar{\alpha}\bar{\beta}^k \leq (1-\omega)\alpha\beta^{\mu_\tau(\xi)}\big\} & \text{if } \Psi_\tau(\xi) > 0. \end{cases} \tag{2.164}$$

In the discrete case, as opposed to the original infinite dimensional algorithm, due to the aforementioned shortcomings of Theorem 2.8 and 2.38, there is no guarantee that the optimization problem solved in order to ν_τ is feasible. Without loss of generality, we say that $\nu_\tau(\xi) = \infty$ for each $\xi \in \mathcal{X}_{\tau,r}$ when there is no feasible solution. Importantly letting $N_0 \in \mathbb{N}$, $\tau_0 \in \mathcal{T}_{N_0}$, and $\xi \in \mathcal{X}_{\tau,r}$, we prove, in Lemma 2.83, that if $\theta(\xi) < 0$ then for each $\eta \in \mathbb{N}$ there exists a finite $N \geq N_0$ such that $\nu_{\sigma_N(\xi)}(\xi, N+\eta)$ is finite. That is, if $\theta(\xi) < 0$, then ν_τ is always finite after a certain discretization quality is reached.

An Implementable Switched System Optimal Control Algorithm

Consolidating our definitions, Algorithm 2.2 describes our numerical method to solve the Switched System Optimal Control Problem. Notice that note that at each step of Algorithm 2.2, $\xi_j \in \mathcal{X}_{\tau_j,p}$. Also, observe the two principal differences between Algorithm 2.1 and Algorithm 2.2.

First, as discussed earlier, ν_τ maybe infinite as is checked in Line 12 of Algorithm 2.2, at which point the discretization precision is increased since we know that if $\theta(\xi) < 0$, then ν_τ is always finite after a certain discretization quality is reached. Second, notice that if θ_τ comes close to zero as is checked in Line 4 of Algorithm 2.2, the discretization precision is increased. To understand why this additional check is required, remember that our goal in this chapter is the construction of an implementable algorithm that constructs a sequence of points by recursive application that converges to a point that satisfies the optimality condition. In particular, θ_τ may come arbitrarily close to zero due to a particular discretization precision that limits the potential descent directions to search among, rather than because it is actually close to a local minimizer of the Switched System Optimal Control Problem. This additional step that improves the discretization precision is included in Algorithm 2.2 to guard against this possibility.

With regards to actual numerical implementation, we make two additional comments. First, a stopping criterion is chosen that terminates the operation of the algorithm if θ_τ is too large. We describe our selection of this parameter in Section 2.6. Second, due to the definitions of $\mathrm{D}J_\tau$ and $\mathrm{D}\psi_{\tau,j,\tau_k}$ for each $j \in \mathcal{J}$ and $k \in \{0, \dots, |\tau|\}$, the optimization required to solve θ_τ is a quadratic program.

For analysis purposes, we define $\Gamma_\tau : \{\xi \in \mathcal{X}_{\tau,p} \mid \nu_\tau(\xi, k_{\max}) < \infty\} \to \mathcal{X}_p$ by:

$$\Gamma_\tau(\xi) = \rho_{\nu_\tau(\xi,k_{\max})}\big(\xi + \beta^{\mu_\tau(\xi)}(g_\tau(\xi) - \xi)\big). \tag{2.165}$$

We say $\{\xi_j\}_{j\in\mathbb{N}}$ is *a sequence generated by Algorithm 2.2* if $\xi_{j+1} = \Gamma_{\tau_j}(\xi_j)$ for each $j \in \mathbb{N}$. We can prove several important properties about the sequence generated by Algorithm 2.2. First, letting $\{N_i\}_{i\in\mathbb{N}}$, $\{\tau_i\}_{i\in\mathbb{N}}$, and $\{\xi_i\}_{i\in\mathbb{N}}$ be the sequences produced by Algorithm 2.2, then, as we prove in Lemma 2.85, there exists $i_0 \in \mathbb{N}$ such that, if $\Psi_{\tau_{i_0}}(\xi_{i_0}) \leq 0$, then $\Psi(\xi_i) \leq 0$ and $\Psi_{\tau_i}(\xi_i) \leq 0$ for each $i \geq i_0$. That is, once Algorithm 2.2 finds a feasible point, every point generated after it remains feasible. Second, as we prove in Theorem 2.87, $\lim_{j\to\infty} \theta(\xi_j) = 0$ for a sequence $\{\xi_j\}_{j\in\mathbb{N}}$ generated by Algorithm 2.2, or Algorithm 2.2 converges to a point that satisfies the optimality condition.

2.5 Implementable Algorithm Analysis

In this section, we derive the various components of Algorithm 2.2 and prove that Algorithm 2.2 converges to a point that satisfies our optimality condition. Our argument proceeds as follows: first, we prove the continuity and convergence of the discretized state, cost, and constraints to their infinite dimensional analogues; second, we construct the components of

Require: $N_0 \in \mathbb{N}$, $\tau_0 \in \mathcal{T}_{N_0}$, $\xi_0 \in \mathcal{X}_{\tau_0,p}$, $\alpha \in (0,1)$, $\bar{\alpha} \in (0,\infty)$, $\beta \in (0,1)$, $\bar{\beta} \in \left(\frac{1}{\sqrt{2}},1\right)$, $\gamma \in (0,\infty)$, $\eta \in \mathbb{N}$, $\Lambda \in (0,\infty)$, $\chi \in \left(0,\frac{1}{2}\right)$, $\omega \in (0,1)$.

Set $j = 0$.
loop
  Compute $\theta_{\tau_j}(\xi_j)$ as defined in Equation (2.158).
  if $\theta_{\tau_j}(\xi_j) > -\Lambda 2^{-\chi N_j}$ **then**
    Set $\xi_{j+1} = \xi_j$.
    Set $N_{j+1} = N_j + 1$.
    Set $\tau_{j+1} = \sigma_{N_{j+1}}(\xi_j)$.
  else
    Compute $g_{\tau_j}(\xi_j)$ as defined in Equation (2.158).
    Compute $\mu_{\tau_j}(\xi_j)$ as defined in Equation (2.162).
    Compute $\nu_{\tau_j}(\xi_j, N_j + \eta)$ as defined in Equation (2.164).
    if $\nu_{\tau_j}(\xi_j, N_j + \eta) = \infty$ **then**
      Set $\xi_{j+1} = \xi_j$.
      Set $N_{j+1} = N_j + 1$.
      Set $\tau_{j+1} = \sigma_{N_{j+1}}(\xi_{j+1})$.
    else
      Set $\xi_{j+1} = \rho_{\nu_{\tau_j}(\xi_j,N_j+\eta)}\big(\xi_j + \beta^{\mu_{\tau_j}(\xi_j)}(g_{\tau_j}(\xi_j) - \xi_j)\big)$.
      Set $N_{j+1} = \max\big\{N_j, \nu_{\tau_j}(\xi_j, N_j + \eta)\big\}$.
      Set $\tau_{j+1} = \sigma_{N_{j+1}}(\xi_{j+1})$.
    end if
  end if
  Replace j by $j + 1$.
end loop

Algorithm 2.2 Numerically Tractable Algorithm for the Switched System Optimal Control Problem

the optimality function and prove the convergence of these discretized components to their infinite dimensional analogues; finally, we prove the convergence of our algorithm.

Continuity and Convergence of the Discretized Components

In this subsection, we prove the continuity and convergence of the discretized state, cost, and constraint. We begin by proving the boundedness of the linear interpolation of the Euler Integration scheme:

Lemma 2.51. *There exists a constant $C > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi \in \mathcal{X}_{\tau,r}$, and $t \in [0,1]$,*

$$\left\|z^{(\xi)}(t)\right\|_2 \leq C \tag{2.166}$$

Proof. We begin by showing the result for each $t \in \tau$. By Condition 1 in Assumption 2.2, together with the boundedness of $\|f(0, x_0, 0, e_i)\|_2$ for each $i \in \mathcal{Q}$, there exists a constant $K > 0$ such that, for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi \in \mathcal{X}_{\tau,r}$, $i \in \mathcal{Q}$, and $k \in \{0, \dots, |\tau|\}$,

$$\left\| f\left(\tau_k, z^{(\xi)}(\tau_k), u(\tau_k), e_i\right) \right\|_2 \leq K\left(\left\| z^{(\xi)}(\tau_k) \right\|_2 + 1\right). \tag{2.167}$$

Employing Equation (2.149) and the Discrete Bellman-Gronwall Inequality (Exercise 5.6.14 in [Pol97]), we have:

$$\begin{aligned} \left\| z^{(\xi)}(\tau_k) \right\|_2 &\leq \|x_0\|_2 + \frac{1}{2^N} \sum_{j=0}^{k} \sum_{i=1}^{q} \left\| f\left(\tau_j, z^{(\xi)}(\tau_j), u(\tau_j), e_i\right) \right\|_2 \\ &\leq (\|x_0\|_2 + 1) \left(1 + \frac{qK}{2^N}\right)^{2^N} \\ &\leq e^{qK} (\|x_0\|_2 + 1), \end{aligned} \tag{2.168}$$

thus obtaining the desired result for each $t \in \tau$.

The result for each $t \in [0, 1]$ follows after observing that, in Equation (2.150), $\left(\frac{t-\tau_k}{\tau_{k+1}-\tau_k}\right) \leq 1$ for each $t \in [\tau_k, \tau_{k+1})$ and $k \in \{0, \dots, |\tau|\}$. □

In fact, this implies that the dynamics, cost, constraints, and their derivatives are all bounded:

Corollary 2.52. *There exists a constant $C > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $j \in \mathcal{J}$, and $\xi = (u, d) \in \mathcal{X}_{\tau,r}$:*

(1) $\left\| f\left(t, z^{(\xi)}(t), u(t), d(t)\right) \right\|_2 \leq C$, $\left\| \frac{\partial f}{\partial x}\left(t, z^{(\xi)}(t), u(t), d(t)\right) \right\|_{i,2} \leq C$, *and* $\left\| \frac{\partial f}{\partial u}\left(t, z^{(\xi)}(t), u(t), d(t)\right) \right\|_{i,2} \leq C$.

(2) $\left| h_0\left(z^{(\xi)}(t)\right) \right| \leq C$, *and* $\left\| \frac{\partial h_0}{\partial x}\left(z^{(\xi)}(t)\right) \right\|_2 \leq C$.

(3) $\left| h_j\left(z^{(\xi)}(t)\right) \right| \leq C$, *and* $\left\| \frac{\partial h_j}{\partial x}\left(z^{(\xi)}(t)\right) \right\|_2 \leq C$.

Proof. The result follows immediately from the continuity of f, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial u}$, h_0, $\frac{\partial h_0}{\partial x}$, h_j, and $\frac{\partial h_j}{\partial x}$ for each $j \in \mathcal{J}$, as stated in Assumptions 2.2 and 2.3, and the fact that the arguments of these functions can be constrained to a compact domain, which follows from Lemma 2.51 and the compactness of U and Σ_r^q. □

Next, we prove that the mapping from the discretized relaxed optimization space to the discretized trajectory is Lipschitz:

Lemma 2.53. *There exists a constant $L > 0$ such that, for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi_1, \xi_2 \in \mathcal{X}_{\tau,r}$ and $t \in [0,1]$:*

$$\|\varphi_{\tau,t}(\xi_1) - \varphi_{\tau,t}(\xi_2)\|_2 \leq L\|\xi_1 - \xi_2\|_{\mathcal{X}}, \tag{2.169}$$

where $\varphi_{\tau,t}(\xi)$ is as defined in Equation (2.153).

Proof. We first prove this result for each $t \in \tau$. For notational convenience we will define $\Delta\tau_j = \tau_{j+1} - \tau_j$. Letting $\xi_1 = (u_1, d_1)$ and $\xi_2 = (u_2, d_2)$, notice that for $j \in \{0, \ldots, |\tau| - 1\}$, by Equation (2.149) and rearranging the terms, there exists $L' > 0$ such that:

$$\begin{aligned}
&\left\|\varphi_{\tau,\tau_{j+1}}(\xi_1) - \varphi_{\tau,\tau_{j+1}}(\xi_2)\right\|_2 - \left\|\varphi_{\tau,\tau_j}(\xi_1) - \varphi_{\tau,\tau_j}(\xi_2)\right\|_2 \leq \\
&\qquad \leq \Delta\tau_j \left\| f\big(\tau_j, \varphi_{\tau,\tau_j}(\xi_1), u_1(\tau_j), d_1(\tau_j)\big) - f\big(\tau_j, \varphi_{\tau,\tau_j}(\xi_2), u_2(\tau_j), d_2(\tau_j)\big)\right\|_2 \\
&\qquad \leq \frac{L'}{2^N}\left\|\varphi_{\tau,\tau_j}(\xi_1) - \varphi_{\tau,\tau_j}(\xi_2)\right\|_2 + L'\Delta\tau_j\big(\|u_1(\tau_j) - u_2(\tau_j)\|_2 + \|d_1(\tau_j) - d_2(\tau_j)\|_2\big),
\end{aligned} \tag{2.170}$$

where the last inequality holds since the vector field f is Lipschitz in all of its arguments, as shown in the proof of Theorem 2.6, and $\Delta\tau_j \leq \frac{1}{2^N}$ by definition of $\mathcal{T}_N$.

Summing the inequality in Equation (2.170) for $j \in \{0, \ldots, k-1\}$ and noting that $\varphi_{\tau,\tau_0}(\xi_1) = \varphi_{\tau,\tau_0}(\xi_2)$:

$$\begin{aligned}
\left\|\varphi_{\tau,\tau_k}(\xi_1) - \varphi_{\tau,\tau_k}(\xi_2)\right\|_2 \leq \frac{L'}{2^N}\sum_{j=0}^{k-1}\left\|\varphi_{\tau,\tau_j}(\xi_1) - \varphi_{\tau,\tau_j}(\xi_2)\right\|_2 + L'\sum_{j=0}^{k-1}\Delta\tau_j\left\|u_1(\tau_j) - u_2(\tau_j)\right\|_2 + \\
+ L'\sum_{j=0}^{k-1}\Delta\tau_j\left\|d_1(\tau_j) - d_2(\tau_j)\right\|_2.
\end{aligned} \tag{2.171}$$

Using the Discrete Bellman-Gronwall Inequality (Exercise 5.6.14 in [Pol97]) and the fact that $\left(1 + \frac{L'}{2^N}\right)^{\frac{L'}{2^N}} \leq e^{L'}$,

$$\begin{aligned}
&\left\|\varphi_{\tau,\tau_k}(\xi_1) - \varphi_{\tau,\tau_k}(\xi_2)\right\|_2 \leq \\
&\qquad \leq L'e^{L'}\left(\sum_{j=0}^{|\tau|-1}\Delta\tau_j\left\|u_1(\tau_j) - u_2(\tau_j)\right\|_2 + \sum_{j=0}^{|\tau|-1}\Delta\tau_j\left\|d_1(\tau_j) - d_2(\tau_j)\right\|_2\right) \\
&\qquad \leq L'e^{L'}\left(\sqrt{\sum_{j=0}^{|\tau|-1}\Delta\tau_j\left\|u_1(\tau_j) - u_2(\tau_j)\right\|_2^2} + \sqrt{\sum_{j=0}^{|\tau|-1}\Delta\tau_j\left\|d_1(\tau_j) - d_2(\tau_j)\right\|_2^2}\right) \\
&\qquad = L\|\xi_1 - \xi_2\|_{\mathcal{X}},
\end{aligned} \tag{2.172}$$

where $L = L'e^{L'}$, and we employed Jensen's Inequality (Equation A.2 in [Mal99]) together the fact that the $\mathcal{X}$–norm of $\xi \in \mathcal{X}_{\tau,r}$ can be written as a finite sum.

The result for any $t \in [0,1]$ follows by noting that $\varphi_{\tau,t}(\xi)$ is a convex combination of $\varphi_{\tau,\tau_k}(\xi)$ and $\varphi_{\tau,\tau_{k+1}}(\xi)$ for some $k \in \{0, \ldots, |\tau| - 1\}$. □

As a consequence, we immediately have the following results:

Corollary 2.54. *There exists a constant $L > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi_1 = (u_1, d_1) \in \mathcal{X}_{\tau,r}$, $\xi_2 = (u_2, d_2) \in \mathcal{X}_{\tau,r}$ and $t \in [0,1]$:*

(1) $\big\|f\big(t, \varphi_{\tau,t}(\xi_1), u_1(t), d_1(t)\big) - f\big(t, \varphi_{\tau,t}(\xi_2), u_2(t), d_2(t)\big)\big\|_2 \leq$
$$\leq L\big(\|\xi_1 - \xi_2\|_{\mathcal{X}} + \|u_1(t) - u_2(t)\|_2 + \|d_1(t) - d_2(t)\|_2\big),$$

(2) $\left\|\dfrac{\partial f}{\partial x}\big(t, \varphi_{\tau,t}(\xi_1), u_1(t), d_1(t)\big) - \dfrac{\partial f}{\partial x}\big(t, \varphi_{\tau,t}(\xi_2), u_2(t), d_2(t)\big)\right\|_{i,2} \leq$
$$\leq L\big(\|\xi_1 - \xi_2\|_{\mathcal{X}} + \|u_1(t) - u_2(t)\|_2 + \|d_1(t) - d_2(t)\|_2\big),$$

(3) $\left\|\dfrac{\partial f}{\partial u}\big(t, \varphi_{\tau,t}(\xi_1), u_1(t), d_1(t)\big) - \dfrac{\partial f}{\partial u}\big(t, \varphi_{\tau,t}(\xi_2), u_2(t), d_2(t)\big)\right\|_{i,2} \leq$
$$\leq L\big(\|\xi_1 - \xi_2\|_{\mathcal{X}} + \|u_1(t) - u_2(t)\|_2 + \|d_1(t) - d_2(t)\|_2\big),$$

where $\varphi_{\tau,t}(\xi)$ is as defined in Equation (2.153)*.*

Proof. The proof of Condition 1 follows by the fact that the vector field f is Lipschitz in all its arguments, as shown in the proof of Theorem 2.6, and applying Lemma 2.53. The remaining conditions follow in a similar fashion. □

Corollary 2.55. *There exists a constant $L > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi_1 = (u_1, d_1) \in \mathcal{X}_{r,\tau}$, $\xi_2 = (u_2, d_2) \in \mathcal{X}_{r,\tau}$, $j \in \mathcal{J}$, and $t \in [0,1]$:*

(1) $\big|h_0\big(\varphi_{\tau,1}(\xi_1)\big) - h_0\big(\varphi_{\tau,1}(\xi_2)\big)\big| \leq L\,\|\xi_1 - \xi_2\|_{\mathcal{X}}$,

(2) $\left\|\frac{\partial h_0}{\partial x}\big(\varphi_{\tau,1}(\xi_1)\big) - \frac{\partial h_0}{\partial x}\big(\varphi_{\tau,1}(\xi_2)\big)\right\|_2 \leq L\,\|\xi_1 - \xi_2\|_{\mathcal{X}}$,

(3) $\big|h_j\big(\varphi_{\tau,t}(\xi_1)\big) - h_j\big(\varphi_{\tau,t}(\xi_2)\big)\big| \leq L\,\|\xi_1 - \xi_2\|_{\mathcal{X}}$,

(4) $\left\|\frac{\partial h_j}{\partial x}\big(\varphi_{\tau,t}(\xi_1)\big) - \frac{\partial h_j}{\partial x}\big(\varphi_{\tau,t}(\xi_2)\big)\right\|_2 \leq L\,\|\xi_1 - \xi_2\|_{\mathcal{X}}$,

where $\varphi_{\tau,t}(\xi)$ is as defined in Equation (2.153)*.*

Proof. This result follows by Assumption 2.3 and Lemma 2.53. □

Even though it is a straightforward consequence of Condition 1 in Corollary 2.55, we write the following result to stress its importance.

Corollary 2.56. *Let $N \in \mathbb{N}$ and $\tau \in \mathcal{T}_N$, then there exists a constant $L > 0$ such that, for each $\xi_1, \xi_2 \in \mathcal{X}_{\tau,r}$:*

$$|J_\tau(\xi_1) - J_\tau(\xi_2)| \leq L\,\|\xi_1 - \xi_2\|_{\mathcal{X}} \tag{2.173}$$

where J_τ is as defined in Equation (2.151)*.*

In fact, Ψ_τ is also Lipschitz continuous:

Lemma 2.57. *Let $N \in \mathbb{N}$ and $\tau \in \mathcal{T}_N$, then there exists a constant $L > 0$ such that, for each $\xi_1, \xi_2 \in \mathcal{X}_r$:*

$$|\Psi_\tau(\xi_1) - \Psi_\tau(\xi_2)| \leq L \left\|\xi_1 - \xi_2\right\|_{\mathcal{X}} \tag{2.174}$$

where Ψ_τ is as defined in Equation (2.152).

Proof. Since the maximum in Ψ_τ is taken over $\mathcal{J} \times k \in \{0, \dots, |\tau|\}$, which is compact, and the maps $(j,k) \mapsto \psi_{\tau,j,\tau_k}(\xi)$ are continuous for each $\xi \in \mathcal{X}_\tau$, we know from Condition 3 in Corollary 2.55 that there exists $L > 0$ such that,

$$\begin{aligned}
\Psi_\tau(\xi_1) - \Psi_\tau(\xi_2) &= \max_{(j,k)\in\mathcal{J}\times\{0,\dots,|\tau|\}} \psi_{\tau,j,\tau_k}(\xi_1) - \max_{(j,k)\in\mathcal{J}\times\{0,\dots,|\tau|\}} \psi_{\tau,j,\tau_k}(\xi_2) \\
&\leq \max_{(j,k)\in\mathcal{J}\times\{0,\dots,|\tau|\}} \psi_{\tau,j,\tau_k}(\xi_1) - \psi_{\tau,j,\tau_k}(\xi_2) \\
&\leq L \left\|\xi_1 - \xi_2\right\|_{\mathcal{X}}.
\end{aligned} \tag{2.175}$$

By reversing ξ_1 and ξ_2, and applying the same argument we get the desired result. □

We can now show the rate of convergence of the Euler Integration scheme:

Lemma 2.58. *There exists a constant $B > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi \in \mathcal{X}_{\tau,r}$, and $t \in [0,1]$:*

$$\left\|z_\tau^{(\xi)}(t) - x^{(\xi)}(t)\right\|_2 \leq \frac{B}{2^N}, \tag{2.176}$$

where $x^{(\xi)}$ is the solution to Differential Equation (2.13) *and $z_\tau^{(\xi)}$ is as defined in Difference Equation* (2.150).

Proof. Let $\xi = (u, d)$, and recall that the vector field f is Lipschitz continuous in all its arguments, as shown in the proof of Theorem 2.6. By applying Picard's Lemma (Lemma 5.6.3 in [Pol97]), we have:

$$\begin{aligned}
\left\|z^{(\xi)}(t) - x^{(\xi)}(t)\right\|_2 &\leq e^L \int_0^1 \left\| \frac{dz^{(\xi)}}{ds}(s) - f\big(s, z^{(\xi)}(s), u(s), d(s)\big) \right\|_2 ds \\
&= e^L \sum_{k=0}^{|\tau|-1} \int_{\tau_k}^{\tau_{k+1}} \Bigg\| f\big(\tau_k, z^{(\xi)}(\tau_k), u(\tau_k), d(\tau_k)\big) + \\
&\qquad - f\left(s, z^{(\xi)}(\tau_k) + \frac{s-\tau_k}{\tau_{k+1}-\tau_k}\big(z^{(\xi)}(\tau_{k+1}) - z^{(\xi)}(\tau_k)\big), u(\tau_k), d(\tau_k)\right) \Bigg\|_2 ds \\
&\leq L e^L \sum_{k=0}^{|\tau|-1} \left(1 + \left\|f\big(\tau_k, z^{(\xi)}(\tau_k), u(\tau_k), d(\tau_k)\big)\right\|_2\right) \left(\int_{\tau_k}^{\tau_{k+1}} |s - \tau_k| ds\right) \\
&\leq \frac{1}{2^N} L e^L (1 + C) \sum_{k=0}^{|\tau|-1} (\tau_{k+1} - \tau_k) = \frac{B}{2^N},
\end{aligned} \tag{2.177}$$

where $C > 0$ is as defined in Condition 1 in Corollary 2.52 and, $B = (1 + C)Le^L$, and we used the fact that $\tau_{k+1} - \tau_k \leq \frac{1}{2^N}$ by definition of $\mathcal{T}_N$ in Equation (2.143). □

Importantly we can show that we can bound the rate of convergence of this discretized cost function. We omit the proof since it follows easily using Assumption 2.3 and Lemma 2.58.

Lemma 2.59. *There exists a constant $B > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, and $\xi \in \mathcal{X}_{\tau,r}$:*

$$|J_\tau(\xi) - J(\xi)| \leq \frac{B}{2^N}, \tag{2.178}$$

where J is as defined in Equation (2.15) and J_τ is as defined in Equation (2.151).

Similarly, we can bound the rate of convergence of this discretized constraint function.

Lemma 2.60. *There exists a constant $B > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, and $\xi \in \mathcal{X}_{\tau,r}$:*

$$|\Psi_\tau(\xi) - \Psi(\xi)| \leq \frac{B}{2^N}, \tag{2.179}$$

where Ψ is as defined in Equation (2.16) and Ψ_τ is as defined in Equation (2.152).

Proof. Let $C > 0$ be as defined in Condition 1 in Corollary 2.5, and let $L > 0$ be the Lipschitz constant as specified in Assumption 2.3. Then, using the definition of $\mathcal{T}_N$ in Equation (2.143), for each $k \in \{0, \ldots, |\tau| - 1\}$ and $t \in [\tau_k, \tau_{k+1}]$,

$$\left|h_j\big(x^{(\xi)}(t)\big) - h_j\big(x^{(\xi)}(\tau_k)\big)\right| \leq L \int_{\tau_k}^{t} \left\|f\big(s, x^{(\xi)}(s), u(s), d(s)\big)\right\|_2 ds \leq \frac{LC}{2^N}. \tag{2.180}$$

Moreover, Condition 3 in Assumption 2.3 together Lemma 2.58 imply the existence of a constant $K > 0$ such that:

$$\left|h_j\big(x^{(\xi)}(\tau_k)\big) - h_j\big(z^{(\xi)}(\tau_k)\big)\right| \leq \frac{K}{2^N}. \tag{2.181}$$

Employing the Triangle Inequality on the two previous inequalities, we know there exists a constant $B > 0$ such that, for each $t \in [\tau_k, \tau_{k+1}]$,

$$\left|h_j\big(x^{(\xi)}(t)\big) - h_j\big(z^{(\xi)}(\tau_k)\big)\right| \leq \frac{B}{2^N}. \tag{2.182}$$

Let $t' \in \arg\max_{t\in[0,1]} h_j\big(x^{(\xi)}(t)\big)$, and let $\kappa(t') \in \{0, \ldots, |\tau| - 1\}$ be such that $t' \in \big[\tau_{\kappa(t')}, \tau_{\kappa(t')+1}\big]$. Then,

$$\max_{t\in[0,1]} h_j\big(x^{(\xi)}(t)\big) - \max_{k\in\{0,\ldots,|\tau|\}} h_j\big(z^{(\xi)}(\tau_k)\big) \leq h_j\big(x^{(\xi)}(t')\big) - h_j\big(z^{(\xi)}(\tau_{\kappa(t')})\big) \leq \frac{B}{2^N}. \tag{2.183}$$

Similarly if $k' \in \arg\max_{k\in\{0,\dots,|\tau|\}} h_j\big(z^{(\xi)}(\tau_k)\big)$, then

$$\max_{k\in\{0,\dots,|\tau|\}} h_j\big(z^{(\xi)}(\tau_k)\big) - \max_{t\in[0,1]} h_j\big(x^{(\xi)}(t)\big) \le h_j\big(z^{(\xi)}(\tau_{k'})\big) - h_j\big(x^{(\xi)}(\tau_{k'})\big) \le \frac{B}{2^N}. \tag{2.184}$$

This implies that:

$$\Psi(\xi) - \Psi_\tau(\xi) \le \max_{j\in\mathcal{J}} \left(\max_{t\in[0,1]} h_j\big(x^{(\xi)}(t)\big) - \max_{k\in\{0,\dots,|\tau|\}} h_j\big(z^{(\xi)}(\tau_k)\big) \right) \le \frac{B}{2^N}, \tag{2.185}$$

$$\Psi_\tau(\xi) - \Psi(\xi) \le \max_{j\in\mathcal{J}} \left(\max_{k\in\{0,\dots,|\tau|\}} h_j\big(z^{(\xi)}(\tau_k)\big) - \max_{t\in[0,1]} h_j\big(x^{(\xi)}(t)\big) \right) \le \frac{B}{2^N}, \tag{2.186}$$

which proves the desired result. □

Derivation of the Implementable Algorithm Terms

Next, we formally derive the components of the discretized optimality function, prove that the discretized optimality function is well defined, and prove the convergence of the discretized optimality function to the optimality function. We begin by deriving the equivalent of Lemma 2.18 for our discretized formulation.

Lemma 2.61. *Let $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi = (u, d) \in \mathcal{X}_{\tau,r}$, $\xi' = (u', d') \in \mathcal{X}_\tau$, and $\varphi_{\tau,t}$ be as defined in Equation* (2.153)*. Then, for each $k \in \{0, \dots, |\tau|\}$, the directional derivative of φ_{τ,τ_k}, as defined in Equation* (2.25)*, is given by*

$$\mathrm{D}\varphi_{\tau,\tau_k}(\xi;\xi') = \sum_{j=0}^{k-1} (\tau_{j+1} - \tau_j)\Phi_\tau^{(\xi)}(\tau_k, \tau_{j+1}) \Bigg(\frac{\partial f}{\partial u}\big(\tau_j, \varphi_{\tau,\tau_j}(\xi), u(\tau_j), d(\tau_j)\big) u'(\tau_j) + \\ + \sum_{i=1}^{q} f\big(\tau_j, \varphi_{\tau,\tau_j}(\xi), u(\tau_j), e_i\big) d_i'(\tau_j) \Bigg), \tag{2.187}$$

where $\Phi_\tau^{(\xi)}(\tau_k, \tau_j)$ is the unique solution of the following matrix difference equation:

$$\Phi_\tau^{(\xi)}(\tau_{k+1}, \tau_j) = \Phi_\tau^{(\xi)}(\tau_k, \tau_j) + (\tau_{k+1} - \tau_k) \frac{\partial f}{\partial x}\big(\tau_k, \varphi_{\tau,\tau_k}(\xi), u(\tau_k), d(\tau_k)\big) \Phi_\tau^{(\xi)}(\tau_k, \tau_j), \\ \forall k \in \{0, \dots, |\tau| - 1\}, \quad \Phi_\tau^{(\xi)}(\tau_j, \tau_j) = I. \tag{2.188}$$

Proof. For notational convenience, let $z^{(\lambda)} = z^{(\xi+\lambda\xi')}$, $u^{(\lambda)} = u + \lambda u'$, and $d^{(\lambda)} = d + \lambda d'$.

Also, let us define $\Delta z^{(\lambda)} = z^{(\lambda)} - z^{(\xi)}$, thus, for each $k \in \{0, \dots, |\tau|\}$,

$$\begin{aligned}\Delta z^{(\lambda)}(\tau_k) &= \sum_{j=0}^{k-1}(\tau_{j+1}-\tau_j)\Big(f\big(\tau_j, z^{(\lambda)}(\tau_j), u^{(\lambda)}(\tau_j), d^{(\lambda)}(\tau_j)\big) - f\big(\tau_j, z^{(\xi)}(\tau_j), u(\tau_j), d(\tau_j)\big)\Big) \\ &= \sum_{j=0}^{k-1}(\tau_{j+1}-\tau_j)\Bigg(\lambda\sum_{i=1}^{q} d_i'(\tau_j) f\big(\tau_j, z^{(\lambda)}(\tau_j), u^{(\lambda)}(\tau_j), e_i\big) + \\ &\qquad + \frac{\partial f}{\partial x}\big(\tau_j, z^{(\xi)}(\tau_j) + \nu_{x,j}\Delta z^{(\lambda)}(\tau_j), u^{(\lambda)}(\tau_j), d(\tau_j)\big)\Delta z^{(\lambda)}(\tau_j) + \\ &\qquad + \lambda\frac{\partial f}{\partial u}\big(\tau_j, z^{(\xi)}(\tau_j), u(\tau_j) + \nu_{u,j}\lambda u'(\tau_j), d(\tau_j)\big)u'(\tau_j)\Bigg),\end{aligned} \tag{2.189}$$

where $\{\nu_{x,j}\}_{j=0}^{|\tau|} \subset [0,1]$ and $\{\nu_{u,j}\}_{j=0}^{|\tau|} \subset [0,1]$.

Let $\{y(\tau_k)\}_{k=0}^{|\tau|}$ be recursively defined as follows:

$$\begin{aligned}y(\tau_{k+1}) = y(\tau_k) + (\tau_{k+1}-\tau_k)\Bigg(&\frac{\partial f}{\partial x}\big(\tau_k, z^{(\xi)}(\tau_k), u(\tau_k), d(\tau_k)\big)y(\tau_k) + \\ &+ \frac{\partial f}{\partial u}\big(\tau_k, z^{(\xi)}(\tau_k), u(\tau_k), d(\tau_k)\big)u'(\tau_k) + \sum_{i=1}^{q} d_i'(\tau_k) f\big(\tau_k, z^{(\xi)}(\tau_k), u(\tau_k), e_i\big)\Bigg), \quad y(\tau_0) = 0.\end{aligned} \tag{2.190}$$

We want to show that $\frac{\Delta z^{(\lambda)}(\tau_k)}{\lambda} \to y(\tau_k)$ as $\lambda \downarrow 0$. Consider:

$$\begin{aligned}\Bigg\|\frac{\partial f}{\partial x}\big(\tau_k, z^{(\xi)}(\tau_k), u(\tau_k), d(\tau_k)\big)y(\tau_k) + &\\ - \frac{\partial f}{\partial x}\big(\tau_k, z^{(\xi)}(\tau_k) + \nu_{x,k}\Delta z^{(\lambda)}(\tau_k), u^{(\lambda)}(\tau_k), d(\tau_k)\big)\frac{\Delta z^{(\lambda)}(\tau_k)}{\lambda}\Bigg\|_2 &\leq \\ \leq L\left\|y(\tau_k) - \frac{\Delta z^{(\lambda)}(\tau_k)}{\lambda}\right\|_2 + L\left(\left\|\Delta z^{(\lambda)}(\tau_k)\right\|_2 + \lambda\left\|u'(\tau_k)\right\|_2\right)\left\|y(\tau_k)\right\|_2,&\end{aligned} \tag{2.191}$$

which follows by Assumption 2.2 and the Triangle Inequality. Also,

$$\begin{aligned}\left\|\left(\frac{\partial f}{\partial u}\big(\tau_k, z^{(\xi)}(\tau_k), u(\tau_k), d(\tau_k)\big) - \frac{\partial f}{\partial u}\big(\tau_k, z^{(\xi)}(\tau_k), u(\tau_k) + \nu_{u,k}\lambda u'(\tau_k), d(\tau_k)\big)\right)u'(\tau_k)\right\|_2 &\leq \\ \leq L\lambda\left\|u'(\tau_k)\right\|_2^2,&\end{aligned} \tag{2.192}$$

and

$$\left\| \sum_{i=1}^{q} d_i'(\tau_k) \Big(f\big(\tau_k, z^{(\xi)}(\tau_k), u(\tau_k), e_i\big) - f\big(\tau_k, z^{(\lambda)}(\tau_k), u^{(\lambda)}(\tau_k), e_i\big) \Big) \right\|_2 \leq \\ \leq L \left\| \Delta z^{(\lambda)}(\tau_k) \right\|_2 + L\lambda \left\| u'(\tau_k) \right\|_2 . \quad (2.193)$$

Hence, using the Discrete Bellman-Gronwall Inequality (Lemma 5.6.14 in [Pol97]) and the inequalities above,

$$\left\| y(\tau_k) - \frac{\Delta z^{(\lambda)}(\tau_k)}{\lambda} \right\|_2 \leq L e^L \sum_{j=0}^{k-1} (\tau_{j+1} - \tau_j) \Bigg(\left(\left\| \Delta z^{(\lambda)}(\tau_j) \right\|_2 + \lambda \left\| u'(\tau_j) \right\|_2 \right) \left\| y(\tau_j) \right\|_2 + \\ + \lambda \left\| u'(\tau_j) \right\|_2^2 + \left\| \Delta z^{(\lambda)}(\tau_j) \right\|_2 + \lambda \left\| u'(\tau_j) \right\|_2 \Bigg) \quad (2.194)$$

where we used the fact that $\left(1 + \frac{L}{2^N}\right)^{\frac{L}{2^N}} \leq e^L$. But, by Lemma 2.53, the right-hand side of Equation (2.194) goes to zero as $\lambda \downarrow 0$, thus obtaining that

$$\lim_{\lambda \downarrow 0} \frac{\Delta z^{(\lambda)}(\tau_k)}{\lambda} = y(\tau_k). \quad (2.195)$$

The result of the first part of the Lemma is obtained after noting that $\mathrm{D}\varphi_{\tau,\tau_k}(\xi;\xi')$ is equal to $y(\tau_k)$ for each $k \in \{0, \ldots, |\tau|\}$. □

Next, we prove that $\mathrm{D}\varphi_{\tau,\tau_k}$ is bounded by proving that Φ^{ξ} is bounded:

Corollary 2.62. *There exists a constant $C > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi \in \mathcal{X}_{\tau,r}$, and $k, l \in \{0, \ldots, |\tau|\}$:*

$$\left\| \Phi_{\tau}^{(\xi)}(\tau_k, \tau_l) \right\|_{i,2} \leq C, \quad (2.196)$$

where $\Phi_{\tau}^{(\xi)}(\tau_k, \tau_l)$ is the solution to the Difference Equation (2.188).

Proof. This follows directly from Equation (2.188) and Condition 1 in Corollary 2.52. □

Corollary 2.63. *There exists a constant $C > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi \in \mathcal{X}_{\tau,r}$, $\xi' \in \mathcal{X}_{\tau}$, and $k \in \{0, \ldots, |\tau|\}$:*

$$\left\| \mathrm{D}\varphi_{\tau,\tau_k}(\xi;\xi') \right\|_2 \leq C \left\| \xi' \right\|_{\mathcal{X}}, \quad (2.197)$$

where $\mathrm{D}\varphi_{\tau,\tau_k}(\xi;\xi')$ is as defined in Equation (2.187).

Proof. This follows by the Cauchy-Schwartz Inequality together with Corollary 2.52 and Corollary 2.62. □

We now show that $\Phi_{\tau}^{(\xi)}$ is in fact Lipschitz continuous.

Lemma 2.64. *There exists a constant $L > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi_1, \xi_2 \in \mathcal{X}_{\tau,r}$, and $k, l \in \{0, \dots, |\tau|\}$:*

$$\left\|\Phi_\tau^{(\xi_1)}(\tau_k, \tau_l) - \Phi_\tau^{(\xi_2)}(\tau_k, \tau_l)\right\|_{i,2} \leq L \left\|\xi_1 - \xi_2\right\|_{\mathcal{X}}, \tag{2.198}$$

where $\Phi_\tau^{(\xi)}$ is the solution to Difference Equation (2.188)*.*

Proof. Let $\xi_1 = (u_1, d_1)$ and $\xi_2 = (u_2, d_2)$. Then, using the Triangle Inequality:

$$\begin{aligned}
&\left\|\Phi_\tau^{(\xi_1)}(\tau_k, \tau_l) - \Phi_\tau^{(\xi_2)}(\tau_k, \tau_l)\right\|_{i,2} \leq \\
&\quad \leq \sum_{i=0}^{k-1} (\tau_{i+1} - \tau_i) \Bigg(\left\| \frac{\partial f}{\partial x}\big(\tau_i, z^{(\xi_2)}(\tau_i), u_2(\tau_i), d_2(\tau_i)\big) \right\|_{i,2} \left\|\Phi_\tau^{(\xi_1)}(\tau_i, \tau_j) - \Phi_\tau^{(\xi_2)}(\tau_i, \tau_j)\right\|_{i,2} + \\
&\quad + \left\| \frac{\partial f}{\partial x}\big(\tau_i, z^{(\xi_1)}(\tau_i), u_1(\tau_i), d_1(\tau_i)\big) - \frac{\partial f}{\partial x}\big(\tau_i, z^{(\xi_2)}(\tau_i), u_2(\tau_i), d_2(\tau_i)\big) \right\|_{i,2} \left\|\Phi_\tau^{(\xi_1)}(\tau_i, \tau_j)\right\|_{i,2} \Bigg).
\end{aligned} \tag{2.199}$$

The result follows by applying Condition 1 in Corollary 2.52, Condition 2 in Corollary 2.54, the same argument used in Equation (2.44), and the Discrete Bellman-Gronwall Inequality (Exercise 5.6.14 in [Pol97]). □

A simple extension of our previous argument shows that $\mathrm{D}\varphi_{\tau,\tau_k}(\xi, \cdot)$ is Lipschitz continuous with respect to its point of evaluation, ξ.

Lemma 2.65. *There exists a constant $L > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi_1, \xi_2 \in \mathcal{X}_{\tau,r}$, $\xi' \in \mathcal{X}_\tau$, and $k \in \{0, \dots, |\tau|\}$:*

$$\left\|\mathrm{D}\varphi_{\tau,\tau_k}(\xi_1; \xi') - \mathrm{D}\varphi_{\tau,\tau_k}(\xi_2; \xi')\right\|_2 \leq L \left\|\xi_1 - \xi_2\right\|_{\mathcal{X}} \left\|\xi'\right\|_{\mathcal{X}}, \tag{2.200}$$

where $\mathrm{D}\varphi_{\tau,\tau_k}$ is as defined in Equation (2.187)*.*

Proof. Let $\xi_1 = (u_1, d_1)$, $\xi_2 = (u_2, d_2)$, and $\xi' = (u', d')$. Then, if $\Delta\tau_k = \tau_{k+1} - \tau_k$, applying the Triangle Inequality:

$$\begin{aligned}
&\left\|\mathrm{D}\varphi_{\tau,\tau_k}(\xi_1; \xi') - \mathrm{D}\varphi_{\tau,\tau_k}(\xi_2; \xi')\right\|_2 \leq \\
&\leq \sum_{j=0}^{k-1} \Delta\tau_j \Bigg(\sum_{i=1}^{q} \left\|\Phi_\tau^{(\xi_1)}(\tau_k, \tau_{j+1}) - \Phi_\tau^{(\xi_2)}(\tau_k, \tau_{j+1})\right\|_{i,2} \left\|f\big(\tau_j, z^{(\xi_1)}(\tau_j), u_1(\tau_j), e_i\big)\right\|_{i,2} \left|d_i'(\tau_j)\right| + \\
&\quad + \left\|\Phi_\tau^{(\xi_2)}(\tau_k, \tau_{j+1})\right\|_{i,2} \left\|f\big(\tau_j, z^{(\xi_1)}(\tau_j), u_1(\tau_j), e_i\big) - f\big(\tau_j, z^{(\xi_2)}(\tau_j), u_2(\tau_j), e_i\big)\right\|_{i,2} \left|d_i'(\tau_j)\right| + \\
&+ \Bigg(\left\|\Phi_\tau^{(\xi_2)}(\tau_k, \tau_{j+1})\right\|_{i,2} \left\| \frac{\partial f}{\partial u}\big(\tau_j, z^{(\xi_1)}(\tau_j), u_1(\tau_j), d_1(\tau_j)\big) - \frac{\partial f}{\partial u}\big(\tau_j, z^{(\xi_2)}(\tau_j), u_2(\tau_j), d_2(\tau_j)\big) \right\|_{i,2} + \\
&\quad + \left\|\Phi_\tau^{(\xi_1)}(\tau_k, \tau_{j+1}) - \Phi_\tau^{(\xi_2)}(\tau_k, \tau_{j+1})\right\|_{i,2} \left\| \frac{\partial f}{\partial u}\big(\tau_j, z^{(\xi_1)}(\tau_j), u_1(\tau_j), d_1(\tau_j)\big) \right\|_{i,2} \Bigg) \left\|u'(\tau_j)\right\|_2 \Bigg)
\end{aligned} \tag{2.201}$$

The result follows by applying Lemma 2.64, Corollary 2.62, Condition 1 in Corollary 2.5, Conditions 1 and 3 in Corollary 2.54, and the same argument used in Equation (2.44). □

Employing these results, we can prove that $\mathrm{D}\varphi_{\tau,\tau_k}(\xi;\xi')$ converges to $\mathrm{D}\varphi_{\tau_k}(\xi;\xi')$ as the discretization is increased:

Lemma 2.66. *There exists $B > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi \in \mathcal{X}_{\tau,r}$, $\xi' \in \mathcal{X}_\tau$ and $k \in \{0,\dots,|\tau|\}$:*

$$\left\|\mathrm{D}\varphi_{\tau_k}(\xi;\xi') - \mathrm{D}\varphi_{\tau,\tau_k}(\xi;\xi')\right\|_2 \leq \frac{B}{2^N}, \tag{2.202}$$

where $D\varphi_{\tau_k}$ and $\mathrm{D}\varphi_{\tau,\tau_k}$ are as defined in Equations (2.49) and (2.187), respectively.

Proof. Let $\xi = (u,d)$, $\xi' = (u',d')$. First, by applying the Triangle Inequality and noticing that the induced matrix norm is compatible, we have:

$$\begin{aligned}
&\left\|\mathrm{D}\varphi_{\tau_k}(\xi;\xi') - \mathrm{D}\varphi_{\tau,\tau_k}(\xi;\xi')\right\|_2 \leq \\
&\leq \sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}} \left\|\Phi^{(\xi)}(\tau_k,s) - \Phi_\tau^{(\xi)}(\tau_k,\tau_{j+1})\right\|_{i,2} \left\|\frac{\partial f}{\partial u}\big(\tau_j, z^{(\xi)}(\tau_j), u(\tau_j), d(\tau_j)\big)\right\|_{i,2} \left\|u'(\tau_j)\right\|_2 + \\
&+ \left\|\Phi^{(\xi)}(\tau_k,s)\right\|_{i,2} \left\|\frac{\partial f}{\partial u}\big(s, x^{(\xi)}(s), u(\tau_j), d(\tau_j)\big) - \frac{\partial f}{\partial u}\big(\tau_j, z^{(\xi)}(\tau_j), u(\tau_j), d(\tau_j)\big)\right\|_{i,2} \left\|u'(\tau_j)\right\|_2 ds + \\
&\quad + \sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}} \sum_{i=1}^{q} \Bigg(\left\|\Phi^{(\xi)}(\tau_k,s)\right\|_{i,2} \left\|f\big(s, x^{(\xi)}(s), u(\tau_j), e_i\big) - f\big(\tau_j, z^{(\xi)}(\tau_j), u(\tau_j), e_i\big)\right\|_2 + \\
&\qquad + \left\|\Phi^{(\xi)}(\tau_k,s) - \Phi_\tau^{(\xi)}(\tau_k,\tau_{j+1})\right\|_{i,2} \left\|f\big(\tau_j, z^{(\xi)}(\tau_j), u(\tau_j), e_i\big)\right\|_2 \Bigg) \left|d_i'(\tau_l)\right| ds.
\end{aligned} \tag{2.203}$$

Second, let $\kappa(t) \in \{0,\dots,|\tau|\}$ such that $t \in [\tau_{\kappa(t)}, \tau_{\kappa(t)+1}]$ for each $t \in [0,1]$. Then, there exists $K > 0$ such that

$$\begin{aligned}
\left\|x^{(\xi)}(s) - z^{(\xi)}\big(\tau_{\kappa(s)}\big)\right\| &\leq \left\|x^{(\xi)}(s) - z^{(\xi)}(s)\right\| + \left\|z^{(\xi)}(s) - z^{(\xi)}\big(\tau_{\kappa(s)}\big)\right\| \\
&\leq \left\|x^{(\xi)}(s) - z^{(\xi)}(s)\right\| + \big(s - \tau_{\kappa(s)}\big)C \\
&\leq \frac{K}{2^N},
\end{aligned} \tag{2.204}$$

where $C > 0$ is as in Condition 1 in Corollary 2.52, and we applied Lemma 2.58 and the definition of $\mathcal{T}_N$ in Equation (2.143).

Third, analogous to our definition of discretized trajectory in Equation (2.150), we define a discretized state transition matrix, $\widetilde{\Phi}_\tau^{(\xi)}$ for each $k \in \{0,\dots,|\tau|\}$ via linear interpolation on the second argument:

$$\widetilde{\Phi}_\tau^{(\xi)}(\tau_k,t) = \sum_{j=0}^{|\tau|-1} \left(\Phi_\tau^{(\xi)}(\tau_k,\tau_j) + \frac{t-\tau_j}{\tau_{j+1}-\tau_j}\left(\Phi_\tau^{(\xi)}(\tau_k,\tau_{j+1}) - \Phi_\tau^{(\xi)}(\tau_k,\tau_j)\right)\right)\pi_{\tau,j}(t). \tag{2.205}$$

where $\tau_{\tau,j}$ is as defined in Equation (2.144). Then there exists a constant $K' > 0$ such that for each $t \in [0,1]$:

$$\begin{aligned}\left\|\Phi^{(\xi)}(\tau_k,t) - \Phi_\tau^{(\xi)}\big(\tau_k,\tau_{\kappa(t)}\big)\right\|_{i,2} &\leq \left\|\Phi^{(\xi)}(\tau_k,t) - \widetilde{\Phi}_\tau^{(\xi)}(\tau_k,t)\right\|_{i,2} + \left\|\widetilde{\Phi}_\tau^{(\xi)}(\tau_k,t) - \Phi_\tau^{(\xi)}\big(\tau_k,\tau_{\kappa(t)}\big)\right\|_{i,2} \\ &\leq \frac{K'}{2^N},\end{aligned} \tag{2.206}$$

where the last inequality follows by an argument identical to the one used in the proof of Lemma 2.58, together with an argument identical to the one used in Equation (2.204).

Finally, the result follows from Equation (2.203) after applying Condition 1 in Corollary 2.52, Corollary 2.19, Conditions 1 and 3 in Corollary 2.54, Equations (2.204) and (2.206), and the same argument as in Equation (2.44). □

Next, we construct the expression for the directional derivative of the discretized cost function and prove that it is Lipschitz continuous.

Lemma 2.67. *Let $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi \in \mathcal{X}_{\tau,r}$, $\xi' \in \mathcal{X}_\tau$, and J_τ be defined as in Equation* (2.151)*. Then the directional derivative of the discretized cost J_τ in the ξ' direction is:*

$$\mathrm{D}J_\tau(\xi;\xi') = \frac{\partial h_0}{\partial x}\big(\varphi_{\tau,1}(\xi)\big)\mathrm{D}\varphi_{\tau,1}(\xi;\xi'). \tag{2.207}$$

Proof. The result follows using the Chain Rule and Lemma 2.61. □

Corollary 2.68. *There exists a constant $L > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi_1, \xi_2 \in \mathcal{X}_{\tau,r}$, and $\xi' \in \mathcal{X}_\tau$:*

$$\left|\mathrm{D}J_\tau(\xi_1;\xi') - \mathrm{D}J_\tau(\xi_2;\xi')\right| \leq L\left\|\xi_1 - \xi_2\right\|_{\mathcal{X}}\left\|\eta\right\|_{\mathcal{X}}, \tag{2.208}$$

where $\mathrm{D}J_\tau$ is as defined in Equation (2.207)*.*

Proof. Notice by the Triangle Inequality and the Cauchy Schwartz Inequality:

$$\begin{aligned}\left|\mathrm{D}J_\tau(\xi_1;\xi') - \mathrm{D}J_\tau(\xi_2;\xi')\right| \leq{}& \left\|\frac{\partial h_0}{\partial x}\big(\varphi_{\tau,1}(\xi_1)\big)\right\|_2 \left\|\mathrm{D}\varphi_{\tau,1}(\xi_1;\eta) - \mathrm{D}\varphi_{\tau,1}(\xi_2;\eta)\right\|_2 + \\ &+ \left\|\frac{\partial h_0}{\partial x}\big(\varphi_{\tau,1}(\xi_1)\big) - \frac{\partial h_0}{\partial x}\big(\varphi_{\tau,1}(\xi_2)\big)\right\|_2 \left\|\mathrm{D}\varphi_{\tau,1}(\xi_2;\eta)\right\|_2.\end{aligned} \tag{2.209}$$

The result then follows by applying Condition 2 in Corollary 2.52, Condition 2 in Corollary 2.55, Corollary 2.63, and Lemma 2.65. □

In fact, the discretized cost function converges to the original cost function as the discretization is increased:

Lemma 2.69. *There exists a constant $B > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi \in \mathcal{X}_{\tau,r}$, and $\xi' \in \mathcal{X}_\tau$:*

$$|\mathrm{D}J_\tau(\xi;\xi') - \mathrm{D}J(\xi;\xi')| \leq \frac{B}{2^N}, \tag{2.210}$$

where $\mathrm{D}J$ is as defined in Equation (2.67) and $\mathrm{D}J_\tau$ is as defined in Equation (2.207).

Proof. Notice by the Triangle Inequality and the Cauchy Schwartz Inequality:

$$\begin{aligned}|\mathrm{D}J_\tau(\xi;\xi') - \mathrm{D}J(\xi;\xi')| \leq \left\|\frac{\partial h_0}{\partial x}\big(\varphi_1(\xi)\big)\right\|_2 \left\|\mathrm{D}\varphi_1(\xi;\xi') - \mathrm{D}\varphi_{\tau,1}(\xi;\xi')\right\|_2 + \\ + \left\|\frac{\partial h_0}{\partial x}\big(\varphi_1(\xi)\big) - \frac{\partial h_0}{\partial x}\big(\varphi_{\tau,1}(\xi)\big)\right\|_2 \left\|\mathrm{D}\varphi_{\tau,1}(\xi;\xi')\right\|_2. \end{aligned} \tag{2.211}$$

Then the result follows by applying Condition 2 in Assumption 2.3, Condition 2 in Corollary 2.5, Lemma 2.58, Lemma 2.66, and Corollary 2.63. □

Next, we construct the expression for the directional derivative of the discretized component functions and prove that they are Lipschitz continuous.

Lemma 2.70. *Let $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi \in \mathcal{X}_{\tau,r}$, $\xi' \in \mathcal{X}_\tau$, $j \in \mathcal{J}$, and ψ_{τ,j,τ_k} be defined as in Equation (2.154). Then the directional derivative of each of the discretized component constraints ψ_{τ,j,τ_k} for each $k \in \{0,\ldots,|\tau|\}$ in the ξ' direction is:*

$$\mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi') = \frac{\partial h_j}{\partial x}\big(\varphi_{\tau,\tau_k}(\xi)\big)\mathrm{D}\varphi_{\tau,\tau_k}(\xi;\xi'). \tag{2.212}$$

Proof. The result is a direct consequence of the Chain Rule and Lemma 2.61. □

Corollary 2.71. *There exists a constant $L > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi_1, \xi_2 \in \mathcal{X}_{\tau,r}$, $\xi' \in \mathcal{X}_\tau$, and $k \in \{0,\ldots,|\tau|\}$:*

$$|\mathrm{D}\psi_{\tau,j,\tau_k}(\xi_1;\xi') - \mathrm{D}\psi_{\tau,j,\tau_k}(\xi_2;\xi')| \leq L\left\|\xi_1 - \xi_2\right\|_{\mathcal{X}}\left\|\xi'\right\|_{\mathcal{X}}, \tag{2.213}$$

where $\mathrm{D}\psi_{\tau,j,\tau_k}$ is as defined in Equation (2.212).

Proof. Notice by the Triangle Inequality and the Cauchy Schwartz Inequality:

$$\begin{aligned}|\mathrm{D}\psi_{\tau,j,\tau_k}(\xi_1;\xi') - \mathrm{D}\psi_{\tau,j,\tau_k}(\xi_2;\xi')| \leq \left\|\frac{\partial h_j}{\partial x}\big(\varphi_{\tau,\tau_k}(\xi_1)\big)\right\|_2 \left\|\mathrm{D}\varphi_{\tau,\tau_k}(\xi_1;\xi') - \mathrm{D}\varphi_{\tau,\tau_k}(\xi_2;\xi')\right\|_2 + \\ + \left\|\frac{\partial h_j}{\partial x}\big(\varphi_{\tau,\tau_k}(\xi_1)\big) - \frac{\partial h_j}{\partial x}\big(\varphi_{\tau,\tau_k}(\xi_2)\big)\right\|_2 \left\|\mathrm{D}\varphi_{\tau,\tau_k}(\xi_2;\xi')\right\|_2. \end{aligned} \tag{2.214}$$

The result then follows by applying Condition 3 in Corollary 2.52, Condition 4 in Corollary 2.55, Corollary 2.63, and Lemma 2.65. □

In fact, the discretized component constraint functions converge to the original component constraint function as the discretization is increased:

Lemma 2.72. *There exists a constant $B > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, $\xi \in \mathcal{X}_{\tau,r}$, $\xi' \in \mathcal{X}_\tau$, $j \in \mathcal{J}$, and $k \in \{0, \ldots, |\tau|\}$:*

$$|\mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi') - \mathrm{D}\psi_{j,\tau_k}(\xi;\xi')| \leq \frac{B}{2^N}, \tag{2.215}$$

where $\mathrm{D}\psi_{j,\tau_k}$ is as defined in Equation (2.70) and $\mathrm{D}\psi_{\tau,j,\tau_k}$ is as defined in Equation (2.212).

Proof. Notice by the Triangle Inequality and the Cauchy Schwartz Inequality:

$$\begin{aligned}|\mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi') - \mathrm{D}\psi_{j,\tau_k}(\xi;\xi')| \leq{}& \left\| \frac{\partial h_j}{\partial x}\big(\varphi_{\tau_k}(\xi)\big) \right\|_2 \left\| \mathrm{D}\varphi_{\tau_k}(\xi;\xi') - \mathrm{D}\varphi_{\tau,\tau_k}(\xi;\xi') \right\|_2 + \\ &+ \left\| \frac{\partial h_j}{\partial x}\big(\varphi_{\tau_k}(\xi)\big) - \frac{\partial h_j}{\partial x}\big(\varphi_{\tau,\tau_k}(\xi)\big) \right\|_2 \left\| \mathrm{D}\varphi_{\tau,\tau_k}(\xi;\xi') \right\|_2 . \end{aligned} \tag{2.216}$$

The result follows by applying Condition 4 in Assumption 2.3, Condition 3 in Corollary 2.5, Lemma 2.58, Lemma 2.66, and Corollary 2.63. □

Given these results, we can begin describing the properties satisfied by the discretized optimality function:

Lemma 2.73. *Let $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, and ζ_τ be defined as in Equation (2.158). Then there exists a constant $L > 0$ such that, for each $\xi_1, \xi_2, \xi' \in \mathcal{X}_{\tau,r}$,*

$$|\zeta_\tau(\xi_1,\xi') - \zeta_\tau(\xi_2,\xi')| \leq L \left\| \xi_1 - \xi_2 \right\|_{\mathcal{X}} . \tag{2.217}$$

Proof. Letting $\Psi_\tau^+(\xi) = \max\{0, \Psi_\tau(\xi)\}$ and $\Psi_\tau^-(\xi) = \max\{0, -\Psi_\tau(\xi)\}$, observe:

$$\begin{aligned}\zeta_\tau(\xi,\xi') = \max\left\{ \mathrm{D}J_\tau(\xi;\xi'-\xi) - \Psi_\tau^+(\xi), \max_{j\in\mathcal{J},\, k\in\{0,\ldots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi) - \gamma\Psi_\tau^-(\xi) \right\} + \\ + \left\|\xi'-\xi\right\|_{\mathcal{X}}. \end{aligned} \tag{2.218}$$

Employing Equation (2.75):

$$\begin{aligned}&\big|\zeta_\tau(\xi_1,\xi') - \zeta_\tau(\xi_2,\xi')\big| \leq \\ &\qquad \leq \max\Big\{ \big|\mathrm{D}J_\tau(\xi_1;\xi'-\xi_1) - \mathrm{D}J_\tau(\xi_2;\xi'-\xi_2)\big| + \big|\Psi_\tau^+(\xi_2) - \Psi_\tau^+(\xi_1)\big|, \\ &\max_{j\in\mathcal{J},\, k\in\{0,\ldots,|\tau|\}} \big|\mathrm{D}\psi_{\tau,j,\tau_k}(\xi_1;\xi'-\xi_1) - \mathrm{D}\psi_{\tau,j,\tau_k}(\xi_2;\xi'-\xi_2)\big| + \gamma\big|\Psi_\tau^-(\xi_2) - \Psi_\tau^-(\xi_1)\big| \Big\} + \\ &\qquad\qquad + \big|\left\|\xi'-\xi_1\right\|_{\mathcal{X}} - \left\|\xi'-\xi_2\right\|_{\mathcal{X}}\big|. \end{aligned} \tag{2.219}$$

We show three results that taken together with the Triangle Inequality prove the desired result. First, by applying the Reverse Triangle Inequality:

$$\big|\|\xi' - \xi_1\|_{\mathcal{X}} - \|\xi' - \xi_2\|_{\mathcal{X}}\big| \leq \|\xi_1 - \xi_2\|_{\mathcal{X}}. \tag{2.220}$$

Second,

$$\begin{aligned}\big|\mathrm{D}J_\tau(\xi_1;\xi'-\xi_1) - \mathrm{D}J_\tau(\xi_2;\xi'-\xi_2)\big| \leq& \\ \leq& \big|\mathrm{D}J_\tau(\xi_1;\xi') - \mathrm{D}J_\tau(\xi_2;\xi')\big| + \big|\mathrm{D}J_\tau(\xi_1;\xi_1) - \mathrm{D}J(\xi_2;\xi_1)\big| + \\ &+ \left|\frac{\partial h_0}{\partial x}\big(\varphi_{\tau,1}(\xi_2)\big)\mathrm{D}\varphi_{\tau,1}(\xi_2;\xi_2-\xi_1)\right| \\ \leq& L\,\|\xi_1 - \xi_2\|_{\mathcal{X}},\end{aligned} \tag{2.221}$$

where $L > 0$ and we employed the linearity of $\mathrm{D}J_\tau$, Corollary 2.68, the fact that ξ' and ξ_1 are bounded since $\xi', \xi_1 \in \mathcal{X}_{\tau,r}$, the Cauchy-Schwartz Inequality, Condition 2 in Corollary 2.52, and Corollary 2.63. Notice that by employing an argument identical to Equation (2.221) and Corollary 2.71, we can assume without loss of generality that $\big|\mathrm{D}\psi_{\tau,j,\tau_k}(\xi_1;\xi'-\xi_1) - \mathrm{D}\psi_{\tau,j,\tau_k}(\xi_2;\xi'-\xi_2)\big| \leq L\,\|\xi_1 - \xi_2\|_{\mathcal{X}}$. Finally, notice that by applying Lemma 2.57, $\Psi_\tau^+(\xi)$ and $\Psi_\tau^-(\xi)$ are Lipschitz continuous. □

Employing these results, we can prove that $\zeta_\tau(\xi;\xi')$ converges to $\zeta(\xi;\xi')$ as the discretization is increased:

Lemma 2.74. *There exists a constant $B > 0$ such that for each $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, and $\xi, \xi' \in \mathcal{X}_{\tau,r}$:*

$$|\zeta_\tau(\xi,\xi') - \zeta(\xi,\xi')| \leq \frac{B}{2^N}, \tag{2.222}$$

where ζ is as defined in Equation (2.31) *and ζ_τ is as defined in Equation* (2.159).

Proof. Let $\Psi^+(\xi) = \max\{0, \Psi(\xi)\}$, $\Psi_\tau^+(\xi) = \max\{0, \Psi_\tau(\xi)\}$, $\Psi^-(\xi) = \max\{0, -\Psi(\xi)\}$, and $\Psi_\tau^-(\xi) = \max\{0, -\Psi_\tau(\xi)\}$. Notice that we can then write:

$$\zeta(\xi,\xi') = \max\left\{\mathrm{D}J(\xi;\xi'-\xi) - \Psi^+(\xi), \max_{j\in\mathcal{J},\, t\in[0,1]} \mathrm{D}\psi_{j,t}(\xi;\xi'-\xi) - \gamma\Psi^-(\xi)\right\} + \|\xi'-\xi\|_{\mathcal{X}}, \tag{2.223}$$

and similarly for $\zeta_\tau(\xi,\xi')$. Employing this redefinition, notice first that by employing an argument identical to the one used in the proof of Lemma 2.60 we can show that there exists a $K > 0$ such that for any positive integer N, $\tau \in \mathcal{T}_N$ and $\xi \in \mathcal{X}_{\tau,r}$:

$$\big|\Psi_\tau^+(\xi) - \Psi^+(\xi)\big| \leq \frac{K}{2^N}, \quad \text{and} \quad \big|\Psi_\tau^-(\xi) - \Psi^-(\xi)\big| \leq \frac{K}{2^N}. \tag{2.224}$$

Let $\kappa(t) \in \{0, \dots, |\tau|\}$ such that $t \in [\tau_{\kappa(t)}, \tau_{\kappa(t)+1}]$ for each $t \in [0,1]$. Then there exists $K' > 0$ such that,

$$\begin{aligned}\left|\mathrm{D}\psi_{j,t}(\xi;\xi'-\xi) - \mathrm{D}\psi_{j,\tau_{\kappa(t)}}(\xi;\xi'-\xi)\right| \leq \\ \leq \left\|\frac{\partial h_j}{\partial x}\big(\varphi_t(\xi)\big) - \frac{\partial h_j}{\partial x}\big(\varphi_{\tau_{\kappa(t)}}(\xi)\big)\right\|_2 \left\|\mathrm{D}\varphi_t(\xi;\xi'-\xi)\right\|_2 + \\ + \left\|\frac{\partial h_j}{\partial x}\big(\varphi_{\tau_{\kappa(t)}}(\xi)\big)\right\|_2 \left\|\mathrm{D}\varphi_t(\xi;\xi'-\xi) - \mathrm{D}\varphi_{\tau_{\kappa(t)}}(\xi;\xi'-\xi)\right\|_2 \\ \leq C'\left(\left\|\varphi_t(\xi) - \varphi_{\tau_{\kappa(t)}}(\xi)\right\|_2 + \left\|\mathrm{D}\varphi_t(\xi;\xi'-\xi) - \mathrm{D}\varphi_{\tau_{\kappa(t)}}(\xi;\xi'-\xi)\right\|_2\right) \\ \leq \frac{K'}{2^N},\end{aligned} \tag{2.225}$$

where $C' > 0$ is a constant obtained after applying Corollary 2.20, Condition 4 in Assumption 2.3, and Condition 3 in Corollary 2.5, and the last inequality follows after noting that both terms can be written as the integral of uniformly bounded functions over an interval of length smaller than 2^{-N}. Thus, by the Triangle Inequality, Lemma 2.72, and Equation (2.225), we know there exists $B > 0$ such that for each $t \in [0,1]$:

$$\left|\mathrm{D}\psi_{j,t}(\xi;\xi'-\xi) - \mathrm{D}\psi_{\tau,j,\tau_{\kappa(t)}}(\xi;\xi'-\xi)\right| \leq \frac{B}{2^N}. \tag{2.226}$$

Moreover, if $t' \in \arg\max_{t\in[0,1]} \mathrm{D}\psi_{j,t}(\xi;\xi'-\xi)$, then

$$\max_{t\in[0,1]} \mathrm{D}\psi_{j,t}(\xi;\xi'-\xi) - \max_{k\in\{0,\dots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi) \leq \mathrm{D}\psi_{j,t'}(\xi;\xi'-\xi) - \mathrm{D}\psi_{\tau,j,\tau_{\kappa(t')}}(\xi;\xi'-\xi) \leq \frac{B}{2^N}. \tag{2.227}$$

Similarly if $k' \in \arg\max_{k\in\{0,\dots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi)$, then

$$\max_{k\in\{0,\dots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi) - \max_{t\in[0,1]} \mathrm{D}\psi_{j,t}(\xi;\xi'-\xi) \leq \mathrm{D}\psi_{\tau,j,\tau_{k'}}(\xi;\xi'-\xi) - \mathrm{D}\psi_{j,\tau_{k'}}(\xi;\xi'-\xi) \leq \frac{B}{2^N}. \tag{2.228}$$

Therefore, by Equation (2.227),

$$\begin{aligned}\max_{j\in\mathcal{J},\, t\in[0,1]} \mathrm{D}\psi_{j,t}(\xi;\xi'-\xi) - \max_{j\in\mathcal{J},\, k\in\{0,\dots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi) \leq \\ \leq \max_{j\in\mathcal{J}}\left(\max_{t\in[0,1]} \mathrm{D}\psi_{j,t}(\xi;\xi'-\xi) - \max_{k\in\{0,\dots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi)\right) \leq \frac{B}{2^N}.\end{aligned} \tag{2.229}$$

and similarly, by Equation (2.228),

$$\begin{aligned}\max_{j\in\mathcal{J},\, k\in\{0,\dots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi) - \max_{j\in\mathcal{J},\, t\in[0,1]} \mathrm{D}\psi_{j,t}(\xi;\xi'-\xi) \leq \\ \leq \max_{j\in\mathcal{J}}\left(\max_{k\in\{0,\dots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi) - \max_{t\in[0,1]} \mathrm{D}\psi_{j,t}(\xi;\xi'-\xi)\right) \leq \frac{B}{2^N},\end{aligned} \tag{2.230}$$

Employing these results and Equation (2.75), observe that:

$$|\zeta_\tau(\xi,\xi') - \zeta(\xi,\xi')| \leq \max\Bigg\{ |\mathrm{D}J_\tau(\xi;\xi'-\xi) - \mathrm{D}J(\xi;\xi'-\xi)| + \left|\Psi^+(\xi) - \Psi^+_\tau(\xi)\right|, \\ \left| \max_{j\in\mathcal{J},\, k\in\{0,\dots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi;\xi'-\xi) - \max_{j\in\mathcal{J},\, t\in[0,1]} \mathrm{D}\psi_{j,t}(\xi;\xi'-\xi) \right| + \gamma \left|\Psi^-(\xi) - \Psi^-_\tau(\xi)\right| \Bigg\}. \quad (2.231)$$

Finally, applying Lemma 2.69 and the inequalities above, we get our desired result. □

ζ_τ is in fact strictly convex just like its infinite dimensional analogue, and its proof is similar to the proof of Lemma 2.31, hence we omit its details.

Lemma 2.75. *Let $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, and $\xi \in \mathcal{X}_{\tau,p}$. Then the map $\xi' \mapsto \zeta_\tau(\xi,\xi')$, as defined in Equation* (2.159)*, is strictly convex.*

Theorem 2.76 is very important since it proves that g_τ, as defined in Equation (2.158), is a well-defined function. Its proof is a consequence of the well-known result that strictly-convex functions in finite-dimensional spaces have unique minimizers.

Theorem 2.76. *Let $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, and $\xi \in \mathcal{X}_{\tau,p}$. Then the map $\xi' \mapsto \zeta_\tau(\xi,\xi')$, as defined in Equation* (2.159)*, has a unique minimizer.*

Employing these results we can prove the continuity of the discretized optimality function. This result is not strictly required in order to prove the convergence of Algorithm 2.2 or in order to prove that the discretized optimality function encodes local minimizers of the Discretized Relaxed Switched System Optimal Control Problem. However, this is a fundamental result from an implementation point of view, since in practice, a computer only produces approximate results, and continuity gives a guarantee that these approximations are at least valid in a neighborhood of the evaluation point.

Lemma 2.77. *Let $N \in \mathbb{N}$ and $\tau \in \mathcal{T}_N$, then the function θ_τ, as defined in Equation* (2.158)*, is continuous.*

Proof. First, we show that θ_τ is upper semi-continuous. Consider a sequence $\{\xi_i\}_{i\in\mathbb{N}} \subset \mathcal{X}_{\tau,r}$ converging to $\xi \in \mathcal{X}_{\tau,r}$, and $\xi' \in \mathcal{X}_{\tau,r}$, such that $\theta_\tau(\xi) = \zeta_\tau(\xi,\xi')$, i.e. $\xi' = g_\tau(\xi)$, where g is defined as in Equation (2.158). Since $\theta_\tau(\xi_i) \leq \zeta_\tau(\xi_i,\xi')$ for all $i \in \mathbb{N}$,

$$\limsup_{i\to\infty} \theta_\tau(\xi_i) \leq \limsup_{i\to\infty} \zeta_\tau(\xi_i,\xi') = \zeta_\tau(\xi,\xi') = \theta_\tau(\xi), \quad (2.232)$$

which proves the upper semi-continuity of θ_τ.

Second, we show that θ_τ is lower semi-continuous. Let $\{\xi'_i\}_{i\in\mathbb{N}} \subset \mathcal{X}_{\tau,r}$ such that $\theta_\tau(\xi_i) = \zeta_\tau(\xi_i,\xi'_i)$, i.e. $\xi'_i = g_\tau(\xi_i)$. From Lemma 2.73, we know there exists a Lipschitz constant $L > 0$ such that for each $i \in \mathbb{N}$, $|\zeta_\tau(\xi,\xi'_i) - \zeta_\tau(\xi_i,\xi'_i)| \leq L\,\|\xi - \xi_i\|_{\mathcal{X}}$. Consequently,

$$\theta_\tau(\xi) \leq \big(\zeta_\tau(\xi,\xi'_i) - \zeta_\tau(\xi_i,\xi'_i)\big) + \zeta_\tau(\xi_i,\xi'_i) \leq L\|\xi - \xi_i\|_{\mathcal{X}} + \theta_\tau(\xi_i). \quad (2.233)$$

Taking limits we conclude that

$$\theta_\tau(\xi) \leq \liminf_{i\to\infty} \theta_\tau(\xi_i), \tag{2.234}$$

which proves the lower semi-continuity of θ_τ, and our desired result. □

Next, we prove that the local minimizers of the Discretized Relaxed Switched System Optimal Control Problem are in fact zeros of the discretized optimality function.

Theorem 2.78. *Let $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, and θ_τ be defined as in Equation* (2.158)*, then:*

(1) *θ_τ is non-positive valued, and*

(2) *If $\xi \in \mathcal{X}_{\tau,p}$ is a local minimizer of the Discretized Relaxed Switched System Optimal Control Problem as in Definition 2.48, then $\theta_\tau(\xi) = 0$.*

Proof. Notice $\zeta_\tau(\xi,\xi) = 0$, therefore $\theta_\tau(\xi) = \min_{\xi'\in\mathcal{X}_{\tau,r}} \zeta_\tau(\xi,\xi') \leq \zeta_\tau(\xi,\xi) = 0$. This proves Condition 1.

To prove Condition 2, we begin by making several observations. Given $\xi' \in \mathcal{X}_{\tau,r}$ and $\lambda \in [0,1]$, using the Mean Value Theorem and Corollary 2.68 we have that there exists $s \in (0,1)$ and $L > 0$ such that

$$\begin{aligned} J_\tau\big(\xi + \lambda(\xi'-\xi)\big) - J_\tau(\xi) &= \mathrm{D}J_\tau\big(\xi + s\lambda(\xi'-\xi); \lambda(\xi'-\xi)\big) \\ &\leq \lambda \mathrm{D}J_\tau\big(\xi; \xi'-\xi\big) + L\lambda^2\|\xi'-\xi\|_{\mathcal{X}}^2. \end{aligned} \tag{2.235}$$

Letting $\mathcal{A}_\tau(\xi) = \big\{(j,k) \in \mathcal{J}\times\{0,\dots,|\tau|\} \mid \Psi_\tau(\xi) = h_j\big(z^{(\xi)}(\tau_k)\big)\big\}$, similar to the equation above, there exists a pair $(j,k) \in \mathcal{A}\big(\xi + \lambda(\xi'-\xi)\big)$ and $s \in (0,1)$ such that, using Corollary 2.71,

$$\begin{aligned} \Psi_\tau\big(\xi + \lambda(\xi'-\xi)\big) - \Psi_\tau(\xi) &\leq \psi_{\tau,j,\tau_k}\big(\xi + \lambda(\xi'-\xi)\big) - \Psi(\xi) \\ &\leq \psi_{\tau,j,\tau_k}\big(\xi + \lambda(\xi'-\xi)\big) - \psi_{\tau,j,\tau_k}(\xi) \\ &= \mathrm{D}\psi_{\tau,j,\tau_k}\big(\xi + s\lambda(\xi'-\xi); \lambda(\xi'-\xi)\big) \\ &\leq \lambda \mathrm{D}\psi_{\tau,j,\tau_k}\big(\xi; \xi'-\xi\big) + L\lambda^2\|\xi'-\xi\|_{\mathcal{X}}^2. \end{aligned} \tag{2.236}$$

We prove Condition 2 by contradiction. That is we assume $\xi \in \mathcal{X}_{\tau,p}$ is a local minimizer of the Discretized Relaxed Switched System Optimal Control Problem and $\theta_\tau(\xi) < 0$ and show that for each $\varepsilon > 0$ there exists $\hat{\xi} \in \{\bar{\xi} \in \mathcal{X}_{\tau,r} \mid \Psi_\tau(\bar{\xi}) \leq 0\} \cap \mathcal{N}_{\tau,\mathcal{X}}(\xi,\varepsilon)$ such that $J_\tau(\hat{\xi}) < J_\tau(\xi)$, where $\mathcal{N}_{\tau,\mathcal{X}}(\xi,\varepsilon)$ is as defined in Equation (2.157), hence arriving at a contradiction.

Before arriving at this contradiction, we make two more observations. First, notice that since $\xi \in \mathcal{X}_p$ is a local minimizer of the Discretized Relaxed Switched System Optimal Control Problem, $\Psi_\tau(\xi) \leq 0$. Second, consider g_τ as defined in Equation (2.158), which exists by Theorem 2.76 and notice that since $\theta_\tau(\xi) < 0$, $g_\tau(\xi) \neq \xi$.

Next, observe that:

$$\theta_\tau(\xi) = \max\left\{\mathrm{D}J_\tau(\xi; g_\tau(\xi)-\xi), \max_{(j,k)\in\mathcal{J}\times\{0,\dots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_k}(\xi; g_\tau(\xi)-\xi) + \gamma\Psi_\tau(\xi)\right\} + \\ + \|g_\tau(\xi)-\xi\|_{\mathcal{X}} < 0. \quad (2.237)$$

For each $\lambda > 0$ by using Equations (2.235) and (2.237) we have:

$$J_\tau(\xi + \lambda(g_\tau(\xi)-\xi)) - J_\tau(\xi) \leq \theta_\tau(\xi)\lambda + 4A^2L\lambda^2, \quad (2.238)$$

where $A = \max\left\{\|u\|_2 + 1 \mid u \in U\right\}$ and we used the fact that $\mathrm{D}J_\tau(\xi; g(\xi)-\xi) \leq \theta_\tau(\xi)$. Hence for each $\lambda \in \left(0, \frac{-\theta_\tau(\xi)}{4A^2L}\right)$:

$$J_\tau(\xi + \lambda(g_\tau(\xi)-\xi)) - J_\tau(\xi) < 0. \quad (2.239)$$

Similarly, for each $\lambda > 0$ by using Equations (2.236) and (2.237) we have:

$$\Psi_\tau(\xi + \lambda(g_\tau(\xi)-\xi)) \leq \Psi_\tau(\xi) + (\theta_\tau(\xi) - \gamma\Psi_\tau(\xi))\,\lambda + 4A^2L\lambda^2, \quad (2.240)$$

where, as in Equation (2.238), $A = \max\left\{\|u\|_2 + 1 \mid u \in U\right\}$ and we used the fact that $\mathrm{D}\psi_{\tau,j,\tau_k}(\xi; g(\xi)-\xi) \leq \theta_\tau(\xi)$. Hence for each $\lambda \in \left(0, \min\left\{\frac{-\theta_\tau(\xi)}{4A^2L}, \frac{1}{\gamma}\right\}\right)$:

$$\Psi_\tau(\xi + \lambda(g_\tau(\xi)-\xi)) \leq (1-\gamma\lambda)\Psi_\tau(\xi) \leq 0. \quad (2.241)$$

Summarizing, suppose $\xi \in \mathcal{X}_{\tau,p}$ is a local minimizer of the Discretized Relaxed Switched System Optimal Control Problem and $\theta_\tau(\xi) < 0$. For each $\varepsilon > 0$, by choosing any

$$\lambda \in \left(0, \min\left\{\frac{-\theta_\tau(\xi)}{4A^2L}, \frac{1}{\gamma}, \frac{\varepsilon}{\|g_\tau(\xi)-\xi\|_{\mathcal{X}}}\right\}\right), \quad (2.242)$$

we can construct a new point $\hat{\xi} = \big(\xi + \lambda(g_\tau(\xi)-\xi)\big) \in \mathcal{X}_{\tau,r}$ such that $\hat{\xi} \in \mathcal{N}_{\tau,\mathcal{X}}(\xi,\varepsilon)$ by our choice of λ, $J_\tau(\hat{\xi}) < J_\tau(\xi)$ by Equation (2.239), and $\Psi_\tau(\hat{\xi}) \leq 0$ by Equation (2.241). Therefore, ξ is not a local minimizer of the Discretized Relaxed Switched System Optimal Control Problem, which is a contradiction and proves Condition 2. □

Finally, we prove that the Discretized Relaxed Switched System Optimal Control Problem consistently approximates the Switched System Optimal Control Problem:

Theorem 2.79. *Let $\{\tau_i\}_{i\in\mathbb{N}}$ and $\{\xi_i\}_{i\in\mathbb{N}}$ such that $\tau_i \in \mathcal{T}_i$ and $\xi_i \in \mathcal{X}_{\tau_i,p}$ for each $i \in \mathbb{N}$. Then*

$$\lim_{i\to\infty} |\theta_{\tau_i}(\xi_i) - \theta(\xi_i)| = 0, \quad (2.243)$$

where θ is as defined in Equation (2.30) *and θ_τ is as defined in Equation* (2.158)*. That is, the Discretized Relaxed Switched System Optimal Control Problem as defined in Equation* (2.156) *is a* consistent approximation *of the Switched System Optimal Control Problem as defined in Equation* (2.18)*, where consistent approximation is defined as in Definition 2.49.*

Proof. First, by Lemma 2.74,

$$\limsup_{i\to\infty} \theta(\xi_i) - \theta_{\tau_i}(\xi_i) \leq \limsup_{i\to\infty} \zeta\big(\xi_i, g(\xi_i)\big) - \zeta_{\tau_i}\big(\xi_i, g_{\tau_i}(\xi_i)\big) \leq \limsup_{i\to\infty} \frac{B}{2^i} = 0, \tag{2.244}$$

where g is as defined in Equation (2.30) and g_τ is as defined in Equation (2.158).

Now, by Condition 2 in Lemma 2.47, we know there exists a sequence $\{\xi_i'\}_{i\in\mathbb{N}}$, with $\xi_i' \in \mathcal{X}_{\tau_i,r}$ for each $i \in \mathbb{N}$, such that $\lim_{i\to\infty} \xi_i' = g(\xi)$. Then, by Lemma 2.74,

$$\begin{aligned}\limsup_{i\to\infty} \theta_{\tau_i}(\xi_i) - \theta(\xi_i) &\leq \limsup_{i\to\infty} \zeta_{\tau_i}(\xi_i, \xi_i') - \zeta\big(\xi_i, g(\xi)\big) \\ &\leq \limsup_{i\to\infty} \big(\zeta_{\tau_i}(\xi_i, \xi_i') - \zeta(\xi_i, \xi_i')\big) + \big(\zeta(\xi_i, \xi_i') - \zeta\big(\xi_i, g(\xi)\big)\big) \\ &\leq \limsup_{i\to\infty} \frac{B}{2^i} + \zeta(\xi_i, \xi_i') - \zeta\big(\xi_i, g(\xi)\big).\end{aligned} \tag{2.245}$$

Employing Equation (2.75):

$$\begin{aligned}\big|\zeta(\xi_i, \xi_i') - \zeta\big(\xi_i, g(\xi)\big)\big| \leq \max\Big\{ &\big|\mathrm{D}J(\xi_i; \xi_i' - \xi_i) - \mathrm{D}J(\xi_i; g(\xi) - \xi_i)\big|, \\ &\max_{j\in\mathcal{J},\, t\in[0,1]} \big|\mathrm{D}\psi_{j,t}(\xi_i; \xi_i' - \xi_i) - \mathrm{D}\psi_{j,t}(\xi_i; g(\xi) - \xi_i)\big|\Big\} + \big|\|\xi_i' - \xi_i\|_{\mathcal{X}} - \|g(\xi) - \xi_i\|_{\mathcal{X}}\big|.\end{aligned} \tag{2.246}$$

Notice, that by applying the reverse Triangle Inequality:

$$\big|\|\xi_i' - \xi_i\|_{\mathcal{X}} - \|g(\xi) - \xi_i\|_{\mathcal{X}}\big| \leq \|\xi_i' - g(\xi)\|_{\mathcal{X}}. \tag{2.247}$$

Next, notice:

$$\begin{aligned}\big|\mathrm{D}J(\xi_i; \xi_i' - \xi_i) - \mathrm{D}J(\xi_i; g(\xi) - \xi_i)\big| &= \big|\mathrm{D}J(\xi_i; \xi_i' - g(\xi))\big| \\ &= \left|\frac{\partial h_0}{\partial x}\big(\varphi_1(\xi_i)\big)\mathrm{D}\varphi_1(\xi_i; \xi_i' - g(\xi))\right| \\ &\leq L\,\|\xi_i' - g(\xi)\|_{\mathcal{X}},\end{aligned} \tag{2.248}$$

where $L > 0$ and we employed the linearity of $\mathrm{D}J$, Condition 2 in Corollary 2.5, and Corollary 2.20. Notice that by employing an argument identical to Equation (2.248), we can assume without loss of generality that $\big|\mathrm{D}\psi_{j,t}(\xi_i; \xi_i' - \xi_i) - \mathrm{D}\psi_{j,t}(\xi_i; g(\xi) - \xi_i)\big| \leq L\,\|\xi_i' - g(\xi)\|_{\mathcal{X}}$. Therefore:

$$\limsup_{i\to\infty} \big|\zeta(\xi_i, \xi_i') - \zeta\big(\xi_i, g(\xi)\big)\big| \leq 0. \tag{2.249}$$

From Equation (2.245), we have $\limsup_{i\to\infty} \big(\theta_{\tau_i}(\xi_i) - \theta(\xi_i)\big) \leq 0$. Notice that

$$\limsup_{i\to\infty} |\theta_{\tau_i}(\xi_i) - \theta(\xi_i)| \geq \liminf_{i\to\infty} |\theta_{\tau_i}(\xi_i) - \theta(\xi_i)| \geq 0. \tag{2.250}$$

Therefore combining our results, we have $\lim_{i\to\infty} |\theta_{\tau_i}(\xi_i) - \theta(\xi_i)| = 0$. □

Convergence of the Implementable Algorithm

In this subsection, we prove that the sequence of points generated by Algorithm 2.2 converges to a point that satisfies the optimality condition. We begin by proving that the Armijo algorithm as defined in Equation (2.162) terminates after a finite number of steps.

Lemma 2.80. *Let $\alpha \in (0,1)$ and $\beta \in (0,1)$. For every $\delta > 0$, there exists an $M_\delta^* < \infty$ such that if $\theta_\tau(\xi) \leq -\delta$ for $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, and $\xi \in \mathcal{X}_{\tau,p}$, then $\mu_\tau(\xi) \leq M_\delta^*$, where θ_τ is as defined in Equation (2.158) and μ_τ is as defined in Equation (2.162).*

Proof. Given $\xi' \in \mathcal{X}$ and $\lambda \in [0,1]$, using the Mean Value Theorem and Corollary 2.68 we have that there exists $s \in (0,1)$ such that

$$\begin{aligned} J_\tau\big(\xi + \lambda(\xi' - \xi)\big) - J_\tau(\xi) &= \mathrm{D}J_\tau\big(\xi + s\lambda(\xi' - \xi); \lambda(\xi' - \xi)\big) \\ &\leq \lambda \mathrm{D}J_\tau\big(\xi; \xi' - \xi\big) + L\lambda^2 \|\xi' - \xi\|_{\mathcal{X}}^2. \end{aligned} \tag{2.251}$$

Let $\mathcal{A}_\tau(\xi) = \{(j,i) \in \mathcal{J} \times \{0, \ldots, |\tau|\} \mid \Psi_\tau(\xi) = \psi_{\tau,j,\tau_i}(\xi)\}$, then there exists a pair $(j,i) \in \mathcal{A}_\tau\big(\xi + \lambda(\xi' - \xi)\big)$ and $s \in (0,1)$ such that, using Corollary 2.28,

$$\begin{aligned} \Psi_\tau\big(\xi + \lambda(\xi' - \xi)\big) - \Psi_\tau(\xi) &\leq \psi_{\tau,j,\tau_i}\big(\xi + \lambda(\xi' - \xi)\big) - \Psi_\tau(\xi) \\ &\leq \psi_{\tau,j,\tau_i}\big(\xi + \lambda(\xi' - \xi)\big) - \psi_{\tau,j,\tau_k}(\xi) \\ &= \mathrm{D}\psi_{\tau,j,\tau_i}\big(\xi + s\lambda(\xi' - \xi); \lambda(\xi' - \xi)\big) \\ &\leq \lambda \mathrm{D}\psi_{\tau,j,\tau_i}\big(\xi; \xi' - \xi\big) + L\lambda^2 \|\xi' - \xi\|_{\mathcal{X}}^2. \end{aligned} \tag{2.252}$$

Now let us assume that $\Psi_\tau(\xi) \leq 0$, and consider g_τ as defined in Equation (2.158). Then

$$\theta_\tau(\xi) = \max\left\{ \mathrm{D}J_\tau(\xi; g_\tau(\xi) - \xi), \max_{(j,i) \in \mathcal{J} \times \{0,\ldots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_i}(\xi; g_\tau(\xi) - \xi) + \gamma \Psi_\tau(\xi) \right\} \leq -\delta, \tag{2.253}$$

and using Equation (2.251),

$$J_\tau\big(\xi + \beta^k(g_\tau(\xi) - \xi)\big) - J_\tau(\xi) - \alpha\beta^k\theta_\tau(\xi) \leq -(1-\alpha)\delta\beta^k + 4A^2L\beta^{2k}, \tag{2.254}$$

where $A = \max\big\{\|u\|_2 + 1 \mid u \in U\big\}$. Hence, for each $k \in \mathbb{N}$ such that $\beta^k \leq \frac{(1-\alpha)\delta}{4A^2L}$ we have that

$$J_\tau\big(\xi + \beta^k(g(\xi) - \xi)\big) - J_\tau(\xi) \leq \alpha\beta^k\theta_\tau(\xi). \tag{2.255}$$

Similarly, using Equations (2.252) and (2.253),

$$\Psi_\tau\big(\xi + \beta^k(g(\xi) - \xi)\big) - \Psi_\tau(\xi) + \beta^k\big(\gamma\Psi_\tau(\xi) - \alpha\theta_\tau(\xi)\big) \leq -\delta\beta^k + 4A^2L\beta^{2k}, \tag{2.256}$$

hence for each $k \in \mathbb{N}$ such that $\beta^k \leq \min\left\{\frac{(1-\alpha)\delta}{4A^2L}, \frac{1}{\gamma}\right\}$ we have that

$$\Psi_\tau\big(\xi + \beta^k(g(\xi) - \xi)\big) - \alpha\beta^k\theta_\tau(\xi) \leq \big(1 - \beta^k\gamma\big)\Psi_\tau(\xi) \leq 0. \tag{2.257}$$

If $\Psi_\tau(\xi) > 0$ then

$$\max_{(j,i)\in\mathcal{J}\times\{0,\dots,|\tau|\}} \mathrm{D}\psi_{\tau,j,\tau_i}(\xi; g_\tau(\xi) - \xi) \leq \theta_\tau(\xi) \leq -\delta, \tag{2.258}$$

thus, from Equation (2.252),

$$\Psi_\tau\big(\xi + \beta^k(g_\tau(\xi) - \xi)\big) - \Psi_\tau(\xi) - \alpha\beta^k\theta_\tau(\xi) \leq -(1-\alpha)\delta\beta^k + 4A^2L\beta^{2k}. \tag{2.259}$$

Hence, for each $k \in \mathbb{N}$ such that $\beta^k \leq \frac{(1-\alpha)\delta}{4A^2L}$ we have that

$$\Psi_\tau\big(\xi + \beta^k(g_\tau(\xi) - \xi)\big) - \Psi_\tau(\xi) \leq \alpha\beta^k\theta_\tau(\xi). \tag{2.260}$$

Finally, let

$$M_\delta^* = 1 + \max\left\{\log_\beta\left(\frac{(1-\alpha)\delta}{4A^2L}\right), \log_\beta\left(\frac{1}{\gamma}\right)\right\}, \tag{2.261}$$

then from Equations (2.255), (2.257), and (2.260), we get that $\mu_\tau(\xi) \leq M_\delta^*$ as desired. □

The proof of the following corollary follows directly from the estimates of M_δ^* in the proof of Lemma 2.80.

Corollary 2.81. *Let $\alpha \in (0,1)$ and $\beta \in (0,1)$. There exists a $\delta_0 > 0$ and $C > 0$ such that if $\delta \in (0, \delta_0)$ and $\theta_\tau(\xi) \leq -\delta$ for $N \in \mathbb{N}$, $\tau \in \mathcal{T}_N$, and $\xi \in \mathcal{X}_{\tau,p}$, then $\mu_\tau(\xi) \leq 1 + \log_\beta(C\delta)$, where θ_τ is as defined in Equation (2.158) and μ_τ is as defined in Equation (2.162).*

Next, we prove a bound between the discretized trajectory for a point in the discretized relaxed optimization space and the discretized trajectory for the same point after projection by ρ_N that we use in a later argument.

Lemma 2.82. *Consider ρ_N defined as in Equation (2.37) and σ_N defined as in Equation (2.163). There exists $K > 0$ such that for each $N_0, N \in \mathbb{N}$, $\tau \in \mathcal{T}_{N_0}$, $\xi = (u,d) \in \mathcal{X}_{r,\tau}$, and $t \in [0,1]$:*

$$\big\|\varphi_{\sigma_N(\xi),t}\big(\rho_N(\xi)\big) - \varphi_{\tau,t}(\xi)\big\|_2 \leq K\left(\left(\frac{1}{\sqrt{2}}\right)^N\big(\mathrm{V}(\xi)+1\big) + \left(\frac{1}{2}\right)^{N_0}\right), \tag{2.262}$$

where $\varphi_{\tau,t}$ is as defined in Equation (2.153) and $\mathrm{V}(\cdot)$ is as defined in Equation (2.3).

Proof. We prove this argument for $t = 1$, but the argument follows identically for all $t \in [0,1]$. Using the Triangle Inequality we have that

$$\begin{aligned}\big\|\varphi_{\sigma_N(\xi),1}\big(\rho_N(\xi)\big) - \varphi_{\tau,1}(\xi)\big\|_2 \leq \big\|\varphi_{\sigma_N(\xi),1}\big(\rho_N(\xi)\big) - \varphi_1\big(\rho_N(\xi)\big)\big\|_2 + \big\|\varphi_1\big(\rho_N(\xi)\big) - \varphi_1(\xi)\big\|_2 + \\ + \big\|\varphi_1(\xi) - \varphi_{\tau,1}(\xi)\big\|_2.\end{aligned} \tag{2.263}$$

Thus, by Theorem 2.38 and Lemma 2.58 there exists K_1, K_2, and K_3 such that

$$\left\|\varphi_{\sigma_N(\xi),1}\big(\rho_N(\xi)\big) - \varphi_{\tau,1}(\xi)\right\|_2 \leq K_1 \left(\frac{1}{\sqrt{2}}\right)^N \big(\mathrm{V}(\xi)+1\big) + \frac{K_2}{2^N} + \frac{K_3}{2^{N_0}}, \tag{2.264}$$

hence the result follows after organizing the constants and noting that $2^{\frac{N}{2}} \leq 2^N$ for each $N \in \mathbb{N}$. □

Using this previous lemma, we can prove that ν_τ is eventually finite for all ξ such that $\theta(\xi) < 0$.

Lemma 2.83. *Let $N_0 \in \mathbb{N}$, $\tau_0 \in \mathcal{T}_{N_0}$, and $\xi \in \mathcal{X}_{\tau,r}$. If $\theta(\xi) < 0$ then for each $\eta \in \mathbb{N}$ there exists a finite $N \geq N_0$ such that $\nu_{\sigma_N(\xi)}(\xi, N+\eta)$ is finite.*

Proof. Recall ν_τ, as defined in Equation (2.164), is infinity only when the optimization problem it solves is not feasible. To simplify our notation, let $\xi' \in \mathcal{X}_{\sigma_N(\xi),r}$ defined by $\xi' = \xi + \beta^{\mu_{\sigma_N(\xi)}(\xi)}\big(g_{\sigma_N(\xi)}(\xi) - \xi\big)$. Then, using Lemma 2.82, for $k \in \mathbb{N}$, $k \in [N, N+\eta]$,

$$\begin{aligned} J_{\sigma_k(\xi')}\big(\rho_k(\xi')\big) - J_{\sigma_N(\xi)}(\xi') &\leq LK\left(\left(\frac{1}{\sqrt{2}}\right)^k (\mathrm{V}(\xi')+1) + \left(\frac{1}{2}\right)^N\right) \\ &\leq LK\left(\frac{1}{\sqrt{2}}\right)^N (\mathrm{V}(\xi')+2)\,, \end{aligned} \tag{2.265}$$

where $\mathrm{V}(\cdot)$ is as defined in Equation (2.3).

Also, from Theorem 2.79 we know that for N large enough,

$$\frac{1}{2}\theta(\xi) \geq \theta_{\sigma_N(\xi)}(\xi). \tag{2.266}$$

Thus, given $\delta > \frac{1}{2}\theta(\xi)$, there exists $N^* \in \mathbb{N}$ such that, for each $N \geq N^*$ and $k \in [N, N+\eta]$,

$$\begin{aligned} J_{\sigma_k(\xi')}\big(\rho_k(\xi')\big) - J_{\sigma_N(\xi)}(\xi') &\leq -\bar{\alpha}\bar{\beta}^N \frac{1}{2}\theta(\xi) \\ &\leq -\bar{\alpha}\bar{\beta}^N \theta_{\sigma_N(\xi)}(\xi). \end{aligned} \tag{2.267}$$

and at the same time

$$\bar{\alpha}\bar{\beta}^N \leq (1-\omega)\alpha\beta^{M_\delta^*} \leq (1-\omega)\alpha\beta^{\mu_{\sigma_N}(\xi)}, \tag{2.268}$$

where M_δ^* is as in Lemma 2.80.

Similarly, given $\mathcal{A}_\tau(\xi) = \{(j,t) \in \mathcal{J} \times [0,1] \mid \Psi_\tau(\xi) = \psi_{\tau,j,t}(\xi)\}$, let $(j,t) \in \mathcal{A}_{\sigma_N(\xi')}(\xi')$. Thus, for $N \geq N^*$, $k \in [N, N+\eta]$, and using Lemma 2.82,

$$\begin{aligned} \Psi_{\sigma_k(\xi')}\big(\rho_k(\xi')\big) - \Psi_{\sigma_N(\xi)}(\xi') &= \psi_{\sigma_k(\xi'),j,t}\big(\rho_k(\xi')\big) - \Psi_{\sigma_N(\xi)}(\xi') \\ &\leq \psi_{\sigma_k(\xi'),j,t}\big(\rho_k(\xi')\big) - \psi_{\sigma_N(\xi),j,t}(\xi') \\ &\leq LK\left(\frac{1}{\sqrt{2}}\right)^N (\mathrm{V}(\xi')+2) \\ &\leq -\bar{\alpha}\bar{\beta}^N \theta_{\sigma_N(\xi)}(\xi). \end{aligned} \tag{2.269}$$

Therefore, for $N \geq N^*$, if $\Psi_{\sigma_N(\xi)}(\xi) \leq 0$, then by Equations (2.267), (2.269), and the inequalities from the computation of $\mu_\tau(\xi)$,

$$J_{\sigma_k(\xi')}\big(\rho_k(\xi')\big) - J_{\sigma_N(\xi)}(\xi) \leq \big(\alpha\beta^{\mu_{\sigma_N}(\xi)} - \bar{\alpha}\bar{\beta}^N\big)\theta_{\sigma_N(\xi)}(\xi), \tag{2.270}$$

$$\Psi_{\sigma_k(\xi')}\big(\rho_k(\xi')\big) \leq \big(\alpha\beta^{\mu_{\sigma_N}(\xi)} - \bar{\alpha}\bar{\beta}^N\big)\theta_{\sigma_N(\xi)}(\xi) \leq 0, \tag{2.271}$$

which together with Equation (2.268) implies that the feasible set is not empty. Similarly, if $\Psi_{\sigma_N(\xi)}(\xi) > 0$, by Equation (2.269),

$$\Psi_{\sigma_k(\xi')}\big(\rho_k(\xi')\big) - \Psi_{\sigma_N(\xi)}(\xi) \leq \big(\alpha\beta^{\mu_{\sigma_N}(\xi)} - \bar{\alpha}\bar{\beta}^N\big)\theta_{\sigma_N(\xi)}(\xi), \tag{2.272}$$

as desired.

Hence for all $N \geq N^*$ the feasible sets of the optimization problems associated with $\nu_{\sigma_N(\xi)}$ are not empty, and therefore $\nu_{\sigma_N(\xi)}(\xi, N+\eta) < \infty$. □

In fact, the discretization precision constructed by Algorithm 2.2 increases arbitrarily.

Lemma 2.84. *Let $\{N_i\}_{i\in\mathbb{N}}$, $\{\tau_i\}_{i\in\mathbb{N}}$, and $\{\xi_i\}_{i\in\mathbb{N}}$ be the sequences generated by Algorithm 2.2. Then $N_i \to \infty$ as $i \to \infty$.*

Proof. Suppose that $N_i \leq N^*$ for all $i \in \mathbb{N}$. Then, by definition of Algorithm 2.2, there exists $i_0 \in \mathbb{N}$ such that $\theta(\xi_i) \leq -\Lambda 2^{-\chi N_i} \leq -\Lambda 2^{-\chi N^*}$ and $\xi_{i+1} = \Gamma_{\tau_i}(\xi_i)$ for each $i \geq i_0$, where Γ_τ is defined in Equation (2.165).

Moreover, by definition of ν_τ we have that if there exists $i_1 \geq i_0$ such that $\Psi_{\tau_{i_1}}(\xi_{i_1}) \leq 0$, then $\Psi_{\tau_i}(\xi_i) \leq 0$ for each $i \geq i_1$. Let us assume that there exists $i_1 \geq i_0$ such that $\Psi_{\tau_{i_1}}(\xi_{i_1}) \leq 0$, then, using Lemma 2.80,

$$\begin{aligned} J_{\tau_{i+1}}(\xi_{i+1}) - J_{\tau_i}(\xi_i) &\leq \big(\alpha\beta^{\mu_{\tau_i}(\xi_i)} - \bar{\alpha}\bar{\beta}^{\nu_{\tau_i}(\xi_i, N_i+\eta)}\big)\theta(\xi_i) \\ &\leq -\omega\alpha\beta^{M^*_{\delta'}}\delta', \end{aligned} \tag{2.273}$$

for each $i \geq i_1$, where $\delta' = \Lambda 2^{-\chi N^*}$. But this implies that $J_{\tau_i}(\xi_i) \to -\infty$ as $i \to \infty$, which is a contradiction since h_0, and therefore J_{τ_i}, is lower bounded.

The argument is completely analogous in the case where the sequence is perpetually infeasible. Indeed, suppose that $\Psi_{\tau_i}(\xi_i) > 0$ for each $i \geq i_0$, then by Lemma 2.80,

$$\begin{aligned} \Psi_{\tau_{i+1}}(\xi_{i+1}) - \Psi_{\tau_i}(\xi_i) &\leq \big(\alpha\beta^{\mu_{\tau_i}(\xi_i)} - \bar{\alpha}\bar{\beta}^{\nu_{\tau_i}(\xi_i, N_i+\eta)}\big)\theta(\xi_i) \\ &\leq -\omega\alpha\beta^{M^*_{\delta'}}\delta', \end{aligned} \tag{2.274}$$

for each $i \geq i_0$, where $\delta' = \Lambda 2^{-\chi N^*}$. But again this implies that $\Psi_{\tau_i}(\xi_i) \to -\infty$ as $i \to \infty$, which is a contradiction since we had assumed that $\Psi_{\tau_i}(\xi_i) > 0$. □

Next, we prove that if Algorithm 2.2 find a feasible point, then every point generated afterwards remains feasible.

Lemma 2.85. *Let $\{N_i\}_{i\in\mathbb{N}}$, $\{\tau_i\}_{i\in\mathbb{N}}$, and $\{\xi_i\}_{i\in\mathbb{N}}$ be the sequences generated by Algorithm 2.2. Then there exists $i_0 \in \mathbb{N}$ such that, if $\Psi_{\tau_{i_0}}(\xi_{i_0}) \leq 0$, then $\Psi(\xi_i) \leq 0$ and $\Psi_{\tau_i}(\xi_i) \leq 0$ for each $i \geq i_0$, where Ψ_τ is as defined in Equation* (2.152).

Proof. Let $\mathcal{I} \subset \mathbb{N}$ be a subsequence defined by

$$\mathcal{I} = \left\{ i \in \mathbb{N} \mid \theta_{\tau_i}(\xi_i) \leq -\frac{\Lambda}{2^{\chi N_i}} \text{ and } \nu_{\tau_i}(\xi_i, N_i + \eta) < \infty \right\}. \tag{2.275}$$

Note that, by definition of Algorithm 2.2, $\Psi(\xi_{i+1}) = \Psi(\xi_i)$ for each $i \notin \mathcal{I}$. Now, for each $i \in \mathcal{I}$ such that $\Psi_{\tau_i}(\xi_i) \leq 0$, by definition of ν_τ in Equation (2.164) together with Corollary 2.81,

$$\begin{aligned} \Psi_{\tau_{i+1}}(\xi_{i+1}) &\leq \left(\alpha\beta^{\mu_{\tau_i}(\xi_i)} - \bar{\alpha}\bar{\beta}^{\nu_{\tau_i}(\xi_i, N_i+\eta)}\right)\theta_{\tau_i}(\xi_i) \\ &\leq -\omega\alpha\beta^{\mu_{\tau_i}(\xi_i)}\frac{\Lambda}{2^{\chi N_i}} \\ &\leq -\omega\alpha\beta C\left(\frac{\Lambda}{2^{\chi N_i}}\right)^2, \end{aligned} \tag{2.276}$$

where $C > 0$. By Lemma 2.60 and the fact that $N_{i+1} \geq N_i$, we have that

$$\begin{aligned} \Psi(\xi_{i+1}) &\leq \frac{B}{2^{N_i}} - \omega\alpha\beta C\left(\frac{\Lambda}{2^{\chi N_i}}\right)^2 \\ &\leq \frac{1}{2^{2\chi N_i}}\left(\frac{B}{2^{(1-2\chi)N_i}} - \omega\alpha\beta C\Lambda^2\right). \end{aligned} \tag{2.277}$$

Hence, if $\Psi_{\tau_{i_1}}(\xi_{i_1}) \leq 0$ for $i_1 \in \mathbb{N}$ such that N_{i_1} is large enough, then $\Psi(\xi_i) \leq 0$ for each $i \geq i_1$.

Moreover, from Equation (2.277) we get that for each $N \geq N_i$ and each $\tau \in \mathcal{T}_N$,

$$\begin{aligned} \Psi_\tau(\xi_{i+1}) &\leq \frac{1}{2^{2\chi N_i}}\left(\frac{B}{2^{(1-2\chi)N_i}} - \omega\alpha\beta C\Lambda^2\right) + \frac{B}{2^N} \\ &\leq \frac{1}{2^{2\chi N_i}}\left(\frac{2B}{2^{(1-2\chi)N_i}} - \omega\alpha\beta C\Lambda^2\right). \end{aligned} \tag{2.278}$$

Thus, if $\Psi_{\tau_{i_2}}(\xi_{i_2}) \leq 0$ for $i_2 \in \mathbb{N}$ such that N_{i_2} is large enough, then $\Psi_\tau(\xi_{i_2}) \leq 0$ for each $\tau \in \mathcal{T}_N$ such that $N \geq N_i$. But note that this is exactly the case when $i_2 + k \notin \mathcal{I}$ for $k \in \{1, \ldots, n\}$, thus we can conclude that $\Psi_{\tau_{i_2+k}}(\xi_{i_2+k}) \leq 0$. Also note that the case of $i \in \mathcal{I}$ is trivially satisfied by the definition of ν_τ.

Finally, by setting $i_0 = \max\{i_1, i_2\}$ we get the desired result. □

Next, we prove θ_τ converges to zero.

Lemma 2.86. *Let $\{N_i\}_{i\in\mathbb{N}}$, $\{\tau_i\}_{i\in\mathbb{N}}$, and $\{\xi_i\}_{i\in\mathbb{N}}$ be the sequences generated by Algorithm 2.2. Then $\theta_{\tau_i}(\xi_i) \to 0$ as $i \to \infty$, where θ_τ is as defined in Equation* (2.158).

Proof. Let us suppose that $\lim_{i\to\infty} \theta_{\tau_i}(\xi_i) \neq 0$. Then there exists $\delta > 0$ such that

$$\liminf_{i\to\infty} \theta_{\tau_i}(\xi_i) < -4\delta, \tag{2.279}$$

and hence, using Theorem 2.79 and Lemma 2.84, there exists an infinite subsequence $\mathcal{K} \subset \mathbb{N}$ defined by

$$\mathcal{K} = \left\{ i \in \mathbb{N} \mid \theta_{\tau_i}(\xi_i) < -2\delta \text{ and } \theta(\xi_i) < -\delta \right\}. \tag{2.280}$$

Let us define a second subsequence $\mathcal{I} \subset \mathbb{N}$ by

$$\mathcal{I} = \left\{ i \in \mathbb{N} \mid \theta_{\tau_i}(\xi_i) \leq -\frac{\Lambda}{2^{\chi N_i}} \text{ and } \nu_{\tau_i}(\xi_i, N_i + \eta) < \infty \right\}. \tag{2.281}$$

Note that by the construction of the subsequence $\mathcal{K}$, together with Lemma 2.83, we get that $\mathcal{K} \cap \mathcal{I}$ is an infinite set.

Now we analyze Algorithm 2.2 by considering the behavior of each step as a function of its membership to each subsequence. First, for each $i \notin \mathcal{I}$, $\xi_{i+1} = \xi_i$, thus $J(\xi_{i+1}) = J(\xi_i)$ and $\Psi(\xi_{i+1}) = \Psi(\xi_i)$. Second, let $i \in \mathcal{I}$ such that $\Psi_{\tau_i}(\xi_i) \leq 0$, then

$$\begin{aligned} J_{\tau_{i+1}}(\xi_{i+1}) - J_{\tau_i}(\xi_i) &\leq \left(\alpha\beta^{\mu_{\tau_i}(\xi_i)} - \bar{\alpha}\bar{\beta}^{\nu_{\tau_i}(\xi_i, N_i+\eta)}\right) \theta_{\tau_i}(\xi_i) \\ &\leq -\omega\alpha\beta^{\mu_{\tau_i}(\xi_i)} \frac{\Lambda}{2^{\chi N_i}} \\ &\leq -\omega\alpha\beta C \left(\frac{\Lambda}{2^{\chi N_i}}\right)^2, \end{aligned} \tag{2.282}$$

where $C > 0$ and the last inequality follows from Corollary 2.81. Recall that $N_{i+1} \geq N_i$, thus using Lemmas 2.59 and 2.84 we have that

$$\begin{aligned} J(\xi_{i+1}) - J(\xi_i) &\leq \frac{2B}{2^{N_i}} - \omega\alpha\beta C \left(\frac{\Lambda}{2^{\chi N_i}}\right)^2 \\ &\leq \frac{1}{2^{2\chi N_i}} \left(\frac{2B}{2^{(1-2\chi)N_i}} - \omega\alpha\beta C\Lambda^2\right), \end{aligned} \tag{2.283}$$

and since $\chi \in \left(0, \frac{1}{2}\right)$, we get that for N_i large enough $J(\xi_{i+1}) \leq J(\xi_i)$. Similarly, if $\Psi_{\tau_i}(\xi_i) > 0$ then

$$\Psi(\xi_{i+1}) - \Psi(\xi_i) \leq \frac{1}{2^{2\chi N_i}} \left(\frac{2B}{2^{(1-2\chi)N_i}} - \omega\alpha\beta C\Lambda^2\right), \tag{2.284}$$

thus for N_i large enough, $\Psi(\xi_{i+1}) \leq \Psi(\xi_i)$. Third, let $i \in \mathcal{K} \cap \mathcal{I}$ such that $\Psi_{\tau_i}(\xi_i) \leq 0$, then, by Lemma 2.80,

$$\begin{aligned} J_{\tau_{i+1}}(\xi_{i+1}) - J_{\tau_i}(\xi_i) &\leq \left(\alpha\beta^{\mu_{\tau_i}(\xi_i)} - \bar{\alpha}\bar{\beta}^{\nu_{\tau_i}(\xi_i, N_i+\eta)}\right) \theta_{\tau_i}(\xi_i) \\ &\leq -2\omega\alpha\beta^{M^*_{2\delta}}\delta, \end{aligned} \tag{2.285}$$

thus, by Lemmas 2.59 and 2.84, for N_i large enough,

$$J(\xi_{i+1}) - J(\xi_i) \leq -\omega\alpha\beta^{M^*_{2\delta}}\delta. \tag{2.286}$$

Similarly, if $\Psi_{\tau_i}(\xi_i) > 0$, using the same argument and Lemma 2.60, for N_i large enough,

$$\Psi(\xi_{i+1}) - \Psi(\xi_i) \leq -\omega\alpha\beta^{M^*_{2\delta}}\delta. \tag{2.287}$$

Now let us assume that there exists $i_0 \in \mathbb{N}$ such that N_{i_0} is large enough and $\Psi_{\tau_{i_0}}(\xi_{i_0}) \leq 0$. Then by Lemma 2.85 we get that $\Psi_{\tau_i}(\xi_i) \leq 0$ for each $i \geq i_0$. But as shown above, either $i \notin \mathcal{K} \cap \mathcal{I}$ and $J(\xi_{i+1}) \leq J(\xi_i)$ or $i \in \mathcal{K} \cap \mathcal{I}$ and Equation (2.286) is satisfied, and since $\mathcal{K} \cap \mathcal{I}$ is an infinite set we get that $J(\xi_i) \to -\infty$ as $i \to \infty$, which is a contradiction as J is lower bounded.

On the other hand, if we assume that $\Psi_{\tau_i}(\xi_i) > 0$ for each $i \in \mathbb{N}$, then either $i \notin \mathcal{K} \cap \mathcal{I}$ and $\Psi(\xi_{i+1}) \leq \Psi(\xi_i)$ or $i \in \mathcal{K} \cap \mathcal{I}$ and Equation (2.287) is satisfied, thus implying that $\Psi(\xi_i) \to -\infty$ as $i \to \infty$. But this is a contradiction since, by Lemma 2.60, this would imply that $\Psi_{\tau_i}(\xi_i) \to -\infty$ as $i \to \infty$.

Finally, both contradictions imply that $\theta_{\tau_i}(\xi_i) \to 0$ as $i \to \infty$ as desired. □

In conclusion, we can prove that the sequence of points generated by Algorithm 2.2 converges to a point that is a zero of θ or a point that satisfies our optimality condition.

Theorem 2.87. *Let $\{N_i\}_{i\in\mathbb{N}}$, $\{\tau_i\}_{i\in\mathbb{N}}$, and $\{\xi_i\}_{i\in\mathbb{N}}$ be the sequences constructed by Algorithm 2.2, then*

$$\lim_{i\to\infty} \theta(\xi_i) = 0, \tag{2.288}$$

where θ is as defined in Equation (2.30)*.*

Proof. This result follows immediately from Lemma 2.86 after noticing that the Discretized Relaxed Switched System Optimal Control Problem is a consistent approximation of the Switched System Optimal Control Problem, as is proven in Theorem 2.79, and applying Theorem 2.50. □

2.6 Examples

In this section, we apply Algorithm 2.2 to calculate an optimal control for four examples. Before describing each example, we begin by describing the numerical implementation of Algorithm 2.2. First, observe that the analysis presented thus far does not require that the initial and final times of the trajectory of switched system be fixed to 0 and 1, respectively. Instead, the initial and final times of the trajectory of the switched system are treated as fixed parameters t_0 and t_f, respectively. Second, we employ a MATLAB implementation of LSSOL from TOMLAB in order to compute the optimality function at each iteration of the algorithm since it is a quadratic program [Hol99]. Third, for each example we employ a stopping

Example	Mode 1	Mode 2	Mode 3
LQR	$\dot{x}(t) = Ax(t) + \begin{bmatrix} 0.9801 \\ -0.1987 \\ 0 \end{bmatrix} u(t)$	$\dot{x}(t) = Ax(t) + \begin{bmatrix} 0.1743 \\ 0.8601 \\ -0.4794 \end{bmatrix} u(t)$	$\dot{x}(t) = Ax(t) + \begin{bmatrix} 0.0952 \\ 0.4699 \\ 0.8776 \end{bmatrix} u(t)$
Tank	$\dot{x}(t) = \begin{bmatrix} 1 - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} 2 - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \end{bmatrix}$	N/A
Quadrotor	$\ddot{x}(t) = \begin{bmatrix} \frac{\sin x_3(t)}{M}(u(t) + Mg) \\ \frac{\cos x_3(t)}{M}(u(t) + Mg) - g \\ 0 \end{bmatrix}$	$\ddot{x}(t) = \begin{bmatrix} g \sin x_3(t) \\ g \cos x_3(t) - g \\ \frac{-Lu(t)}{I} \end{bmatrix}$	$\ddot{x}(t) = \begin{bmatrix} g \sin x_3(t) \\ g \cos x_3(t) - g \\ \frac{Lu(t)}{I} \end{bmatrix}$
Needle	$\dot{x}(t) = \begin{bmatrix} \sin\big(x_5(t)\big)u_1(t) \\ -\cos\big(x_5(t)\big)\sin\big(x_4(t)\big)u_1(t) \\ \cos\big(x_4(t)\big)\cos\big(x_5(t)\big)u_1(t) \\ \kappa\cos\big(x_6(t)\big)\sec\big(x_5(t)\big)u_1(t) \\ \kappa\sin\big(x_6(t)\big)u_1(t) \\ -\kappa\cos\big(x_6(t)\big)\tan\big(x_5(t)\big)u_1(t) \end{bmatrix}$	$\dot{x}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_2(t) \end{bmatrix}$	N/A

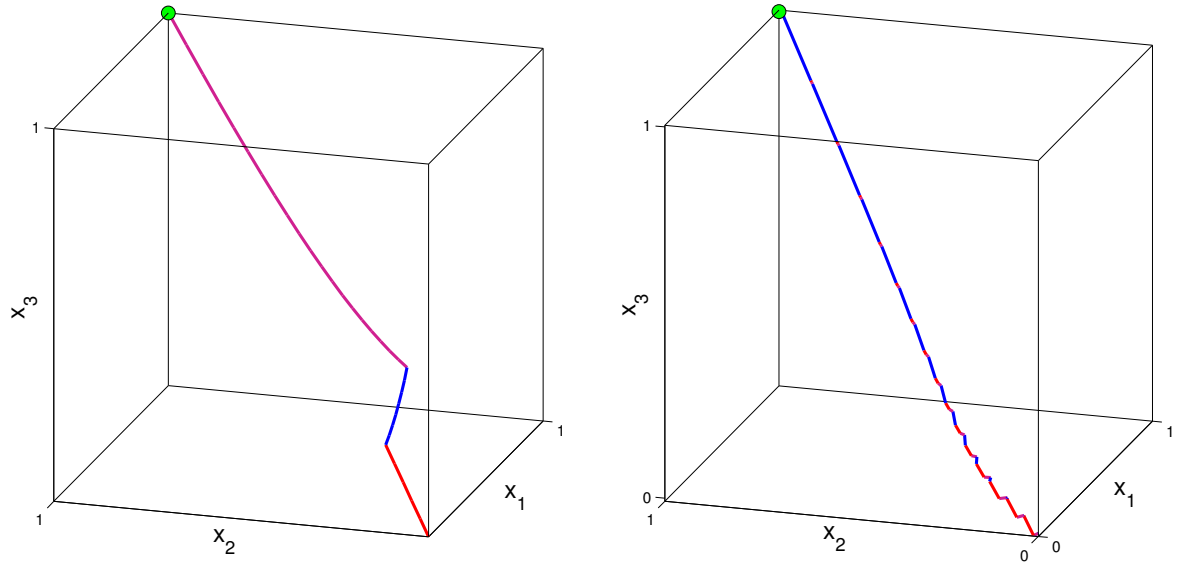
Table 2.1: The dynamics of each of the modes of the switched system examples.

Example	$L(x(t), u(t), t)$	$\varphi\left(x^{(\xi)}(t_f)\right)$	U	γ	α	β	$\bar{\alpha}$	$\bar{\beta}$	Λ	χ	ω	t_0	t_f
LQR	$0.01 \cdot (u(t))^2$	$\left\| \begin{bmatrix} x_1(t_f) - 1 \\ x_2(t_f) - 1 \\ x_3(t_f) - 1 \end{bmatrix} \right\|_2^2$	$u(t) \in [-20, 20]$	1	0.1	0.87	0.005	0.72	10^{-4}	$\frac{1}{4}$	10^{-6}	0	2
Tank	$2 \cdot (x_2(t) - 3)^2$	0	N/A	100	0.01	0.75	0.005	0.72	10^{-4}	$\frac{1}{4}$	10^{-6}	0	10
Quadrotor	$5 \cdot (u(t))^2$	$\left\| \begin{bmatrix} \sqrt{5} \cdot (x_1(t_f) - 6) \\ \sqrt{5} \cdot (x_2(t_f) - 1) \\ \sin\left(\frac{x_3(t_f)}{2}\right) \end{bmatrix} \right\|_2^2$	$u(t) \in [0, 10^{-3}]$	10	0.01	0.80	5×10^{-4}	0.72	10^{-4}	$\frac{1}{4}$	10^{-6}	0	7.5
Needle	$0.01 \cdot \left\| \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right\|_2^2$	$\left\| \begin{bmatrix} x_1(t_f) + 2 \\ x_2(t_f) - 3.5 \\ x_3(t_f) - 10 \end{bmatrix} \right\|_2^2$	$u_1(t) \in [0, 5]$ $u_2(t) \in [\frac{-\pi}{2}, \frac{\pi}{2}]$	100	0.002	0.72	0.001	0.71	10^{-4}	$\frac{1}{4}$	0.05	0	8

Table 2.2: The parameters and cost functions used for each of the examples in the implementation of Algorithm 2.2.

Example	Initial Continuous Input, $\forall t \in [t_0, t_f]$	Initial Discrete Input, $\forall t \in [t_0, t_f]$	Algorithm 2.2 Computation Time	Algorithm 2.2 Final Cost	MIP Computation Time	MIP Final Cost
LQR	$u(t) = 0$	$d(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	9.827[s]	1.23×10^{-3}	753.0[s]	1.89×10^{-3}
Tank	N/A	$d(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	32.38[s]	4.829	119700[s]	4.828
Quadrotor	$u(t) = 5 \times 10^{-4}$	$d(t) = \begin{bmatrix} 0.33 \\ 0.34 \\ 0.33 \end{bmatrix}$	8.350[s]	0.128	2783[s]	0.165
Needle	$u(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$d(t) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$	62.76[s]	0.302	did not converge	did not converge

Table 2.3: The initialization parameters used for each of the examples in the implementation of Algorithms 2.2 and MIP, together with the computation time and the final optimal values.



(a) MIP

(b) Algorithm 2.2

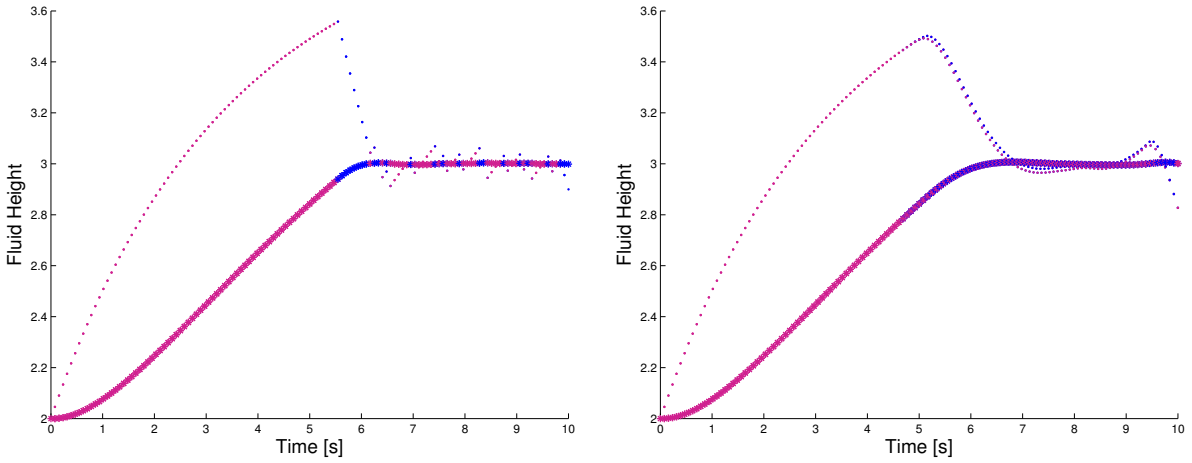
Figure 2.3: Optimal trajectories for each of the algorithms where the point $(1, 1, 1)$ is drawn in green, and where the trajectory is drawn in blue when in mode 1, in purple when in mode 2, and in red when in mode 3.

criterion that terminates Algorithm 2.2, if θ_τ becomes too large. Each of these stopping criteria is described when we describe each example. Next, for the sake of comparison we compare the performance of Algorithm 2.2 on each of the examples to a traditional Mixed Integer Program (MIP). To perform this comparison, we employ a TOMLAB implementation of a MIP described in [FL94] which mixes branch and bound steps with sequential quadratic programming steps. Finally, all of our comparisons are performed on an Intel Xeon, 6 core, 3.47 GHz, 100 GB RAM machine.

Constrained Switched Linear Quadratic Regulator (LQR)

Switched Linear Quadratic Regulator (LQR) examples have been used to illustrate the utility of a variety of proposed optimal control algorithms [EWA06; XA02]. We consider an LQR system in three dimensions, with three discrete modes, and a single continuous input. The dynamics in each mode are as described in Table 2.1 where:

$$A = \begin{bmatrix} 1.0979 & -0.0105 & 0.0167 \\ -0.0105 & 1.0481 & 0.0825 \\ 0.0167 & 0.0825 & 1.1540 \end{bmatrix}. \tag{2.289}$$



(a) MIP (b) Algorithm 2.2

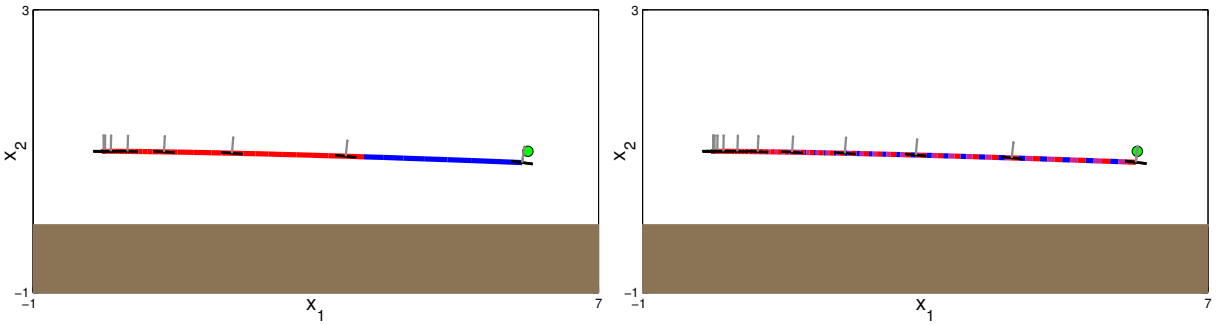
Figure 2.4: Optimal trajectories for each of the algorithms where $x_1(t)$ is drawn using points, $x_2(t)$ is drawn using stars, and each state trajectory is drawn in blue when in mode 1 and in purple when in mode 2.

The system matrix is purposefully chosen to have three unstable eigenvalues and the control matrix in each mode is only able to control along single dimension. Hence, while the system and control matrix in each mode is not a stabilizable pair, the system and all the control matrices taken together simultaneously is stabilizable and is expected to appropriately switch between the modes to reduce the cost. The objective of the optimization is to have the trajectory of the system at time t_f be at $(1, 1, 1)$ while minimizing the input required to achieve this task. This objective is reflected in the chosen cost function which is described in Table 2.2.

Algorithm 2.2 and the MIP are initialized at $x_0 = (0, 0, 0)$ with continuous and discrete inputs as described in Table 2.3 with 16 equally spaced samples in time. Algorithm 2.2 took 11 iterations, ended with 48 time samples, and terminated after the optimality condition was bigger than -10^{-2}. The result of both optimization procedures is illustrated in Figure 2.3. The computation time and final cost of both algorithms can be found in Table 2.3. Notice that Algorithm 2.2 is able to compute a lower cost continuous and discrete input when compared to the MIP and is able to do it more than 75 times faster.

Double Tank System

To illustrate the performance of Algorithm 2.2 when there is no continuous input present, we consider a double-tank example. The two states of the system correspond to the fluid levels of an upper and lower tank. The output of the upper tank flows into the lower tank,



(a) MIP (b) Algorithm 2.2

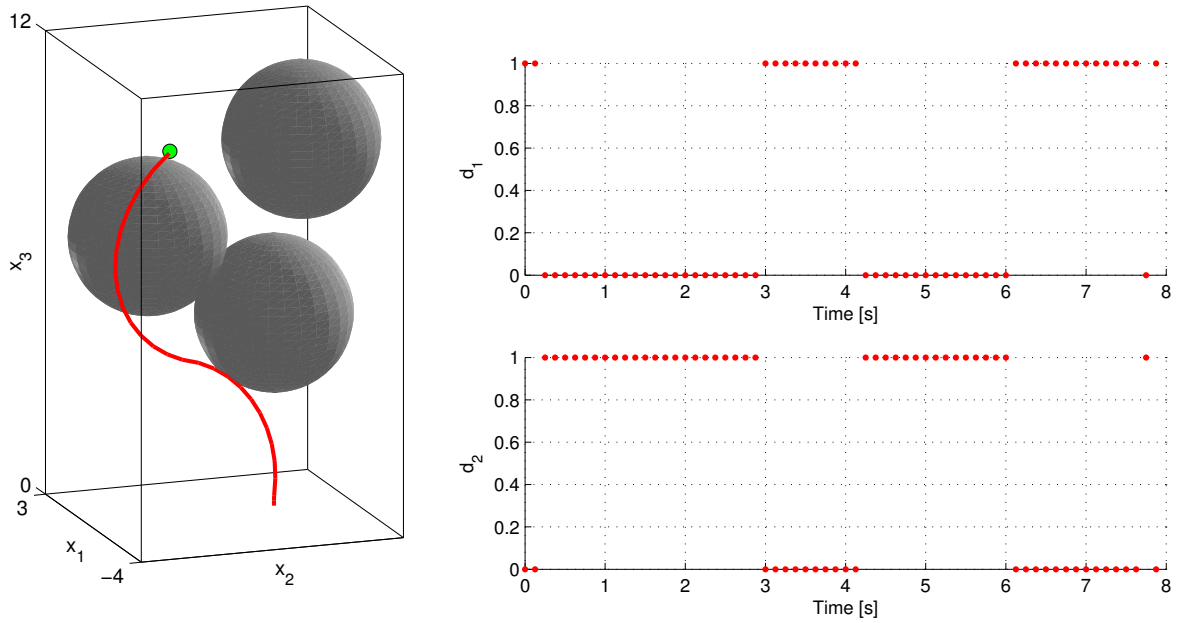
Figure 2.5: Optimal trajectories for each of the algorithms where the point $(6, 1)$ is drawn in green, the trajectory is drawn in blue when in mode 1, in purple when in mode 2, and in red when in mode 3. Also, the quadrotor is drawn in black and the normal direction to the frame is drawn in gray.

the output of the lower tank exits the system, and the flow into the upper tank is restricted to either 1 or 2. The dynamics in each mode are then derived using Toricelli's Law and are describe in Table 2.1. The objective of the optimization is to have the fluid level in the lower tank track 3 and this is reflected in the chosen cost function described in Table 2.2.

Algorithm 2.2 and the MIP are initialized at $x_0 = (0, 0)$ with a discrete input described in Table 2.3 with 128 equally spaced samples in time. Algorithm 2.2 took 67 iterations, ended with 256 time samples, and terminated after the optimality condition was bigger than -10^{-2}. The result of both optimization procedures is illustrated in Figure 2.4. The computation time and final cost of both algorithms can be found in Table 2.3. Notice that Algorithm 2.2 is able to compute a comparable cost discrete input compared to the MIP and is able to do it nearly 3700 times faster.

Quadrotor Helicopter Control

Next, we consider the optimal control of a quadrotor helicopter in 2D using a model described in [Gil+11]. The evolution of the quadrotor can be defined with respect to a fixed 2D reference frame using six dimensions where the first three dimensions represent the position along a horizontal axis, the position along the vertical axis and the roll angle of the helicopter, respectively, and the last three dimensions represent the time derivative of the first three dimensions. We model the dynamics as a three mode switched system (the first mode describes the dynamics of going up, the second mode describes the dynamics of moving to the left, and the third mode describes the dynamics of moving to the right) with a single input as described in Table 2.1 where $L = 0.3050$ meters, $M = 1.3000$ kilograms, $I = 0.0605$ kilogram meters squared, and $g = 9.8000$ meters per second squared. The objective of the optimization is to have the trajectory of the system at time t_f be at position $(6, 1)$ with a



(a) Algorithm 2.2 Final Result

(b) Algorithm 2.2 Final Discrete Input

Figure 2.6: Optimal trajectory and optimal discrete inputs generated by Algorithm 2.2, where the point $(-2, 3.5, 10)$ is drawn in green and obstacles are drawn in gray.

zero roll angle while minimizing the input required to achieve this task. This objective is reflected in the chosen cost function which is described in Table 2.2. A state constraint is added to the optimization to ensure that the quadrotor remains above ground.

Algorithm 2.2 and the MIP are initialized at position $(0, 1)$ with a zero roll angle, with zero velocity, with continuous and discrete inputs as described in Table 2.3, and with 64 equally spaced samples in time. Algorithm 2.2 took 31 iterations, ended with 192 time samples, and terminated after the optimality condition was bigger than -10^{-4}. The result of both optimization procedures is illustrated in Figure 2.5. The computation time and final cost of both algorithms can be found in Table 2.3. Notice that Algorithm 2.2 is able to compute a lower cost continuous and discrete input when compared to the MIP and is able to do it more than 333 times faster.

Bevel-Tip Flexible Needle

Bevel-tip flexible needles are asymmetric needles that move along curved trajectories when a forward pushing force is applied. The 3D dynamics of such needles has been described in [KC07] and the path planning in the presence of obstacles has been heuristically considered in [Dui+08]. The evolution of the needle can be defined using six dimensions where the first

three dimensions represent the position of the needle relative to the point of entry and the last three dimensions represent the yaw, pitch and roll of the needle relative to the plane, respectively. As suggested by [Dui+08], the dynamics of the needle are naturally modeled as a two mode (the first mode describes the dynamics of going forward while the second mode describes the dynamics of the needle turning) switched system as described in Table 2.1 with two continuous inputs: u_1 representing the insertion speed and u_2 representing the rotation speed of the needle and where κ is the curvature of the needle and is equal to .22 inverse centimeters. The objective of the optimization is to have the trajectory of the system at time t_f be at position $(-2, 3.5, 10)$ while minimizing the input required to achieve this task. This objective is reflected in the chosen cost function which is described in Table 2.2. A state constraint is added to the optimization to ensure that the needle remains outside of three spherical obstacles centered at $(0\,,\,0\,,\,5)$, $(1\,,\,3\,,\,7)$, and $(-2\,,\,0\,,\,10)$ all with radius 2.

Algorithm 2.2 and the MIP are initialized at position $(0, 0, 0)$ with continuous and discrete input described in Table 2.3 with 64 equally spaced samples in time. Algorithm 2.2 took 103 iterations, ended with 64 time samples, and terminated after the optimality condition was bigger than -10^{-3}. The computation time and final cost of both algorithms can be found in Table 2.3. The MIP was unable to find any solution. The result of Algorithm 2.2 is illustrated in Figure 2.6.

Chapter 3

Metrization and Numerical Integration of Hybrid Dynamical Systems

Hybrid dynamical systems provide natural models for systems whose dynamics involve both continuous and discrete transitions. Critical to the study of such systems is numerical simulation. Two approaches to numerical simulation have been considered in the hybrid systems literature. The first method, *event detection*, aims to approximate the instant in time when a trajectory crosses a switching surface by constructing a polynomial approximation to the trajectory and then employing a root-finding scheme [Car78; EKP01; SGB91]. Unfortunately no proof exists that the approximation generated using this method converges to the actual trajectory. The second method, *time stepping*, uses a variable-step integrator to place events at sample times of the discrete approximation [Bro+02]. Convergence results exist for this method but only for the particular case of mechanical systems with impact [PS03b; PS03a].

Here we present a numerical integration algorithm to simulate hybrid dynamical systems whose continuous states evolve on smooth manifolds. First, we relax switching surfaces by attaching an ε-sized strip in a manner similar to the technique involved in regularizing Zeno executions [JEL99]. We then extend the vector field and distance metric from each domain onto these strips to obtain a *relaxed hybrid dynamical system*. In a manner similar to the construction of the *hybrifold* [Sim+05] and *hybrid colimit* [AS05], we identify subsets of the relaxed domains to construct a single metric space and develop our numerical integration scheme on this space. Importantly, we prove that the discrete approximation generated by our algorithm converges to the original trajectory in this space. We formulate our result in the context of autonomous vector fields to simplify the exposition, but under reasonable assumptions the same result can be applied on controlled vector fields.

Our contributions are twofold: first, in Section 3.2 we construct a metric space which contains the domains of a hybrid system and supports convergence analysis; second, in Section 3.3 we develop a discrete approximation technique and prove that this approximation converges to the original trajectory. Section 3.1 describes the notation used throughout the

chapter and Section 3.4 contains an example illustrating the numerical integration scheme. The contents of this chapter are based on the results presented in [Bur+11].

3.1 Preliminaries

Smooth Manifolds

We begin by introducing the standard mathematical objects used throughout this chapter. An extended introduction to the ideas presented herein related to manifolds can be found in [Lee03].

Definition 3.1. *A* topological n–dimensional manifold *is a space with the following properties:*

(1) *M is a Hausdorff space.*

(2) *M is second countable.*

(3) *M is locally equivalent to a subset of $\mathbb{R}^n$, i.e. for each $p \in M$ there exists a neighborhood U_p of p and a function $\varphi_p : M \to \mathbb{R}^n$ such that $\varphi_p|_{U_p}$ is a homeomorphism.*

The last condition implies the existence of a collection of pairs of neighborhoods and functions. We call this collection the charts of M *and denote it by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ for some index set $\mathcal{A}$.*

Definition 3.2. *A* smooth n–dimensional manifold*, or simply a manifold, is a topological manifold, as described in Definition 3.1, such that for each $\alpha, \beta \in \mathcal{A}$ the map $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.*

Definition 3.3. *A* manifold with boundary *is a topological manifold where the range of the charts $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ is not $\mathbb{R}^n$, as described in Definition 3.1, but*

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}. \tag{3.1}$$

We define the boundary of M, denoted by ∂M, as the union of the preimages of all charts of the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.

Note that the definition of manifold with boundary is a generalization of the original definition of manifold, in the sense that every manifold can be seen as a manifold with boundary, where the boundary is the null set.

Definition 3.4. *An* embedded k–dimensional submanifold *of a manifold M is a subset $S \subset M$ where for each $p \in S$ there exists a neighborhood U_p of p and a function $\varphi_p : M \to R^n$ such that $\varphi_p(U_p \cap S) \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{k+1} = \cdots = x_n = 0\}$. We also say that S has* codimension *equal to $(n - k)$.*

Given a manifold with boundary M, ∂M is clearly an embedded $(n-1)$–dimensional submanifold, as described in Definition 3.3.

Definition 3.5. *Given a smooth manifold M, we say that $C^\infty(M)$ is the set of all the infinitely smooth functions from M to $\mathbb{R}$.*

Definition 3.6. *Given a manifold M, the* tangent space *at $p \in M$, denoted by T_pM, is the set of all maps $V : C^\infty(M) \to \mathbb{R}$ such that, for any $f, g \in C^\infty(M)$,*

$$V(fg) = f(p)V(g) + g(p)V(f). \tag{3.2}$$

If the manifold is $\mathbb{R}^n$, then, given $p \in \mathbb{R}^n$, $T_p\mathbb{R}^n$ is exactly the set of all directional derivatives of smooth functions evaluated at p, i.e. for each $V \in T_pM$ there exists a direction $d \in \mathbb{R}^n$ such that

$$V(f) = \lim_{\lambda \to 0} \frac{f(p + \lambda d) - f(p)}{\lambda}. \tag{3.3}$$

As shown in Section 3 in [Lee03], Definition 3.6 is the right definition for the tangent space in the sense that we can intuitively match every element in T_pM with a vector belonging to a tangential plane to $p \in M$. Also, given a smooth curve $\gamma : \mathbb{R} \to M$, a natural definition for its derivative at each $t \in \mathbb{R}$, denoted by $\dot{\gamma}(t)$, is exactly a vector in $T_{\gamma(t)}M$.

The most important result obtained from Definition 3.6 is that for each $p \in M$, T_pM is an n–dimensional vector space, thus isomorphic to $\mathbb{R}^n$ (Lemmas 3.9 and 3.10 in [Lee03]).

Definition 3.7. *The* tangent bundle *of a manifold M, denoted by TM, is the disjoint union of all the tangent spaces, i.e. $TM = \coprod_{p \in M} T_pM$. In other words, the elements in TM are pairs of the form (p, V) where $p \in M$ and $V \in T_pM$.*

Definition 3.8. *A* vector field *in a manifold M is a map $F : M \to TM$ such that, for each $p \in M$, $F(p) \in T_pM$.*

Definition 3.9. *Given two manifolds M and N and a smooth function $F : M \to N$, the* pushforward *at $p \in M$, denoted by $F_*|_p : T_pM \to T_{F(p)}N$, is defined as $\big(F_*|_p(V)\big)(f) = V(f \circ F)$.*

In practice, the pushforward can be understood as the Jacobian matrix of F evaluated at p, taking vectors from T_pM to $T_{F(p)}N$.

Definition 3.10. *Given a smooth vector field $F : M \to TM$ and a point $p \in M$, the* integral curve of F *with initial condition p, denoted $x : I \to M$, where $I \subset \mathbb{R}$ is an interval containing the origin, is a curve satisfying:*

$$\dot{x}(t) = F\big(x(t)\big), \quad x(0) = p, \quad \forall t \in I. \tag{3.4}$$

Moreover, we say that x is a maximal integral curve of V *if for any other integral curve $\tilde{x} : \widetilde{I} \to M$ of F with initial condition p, $\widetilde{I} \subset I$.*

Theorem 17.8 in [Lee03] proves all the fundamental results for integral curves, namely that for every smooth vector field there exists a unique maximal smooth integral curve. Also, Theorem 17.11 in [Lee03] proves that every smooth vector field in a compact manifold has a maximal integral curve whose domain is $\mathbb{R}$, which is a particular case of the well known result for ordinary differential equations about the existence and uniqueness of solutions when the vector field is Lipschitz continuous (see Section 2.4.1 in [Vid02] for a standard version of this result).

Definition 3.11. *A* Riemannian metric at $p \in M$ *is a smooth bilinear map* $g_p : T_pM \times T_pM \to \mathbb{R}$, *such that* $g_p(V,W) = g_p(W,V)$ *for each* $V, W \in T_pM$, *and* $g_p(V,V) > 0$ *whenever* $V \neq 0$.

A Riemannian metric on M *is a collection of Riemannian metrics at each point in* M *forming a smooth map* $g : TM \times TM \to \mathbb{R}$.

A Riemannian manifold *is a pair* (M, g) *where* M *is a manifold (possibly with boundary) and* g *is a Riemannian metric on* M.

We can define an *induced distance* by the Riemannian metric on a Riemannian manifold, denoted $d : M \times M \to [0, \infty)$, to be the infimum of the length of piece–wise smooth curves between the arguments of d, i.e. if we define the length of a curve $\gamma : [0,T] \to M$ by:

$$L(\gamma) = \int_0^T \sqrt{g_{\gamma(t)}\big(\dot{\gamma}(t), \dot{\gamma}(t)\big)}dt, \tag{3.5}$$

then the metric d is defined by:

$$d(p,q) = \inf\{L(\gamma) \mid \gamma : [0,T] \to M \text{ piece–wise smooth, } \gamma(0) = p, \text{ and } \gamma(T) = q\}. \tag{3.6}$$

A fundamental result in Riemannian manifolds is that its induced metric d generates the topology of M (see Lemma 6.2 in [Lee97] for manifolds without boundary, and Lemma 2 in [AA81] for manifold with boundary). Also note that given $V \in T_pM$, the map $\|V\|_g = \sqrt{g_p(V,V)}$ is indeed a norm in T_pM.

Definition 3.12. *Given* $W_p \subset T_pM$, *a neighborhood of* $0 \in T_pM$, *we say that a* retraction at $p \in M$ *is a map* $\beta_p : W_p \to M$ *that is differentiable at the origin satisfying:*

(1) $\beta_p(0) = p$.

(2) $(\beta_p)_*|_0 \equiv \mathrm{id}_{T_pM}$, *where we use the canonical identification of every element in* $T_0(T_pM)$ *to its analogous in* T_pM, *and* $\mathrm{id}_{T_pM} : T_pM \to T_pM$ *is the identity function.*

A retraction on M *is a collection of retractions at each point in* M *forming a map* $\beta : W \to M$, *where* $W \subset TM$ *is the disjoint union of neighborhoods* W_p.

The intuition behind retractions is closely related to the concept of first-order approximations to curves. Indeed, given a smooth curve $\gamma : \mathbb{R} \to M$, $t \in \mathbb{R}$, its tangent vector $V = \dot{\gamma}(t) \in T_pM$ at t, and a retraction $\beta_{\gamma(t)}$, a first-order approximation of $\gamma(t+\lambda)$, for small values of λ, is $\beta_{\gamma(t)}(\lambda V)$. Moreover, if $M = \mathbb{R}^n$ then the canonical retraction is $\beta_p(V) = p+V$.

Retractions are defined to preserve first derivatives around the origin, as stated in Condition 2 in Definition 3.12. Even though we do not make use of it in our result, the most important retraction in a Riemannian manifold is the Exponential Map (see Section 5 in [Lee97]) which preserves the derivatives of any order.

Hybrid Dynamical Systems

Motivated by the definitions of hybrid systems presented in [BBM98; JEL99; Lyg+99; Sim+05], we begin by defining the class of hybrid systems of interest.

Definition 3.13. *A* hybrid dynamical system *is a tuple* $\mathcal{H} = (\mathcal{J}, \Gamma, \mathcal{D}, \mathcal{B}, \mathcal{F}, \mathcal{G}, \mathcal{R})$*, where:*

- $\mathcal{J}$ *is a finite set indexing the discrete states of* $\mathcal{H}$*.*
- $\Gamma \subset \mathcal{J} \times \mathcal{J}$ *is the set of edges, forming a directed graph structure over* $\mathcal{J}$*.*
- $\mathcal{D} = \{D_j\}_{j\in\mathcal{J}}$ *is the set of domains, where each* D_j *is a compact connected smooth* n_j*-dimensional Riemannian manifold with boundary, with Riemannian metric* g_j *and induced distance* d_j*.*
- $\mathcal{B} = \{\beta_j\}_{j\in\mathcal{J}}$ *is the set of retractions, where* β_j *is a retraction defined on* D_j*.*
- $\mathcal{F} = \{F_j\}_{j\in\mathcal{J}}$ *is the set of vector fields, where each* F_j *is a vector field defined on* D_j*.*
- $\mathcal{G} = \{G_e\}_{e\in\Gamma}$ *is the set of guards, where* $G_{(j,j')} \subset \partial D_j$ *is a guard in mode* $j \in \mathcal{J}$ *which defines a transition to mode* $j' \in \mathcal{J}$*.*
- $\mathcal{R} = \{R_e\}_{e\in\Gamma}$ *is the set of reset maps, where* $R_{(j,j')} : G_{(j,j')} \to \partial D_{j'}$ *is a continuous map.*

A diagram presenting a three mode hybrid system is shown in Figure 3.1.

We make the following assumptions on the vector fields, guards, reset maps, and retractions:

Assumption 3.14. *Given* $j \in \mathcal{J}$*,* F_j *is Lipschitz continuous. That is, for each chart* $(U_\alpha, \varphi_\alpha)$ *of* D_j*, the function* $\widetilde{F}_j : U_\alpha \to T\mathbb{R}^{n_j}$*, defined by* $\widetilde{F}_j(p) = \big((\varphi_\alpha)_* \circ F_j \circ \varphi_\alpha^{-1}\big)(p)$*, is Lipschitz continuous.*

Assumption 3.15. *The guards do not intersect. That is, for each pair of edges* $e_1, e_2 \in \Gamma$*, with* $e_1 \neq e_2$*,* $G_{e_1} \cap G_{e_2} = \emptyset$*.*

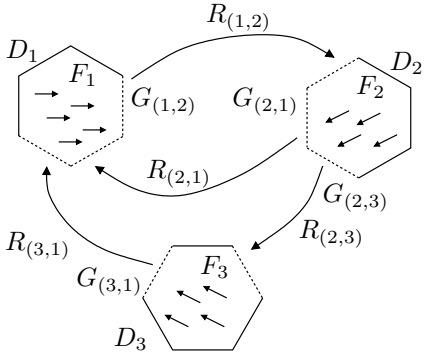


Figure 3.1: Diagram of a hybrid dynamical system with three modes.

Assumption 3.16. *The guards are closed embedded submanifolds with codimension* 1*. Also, the image of each reset map is a closed set.*

Note that, since reset maps are continuous, Assumption 3.16 implies that the guards are also closed sets.

Assumption 3.17. *The pushforward of each retraction in each chart is Lipschitz with respect to its point of evaluation, i.e. given a chart φ_α on D_j and a neighborhood of the origin $\widetilde{W}_p \subset T_p\mathbb{R}^{n_j}$, the function $\widetilde{\beta}_p(V) : \widetilde{W}_p \to \mathbb{R}^{n_j}$, defined by $\widetilde{\beta}_p(V) = \big(\varphi_\alpha \circ (\beta_j)_p \circ (\varphi_\alpha)_*^{-1}\big)(V)$, has a pushforward, denoted by $(\widetilde{\beta}_p)_*|_V$ that is Lipschitz with respect to its point of evaluation V.*

Assumptions 3.14 and 3.15 are sufficient to ensure the existence and uniqueness of executions of hybrid dynamical systems as we prove in Lemma 3.32. Assumption 3.16 allows us to metrize our relaxed domain in Section 3.2. Assumption 3.17 is critical in ensuring that the numerical integration scheme described in Section 3.3 is properly defined.

3.2 Relaxation and Metrization of Hybrid Dynamical Systems

Rather than approximating the time instant when a trajectory intersects a guard, we prove convergence of the numerical integration scheme described in the next section by relaxing the hybrid dynamical system. First, we relax hybrid domains along their guards and extend the definition of the domain's metric, vector field, and retraction onto this relaxation. Next, we attach the disparate domains to each other via a topological quotient and construct a single metric space in which we can prove convergence.

The main mathematical tool behind the final result in this section is that we can create an equivalence relation that relates each relaxed guard, G_e^ε, with its image through its

corresponding reset map, $R_e^\varepsilon\big(G_e^\varepsilon\big)$. Formally, given a topological space $\mathcal{S}$ and a function $f : A \to B$, $A, B \subset \mathcal{S}$, we define the following equivalence relation:

$$\Lambda_f = \big\{(a,b) \in \mathcal{S} \times \mathcal{S} \mid a \in f^{-1}(b), \text{ or } b \in f^{-1}(a), \text{ or } a = b\big\}, \tag{3.7}$$

and we say that $a, b \in \mathcal{S}$ *are related*, denoted by $a \overset{f}{\sim} b$, if $(a,b) \in \Lambda_f$. It is clear that Λ_f defines an equivalence relation, since it is reflexive, symmetric, and transitive. We denote the *quotient of* $\mathcal{S}$ *under* Λ_f by $\frac{\mathcal{S}}{\Lambda_f}$. Also, in this case we say that the *equivalence class of* $x \in \mathcal{S}$ is $[x]_f = \Big\{a \in \mathcal{S} \mid a \overset{f}{\sim} x\Big\}$.

Another useful tool from graph theory is the concept of neighborhood in a graph. Using the same notation as in Definition 3.13, given a node $j \in \mathcal{J}$, we say that the *neighborhood of* j is

$$\mathcal{N}_j = \{e \in \mathcal{J} \times \mathcal{J} \mid \exists j' \in \mathcal{J} \text{ s.t. } e = (j, j') \in \Gamma\}. \tag{3.8}$$

We begin by defining the relaxation of D_j, a single domain of a hybrid system. The relaxation is accomplished by first "stretching" the guards belonging to D_j by attaching an ε-sized strip along each guard. As we show in the next section, this eliminates the need to exactly detect guard satisfaction, which is fundamental in order to create an implementable computational scheme.

Definition 3.18. *Let $\mathcal{H}$ be a hybrid dynamical systems as in Definition 3.13. Given $e \in \Gamma$, we say that $S_e^\varepsilon = G_e \times [0, \varepsilon]$ is the* strip *associated to guard G_e. Given $j \in \mathcal{J}$, let $\iota_j : \coprod_{e \in \mathcal{N}_j} G_e \to \coprod_{e \in \mathcal{N}_j} S_e^\varepsilon$ be the canonical identification $\iota_j(p) = (p, 0)$ in its corresponding strip.*

Then, the relaxation of D_j *is defined by:*

$$D_j^\varepsilon = \frac{D_j \coprod \Big(\coprod_{e \in \mathcal{N}_j} S_e^\varepsilon\Big)}{\Lambda_{\iota_j}}, \tag{3.9}$$

where Λ_{ι_j} is defined as in Equation (3.7)*.*

Note that, since G_e, for $e \in \mathcal{N}_j$, is a closed embedded submanifold of ∂D_j by Assumption 3.16, S_e^ε is a compact smooth manifold.

A point on a strip S_e^ε of D_j is defined using n_j coordinates $(z_1, \ldots, z_{n_j - 1}, \tau)$, shortened (ζ, τ), where the final coordinate, τ, is called the *transverse coordinate* and is the distance along the interval $[0, \varepsilon]$. An illustration of the quotient process taking place in Definition 3.18 together with the definition of coordinates in each strip is shown in Figure 3.2. In the definitions and results below we slightly abuse the notation in the following way: given $j \in \mathcal{J}$, $e \in \mathcal{N}_j$, and $p \in G_e$, we say that $(p, 0) \in S_e^\varepsilon$, where what we mean is that $\iota_j(p) \in S_e^\varepsilon$, with ι_j as in Definition 3.18.

We can construct coordinate charts for relaxations by extending the existing coordinate charts in our original space.

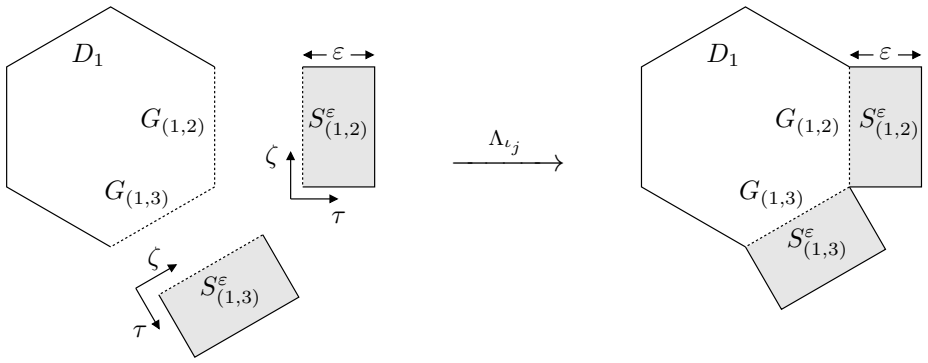


Figure 3.2: Left: disjoint union of D_j and all its strips $\{S^\varepsilon_e\}_{e\in\mathcal{N}_j}$. Right: relaxed domain D^ε_j after taking the quotient with the relation Λ_{ι_j}.

Definition 3.19. *Let $\alpha \in \mathcal{A}$ and $(U_\alpha, \varphi_\alpha)$ be a chart on D_j. If U_α is not a boundary chart, then its relaxation $(U^\varepsilon_\alpha, \varphi^\varepsilon_\alpha)$ equals the original chart. Otherwise, let $\iota_\alpha : \partial U_\alpha \to \partial U_\alpha \times \{0\}$ be the canonical identification $\iota_\alpha(p) = (p, 0)$.*

Then the relaxation of U_α *is:*

$$U^\varepsilon_\alpha = \frac{U_\alpha \coprod \big(\partial U_\alpha \times [0,\varepsilon]\big)}{\Lambda_{\iota_\alpha}}, \tag{3.10}$$

where Λ_{ι_α} is as defined in Equation (3.7)*, and the* relaxation of φ_α *is:*

$$\varphi^\varepsilon_\alpha(x) = \begin{cases} \varphi_\alpha(x) & \text{if } x \in U_\alpha, \\ \big(\varphi_\alpha(\zeta), \tau\big) & \text{if } x = (\zeta,\tau) \in (\partial U_\alpha \times [0,\varepsilon]). \end{cases} \tag{3.11}$$

Note that $\varphi^\varepsilon_\alpha|_{U^\varepsilon_\alpha}$ is a homeomorphism, and that the relaxation of U_α is done in the same spirit as the relaxation of D_j in Definition 3.18.

Next, we develop a metric on each relaxed domain. We aim to endow each relaxed domain D^ε_j with a metric $d^\varepsilon_j : D^\varepsilon_j \times D^\varepsilon_j \to [0,\infty)$ which restricts to d_j on D_j. To achieve this, we first define a metric on each strip and then prove that the metric induced by the quotient structure of the relaxation is actually a metric on D^ε_j with the desired property.

Definition 3.20. *Let $j \in \mathcal{J}$ and $e \in \mathcal{N}_j$, where $\mathcal{N}_j$ is as in Equation* (3.8)*. Then, the metric $d_{S^\varepsilon_e} : S^\varepsilon_e \times S^\varepsilon_e \to [0,\infty)$ on the strip S^ε_e is:*

$$d_{S^\varepsilon_e}\big((\zeta,\tau),(\zeta',\tau')\big) = d_j(\zeta,\zeta') + |\tau - \tau'|\,. \tag{3.12}$$

Before we continue we will define the simplest possible metric for the disjoint union of sets in metric spaces.

Definition 3.21. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sets in metric spaces, with metric d_1 and d_2 respectively. Then we say that $\widetilde{d} : \mathcal{S}_1 \coprod \mathcal{S}_2 \times \mathcal{S}_1 \coprod \mathcal{S}_2 \to [0, \infty]$ is the* disjoint metric *of $\mathcal{S}_1 \coprod \mathcal{S}_2$, defined by:*

$$\widetilde{d}(x, y) = \begin{cases} d_1(x, y) & \text{if } x, y \in \mathcal{S}_1, \\ d_2(x, y) & \text{if } x, y \in \mathcal{S}_2, \\ \infty & \text{otherwise.} \end{cases} \tag{3.13}$$

Using Definition 3.21 we can define $\widetilde{d_j^\varepsilon}$ to be the disjoint metric of $D_j \coprod \left(\coprod_{e \in \mathcal{N}_j} S_e^\varepsilon \right)$. Then the function $d_j^\varepsilon : D_j^\varepsilon \times D_j^\varepsilon \to [0, \infty)$ defined by:

$$d_j^\varepsilon(x, y) = \inf \left\{ \sum_{i=1}^{k} \widetilde{d_j^\varepsilon}(p_i, q_i) \mid k \in \mathbb{N},\ x = p_1,\ y = q_k,\ q_i \overset{\iota_j}{\sim} p_{i+1}\ \forall i \in \{1, \ldots, k-1\} \right\}, \tag{3.14}$$

where ι_j is as in Definition 3.18, is a semi-metric on D_j^ε, i.e. it is non-negative, symmetric, and satisfies the Triangle Inequality (Definition 3.1.12 in [BBI01]), but in general, $d_j^\varepsilon(x, y) = 0$ may not imply $x \overset{\iota_j}{\sim} y$. The following theorem establishes that, in fact, d_j^ε is a metric on D_j^ε.

Theorem 3.22. *For each $j \in \mathcal{J}$, the function d_j^ε, as in Equation* (3.14)*, is a metric on D_j^ε, as in Definition 3.18.*

Proof. We already know that d_j^ε is a semi-metric, so all we must show is that $[x]_{\iota_j} = [y]_{\iota_j}$ whenever $d_j^\varepsilon\left([x]_{\iota_j}, [y]_{\iota_j}\right) = 0$, where ι_j is as in Definition 3.18.

Let $e \in \mathcal{N}_j$. Each $x \in D_j \setminus G_e$ has a d_j-ball that is disjoint from G_e, since G_e is closed by Assumption 3.16, therefore $[x]_{\iota_j} = \{x\}$ is a singleton. Similarly each $x \in S_e^\varepsilon \setminus (G_e \times \{0\})$ has a $d_{S_e^\varepsilon}$-ball which is disjoint from $G_e \times \{0\}$, therefore $[x]_{\iota_j} = \{x\}$ is a singleton. Finally, each $x \in G_e$ has a d_j-ball and a $d_{S_e^\varepsilon}$-ball (defined in their appropriate space) disjoint from any other $y \in G_e$, therefore $[x]_{\iota_j} = \{x, (x, 0)\}$.

This argument is true for each $e \in \mathcal{N}_j$, and thus establishes that d_j^ε is a metric on D_j^ε. □

The following Lemma, whose proof we omit, gives us a basic estimate of the relaxed metric d_j^ε.

Lemma 3.23. *Let d_j^ε be defined as in Equation* (3.14)*, and let $e \in \mathcal{N}_j$, where $\mathcal{N}_j$ is as in Equation* (3.7)*.*

Then, given $x, y \in D_j$, $d_j^\varepsilon(x, y) \leq d_j(x, y)$. Similarly, given $x, y \in S_e^\varepsilon$, $d_j^\varepsilon(x, y) \leq d_{S_e^\varepsilon}(x, y)$.

Next, we extend the vector field onto the strip:

Definition 3.24. *Given $j \in \mathcal{J}$, for each $e \in \mathcal{N}_j$, where $\mathcal{N}_j$ is as in Equation* (3.7)*, let the vector field on the strip S_e^ε, denoted $F_{S_e^\varepsilon}$, be the unit vector pointing outward along the transverse direction, i.e. $F_{S_e^\varepsilon}(\zeta, \tau) = \frac{\partial}{\partial \tau}$.*

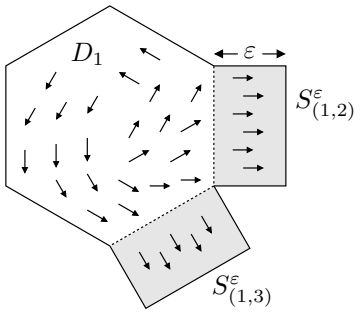


Figure 3.3: Relaxed vector field F_j^ε on D_j^ε.

Then, the relaxation of F_j *is:*

$$F_j^\varepsilon(x) = \begin{cases} F_j(x) & \text{if } x \in D_j, \\ F_{S_e^\varepsilon}(x) & \text{if } x \in G_e \times (0, \varepsilon],\ \forall e \in \mathcal{N}_j. \end{cases} \tag{3.15}$$

Note that the relaxation of the vector field is generally not continuous along each G_e, for $e \in \mathcal{N}_j$. As we will show in Lemma 3.33, this discontinuous vector field does not lead to executions described as Filippov solutions of a switched differential equation [Fil88], since the vector field on the strips always points away of the guard. An illustration of the relaxed vector field F_j^ε on D_j^ε is shown in Figure 3.3.

In contrast to the vector field, which we explicitly extend throughout the strip, we do not require an explicit form for our relaxed retractions. Instead, we require that any relaxed retraction centered at points sufficiently close to a guard has a range that includes the strip.

Definition 3.25. *Let $j \in \mathcal{J}$, and $e \in \mathcal{N}_j$, where $\mathcal{N}_j$ is as in Equation* (3.7)*. Also, let $p \in D_j \setminus \partial D_j$, and $\beta_p : W_p \to D_j$ a retraction on p, as in Definition 3.12.*

If $\beta_p^{-1}\big(G_{(j,j')}\big) \cap W_p = \emptyset$, then any relaxation of β_p equals β_p. Otherwise, we say that a relaxation of β_p *is any differentiable function $\beta_p^\varepsilon : U_p \to D_j^\varepsilon$, with U_p an open set containing W_p, so that β_p^ε agrees with β_p on W_p.*

The relaxation of β_j*, denoted by β_j^ε, is just the collection of relaxations of β_p^ε on the interior of D_j.*

Constructing a relaxed retraction is always possible by means of local relaxed coordinate charts, but we omit the proof since it is outside the scope of our result. Note in particular that if the domain D_j is a subset of $\mathbb{R}^{n_j}$, then a relaxation of a retraction $\beta_p(v) = p + v$ could be constructed by simply extending the domain of β_p and setting $\beta_p^\varepsilon(v) = p + v$ for each v in the new domain.

We simultaneously relax each hybrid domain to define the relaxation of a hybrid dynamical system and then attach the disparate domains of the relaxed system together to construct a metric space.

Definition 3.26. *Let $\mathcal{H}$ be a hybrid system as in Definition 3.13. The* relaxation of $\mathcal{H}$ *is a tuple $\mathcal{H}^\varepsilon = (\mathcal{J}, \Gamma, \mathcal{D}^\varepsilon, \mathcal{B}^\varepsilon, \mathcal{F}^\varepsilon, \mathcal{G}^\varepsilon, \mathcal{R}^\varepsilon)$, where:*

- $\mathcal{D}^\varepsilon = \left\{D_j^\varepsilon\right\}_{j\in\mathcal{J}}$ *is the set of relaxations of the domains $\mathcal{D}$, as in Definition 3.18, with metrics $\left\{d_j^\varepsilon\right\}_{j\in\mathcal{J}}$, as in Equation* (3.14).
- $\mathcal{B}^\varepsilon = \left\{\beta_j^\varepsilon\right\}_{j\in\mathcal{J}}$ *is a set of relaxations of the retractions $\mathcal{B}$, as in Definition 3.25.*
- $\mathcal{F}^\varepsilon = \left\{F_j^\varepsilon\right\}_{j\in\mathcal{J}}$ *is the set of relaxations of the vector fields $\mathcal{F}$, as in Definition 3.24.*
- $\mathcal{G}^\varepsilon = \{G_e^\varepsilon\}_{e\in\Gamma}$ *is the set of relaxations of the guards $\mathcal{G}$, where, given $e = (j, j')$, each guard $G_e \subset \partial D_j$ is relaxed to $G_e^\varepsilon = G_e \times \{\varepsilon\} \subset \partial D_j^\varepsilon$.*
- $\mathcal{R}^\varepsilon = \{R_e^\varepsilon\}_{e\in\Gamma}$ *is the set of relaxations of the reset maps $\mathcal{R}$, where, given $e = (j, j')$, $R_e^\varepsilon : G_e^\varepsilon \to \partial D_{j'}$ is defined by $R_e^\varepsilon(\zeta, \tau) = R_e(\zeta)$.*

Now we can define a new metric space where the executions of hybrid systems can be naturally described. This definition is, in practice, completely equivalent to the *hybrid colimit* presented in [AS05].

Definition 3.27. *Let $\mathcal{H}^\varepsilon$ be a relaxed hybrid system as in Definition 3.26. Also, let*

$$R^\varepsilon : \coprod_{e\in\Gamma} G_e^\varepsilon \to \coprod_{j\in\mathcal{J}} D_j^\varepsilon, \tag{3.16}$$

the function that represents all reset maps as a unified function, i.e. $R^\varepsilon(p) = R_e^\varepsilon(p)$ whenever $p \in G_e^\varepsilon$.

Then, the relaxed hybrid quotient space *of a relaxed hybrid dynamical system $\mathcal{H}^\varepsilon$ is:*

$$\mathcal{M}^\varepsilon = \frac{\coprod_{j\in\mathcal{J}} D_j^\varepsilon}{\Lambda_{R^\varepsilon}}, \tag{3.17}$$

where Λ_{R^ε} is as in Equation (3.7).

The illustration in Figure 3.4 shows the details about the construction in Definition 3.27.

As we did in the definition of the relaxed metric d_j^ε, we say that $\widetilde{\mu}^\varepsilon$ is the disjoint metric on $\coprod_{j\in\mathcal{J}} D_j^\varepsilon$ based on the individual metrics $\left\{d_j^\varepsilon\right\}_{j\in\mathcal{J}}$, as in Definition 3.21. Then the function $\mu^\varepsilon : \mathcal{M}^\varepsilon \times \mathcal{M}^\varepsilon \to [0, \infty)$ defined by:

$$\mu^\varepsilon(x, y) = \inf\left\{\sum_{i=1}^{k} \widetilde{\mu}^\varepsilon(p_i, q_i) \mid k \in \mathbb{N},\ x = p_1,\ y = q_k,\ q_i \overset{R^\varepsilon}{\sim} p_{i+1}\ \forall i \in \{1, \ldots, k-1\}\right\}, \tag{3.18}$$

is a semi-metric on $\mathcal{M}^\varepsilon$, where R^ε is as in Definition 3.27. In general, $\mu^\varepsilon(x, y) = 0$ may not necessarily imply $x \overset{R^\varepsilon}{\sim} y$. The following theorem establishes the fact that μ^ε is a metric on $\mathcal{M}^\varepsilon$.

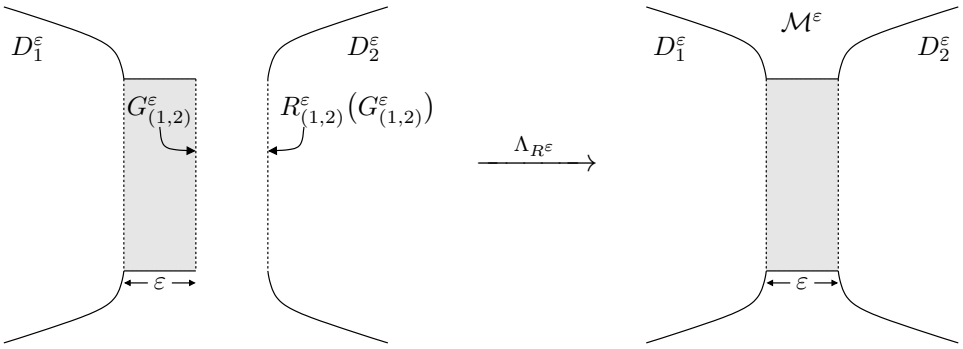


Figure 3.4: Left: disjoint union of D_1^ε and D_2^ε. Right: relaxed hybrid quotient space $\mathcal{M}^\varepsilon$ after taking the quotient with the relation Λ_{R^ε}.

Theorem 3.28. *The function μ^ε is a metric on $\mathcal{M}^\varepsilon$.*

Proof. Since, by Assumption 3.16, the image of each reset map is closed, and since we can obtain metric neighborhoods separating distinct points in $R_e^\varepsilon\big(G_e^\varepsilon\big)$ for each $e \in \Gamma$, the result follows by a similar argument to the one presented in the proof of Theorem 3.22. □

As we did in Lemma 3.23, the next Lemma shows that we can estimate the value of μ^ε using each relaxed metric d_j^ε within a domain. We omit the proof since it follows easily from the definition of the metric as the infimum of all possible paths.

Lemma 3.29. *Let μ^ε be defined as in Equation* (3.18)*, and let $j \in \mathcal{J}$. Then given $x, y \in D_j^\varepsilon$, $\mu^\varepsilon(x, y) \leq d_j^\varepsilon(x, y)$.*

Perhaps the most relevant property of μ^ε is that if $y = R_e^\varepsilon(x)$ for some $x \in G_e^\varepsilon$ and $e \in \Gamma$, then $\mu^\varepsilon(x, y) = 0$, which is exactly what we would expect to happen if we want the trajectories of hybrid systems to be continuous. Exploiting this property, we define a metric between curves on $\mathcal{M}^\varepsilon$.

Definition 3.30. *Let $I \subset [0, \infty)$ a bounded interval, and μ^ε as in Equation* (3.18)*. Then, given any two curves $x, x' : I \to \mathcal{M}^\varepsilon$ we define*

$$\rho_I^\varepsilon\big(x, x'\big) = \sup\big\{\mu^\varepsilon\big(x(t), x'(t)\big) \mid t \in I\big\}. \tag{3.19}$$

Our choice of the supremum among point-wise distances in ρ_I^ε is inspired in the sup-norm for continuous real functions, since, as we show in Section 3.3, the executions of relaxed hybrid systems are continuous.

Require: $t = 0$, $j \in \mathcal{J}$, and $p \in D_j$.
1: Set $x(0) = p$.
2: **loop**
3: Compute $\gamma : J \to D_j$, the maximal integral curve of F_j such that $\gamma(t) = x(t)$.
4: Set $x(s) = \gamma(s)$ for each $s \in J \cap [t, \infty)$.
5: Let $t' = \sup J$. ▷ Note that if t' is finite, then $x(t') \in \partial D_j$.
6: **if** $t' = \infty$ or $\nexists e \in \mathcal{N}_j$ such that $x(t') \in G_e$ **then**
7: Stop.
8: **end if**
9: Let $(j, j') \in \mathcal{N}_j$ such that $x(t') \in G_{(j,j')}$. ▷ Now we perform a mode transition.
10: Replace the value of $x(t')$ with $R_{(j,j')}\big(x(t')\big)$.
11: Set $t = t'$ and $j = j'$.
12: **end loop**

Algorithm 3.1 Execution of a hybrid system $\mathcal{H}$.

3.3 Relaxed Executions and Discrete Approximations

This section contains our main result: discrete approximations to trajectories of hybrid dynamical systems, constructed using a modified version of the Forward Euler Algorithm, converge to the actual trajectories. First, we define executions of hybrid dynamical systems and relaxed hybrid dynamical systems. Next, we define our discrete approximation scheme on our relaxed quotient space $\mathcal{M}^\varepsilon$. Finally, we prove that the discrete approximations of executions of the relaxed hybrid dynamical system converge to the executions of the original hybrid dynamical system.

Execution of a Hybrid System

We begin by defining an *execution* of a hybrid dynamical system. This definition agrees with the standard intuition about executions of hybrid systems, i.e. the execution evolves as a standard dynamical system until a guard is reached, in which case a "jump" occurs via a reset map to a new hybrid domain. The main purpose of writing a new definition is to clarify technical details that will become relevant in the proofs below.

Let $\mathcal{H}$ be a hybrid system as described in Definition 3.13. Then, Algorithm 3.1 defines the execution of $\mathcal{H}$ by construction. Note that, since each domain in $\mathcal{D}$ is compact and each vector field in $\mathcal{F}$ is Lipschitz continuous by Assumption 3.14, every maximal integral curve of the vector field F_j, as in Definition 3.10, either stops in finite time at a point $p \in \partial D_j$ or it continues in D_j indefinitely (this is a simple extension of Theorem 7.10 in [Lee03]). This fact is fundamental in Step 6, since it narrows down to only two possible cases after we compute a maximal integral curve. A resulting execution from Algorithm 3.1, denoted x, is a piece–wise continuous function defined from an interval $I \subset [0, \infty)$ to $\coprod_{j \in \mathcal{J}} D_j$. Figure 3.5

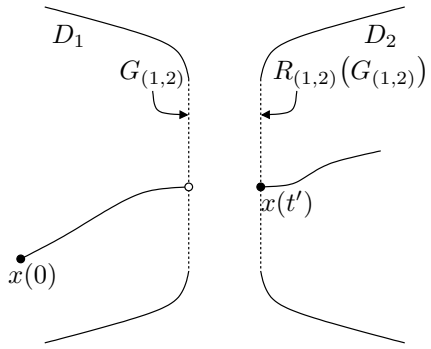


Figure 3.5: Mode transition of execution x in a two-mode hybrid dynamical system.

shows an example of the mode transition of an execution.

Formally, given x as constructed by Algorithm 3.1 and $t \in I$, $x(t) \in \coprod_{j\in\mathcal{J}} D_j$, but we abuse notation and say that $x(t) \in \mathcal{M}^\varepsilon$ when we should say that $[x]_{R^\varepsilon} \in \mathcal{M}^\varepsilon$, where R^ε is as in Definition 3.27. We make use of this fact when comparing a hybrid execution with its corresponding relaxed execution, which we define in Algorithm 3.2.

With our definition of execution of a hybrid system in place, we can define an important class of executions that only exist in hybrid systems.

Definition 3.31. *A* Zeno execution *is an execution where there exists an infinite number of discrete transitions in a finite amount of time. Hence, there exists $T \in \mathbb{R}$ so that the execution is only defined on $I = [0, T)$.*

Zeno executions are important because, among other properties, they are very hard to approximate numerically since, in practice, simulating an infinite number of discrete transitions takes an infinite amount of time in any computer.

Now we show that Algorithm 3.1 produces well-defined trajectories.

Lemma 3.32. *Let $\mathcal{H}$ be a hybrid system as in Definition 3.13, $j \in \mathcal{J}$, and $p \in D_j$. Then, Algorithm 3.1 produces a unique maximal trajectory.*

Proof. Let $x : I \to \coprod_{j\in\mathcal{J}} D_j$ be the execution produced by Algorithm 3.1. First note that for each $t \in I$, $x(t)$ is uniquely defined. Indeed, either $x(t)$ is defined by a maximal integral curve in Step 3, which is unique by Assumption 3.14, or it is defined as the image of a reset map in Step 10, which is also unique by Assumption 3.15.

Now, Algorithm 3.1 can produce only three types of executions:

- x has a finite number of mode transitions and $I = [0, \infty)$.
- x has a finite number of mode transitions, $I = [0, T]$ for some $T > 0$, and $x(T) \in \partial D_j$ for some $j \in \mathcal{J}$.

Require: $t = 0$, $j \in \mathcal{J}$, and $p \in D_j$.
1: Set $x^\varepsilon(0) = p$.
2: **loop**
3: Compute $\gamma : J \to D_j$, the maximal integral curve of F_j such that $\gamma(t) = x^\varepsilon(t)$.
4: Set $x^\varepsilon(s) = \gamma(s)$ for each $s \in J \cap [t, \infty)$.
5: Let $t' = \sup J$. ▷ Note that if t' is finite, then $x^\varepsilon(t') \in \partial D_j$.
6: **if** $t' = \infty$ or $\nexists e \in \mathcal{N}_j$ such that $x(t') \in S^\varepsilon_e$ **then**
7: Stop.
8: **end if**
9: Let $(j, j') \in \mathcal{N}_j$ such that $x(t') = (q, 0) \in S^\varepsilon_{(j,j')}$.
▷ Now we perform a mode transition.
10: Set $x^\varepsilon(t' + \tau) = (q, \tau)$ for each $\tau \in [0, \varepsilon]$.
▷ Note that $[x^\varepsilon(t' + \varepsilon)]_{R^\varepsilon} = \left[R^\varepsilon_{(j,j')}\big(x^\varepsilon(t' + \varepsilon)\big)\right]_{R^\varepsilon}$.
11: Set $t = t' + \varepsilon$ and $j = j'$.
12: **end loop**

Algorithm 3.2 Relaxed execution of a relaxed hybrid system $\mathcal{H}^\varepsilon$.

- x is a Zeno execution and $I = [0, T)$ for some $T > 0$.

In either case we obtain a maximal trajectory, since its time domain cannot be extended. □

Relaxed Execution of a Hybrid System

Now we define the *relaxed execution* of a hybrid dynamical system. The main idea here is that, once the execution reaches a guard, we continue integrating over the strip with its trivial vector field $F_{S^\varepsilon_e}$, as in Definition 3.24.

Let $\mathcal{H}$ be a hybrid system, as in Definition 3.13, and $\mathcal{H}^\varepsilon$ its relaxation, as in Definition 3.26. Then, Algorithm 3.2 defines a relaxed execution of $\mathcal{H}^\varepsilon$ by construction. The resulting relaxed execution, denoted x^ε, is continuous function defined from an interval $I \subset [0, \infty)$ to $\mathcal{M}^\varepsilon$, as in Definition 3.27. Note that Algorithm 3.2 is only defined for initial conditions in D_j for some $j \in \mathcal{J}$ since the strips are artificial objects which do not appear in the original model of the system.

A very important part of Algorithm 3.2 is Step 10, since it allow us to "connect" each mode transition, forming a continuous curve in the relaxed quotient space $\mathcal{M}^\varepsilon$. It is also important that our definition for the relaxed execution over the strip S^ε_e, also in Step 10, is exactly equal to the maximal integral curve of $F_{S^\varepsilon_e}$. In other words, we could have written a shorter version of Algorithm 3.2 by solving the maximal integral curve of the relaxed vector field F^ε_j in Step 3, at the expense of making it harder to understand. Figure 3.6 shows an example of the relaxed mode transition produced by this Algorithm.

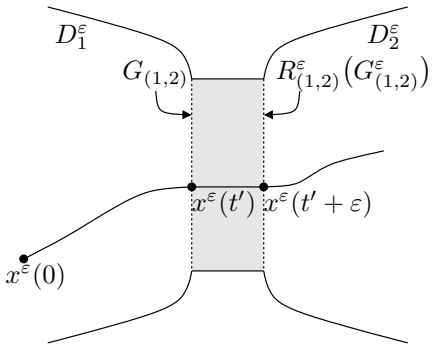


Figure 3.6: Relaxed mode transition of the relaxed execution x^ε in a two-mode hybrid dynamical system.

Note that, given a hybrid system $\mathcal{H}$, its relaxation $\mathcal{H}^\varepsilon$, and an initial condition $p \in D_j$, the relaxed execution of $\mathcal{H}^\varepsilon$ produced by Algorithm 3.2 is simply a delayed version of the execution of $\mathcal{H}$ produced by Algorithm 3.1, since the relaxed version has to spend additional time on each mode transition. Also, note that our definition of relaxed execution is compatible with the execution of a regularized hybrid system as defined in [JEL99].

We omit the proof of the following Lemma since it is similar to the proof of Lemma 3.32.

Lemma 3.33. *Let $\mathcal{H}$ be a hybrid system as in Definition 3.13, and let $\mathcal{H}^\varepsilon$ be its relaxation as in Definition 3.26. Also, let $j \in \mathcal{J}$ and $p \in D_j$.*

Then, Algorithm 3.2 produces a unique maximal trajectory.

Next, we define the types of trajectories that can be approximated.

Definition 3.34. *Let $\mathcal{H}$ be a hybrid system, as in Definition 3.13, and $\mathcal{H}^\varepsilon$ be its relaxation, as in Definition 3.26. Given $j \in \mathcal{J}$ and $p \in D_j$, let us denote the execution of $\mathcal{H}$ with initial condition p by x_p, and similarly let us denote the relaxed execution of $\mathcal{H}^\varepsilon$ with initial condition p by x_p^ε.*

Then, we say that x_p is orbitally stable *at $p' \in \mathcal{M}^\varepsilon$ if, for each $t \in I$, the map $p \mapsto x_p^\varepsilon(t)$ is continuous at p'.*

Orbitally stable executions are exactly the type of execution that can be approximated in a hybrid dynamical system [Lyg+03]. Indeed, if an execution is not orbitally stable then there exists a time t' such that, if we initialize another execution arbitrarily close to $x^\varepsilon(t')$, the executions will have different sequences of discrete transitions. Classical dynamical systems are always orbitally stable (see Theorem 17.8 in [Lee03] for a stronger version of this result), hence non-orbitally stable executions are, together with Zeno executions, two of the consequences of allowing discrete transitions as time evolves. Figure 3.7 shows the case of a non-orbitally stable execution due to a discrete transition in the boundary of a guard.

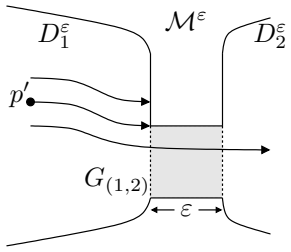


Figure 3.7: Non-orbitally stable execution at p' of a two-mode hybrid dynamical system.

Even though we do not deal with this problem in our result, we can mention a third consequence of discrete mode transitions: the evolution of a hybrid system cannot be reversed in time, as opposed to classical dynamical systems where we can always solve a differential equation either forward or backwards in time. This problem is one of the biggest problems in the computation of the optimal control of hybrid systems, since the costate is usually calculated as the solution of a differential equation backwards in time (more details can be found in [Sus99b; SC07; HT11]).

Before we state our result proving the convergence of relaxed executions as $\varepsilon \to 0$, we introduce a particular type of Zeno executions of interest.

Definition 3.35. *Let $\mathcal{H}$ be a hybrid system, as in Definition 3.13, $j \in \mathcal{J}$, and $p \in D_j$ such that $x : [0, T) \to \mathcal{M}^\varepsilon$, the execution of $\mathcal{H}$ with initial condition p as constructed by Algorithm 3.1, is a Zeno execution as in Definition 3.31.*

We say that x accumulates on $p' \in \mathcal{M}^\varepsilon$ *if there exists $C > 0$ such that for each $r > 0$ there exists $\delta > 0$ satisfying:*

$$\sup_{t \in [T-\delta, T)} \mu^\varepsilon\big(p', x(t)\big) \leq C\varepsilon + r. \tag{3.20}$$

In practice, if a Zeno executions accumulates then it is impossible for this execution to "fill" a portion of the hybrid space while having an infinite number of discrete transitions. Examples of Zeno executions that do not accumulate can be found in [Zha+01]. Figure 3.8 shows a Zeno execution that accumulates on p'. Note that p' must belong to a guard of the hybrid system. Also, the choice of p' is not unique in $\mathcal{M}^\varepsilon$ for $\varepsilon > 0$, since we can always choose the image of p' via a reset map.

Now we state our first convergence theorem.

Theorem 3.36. *Let $\mathcal{H}$ be a hybrid system, as in Definition 3.13, and $\mathcal{H}^\varepsilon$ be its relaxation, as in Definition 3.26. Also, let $j \in \mathcal{J}$, $p \in D_j$, $x : I \to \mathcal{M}^\varepsilon$ be the execution of $\mathcal{H}$ with initial condition p, constructed by Algorithm 3.1, and x^ε be the relaxed execution of $\mathcal{H}^\varepsilon$ with initial condition p, constructed by Algorithm 3.2.*

If x has a finite number of discrete transitions or it is a Zeno execution that accumulates, then for each $J \subset I$, where J is bounded interval,

$$\lim_{\varepsilon \to 0} \rho_J^\varepsilon(x, x^\varepsilon) = 0, \tag{3.21}$$

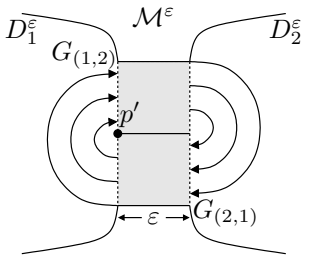


Figure 3.8: Zeno execution accumulating on p' for a two mode hybrid dynamical system.

where ρ_J^ε is as in Definition 3.30.

Proof. First, let us consider the case when x undergoes a finite number of discrete transitions on J, thus without loss of generality we can assume that $J = [0, T]$ for some $T > 0$. Let $\{t_k\}_{k=0}^{m+1}$ be the sequence of times at which discrete mode transitions occur, with $t_0 = 0$ and $t_{m+1} = T$, and let $\{j_k\}_{k=0}^{m}$ be the sequence of domains visited by the execution. Also, let $\varepsilon > 0$ such that

$$(m+1)\varepsilon < \sup\{t_{k+1} - t_k \mid t_{k+1} > t_k,\ k \in \{0, \dots, m\}\}. \tag{3.22}$$

Since the function $x|_{[t_k, t_{k+1})}$ is defined on D_{j_k}, then using Lemmas 3.23 and 3.29 and the definition of d_j in Equation (3.6), for each $t, t' \in [t_k, t_{k+1})$, $t' \geq t$,

$$\mu^\varepsilon\big(x(t), x(t')\big) \leq d_{j_k}\big(x(t), x(t')\big) \leq \int_t^{t'} \sqrt{g_{j_k}\big(\dot{x}(s), \dot{x}(s)\big)}ds \leq L(t' - t), \tag{3.23}$$

where $L = \sup\left\{\|F_j(p)\|_{g_j} \mid p \in D_j,\ j \in \mathcal{J}\right\}$, which is bounded since each D_j is compact.

Note that by definition, $x|_{[t_0,t_1)} = x^\varepsilon|_{[t_0,t_1)}$. Let $k \in \{1, \dots, m\}$ such that $t_{k+1} > t_k$. Then we compute a bound on the distance $\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big)$ for $t \in [t_k, t_{k+1})$, by considering three cases:

(1) $t \in [t_k, t_k + (k-1)\varepsilon)$: In this case $x^\varepsilon(t) \in D_{j_{k-1}}$ and $x(t) \in D_{j_k}$ by our choice of ε in Equation (3.22). Moreover, $x^\varepsilon(t) = x\big(t - (k-1)\varepsilon\big)$. If we define $x(t_k^-) = \lim_{t \to t_k^-} x(t)$, then by Equation (3.23):

$$\begin{aligned}\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big) &\leq \mu^\varepsilon\big(x\big(t - (k-1)\varepsilon\big), x(t_k^-)\big) + \mu^\varepsilon\big(x(t_k^-), x(t_k)\big) + \mu^\varepsilon\big(x(t_k), x(t)\big) \\ &\leq L(k-1)\varepsilon + \varepsilon + L(k-1)\varepsilon.\end{aligned} \tag{3.24}$$

(2) $t \in [t_k + (k-1)\varepsilon, t_k + k\varepsilon)$: In this case $x^\varepsilon(t) \in S^\varepsilon_{(j_{k-1}, j_k)}$ and $x(t) \in D_{j_k}$, hence:

$$\begin{aligned}\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big) &\leq \mu^\varepsilon\big(x^\varepsilon(t), x(t_k)\big) + \mu^\varepsilon\big(x(t_k), x(t)\big) \\ &\leq \varepsilon + Lk\varepsilon.\end{aligned} \tag{3.25}$$

(3) $t \in [t_k + k\varepsilon, t_{k+1})$: In this case $x^\varepsilon(t), x(t) \in D_{j_k}$, hence $x^\varepsilon(t) = x(t - k\varepsilon)$ and:

$$\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big) \leq Lk\varepsilon. \tag{3.26}$$

If there is a sequence of ℓ consecutive jumps such that $t_k = \cdots = t_{k+\ell-1}$ then the argument is analogous, noting that in the case 2 the interval of interest is $[t_k+(k-1)\varepsilon, t_k+(k+\ell)\varepsilon)$, with a bound $\ell\varepsilon+L(k+\ell)\varepsilon$. Also, note that $\mu^\varepsilon\big(x^\varepsilon(t_{m+1}), x(t_{m+1})\big) \leq \mu^\varepsilon\big(x^\varepsilon(t^-_{m+1}), x(t^-_{m+1})\big)+\varepsilon$, this last bound being an equality if there is a discrete transition exactly at $t = t_{m+1}$. Therefore, putting together all the bounds above we have that for each $t \in [0, T]$:

$$\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big) \leq (2L + 1)m\varepsilon, \tag{3.27}$$

which proves the theorem if the number of transitions is finite.

Next, let us consider the case when x is a Zeno execution that accumulates on p', thus $I = [0, T)$. As we did above, let $\{t_k\}_{k=0}^\infty$ be the sequence of times at which discrete mode transitions occur, and let $\{j_k\}_{k=0}^\infty$ be the sequence of domains visited by the execution. Let $r > 0$, and note that as a consequence of Definition 3.35 there exists $C > 0$ and $\delta_r > 0$ such that $\mu^\varepsilon\big(p', x(t)\big) \leq C\varepsilon + r$ for each $t \in [T - \delta_r, T)$. Let us denote by $n(\delta)$ the number of discrete transitions on the interval $[0, T - \delta]$, and let $\varepsilon > 0$ such that:

$$n(\delta_r)\varepsilon < \min\big\{t_{n(\delta_r)+1} - t_{n(\delta_r)}, \delta_{2r}\big\}, \tag{3.28}$$

where without loss of generality we have assumed that $t_{m(\delta_r)+1} > t_{m(\delta_r)}$.

From this choice of ε we get that both x^ε and x are in the same domain at $t = T - \delta_r$, hence $x^\varepsilon(t) = x\big(t - n(\delta_r)\varepsilon\big)$, and since $n(\delta_r)\varepsilon < \delta_{2r}$, then

$$\mu^\varepsilon\big(p', x^\varepsilon(t)\big) \leq C\varepsilon + 2r, \quad \forall t \in [T - \delta_r, T)\,. \tag{3.29}$$

The conclusion follows since there exists $C' > 0$ such that, using Equations (3.27) and (3.28),

$$\mu^\varepsilon\big(x^\varepsilon(t), x(t)\big) \leq C'\left(t_{n(\delta_r)+1} - t_{n(\delta_r)} + \varepsilon\right) + 2r, \quad \forall t \in [0, T), \tag{3.30}$$

But $r > 0$ was arbitrary and $t_{n(\delta_r)+1} - t_{n(\delta_r)} \to 0$ as $r \to 0$ since the number of discrete transitions goes to infinity, hence $\rho^\varepsilon_{[0,T)}(x^\varepsilon, x) \to 0$ as $\varepsilon \to 0$ as desired. □

Discrete Approximations

Now we can define the *discrete approximation of a relaxed execution*. As mentioned above, our discrete approximation scheme is based on the Forward Euler integration approximation for ODE's, modified to be applicable on Riemannian manifolds by using retractions, as in Definition 3.12.

Given a hybrid system $\mathcal{H}$, as in Definition 3.13, and its relaxation $\mathcal{H}^\varepsilon$, as in Definition 3.26. Then, Algorithm 3.3 defines a discrete approximation of a relaxed execution of $\mathcal{H}^\varepsilon$. The

Require: $h > 0$, $k = 0$, $j \in \mathcal{J}$, and $p \in D_j$.
1: Set $t_0 = 0$ and $z^{(\varepsilon,h)}(0) = p$.
2: **loop**
3: **if** $\nexists t > 0$ such that $tF_j\big(z^{(\varepsilon,h)}(t_k)\big)$ is in the domain of β_j^ε **then**
4: Stop.
5: **end if**
6: Find smallest n such that $\frac{h}{2^n}F_j\big(z^{(\varepsilon,h)}(t_k)\big)$ is in the domain of β_j^ε.
7: Set $t_{k+1} = t_k + \frac{h}{2^n}$.
8: Set $z^{(\varepsilon,h)}(t) = \beta_j^\varepsilon\big(tF_j\big(z^{(\varepsilon,h)}(t_k)\big)\big)$ for each $t \in [t_k, t_{k+1}]$.
9: **if** $\exists e \in \mathcal{N}_j$ such that $z^{(\varepsilon,h)}(t_{k+1}) \in S_e^\varepsilon$ **then**
10: Let $(j, j') \in \mathcal{N}_j$ such that $z^{(\varepsilon,h)}(t_{k+1}) = (q, \tau) \in S^\varepsilon_{(j,j')}$.
▷ Now we perform a mode transition.
11: Set $t_{k+2} = t_{k+1} + \varepsilon - \tau$.
12: Set $z^{(\varepsilon,h)}(t) = (q, t - t_{k+1} + \tau)$ for each $t \in [t_{k+1}, t_{k+2}]$.
▷ Note that $\big[z^{(\varepsilon,h)}(t_{k+2})\big]_{R^\varepsilon} = \Big[R^\varepsilon_{(j,j')}\big(z^{(\varepsilon,h)}(t_{k+2})\big)\Big]_{R^\varepsilon}$.
13: Set $k = k + 2$ and $j = j'$.
14: **else**
15: Set $k = k + 1$.
16: **end if**
17: **end loop**

Algorithm 3.3 Discrete approximation of the execution of a relaxed hybrid system $\mathcal{H}^\varepsilon$.

resulting discrete approximation, for a step size $h > 0$, denoted by $z^{(\varepsilon,h)}$, is a function from a closed interval $I \subset [0, \infty)$ to $\mathcal{M}^\varepsilon$, as in Definition 3.27.

We now make a few remarks about Algorithm 3.3. First, the condition in Step 3 can only be satisfied, i.e. the Algorithm only stops, if $z^{(\varepsilon,h)}(t_k) \in \partial D_k$ and $F_j\big(z^{(\varepsilon,h)}(t_k)\big)$ is outward-pointing, since otherwise the domain of β_j^ε contains a Euclidean neighborhood of the origin. The intuition about outward-pointing vectors in the tangent space is very simple, these are exactly the vectors that generate curves going "out" of the manifold. More details can be found in Section 13 in [Lee03], particularly in Lemma 13.5. Second, Step 6 effectively implements a variable step size Forward Euler scheme for Riemannian manifolds, where we only reduce the step size when we are close to the boundary of D_j^ε. Third, in Step 8, note that if $D_j \subset \mathbb{R}^{n_j}$ then the canonical retraction $(\beta_j)_p(V) = p + V$ produces exactly the Forward Euler approximation as expected. Fourth, the function $z^{(\varepsilon,h)}$ is continuous on $\mathcal{M}^\varepsilon$.

But perhaps the most important comment we can make about Algorithm 3.3 is related with Step 12. Indeed, similarly to Algorithm 3.2, the curve assigned to $z^{(\varepsilon,h)}$ in this Step is exactly the maximal integral curve of $F_{S_e^\varepsilon}$ in the strip. We explain the importance of this fact within the context of existing algorithms. As mentioned in the introduction to this Chapter, the hardest problem to solve in the numerical integration of hybrid dynamical systems is

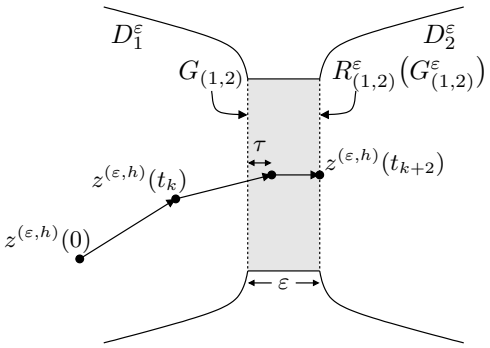


Figure 3.9: Discrete approximation $z^{(\varepsilon,h)}$ of a relaxed execution in a two-mode hybrid dynamical system.

finding the guard with enough accuracy so that the reset could be applied without errors. By relaxing the guards using strips, and then endowing the strips with a trivial vector field, we can implement a scheme that does not need to find exact guard, while still evaluating the reset maps at exactly the points belonging to the guards. Our relaxation does introduce an error in the approximation, but as we show in Theorem 3.37, the error is of order ε.

Figure 3.9 shows a discrete approximation produced by Algorithm 3.3 as it performs a mode transition.

Theorem 3.37. *Let $\mathcal{H}$ be a hybrid dynamical system, as in Definition 3.13, $\mathcal{H}^\varepsilon$ its relaxation, as in Definition 3.26. Also, let $j \in \mathcal{J}$, $p \in D_j$, and $I = [0,T]$ for some $T > 0$.*

If the relaxed execution of $\mathcal{H}^\varepsilon$ with initial condition p, constructed by Algorithm 3.2, has a domain that contains I and is orbitally stable, as in Definition 3.34, then there exists $C > 0$ such that

$$\lim_{h\to 0} \rho_I^\varepsilon\big(x^\varepsilon, z^{(\varepsilon,h)}\big) = C\varepsilon, \tag{3.31}$$

where ρ_I^ε is as in Definition 3.30.

Proof. We complete this proof in incremental steps, first showing the convergence of our modified Forward Euler scheme in Riemannian manifolds for a single domain, and then showing that our relaxation of the discrete transitions converge on the relaxed quotient space $\mathcal{M}^\varepsilon$.

In order to simplify the first part of our argument, let us assume that the hybrid system has only one domain and no discrete transitions, i.e. $\mathcal{D} = \{D\}$, $\mathcal{F} = \{F\}$, $\mathcal{B} = \{\beta\}$, $\mathcal{J} = \{1\}$, $\Gamma = \mathcal{G} = \mathcal{R} = \emptyset$. Let $x^\varepsilon : [0,T] \to D$ a relaxed execution, constructed by Algorithm 3.2, where without loss of generality we are assuming that $[0,T]$ belongs to the domain of the relaxed execution. Similarly, let $z^{(\varepsilon,h)} : [0,T] \to D$ a discrete approximation, constructed using Algorithm 3.3. Note that we have not made any assumptions about the initial conditions of x^ε or $z^{(\varepsilon,h)}$. Also, since we have a single domain, x and x^ε are identical.

Let $\{t_k\}_{k=0}^{\ell} \subset [0,T]$ be the sequence of time samples derived from Algorithm 3.3, where $t_0 = 0$ and we remove the dependence of each t_k and ℓ on h for notational convenience. Note that ℓ does not need to be finite. Given a chart (U,φ), let us also define $\widetilde{\beta} : \widetilde{W} \to \mathbb{R}^n$, $\widetilde{W} \subset T\mathbb{R}^n$, and $\widetilde{F} : \widetilde{U} \to T\mathbb{R}^n$, $\widetilde{U} \subset \mathbb{R}^n$, such that

$$\begin{aligned} \widetilde{\beta}_p(V) &= \left(\varphi \circ \beta_p \circ \varphi_*^{-1}\right)(V), \\ \widetilde{F}(q) &= \left(\varphi_* \circ F \circ \varphi^{-1}\right)(q), \end{aligned} \tag{3.32}$$

and similarly denote $\tilde{x}^{\varepsilon} = \varphi \circ x$ and $\tilde{z}^{(\varepsilon,h)} = \varphi \circ z^{(\varepsilon,h)}$.

Let us assume that given $i, i' \in \{0,\dots,\ell\}$ such that $t_i \le t_{i'}$, both $x^{\varepsilon}|_{[t_i,t_{i'})}$ and $z^{(\varepsilon,h)}|_{[t_i,t_{i'})}$ belong to the same chart. Then, using Picard's Lemma (Lemma 5.6.3 in [Pol97]), for each $t \in [t_i, t_{i'})$:

$$\left\|\tilde{x}^{\varepsilon}(t) - \tilde{z}^{(\varepsilon,h)}(t)\right\| \le e^{L(t_{i'}-t_i)}\left(\left\|\tilde{x}^{\varepsilon}(t_i) - \tilde{z}^{(\varepsilon,h)}(t_i)\right\| + \sum_{k=i}^{i'-1}\int_{t_k}^{t_{k+1}} \left\|\widetilde{F}_k - \left(\widetilde{\beta}_k\right)_*|_{(s-t_k)\widetilde{F}_k}\widetilde{F}_k\right\| ds\right), \tag{3.33}$$

where L is the Lipschitz constant of $\widetilde{F}$ as defined in Assumption 3.14, $\widetilde{F}_k = \widetilde{F}\left(\tilde{z}^{(\varepsilon,h)}(t_k)\right)$, and $\widetilde{\beta}_k = \widetilde{\beta}_{\tilde{z}^{(\varepsilon,h)}(t_k)}$. Now, since the domain D is compact, $\widetilde{\beta}_*$ is Lipschitz by Assumption 3.17, and since $\widetilde{\beta}_*|_0 = I_n$ as in Definition 3.12, where n is the dimension of D, then there exists $C > 0$ such that:

$$\left\|\widetilde{F}_k - \left(\widetilde{\beta}_k\right)_*|_{(s-t_k)\widetilde{F}_k}\widetilde{F}_k\right\| \le C\left(s - t_k\right), \tag{3.34}$$

and therefore, for all $t \in [t_i, t_{i'})$,

$$\left\|\tilde{x}(t) - \tilde{z}^{h}(t)\right\| \le e^{L(t_{i'}-t_i)}\left(\left\|\tilde{x}^{\varepsilon}(t_i) - \tilde{z}^{(\varepsilon,h)}(t_i)\right\| + \frac{1}{2}Ch\right). \tag{3.35}$$

Since D is compact, there exists a finite set of charts $\{(U_i,\varphi_i)\}_{i=1}^{\nu}$ such that $\{U_i\}_{i=1}^{\nu}$ form a cover of D. Without loss of generality, we can assume that U_i is closed for each $i \in \{1,\dots,\nu\}$, Let r_0 be defined by:

$$r_0 = \inf_{i\in\{1,\dots,\nu\}} \inf_{q\in\partial U_i} \sup_{j\neq i} \inf_{q'\in\partial U_j\cap U_i} \left\|\varphi_i(q) - \varphi_i(q')\right\|, \tag{3.36}$$

and note that $r_0 > 0$ since every point at a boundary of a chart is at a positive distance from another boundary, because the boundaries are closed and there are only a finite number of them. Then, for each $a \in \varphi_i(U_i)$ there exists a neighborhood of a with radius at least r_0 contained in some $\varphi_j(U_j)$, $j \neq i$.

Let (U_i,φ_i) be a chart. If $q, q' \in U_i$ then, given the curve $\alpha(t) = \varphi_i^{-1}\left((1-t)\varphi_i(q) + t\varphi_i(q')\right)$ for $t \in [0,1]$, there exists $C > 0$ such that:

$$d_j(q,q') \le \int_0^1 \sqrt{g\left(\dot{\alpha}(t),\dot{\alpha}(t)\right)}dt \le C\|\varphi_i(q) - \varphi_i(q')\|, \tag{3.37}$$

where C is in practice the supremum of $\|(\varphi_i)_*\|$, over all $i \in \{1, \dots, \nu\}$ and all points in the compact domain D.

Now we can finish the argument for executions on a manifold. Suppose that $x^\varepsilon(0)$ and $z^{(\varepsilon,h)}(0)$ are in the same chart and that $\left\|\tilde{x}^\varepsilon(0) - \tilde{z}^{(\varepsilon,h)}(0)\right\| \leq \delta$, then there exists $s > 0$ such that $x^\varepsilon|_{[0,s)}$ and $z^{(\varepsilon,h)}|_{[0,s)}$ are in the same chart. Hence, by Equation 3.35,

$$d\big(x^\varepsilon(s), z^{(\varepsilon,h)}(s)\big) \leq e^{LT}Ch + \delta. \tag{3.38}$$

Without loss of generality assume that either $x^\varepsilon(s)$ or $z^{(\varepsilon,h)}(s)$ is at the boundary of a chart. If h and δ are small enough then this distance is smaller than $\frac{r_0}{2}$, and in that case there exists $s' > s$ such that, for an interval $[s, s')$, both functions are again in the same chart domain U_i. Also note that, by a similar argument to the one used in Equation (3.23), $s' - s \geq \frac{r_0}{2L}$. Therefore, since there exists a lower bound in the time spent in each chart, given δ small enough we can be certain that both executions cross the same finite charts. If the executions cover N different charts in their execution then, by Equation 3.35, for each $t \in [0, T]$,

$$d\big(x^\varepsilon(t), z^{(\varepsilon,h)}(t)\big) \leq e^{LT}NC(\delta + h). \tag{3.39}$$

Now let us consider the case of relaxed domains in our original relaxed hybrid dynamical system $\mathcal{H}^\varepsilon$. Let $q \in D_j^\varepsilon$ be in the same chart (U, φ) as $p \in D_j$, and assume $\|\varphi(p) - \varphi(q)\| \leq \delta$. Then, if $z^{(\varepsilon,h)}$ is the discrete approximation starting at q, Equation (3.39) is satisfied for the distance between x^ε and $z^{(\varepsilon,h)}$ as long as they do not transition onto a strip.

Suppose that there exists t' such that $x^\varepsilon(t') \in G_{(j,j')}$ for some $(j, j') \in \mathcal{N}_j$, where $\mathcal{N}_j$ is as in Equation (3.8). Since x^ε is assumed orbitally stable, there exists δ small enough such that $z^{(\varepsilon,h)}(t'_h) \in G_{(j,j')}$ for some $t'_h \in [0, T]$. Let $k' \in \mathbb{N}$ be such that $t'_h \in [t_{k'}, t_{k'+1})$, where we remove the dependence of k' on h for notational convenience. Since by construction x^ε crosses the guard at a unique point and so does $z^{(\varepsilon,h)}$, then $t'_h \to t'$ as $h \to 0$. Thus, for each $\delta' > 0$ there exists h small enough such that $|t' - t_{k'+1}| \leq \delta' + h$.

Let us define the following times:

$$\begin{aligned} \sigma_m &= \min\big\{t_{k'+1}, t'\big\}, & \sigma_M &= \max\big\{t_{k'+1}, t'\big\}, \\ \omega_m &= \min\big\{t_{k'+2}, t' + \varepsilon\big\}, & \omega_M &= \max\big\{t_{k'+2}, t' + \varepsilon\big\}. \end{aligned} \tag{3.40}$$

Then on the interval $[0, \sigma_m)$ we can still use the bound in Equation (3.39). On the interval $[\sigma_m, \sigma_M)$ one execution has transitioned into a strip, while the other is still governed by the vector field on D_j. On the interval $[\sigma_M, \omega_m)$ both executions are inside the strip, and on the interval $[\omega_m, \omega_M)$ one execution has transitioned to a new domain, while the second is still on the strip. After time ω_M both executions are in a new domain, and we can repeat the process. Therefore, we need to find bounds for the distance between x^ε and $z^{(\varepsilon,h)}$ on each of these intervals.

Due to the compactness of each domain D_j, we know that the relaxed vector fields F_j^ε and their coordinate representations $\widetilde{F}_j^\varepsilon$ are bounded. Similarly, the pushforward of the relaxed

retractions $(\beta_j^\varepsilon)_*$ are also bounded. Then, using Equation (3.23) as we did in the proof of Theorem 3.36, together with Equation (3.39), there exists $C > 0$ such that:

$$\mu^\varepsilon\big(x^\varepsilon(\sigma_M), z^{(\varepsilon,h)}(\sigma_M)\big) \leq C(\delta + \delta' + h). \tag{3.41}$$

Also

$$\mu^\varepsilon\big(x^\varepsilon(\omega_m), z^{(\varepsilon,h)}(\omega_m)\big) \leq \mu^\varepsilon\big(x^\varepsilon(\sigma_M), z^{(\varepsilon,h)}(\sigma_M)\big) + 2\varepsilon, \tag{3.42}$$

and, using the same argument, there exists another $C > 0$ such that

$$\mu^\varepsilon\big(x^\varepsilon(\omega_M), z^{(\varepsilon,h)}(\omega_M)\big) \leq \mu^\varepsilon\big(x^\varepsilon(\omega_m), z^{(\varepsilon,h)}(\omega_m)\big) + C(\delta' + h), \tag{3.43}$$

because $|t' + \varepsilon - t_{k'+2}| \leq \delta' + 2h$ by the construction in Algorithm 3.3.

At this point the generalization to relaxed executions defined on $\mathcal{M}^\varepsilon$ and their discrete approximations follows by noting that they have the same initial condition, they perform a finite number of discrete jumps on any bounded interval, the number of discrete modes is finite, and that δ' can be chosen arbitrarily small. With that information we can construct a constant $C > 0$ such that, for each $t \in [0, T]$,

$$\mu^\varepsilon\big(x^\varepsilon(t), z^{(\varepsilon,h)}(t)\big) \leq C(\varepsilon + h), \tag{3.44}$$

therefore proving the theorem. □

Now we can state the main result in this Section, which is simply a Corollary of Theorems 3.36 and 3.37.

Corollary 3.38. *Let $\mathcal{H}$ be a hybrid dynamical system, as in Definition 3.13, and $\mathcal{H}^\varepsilon$ be its relaxation, as in Definition 3.26. Let $j \in \mathcal{J}$, $p \in D_j$, and x be the execution of $\mathcal{H}$ with initial condition p constructed by Algorithm 3.1, and let $I \subset [0, \infty)$ be a bounded interval contained in the domain of x. Also, let $z^{(\varepsilon,h)}$ be the discrete approximation with initial condition p constructed by Algorithm 3.3.*

If the following conditions are satisfied:

(1) *x has a finite number of mode transitions, or x is a Zeno execution that accumulates, as in Definition 3.31,*

(2) *x is orbitally stable, as in Definition 3.34,*

then

$$\lim_{\substack{\varepsilon \to 0 \\ h \to 0}} \rho_I^\varepsilon\big(x, z^{(\varepsilon,h)}\big) = 0. \tag{3.45}$$

Proof. Note that, by Theorem 3.36 together with the Triangle Inequality, this corollary is equivalent to prove that $\rho_I^\varepsilon\big(x^\varepsilon, z^{(\varepsilon,h)}\big) \to 0$ as both $\varepsilon, h \to 0$.

Hence we show that $\rho_I^\varepsilon\big(x^\varepsilon, z^{(\varepsilon,h)}\big)$ converges uniformly on h as $\varepsilon \to 0$. Using an argument similar to the one in the proof of Theorem 7.9 in [Rud64], proving the uniform convergence on h is equivalent to showing:

$$\lim_{h\to 0} \limsup_{\varepsilon \to 0} \rho_I^\varepsilon\big(x^\varepsilon, z^{(\varepsilon,h)}\big) = 0 \tag{3.46}$$

but this is clearly true by Theorem 3.37, therefore obtaining our desired result. □

3.4 Implementation and Example

In this section, we describe the information required to implement Algorithm 3.3, and then present an example.

To simplify the exposition, we will consider the case where the hybrid system is comprised of a single domain and a single guard, i.e. $\mathcal{J} = \{1\}$, $\Gamma = \{(1,1)\}$, $\mathcal{D} = \{D\}$, $\mathcal{B} = \{\beta\}$, $\mathcal{F} = \{F\}$, $\mathcal{G} = \{G\}$, and $\mathcal{R} = \{R\}$. There is no loss of generality in specializing the discussion to this particular case, as the only difference with the general case is the choice of proper guard before performing a relaxed mode transition.

There are two main issues to consider before implementing our numerical scheme. First, the algorithm needs a collection of charts whose domains form a cover of D, and a way to determine which charts contain a given point in the domain. Since D is compact, only a finite number of charts is required. Note that if D admits a single chart, as in the case of D being a subset of $\mathbb{R}^n$ where the chart is trivial, then the implementation is greatly simplified. Second, in some problems the boundary of the manifold is not described as the preimage of the set $\{(x_1, \ldots, x_n) \mid x_n = 0\}$ under a boundary chart, but rather as the zero section of a smooth function $\lambda : D \to \mathbb{R}$, i.e. $\partial D = \lambda^{-1}(0)$ and $\lambda_*(x) \neq 0$ for all $x \in \partial D$. In this case, since D is compact, and given ε sufficiently small, the value of λ can be used as the transverse coordinate on the strip $S^\varepsilon = G \times [0, \varepsilon]$, which can be used in place of the boundary charts.

We illustrate the implementation[1] of our numerical scheme to approximate trajectories of a double pendulum with a mechanical stop. Figure 3.10 shows the illustration of the double pendulum and the parameters of the problem. Prior to an impact with the mechanical stop, i.e. while $\theta_2 > 0$ and the system is *unconstrained*, the system has two angular degrees of freedom, $q = (\theta_1, \theta_2) \in \mathbb{R}^2$ and the dynamics are Lagrangian, i.e. they have the form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \frac{\partial V}{\partial q}(q) = 0, \tag{3.47}$$

where $M(q) \in \mathbb{R}^{2\times 2}$ is the mass matrix, $C(q, \dot{q}) \in \mathbb{R}^{2\times 2}$ is the Coriolis matrix, and $V(q) \in \mathbb{R}$ is the potential energy. We refer the reader to [OA09] for the explicit expressions of these functions. When the second link collides with the mechanical stop, i.e. when $\theta_2 = 0$, the

[1] Code is available at `http://purl.org/sburden/cdc2011`

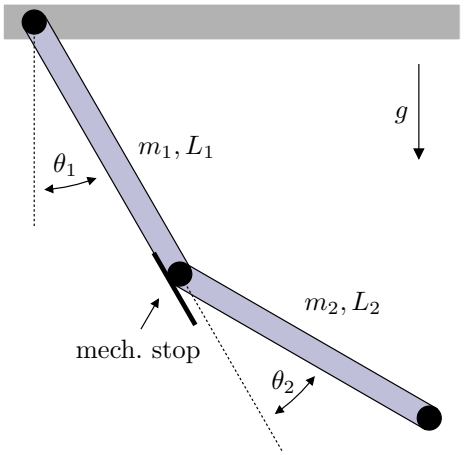


Figure 3.10: Planar double pendulum with mechanical stop. θ_1 gives the angle of the first link with respect to vertical, and θ_2 gives the angle of the second link with respect to the first. The link masses are m_1 and m_2, and their lengths are L_1 and L_2. A gravitational force points downward with constant g.

system becomes *constrained*. During the impact the velocities are updated according to the following reset map:

$$\left(\dot{\theta}_1, \dot{\theta}_2\right) \mapsto \left(\dot{\theta}_1 - (1+c)\dot{\theta}_2 \frac{\left(M(q)^{-1}\right)_{1,2}}{\left(M(q)^{-1}\right)_{2,2}}, -c\,\dot{\theta}_2\right), \tag{3.48}$$

where $c \in [0,1]$ is the *coefficient of restitution*. After impact, the system is re-initialized in a different discrete mode depending on the value of this constant. If $c > 0$, the system "bounces", i.e. it is reset to the unconstrained mode and simulation continues as before. If $c = 0$, the system enters the constrained state until the virtual force required to enforce the constraint $\theta_2 = 0$, defined by:

$$\lambda(q, \dot{q}) = -\frac{\left(M(q)^{-1}\right)_{2,\cdot}}{\left(M(q)^{-1}\right)_{2,2}} \left(C(q,\dot{q})\dot{q} + \frac{\partial V}{\partial q}(q)\right). \tag{3.49}$$

becomes non-positive.

As shown in [OA09], when $\theta_2 = 0$ either $\lambda > 0$ and $\ddot{\theta}_2 = 0$ (i.e. the constraint is maintained), or $\lambda = 0$ and $\ddot{\theta}_2 > 0$ (i.e. the system transitions to unconstrained motion). Thus the description of the hybrid dynamics of the system is consistent.

An illustration of the execution with different values for the coefficient restitution are shown in Figures 3.11 and 3.12. Observe that in either instance there is an epsilon sized delay due to the addition of the strip. In particular, notice that in the case of Zeno the strips begin to accumulate.

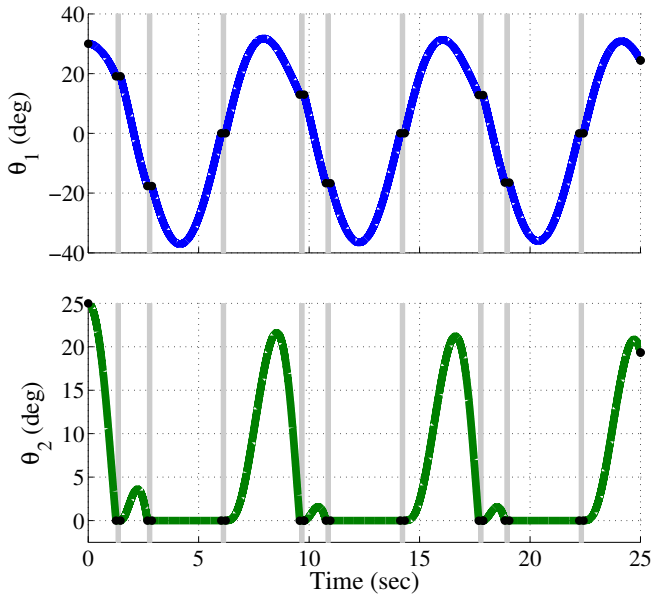


Figure 3.11: Trajectory of double pendulum with $c = 0$, $h = 0.001$, $\varepsilon = 0.2$, and initial condition $\left(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2\right) = (30°, 25°, 0, 0)$. Vertical gray bars indicate when the simulation resides in the strip.

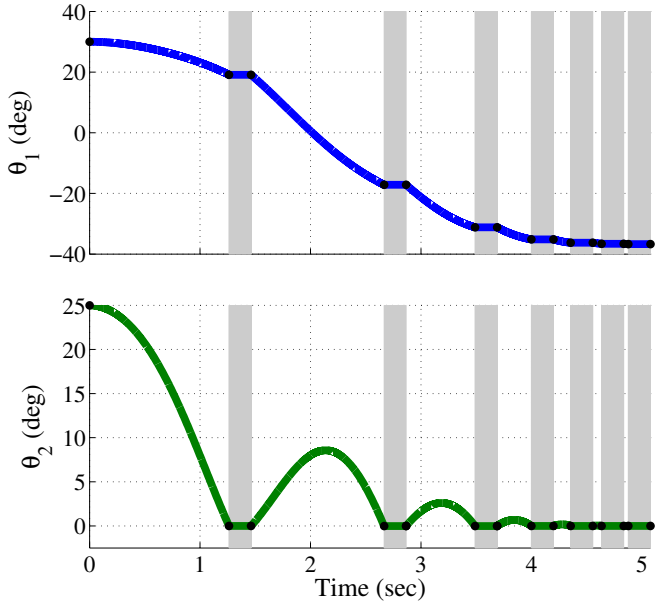


Figure 3.12: Zeno trajectory of double pendulum with $c = 0.5$, $h = 0.001$, $\varepsilon = 0.2$, and initial condition $\left(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2\right) = (30°, 25°, 0, 0)$. Vertical gray bars indicate when the simulation resides in the strip.

Chapter 4

Pseudospectral Approximations for Optimal Control

Computational tools for the calculation of the optimal control of a continuous–time dynamical system using nonlinear programming have existed for a long time in the literature [CCP70]. They have allowed us to address general forms of optimal control problems, ranging from state–constrained problems [Pol97] to receding–horizon control problems [May+00; SB00], including many important applications such as path planning in aerospace control [Tre12].

On the other hand, nonlinear programming lacks the nice theoretical properties of convex programming, namely the ability to always find a global minimizer, and the existence of polynomial–time algorithms that solve most interesting classes of convex programming problems [BV04; Kar84; Roc70]. But from an optimal control perspective we do not need to focus purely on the complexity of the numerical nonlinear programming algorithms, since what is relevant is the interaction between the discretization scheme, used to transform the optimal control problem into a nonlinear programming problem, and the numerical algorithm used to solve such nonlinear programming problem. For example, it is a well known fact that using a Forward Euler scheme for the discretization of the optimal control problem leads to a set of equality constraints whose derivatives are sparse, thus greatly increasing the speed of computation when used with sparse nonlinear programming solvers [Bet10; Pol97]. Similarly, research has also been done to study the interaction of the Runge–Kutta method applied to the discretization of optimal control problems [SP96].

In recent years, a relatively new family of techniques, called pseudospectral methods, originally used to approximate partial differential equations [Can+88; For96; ST89; Ste73], found a strong support in the optimal control community [Ben05; EKR95; Hun07; RF03]. The main feature of pseudospectral methods is that, for smooth functions, they have a convergence rate that is faster than any polynomial [SW05]. This result is impressive when compared to Forward Euler and Runge–Kutta schemes, which have a polynomial rate of convergence. From an implementation point of view, a faster rate of convergence means that less samples are required to achieve a desired error level. For the discretization of optimal

control problems it has been found experimentally that one can get a reduction of an order of magnitude in the number of samples used to solve a particular problem (for example, see Table 1 in [GKR06]). This last fact cannot be overlooked, since the time it takes to compute each iteration in a nonlinear programming solver is a (usually polynomial) function of the number of variables in the problem.

A fundamental question must be asked when an infinite dimensional optimization problem is approximated by a finite dimensional problem, such as the case with optimal control problems being approximated by nonlinear programming problems: do the global minimizers of the finite dimensional problems converge to global minimizers of the original infinite dimensional problem as the discretization becomes more accurate? Similarly, we can ask the analogous question about the convergence of local minimizers. The former question can be addressed using the concept of Epi–convergence, as explained in Chapter 7 in [RW98], while the latter can be addressed using the concept of Consistent Approximations, defined in Chapter 3 in [Pol97].

For pseudospectral methods, the convergence of global minimizers has been addressed in the literature. In [KRG07] the authors present an overview of the state of the art in terms of convergence of global minimizers. The main result in that paper states that, whenever the underlying dynamical system has a single input, is affine in the control, has relative degree n (see Definition 9.1 in [Sas99] for a definition of relative degree), and its optimal control is smooth, then the global minimizers converge. The problem of global minimizer convergence when the optimal control is not smooth is address in [KGR05; RF04]. Both papers prove the convergence for particular classes of dynamical systems and problem formulations, the former for feedback–linearizable nonlinear systems, and the latter for problems where one can find the discontinuities in the control a priori. In [Gon+07] the authors prove the convergence of global minimizers of both primal and dual pseudospectral approximations, also for the case when the optimal control is smooth. In [RZL11] the authors show that a L^2–norm based discretization, as opposed to the standard L^∞–norm based discretization usually used in the literature, provides convergence of the value function of the optimal control problem, provided the optimal control itself is smooth. All these results show that this topic has been an active area of research in recent years, but the general problem of convergence of local minimizers for general nonlinear systems, and more importantly, without assumptions about the smoothness of the optimal control, remains open.

In this chapter we take the first step towards this result. Here we show that, for a particular pseudospectral approximation of the optimal control problem, we can prove the convergence of global minimizers whenever the optimal control is of bounded variation. Our formulation is special because we decouple the discretization of the signals, i.e. the state and control, from the discretization of the differential equation, eventually showing that the differential equation needs to be sampled at a higher rate than the signals. Also, we are hopeful that this new formulation will allow us in the future to prove the convergence of local minimizers, and at the same time formulate new adaptive algorithms that iterate both on the optimization space and on the discretization level.

4.1 Preliminaries

We begin this section by introducing the functional spaces that are the foundation to our optimization space. Throughout this chapter every integral will be computed with respect to the Lebesgue measure.

Given $n \in \mathbb{N}$, we endow the real vector space $\mathbb{R}^n$ with the 2-norm, thus making $(\mathbb{R}^n, \|\cdot\|_2)$ a normed vector space. Consider $f\colon [-1,1] \to \mathbb{R}^n$. We say that $f \in L^2([-1,1],\mathbb{R}^n)$ if

$$\|f\|_{L^2} = \left(\int_{-1}^{1} \|f(t)\|_2^2 \, dt\right)^{\frac{1}{2}} < \infty. \tag{4.1}$$

A key feature of $L^2([-1,1],\mathbb{R}^n)$ is that it is a Hilbert space together with the following dot product:

$$(f,g) \mapsto \langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt, \tag{4.2}$$

and note that $\sqrt{\langle f, f\rangle} = \|f\|_{L^2}$.

Also, we say that $f \in L^\infty([-1,1],\mathbb{R}^n)$ if

$$\|f\|_{L^\infty} = \inf\{\alpha \in [0,\infty) \mid \|f(t)\|_2 \leq \alpha \text{ for almost every } t \in [-1,1]\} < \infty. \tag{4.3}$$

Note that the definition in Equation (4.3) is very similar to the definition of supremum. This similarity motivates most authors to define it as the *essential supremum*, hence:

$$\|f\|_{L^\infty} = \operatorname{ess\,sup}\{\|f(t)\|_2 \mid t \in [-1,1]\}. \tag{4.4}$$

Let P be the set of finite partitions of $[-1,1]$. We define the *total variation of* f by:

$$\mathrm{V}(f) = \sup\left\{\sum_{i=0}^{k-1} \|f(t_{i+1}) - f(t_i)\|_1 \mid k \in \mathbb{N},\ \{t_i\}_{k=0}^{k} \in P\right\}, \tag{4.5}$$

and we say that f is of *bounded variation* if $\mathrm{V}(f) < \infty$. Moreover, we define $BV([-1,1],\mathbb{R}^n)$ as the vector space of all functions of bounded variation.

As we did in Chapter 2, we now introduce a very important relation between functions of bounded variation and the existence of weak derivatives. We say that f has a *weak derivative* if there exists a Radon signed measure μ over $[-1,1]$ such that, for each smooth bounded function v with $v(-1) = v(1) = 0$,

$$\int_{-1}^{1} f(t)\dot{v}(t)dt = -\int_{-1}^{1} v(t)d\mu(t). \tag{4.6}$$

Moreover, we say that $\dot{f} = \frac{d\mu}{dt}$, where the derivative is taken in the Radon–Nikodym sense, is the weak derivative of f. Note that $\dot{f}$ is in general a distribution, thus it only makes sense as an element in the dual space of L^1, when considering L^1 as a Banach space. Perhaps the most common example of weak derivative is the Dirac Delta distribution, which is the weak derivative of the Step Function. The following result is fundamental in our analysis of functions of bounded variation:

Theorem 4.1 (Exercise 5.1 in [Zie89]). *If $f \in BV([-1,1],\mathbb{R}^n)$, then f has a weak derivative, denoted $\dot{f}$. Moreover,*

$$\mathrm{V}(f) = \int_{-1}^{1} \left\|\dot{f}(t)\right\|_1 dt. \tag{4.7}$$

We omit the proof of this result since it is beyond the scope of this chapter. More details about the functions of bounded variation and weak derivatives can be found in Sections 3.5 and 9 in [Fol99] and Section 5 in [Zie89].

We will consider optimal control problems in which the dynamical system is defined by a controlled vector field $h\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ as follows,

$$\dot{x}(t) = h\big(x(t),u(t)\big), \quad \text{for almost every } t \in [-1,1], \quad x(-1) = \xi, \tag{4.8}$$

where $\xi \in \mathbb{R}^n$ is the *initial condition*, $u\colon [-1,1] \to \mathbb{R}^m$ is the *control*, and $x\colon [-1,1] \to \mathbb{R}^n$ is the *state* of the dynamical system. To ensure the existence and uniqueness of the solutions of Differential Equation (4.8) we make the following assumption.

Assumption 4.2. *The function h is differentiable in both x and u. Also, h and its partial derivatives are Lipschitz continuous in both of their arguments, i.e. there exists $L > 0$ such that for each $x_1, x_2 \in \mathbb{R}^n$ and $u_1, u_2 \in \mathbb{R}^m$,*

(1) $\|h(x_1,u_1) - h(x_2,u_2)\|_2 \le L\,(\|x_1 - x_2\|_2 + \|u_1 - u_2\|_2)$.

(2) $\left\|\frac{\partial h}{\partial x}(x_1,u_1) - \frac{\partial h}{\partial x}(x_2,u_2)\right\|_2 \le L\,(\|x_1 - x_2\|_2 + \|u_1 - u_2\|_2)$.

(3) $\left\|\frac{\partial h}{\partial u}(x_1,u_1) - \frac{\partial h}{\partial u}(x_2,u_2)\right\|_2 \le L\,(\|x_1 - x_2\|_2 + \|u_1 - u_2\|_2)$.

We say that f is *absolutely continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite set of disjoint intervals $\{(a_i,b_i)\}_{i=1}^k \subset [-1,1]$,

$$\sum_{i=1}^{k} (b_i - a_i) < \delta \quad \Rightarrow \quad \sum_{i=1}^{k} \|f(b_i) - f(a_i)\|_2 < \varepsilon. \tag{4.9}$$

We also define the vector space of all absolutely continuous functions from $[-1,1]$ to $\mathbb{R}^n$ by $\mathcal{AC}([-1,1],\mathbb{R}^n)$. More details about the definition of absolutely continuous functions and their properties can be found in Section 3.5 in [Fol99].

We state the following theorem without proof since it is a standard result in the literature. A standard version of this proof for existence and uniqueness can be found in Section 2.4.1 in [Vid02]. A version of the proof of absolute continuity can be found in Theorem 3.35 in [Fol99].

Theorem 4.3. *Let $\xi \in \mathbb{R}^n$ and $u\colon [-1,1] \to \mathbb{R}^m$ such that $\|u\|_{L^\infty} < \infty$. Then, Differential Equation (4.8) has a unique solution $x\colon [-1,1] \to \mathbb{R}^n$. Moreover, x is absolutely continuous.*

Whenever we want to make explicit the relation between the pair (ξ, u) and the solution of Differential Equation (4.8), we write its unique solution by $x^{(\xi,u)}$.

We define the *set of admissible controls* by:

$$\mathcal{U} = \left\{ u \in L^2([-1,1],\mathbb{R}^m) \cap BV([-1,1],\mathbb{R}^m) \mid \sup_{t\in[-1,1]} \|u(t)\|_2 < \infty \right\}, \tag{4.10}$$

and we endow it with the L^2-norm, as defined in Equation (4.1). From Equation (4.10) it is clear that $\mathcal{U} \subset L^\infty([-1,1],\mathbb{R}^m) \subset L^2([-1,1],\mathbb{R}^m)$.

Lemma 4.4. *Let $\xi \in \mathbb{R}^n$, $u \in \mathcal{U}$, and let x be the unique solution of Differential Equation (4.8) with initial condition ξ and control u. Then $\dot{x} \in BV([-1,1],\mathbb{R}^n)$.*

Proof. From Condition 1 in Assumption 4.2 we know that there exists $C > 0$ such that, for each $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$:

$$\|h(x,u)\|_2 \leq C(\|x\|_2 + \|u\|_2 + 1), \tag{4.11}$$

and that same condition together with the definition of total variation in Equation (4.5) imply that:

$$\mathrm{V}(\dot{x}) \leq L\big(\mathrm{V}(x) + \mathrm{V}(u)\big). \tag{4.12}$$

Now, it follows by Hölder's Inequality, together with Condition 1 in Assumption 4.2 and Theorem 4.1, that:

$$\mathrm{V}(x) = \int_{-1}^{1} \big\|h\big(x(t),u(t)\big)\big\|_2 \, dt \leq \sqrt{2}L\big(\|x\|_{L^2} + \|u\|_{L^2} + 1\big). \tag{4.13}$$

The conclusion follows since $x \in L^2([-1,1],\mathbb{R}^n)$, because it is a continuous function in a bounded domain, and $u \in L^2([-1,1],\mathbb{R}^m) \cap BV([-1,1,\mathbb{R}^m)$ by definition of $\mathcal{U}$ in Equation (4.10). □

Lemma 4.4 and Theorem 4.3 introduce a natural definition for our space of solutions of Differential Equation (4.8). We define the space that contains the solutions of Differential Equation (4.8) with inputs taken from the set $\mathcal{U}$ and initial conditions from $\mathbb{R}^n$ by:

$$\mathcal{X} = \left\{ x \in \mathcal{AC}([-1,1],\mathbb{R}^n) \mid \dot{x} \in BV([-1,1],\mathbb{R}^n), \text{ and } \sup_{t\in[-1,1]} \|\dot{x}(t)\|_2 < \infty \right\}. \tag{4.14}$$

and we endow $\mathcal{X}$ with the following norm:

$$\|x\|_{\mathcal{X}} = \|x\|_{L^\infty} + \|\dot{x}\|_{L^2}. \tag{4.15}$$

Note that, since $x \in \mathcal{X}$ is a continuous function, $\|x\|_{L^\infty} = \sup_{t\in[-1,1]} \|x(t)\|_2$.

Polynomial Approximations

Let $N \in \mathbb{N}$. We say that $\{t_{N,k}\}_{k=0}^{N} \subset [-1,1]$ is a set of *collocation nodes* if

$$-1 \leq t_{N,0} < t_{N,1} < \cdots < t_{N,N} \leq 1. \tag{4.16}$$

Given a set of collocation nodes we can define a unique set of *Lagrange interpolation polynomials* of order N, denoted by $\{b_{N,k}\}_{k=0}^{N}$, $b_{N,k}\colon [-1,1] \to \mathbb{R}$, satisfying the following condition for each $k, j \in \{0,\ldots,N\}$:

$$b_{N,k}(t_{N,j}) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

It is not hard to conclude from Equation (4.17) that, for each $k \in \{0,\ldots,N\}$,

$$b_{N,k}(t) = \frac{\prod_{i\neq k}(t - t_{N,i})}{\prod_{i\neq k}(t_{N,k} - t_{N,i})} \tag{4.18}$$

Note that $\{b_{N,k}\}_{k=0}^{N}$ is linearly independent, hence it is a basis for the vector space of polynomials of order N.

Furthermore, given a set of collocation nodes $\{t_{N,k}\}_{k=0}^{N}$ there exists a unique set of reals $\{\omega_{N,k}\}_{k=0}^{N}$ such that the Gauss quadrature integral approximation is exact for polynomials of order at most N, i.e. such that for each polynomial p of order at most N,

$$\int_{-1}^{1} p(t)dt = \sum_{k=0}^{N} \omega_{N,k}\, p(t_{N,k}). \tag{4.19}$$

The proof of uniqueness of the constants $\{\omega_{N,k}\}_{k=0}^{N}$ is outside the scope of this chapter, but we will mention that, in practice,

$$\omega_{N,k} = \int_{-1}^{1} b_{N,k}(t)dt. \tag{4.20}$$

More details about these results can be found in Sections 2.7.1 and 2.7.2 in [DR84], and in [Pat68].

At this point we can construct a subset of $\mathcal{X}$ consisting of polynomial approximations of its functions in the following way. Define the set $\mathcal{X}_N$ as the subset of $\mathcal{X}$ containing all functions that are a linear combination of the polynomials $(b_{N,k})_{k=0}^{N}$, i.e.

$$\mathcal{X}_N = \Big\{ x \in \mathcal{X} \mid x_i \in \operatorname{span}\{b_{N,k}\}_{k=0}^{N} \text{ for each } i \in \{1,\ldots,n\} \Big\}. \tag{4.21}$$

In a similar way, we can define $\mathcal{U}_N$ by

$$\mathcal{U}_N = \Big\{ u \in \mathcal{U} \mid u_i \in \operatorname{span}\{b_{N,k}\}_{k=0}^{N} \text{ for each } i \in \{1,\ldots,m\} \Big\}. \tag{4.22}$$

Definition 4.5. *Let $\{t_{N,k}\}_{k=0}^{N}$ be a set of collocation nodes. We define the* interpolation operator*, denoted I_N, such that when applied to a function f, it returns following unique polynomial:*

$$(I_N f)(t) = \sum_{k=0}^{N} f(t_{N,k}) b_{N,k}(t). \tag{4.23}$$

It is clear from Definition 4.5 that $(I_N f)(t_{N,k}) = f(t_{N,k})$ for each $k \in \{0, \ldots, N\}$.

Since the derivative of a polynomial of degree N is another polynomial of degree $N-1$, such derivative can also be written as an expansion of the basis of polynomials of order smaller than N formed by $\{b_{N,k}\}_{k=0}^{N}$. Indeed, there exists a unique set of constants $\{d_{N,k,j}\}_{k,j=0}^{N}$ such that, for each $x_N = \sum_{k=0}^{N} \bar{x}_k b_{N,k} \in \mathcal{X}_N$,

$$\dot{x}_N(t) = \sum_{k=0}^{N} \left(\sum_{j=0}^{N} d_{N,k,j}\, \bar{x}_j \right) b_{N,k}(t). \tag{4.24}$$

Note that if we write $\dot{x}_N = \sum_{k=0}^{N} \bar{y}_k\, b_{N,k}$ then, given $D_N \in \mathbb{R}^{(n+1)\times(n+1)}$ defined by $[D_N]_{k,j} = d_{N,k,j}$, we have that:

$$\begin{pmatrix} \bar{y}_0 \\ \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{pmatrix} = D_N \begin{pmatrix} \bar{x}_0 \\ \bar{x}_1 \\ \vdots \\ \bar{x}_N \end{pmatrix} \tag{4.25}$$

As we mention above, the constants $\{\omega_{N,k}\}_{k=0}^{N}$ and $\{d_{N,k,j}\}_{k,j=0}^{N}$, as well as the polynomials $\{b_{N,k}\}_{k=0}^{N}$, are completely determined by the choice of collocation nodes $\{t_{N,k}\}_{k=0}^{N}$. There are several, well established, choices for the collocation nodes based on the roots of orthogonal polynomials for some measure in $[-1, 1]$. We present some of these choices of collocation nodes, and their associated coefficients for approximate integrals and derivatives, in the next section.

Types of Collocation Nodes

As we mentioned above, given a set of collocation nodes $\{t_{N,k}\}_{k=0}^{N}$ we get three unique sets that are key parts of pseudospectral approximations: a collection of Lagrange polynomials $\{b_{N,k}\}_{k=0}^{N}$ that define the interpolation operator, a collection of coefficients $\{\omega_{N,k}\}_{k=0}^{N}$ that define the integral approximation using Gauss quadrature, and a collection of coefficients $\{d_{N,k,j}\}_{k,j=0}^{N}$ that define the derivative matrix for polynomials of order N.

Before we present more details we need to define the set of Legendre polynomials. We say that $\{p_k\}_{k=0}^{\infty}$, where $p_k : [-1, 1] \to \mathbb{R}$ for each $k \in \{0, \ldots, N\}$, is the set of *Legendre polynomials*, defined by the following recursion:

$$(k+1)\, p_{k+1}(t) = (2k+1)\, t\, p_k(t) - k\, p_{k-1}(t), \quad \forall k \geq 2. \tag{4.26}$$

where $p_0(t) = 1$ and $p_1(t) = t$. Among the many properties of Legendre polynomials, the most important one from our point of view is that they form an orthogonal basis for $L^2([-1,1],\mathbb{R})$, using the dot product defined in Equation (4.2).

We now present two types of collocation nodes that are particularly interesting from an implementation point of view since they are derived using integrals with the Lebesgue measure. In particular, the Lebesgue–Gauss–Lobatto rule is commonly used to approximate differential equations in optimal control problems since it includes collocation nodes at $t = -1$ and $t = 1$.

The derivations of the formulas included below is outside the scope of this chapter. Moreover, to the best of our knowledge, the details of the derivations of these formulas cannot be found in a single publication. Nevertheless, they can be collected from [Can+88; Can+06; DR84; For96; Fun92; STW11].

Legendre–Gauss (LG)

Set of collocation nodes:

$$\{t_k\}_{k=0}^{N} = \big\{t \in [-1,1] \mid p_{N+1}(t) = 0\big\}. \tag{4.27}$$

Lagrange polynomials:

$$b_{N,k}(t) = \frac{p_{N+1}(t)}{\big(t - t_{N,k}\big)\dot{p}_{N+1}(t_{N,k})}. \tag{4.28}$$

Gauss quadrature coefficients:

$$\omega_{N,k} = \frac{2}{\big(1 - t_{N,k}^2\big)\big(\dot{p}_{N+1}(t_{N,k})\big)^2}. \tag{4.29}$$

Derivative matrix coefficients:

$$d_{N,k,j} = \begin{cases} \dfrac{\dot{p}_{N+1}(t_{N,k})}{\big(t_{N,k} - t_{N,j}\big)\dot{p}_{N+1}(t_{N,j})} & \text{if } k \neq j, \\ \dfrac{t_{N,k}}{1 - t_{N,k}^2} & \text{if } k = j. \end{cases} \tag{4.30}$$

Legendre–Gauss–Lobatto (LGL)

Set of collocation nodes:

$$\{t_k\}_{k=0}^{N} = \big\{t \in [-1,1] \mid (1 - t^2)\dot{p}_N(t) = 0\big\}. \tag{4.31}$$

Lagrange polynomials:

$$b_{N,k}(t) = \frac{(t^2 - 1)\dot{p}_N(t)}{N(N+1)\big(t - t_{N,k}\big)p_N(t_{N,k})}. \tag{4.32}$$

Gauss quadrature coefficients:

$$\omega_{N,k} = \frac{2}{N(N+1)\big(p_N(t_{N,k})\big)^2}. \tag{4.33}$$

Derivative matrix coefficients:

$$d_{N,k,j} = \begin{cases} \dfrac{p_N(t_{N,k})}{\big(t_{N,k} - t_{N,j}\big)p_N(t_{N,j})} & \text{if } k \neq j, \\ -\dfrac{N(N+1)}{4} & \text{if } k = j = 0, \\ \dfrac{N(N+1)}{4} & \text{if } k = j = N, \\ 0 & \text{if } k = j \in \{1, \ldots, N-1\}. \end{cases} \tag{4.34}$$

Optimal Control Problem Definition

Since we need to relate the solutions of the optimal control problem with the approximation using pseudospectral methods, we find it convenient to define the optimal control problem of interest in an unconventional form. Given the functions $f^0 \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and $f^j \colon \mathbb{R}^n \to \mathbb{R}$ for each $j \in \{1, \ldots, q\}$, we define the optimal control problem P_0 as follows:

$$\begin{aligned} (P_0) \qquad \min\Big\{ & f^0\big(\xi, x(1)\big) \mid \xi \in \mathbb{R}^n,\ x \in \mathcal{X},\ u \in \mathcal{U}, \\ & \int_{-1}^{1} \big\| \dot{x}(t) - h\big(x(t), u(t)\big) \big\|_2^2 \, dt = 0,\ x(-1) = \xi, \\ & f^j\big(x(t)\big) \leq 0,\ \forall t \in [-1, 1],\ \forall j \in \{1, \ldots, q\} \Big\}. \end{aligned} \tag{4.35}$$

Note that the main difference with the standard optimal control notation lies in our formulation for the differential equation constraint. The formulation for the differential equation constraint in Equation (4.35) is completely equivalent to the original formulation in Equation (4.8), thus our decision is purely based on our desire to simplify the understanding of our pseudospectral approximation for Problem P_0. Also, to simplify our notation, we define $\mathcal{F}_0 \subset \mathbb{R}^n \times \mathcal{X} \times \mathcal{U}$ as the feasible set of Problem P_0, i.e. the set of all points satisfying the constraints in Equation (4.35).

We now introduce a Lipschitz continuity assumption for the cost and state constraint functions in Equation (4.35).

Assumption 4.6. *For each $j \in \{0, \ldots, q\}$, f^j is differentiable. Also, each f^j and its partial derivatives are Lipschitz continuous in both of their arguments, i.e. there exists $L > 0$ such that for each $\xi_1, \xi_2, x_1, x_2 \in \mathbb{R}^n$ and $j \in \{1, \ldots, q\}$,*

(1) $\|f^0(\xi_1, x_1) - f^0(\xi_2, x_2)\|_2 \leq L\left(\|\xi_1 - \xi_2\|_2 + \|x_1 - x_2\|_2\right)$.

(2) $\left\| \frac{\partial f^0}{\partial \xi}(\xi_1, x_1) - \frac{\partial f^0}{\partial \xi}(\xi_2, x_2) \right\|_2 \leq L \left(\|\xi_1 - \xi_2\|_2 + \|x_1 - x_2\|_2 \right)$.

(3) $\left\| \frac{\partial f^0}{\partial x}(\xi_1, x_1) - \frac{\partial f^0}{\partial x}(\xi_2, x_2) \right\|_2 \leq L \left(\|\xi_1 - \xi_2\|_2 + \|x_1 - x_2\|_2 \right)$.

(4) $\|f^j(x_1) - f^j(x_2)\|_2 \leq L \|x_1 - x_2\|_2$.

(5) $\left\| \frac{\partial f^j}{\partial x}(x_1) - \frac{\partial f^j}{\partial x}(x_2) \right\|_2 \leq L \|x_1 - x_2\|_2$.

We will restrict our analysis to problems where the minimum is achieved, and therefore the feasible set is not empty. This condition is summarized in the following assumption.

Assumption 4.7. *There exists a triple* $(\xi^*, x^*, u^*) \in \mathcal{F}_0$ *such that for each* $(\xi, x, u) \in \mathcal{F}_0$,

$$f^0\big(\xi^*, x^*(1)\big) \leq f^0\big(\xi, x(1)\big). \tag{4.36}$$

Using the sets $\mathcal{X}_N$ and $\mathcal{U}_N$, as defined in Equations (4.21) and (4.22), we can define the following pseudospectral approximation of Problem P_0. Let $N \in \mathbb{N}$, $\{t_{N,k}\}_{k=0}^N$ a set of collocation nodes as in Equation (4.16), and $\{\omega_{N,k}\}_{k=0}^N$ its related Gauss quadrature coefficients, as defined in Equation (4.19). Moreover, let $\delta > 0$, $M = \lceil N^{4+\delta} \rceil$, and $\{\pi_k\}_{k=0}^\infty$, a strictly increasing divergent sequence. We define our pseudospectral approximation of Problem P_0, as defined in Equation (4.35), as follows:

$$\begin{aligned}(P_{N,\delta}) \ \min \Bigg\{ & f^0\big(\xi, x(1)\big) + \pi_N \varepsilon \mid \xi \in \mathbb{R}^n,\ x \in \mathcal{X}_N,\ u \in \mathcal{U}_N,\ \varepsilon \in [0, \infty), \\ & \sum_{k=0}^{M} \omega_{M,k} \left\| \dot{x}(t_{M,k}) - h\big(x(t_{M,k}), u(t_{M,k})\big) \right\|_2^2 \leq \left(N^{-\frac{\delta}{3}} + \varepsilon\right)^2, \\ & x(-1) = \xi, \\ & f^j\big(x(t_{N,k})\big) \leq N^{-\frac{1}{3}} + \varepsilon,\ \forall k \in \{0, \ldots, N\},\ \forall j \in \{1, \ldots, q\} \Bigg\}\end{aligned} \tag{4.37}$$

Note that the time derivative of $x \in \mathcal{X}_N$ in Equation (4.37), $\dot{x}$, can be explicitly computed as explained in Equation (4.24). As we did with Problem P_0, we define $\mathcal{F}_{N,\delta} \subset \mathbb{R}^n \times \mathcal{X} \times \mathcal{U} \times [0, \infty)$ as the feasible set of Problem $P_{N,\delta}$, i.e. the set of all points satisfying the constraints in Equation (4.37).

4.2 Global Minimizer Convergence of Pseudospectral Approximations

In this section we show that for each $\delta > 0$, every sequence of global minimizers of Problems $P_{N,\delta}$, indexed by $N \in \mathbb{N}$, accumulates in set of global minimizers of Problem P_0.

Feasibility of Pseudospectral Approximation

We begin with a lemma that provides useful estimates for the norm and total variation of a solution of Differential Equation (4.8) given its initial condition and control.

Lemma 4.8. *There exists a constant $C > 0$ such that, for each tuple $(\xi, x, u) \in \mathcal{F}_0$:*

(1) $\|x\|_{L^\infty} \leq C\big(\|\xi\|_2 + \|u\|_{L^2} + 1\big)$,

(2) $\mathrm{V}(x) \leq C\big(\|\xi\|_2 + \|u\|_{L^2} + 1\big)$,

where $\mathcal{F}_0$ is the feasible set of Problem P_0, as defined in Equation (4.35).

Proof. Let $t \in [-1, 1]$. Then, by Differential Equation (4.8) and Condition 1 in Assumption 4.2,

$$\|x(t)\|_2 \leq \|\xi\|_2 + L \int_{-1}^{1} \big(\|x(s)\|_2 + \|u(s)\|_2 + 1\big) ds. \tag{4.38}$$

Thus, applying the Bellman-Gronwall Inequality (Lemma 5.6.4 in [Pol97]),

$$\|x(t)\|_2 \leq e^{2L}\big(\|\xi\|_2 + 2L\,\|u\|_{L^2} + 2L\big), \tag{4.39}$$

proving the inequality in Condition 1.

The inequality in Condition 2 follows after noting that, by Theorem 4.1, and as we computed in Equation (4.13) in the proof of Lemma 4.4:

$$\mathrm{V}(x) = \int_{-1}^{1} \big\|h\big(x(t), u(t)\big)\big\|\, dt \leq \sqrt{2}L\big(\|x\|_{L^2} + \|u\|_{L^2} + 1\big). \tag{4.40}$$

Therefore, using Equation (4.39) we obtain the desired result. □

The following lemma presents a rate of convergence for our polynomial approximation for functions of bounded variation.

Lemma 4.9. *There exists $N_0 \in \mathbb{N}$ and $C > 0$ such that, for each $f \in L^2([-1, 1], \mathbb{R}) \cap BV([-1, 1], \mathbb{R})$, we can find a polynomial p of order $N \geq N_0$ satisfying:*

$$\|f - p\|_{L^2} \leq C \frac{1}{\sqrt{N}}\, \mathrm{V}(f)\,, \tag{4.41}$$

where $\mathrm{V}(\cdot)$ is as defined in Equation (4.5).

Proof. Our proof will be divided in two parts. First, we prove that we can approximate f using a trigonometric polynomial using a truncated Fourier series expansion. Then, we approximate that trigonometric polynomial using its Taylor expansion.

Let $c_k\colon [-1,1] \to \mathbb{R}$ and $s_k\colon [-1,1] \to \mathbb{R}$ by $c_k(t) = \cos(2\pi kt)$ and $s_k(t) = \sin(2\pi kt)$. Note that the set $\{c_k, s_k\}_{k=1}^{\infty}$ forms an orthonormal basis for $L^2([-1,1],\mathbb{R})$ using the dot product defined in Equation (4.2). Then, by Theorem 5.27 in [Fol99],

$$f = \langle f, \mathbb{1}\rangle + \sum_{k=1}^{\infty} \langle f, c_k\rangle\, c_k + \langle f, s_k\rangle\, s_k. \tag{4.42}$$

where $\mathbb{1}$ is the function identical to one for all $t \in [-1,1]$. Let us define the truncated Fourier series expansion of order M for f by:

$$S_M f = \langle f, \mathbb{1}\rangle + \sum_{k=1}^{M} \langle f, c_k\rangle\, c_k + \langle f, s_k\rangle\, s_k. \tag{4.43}$$

Now, using the identity in Equation (4.6),

$$|\langle f, c_k\rangle| = \left| \int_{-1}^{1} \dot{f}(t) \frac{1}{2\pi k} \sin(2\pi kt) dt \right| \leq \frac{1}{2\pi k}\, \mathrm{V}(f)\,, \tag{4.44}$$

and using the same argument,

$$|\langle f, s_k\rangle| \leq \frac{1}{2\pi k}\, \mathrm{V}(f)\,. \tag{4.45}$$

Thus, by Parseval's Identity (Theorem 5.27 in [Fol99]),

$$\begin{aligned} \|f - S_M f\|_{L^2}^2 &= \sum_{k=M+1}^{\infty} |\langle f, c_k\rangle|^2 + |\langle f, s_k\rangle|^2 \\ &\leq \frac{1}{2\pi^2}\, \mathrm{V}(f)^2 \sum_{k=M+1}^{\infty} \frac{1}{k^2} \\ &\leq \frac{1}{2\pi^2}\, \mathrm{V}(f)^2\, \frac{1}{M}. \end{aligned} \tag{4.46}$$

Let us now consider the Taylor expansion of order N for $S_M f$ at $t = 0$, denoted $P_N(S_M f)$. By the Lagrange Remainder Formula (see Section 1.11.4 in [DR84] one of its expressions),

$$\begin{aligned} \|S_M f - P_N(S_M f)\|_{L^\infty} &\leq \frac{1}{(N+1)!} \left| \frac{d^{N+1} S_M f}{dt^{N+1}}(t') \right| \left| t^{N+1} \right| \\ &\leq \frac{1}{(N+1)!} \left\| \frac{d^{N+1} S_M f}{dt^{N+1}} \right\|_{L^\infty} \end{aligned} \tag{4.47}$$

where we used the fact that $\left|t^{N+1}\right| \leq 1$ for each $t \in [-1,1]$, and $t' \in (-1,1)$. But note that, using Equations (4.44) and (4.45),

$$\begin{aligned}\left|\frac{d^{N+1}}{dt^{N+1}}(S_M f)(t)\right| &= \left|\sum_{k=1}^{M} \langle f, c_k\rangle \frac{d^{N+1}c_k}{dt^{N+1}}(t) + \langle f, s_k\rangle \frac{d^{N+1}s_k}{dt^{N+1}}(t)\right| \\ &\leq \sum_{k=1}^{M} |\langle f, c_k\rangle| (2\pi k)^{N+1} + |\langle f, s_k\rangle| (2\pi k)^{N+1} \\ &\leq 2\,\mathrm{V}(f)\,(2\pi)^N \sum_{k=1}^{M} k^N \\ &\leq 2\,\mathrm{V}(f)\,(2\pi)^N M^{N+1}.\end{aligned} \tag{4.48}$$

Also, by Stirling's Approximation (Equation (17.42) in [CT06]),

$$\frac{1}{(N+1)!} < \frac{1}{\sqrt{2\pi(N+1)}} \left(\frac{e}{N+1}\right)^{N+1}. \tag{4.49}$$

Hence, from Equations (4.47), (4.47), and (4.49) we know that there exists $C' > 0$ such that

$$\|S_M f - P_N(S_M f)\|_{L^2} \leq C'\,\mathrm{V}(f) \left(\frac{2\pi e M}{N+1}\right)^{N+1}. \tag{4.50}$$

Finally, given $N \geq (3\pi e - 1)$, if we choose $M = \left\lfloor \frac{N+1}{3\pi e} \right\rfloor$ then by the Triangle Inequality and Equations (4.46) and (4.50) we get the desired result. □

Note that, from the proof of Lemma 4.9 we can conclude that a good estimate for N_0 is 25, which is important from an implementation point of view.

Besides providing a clear bound for the rate of convergence of polynomial approximations to functions of bounded variation, Lemma 4.9 also proves that $\bigcup_{N\in\mathbb{N}} \mathcal{X}_N$ and $\bigcup_{N\in\mathbb{N}} \mathcal{U}_N$ are dense in $\mathcal{X}$ and $\mathcal{U}$, respectively. In particular, we get the following corollary.

Corollary 4.10. *There exists $C > 0$ and $N_0 \in \mathbb{N}$ such that, for each tuple $(\xi, x, u) \in \mathcal{F}_0$ and each $N \in \mathbb{N}$, $N \geq N_0$, we can find a pair $(x_N, u_N) \in \mathcal{X}_N \times \mathcal{U}_N$ satisfying the following conditions:*

(1) $\|u - u_N\|_{L^2} \leq C\dfrac{1}{\sqrt{N}}\,\mathrm{V}(u)$,

(2) $\|x - x_N\|_{\mathcal{X}} \leq C\dfrac{1}{\sqrt{N}}\big(\|\xi\|_2 + \|u\|_{L^2} + 1\big)$,

where $\mathrm{V}(\cdot)$ is as defined in Equation (4.5), and $\mathcal{F}_0$ is the feasible set of Problem P_0, as defined in Equation (4.35).

Proof. Since $\mathcal{U} \subset L^2([-1,1],\mathbb{R}^m) \cap BV([-1,1],\mathbb{R}^m)$, then by Lemma 4.9 there exists $u_N \in \mathcal{U}_N$ that satisfies the Condition (1).

Now, Lemma 4.4 indicates that $\dot{x} \in BV([-1,1],\mathbb{R}^n)$. Hence, again using Lemma 4.9, for each $N \geq N_0$ there exists $C' > 0$ and $p \in \mathcal{X}_{N-1}$ such that

$$\|\dot{x} - p\|_{L^2} \leq C' \frac{1}{\sqrt{N}} \mathrm{V}(x)\,. \tag{4.51}$$

Let $x_N \in \mathcal{X}_N$ defined by

$$x_N(t) = \xi + \int_{-1}^{t} p(s)ds, \tag{4.52}$$

where it is important to note that x_N is indeed a polynomial or order at most N. Then, there exists $C' > 0$ such that for each $t \in [-1,1]$,

$$\|x(t) - x_N(t)\|_2 \leq \int_{-1}^{1} \|\dot{x}(s) - p(s)\|_2 \, ds \leq C' \frac{1}{\sqrt{N}} \mathrm{V}(x)\,, \tag{4.53}$$

where the last inequality follows by Hölder's Inequality. The conclusion follows from Equations (4.51) and (4.53), the definition of the $\mathcal{X}$-norm in Equation (4.15), and Condition 2 in Lemma 4.8. □

The following lemma states that given an element in the feasible set of Problem P_0, we can find an element in the feasible set of Problem $P_{N,\delta}$ for N large enough.

Lemma 4.11. *For each $N \in \mathbb{N}$, let $\{t_{N,k}\}_{k=0}^{N}$ be the set of either Legendre–Gauss or Legendre–Gauss–Lobatto collocation nodes, as defined in Equations (4.27) and (4.31), respectively. Also, let $\{\omega_{N,k}\}_{k=0}^{N}$ be its associated set of Gauss quadrature coefficients.*

Then there exists $C > 0$ and $N_0 \in \mathbb{N}$ such that, given $(\xi, x, u) \in \mathcal{F}_0$, $N \geq N_0$, and $\delta > 0$, we can find $(x_N, u_N) \in \mathcal{X}_N \times \mathcal{U}_N$ satisfying:

(1) $\Big(\sum_{k=0}^{M} \omega_{M,k} \big\|\dot{x}_N(t_{M,k}) - h\big(x_N(t_{M,k}), u_N(t_{M,k})\big)\big\|_2^2\Big)^{\frac{1}{2}} \leq CN^{-\frac{\delta}{2}}\big(\|\xi\|_2 + \|u\|_{L^2} + \mathrm{V}(u) + 1\big)$,

(2) $f^j\big(x_N(t_{N,k})\big) \leq CN^{-\frac{1}{2}}\big(\|\xi\|_2 + \|u\|_{L^2} + 1\big), \quad \forall k \in \{0,\ldots,N\}, \quad \forall j \in \{1,\ldots,q\}$,

where $M = \lceil N^{4+\delta} \rceil$ and $\mathrm{V}(\cdot)$ is as defined in Equation (4.5).

Proof. Let $N \geq N_0$. Since x is feasible and f^j is Lipschitz continuous, as stated in Condition 4 in Assumption 4.6, there exists $C_1 > 0$ such that for each $j \in \{1,\ldots,q\}$ and $k \in \{0,\ldots,N\}$:

$$f^j\big(x_N(t_{N,k})\big) \leq \big|f^j\big(x(t_{N,k})\big) - f^j\big(x_N(t_{N,k})\big)\big| \leq C_1 \frac{1}{\sqrt{N}}\big(\|\xi\|_2 + \|u\|_{L^2} + 1\big), \tag{4.54}$$

where the last inequality follows by Condition 2 in Corollary 4.10.

Let $e_N\colon [-1,1] \to \mathbb{R}^n$ be defined by $e_N(t) = \dot{x}_N(t) - h\big(x_N(t), u_N(t)\big)$. Then, by Condition 1 in Assumption 4.2 together with Condition 2 in Corollary 4.10, there exists $C_2 > 0$ such that:

$$\begin{aligned}\|e_N\|_{L^2} &\leq \|\dot{x}_N - \dot{x}\|_{L^2} + \left(\int_{-1}^{1} \big\|h\big(x(s),u(s)\big) - h\big(x_N(s),u_N(s)\big)\big\|_2^2\, ds\right)^{\frac{1}{2}} \\ &\leq C_1 \frac{1}{\sqrt{N}}\big(\|\xi\|_2 + \|u\|_{L^2} + 1\big) + 2L\big(\|x - x_N\|_{L^2} + \|u - u_N\|_{L^2}\big) \\ &\leq C_2 \frac{1}{\sqrt{N}}\big(\|\xi\|_2 + \|u\|_{L^2} + \mathrm{V}(u) + 1\big).\end{aligned} \tag{4.55}$$

Now let $M \in \mathbb{N}$. Then, since the collocation nodes are either Legendre–Gauss or Legendre–Gauss–Lobatto, by Theorem 6.6.1 in [Fun92] we get that there exists $C_3 > 0$ such that:

$$\begin{aligned}\|e_N - I_M e_N\|_{L^2} &\leq C_3 \frac{1}{\sqrt{M}} \|\dot{e}_N\|_{L^2} \\ &\leq C_3 \frac{1}{\sqrt{M}} \left(\|\ddot{x}_N\|_{L^2} + \left\|\frac{dh}{dx}\right\|_{L^\infty} \|\dot{x}_N\|_{L^2} + \left\|\frac{dh}{du}\right\|_{L^\infty} \|\dot{u}_N\|_{L^2}\right).\end{aligned} \tag{4.56}$$

In order to obtain a bound for the inequality in Equation (4.56) note that, by Condition 2 in Corollary 4.10, Condition 1 in Assumption 4.2, and Lemma 4.8, there exists C_4 such that:

$$\begin{aligned}\|\dot{x}_N\|_{L^2} &\leq C_1 \frac{1}{\sqrt{N}}\big(\|\xi\|_2 + \|u\|_{L^2} + 1\big) + \|\dot{x}\|_{L^2} \\ &\leq C_4 \left(\frac{1}{\sqrt{N}} + 1\right)\big(\|\xi\|_2 + \|u\|_{L^2} + 1\big).\end{aligned} \tag{4.57}$$

Also, note that by Condition 1 in Assumption 4.2:

$$\left\|\frac{dh}{dx}\right\|_{L^\infty} \leq L, \quad \text{and} \quad \left\|\frac{dh}{du}\right\|_{L^\infty} \leq L, \tag{4.58}$$

Moreover, Theorem 6.3.2 in [Fun92] states that there exists $C_4 > 0$ such that for each polynomial p of order N:

$$\|\dot{p}\|_{L^2} \leq C_4 N^2 \|p\|_{L^2}. \tag{4.59}$$

Therefore, from Equations (4.57), (4.58), and (4.59), we get the following estimate:

$$\|e_N - I_M e_N\|_{L^2} \leq C_5 \frac{N^2}{\sqrt{M}}\big(\|\xi\|_2 + \|u\|_{L^2} + \mathrm{V}(u) + 1\big), \tag{4.60}$$

for some $C_5 > 0$. The proof is complete by noting that there exists $C_6 > 0$ such that:

$$\left(\sum_{k=0}^{M} \omega_{M,k} \|e_N(t_{M,k})\|_2^2\right)^{\frac{1}{2}} = \left(\sum_{k=0}^{M} \omega_{M,k} \|I_M e_N(t_{M,k})\|_2^2\right)^{\frac{1}{2}} \leq C_6 \|I_M e_N\|_{L^2}, \tag{4.61}$$

where the last inequality follows from Theorem 3.8.2 in [Fun92]. Therefore, using the bounds in Equations (4.55) and (4.60), we get that there exists $C_7 > 0$ such that:

$$\left(\sum_{k=0}^{M} \omega_{M,k} \left\| e_N(t_{M,k}) \right\|^2 \right)^{\frac{1}{2}} \leq C_7 \left(\frac{1}{\sqrt{N}} + \frac{N^2}{\sqrt{M}} \right), \tag{4.62}$$

thus, if we choose $M = \left\lceil N^{4+\delta} \right\rceil$ for any $\delta > 0$, we get the desired result. □

Now we can state the main result in this chapter, which is that global minimizers of any sequence of Problems $P_{N,\delta}$ as $N \to \infty$,

Theorem 4.12. *For each $N \in \mathbb{N}$, let $\{t_k\}_{k=0}^{N}$ be a set of either Legendre–Gauss or Legendre–Gauss–Lobatto collocation nodes, as defined in Equations* (4.27) *and* (4.31)*, respectively. Also, given $\delta > 0$, let $\{(\xi_N, x_N, u_N, \varepsilon_N)\}_{N\in\mathbb{N}}$ be a sequence of minimizers of Problems $\{P_{N,\delta}\}_{N\in\mathbb{N}}$, as defined in Equation* (4.37)*.*

Then any accumulation point of this sequence is of the form $\left(\hat{\xi}, \hat{x}, \hat{u}, 0\right) \in \mathbb{R}^n \times \mathcal{X} \times \mathcal{U} \times [0, \infty)$, where the tuple $\left(\hat{\xi}, \hat{x}, \hat{u}\right)$ is a minimizer of Problem P_0, as defined in Equation (4.35)*.*

Proof. Without loss of generality, let us assume that $(\xi_N, x_N, u_N, \varepsilon_N) \to \left(\hat{\xi}, \hat{x}, \hat{u}, \hat{\varepsilon}\right)$. We will first show that $\hat{\varepsilon} = 0$. Recall that $\hat{\varepsilon} \geq 0$ since, by definition of the constraints of Problem $P_{N,\delta}$, $\varepsilon_N \geq 0$ for each $N \in \mathbb{N}$.

Let us assume that $\hat{\varepsilon} > 0$. Let $\left(\tilde{\xi}, \tilde{x}, \tilde{u}\right) \in \mathbb{R}^n \times \mathcal{X} \times \mathcal{U}$ be a minimizer of Problem (P_0). Then there exists a sequence $\left\{\left(\bar{\xi}_N, \bar{x}_N, \bar{u}_N, \bar{\varepsilon}_N\right)\right\}_{N\in\mathbb{N}} \subset \mathcal{F}_{N,\delta}$, where $\mathcal{F}_{N,\delta}$ is the feasible set of Problem $P_{N,\delta}$, converging to $\left(\tilde{\xi}, \tilde{x}, \tilde{u}, 0\right)$. Indeed, for any $C', \delta' > 0$ there exists N_0 such that for each $N \geq N_0$:

$$\begin{aligned} C' N^{-\frac{\delta'}{2}} \left(\left\| \tilde{\xi} \right\|_2 + \left\| \tilde{u} \right\|_{L^2} + \mathrm{V}(\tilde{u}) + 1 \right) &\leq N^{-\frac{\delta'}{3}}, \\ C' N^{-\frac{1}{2}} \left(\left\| \tilde{\xi} \right\|_2 + \left\| \tilde{u} \right\|_{L^2} + 1 \right) &\leq N^{-\frac{1}{3}}. \end{aligned} \tag{4.63}$$

Hence, by Lemma 4.11, for $N \geq N_0$ we can choose $\left(\bar{\xi}_N, \bar{x}_N, \bar{u}_N\right)$ such that $\left(\bar{\xi}_N, \bar{x}_N, \bar{u}_N, 0\right) \in \mathcal{F}_{N,\delta}$.

Let $\alpha > 0$, and let $N_1 \in \mathbb{N}$, $N_1 \geq N_0$, such that for each $N \geq N_1$,

$$f^0\left(\bar{\xi}_N, \bar{x}_N(1)\right) \leq f^0\left(\tilde{\xi}, \tilde{x}(1)\right) + \alpha. \tag{4.64}$$

Also, let $N_2 \in \mathbb{N}$, $N_2 \geq N_1$, such that for each $N \geq N_2$

$$f^0\left(\hat{\xi}, \hat{x}(1)\right) + \pi_N \hat{\varepsilon} \geq f^0\left(\tilde{\xi}, \tilde{x}(1)\right) + 4\alpha. \tag{4.65}$$

Since the map $(\xi, x) \mapsto f^0\left(\xi, x(1)\right)$ is continuous with the topology of $\mathbb{R}^n \times \mathcal{X}$, there exists a neighborhood $\mathcal{V} \subset \mathbb{R}^n \times \mathcal{X} \times \mathcal{U} \times [0, \infty)$ of $\left(\hat{\xi}, \hat{x}, \hat{u}, \hat{\varepsilon}\right)$ such that for each tuple $(\xi, x, u, \varepsilon) \in \mathcal{V}$

$$f^0\left(\xi, x(1)\right) + \pi_N \varepsilon \geq f^0\left(\tilde{\xi}, \tilde{x}(1)\right) + 2\alpha. \tag{4.66}$$

But we assumed that $(\xi_N, x_N, u_N, \varepsilon_N) \to \big(\hat{\xi}, \hat{x}, \hat{u}, \hat{\varepsilon}\big)$, hence there exists $N_3 \in \mathbb{N}$, $N_3 \geq N_2$, such that $(\xi_N, x_N, u_N, \varepsilon_N) \in \mathcal{V}$ for each $N \geq N_3$. Thus:

$$f^0\big(\xi_N, x_N(1)\big) + \pi_N \hat{\varepsilon} \geq f^0\big(\tilde{\xi}, \tilde{x}(1)\big) + 2\alpha > f^0\big(\bar{\xi}_N, \bar{x}_N(1)\big), \tag{4.67}$$

which contradicts the fact that $(\xi_N, x_N, u_N, \varepsilon_N)$ is a minimizer of Problem $P_{N,\delta}$. Therefore $\hat{\varepsilon} = 0$.

Now we prove that $\big(\hat{\xi}, \hat{x}, \hat{u}\big)$ is a minimizer of Problem P_0. Suppose not, then:

$$f^0\big(\tilde{\xi}, \tilde{x}(1)\big) < f^0\big(\hat{\xi}, \hat{x}(1)\big), \tag{4.68}$$

and therefore, again by continuity of the map $(\xi, x) \mapsto f^0\big(\xi, x(1)\big)$, there exists $N_4 \in \mathbb{N}$, $N_4 \geq N_3$, such that:

$$f^0\big(\bar{\xi}_N, \bar{x}_N(1)\big) < f^0\big(\xi_N, x_N(1)\big). \tag{4.69}$$

Recall that $\bar{\varepsilon}_N = 0$ for each $N \geq N_4$. Since $(\xi_N, x_N, u_N, \varepsilon_N)$ is a minimizer of Problem $P_{N,\delta}$, then the following inequalities are satisfied:

$$f^0\big(\bar{\xi}_N, \bar{x}_N(1)\big) < f^0\big(\xi_N, x_N(1)\big) + \pi_N \varepsilon_N \leq f^0\big(\bar{\xi}_N, \bar{x}_N(1)\big), \tag{4.70}$$

but this is clearly a contradiction, hence proving the theorem. □

Chapter 5

Conclusion

We have presented three results in the intersection of hybrid dynamical systems and optimal control, with applications to Cyber–Physical Systems. The first result, in Chapter 2, is an algorithm that computes the optimal control of switched hybrid dynamical systems with state constraints. The second result, in Chapter 3, is an algorithm that simulates autonomous hybrid dynamical systems using a novel relaxation technique. The third result, in Chapter 4, is an algorithm that computes the optimal control of nonlinear dynamical systems with states constraints using pseudospectral approximations. These algorithms where developed motivated by the some difficulties faced by Cyber–Physical applications, such as the problem of generalizing dynamical models to account for both continuous and discrete state variables, and the problem of improving the speed of computation so that large–scale problems can be addressed.

Each of these results opens the doors to many research possibilities. First of all, we have not tested any of these algorithms using experimental data. Algorithms tend to perform better with simulated data than with experimental data, hence using them in real–life applications is fundamental to understand their weaknesses. In particular, the algorithm in Chapter 2 has many promising properties, as shown in Section 2.6, that make us think it will be applicable to a wide range of applications. Second, in Chapter 3 we construct a hybrid system metric, i.e. a metric that takes into account mode transitions to measure distances within a hybrid dynamical system, which has the potential to solve many long standing problems in hybrid system theory. Our result uses this hybrid system metric to prove that our simulations converge to real trajectories, but at the same time it can be used to answer questions about the topology of the hybrid system, such as finding the families of trajectories that are not orbitally stable, or showing that some trajectories are Zeno. Also, we expect to extend our result on Chapter 3 to hybrid systems with controlled inputs, both continuous and discrete. Third, as explained in Chapter 4, our result in pseudospectral approximations is just a first step towards finding an numerical method for the computation of optimal control, based on pseudospectral methods, that can be provably convergent for practical problems. This is very relevant for real–life applications, since using this new algorithm will allow us to implement exciting experiments solving optimal control problems in real–time.

Finally, we firmly believe that addressing the challenges faced today in the context of Cyber–Physical Systems will produce dramatic improvements to our society. The incorporation of new distributed sources of energy to the electric power grid, autonomous cars that guarantee our safety while increasing their performance, and the development of prosthetic devices, are just a few of the applications where control theory for Cyber–Physical Systems plays a fundamental part. The results presented in this thesis are a small step towards undertaking these huge problems, nevertheless we are certain that they are steps in the right direction, and we hope that they help inspire even better results in the thriving Cyber–Physical research community.

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